VARIATIONS OF MIXED HODGE STRUCTURE ARISING FROM CUBIC HYPEREQUISINGULAR FAMILIES OF COMPLEX PROJECTIVE VARIETIES, II

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Introduction

This is the continuation of our previous paper [8]. We inherit the notation and terminology of it. In [8] we have introduced the notion of *cubic hyperequisingular families of complex projective varieties* ([8, Defintion 2.1]) and proved, for these families, the cohomological descent of R-module sheaves (R:a commutative ring) and relative de Rham complexes ([8, Theorem 2.7, Theorem 2.9, Theorem 2.10, Corollary 2.11]). The purpose of this paper is, using these results, to prove the following theorem.

Main Theorem. *Let $\mathfrak{X}_\cdot \xrightarrow{a_\cdot} \mathfrak{X} \xrightarrow{\pi} M$ be a cubic hyperequisingular family of complex projective varieties, parametrized by a complex manifold M. We define $R^\ell_{\mathbb{Z}}(\pi) := R^\ell\pi_*\mathbb{Z}_{\mathfrak{X}}$ modulo torsion $(0 \leq \ell \leq 2(dim\mathfrak{X}\text{-}dimM))$, $R^\ell_{\mathbb{Q}}(\pi) := R^\ell_{\mathbb{Z}}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $R^\ell_{\mathcal{O}}(\pi) := R^\ell\pi_*(\pi^\cdot\mathcal{O}_M) \simeq \mathbb{R}^\ell\pi_*(DR^\cdot_{\mathfrak{X}/M})$, where $\pi^\cdot\mathcal{O}_M$ is the topological inverse of the structure sheaf of M by the map $\pi : \mathfrak{X} \to M$ and $DR^\cdot_{\mathfrak{X}/M}$ the cohomological relative de Rham complex of the family $\pi : \mathfrak{X} \to M$. Then there exist a family of increasing sub-local systems $\mathbb{W}$ (weight filteration) on $R^\ell_{\mathbb{Q}}(\pi)$ and a family of decreasing holomorphic subbundles $\mathbb{F}$ (Hodge filteration) on $R^\ell_{\mathcal{O}}(\pi)$ such that $\{R^\ell_{\mathbb{Z}}(\pi), (R^\ell_{\mathbb{Q}}(\pi), \mathbb{W}), (R^\ell_{\mathcal{O}}(\pi), \mathbb{W}, \mathbb{F})\}$ is a variation of mixed Hodge structure (For definition see Definition 3.3 below).*

The proof will proceed in two steps. In paragraph 3.1 we shall prove the existence of weight filteration $\mathbb{W}$ and Hodge filteration $\mathbb{F}$ on $R^\ell_{\mathbb{Q}}(\pi)$ and $R^\ell_{\mathcal{O}}(\pi)$, respectively, and in paragraph 3.2 we shall prove *"Griffiths Transversality"*. Now let us explain the outline of the proofs.

By Theorem 2.9 and Theorem 2.10 in [8], we have an isomorphism

$$\pi^\cdot\mathcal{O}_M \approx DR^\cdot_{\mathfrak{X}/M} \approx s(a_{1\cdot *}\Omega^\cdot_{\mathfrak{X}_\cdot/M})[1]$$

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in $D^+(\mathfrak{X},\mathbb{C})$, the derived category of lower bounded complexes of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{X}$, where $\Omega^{\cdot}_{\mathfrak{X}./M}$ is the relative de Rham complex of the n-cubic object $\mathfrak{X}. \xrightarrow{\pi.:=\pi\cdot a.} M$ of smooth families of complex manifolds, parametrized by M, $\mathfrak{X}. \xrightarrow{a_1.} \mathfrak{X}\times\square_n \xrightarrow{a_2.} \mathfrak{X}$ is the natural factorization of $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X}$ (cf. [8, (1.6) and (1.9)]), and $s(a_{1\cdot *}\Omega^{\cdot}_{\mathfrak{X}./M})$ is the simple complex associated to the n-cubic object $a_{1\cdot *}\Omega^{\cdot}_{\mathfrak{X}./M}$ of sheaves of $\mathbb{C}$-vector spaces ([8, Definition 1.18]). By this ismorphism we have

$$R^{\ell}_{\mathcal{O}}(\pi) := R^{\ell}\pi_*(\pi^{\cdot}\mathcal{O}_M) \simeq \mathbb{R}^{\ell}\pi_*(s(a_{1\cdot *}\Omega^{\cdot}_{\mathfrak{X}./M})[1]).$$

To compute the hyper-direct image $\mathbb{R}^{\ell}\pi_*(s(a_{1\cdot *}\Omega^{\cdot}_{\mathfrak{X}./M})[1])$, we shall use the fine resolution $\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}./M}$ of $\Omega^{\cdot}_{\mathfrak{X}./M}$, where $\mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M}$ are the sheaves of C^∞ relative differential forms of type (r,s) on $\mathfrak{X}_\alpha$ $(\alpha\in\square_n)$. Then the natural homomorphism

$$s(a_{1\cdot *}\Omega^{\cdot}_{\mathfrak{X}./M})[1] \to s(a_{.1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}./M})[1]$$

is an isomorphism in $D^+(\mathfrak{X},\mathbb{C})$, where $tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\alpha/M}$ is the simple complex associated to the double complex $\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\alpha/M}$ for each $\alpha\in\square_n$; and $s(a_{.1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}./M})[1]$ is π_*-acyclic. Hence we have

$$R^{\ell}_{\mathcal{O}}(\pi) \simeq H^{\ell}(\pi_* s(a_{.1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}./M})[1]).$$

The simple complex $s(a_{.1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}./M})[1]$, and so $\pi_*(s(a_{.1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}./M})[1])$, has naturally two filterations W and F, where W is defined over $\mathbb{Q}$. We shall caluculate the spectral sequence asoociated to these filterations, abuting to $R^{\ell}_{\mathcal{O}}(\pi)$, and prove that the specral sequence associated to W degenerates at E_2-term and the one associated to F degenerates at E_1-term. From these the existence of weight filteration $\mathbb{W}$ and Hodge filteration $\mathbb{F}$ on $R^{\ell}_{\mathbb{Q}}(\pi)$ and $R^{\ell}_{\mathcal{O}}(\pi)$, respectively, follows (Theorem 3.5).

To prove the Griffiths transversality, we shall use a system of very special Stein coverings $\{\mathcal{V}_\alpha\}_{\alpha\in\square_n^+}$ of $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X}$, where $\mathfrak{V}_\alpha := \{\mathcal{V}^{(\alpha)}_i\}_{i\in\Lambda_\alpha}$ is a Stein covering of $\mathfrak{X}_\alpha$ for each $\alpha\in\square_n^+$, subject to some "analytically trivial condition" (cf. [8, (2.19)]). The existence of such a system $\{\mathcal{V}_\alpha\}_{\alpha\in\square_n^+}$ is guaranteed by the assumption that the family $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X}$ is "analytically locally trivial". We take the Cěch resolution $C^{\cdot}(\mathcal{V}_\alpha,\Omega^{\cdot}_{\mathfrak{X}_\alpha/M})$ of the complex $\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}$ with respect to the covering $\mathcal{V}_\alpha$ for each $\alpha\in\square_n$. Then the natural homomorphism

$$s(a_{.1*}\Omega^{\cdot}_{\mathfrak{X}./M})[1] \to s(a_{.1*}tot\,C^{\cdot}(\mathcal{V}.,\Omega^{\cdot}_{\mathfrak{X}./M}))[1]$$

is an isomorphism in $D^+(\mathfrak{X},\mathbb{C})$; and
$s(a_{.1*}tot\,C^{\cdot}(\mathcal{V}.,\Omega^{\cdot}_{\mathfrak{X}./M}))[1]$ is π_*-acyclic. Hence we have

$$R^{\ell}_{\mathcal{O}}(\pi) \simeq H^{\ell}(\pi_* s(a_{.1*}tot\,C^{\cdot}(\mathcal{V},\Omega^{\cdot}_{\mathfrak{X}./M}))[1])$$

By use of this isomorhism, following the method of Katz and Oda in [6], we shall calculate the Gauss-Mannin connection ∇ on $R^{\ell}_{\mathcal{O}}(\pi)$. From this the Griffiths transversality follows. We should mention that the assumption that the family $\mathfrak{X}_{\cdot} \xrightarrow{a_{\cdot}} \mathfrak{X} \xrightarrow{\pi} M$ is analytically locally trivial is crucial so that this procedure can be carried out in our arguments.

§3 Variations of mixed Hodge structure

3.1 Weight and Hodge filterations on the higher direct image sheaf of cohomology

We begin with recalling the definitions of *Hodge structure, mixed Hodge structure* and *variations of mixed Hodge structure*. Let A be equal to $\mathbb{Z}$ or $\mathbb{Q}$. For an A-module H_A, the complex conjugation can be defined on $H_{\mathbb{C}} := \mathbb{C} \otimes_A H_A$. A filteration $F = \{F^p\}$ of $H_{\mathbb{C}}$ by $\mathbb{C}$-vector subspaces admits its conjugate filteration $\overline{F}$ such that $(\overline{F})^p = \overline{F^p}$.

3.1 Definition. An A–Hodge structure of weight ℓ consists of

(i) an A–module of finite type H_A, and

(ii) a finite decreasing filteration $F = \{F^p\}$ of $H_{\mathbb{C}}$ by $\mathbb{C}$-vector subspaces (Hodge filteration) such that $Gr_F^p Gr_{\overline{F}}^q (H_{\mathbb{C}}) = 0$ for $p + q \neq \ell$.

The relation above implies that the subspaces

$$H^{p,q} := F^p \cap \overline{F^q}$$

give a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=\ell} H^{p,q}, \qquad H^{p,q} = \overline{H^{q,p}}.$$

3.2 Definition. A mixed Hodge structure consists of

(i) a free $\mathbb{Z}$-module of finite type $H_{\mathbb{Z}}$,

(ii) a finite increasing filteration $W = \{W_q\}$ of $H_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ by $\mathbb{Q}$-vector subspaces (weight filteration), and

(iii) a finite decreasing filteration $F = \{F^p\}$ of $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ by $\mathbb{C}$-vector subspaces (Hodge filteration), satisfying the condition that

$$Gr_q^W(H) := (Gr_q^W(H_{\mathbb{Q}}), Gr_q^W(H_{\mathbb{C}}), F)$$

forms a $\mathbb{Q}$-Hodge structure of weight q for every q.

3.3 Definition. A variation of mixed Hodge structure on a complex manifold M consists of

(i) a local system $V^{\mathbb{Z}}$ of free $\mathbb{Z}$–module of finite type on M,

(ii) a finite increasing filteration $\mathbb{W} = \{W_q\}$ of $V^{\mathbb{Q}} := V^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ by sub-local systems of $\mathbb{Q}$–vector spaces, and

(iii) a finite decreasing filteration $\mathbb{F} = \{\mathcal{F}^p\}$ of $\mathcal{V} := V^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_M$ by holomorphic subbundles, satisfying

(a)(Griffiths transversality)

$\nabla \mathcal{F}^p \subset \Omega_M^1 \otimes \mathcal{F}^{p-1}$,

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where ∇ denotes the Gauss-Mannin connection on $\mathcal{V}$, and

(b) $(V^{\mathbb{Z}}, \mathbb{W}, \mathbb{F})$ defines a mixed Hodge structuer at each point $t \in M$.

Let $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$ be a cubic hyperequisingular family of complex projective varieties, parametrized by a complex manifold M. We denote by $R^{\ell}_{\mathbb{Z}}(\pi)$ the ℓ-th higher direct image sheaf of the constant sheaf $\mathbb{Z}$ on $\mathfrak{X}$. We are now going to define a finite increasing filteration $\mathbb{W} = \{W_q\}$ on $R^{\ell}_{\mathbb{Q}}(\pi) := R^{\ell}_{\mathbb{Z}}(\pi) \otimes \mathbb{Q}$ and a finite decreasing filteration $\mathcal{F} = \{\mathcal{F}^p\}$ on $R^{\ell}_{\mathcal{O}}(\pi) := R^{\ell}\pi_*(\pi^{\cdot}\mathcal{O}_M)$, where $\pi^{\cdot}\mathcal{O}_M$ denotes the topological inverse of the structure sheaf of M: First let us notice that

$$
\begin{aligned}
R^{\ell}_{\mathbb{Q}}(\pi) &\simeq R^{\ell}\pi_*(\mathbb{Q}_{\mathfrak{X}}) \\
&\simeq \mathbb{R}^{\ell}\pi_*(\mathbb{R}a._*(\mathbb{Q}_{\mathfrak{X}.})) \ \ \text{(by [8, Theorem 2.7])} \\
&\simeq \mathbb{R}^{\ell}\pi_*(\mathbb{R}a_{2*}(\mathbb{R}a_{\cdot 1*}(\mathbb{Q}_{\mathfrak{X}.})) \ \ \text{(by [5, Corollary 5.13])} \\
&\simeq \mathbb{R}^{\ell}\pi_*(s(a_{\cdot 1*}\mathbb{Q}_{\mathfrak{X}.})[1]) \ \ \text{(by [8, (1.10)])}
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
R^{\ell}_{\mathcal{O}}(\pi) &\simeq \mathbb{R}^{\ell}\pi_*(\mathbb{R}a._*(\pi^{\cdot}_{\cdot}\mathcal{O}_M)) \ \ \text{(by [8, Theorem 2.5])} \\
&\simeq \mathbb{R}^{\ell}\pi_*(\mathbb{R}a._*(\Omega^{\cdot}_{\mathfrak{X}./M})) \\
&\simeq \mathbb{R}^{\ell}\pi_*(\mathbb{R}a_{\cdot 2*}(\mathbb{R}a_{\cdot 1*}(\Omega^{\cdot}_{\mathfrak{X}./M})) \\
&\simeq \mathbb{R}^{\ell}\pi_*(s(a_{\cdot 1*}\Omega^{\cdot}_{\mathfrak{X}./M}[1]),
\end{aligned} \tag{3.2}
$$

where $\pi_\alpha := \pi \cdot a_\alpha$ ($\alpha \in \square_n$, n=the length of the cubic hyperresolution $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X}$) and the second isomorphism in (3.2) comes from that $\pi^{\cdot}_{\alpha}\mathcal{O}_M \to \Omega^{\cdot}_{\mathfrak{X}_\alpha/M}$ is a quasi-isomorphism for every $\alpha \in \square_n$ (reference [2] in [8], p.15, 2.23.2).

3.4 Remark. Since $\pi^{\cdot}\mathcal{O}_M$ has an injective resolution in the category of $\pi^{\cdot}\mathcal{O}_M$-modules, $R^{\ell}_{\mathcal{O}}(\pi) := R^{\ell}\pi_*(\pi^{\cdot}\mathcal{O}_M)$ has naturally the structure of $\mathcal{O}_M$-modules. Furhtermore, we claim that it is isomorphic to $(R^{\ell}\pi_*\mathbb{C}_{\mathfrak{X}}) \otimes_{\mathbb{C}} \mathcal{O}_M$ as $\mathcal{O}_M$-modules: to prove this, since there is naturally a $\mathcal{O}_M$-module homomorphism $(R^{\ell}\pi_*\mathbb{C}_{\mathfrak{X}}) \otimes \mathcal{O}_M \to R^{\ell}\pi_*\pi^{\cdot}\mathcal{O}_M$, it suffices to show that

$$(R^{\ell}\pi_*\mathbb{C}_{\mathfrak{X}})_t \otimes \mathcal{O}_{M,t} \to (R^{\ell}\pi_*\pi^{\cdot}\mathcal{O}_M)_t \tag{3.3}$$

is an isomorphism for any $t \in M$. While, we have

$$(R^{\ell}\pi_*\pi^{\cdot}\mathcal{O}_M)_t \simeq H^{\ell}(\pi^{-1}(t), \pi^{\cdot}\mathcal{O}_M) \tag{3.4}$$

for any $t \in M$, since $\pi : \mathfrak{X} \to M$ is proper ([6, p.176, Theorem 6.2]). Furthermore,

$$H^{\ell}(\pi^{-1}(t), \pi^{\cdot}\mathcal{O}_M) \simeq H^{\ell}(\pi^{-1}(t), \mathbb{C}) \otimes \mathcal{O}_{M,t}, \tag{3.5}$$

since $\pi^{\cdot}\mathcal{O}_M$ equals to the constant sheaf $\mathcal{O}_{M,t}$ on $\pi^{-1}(t)$; and

$$H^\ell(\pi^{-1}(t),\mathbb{C})\otimes\mathcal{O}_{M,t}\simeq(R^\ell\pi_*\mathbb{C})_t\otimes\mathcal{O}_{M,t}, \tag{3.6}$$

since the family $\pi:\mathfrak{X}\to M$ is topologically "locally trivial" over M ([6, p.205, Theorem 1.6]). Then (3.3) follows from (3.4), (3.5) and (3.6).

We define an increasing filteration $\mathbb{W}=\{W_{-q}\}$ on the simple complex $s(a_{\cdot 1*}\mathbb{Q}_{\mathfrak{X}_\cdot})[1]$ by

$$W_{-q}(s(a_{\cdot 1*}\mathbb{Q}_{\mathfrak{X}_\cdot})[1]):=\sigma_{|\alpha|\geq q+1}s(a_{\alpha 1*}\mathbb{Q}_{\mathfrak{X}_\alpha})\ \ (q\geq 0), \tag{3.7}$$

where $\sigma_{|\alpha|\geq q+1}s(a_{\alpha 1*}\mathbb{Q}_{\mathfrak{X}_\alpha}):=\sigma_{\geq q}(K)$ if we put $K=s(a_{\cdot 1*}\mathbb{Q}_{\mathfrak{X}_\cdot})[1]$. In general, the subcomplex $\sigma_{\geq q}(K)$ of a complex K is defined as follows:

$$\sigma_{\geq q}(K)^n=\begin{cases}0 & \text{if } n<q\\ K^n & \text{if } n\geq p\end{cases}$$

The filteration of K defined by these subcomplexes is called *stupid filteration*. Using this filteration, we define an increasing filteration on $R^\ell_{\mathbb{Q}}(\pi)$ by

$$W_{-q}(R^\ell_{\mathbb{Q}}(\pi)):=Im\{\mathbb{R}^\ell\pi_a st(W_{-q}(s(a_{\cdot 1*}\mathbb{Q}_{\mathfrak{X}}\cdot)))\to\mathbb{R}^\ell\pi_*(s(a_{\cdot 1*}\mathbb{Q}_{\mathfrak{X}}\cdot))\}. \tag{3.8}$$

Here we identify $R^\ell_{\mathbb{Q}}(\pi)$ with $\mathbb{R}^\ell\pi_*(s(a_{\cdot 1*}\mathbb{Q}_{\mathfrak{X}_\cdot}))$ by the isomorphism in (3.1). Next we define an increasing filteration $\mathbb{W}=\{W_q\}$ and a decreasing one $\mathbb{F}=\{F_q\}$ on the simple complex $s(a_{\cdot 1*}\Omega^{\cdot}_{\mathfrak{X}_\cdot/M}[1])$ by

$$\begin{aligned}&W_{-q}(s(a_{\cdot 1*}\Omega^{\cdot}_{\mathfrak{X}_\cdot/M})[1]):=\sigma_{|\alpha|\geq q+1}s(a_{\alpha 1*}\Omega^{\cdot}_{\mathfrak{X}_\alpha/M})\ \ (q\geq 0)\ \text{ and}\\ &F^p(s(a_{\cdot 1*}\Omega^{\cdot}_{\mathfrak{X}_\cdot/M})[1]):=\sigma_{k\geq p}s(a_{\cdot 1*}\Omega^k_{\mathfrak{X}_\cdot/M})[1]\ \ (p\geq 0)\end{aligned} \tag{3.9}$$

Using these filterations, we define an increasing filteration and a decreasing one on $R^\ell_{\mathcal{O}}(\pi)$ as follows:

$$\begin{aligned}&W_{-q}(R^\ell_{\mathcal{O}}(\pi))\\ &:=Im\{\mathbb{R}^\ell\pi_*(W_{-q}s(a_{\cdot 1*}\Omega^{\cdot}_{\mathfrak{X}_\cdot/M})[1]))\to\mathbb{R}^\ell\pi_*(s(a_{\cdot 1*}\Omega^{\cdot}_{\mathfrak{X}_\cdot/M})[1])\},\end{aligned} \tag{3.10}$$

(3.11)

$$F^p(R^\ell_{\mathcal{O}}(\pi)):=Im\{\mathbb{R}^\ell\pi_*(F^p(s(a_{\cdot 1*}\Omega_{\mathfrak{X}_\cdot/M})[1]))\to\mathbb{R}\pi_*(s(a_{\cdot 1*}\Omega_{\mathfrak{X}_\cdot/M})[1])\}$$

Using the same letter as for $\mathbb{R}^\ell_{\mathbb{Q}}(\pi)$ as to the filteration $\mathbb{W}$ is justified by the fact that the filteration $\mathbb{W}$ on $R^\ell_{\mathbb{Q}}(\pi)$ is mapped to $\mathbb{W}$ on $R^\ell_{\mathcal{O}}(\pi)$ through the isomorphism $R^\ell_{\mathbb{Q}}(\pi)\otimes_{\mathbb{Q}}\mathcal{O}_M\simeq R^\ell_{\mathcal{O}}(\pi)$.

3.5 Theorem. *Let $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$ be a cubic hyperequisingular family of complex projective varieties, parametrized by a complex manifold M (For definition see [8, Definition 2.4]). Then, with the same notation as above, there exist finite increasing filteration $\mathbb{W} = \{W_{-q}\}$ of $R^\ell_{\mathbb{Q}}(\pi)(0 \leq \ell \leq 2(dim\mathfrak{X}\text{-}dimM))$ by sub-local systems of $\mathbb{Q}$-vector spaces and a finite decreasing filteration $\mathbb{F} = \{F^p\}$ of $R^\ell_{\mathcal{O}}(\pi)$ by holomorphic subbundles such that $(R^\ell_{\mathbb{Z}}(\pi), \mathbb{W}[\ell], \mathbb{F})$ defines a mixed Hodge structure at each point $t \in M$, where $\mathbb{W}[\ell]_q$ denotes the shift of the filteration degree to the right by ℓ, i.e., $\mathbb{W}[\ell]_q := \mathbb{W}_{q-\ell}$. Furthermore, there are spectral sequences*

$$
{}_W E_1^{p,q} \simeq \oplus_{|\alpha|=p+1} R^q \pi_{\alpha *} \mathbb{Q}_{\mathfrak{X}_\alpha/M} \Rightarrow_W E_\infty^{p,q} = Gr^W_{-p}(R^{p+q}_{\mathbb{Q}}(\pi)),
$$
$$
{}_F E_1^{p,q} = \mathbb{R}^q \pi_*(s(a_{.1*}\Omega^p_{\mathfrak{X}./M})[1]) \Rightarrow_F E_\infty^{p,q} = Gr^p_F(R^{p+q}_{\mathcal{O}}(\pi))
$$

with ${}_W E_2^{p,q} =_W E_\infty^{p,q}$, ${}_F E_1^{p,q} =_F E_\infty^{p,q}$.

The proof of the theorem will be accomplished after several lemmas. By (3.1) we have isomorphisms

$$
R^\ell_{\mathbb{Q}}(\pi) := R^\ell \pi_* \mathbb{Q}_{\mathfrak{X}} \simeq \mathbb{R}^\ell \pi_*(s(a_{.1*}\mathbb{Q}_{\mathfrak{X}.})[1] \quad \text{and}
$$
$$
H^\ell(X_t, \mathbb{Q}) \simeq \mathbb{H}^\ell(X_t, s(a_{t\cdot *}\mathbb{Q}_{X_t.})[1]) \quad \text{for any } t \in M
$$

To compute $\mathbb{R}^\ell\pi_*(s(a_{.1*}\mathbb{Q}_{\mathfrak{X}.}))$ and $\mathbb{H}^\ell(X_t, s(a_{t\cdot *}\mathbb{Q}_{X_t.}))$, we take the canonical resolution $C^\cdot(\mathbb{Q}_{\mathfrak{X}_\alpha})$ of $\mathbb{Q}_{\mathfrak{X}_\alpha}$ for each $\alpha \in \square_n$ and put

$$
K_1 := s(a_{.1*}C^\cdot(\mathbb{Q}_{\mathfrak{X}.}))[1]),
$$
$$
K := \pi_* K_1, \quad \text{and}
$$
$$
K_t := s(a_{t\cdot 1*}C^\cdot(\mathbb{Q}_{X_t.}))[1] \quad (t \in M).
$$

3.6 Lemma. $\quad R^\ell_{\mathbb{Q}}(\pi) \simeq H^\ell(K)$

Proof. It suffices to show that K_1 is a π_*-acyclic, i.e., $R^p\pi_*(K_1^q) = 0$ for $p \geq 1, q \geq 0$, because

$$
\mathbb{Q}_{\mathfrak{X}} \longrightarrow s(a_{.1*}\mathbb{Q}_{\mathfrak{X}.})[1] \rightarrow s(a_{.1*}C^\cdot(\mathbb{Q}_{\mathfrak{X}.}))[1]
$$

are isomorphisms in $D^+(\mathfrak{X}, \mathbb{Q})$. Since

$$
K_1^q = \otimes_{|\alpha|+k=q+1} a_{\alpha 1 *} C^k(\mathbb{Q}_{\mathfrak{X}_\alpha}),
$$

we have

$$
R^p\pi_*(K_1^q) = \otimes_{|\alpha|+k=q+1} R^p \pi * (a_{\alpha 1*} C^k(\mathbb{Q}_{\mathfrak{X}_\alpha})).
$$

Since the direct images of flabby sheaves are also flabby, $a_{\alpha 1*}C^k(\mathbb{Q}_{\mathfrak{X}_\alpha})$ are flabby. Hence $R^p\pi_*(a_{\alpha 1*}C^k(\mathbb{Q}_{\mathfrak{X}_\alpha}) = 0$ for all p, α, k with $p \geq 1, |\alpha| + k = q+1$, and so $R^p\pi_a st(K_1^q) = 0$ for $p \geq 1, q \geq 0$ as requied.

Q.E.D.

We denote by $\mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M}$ ($\alpha \in \square_n$) the sheaf of C^∞ relative differential forms of type (r,s) on $\mathfrak{X}_\alpha$ and by $tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\alpha/M}$ the simple complex associated to this double complex. Then the natural map

$$s(a_{\cdot 1*}\Omega^{\cdot}_{\mathfrak{X}_\cdot/M})[1] \to s(a_{\cdot 1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\cdot/M})[1] \tag{3.12}$$

is an isomorphism in $D^+(\mathfrak{X}, \mathbb{C})$. Hence by (3.1) and (3.12) we have isomorphisms

$$\begin{aligned} R^\ell_{\mathcal{O}}(\pi) &\simeq \mathbb{R}^\ell\pi_* s(a_{\cdot 1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\cdot/M})[1], \quad \text{and} \\ H^\ell(X_t, \mathbb{C}) &\simeq \mathbb{H}^\ell(X_t, s(a_{t\cdot 1*}tot\,\mathcal{A}^{\cdot\cdot}_{X_{t\cdot}})[1]) \end{aligned} \tag{3.13}$$

The latter is obtained from the former by putting $M = \{t\}$ (one point). We put

$$\begin{aligned} &L_1 := s(a_{\cdot 1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\cdot/M})[1]), \\ &L := \pi_* L_1, \qquad \text{and} \\ &L_t := s(a_{t\cdot *}tot\,\mathcal{A}^{\cdot\cdot}_{X_{t\cdot}})[1] \quad (t \in M). \end{aligned}$$

3.7 Lemma. $\quad R^\ell_{\mathcal{O}}(\pi) \simeq H^\ell(L)$

Proof. By (3.13) it sufices to show that L is π_*-acyclic. Since

$$L_1^q = \bigoplus_{|\alpha|+r+s=q+1} a_{\alpha 1*}\mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M},$$

we have

$$R^p\pi_*(L_1^q) = \bigoplus_{|\alpha|+r+s=q+1} R^p\pi_*(a_{\alpha 1*}\mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M}).$$

Since the direct images of fine sheaves are also fine, $a_{\alpha*}\mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M}$ are fine. Hence $R^p\pi_*(a_{\alpha*}\mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M}) = 0$ for all p, α, r, s with $p \geq 1, |\alpha| + r + s = q+1$, and so $R^p\pi_*(L_1^q) = 0$ for $p \geq 1, q \geq 0$.

Q.E.D.

The increasing filteration $\mathbb{W} = \{W_{-q}\}$ on the complex $s(a_{\cdot 1*}\mathbb{Q}_{\mathfrak{X}_\cdot})[1]$ defined in (3.8) induces filterations on K, K_t, L and L_t as follows:

$$\begin{aligned}
W_{-q}(K) &:= \pi_*(\sigma_{|\alpha|\geq q+1} s(a_{\alpha 1*}C^\cdot(\mathbb{Q}_{\mathfrak{X}_\alpha})[1]),\\
W_{-q}(K_t) &:= \sigma_{|\alpha|\geq q+1} s(a_{t\alpha 1*}C^\cdot(\mathbb{Q}_{X_t\alpha}))[1],\\
W_{-q}(L) &:= \pi_*(\sigma_{|\alpha|\geq q+1} s(a_{\alpha 1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\alpha/M})[1]), \text{ and}\\
W_{-q}(L_t) &:= \sigma_{|\alpha|\geq q+1} s(a_{t\alpha 1*}tot\,\mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\alpha/M})[1].
\end{aligned}$$

The decreasing filteration $\mathbb{F} = \{F^p\}$ on the complex $s(a_{\cdot 1*}\Omega^\cdot_{\mathfrak{X}_\cdot/M})[1]$ defined in (3.10) induces filterations on L and L_t as follows:

$$F^p(L) := \pi_* s(a_{\cdot 1*}tot(\sigma_{q\geq p}\mathcal{A}^{q\cdot}_{\mathfrak{X}_\cdot/M}))[1] \text{ and} \tag{3.14}$$

$$F^p(L_t) := s(a_{t\cdot 1*}tot(\sigma_{q\geq p}\mathcal{A}^{q\cdot}_{X_t\cdot}))[1].$$

Using the filteration $\mathbb{W}$ on K defined above, we define a filteration on $H^\ell(K)$ by

$$W_{-q}(H^\ell(K)) := Im\{H^\ell(W_{-q}(K)) \to H^\ell(K)\}$$

Notice that the filteration of $\mathbb{R}^\ell_{\mathbb{Q}}(\pi)$ defined in (3.9) is mapped to this filteration by the isomprphism in Lemma 3.6. Similary, using the filteration $\mathbb{W}$ and $\mathbb{F}$ on L we define filterations on $H^\ell(L)$ by

$$W_{-q}(H^\ell(L)) := Im\{H^\ell(W_{-q}(L) \to H^\ell(L)\}, \tag{3.15}$$

$$F^p(H^\ell(L)) := Im\{H^\ell(F^p(L)) \to H^\ell(L)\}. \tag{3.16}$$

The filterations defined for $R^\ell_{\mathcal{O}}(\pi)$ in (3.10) and (3.11) are mapped to thses filterations by the isomorphism in Lemma 3.7. Furthermore, the filterations $\mathbb{W}$ of $H^\ell(K)$ and $H^\ell(L)$ defined above correspond each other through the isomorphisms

$$H^\ell(K)\otimes_{\mathbb{Q}}\mathcal{O}_M \simeq R^\ell_{\mathbb{Q}}(\pi)\otimes_{\mathbb{Q}}\mathcal{O}_M \simeq R^\ell_{\mathcal{O}}(\pi) \simeq H^\ell(L).$$

We denote by $\{E_r(K,W), d_r\}$ the spectral sequence of the complex K with respect to the filteration $\mathbb{W}$, abuting to

$$E^{p,q}_\infty(K,W) = Gr^W_{-p}(H^{p+q}(K)).$$

In the subsequence we will use the notation

$$H(A^{p-1} \xrightarrow{d^{p-1}} A^p \xrightarrow{d^p} A^{p+1})$$

for a complex $A^\cdot$, which means the cohomplogy at the middle term A^p. When we use this notation we often omit the differentials to avoid annoyance.

3.8 Lemma..

$$E_1^{p,q}(K,W) \simeq \bigoplus_{|\alpha|=p+1} R^q\pi_{\alpha *}\mathbb{Q}_{\mathfrak{X}_\alpha} \tag{3.17}$$

and this spectral sequence is degenerated at E_2-terms., i.e.,

$$E_2^{p,q}(K,W) \simeq E_3^{p,q}(K,W) \simeq \cdots \simeq E_\infty^{p,q}(K,W)$$

Proof. By definition

$$E_0^{p,q}(K,W) := Gr_{-p}^W(K^{p+q}) = \otimes_{|\alpha|=p+1}\pi_{\alpha *}C^q(\mathbb{Q}_{\mathfrak{X}_\alpha}).$$

Since

$$E_1^{p,q}(K,W) = H(E_0^{p,q-1} \xrightarrow{d_0} E_0^{p,q}(K,W) \xrightarrow{d_0} E_0^{p,q+1}(K,W)),$$

we obtain (3.17). From this it follows that

$$\begin{aligned} &E_2^{p,q}(K,W) \simeq \\ (3.18)\quad & \\ &H(\otimes_{|\alpha|=p}R^q\pi_{\alpha *}\mathbb{Q}_{\mathfrak{X}_\alpha} \longrightarrow \otimes_{|\alpha|=p+1}R^q\pi_{\alpha *}\mathbb{Q}_{\mathfrak{X}_\alpha} \longrightarrow \otimes_{|\alpha|=p+2}R^q\pi_{\alpha *}\mathbb{Q}_{\mathfrak{X}_\alpha}).\end{aligned}$$

Since $\mathfrak{X}. \to \mathfrak{X} \to M$ is C^∞ trivial ([8, Proposition 2.5]), $E_2^{p,q}(K,W)$ is a local system on M, and

$$\begin{aligned} &E_2^{p,q}(K,W)_t \\ (3.19) \simeq\; &H(\otimes_{|\alpha|=p}H^q(X_{t\alpha},\mathbb{Q}) \to \otimes_{|\alpha|=p+1}H^q(X_{t\alpha},\mathbb{Q}) \to \otimes_{|\alpha|=p+2}H^q(X_{t\alpha},\mathbb{Q})) \\ &\simeq E_2^{p,q}(K_t,W) \text{ for any } t \in M.\end{aligned}$$

The data: $K_{t\mathbb{Z}} := \mathbb{Z}_{X_t}$, (K_t,W), $K_{t\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \approx K_t$, (L_t,W,F), $(K_t,W)\otimes_{\mathbb{Q}} \mathbb{C} \approx (L_t,W)$, is a *cohomological mixed Hodge complex* in the sense of Deligne (For definition see [2, (8.1.6)]). Hence the spectral sequence of $\{E_r(K_t,W),d_r\}$ is degenerated at E_2-terms ([3, p.48, Théorème 3.2.1 (Deligne), (vi)). Therefore, by (3.19), the restriction of $d_2 : E_2^{p,q} \to E_2^{p,q}(K,W)$ to each fiber $E_2^{p,q}(K,W)_t$ is zero map. This implies that $d_2 : E_2^{p,q}(K,W) \to E_2^{p+2,q-1}(K,W)$ is zero map, since $E_2^{p,q}(K,W)$ is a local syatem on M.

Q.E.D.

3.9 Lemma..

(1) $E_1^{p,q}(L,W) \simeq E_1^{p,q}(K,W) \otimes_{\mathbb{Q}} \mathcal{O}_M$

$\simeq \oplus_{|\alpha|=p+1} R^q \pi_{\alpha *} \pi_\alpha^{\cdot} \mathcal{O}_M \simeq \oplus_{|\alpha|=p+1} \mathbb{R}^q \pi_{\alpha *}(\Omega^{\cdot}_{\mathfrak{X}_\alpha/M})$

(2) $E_2^{p,q}(L,W) \simeq E_2^{p,q}(K,W) \otimes_{\mathbb{Q}} \mathcal{O}_M \simeq$

$H(\oplus_{|\alpha|=p} \mathbb{R}^q \pi_{\alpha *}(\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}) \to \oplus_{|\alpha|=p+1} \mathbb{R}^q \pi_{\alpha *}(\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}) \to \oplus_{|\alpha|=p+2} \mathbb{R}^q \pi_{\alpha *}(\Omega^{\cdot}_{\mathfrak{X}_\alpha/M})).$ ∎

Proof. By definition

$$E_0^{p,q}(L,W) := Gr^W_{-p}(L^{p+q}) = \oplus_{|\alpha|=p+1} \pi_{\alpha *}(\oplus_{r+s=q} \mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M}).$$

Since

$$E_1^{p,q}(L,W) = H(E_0^{p,q-1}(L,W) \to E_0^{p,q}(L,W) \to E_0^{p,q+1}(L,W)),$$

and $tot\, \mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\alpha/M}$ is a $\pi_{\alpha *}$ -acyclic resolution of $\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}$, we have

$$E_1^{p,q}(L,W) \simeq \oplus_{|\alpha|=p+1} \mathbb{R}^q \pi_{\alpha *}(\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}) \simeq \oplus_{|\alpha|=p+1} R^q \pi_{\alpha *} \pi_\alpha^{\cdot} \mathcal{O}_M.$$

Hence, by (3.17) and Remark 3.5,

$$E_1^{p,q}(L,W) \simeq E_1^{p,q}(K,W) \otimes_{\mathbb{Q}} \mathcal{O}_M.$$

The assertion (2) follows from (1) and (3.18).

Q.E.D.

3.10 Lemma.

(3.20) $$E_1^{p,q}(L,F) \simeq \mathbb{R}^q \pi_*(s(a_{\cdot 1 *} \Omega^p_{\mathfrak{X}\cdot/M})[1])$$

and this is a locally free $\mathcal{O}_M$-module on M. Further, the spectral sequence $\{E_r^{p,q}(L,F), d_r\}$ is degenerated at E_1-terms.

Proof. By definition

$$E_0^{p,q}(L,F) := Gr^p_F(L^{p+q}) = \pi_*(\oplus_{|\alpha|+s=q+1} s(a_{\cdot 1 *} \mathcal{A}^{p,s}_{\mathfrak{X}\cdot/M})[1]).$$

Hence

(3.21) $$E_1^{p,q}(L,F) \simeq H^q(\pi_* s(a_{\cdot 1 *} \mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M})[1])$$

By the same reasoning as in the proof of Lemma 3.8, the simple complex

$s(a_{\cdot 1*}\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M}[1])$ is π_*-acyclic. Hence

$$H^q(\pi_* s(a_{\cdot 1*}\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M})[1]) \simeq \mathbb{R}^q\pi_*(s(a_{\cdot 1*}\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M})[1]) \tag{3.22}$$

While, since $\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M}$ gives a fine resolution of $\Omega^p_{\mathfrak{X}\cdot/M}$ for each $\alpha \in \Box_n$, the natural morphism

$$s(a_{\cdot 1*}\Omega^p_{\mathfrak{X}\cdot/M})[1] \to s(a_{\cdot 1*}\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M})[1]$$

is an isomorphism in $D^+(\mathfrak{X}, \mathbb{C})$. Hence we have

$$\mathbb{R}\pi_*(s(a_{\cdot 1*}\Omega^p_{\mathfrak{X}\cdot/M})[1]) \simeq \mathbb{R}^q\pi_*(s(a_{\cdot 1*}\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M})[1]) \tag{3.23}$$

By (3.21), (3.22) and (3.23), we obtain (3.20). Since

$$E_1^{p,q}(L,F)_t \simeq$$

$$H^q(\Gamma(X_t, s(a_{t\cdot *}\mathcal{A}^{p,q-1}_{X_t\cdot})[1]) \to \Gamma(X_t, s(a_{t\cdot *}\mathcal{A}^{p,q}_{X_t\cdot})[1]) \to \Gamma(X_t, s(a_{t\cdot 1*}\mathcal{A}^{p+1,q}_{X_t\cdot})[1])\,)$$

for any $t \in M$, and since $\mathfrak{X}\cdot \xrightarrow{\pi\cdot := \pi\cdot a\cdot} M$ is C^∞ trivial, $dim_{\mathbb{C}} E_1^{p,q}(L,F)_t$ is independent of $t \in M$. Hence, in order to prove the lemma it suffices to show that $\mathbb{R}\pi_*(s(a_{\cdot 1*}\Omega^p_{\mathfrak{X}\cdot/M}[1]))$ is a coherent $\mathcal{O}_M$-module. For this end, taking into account the isomorphism in (3.22), we define a filteration $'\mathbb{F} = \{'F^q\}$ of the complex $\pi_*(s(a_{\cdot 1*}\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M})[1])$ by

$$'F(\pi_*(s(a_{\cdot 1*}\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M})[1])) := \sigma_{k\geq q}\pi_*(s(a_{\cdot 1*}\mathcal{A}^{p,k}_{\mathfrak{X}\cdot/M}[1]))$$

and consider the spectral sequence of $\pi_*(s(a_{\cdot 1*}\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M}))[1]$ with respect to this filteration , abuting to

$$\begin{aligned} {}_{'F}E_\infty^{p,q} &= Gr_r^{'F}(H^{r+s}(\pi_*(s(a_{\cdot 1*}\mathcal{A}^{p\cdot}_{\mathfrak{X}\cdot/M})[1])) \\ &\simeq Gr_r^{'F}(\mathbb{R}^{r+s}\pi_*(s(a_{\cdot 1*}\Omega^p_{\mathfrak{X}\cdot/M})[1])). \end{aligned}$$

Then we have

$$\begin{aligned} {}_{'F}E_0^{r,s} &= \oplus_{|\alpha|=s+1}\pi_* s(a_{\alpha 1*}\mathcal{A}^{p,r}_{\mathfrak{X}_\alpha/M}), \\ {}_{'F}E_1^{r,s} &= H^s(\pi_* s(a_{\cdot 1*}\mathcal{A}^{p,r}_{\mathfrak{X}_\alpha/M}[1]), \end{aligned}$$

and

$$\begin{aligned} &{}_{'F}E_2^{r,s} \simeq \\ &H^s(\,\pi_* H(s(a_{\cdot 1*}\mathcal{A}^{p,r-1}_{\mathfrak{X}\cdot/M}[1] \to s(a_{\cdot 1*}\mathcal{A}^{p,r}_{\mathfrak{X}\cdot/M})[1] \to s(a_{\cdot 1*}\mathcal{A}^{p,r+1}_{\mathfrak{X}\cdot/M})[1]\,) \\ &\simeq R^s\pi_*(H^r(s(a_{\cdot 1*}\Omega^p_{\mathfrak{X}\cdot/M})). \end{aligned} \tag{3.24}$$

Since $H^r(s(a_{\cdot 1*}\Omega^p_{\mathfrak{X}\cdot/M}))$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module, by (3.24) ${}_{'F}E_2^{r,s}$ is a coherent $\mathcal{O}_M$- module. Hence so is ${}_{'F}E_\infty^{r,s}$ for any (r,s), which implies that $\mathbb{R}^q\pi_*(s(a_{\cdot 1*}\Omega^p_{\mathfrak{X}\cdot/M})[1]$ is a coherent $\mathcal{O}_M$-module as requied. To prove the degeneracy of the spectral sequence $\{E_r^{p,q}(L,F),d_r\}$ at E_1-terms, it suffices to show the degeneracy of the spectral sequence

$$\{E_r^{p,q}(L,F)_t,d_r\} \simeq \{E_r^{p,q}(L_t,F),d_{tr}\}$$

at E_1-terms, because $E_1^{r,q}(L,F)$ is a locally free $\mathcal{O}_M$-module by Lemma 3.10. While the degeneracy of this spectral sequence follows from the same theorem due to Deligne, which we have used to show the degeneracy of the spectral sequence $\{E_r^{p,q}(K_t,W),d_{tr}\}$ at E_2-terms in the proof of Lemma 3.9. This completes the proof of Lemma 3.10.

Q.E.D.

The filteration $\mathbb{F}$ on L defined in (3.14) induces filterations on each $E_r(L,W)$ $(r\geq 0)$ in three different ways. The first two filterations are defined as follows:

$$F_d^p(E_r(L,W)) := Im\{E_r(F^pL,W)\to E_r(L,W)\} \text{ (direct filteration)}$$
$$F_{d*}^p(E_r(L,W)) := Ker\{E_r(L,W)\to E_r(L/F^pL,W)\},$$

where $\{E_r(F^pL,W),d_r\}$ stands for the spectral sequence of the subcomplex F^pL of L with respect to the filteration induced by $\mathbb{W}$ and so on. From the definition it follows that $F_d^p = F_{d*}^p$ on $E_0(L,W)$ and $E_1(L,W)$ (cf. [3, Lemma 2.4.2]). The third one, which we call *"recurent filteration"* and denote by F_{rec}, is defined inductively as follows: on $E_0^{r,s}(L,W)$ we define

$$F_{rec}^p(E_0^{r,s}(L,W)) := F_d^p(E_0^{r,s}(L,W)) = F_{d*}^{r,s}(E_0^{r,s}(L,W)).$$

If F_{rec} has already been defined on $E_r(L,W)$, we define F_{rec} to be the natural one induced by F_{rec} on $E_{r+1}(L,W)$ through the relation

$$E_{r+1}^{p,q} = H(E_i^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}).$$

The relation among these three filterations is

$$F_d(E_r^{p,q}) \subset F_{\text{rec}}(E_r^{p,q}) \subset F_{d*}(E_r^{p,q}).$$

(cf. [3, Proposition(2.4.5), (iii)). Since there is a natural number r_0 such that

$$E_{r_0}^{p,q}(L,W) = E_{r_0+1}^{p,q}(L,W) = \cdots = E_\infty^{p,q}(L,W),$$

the filterations F_d, F_{d*} on each $E_r(L,W)$ induces the filterations on $E_\infty(L,W)$, which we denote by $F_d(E_\infty(L,W)), F_{d*}(E_\infty(L,W))$. Then we have

$$F_d(E_\infty^{p,q}(L,W)) \subset F(E_\infty^{p,q}(L,W)) \subset F_{d*}(E_\infty^{p,q}(L,W)), \tag{3.25}$$

where $F(E_\infty^{p,q}(L,W))$ is the filteration induced by that on $E_\infty^{p+q}(L) = H^{p+q}(L)$ defined in (3.16) ([3, Proposition 2.4.5, (iv)). Now we are going to prove that

$$F_d(E_r^{p,q}(L,W)) = F_{\mathrm{rec}}(E_r^{p,q}(L,W)) = F_{d*}(E_r^{p,q}(L,W)) \tag{3.26}$$

for any $r \geq 0$, which implies

$$F(E_\infty^{p,q}(L,W)) = F_{\mathrm{rec}}(E_\infty^{p,q}(L,W)), \tag{3.27}$$

because of (3.26). This fact will be used to compute $F^q(Gr_{-p}^{W[\ell]}(R_{\mathcal{O}}^\ell(\pi))$ later.

3.11 Lemma. *The differential $d_0 : E_0(L,W) \to E_0(L,W)$ is strictly compatible with the filteration $F_d = F_{d*}$, i.e., $d_0(F_d^p) = Im d_0 \cap F_d^p$ for any p.*

Proof. We will prove the equivalent assertion that the spectral sequence associated to the filtered complex $(E_0^{p\cdot}(L,W), F_d)$ is degenerated at E_1-term for every p. Since

$$E_0^{p,q}(L,W) = \oplus_{|\alpha|=p+1} \oplus_{k+\ell=q} (\pi_{\alpha *}\mathcal{A}^{k,\ell}_{\mathfrak{X}_\alpha/M}),$$

we have

$$\begin{aligned} E_0^{r,s}(E_0^{p\cdot}(L,W),F_d) &= Gr^r_{F_d}(\oplus_{|\alpha|=p+1} \oplus_{k+\ell=r+s} \pi_{\alpha *}\mathcal{A}^{k,\ell}_{\mathfrak{X}_\alpha/M}) \\ &= \oplus_{|\alpha|=p+1}\pi_{\alpha *}\mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M} \end{aligned}$$

and

$$\begin{aligned} d_0 &: E_0^{r,s}(E_0^{p\cdot}(L,W),F_d) = \oplus_{|\alpha|=p+1}\pi_{\alpha *}\mathcal{A}^{r,s}_{\mathfrak{X}_\alpha/M} \\ &\to E_0^{r,s+1}(E_0^{p\cdot}(L,W),F_d) = \oplus_{|\alpha|=p+1}\pi_{\alpha *}\mathcal{A}^{r,s+1}_{\mathfrak{X}_\alpha/M}. \end{aligned}$$

Therefore

$$E_1^{r,s}(E_0^{p\cdot}(L,W),F_d) = \oplus_{|\alpha|=p+1}R^s\pi_{\alpha *}\Omega^r_{\mathfrak{X}_\alpha/M} \quad \text{and}$$

$$\begin{aligned} d_1 &: E_1^{r,s}(E_0^{p\cdot}(L,W),F_d) = \oplus_{|\alpha|=p+1}R^s\pi_{\alpha *}\Omega^r_{\mathfrak{X}_\alpha/M} \\ &\to E_1^{r+1,s}(E_0^{p\cdot}(L,W),F_d) = \oplus_{|\alpha|=p+1}R^s\pi_{\alpha *}\Omega^{r+1}_{\mathfrak{X}_\alpha/M}. \end{aligned} \tag{3.28}$$

While, in the commutative duagram

$$(3.29)\qquad \begin{array}{ccc} (\oplus_{|\alpha|=p+1} R^{s+1}\pi_{\alpha *}\Omega^{r}_{\mathfrak{X}_\alpha/M})_t & \longrightarrow & \oplus_{|\alpha|=p+1} H^s(X_{\alpha t}, \Omega^{r}_{X_{\alpha t}}) \\ {\scriptstyle d_{1t}}\downarrow & & \downarrow{\scriptstyle d} \\ (\oplus_{|\alpha|=p+1} R^{s+1}\pi_{\alpha *}\Omega^{r+1}_{\mathfrak{X}_\alpha/M})_t & \longrightarrow & \oplus_{|\alpha|=p+1} H^s(X_{\alpha t}, \Omega^{r+1}_{X_{\alpha t}}) \end{array}$$

for each $t \in M$, the right vertical arow is zero map, because $X_{\alpha t}$ is algebraic. Hence so is d_{1t} for any $t \in M$. This implies $d_1 : E_1^{r,s}(E_0^{p\cdot}(L,W), F_d) \to E_1^{r,s+1}(E_0^{p\cdot}(L,W), F_d)$ is zero map as required, since $\oplus_{|\alpha|=p+1} R^{s+1}\pi_{\alpha *}\Omega^{k}_{\mathfrak{X}_\alpha/M}$ (k=r, r+1) are locally free sheaf on M.

Q.E.D.

3.12 Lemma. *The differential $d_1 : E_1(L,W) \to E_1(L,W)$ is strictly compatible with the filteration $F_d = F_{d*} = F_{\text{rec}}$. Hence $F_d = F_{d*} = F_{\text{rec}}$ on $E_r^{p,q}(L,W)$ $(r \geq 2)$.*

Proof. We will prove the equivalent assertion that the spectral sequence associated to the filtered complex $(E_1^{\cdot q}(L,W), F_d)$ is degenerated at E_1-term for every q. Since

$$\begin{aligned} E_1^{p,q}(L,W) &= H^q(E_0^{p\cdot}(L,W)) = H^q(\oplus_{|\alpha|=p+1}\pi_{\alpha *} tot\, \mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\alpha/M}) \\ &= \oplus_{|\alpha|=p+1} H^q(\pi_{\alpha *} tot\, \mathcal{A}^{\cdot\cdot}_{\mathfrak{X}_\alpha/M}) \simeq \oplus_{|\alpha|=p+1} R^q\pi_{\alpha *}(\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}) \end{aligned}$$

and
$$\begin{aligned} F_d^r(E_1^{p,q}(L,W)) &= Im\{H^q(E_0^{p\cdot}(F_d^r(K)) \to H^q(E_0^{p\cdot})\} \\ &= \oplus_{|\alpha|=p+1} H^q(\pi_{\alpha *} \oplus_{k\geq r} tot\, \mathcal{A}^{k\cdot}_{\mathfrak{X}_\alpha/M}) \text{ (by Lemma 3.11)} \\ &\simeq \oplus_{|\alpha|=p+1} \oplus_{k\geq r} R^q\pi_{\alpha *}\Omega^{k}_{\mathfrak{X}_\alpha/M}, \end{aligned}$$

we have

$$\begin{aligned} E_0^{r,s}(E_1^{\cdot q}(L,W), F_d) &= F_d^r(E_1^{r+s,q}(L,W))/F_d^{r+1}(E_1^{r+s,q}(L,W)) \\ &\simeq \oplus_{|\alpha|=r+s+1} R^q\pi_{\alpha *}\Omega^{r}_{\mathfrak{X}_\alpha/M} \end{aligned}$$

and
$$\begin{aligned} d_0 : E_0^{r,s}(E_1^{\cdot q}(L,W), F_d) &\simeq \oplus_{|\alpha|=r+s+1} R^q\pi_{\alpha *}\Omega^{r}_{\mathfrak{X}_\alpha/M} \\ &\to E_0^{r,s+1}(E_1^{\cdot q}(L,W), F_d) \simeq \oplus_{|\alpha|=r+s+2} R^q\pi_{\alpha *}\Omega^{r}_{\mathfrak{X}_\alpha/M}. \end{aligned}$$

Therefore the differential d_1 is as follows:

$E_1^{r,s}(E_1^{\cdot q}(L,W), F_d) \simeq$

$$
\begin{aligned}
&H(\textstyle\bigoplus_{|\alpha|=r+s} R^q\pi_{\alpha*}\Omega^r_{\mathfrak{X}_\alpha/M} \to \bigoplus_{|\alpha|=r+s+1} R^q\pi_{\alpha*}\Omega^r_{\mathfrak{X}_\alpha/M} \\
&\qquad\qquad \to \textstyle\bigoplus_{|\alpha|=r+s+2} R^q\pi_{\alpha*}\Omega^r_{\mathfrak{X}_\alpha/M}) \\
&\qquad\qquad\qquad \big\downarrow d_1 \\
&E_1^{r+1,s}(E_1^{\cdot q}(L,W),F_d) \simeq \\
&H(\textstyle\bigoplus_{|\alpha|=r+s} R^q\pi_{\alpha*}\Omega^{r+1}_{\mathfrak{X}_\alpha/M} \to \bigoplus_{|\alpha|=r+s+1} R^q\pi_{\alpha*}\Omega^{r+1}_{\mathfrak{X}_\alpha/M} \\
&\qquad\qquad \to \textstyle\bigoplus_{|\alpha|=r+s+2} R^q\pi_{\alpha*}\Omega^{r+1}_{\mathfrak{X}_\alpha/M}).
\end{aligned}
$$

This differential d_1 is nothing but the one induced by the relative exterior differential $d_{\mathfrak{X}_\alpha/M} : \Omega^r_{\mathfrak{X}_\alpha/M} \to \Omega^{r+1}_{\mathfrak{X}_\alpha/M}$. While the morphism $R^q\pi_{\alpha*}\Omega^r_{\mathfrak{X}_\alpha/M} \to R^q\pi_{\alpha*}\Omega^{r+1}_{\mathfrak{X}_\alpha/M}$ induced by $d_{\mathfrak{X}_\alpha/M}$ is zero map because of the same reasoning as in (3.29). Hence $d_1 : E_1^{r,s}(E_1^{\cdot q}(L,W),F_d) \to E_1^{r+1,s}(E_1^{\cdot q}(L,W),F_d)$ is zero map. This completes the former part of the lemma. The latter part follows from Théoreme 2.4.9, (i) in [3], Lemma 3.8 and Lemma 3.9.

Q.E.D.

3.13 Lemma.

$$F^q(Gr_{-p}^{W[\ell]}(R^\ell_{\mathcal{O}}(\pi)))_t \simeq F^q(Gr_{-p}^{W[\ell]}(H^\ell(X_t,\mathbb{C}))$$

for any $t \in M$.

Proof. By Lemma 3.8 and Lemma 3.9,

$$
\begin{aligned}
&Gr_p^{W[\ell]}(R^\ell_{\mathcal{O}}(\pi)) \simeq Gr^W_{p-\ell}(R^\ell_{\mathcal{O}}(\pi)) \simeq E_2^{\ell-p,p}(L,W) \\
&\simeq H(\oplus_{|\alpha|=\ell-p}\mathbb{R}^p\pi_{\alpha*}(\Omega^\cdot_{\mathfrak{X}_\alpha/M}) \to \oplus_{|\alpha|=\ell-p+1}\mathbb{R}^p\pi_{\alpha*}(\Omega^\cdot_{\mathfrak{X}_\alpha/M}) \\
&\qquad \to \oplus_{|\alpha|=\ell-p+2}\mathbb{R}^p\pi_{\alpha*}(\Omega^\cdot_{\mathfrak{X}_\alpha/M})
\end{aligned}
$$

By Lemma 3.12 and (3.27),

$$
\begin{aligned}
&F^q(Gr_p^{W[\ell]}(R^\ell_{\mathcal{O}}(\pi))) \simeq F^q_{\mathrm{rec}}(E_2^{\ell-p,p}(L,W)) \\
&\simeq H(\oplus_{|\alpha|=\ell-p}\mathbb{R}^p\pi_{\alpha*}(F^q(\Omega^\cdot_{\mathfrak{X}_\alpha/M})) \to \oplus_{|\alpha|=\ell-p+1}\mathbb{R}^p\pi_{\alpha*}(F^q(\Omega^\cdot_{\mathfrak{X}_\alpha/M}) \\
&\qquad \to \oplus_{|\alpha|=\ell-p+2}\mathbb{R}^p\pi_{\alpha*}(F^q(\Omega^\cdot_{\mathfrak{X}_\alpha/M}))
\end{aligned}
$$

Therefore, since $\mathfrak{X}. \xrightarrow{b.:=\pi\cdot a.} M$ is C^∞ trivial ([8, Proposition 2.5]),

$$
\begin{aligned}
&F^q(Gr_p^{W[\ell]}(R^\ell_{\mathcal{O}}(\pi))_t \\
&\simeq H(\oplus_{|\alpha|=\ell-p}\mathbb{H}^p(F^q(\Omega^\cdot_{X_{\alpha t}})) \to \oplus_{|\alpha|=\ell-p+1}\mathbb{H}^p(F^q(\Omega^\cdot_{X_{\alpha t}})) \\
&\qquad \to \oplus_{|\alpha|=\ell-p+2}\mathbb{H}^p(F^q(\Omega^\cdot_{X_{\alpha t}})) \\
&\simeq F^q_{\mathrm{rec}}(Gr_p^{W[\ell]}(H^\ell(X_t,\mathbb{C}))) \simeq F^q(Gr_p^{W[\ell]}(H^\ell(X_t,\mathbb{C})))
\end{aligned}
$$

where the last isomorphism is obtained from (3.27) by putting $M = \{t\}$ (one point).

Q.E.D.

Now Theorem 3.6 follows from Lemma 3.9, Lemma 3.11 and Lemma 3.14.

3.2 Griffiths transversality

The purpose of this paragraph is to prove the following theorem.

3.14 Theorem. *(Griffiths transversality)*
In the same setting and with the same notations as in Theorem 3.6, we have

$$\nabla F^p(R^\ell_{\mathcal{O}}(\pi)) \subset \Omega^1_M \otimes F^{p-1}(R^\ell_{\mathcal{O}}(\pi)),$$

where ∇ is the Gauss-Mannin connection on $R^\ell_{\mathcal{O}}(\pi)$.

The proof will proceed in the following three steps:
(I) Definition of an integrable connection $\nabla : R^\ell_{\mathcal{O}}(\pi) \longrightarrow \Omega^1_M \otimes R^\ell_{\mathcal{O}}(\pi)$.
(II) "Explicit" calculation of the connection.
(III) Proof of $Ker\nabla = R^n_{\mathbb{C}}(\pi) := R^\ell_{\mathbb{Z}}(\pi) \otimes_{\mathbb{Z}} \mathbb{C}$.

Step I: Definition of an integrable connection $\nabla : R^\ell_{\mathcal{O}}(\pi) \longrightarrow \Omega^1_M \otimes R^\ell_{\mathcal{O}}(\pi)$.

We modify the proof in the case of a smooth family of algebraic manifolds in [7]. Since the family of algebraic manifolds $\pi_\alpha : \mathfrak{X}_\alpha \longrightarrow M$ is smooth for every $\alpha \in \square_n$ (n=the length of a cubic hyperresolution $X_{t\cdot} \xrightarrow{a.} X_t \xrightarrow{\pi} M$ of a fiber of the family $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$), the sequence

$$0 \to \pi^*_\alpha(\Omega^1_M) \longrightarrow \Omega^1_{\mathfrak{X}_\alpha} \longrightarrow \Omega^1_{\mathfrak{X}_\alpha/M} \longrightarrow 0 \ \ (\alpha \in \square_n)$$

is exact. Since

$$E^*_{\alpha\beta}(\Omega^{\cdot-p}_{\mathfrak{X}_\alpha} \otimes_{\mathcal{O}_{\mathfrak{X}_\alpha}} \pi^*_\alpha \Omega^p_M) \subset \Omega^{\cdot-p}_{\mathfrak{X}_\beta} \otimes_{\mathcal{O}_{\mathfrak{X}_\beta}} \pi^*_\beta \Omega^p_M$$

for every integer p with $0 \le p \le m$ $(m = dimM)$ and for every pair (α, β) of $\alpha, \beta \in \square_n$ with $\alpha \longrightarrow \beta$ in the category $\square_n$, $\{\Omega^{\cdot-p}_{\mathfrak{X}_\alpha} \otimes_{\mathcal{O}_{\mathfrak{X}_\alpha}} \pi^*_\alpha \Omega^p_M\}_{\alpha \in \square_n}$ constitutes a subcomlex of sheaves of $\Omega^\cdot_{\mathfrak{X}.}$. Hence the complex $s(a_{\cdot 1*}\Omega^\cdot_{\mathfrak{X}.})$ admits a canonical filteration

$$s(a_{\cdot 1*}\Omega^\cdot_{\mathfrak{X}.}) = F^0(s(a_{\cdot 1*}\Omega^\cdot_{\mathfrak{X}.})) \supset F^1(s(a_{\cdot 1*}\Omega^\cdot_{\mathfrak{X}.})) \supset F^2(s(a_{\cdot 1*}\Omega^1_{\mathfrak{X}.})) \supset \cdots\cdots$$

where

$$F^p = F^p(s(\pi_{\cdot *}\Omega^\cdot_{\mathfrak{X}.}))$$

$$= Im[s(a_{\cdot 1*}(\Omega_{\mathfrak{X}\cdot}^{\cdot -p} \otimes_{\mathfrak{X}\cdot} \pi_\cdot^* \Omega_M^p)) \to s(a_{\cdot 1*}\Omega_{\mathfrak{X}\cdot}^\cdot)]$$

The associated graded objects of this filteration are given by

$$gr^p = gr^p(s(a_{\cdot 1*}\Omega_{\mathfrak{X}\cdot}^\cdot)) := F^p/F^{p+1}$$
$$\cong s(a_{\cdot 1*}(\pi_\cdot^*(\Omega_M^p) \otimes_{\mathcal{O}_{\mathfrak{X}\cdot}} \Omega_{\mathfrak{X}\cdot/M}^{\cdot -p}))$$

Therefore the spectral sequence which comes from the filtration $\{F^p\}$ and abuts to the graded objects of $H^\cdot(\mathbb{R}\pi_*(s(a_{\cdot 1*}\Omega_{\mathfrak{X}\cdot}^\cdot)))$ associated to the filteration on it induced by $\{F^p\}$ is as follows:

$$\begin{aligned} E_1^{p,q} &:= \mathbb{R}^{p+q}\pi_*(gr^p) = \mathbb{R}^{p+q}(s(a_{\cdot 1*}((\pi_\cdot^*\Omega_M^p) \otimes_{\mathcal{O}_{\mathfrak{X}\cdot}} \Omega_{\mathfrak{X}\cdot/M}^{\cdot -p}))) \\ &\simeq \mathbb{R}^q\pi_*(s(a_{\cdot 1*}((\pi_\cdot^*\Omega_M^p \otimes_{\mathcal{O}_{\mathfrak{X}\cdot}} \Omega_{\mathfrak{X}\cdot/M}^\cdot))) \\ &\simeq \Omega_M^p \otimes_{\mathcal{O}_M} \mathbb{R}^q\pi_*(s(a_{\cdot 1*}\Omega_{\mathfrak{X}\cdot/M}^\cdot))) \simeq \Omega_M^p \otimes_{\mathcal{O}_M} R_{\mathcal{O}}^q(\pi) \\ &\Longrightarrow E_\infty^{p,q} = Gr^p(\mathbb{R}^{p+q}\pi_*(s(a_{\cdot 1*}\Omega_{\mathfrak{X}\cdot}^\cdot))) \end{aligned}$$

Since the filteration on $s(a_{\cdot 1*}\Omega_{\mathfrak{X}\_}^\cdot)$ is compatible with the exterior product, i.e., $F^i \wedge F^j \subset F^{i+j}$, and since the sequence of functor $\mathbb{R}^q\pi_*$ is multiplicative, the spectral sequence has a product structure; that is, there are pairings

$$E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$$

for each p, q, p', q' and r, sendinfg (e, e') to $e \cdot e'$ where e, e' are sections of $E_r^{p,q}$ and $E_r^{p',q'}$, respectively, over an open subset of M. This pairing satisfies

$$\begin{aligned} &e \cdot e' = (-1)^{(p+q)(p'+q')} e' \cdot e, \qquad \text{and} \\ &d_r(e \cdot e') = d_r(e) \cdot e' + (-1)^{p+q} e \cdot d_r(e'). \end{aligned}$$

The E_1 terms of the spectral sequence are as follows:

$$0 \to R_{\mathcal{O}}^q(\pi) \xrightarrow{d_1^{0,q}} \Omega_M^1 \otimes R_{\mathcal{O}}^q(\pi) \xrightarrow{d_1^{1,q}} \Omega_M^2 \otimes R_{\mathcal{O}}^q(\pi) \to \cdots .$$

To show that $d_1^{0,q}$ is a connection, let us consider the pairing

$$E_1^{0,0} \times E_1^{0,q} \to E_1^{0,q},$$

which satisfies

$$d_1^{0,q}(\omega \cdot e) = d_1^{0,0}\omega \cdot e + \omega \cdot d_1^{0,q} e, \tag{3.30}$$

where ω, e are sections of $E_1^{0,0} \simeq R_{\mathcal{O}}^0(\pi) \simeq \mathcal{O}_M$ and $E_1^{0,q} = \mathbb{R}_{\mathcal{O}}^q(\pi)$, respectively, over an open subset of M. Since

$$d_1^{0,0} : E_1^{0,0} \simeq \mathcal{O}_M \to E_1^{1,0} = \Omega_M^1$$

is nothing but the exterior differentiation d_M on M, (3.30) shows that $d_1^{0,q}$ is certainly a connection. Furthermore, since

$$d_1^{0,q} : E_1^{1,q} = \Omega_M^1 \otimes R_{\mathcal{O}}^q(\pi) \longrightarrow E_1^{2,q} = \Omega_M^2 \otimes R_{\mathcal{O}}^q(\pi)$$

is equal to $d_M \otimes 1$, $d_1^{1,q} \cdot d_1^{0,q} = 0$ shows that $d_1^{0,q}$ is an integrable connection. We denote $d_1^{o,q}$ by ∇ in the subsequence.

Step(II):"Explicit" caluculation of the connection.

First, we mention a general fact that, in the spectral sequence of a filtered object, the differential

$$d_1^{p,q} : E_1^{p,q} = \mathbb{R}^{p+q}\pi_*(gr^p) \longrightarrow E_1^{p+1,q} = \mathbb{R}^{p+q+1}\pi_*(gr^{p+1})$$

is the connecting homomorphism of the functor $\mathbb{R}^q\pi_*$ for the exact sequence

$$0 \longrightarrow gr^{p+1} \longrightarrow F^p/F^{p+2} \longrightarrow gr^p \longrightarrow 0$$

Using this fact, we shall calculate explicitly the connection

$$\begin{aligned} d_1^{0,q} : E_1^{0,q} &= \mathbb{R}^q\pi_*(gr^0) \simeq R_{\mathcal{O}}^q(\pi) \\ &\longrightarrow E_1^{1,q} = \mathbb{R}^{q+1}\pi_*(gr^1) \simeq \Omega_M^1 \otimes R_{\mathcal{O}}^q(\pi) : \end{aligned}$$

In the calculation we will use a special system of Stein coverings $\{\mathcal{U}_\alpha\}_{\alpha\in\square_n}$ of $\mathfrak{X}.$, whose existence is guaranteed by the *analytically local triviality* of the cubic hyperequisingular family $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$. Using this special covering of $\mathfrak{X}.$, we will explicitly describe the map

$$\begin{aligned} \nabla : \mathbb{H}^\ell(\pi^{-1}(U), s(a_{.1*}\Omega^{\cdot}_{\mathfrak{X}./M})[1]) \\ \longrightarrow \Gamma(U, \Omega_M^1) \otimes_{\Gamma(U,\mathcal{O}_M)} \mathbb{H}^\ell(\pi^{-1}(U), s(a_{.1*}\Omega^{\cdot}_{\mathfrak{X}./M})[1]) \end{aligned}$$

for a sufficiently small open subset U of M. We take a point $o \in M$ and put

$$X_\alpha := (\pi \cdot a_\alpha)^{-1}(o), \;\; X := \pi^{-1}(o).$$

By the definition of an n-cubic hyperequisingular family $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$, it is analytically locally trivial. Hence, schrinking M sufficiently small around o, we are allowed to assume that there is a system of Stein coverings $\mathcal{U}_\alpha := \{U_i^{(\alpha)}\}_{i\in\Lambda_\alpha}$ of $\mathfrak{X}_\alpha$ $(\alpha \in \square_n^+)$, which is subject to the following requirements:

(i) for each pair (α, β) of elements of $Ob(\square_n^+)$ with $\alpha \longrightarrow \beta$ in $\square_n^+$, there is a map $\lambda_{\alpha\beta} : \Lambda_\beta \longrightarrow \Lambda_\alpha$ such that;

(a) if α, β, γ are elements of $Ob(\square_n^+)$ with $\alpha \longrightarrow \beta \longrightarrow \gamma$ in $\square_n^+$, then $\lambda_{\alpha\gamma} = \lambda_{\alpha\beta} \cdot \lambda_{\beta\gamma}$, and

(b) $e_{\alpha\beta}(U_i^{(\beta)}) \subset U^{(\alpha)}_{\lambda_{\alpha\beta}(i)}$ for any $i \in \Lambda_\beta$, where $e_{\alpha\beta} : X_\beta \longrightarrow X_\alpha$ is

(3.31) a holomorphic map corresponding to an arrow $\alpha \longrightarrow \beta$ in $\square_n^+$,

(ii) if we define $V_i^{(\alpha)} := U_i^{(\alpha)} \times M$ for $\alpha \in Ob(\square_n^+)$ and $i \in \Lambda_\alpha$, then $\mathcal{V} := \{V_i^{(\alpha)}\}$ is a Stein covering of $\mathfrak{X}_\alpha$ for every $\alpha \in Ob(\square_n^+)$, and

(iii) $E_{\alpha\beta|V_i^\beta} : V_i^{(\beta)} \longrightarrow V^\alpha_{\lambda_{\alpha\beta}(i)}$ is equal to $e_{\alpha\beta|U_i^{(\beta)}} \times id_M$, where $E_{\alpha\beta} : \mathfrak{X}_\beta \longrightarrow \mathfrak{X}_\alpha$ is a holomorphic map over M corresponding to an arrow $\alpha \longrightarrow \beta$ in $\square_n^+$ for $\alpha \in Ob(\square_n^+)$ and $i \in \Lambda_\alpha$,

(iv) $\pi_{\alpha|V_i^{(\alpha)}} = Pr_M : V_i^{(\alpha)} := U_i^{(\alpha)} \times M \longrightarrow M$(projection to M, where $\pi_\alpha := \pi \cdot a_\alpha$ and $\pi_0 = \pi$.

Note that the existence of a system of Stein coverings of an n-cubic hyperequisingular family $\mathfrak{X}_. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$ as above also relies on the fact that, for a holomorphic map $f : X \longrightarrow Y$ between complex spaces, the intersection $f^{-1}(U) \cap V$ of the inverse image of a Stein subset U of Y by f and a Stein subset V of X is Stein ([4, p.33, Chapter, I, §4, 4]).

We take such a system of Stein coverings of $\mathfrak{X}_. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$ and fix it. In the following we will always caluculate with respect to this coverings unless otherwise mentioned. Let $\{\mathcal{K}^\cdot_\alpha, d_\alpha\}_{\alpha\in\square_n}$ be a bounded complex of sheaves of coherent analytic sheaves on $\mathfrak{X}_.$. Let $\{C^\cdot(\mathcal{U}_\alpha, \mathcal{K}^p_\alpha), \delta_\alpha\}$ be the Cěch complex consisting of alternating cochains with values in K^p_α, respecting the Stein covering $\mathcal{U}_\alpha$; that is,

$$C^q(\mathcal{U}_\alpha, \mathcal{K}^p_\alpha) = \oplus_{\substack{(i_0\cdots i_q)\in\Lambda_\alpha^{q+1}\\ i_0\leq\cdots\leq i_q}}\Gamma(U_{i_0}\cap\cdots\cap U_{i_q}, \mathcal{K}^p_\alpha)$$

and the coboundary map $\delta^q_\alpha : C^q(\mathcal{U}_\alpha, \mathcal{K}^p_\alpha) \longrightarrow C^{q+1}(\mathcal{U}_\alpha, \mathcal{K}^p_\alpha)$ is defined by

$$(\delta_\alpha\beta)(\alpha;p;i_0\cdots i_{q+1}) = (-1)^{p+|\alpha|}\sum_{j=0}^{q+1}(-1)^j\beta(i_0\cdots \check{i_j}\cdots i_{q+1})$$

for $\beta = \{\beta(\alpha;p;i'_0\cdots i'_q)\} \in C^q(\mathcal{U}_\alpha, \mathcal{K}^p_\alpha)$, where $\beta(\alpha;p;i'_0\cdots i'_q) \in \Gamma(U_{i'_0}\cap\cdots\cap U_{i'_q}, \mathcal{K}^p_\alpha)$ for $(i'_0,\cdots,i'_q) \in \Lambda^{q+1}_\alpha$ and $|\alpha| = \alpha_0+\cdots+\alpha_n$ for $\alpha \in \square_n$. The pre-sheaf

$$U \longmapsto C^q(\mathcal{U}_\alpha\cap U, \mathcal{K}^p_\alpha) := \oplus_{\substack{(i_0\cdots i_q)\in\Lambda_\alpha^{q+1}\\ i_0\leq\cdots\leq i_q}}\Gamma(U_{i_0}\cap\cdots\cap U_{i_q}\cap U, \mathcal{K}^p_\alpha),$$

where U is an open subset of $\mathfrak{X}_\alpha$, define a sheaf, which we denote by $\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}^p_\alpha)$. We associate to the double complex of abelian sheaves $\mathcal{C}^\cdot(\mathcal{U}_\alpha, \mathcal{K}^\cdot_\alpha)$ a simple complex $tot\,\mathcal{C}^\cdot(\mathcal{U}_\alpha, \mathcal{K}^\cdot_\alpha)$ defined as followed:

$$tot\,\mathcal{C}^\cdot(\mathcal{U}_\alpha, \mathcal{K}^\cdot_\alpha))^r := \oplus_{p+q=r}\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}^p_\alpha),$$
$$d^{(r)}_\alpha := \oplus_{p+q=r}(-1)^{|\alpha|}(d'^p_\alpha + \delta^{p,q}_\alpha) : tot\,\mathcal{C}^\cdot(\mathcal{U}_\alpha, \mathcal{K}^\cdot_\alpha)^r \longrightarrow tot\,(\mathcal{C}^\cdot(\mathcal{U}_\alpha, \mathcal{K}^\cdot_\alpha))^{r+1},$$

where $\delta_\alpha^{p,q}$ is the Cěch coboudary map $\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p) \to \mathcal{C}^{q+1}(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p)$ and $d_\alpha'^p$ is the map $\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p) \to \mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^{p+1})$ induced by the differential of the complex $\mathcal{K}_\alpha^\cdot$. Obviously, $\{tot\, \mathcal{C}^\cdot(\mathcal{U}_\alpha, \mathcal{K}_\alpha^\cdot), d_\alpha\}_{\alpha \in \square_n}$ defines a complex of abelian sheaves on $\mathfrak{X}.$ which is quasi-isomorphic to $\{\mathcal{K}_\alpha^\cdot, d_\alpha'\}_{\alpha\in\square_n}$, because $\{tot\, \mathcal{C}^\cdot(\mathcal{U}_\alpha, \mathcal{K}_\alpha^\cdot), d_\alpha\}$ is quasi-isomorphic to $(\mathcal{K}_\alpha^\cdot, d_\alpha)$ for every $\alpha \in \square_n$.

3.15 Proposition. *The simple complex of abelian sheaves $s(a_{\cdot 1*} tot\, \mathcal{C} \cdot (\mathcal{U}_\cdot, \mathcal{K}_\cdot^\cdot)))$ is π_*-acyclic. Hence*

$$\mathbb{H}^k(\mathfrak{X}, s(a_{\cdot 1*}\mathcal{K}_\cdot^\cdot)) \simeq H^k(s(tot\, \mathcal{C}^\cdot(\mathcal{U}_\cdot, \mathcal{K}_\cdot^\cdot))) \text{ for } k \geq 0,$$

*where $s(tot\,(\mathcal{C}^\cdot(\mathcal{U}_\cdot, \mathcal{K}_\cdot^\cdot))))$ is the simple complex of abelian groups associated to $(n+p)$-ple complex $tot\, \mathcal{C}^\cdot(\mathcal{U}_\cdot, \mathcal{K}_\cdot^\cdot)$ (*cf. [8, Definition 1.18]*).*

Proof. To prove that $s(a_{\cdot 1*} tot\, \mathcal{C}^\cdot(\mathcal{U}_\cdot, \mathcal{K}_\cdot^\cdot))$ is π_*-acyclic, it suffices to show that

$$H^k(\pi^{-1}(U), s(a_{\cdot 1*} tot\, \mathcal{C}^\cdot(\mathcal{U}_\cdot, \mathcal{K}_\cdot^\cdot)))^r) = 0 \quad (k \geq 1, r \in \mathbb{Z}). \tag{3.32}$$

Let us notice that

$$s(a_{\cdot 1*} tot\, \mathcal{C}^\cdot(\mathcal{U}_\cdot, \mathcal{K}_\cdot^\cdot))^r = \oplus_{|\alpha|+p+q=r} a_{\alpha *}\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p),$$

hence

$$\begin{aligned} &H^k(\pi^{-1}(U), s(a_{\cdot 1*} tot\, \mathcal{C}^\cdot(\mathcal{U}_\cdot, \mathcal{K}_\cdot^\cdot))^r) \\ &= \oplus_{|\alpha|+p+q=m} H^k(\pi^{-1}(U), a_{\alpha *}\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p)). \end{aligned} \tag{3.33}$$

Concerning the holomorphic map $a_{\alpha|\pi_\alpha^{-1}(U)} : \pi_\alpha^{-1}(U) \to \pi^{-1}(U)$, we have the Leray spectral sequence

$$\begin{aligned} E_2^{s,k-s} &= H^k(\pi^{-1}(U), R^s a_{\alpha *}\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p)) \\ &\to E_\infty^k = H^k(\pi_\alpha^{-1}(U), \mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p)). \end{aligned}$$

From this it follows that

$$H^k(\pi^{-1}(U), a_{\alpha *}\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p)) \simeq H^k(\pi_\alpha^{-1}(U), \mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p)),$$

since $R^s a_{\alpha *}\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p) = 0$ for $s \geq 1$. On the other hand,

$$H^k(\pi_\alpha^{-1}(U), \mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p)) = 0 \text{ for } k \geq 1;$$

hence

$$H^k(\pi^{-1}(U), a_{\alpha *}\mathcal{C}^q(\mathcal{U}_\alpha, \mathcal{K}_\alpha^p)) = 0 \text{ for } k \geq 1$$

Consequently, by (3.33) we obtain (3.32). The latter part of the proposition follows from the facts that the natural map $s(a_{.1*}\mathcal{K}^{\cdot}_{.}) \to s(a_{.1*}tot\,(\mathcal{C}^{\cdot}(\mathcal{U}_{.}, \mathcal{K}^{\cdot}_{.})))$ is an isomorphism in $D^{+}(\mathfrak{X}, \underline{Ab})$ and so that

$$\mathbb{H}^{k}(\mathfrak{X}, s(a_{.1*}\mathcal{K}^{\cdot}_{.})) \simeq H^{k}(\mathfrak{X}, \Gamma(\mathfrak{X}, s(a_{.1*}tot\,\mathcal{C}^{\cdot}(\mathcal{U}_{.}, \mathcal{K}^{\cdot}_{.})))).$$

Q.E.D.

By Proposition 3.16 the explicit description of

$$\begin{aligned}\nabla : \mathbb{H}^{q}(M, \pi_{*}s(a_{.1*}\Omega^{\cdot}_{\mathfrak{X}./M})[1]) \\ \longrightarrow \Gamma(M, \Omega^{1}_{M}) \otimes_{\Gamma(M,\mathcal{O}_{M})} \mathbb{H}^{q}(M, \pi_{*}s(a_{.1*}\Omega^{\cdot}_{\mathfrak{X}./M})[1])\end{aligned} \tag{3.34}$$

is reduced to that of

$$\begin{aligned}\nabla : H^{q}(s(tot\,C^{\cdot}(\mathcal{U}_{.}, \Omega_{\mathfrak{X}./M}{}^{\cdot}))[1]) \\ \longrightarrow \Gamma(M, \Omega^{1}_{M}) \otimes_{\Gamma(M,\mathcal{O}_{M})} H^{q}(s(tot\,C^{\cdot}(\mathcal{U}_{.}, \Omega^{\cdot}_{\mathfrak{X}./M}))[1]).\end{aligned}$$

In the subsequence we will use the notation

$$K^{\cdot}(\mathcal{F}^{\cdot}_{.}) := s(tot(C^{\cdot}\,\mathcal{U}_{.}, \mathcal{F}^{\cdot}_{.}))$$

for a complex of abelian sheaves $\mathcal{F}^{\cdot}_{.}$ on $\mathfrak{X}_{.}$. With this notation we have the following exact sequences of abelian groups.

$$0 \to K^{\cdot}(F^{1}(\Omega^{\cdot}_{\mathfrak{X}.})) \to K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}.}) \to K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M}) \to 0 \tag{3.35}$$

$$0 \to K^{\cdot}(Gr^{1}(\Omega^{\cdot}_{\mathfrak{X}.})) \to K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}.}/F^{2}(\Omega^{\cdot}_{\mathfrak{X}.})) \to K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M}) \to 0. \tag{3.36}$$

For $\alpha \in \square_{n}$ and $i \in \Lambda_{\alpha}$, we denote by $(x^{(\alpha)}_{i_1}, \cdots, x^{(\alpha)}_{i_{n_\alpha}})$ a local coordinate system on $U^{(\alpha)}_{i}$, where $n_{\alpha} = dim\,U^{(\alpha)}_{i}$. We denote by $(t_1, \cdots, t_m)$ a local coordinate system on M. We decompose the exterior differentiation $d_{\mathfrak{X}_\alpha}$ on $\mathfrak{X}_{\alpha}$ as

$$d_{\mathfrak{X}_\alpha} = d_{M} + d_{U^{(\alpha)}_{i}} \tag{3.37}$$

on each $V^{(\alpha)}_{i} = U^{(\alpha)}_{i} \times M$, where d_M is the differentiation with respect to $(t_1, \cdots, t_m)$ and $d^{(\alpha)}_{U_i}$ with respect to $(x^{(\alpha)}_{i_1}, \cdots, x^{(\alpha)}_{i_{n_\alpha}})$. We define

$$\phi^{(\alpha)}_{i} : \Omega^{\cdot}_{\mathfrak{X}_\alpha/M|V^{(\alpha)}_{i}} \longrightarrow \Omega^{\cdot}_{\mathfrak{X}_\alpha|V^{(\alpha)}_{i}}$$

by

$$\phi_i^{(\alpha)}([\sum_{j_1\le\cdots\le j_p} a^{(\alpha)}_{j_1\cdots j_p}(x,t)dx^{(\alpha)}_{ij_1}\wedge\cdots dx^{(\alpha)}_{ij_p}])$$
$$= \sum_{j_1\le\cdots\le j_p} a^{(\alpha)}_{j_1\cdots j_p}(x,t)dx^{(\alpha)}_{ij_1}\wedge\cdots dx^{(\alpha)}_{ij_p},$$

where $[\sum_{j_1\le\cdots\le j_p} a^{(\alpha)}_{j_1\cdots j_p}(x,t)dx^{(\alpha)}_{ij_1}\wedge\cdots dx^{(\alpha)}_{ij_p}]$ is a local cross-section of the sheaf $\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}$ over an open subset V^i_α, represented by a differential form

$$\sum_{j_1\le\cdots\le j_p} a^{(\alpha)}_{j_1\cdots j_p}(x,t)dx^{(\alpha)}_{ij_1}\wedge\cdots dx^{(\alpha)}_{ij_p}$$

involving $dx^{(\alpha)}_1,\cdots,dx^{(\alpha)}_{n_\alpha}$ only. In the following we will omit the proofs from Lemma 3.17 through Lemma 3.20, as they are simple calculations.

3.16 Lemma. *$\phi_i^{(\alpha)}$ splits the exact sequences $\mathcal{O}^{(\alpha)}_{U_i}$-modules*

$$0\to F^1(\Omega^{\cdot}_{\mathfrak{X}_\alpha})_{|U_\alpha}\to\Omega^{\cdot}_{\mathfrak{X}_\alpha|U_\alpha}\to\Omega^{\cdot}_{\mathfrak{X}_\alpha/M|U_\alpha}\to 0$$

and satisfies

$$\phi_i^{(\alpha)}\cdot d_{\mathfrak{X}_\alpha/M}=d_{U_i^{(\alpha)}},$$

where $d_{\mathfrak{X}_\alpha/M}$ is the differential of the complex $\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}$, i.e., the relative exterior differentiation.

Hereafter we use the notation $\beta(\alpha;p;i_0\cdots i_q)$ ($\alpha\in\square_n$, p is a positive integer and $i_0,\cdots,i_1\in\Lambda_\alpha$) ,which represents the component of $\beta\in s(totC^{\cdot}(\mathcal{U}.,\mathcal{K}^{\cdot}_{.}))^r$ (r $=|\alpha|+p+q$) lying in $\Gamma(U^{(\alpha)}_{i_0}\cap\cdots\cap U^{(\alpha)}_{i_q},\mathcal{K}^p_\alpha)$, for a complex of abelian sheaves $\mathcal{K}^{\cdot}_{.}$ is a complex on $\mathfrak{X}.$. We define

$$\phi: K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M})\to K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}.})$$

by

$$(\phi\beta)(\alpha;p;i_0\cdots i_q):=\phi^{(\alpha)}_{i_0}(\beta(\alpha;p;i_0\cdots i_q))$$

for $\beta\in K^{\cdot}(\Omega_{\mathfrak{X}./M})$.

3.17 Lemma. *ϕ splits the exact sequence of abelian groups*

$$0\to K^{\cdot}(F^1(\Omega^{\cdot}_{\mathfrak{X}.}))\to K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}.})\to K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M})\to 0$$

Define $J: K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M})\to K^{\cdot+1}(\Omega^{\cdot}_{\mathfrak{X}.})$ by

$$(J\beta)(\alpha;p;i_0\cdots i_q):=(-1)^{p+|\alpha|+1}(\phi^{(\alpha)}_{i_0}-\phi^{(\alpha)}_{i_1})(\beta(\alpha;p;i_1\cdots i_q))$$

Then we have $J(K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M}))\subset K^{\cdot}(F^1(\Omega^{\cdot}_{\mathfrak{X}.}))$

3.18 Lemma. $\delta\phi - \phi\delta = J$

,where δ *is the Cěch coboundary map.*

Define the *total Lie derivarive* $L_M : K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}.}) \to K^{\cdot+1}(\Omega^{\cdot}_{\mathfrak{X}.})$ with respect to the parameters of M by

$$(L_M\beta)(\alpha; p; i_0 \cdots i_q) := (-1)^{|\alpha|} d_M(\beta(\alpha; p; i_0 \cdots i_q)$$

Note that $L_M(K^{\cdot}(F^i(\Omega^{\cdot}_{\mathfrak{X}.})) \subset K^{\cdot}(F^{i+1}(\Omega^{\cdot}_{\mathfrak{X}.}))$. We denote by

$$d_{\mathfrak{X}.} : K^{\cdot} \to K^{\cdot+1}(\Omega^{\cdot}_{\mathfrak{X}.})$$

the morphism of $\mathbb{C}$-vector spaces induced by exterior differentiations $d_{\mathfrak{X}_\alpha}$: $\Omega^{\cdot}_{\mathfrak{X}_\alpha} \to \Omega^{\cdot}_{\mathfrak{X}_\alpha}$, and by

$$d_{\mathfrak{X}./M} : K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M}) \to K^{\cdot+1}(\Omega^{\cdot}_{\mathfrak{X}./M}).$$

the one induced by relative exterior differentiations $d_{\mathfrak{X}./M} : \Omega^{\cdot}_{\mathfrak{X}_\alpha/M} \to \Omega^{\cdot+1}_{\mathfrak{X}./M}$. Combining Lemma 3.16 and (3.37), we obtain

3.19 Lemma. $(-1)^{|\alpha|} d_{\mathfrak{X}.}\phi = L_M\phi + (-1)^{|\alpha|}\phi\, d_{\mathfrak{X}./M}$.

We define by

$$(3.38) \qquad D^{(r)} := \oplus_{|\alpha|+p+q=r}\{\sum_{1\le j\le n} (-1)^{\varepsilon_j} d_j^{(\alpha,p,q)*} + (-1)^{|\alpha|} d_{\mathfrak{X}_\alpha}^{(p,q)} + \delta^{(\alpha,p,q)}\},$$

the differential map $K^r(\Omega^{\cdot}_{\mathfrak{X}.}) \to K^{r+1}(\Omega^{\cdot}_{\mathfrak{X}.})$ of complexes of $\mathbb{C}$-vector spaces, where

$$d_j^{(\alpha,p,q)*} : C^q(\mathcal{U}_\alpha, \Omega^p_{\mathfrak{X}_\alpha}) \to C^q(\mathcal{U}_{\alpha+e_j}, \Omega^p_{\mathfrak{X}_{\alpha+e_j}}) \ (e_j = (0 \cdots \check{1} \cdots 0))$$

is the map induced by the holomorphic map $E_{\alpha\alpha+e_j} : \mathfrak{X}_{\alpha+e_j} \to \mathfrak{X}_\alpha$ over M corresponding to an arrow $\alpha \to \alpha + e_j$ in $\square_n$,

$$\varepsilon_j = \alpha_0 + \alpha_1 + \cdots + \alpha_{j-1} \ (1 \le j \le n) \text{ for } \alpha = (\alpha_0 \cdots \alpha_n) \in \square_n,$$

$d_{\mathfrak{X}_\alpha}^{(p,q)} : C^q(\mathcal{U}_\alpha, \Omega^p_{\mathfrak{X}_\alpha}) \to C^q(\mathcal{U}_\alpha, \Omega^{p+1}_{\mathfrak{X}_\alpha})$ the exterior differentiation on $\mathfrak{X}_\alpha$,

and $\delta^{(\alpha,p,q)} : C^q(\mathcal{U}_\alpha, \Omega^p_{\mathfrak{X}_\alpha}) \to C^{q+1}(\mathcal{U}_\alpha, \Omega^p_{\mathfrak{X}_\alpha})$ the Cěch coboundary map.

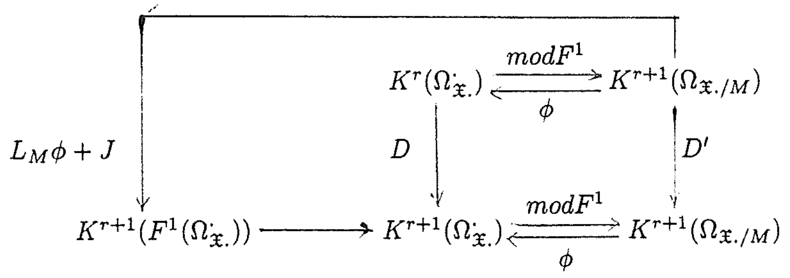
Similarly, we define the differential map $K^m(\Omega^{\cdot}_{\mathfrak{X}./M}) \to K^{m+1}(\Omega^{\cdot}_{\mathfrak{X}./M})$ for $\Omega^{\cdot}_{\mathfrak{X}./M}$ instead of $\Omega^{\cdot}_{\mathfrak{X}.}$ by

$$(3.39) \quad D'^{(r)} := \oplus_{|\alpha|+p+q=r}\{\sum_{1\le j\le n} (-1)^{\varepsilon_j} d_j^{(\alpha,p,q)\prime *} + (-1)^{|\alpha|} d_{\mathfrak{X}_\alpha/M}^{(p,q)} + \delta^{(\alpha,p,q)\prime}\}.$$

Combining Lemma 3.18 and Lemma 3.19, we find

Lemma 3.20.

$$D\phi - \phi D' = L_M\phi + J$$



Thus the connecting homomorphism associated to the exact sequence (3.35) is induced by the morphism of $\mathbb{C}$*-vector spaces*

$$L_M\phi + J: \ K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M}) \to K^{\cdot+1}(F^1(\Omega^{\cdot}_{\mathfrak{X}.})).$$

Proof. We denote by D and D' the differentials of the simple complexes of $\mathbb{C}$-vector spaces $K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}.})$ and $K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M})$, respectively, defined in (3.38) and (3.39). Since

$$D = \textstyle\sum_{1\leq j\leq n}(-1)^{\varepsilon_j}d_j^* + (-1)^{|\alpha|}d^*_{\mathfrak{X}_\alpha}\delta_\alpha \text{ on } C^{\cdot}(\mathcal{U}_\alpha, \Omega^{\cdot}_{\mathfrak{X}_\alpha}) \text{ and,}$$
$$D' = \textstyle\sum_{1\leq j\leq n}(-1)^{\varepsilon_j}d_j^* + (-1)^{|\alpha|}d_{\mathfrak{X}_\alpha/M} + \delta_\alpha \text{ on } C^{\cdot}(\mathcal{U}_\alpha, \Omega^{\cdot}_{\mathfrak{X}_\alpha/M}),$$

we have

$$\begin{aligned} D\phi - \phi D' &= (\textstyle\sum(-1)^{\varepsilon_j}d_j^*)\phi - \phi(\textstyle\sum(-1)^{\varepsilon_j}d_j^*) \\ &\quad +(-1)^{|\alpha|}(d_{\mathfrak{X}_\alpha}\phi - \phi d_{\mathfrak{X}_\alpha/M}) + (\delta_\alpha\phi - \phi\delta_\alpha) \end{aligned} \tag{3.40}$$

on $C^{\cdot}(\mathcal{U}_\alpha, \Omega^{\cdot}_{\mathfrak{X}_\alpha/M})$. From the requirement (iii) in (3.31), satisfied by the map $E_{\alpha\alpha+e_j}$ $(0 \leq j \leq n)$, it follows that

$$(\sum(-1)^{\varepsilon_j}d_j^*)\phi - \phi(\sum(-1)^{\varepsilon_j}d_j^*) = 0.$$

Therefore we obtain Lemma 3.20 by Lemma 3.18, Lemma 3.19 and (3.40).

Q.E.D.

On each $V_i^{(\alpha)}$, we define the *total interior product* with respect to the parameters of M

$$I^i_{(\alpha)}: \Omega^{\cdot}_{\mathfrak{X}_\alpha|V_i^{(\alpha)}} \to \Omega^{\cdot}_{\mathfrak{X}_\alpha|V_i^{(\alpha)}}$$

by

$$I^i_{(\alpha)}(\sum_{\substack{j_1<\cdots<j_r\\ k_1<\cdots<k_r\\ r+s=p\\ 0\leq r,s}} a_{j_1\cdots j_r k_1\cdots k_s}(x,t)dx^{(\alpha)}_{ij_1}\wedge\cdots\wedge dx_i j^{(\alpha)}_r\wedge dt_{k_1}\wedge\cdots dt_{k_s})$$
$$=\sum_{\substack{j_1<\cdots<j_r\\ k_1<\cdots<k_r\\ r+s=p\\ 0\leq r,s}} s\cdot a_{j_1\cdots j_r k_1\cdots k_s}(x,t)dx^{(\alpha)}_{ij_1}\wedge\cdots\wedge dx^{(\alpha)}_{ij_r}\wedge dt_{k_1}\wedge\cdots dt_{k_s})$$

for a local holomorphic p-form on $V_i^{(\alpha)}$. When $p=0$, we put $I_i^{(\alpha)}\equiv 0$. Notice

$$I^i_{(\alpha)}(\Omega^{\cdot}_{\mathfrak{X}_\alpha|V_i^{(\alpha)}})\subset F^1(\Omega^{\cdot}_{\mathfrak{X}_\alpha})_{|V_i^{(\alpha)}}.$$

Define

$$\lambda : K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}.})\to K^{\cdot+1}(\Omega^{\cdot}_{\mathfrak{X}.})$$

by $$(\lambda\beta)(\alpha;p;i_0\cdots i_q):=(-1)^{p+|\alpha|}(I^{i_0}_{(\alpha)}-I^{i_1}_{(\alpha)})(\beta(\alpha;p;i_1\cdots i_q))$$

Lemma 3.21.

$$\lambda\phi\equiv J\ mod\ K^{\cdot}(F^2(\Omega^{\cdot}_{\mathfrak{X}.}))$$

Proof. It suffices to show that, for $\beta(\alpha;p;i_1\cdots i_q)\in\Gamma(U^{(\alpha)}_{i_1}\cap\cdots\cap U^{(\alpha)}_{i_q},\ \Omega^p_{\mathfrak{X}_\alpha/M})$ of the form

$$\beta(\alpha;p;i_1\cdots i_q)=[\mu dx^{(\alpha)}_{i_1j_1}\wedge\cdots\wedge dx^{(\alpha)}_{i_1j_p}],$$

where μ is a local holomorphic function on $U^{(\alpha)}_{i_1}\cap\cdots\cap U^{(\alpha)}_{i_q}$,

$$\lambda(\phi\beta)(\alpha;p;i_0i_1\cdots i_p)-J\beta(\alpha;p;i_0i_1\cdots i_p)\in F^2(\Omega^p_{\mathfrak{X}_\alpha})\tag{3.41}$$

holds. Since

$$\mu dx^{(\alpha)}_{i_1j_1}\wedge\cdots\wedge dx^{(\alpha)}_{i_1j_p}$$
$$\equiv\mu(\sum_{j'_1<\cdots<j'_p}\frac{\partial(x^{(\alpha)}_{i_1j_1},\cdots,x^{(\alpha)}_{i_1j_p})}{\partial(x^{(\alpha)}_{i_0j'_1},\cdots,x^{(\alpha)}_{i_0j'_p})}dx^{(\alpha)}_{i_0j'_1}\wedge\cdots\wedge dx^{(\alpha)}_{i_0j'_p}$$
$$+\sum_{k=1}^m\sum_{j'_1<\cdots<j'_{p-1}}\frac{\partial(x^{(\alpha)}_{i_1j_1},\cdots,x^{(\alpha)}_{i_1j_p})}{\partial(x^{(\alpha)}_{i_0j'_1},\cdots,x^{(\alpha)}_{i_0j'_{p-1}}t_k)}dx^{(\alpha)}_{i_0j'_1}\wedge\cdots\wedge dx^{(\alpha)}_{i_0j'_{p-1}}\wedge dt_k)$$
$$\mod\quad F^2(\Omega^p_{\mathfrak{X}_\alpha}),$$

$$(3.42)\qquad J\beta(\alpha;p;i_0i_1\cdots i_p)=(-1)^{p+|\alpha|+1}(\phi_{i_0}^{(\alpha)}-\phi_{i_1}^{(\alpha)})\beta(\alpha;p;i_1\cdots i_q)\equiv$$

$$(-1)^{p+|\alpha|+1}\mu(-\sum_{k=1}^m\sum_{j_1'<\cdots<j_{p-1}'}\frac{\partial(x_{i_1j_1}^{(\alpha)},\cdots,x_{i_1j_p}^{(\alpha)})}{\partial(x_{i_0j_1'}^{(\alpha)},\cdots,x_{i_0j_{p-1}'}^{(\alpha)}\ t_k)}dx_{i_0j_1'}^{(\alpha)}\wedge\cdots\wedge dx_{i_0j_{p-1}'}^{(\alpha)}\wedge dt_k)$$

$$\mod F^2(\Omega_{\mathfrak{X}_\alpha}^p)$$

On the other hand,

$$\lambda(\phi\beta)(\alpha;p;i_0i_1\cdots i_p)$$

$$=(-1)^{p+|\alpha|}(I_{(\alpha)}^{i_0}-I_{(\alpha)}^{i_1})(\mu dx_{i_1j_1}^{(\alpha)}\wedge\cdots\wedge dx_{i_1j_p}^{(\alpha)})$$

$$(3.43)\qquad =(-1)^{p+|\alpha|}I_{(\alpha)}^{i_0}(\mu(\sum_{j_1'<\cdots<j_p'}\frac{\partial(x_{i_1j_1}^{(\alpha)},\cdots,x_{i_1j_p}^{(\alpha)})}{\partial(x_{i_0j_1'}^{(\alpha)},\cdots,x_{i_0j_p'}^{(\alpha)})}dx_{i_0j_1'}^{(\alpha)}\wedge\cdots\wedge dx_{i_0j_p'}^{(\alpha)}$$

$$+\sum_{k=1}^m\sum_{j_1'<\cdots<j_{p-1}'}\frac{\partial(x_{i_1j_1}^{(\alpha)},\cdots,x_{i_1j_p}^{(\alpha)})}{\partial(x_{i_0j_1'}^{(\alpha)},\cdots,x_{i_0j_{p-1}'}^{(\alpha)}\ t_k)}dx_{i_0j_1'}^{(\alpha)}\wedge\cdots\wedge dx_{i_0j_{p-1}'}^{(\alpha)}\wedge dt_k)+\cdots))\equiv$$

$$(-1)^{p+|\alpha|}\mu(\sum_{k=1}^m\sum_{j_1'<\cdots<j_{p-1}'}\frac{\partial(x_{i_1j_1}^{(\alpha)},\cdots,x_{i_1j_p}^{(\alpha)})}{\partial(x_{i_0j_1'}^{(\alpha)},\cdots,x_{i_0j_{p-1}'}^{(\alpha)}\ t_k)}dx_{i_0j_1'}^{(\alpha)}\wedge\cdots\wedge dx_{i_0j_{p-1}'}^{(\alpha)}\wedge dt_k)$$

$$\mod F^2(\Omega_{\mathfrak{X}_\alpha}^p)$$

Then (3.41) follows from (3.42) and (3.43).

Q.E.D.

By Lemma 3.20 and Lemma 3.21 we infer that the connecting homomorphism associated to the exact sequence in (3.35) is induced from

$$K^\cdot(\Omega_{\mathfrak{X}./M}^\cdot)\xrightarrow{\phi}K^\cdot(\Omega_{\mathfrak{X}.}^\cdot)\xrightarrow{L_M+\lambda}K^\cdot(F^1(\Omega_{\mathfrak{X}.}^\cdot))\xrightarrow{mod\ F^2}K^\cdot(Gr^1(\Omega_{\mathfrak{X}.}^\cdot)).$$

Furthermore, since

$$K^\cdot(\Omega_{\mathfrak{X}./M}^\cdot)\xrightarrow{\phi}K^\cdot(\Omega_{\mathfrak{X}.}^\cdot)/K^\cdot(F^1(\Omega_{\mathfrak{X}.}^\cdot))\equiv K^\cdot(\Omega_{\mathfrak{X}./M}^\cdot)$$

is the identity map and

$$(L_M+\lambda)(K^\cdot(F^1(\Omega_{\mathfrak{X}.}^\cdot))\subset K^\cdot(F^2(\Omega_{\mathfrak{X}.}^\cdot)),$$

we conclude that this connecting homomorphism is induced from $L_M+\lambda: K^\cdot(\Omega_{\mathfrak{X}.}^\cdot)\longrightarrow K^\cdot(F^1(\Omega_{\mathfrak{X}.}^\cdot))$ by passing to quotients, i.e.,

$$K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}./M}) = K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}.})/K^{\cdot}(F^1(\Omega^{\cdot}_{\mathfrak{X}.}))$$

(3.44)
$$\xrightarrow{L_M+\lambda} K^{\cdot}(F^1(\Omega^{\cdot}_{\mathfrak{X}.}))/K^{\cdot}(F^2(\Omega^{\cdot}_{\mathfrak{X}.})) = K^{\cdot}(Gr^1(\Omega^{\cdot}_{\mathfrak{X}.})).$$

Lemma 3.22.
$$(L_M + \lambda)D + D(L_M + \lambda) = 0$$

Proof. The proof will be accomplished after proving several claims.

Claim 1.

$$d_j^* L_M + L_M d_j^* = 0 \quad and$$
$$d_j^* \lambda + \lambda d_j^* = 0 \quad (0 \leq j \leq n)$$

Proof. Let $\omega \in K^r(\Omega^{\cdot}_{\mathfrak{X}.}) := \oplus_{|\alpha|+p+q=r} C^q(\mathcal{U}_\alpha, \Omega^{\cdot p}_{\mathfrak{X}_\alpha})$ and let $\beta \in \Box_n$ be such that there exist $\alpha \in \Box_n$ with $\beta = \alpha + e_j$, where $e_j = (0 \cdots \overset{j}{1} \cdots 0)$. Then

$$(L_M d_j^* \omega)(\beta; p; i_0 \cdots i_q) == (-1)^{|\beta|} d_M(E^*_{\alpha\beta}\omega(\alpha; p; \lambda_{\alpha\beta}(i_0), \cdots, \lambda_{\alpha\beta}(i_q)),$$

where $E_{\alpha\beta} : \mathfrak{X}_\beta \longrightarrow \mathfrak{X}_\alpha$ and $\lambda_{\alpha\beta} : \Lambda_\beta \longrightarrow \Lambda_\alpha$ are such which have been defined in (3.31). On the other hand,

$$\begin{aligned}&(d_j^* L_M \omega)(\beta; p; i_0 \cdots i_q)\\ &= E^*_{\alpha\beta}(L_M\omega)(\alpha; p; \lambda_{\alpha\beta}(i_0), \cdots, \lambda_{\alpha\beta}(i_q))\\ &= E^*_{\alpha\beta}((-1)^{|\alpha|} d_M \omega(\alpha; p; \lambda_{\alpha\beta}(i_0), \cdots, \lambda_{\alpha\beta}(i_q))\end{aligned}$$

Since $E_{\alpha\beta} = e_{\alpha\beta} \times id_M$ on $V^{(\beta)}_{i_0} \cap \cdots \cap V^{(\beta)}_{i_q}$, $d_M E^*_{\alpha\beta} = E^*_{\alpha\beta} d_M$. Hence

$$(-1)^{|\beta|}(L_M d_j^* \omega)(\beta; p; i_0 \cdots i_q) = (-1)^{|\alpha|}(d_j^* L_M \omega)(\beta; p; i_0 \cdots i_q)$$

, which means $d_j^* L_M + L_M d_j^* = 0$ as $|\beta| = |\alpha| + 1$. Similarly, we can show that $d_j^* \lambda + \lambda d_j^* = 0$.

Q.E.D.

Claim 2.
$$\delta L_M + L_M \delta = 0 \quad and \quad \delta\lambda + \lambda\delta = 0$$

Proof. The first identity is trivial. We are going to show the second identity. Let $\omega \in K^r(\Omega^{\cdot}_{\mathfrak{X}.}) = \oplus_{|\alpha|+p+q-1=r} C^{q-1}(\mathcal{U}_\alpha, \Omega^p_{\mathfrak{X}_\alpha})$. Then $\lambda\delta\omega \in K^{r+2}(\Omega^{\cdot}_{\mathfrak{X}.}) = \oplus_{|\alpha|+p+q+1} C^{q+1}(\mathcal{U}_\alpha, \Omega^p_\alpha)$ is giben by

$$(\lambda\delta\omega)(\alpha;p;i_0\cdots i_{q+1}) = (-1)^{p+|\alpha|}(I^{i_0}_{(\alpha)} - I^{i_1}_{(\alpha)}(\delta\omega)(\alpha;p;i_1\cdots i_{q+1})$$

(3.45)

$$= \textstyle\sum_{j=1}^{q+1}(-1)^{j-1}(I^{i_0}_{(\alpha)} - I^{i_1}_{(\alpha)})\omega(\alpha;p;i_1\cdots \check{i}_j\cdots i_{q+1}).$$

On the other hand,

$$(\delta\lambda\omega)(\alpha;p;i_0\cdots i_{q+1}) = (-1)^{p+|\alpha|}\textstyle\sum_{j=0}^{q+1}(-1)^j(\lambda\omega)(\alpha;p;i_0\cdots\check{i}_j\cdots i_{q+1})$$

$$= (I^{i_1}_{(\alpha)} - I^{i_2}_{(\alpha)})\omega(\alpha;p;i_2\cdots i_{q+1}) - (I^{i_0}_{(\alpha)} - I^{i_2}_{(\alpha)})\omega(\alpha;p;i_2\cdots i_{q+1})$$

(3.46)

$$+\textstyle\sum_{j=2}^{q+1}(-1)^j(I^{i_0}_{(\alpha)} - I^{i_1}_{(\alpha)})\omega)(\alpha;p;i_1\cdots\check{i}_j\cdots i_{q+1}).$$

$$= -\{\textstyle\sum_{j=1}^{q+1}(-1)^{j-1}(I^{i_0}_{(\alpha)} - I^{i_1}_{(\alpha)})\omega)(\alpha;p;i_1\cdots\check{i}_j\cdots i_{q+1})\}.$$

From (3.45) and (3.46) it follows that

$$\{(\lambda\delta+\delta\lambda)\omega\}(\alpha;p;i_0\cdots i_{q+1}) = 0.$$

Hence $\lambda\delta+\delta\lambda = 0$ as required.

Q.E.D.

Claim 3.

$$L_M d_{\mathfrak{X}.} + d_{\mathfrak{X}.}L_M = 0$$

Proof. Let $\omega \in K^r(\Omega^{\cdot}_{\mathfrak{X}.}) := \oplus_{|\alpha|+(p-2)+q=r}C^q(\mathcal{U}_\alpha, \Omega^{p-2}_{\mathfrak{X}_\alpha})$. Then $L_M d.\omega \in K^{r+2}(\Omega^{\cdot}_{\mathfrak{X}.}) = \oplus_{|\alpha+p+q=r+2}C^q(\mathcal{U}_\alpha, \Omega^p_{\mathfrak{X}_\alpha})$ is given by

$$\begin{aligned}
&(L_M d_{\mathfrak{X}.}\omega)(\alpha;p;i_0\cdots i_q)\\
&= (-1)^{|\alpha|}d_M(d_{\mathfrak{X}.}\omega)(\alpha;\text{p-1};i_0\cdots i_q))\\
&= (-1)^{|\alpha|+1}d_{\mathfrak{X}.}(d_M\omega)(\alpha;\text{p-1};i_0\cdots i_q)\\
&= (-1)d_{\mathfrak{X}.}(L_M\omega)(\alpha;\text{p-1};i_0\cdots i_q)\\
&= -(d_{\mathfrak{X}.}L_M\omega)(\alpha;p;i_0\cdots i_q)
\end{aligned}$$

Hence we have done.

Q.E.D.

Claim 4. *On each* $V_i^{(\alpha)}$ $(\alpha \in \square_n, i \in \Lambda_\alpha)$,

$$I^i_{(\alpha)} d_{\mathfrak{X}_\alpha} - d_{\mathfrak{X}_\alpha} I^i_{(\alpha)} = d^\alpha_M$$

holds.

Proof. It suffices to show that for a local holomorphic form ω on $V_i^{(\alpha)}$ of the form

$$\omega = \mu dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx^{(\alpha)}_{ij_p} \wedge dt_{k_1} \wedge \cdots dt_{k_q}),$$

where μ is a local holomorphic function,

$$I^i_{(\alpha)} d_{\mathfrak{X}_\alpha} \omega - d_{\mathfrak{X}_\alpha} I^i_{(\alpha)} \omega = d^\alpha_M \omega$$

holds. Meanwhile,

$$d_{\mathfrak{X}_\alpha} \omega = \sum_{j \notin (j_1 \cdots j_p)} \frac{\partial \mu}{\partial x^{(\alpha)}_{ij}} dx^{(\alpha)}_{ij} \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx_i j_p^{(\alpha)} \wedge dt_{k_1} \wedge \cdots dt_{k_q}$$

$$+ \sum_{j \notin (k_1 \cdots k_q)} \frac{\partial \mu}{\partial t_j} dt_j \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx_i j_p^{(\alpha)} \wedge dt_{k_1} \wedge \cdots dt_{k_q}.$$

Hence

$$I^i_{(\alpha)} (d_{\mathfrak{X}_\alpha} \omega) = q(\sum_{j \notin (j_1 \cdots j_p)} \frac{\partial \mu}{\partial x^{(\alpha)}_{ij}} dx^{(\alpha)}_{ij} \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx_i j_p^{(\alpha)} \wedge dt_{k_1} \wedge \cdots dt_{k_q})$$

(3.47)

$$+(q+1)(\sum_{j \notin (k_1 \cdots k_q)} \frac{\partial \mu}{\partial t_j} dt_j \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx_i j_p^{(\alpha)} \wedge dt_{k_1} \wedge \cdots dt_{k_q})$$

On the other hand,

$$I^i_{(\alpha)} \omega = q \mu dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx_i j_p^{(\alpha)} \wedge dt_{k_1} \wedge \cdots dt_{k_q}).$$

Hence

$$d_{\mathfrak{X}_\alpha} (I^i_{(\alpha)} \omega) = q(\sum_{j \notin (j_1 \cdots j_p} \frac{\partial \mu}{\partial x^{(\alpha)}_{ij}} dx^{(\alpha)}_{ij} \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx_i j_p^{(\alpha)} \wedge dt_{k_1} \wedge \cdots dt_{k_q})$$

(3.48)

$$+q(\sum_{j \notin (k_1 \cdots k_p} \frac{\partial \mu}{\partial t_j} dt_j \wedge dx^{(\alpha)}_{ij_1} \wedge \cdots \wedge dx_i j_p^{(\alpha)} \wedge dt_{k_1} \wedge \cdots dt_{k_q})$$

From (3.47) and (3.48) it follows that

$$I^i_{(\alpha)} (d_{\mathfrak{X}_\alpha} \omega) - d_{\mathfrak{X}_\alpha} (I^i_{(\alpha)} \omega) = d^\alpha_M \omega.$$

Q.E.D.

Claim 5.

$$\lambda d_{\mathfrak{X}.} + d_{\mathfrak{X}.}\lambda = 0$$

Proof. Let $\omega \in K^r(\Omega^{\cdot}_{\mathfrak{X}.}) := \oplus_{|\alpha|+(p-1)+(q-1)=r} C^{q-1}(\mathcal{U}_\alpha, \Omega^{p-1}_{\mathfrak{X}_\alpha})$. Then $\lambda d_{\mathfrak{X}.}\omega \in K^{r+2}(\Omega^{\cdot}_{\mathfrak{X}.}) = \oplus_{|\alpha|+p+q=r+2} C^q(\mathcal{U}_\alpha, \Omega^p_\alpha)$ is given by

$$\begin{aligned}
&(\lambda d_{\mathfrak{X}.}\omega)(\alpha; p; i_0 \cdots i_q) \\
&= (-1)^{p+|\alpha|}(I^{i_0}_{(\alpha)} - I^{(i_1)}_{(\alpha)})(d_{\mathfrak{X}_\alpha}\omega)(\alpha; p; i_0 \cdots i_q) \\
&= (-1)^{p+|\alpha|} d_{\mathfrak{X}_\alpha}((I^{i_0}_{(\alpha)} - I^{(i_1)}_{(\alpha)})\omega)(\alpha; p; i_0 \cdots i_q) \\
&= -d_{\mathfrak{X}_\alpha}((-1)^{p-1+|\alpha|}(I^{i_0}_{(\alpha)} - I^{(i_1)}_{(\alpha)})\omega))(\alpha; p; i_0 \cdots i_q) \ (\mathit{Claim}\ 4) \\
&= -d_{\mathfrak{X}_\alpha}(\lambda\omega)(\alpha; p; i_0 \cdots i_q)
\end{aligned}$$

Hence

$$((\lambda d_{\mathfrak{X}.} + d_{\mathfrak{X}.}\lambda)\omega)(\alpha; p; i_0 \cdots i_q) = 0$$

This means $\lambda d_{\mathfrak{X}.} + d_{\mathfrak{X}.}\lambda = 0$.

Q.E.D.

Proof of Lemma 3.22. Now we are ready to prove Lemma 3.22. By Claim 1, Claim 2, Claim 3 and Claim 5, we have

$$\begin{aligned}
&L_M D^{(r)} \\
&= L_M\{(\oplus_{\substack{|\alpha|+p+q \\ =r}}\{\textstyle\sum_{1\le j\le n}(-1)^{\epsilon_j} d_j^{(\alpha,p,q)*} + (-1)^{|\alpha|} d^{(p,q)}_{\mathfrak{X}_\alpha} + \delta^{(\alpha,p,q)}\} \\
&= \{\oplus_{\substack{|\alpha|+p+q \\ =r}}(\textstyle\sum_{1\le j\le n}(-1)^{\epsilon_j} L_M d_j^{(\alpha,p,q)*} + (-1)^{|\alpha|} d_M d^{(p,q)}_{\mathfrak{X}_\alpha} + L_M \delta^{(\alpha,p,q)})\} \\
&= (-1)\{(\oplus_{\substack{|\alpha|+p \\ +q=r}}(\textstyle\sum_{1\le j\le n}(-1)^{\epsilon_j} d_j^{(\alpha,p+1,q)*} L_M + (-1)^{|\alpha|} d^{(p,q)}_{\mathfrak{X}_\alpha} L_M + \delta^{(\alpha,p+1,q)} L_M)\} \\
&= \{(\oplus_{\substack{|\alpha|+p+q \\ =r+1}}(\textstyle\sum_{1\le j\le n}(-1)^{\epsilon_j} d_j^{(\alpha,p,q)*} + (-1)^{|\alpha|} d^{(p,q)}_{\mathfrak{X}_\alpha} + \delta^{(\alpha,p,q)}\} L_M \\
&= (-1) D^{(r+1)} L_M,
\end{aligned}$$

and similarly $\lambda D^{(r)} = (-1) D^{(r+1)}\lambda$. Therefore,

$$(L_M + \lambda) D^{(r)} = (-1) D^{(r+1)} (L_M + \lambda).$$

Q.E.D.

Consequently, by (3.36), (3.44) and Lemma 3.22, we conclude that the connection

$$\begin{aligned}\nabla : \mathbb{H}^q(M, s(a_{\cdot 1*}\Omega^{\cdot}_{\mathfrak{X}_\cdot/M})) &\simeq H^q(K^{\cdot}(\Omega_{\mathfrak{X}_\cdot/M})^{\cdot})[1]) \\ &\to \Gamma(M, \Omega^1_M) \otimes_{\Gamma(M,\mathcal{O}_M)} \mathbb{H}^q(M, s(a_{\cdot 1*}\Omega^{\cdot}_{\mathfrak{X}_\cdot/M})[1]) \simeq H^q(K^{\cdot}(Gr^1(\Omega^{\cdot}_{\mathfrak{X}_\cdot})[1]) \\ &\simeq \Gamma(M, \Omega^1_M) \otimes_{\Gamma(M,\mathcal{O}_M)} H^q(K^{\cdot}(\Omega^{\cdot}_{\mathfrak{X}_\cdot/M})[1])\end{aligned}$$

is nothing but the homomorphism induced by $L_M + \lambda$ in (3.44). We should note that $L_M + \lambda$ is independent on choice of a system of Stein coverings of $\mathfrak{X}_\cdot$ subject to the conditions (i) through (iv) in (3.31) (cf. [1, p.220, (3.7.1)]).

Step(III) Proof of $Ker\nabla = Im\{R^\ell_{\mathbb{C}}(\pi) \to R^\ell_{\mathcal{O}}(\pi)\}$

Let $\mathcal{O}_\infty$ be the sheaf of germs of C^∞ functions on M. If $K^{\cdot}(\Omega^{\cdot}_{\infty\mathfrak{X}_\cdot/M})$ is a C^∞ analogue of $K^{\cdot}(\Omega_{\mathfrak{X}_\cdot/M})$ constructed by use of the complex of C^∞ $\mathbb{C}$-valued relative differential forms on $\mathfrak{X}_\cdot$ over M, then locally we have

$$R^\ell_{\mathcal{O}_\infty}(\pi) := H^\ell(K^{\cdot}(\Omega^{\cdot}_{\infty\mathfrak{X}_\cdot/M})) \tag{3.49}$$

Furthermore, we can define the C^∞ analogue

$$\nabla_\infty : R^\ell_{\mathcal{O}_\infty}(\pi) \to \Omega^1_M \otimes R^\ell_{\mathcal{O}_\infty}(\pi) \tag{3.50}$$

of the connection $\nabla : R^\ell_{\mathcal{O}}(\pi) \to \Omega^1_M \otimes R^\ell_{\mathcal{O}}(\pi)$ so that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^\ell_{\mathbb{C}}(\pi) & \longrightarrow & R^\ell_{\mathcal{O}}(\pi) & \xrightarrow{\nabla} & \Omega^1_M \otimes R^\ell_{\mathcal{O}}(\pi) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R^\ell_{\mathbb{C}}(\pi) & \longrightarrow & R^\ell_{\mathcal{O}_\infty}(\pi) & \xrightarrow{\nabla_\infty} & \Omega^1_{\infty M} \otimes R^\ell_{\mathcal{O}_\infty}(\pi) \end{array}$$

Therefore it suffices to show that

$$Ker\nabla_\infty = Im\{R^\ell_{\mathbb{C}}(\pi) \to R^\ell_{\mathcal{O}_\infty}(\pi)\} \tag{3.51}$$

Since $\Omega_{\infty\mathfrak{X}_\alpha/M^p}$ $(0 \le p \le dim_{\mathbb{R}}\mathfrak{X}_\alpha, \alpha \in \square_n)$ are fine sheaves, the explicite calculation of ∇_∞ in terms of $H^n(K^{\cdot}(\Omega^{\cdot}_{\infty\mathfrak{X}_\cdot/M}))$ remain valid verbally for all coverings of $\mathfrak{X}_\cdot$ which are subject to the conditions (i) through (iv) in (3.31), but not Stein open coverings. Since, by [1, Proposition 2.5], the family $\mathfrak{X}_\cdot \xrightarrow{\pi_\cdot} M$ $(\pi_\cdot = \pi \cdot a_\cdot)$ is C^∞ trivial at any point of M, we may take $\mathcal{U}_\alpha = \{\mathfrak{X}_\alpha\}$ for all $\alpha \in \square_n$ to calculate $H^n(K^{\cdot}(\Omega^{\cdot}_{\infty\mathfrak{X}_\cdot/M}))$ and

$$\nabla_\infty : H^\ell(K^{\cdot}(\Omega^{\cdot}_{\infty\mathfrak{X}_\cdot/M})) \to \Gamma(M, \Omega^1_{\infty M}) \otimes H^\ell(K^{\cdot}(\Omega^{\cdot}_{\infty\mathfrak{X}_\cdot/M})).$$

We fix this covering. Here we use the following symbols:

$$\begin{aligned}&K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}./M}) := s(\Gamma(\mathfrak{X}.,\Omega^\cdot_{\infty\mathfrak{X}./M}))\\&K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}.}) := s(\Gamma(\mathfrak{X}.,\Omega_{\infty\mathfrak{X}.})),\\&K^\cdot(F^k(\Omega^\cdot_{\infty\mathfrak{X}.})) := s(\Gamma(\mathfrak{X}.,F^p(\Omega_{\infty\mathfrak{X}.})) \quad \text{for } p\geq 1,\\&\text{etc..}\end{aligned}$$

Locally the connection ∇_∞ in (3.50) is induced from

$$L_M+\lambda : K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}./M}) \longrightarrow K^\cdot(F^1(\Omega^\cdot_{\infty\mathfrak{X}./M}))$$

by passing to quotients, i.e.,

$$\begin{aligned}&K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}./M}) \cong K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}.})/K^\cdot(F^1(\Omega^\cdot_{\infty\mathfrak{X}.}))\\&\xrightarrow{L_M+\lambda} K^\cdot(F^1(\Omega^\cdot_{\infty\mathfrak{X}.}))/K^\cdot(F^2(\Omega^\cdot_{\infty\mathfrak{X}.})) \cong \Gamma(M,\Omega^1_{\infty M})\otimes_{\Gamma(M,\mathcal{O}_\infty)}K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}./M}). \blacksquare\end{aligned}$$

Note that λ is in fact zero map in this case, because we may take $\mathcal{V}_\alpha = \{\mathfrak{X}_\alpha\}$ for any $\alpha\in\square_n$. We have the C^∞ analogue of the exact sequence in (3.36):

$$\begin{aligned}&0\longrightarrow K^\cdot(F^1(\Omega^\cdot_{\infty\mathfrak{X}.}))/K^\cdot(F^2(\Omega^\cdot_{\infty\mathfrak{X}./M}))\\(3.52)\qquad&\longrightarrow K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}.})/K^\cdot(F^2(\Omega^\cdot_{\infty\mathfrak{X}.})) \xrightarrow{P} K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}.})/K^\cdot(F^1(\Omega^\cdot_{\infty\mathfrak{X}./M})) \longrightarrow 0.\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\| \\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}./M})\end{aligned}$$

Remember that ∇_∞ comes from the connecting homomorphism of long exact sequences of cohomology associated to this exact sequence. On the other hand, we have

$$\begin{aligned}R^\ell_{\mathbb{C}}(\pi) &\cong \mathbb{R}^\ell\pi_*(s(a_{\cdot 1*}\mathbb{C}_{\mathfrak{X}}.)[1])\\&\cong \mathbb{R}^\ell_*(s(a_{\cdot 1*}\Omega^\cdot_{\infty\mathfrak{X}.})[1])\\&\cong H^\ell(K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}.})[1]) \quad \text{(locally)}\end{aligned}$$

Therefore, since the inclusion map $R^\ell_{\mathbb{C}}(\pi)\longrightarrow R^\ell_{\mathcal{O}_\infty}(\pi)$ is induced from the composite of the projection map

$$K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}.}) \longrightarrow K^\cdot(\Omega^\cdot_{\infty\mathfrak{X}.})/K^\cdot(F^2(\Omega^\cdot_{\infty\mathfrak{X}.}))$$

and the map P in (3.52), we infer that

$$Im\{R^{\ell}_{\mathbb{C}}(\pi) \to R^{\ell}_{\mathcal{O}_\infty}(\pi)\} \subset Ker\nabla_\infty \tag{3.53}$$

Hence,

$$rank_{\mathcal{O}_\infty} R^{\ell}_{\mathcal{O}_\infty}(\pi) \geq rank_{\mathbb{C}} Ker\nabla_\infty \geq rank_{\mathbb{C}} Im\{R^{\ell}_{\mathbb{C}}(\pi) \to R^{\ell}_{\mathcal{O}_\infty}(\pi)\}$$
$$= rank_{\mathcal{O}_\infty} R^{\ell}_{\mathcal{O}_\infty}(\pi)$$

From this (3.51) follows, and so $Ker\nabla = Im\{R^{\ell}_{\mathbb{C}}(\pi) \to R^{\ell}_{\mathcal{O}}(\pi)\}$. This means that horizontal local cross-sections of $R^{n}_{\mathcal{O}}(\pi)$ with respect to ∇ coinsides with local cross-sections of $R^{n}_{\mathbb{C}}$. That is, ∇ is the Gauss-Mannin connection on $R^{n}_{\mathcal{O}}(\pi)$.

The fact $\nabla F^p(R^n_{\mathbb{C}}(\pi)) \subset \Omega^1_M \otimes F^{p-1}(R^n_{\mathbb{C}}(\pi))$ follows from the explicit calculation of ∇ in *StepII*. This complete the proof of Theorem (3.14).

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