VARIATIONS OF MIXED HODGE STRUCTURE ARISING FROM CUBIC HYPEREQUISINGULAR FAMILIES OF COMPLEX PROJECTIVE VARIETIES, I

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Introduction

As far as we know the first paper which has treated variations of mixed Hodge structure from the view point of *infinitesimal mixed Torelli problem* is [14]. This Usui's paper has discussed such variations of mixed Hodge structure that arise from logarithmic deformation families of pairs (X, D), where X are complete nonsingular complex algebraic varieties and D are divisors with normal crossing on X. In this our paper, concerning complex projective singular cases we would like to propose a notion of *cubic hyperequisingular families of complex projective varieties* as families from which there arise naturally variations of mixed Hodge structure. To define such families we use cubic hyperresolutions of complex projective varieties due to V. Navarro Aznar, F. Guillén et al.([7]). The very rough defintion of the family is as follows: for a given complex projective variety X we take its cubic hyperresolution $X_. \to X$ and fix it. A cubic hyperequisingular family of complex projective varieties is defined to be an analytically "*locally trivial*" deformation family of a cubic object $X_. \to X$ (cf. Defintion2.4 below).

The initial motivation of this notion was related to the infinitesimal mixed Torelli problem for certain algebraic varieties. In [12] we have introduced the notion of *analytic subvarieties with locally stable paramtrizations of a complex manifold*, which is a unification and a generalization of closed complex analytic subsets of normal crossing (not necessarily of pure dimension) and analytic subvarieties with "*ordinary singularities*"; and showed that, for a given compact complex manifold Y, there exists a universal family $\tilde{\pi} : \tilde{\mathfrak{Z}}(Y) \to E(Y)$ for locally trivial families, i.e., families which are locally products at every ponit of their total space, of analytic subvarieties with locally stable parametrizations of Y, parametrized by (possibly nonreduced) complex spaces. A remarkable fact on this family is that it is C^∞ trivial at a non-singular point of $E(Y)_{red}$ (the reduction of $E(Y)$) and the C^∞ type of the fiber is constant over each connected component of $E(Y)$. Therefore we might expect that if Y is a complex projective manifold there arises naturally a variation of mixed Hodge structure from the family $\tilde{\pi} : \tilde{\mathfrak{Z}}(Y) \to E(Y)$ at a non-singular point of $E(Y)_{red}$. In the procedure of the trial to find out how to describe its variation of mixed Hodge structure, we have come to the notion of a cubic hyperequisingular family of complex projective varieties. Since we have not taken into consideration "*polarization*" of the family, embeddings are inessential for this notion. Hence we shall treat the family, forgetting its embedding.

Unfortunately we cannot so far prove that cubic hyperequisingular families of complex projective varieties can always be obtained from locally trivial families of complex projective varieties with "*ordinary singularites*" of any dimension, where we say that a complex projective variety is with "*ordinary singularities*" if it is locally isomorphic to one of the germs of pure dimensional hypersurfaces with locally stable parametrizations in a complex manifold. We can do this just for locally trivial families of complex projective varieties with "*ordinary singularities*" of dimension ≤ 3, i.e., complex projective varieties with "*ordinary* *singularities"* in classical sense, as well as for locally trivial families of complex projective varieties with normal crossings of any dimension. Hence we may at least say that the notion of cubic hyperequisingular family of complex projective varieties is non-empty in any dimension.

The arrangement of this paper is as follows: In §1 we shall review, for the readers' convenience, the definition of cubic hyperresolution of an algebraic variety and the basic facts about it, which are due to V. Navarro Aznar, F. Guillén et al. In §2 we shall give the definition of cubic hyperequisingular families of complex projective varieties and prove the relative version of cohomological descent of R-mudule sheaves (R: a commutative ring) and de Rham complexes for these families. In §3, using the results in §2, we shall prove that there arises naturally a variation of mixed Hodge structure from a cubic hyperequisingular family of complex projective varieties. The method is to extend the arguments in [3] (we also refer to [4]) and [11] to the relative case of cubic hyperequisingular families of complex projective manifolds. In §4 we shall give a formulation of the Kodaira-Spencer map for a cubic hyperequisingular family of complex projective varieties and prove a formula which relate the Kodaira-Spencer map and infinitesimal period map, i.e., the differential of the canonical map from the parameter space of a cubic hyperequisingular family of complex projective varieties to the moduli variety of mixed Hodge structure. In §5 we shall show that there arise naturally cubic hyperequisingular families of complex projective varieties from locally trivial families of complex projective varieties with "*ordinary singularities*" of dimension ≤ 3 as well as from locally trivial families of complex projective varieties with normal crossings of any dimension, and discuss an infinitesimal mixed Torelli type problem for surfaces with ordinary singularities.

Throughout this paper our method is basically "*complex analytic*" and we always consider algebraic manifolds and algebraic varieties over the complex number field as complex manifols and complex analytic varieties, where we use the term of analytic varieties in the sense of reduced complex spaces (possibly not irreducible). The paper will be devided into three parts. Part I has contained §1, §2, Part II will contain §3, and Part III §4, §5.

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§1 Preliminaries

In this section we shall briefly review the minimum of the theory of cubic hyperresolution of complex projective varieties due to V. Navarro Aznar, F. Guillèn et al., which will be needed later. For detailes we refer to [7].

1.1 Sheaf theory on diagrams of topological spaces

Let I be a finite ordered set. We always think of I as a category whose $Ob(I)$ are all of elements of I and the set of homomorphisms $Hom_I(i,j)$ $(i,j \in I)$ are defined by

$$Hom_I(i,j) = \begin{cases} i \to j & (\textit{an arrow from } i \textit{ to } j) \quad \text{if } i \leq j \\ \emptyset & \text{otherwise} \end{cases}$$

In the following we denote finite ordered sets by $I, J, \cdots$ etc.. Let $\mathfrak{C}$ be a category.

1.1 Definition. We call a functor $X. : I^\circ \to \mathfrak{C}$ a *diagram of* $\mathfrak{C}$ *of type* I, or shortly an *I-object of* $\mathfrak{C}$, where $\circ$ stands for the dual category

Let $X.$ and $Y.$ be an I-object and a J-object of $\mathfrak{C}$ respectively, and let $\varphi : I \to J$ be a functor. We denote by $\varphi^* Y.$, or $Y. \times_J I$, the diagram of $\mathfrak{C}$ of type I defined by the composite of the functors $Y. \circ \varphi$.

1.2 Definition. We define a *morphism* $\Phi.$ *from* $X.$ *to* $Y.$ *over* $\varphi : I \to J$ to be a natural transformation $\Phi. : X. \Longrightarrow \varphi^* Y.$. If $I = J$ and φ is the identity on I, we simply call $\Phi.$ an *I-morphism.*

For an I-object $X.$ of $\mathfrak{C}$ we use the following notations:

$$X_i := X.(i) \in Ob(\mathfrak{C}) \quad \textit{for } i \in I,$$

$$X_{ij} := X.(i \to j) \in Hom_{\mathfrak{C}}(X_j, X_i) \quad \textit{for} \quad i,j \in I \quad \textit{with} \quad i \leq j.$$

For a morphism $\Phi. : X. \to Y.$ from an I-object $X.$ of $\mathfrak{C}$ to a J-object $Y.$ of $\mathfrak{C}$ over a functor $\varphi : I \to J$, we denote by $\Phi_i : X_i \to Y_{\varphi(i)}$ the element of $Hom_{\mathfrak{C}}(X_i, Y_{\varphi(i)})$ corresponding to $i \in I$. We denote an I°-object of $\mathfrak{C}$ by $X^\cdot : I \to \mathfrak{C}$. For an I°-object of $\mathfrak{C}$ and a morphism $\Phi : X^\cdot \to Y^\cdot$ from an I°-object of $\mathfrak{C}$ to a J°-object of $\mathfrak{C}$ over a functor $\varphi : I \to J$, we use the notation X^i, X^{ij} and Φ^i instead of X_i, X_{ij} and Φ_i.

1.3 Definition. Let S be an object of $\mathfrak{C}$ and $X.$ an I-object of $\mathfrak{C}$. We think of S as a $\{*\}$-object of $\mathfrak{C}$, where $\{*\}$ is the *"ponctuel"* category, i.e., the category consisting of one point. We call a morphism $X. \to S$ over the trivial functor $I \to \{*\}$ an *augmentation of* $X.$ *to* S, and we call an I-object $X.$ of $\mathfrak{C}$ with an augmentation $X. \to S$ an *I-object of* $\mathfrak{C}$ *augmented toward* S.

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We denote by $\underline{\mathfrak{Cat}}$ the large category of all (small) categories. Let K be an I-object of $\underline{\mathfrak{Cat}}$ (resp. I°-object of $\underline{\mathfrak{Cat}}$).

1.4 Definition. The total category $tot(K)$ of an I-object (resp. I°-object) K of $\underline{\mathfrak{Cat}}$ is defined to be the category whose objects $Ob(tot(K))$ and the set of homomorphisms $Hom_{tot(K)}(\ ,\ )$ are defined as follows:

$$Ob(tot(K)) := \{(i,x) \mid i \in I,\ x \in K_i\}$$
$$(resp.\ Ob(tot(K)) := \{(i,x) \mid i \in I,\ x \in K^i\})$$

$$Hom_{tot(K)}((i,x),(j,y)) = \begin{cases} Hom_{K_i}(x, K_{ij}(y)) & i \leq j \\ (resp.\ Hom_{K^j}(K^{ij}(x), y)) & \\ \emptyset & \text{otherwise.} \end{cases}$$

We denote by (Top) the category of topological spaces. Recall that we can identify a topological space with the category of open subsets of X, and a continuous map $f : X \to Y$ between topological spaces with a functor f^{-1} between the categories of open subsets of Y and X. This fact allows us to identify $(Top)^\circ$ with a subcategory of $\underline{\mathfrak{Cat}}$. In the following we shall always make this identification unless otherwise specifically mentioned. For the sake of brevity we call a *diagrm of topological spaces* $X. : I \to (Top)$, i.e., an I-object of (Top), an *I-topological space.* For an I-topological space $X. : I \to (Top)$, regarding $X.$ as a functor $X. : I^\circ \to \underline{\mathfrak{Cat}}$ with the identification of Top° as a subcatogory of $\underline{\mathfrak{Cat}}$, we get the total category $tot(X.)$.

1.5 Definition. A *R-module presheaf* $F^\cdot$ on an I-topological space $X. : I \to (Top)$ is defined to be a contravariant functor from the total category $tot(X.)$ to the category of R-modules, where R is a commutative ring.(We shall mainly be concerned with the cases $R = \mathbb{Z}, \mathbb{Q}$ *or* $\mathbb{C}$). We say a R-module presheaf $F^\cdot$ on an I-topological space $X. : I \to (Top)$ is a *R-module sheaf* if the presheaves F^i on X_i, defined by $F^\cdot$, are sheaves for all $i \in I$. For $R-$module (pre)sheaves $F^\cdot$ and $G^\cdot$ on $X.$, a *morphism from $F^\cdot$ to $G^\cdot$* is defined to be a natural transformation from $F^\cdot$ to $G^\cdot$.

We denote by $\mathcal{M}(X., R)$ the category of $R-$module sheaves on $X.$. Let $\Phi. : X. \to Y.$ be a morphism from an I-topological space $X.$ to a J-topological space $Y.$ over a functor $\varphi : I \to J$.

1.6 Definition. For a R-module sheaf $G^\cdot$ on $Y.$ we define its inverse image $\Phi^*_\cdot G^\cdot \in \mathcal{M}(X., R)$ by $\Phi.$ as follows:

$$(\Phi^*_\cdot G^\cdot)^i := \Phi^*_i(G^{\varphi(i)}) \quad \text{for} \quad i \in Ob(I).$$

For a R-module sheaf $F^\cdot$ on $X.$ we define its direct image $\Phi._* F^\cdot \in \mathcal{M}(Y., R)$ by $\Phi.$ as follows:

$$(\Phi_{\cdot *}F^{\cdot})^j := \varprojlim_{i\in j\backslash\varphi} (Y_{j\varphi(i)})_* F^i \text{ for } \quad j \in Ob(J),$$

where

$$(1.1) \qquad j\backslash\varphi := \{i \in Ob(I) \mid j \le \varphi(i) \quad \text{in} J\}.$$

Thus we have functors

$$(1.2) \qquad \Phi_{\cdot}^{*} : \mathcal{M}(Y_{\cdot}, R) \to \mathcal{M}(X_{\cdot}, R) \quad \text{and}$$

$$(1.3) \qquad \Phi_{\cdot *} : \mathcal{M}(X_{\cdot}, R) \to \mathcal{M}(Y_{\cdot}, R),$$

for a morphism $\Phi_{\cdot} : X_{\cdot} \to Y_{\cdot}$ from an I-topological space $X_{\cdot}$ to a J-topological space $Y_{\cdot}$ over a functor $\varphi : I \to J$. We can easily see that $\Phi_{\cdot *}$ is the right adjoint of $\Phi_{\cdot}^{*}$. Now we are going to describe injective objects of the category $\mathcal{M}(X_{\cdot}, R)$.

Let I^{disc} be the discrete category associated to I; $e : I^{\text{disc}} \to I$ the inclusion functor; $X_{\cdot}^{\text{disc}}$ the I^{disc}-topological space $X_{\cdot} \times_I I^{\text{disc}}$; and $E_{\cdot} : X_{\cdot}^{disc} \to X_{\cdot}$ the natural morphism over $e : I^{disc} \to I$. Since the inverse image functor $E_{\cdot}^{*} : \mathcal{M}(X_{\cdot}, R) \to \mathcal{M}(X_{\cdot}^{\text{disc}}, R)$ is exact, its right adjoint $E_{\cdot *} : \mathcal{M}(X_{\cdot}^{\text{disc}}, R) \to \mathcal{M}(X_{\cdot}, R)$ preserves injective objects. A R-module sheaf $K^{\cdot} \in \mathcal{M}(X_{\cdot}^{\text{disc}}, R)$ whose $K^i \in \mathcal{M}(X_{\cdot}, R)$ are injective for all $i \in I$ is injective in $\mathcal{M}(X_{\cdot}^{\text{disc}}, R)$. Hence the direct image $E_{\cdot *}K^{\cdot} \in \mathcal{M}(X_{\cdot}, R)$ of such $K^{\cdot} \in \mathcal{M}(X_{\cdot}^{\text{disc}}, R)$ by $E_{\cdot} : X_{\cdot}^{\text{disc}} \to X_{\cdot}$ is injective in $\mathcal{M}(X_{\cdot}, R)$. By this description of injective objects of $\mathcal{M}(X_{\cdot}, R)$, we conclude that $\mathcal{M}(X_{\cdot}, R)$ has enough injective objects.

We denote by $C^{+}(X_{\cdot}, R)$ the category of lower bounded complexes of R-module sheaves on $X_{\cdot}$ and $D^{+}(X_{\cdot}, R)$ the derived category obtained by localization of the category $C^{+}(X_{\cdot}, R)$ with respect to quasi-isomprphisms, i.e., morphisms $u^{\cdot} : F^{\cdot} \to G^{\cdot}$ of complexes of R-module sheaves on $X_{\cdot}$ such that $u^i : F^i \to G^i$ are quasi-isomorphisms of complexes of R-module sheaves on X_i for all $i \in Ob(I)$. Since $\mathcal{M}(X_{\cdot}, R)$ has enough injective objects, we may identify $D^{+}(X_{\cdot}, R)$ with $K^{+}(X_{\cdot}, Inj.R)$ the homotopy category of lower bounded complexes of injective R-module sheaves on $X_{\cdot}$([10, p.435, Proposition 2.7]). For a morphism $\Phi_{\cdot} : X_{\cdot} \to Y_{\cdot}$ between diagram of topological spaces, since the functor $\Phi_{\cdot}^{*}$ in (1.2) is exact, it trivially defines a functor

$$(1.4) \qquad \Phi_{\cdot}^{*} : D^{+}(Y_{\cdot}, R) \to D^{+}(X_{\cdot}, R)$$

Furthermore, since the functor $\Phi_{\cdot *}$ in (1.3) is left exact, it admits the right derived functor

$$\mathbb{R}\Phi._{*} : D^{+}(X., R) \to D^{+}(Y., R). \tag{1.5}$$

For better understanding of this map the following factorization of $\Phi. : X. \to Y.$ is convenient:

$$\begin{array}{ccc} X. & \xrightarrow{\Phi.} & Y. \\ & {\scriptstyle \Phi_1.}\searrow \quad \nearrow{\scriptstyle \Phi_2.} & \\ & Y. \times_J I & \end{array} \tag{1.6}$$

where $\Phi_1.$ is an I-morphism of I-topological spaces defined by $(\Phi_1.)_i = \Phi._i$ for $i \in Ob(I)$, and $\Phi_2.$ is the natural φ-morphism from the I-topological space $Y. \times_J I$ to the J-topological space $Y.$. For lower bounded complexes of R-module sheaves $F^{\cdot}$ and $F'^{\cdot}$ on $X.$ and $Y. \times_J I$, respectively, we have

$$(\mathbb{R}\Phi_1._{*} F^{\cdot})^i = (\mathbb{R}\Phi_i)_{*} F^i \text{ for } j \in Ob(I), \quad \text{and} \tag{1.7}$$

$$(\mathbb{R}\Phi_2._{*} F'^{\cdot})^j = \mathbb{R}\varprojlim_{i \in j\backslash\varphi} F'^i \text{ for } j \in Ob(J), \tag{1.8}$$

where $j\backslash\varphi$ is the same as in (1.1). Therefore, since $(\mathbb{R}\Phi.)_{*} = (\mathbb{R}\Phi_2)_{*} \cdot (\mathbb{R}\Phi_1)_{*}$, we have

$$(\mathbb{R}\Phi._{*} F^{\cdot})^j = \mathbb{R} \varprojlim_{i \in j\backslash\varphi} (\mathbb{R}\Phi_i)_{*} F^i \text{ for } i \in Ob(I)$$

for a lower bounded complex of R-module sheaves $F^{\cdot}$ on X.

1.2 Cubic hyperresolutions of complex projective varieties

We denote by $\mathbb{Z}$ the integer ring.

1.7 Definition. For $n \in \mathbb{Z}$ with $n \geq 0$ the *augmented n-cubic category*, denoted by $\square_n^+$, is defined to be a category whose objects $Ob(\square_n^+)$ and homomorphisms $Hom_{\square_n^+}(\alpha, \beta)$ $(\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n), \beta = (\beta_0, \beta_1, \cdots, \beta_n) \in Ob(\square_n^+))$ are given as follows:

$$Ob(\square_n^+) := \{\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^{n+1} \mid 0 \leq \alpha_i \leq 1 \text{ for } 0 \leq i \leq n\},$$

$$Hom_{\square_n^+}(\alpha, \beta) := \begin{cases} \alpha \to \beta \text{ (an arrow from } \alpha \text{ to } \beta) & \text{if } \alpha_i \leq \beta_i \text{ for } 0 \leq i \leq n \\ \emptyset & \text{otherwise} \end{cases}$$

For $n = -1$ we understand $\Box^+_{-1}$ is the "ponctuel" category $\{*\}$, i.e., the category consisting of one point. For $n \geq 0$ the *n-cubic category*, denoted by $\Box_n$, is defined to be a full subcategory of $\Box^+_n$ with $Ob(\Box_n) = Ob(\Box^+_n) - \{(0, \cdots, 0)\}$.

Notice that $Ob(\Box^+_n) - \{(0, \cdots, 0)\}$ (resp. $Ob(\Box_n)$) can be considered as a finite ordred set whose order is defined by $\alpha \leq \beta \Longleftrightarrow \alpha \to \beta$ for $\alpha, \beta \in Ob(\Box^+_n)$ (resp. $Ob(\Box_n)$). Let $\mathcal{C}$ be a category.

1.8 Definition. A $\Box^+_n$-object $X.^+ : (\Box^+_n)^\circ \to \mathcal{C}$ (resp. a $\Box_n$-object $X. : (\Box_n)^\circ \to \mathcal{C}$) of $\mathcal{C}$ is called an *augmented n-cubic object of* $\mathcal{C}$ (resp. an *n-cubic object of* $\mathcal{C}$).

1.9 Remark. Notice that an augmented n-cubic object $X.^+ : (\Box^+_n)^\circ \to \mathcal{C}$ of $\mathcal{C}$ can be identified with an n-cubic object $X. := X^+_{.|\Box_n} : (\Box_n)^\circ \to \mathcal{C}$ of $\mathcal{C}$ with an augmentation toward $X^+_{(0,\cdots,0)}$ (cf. Definition 1.3).

In the following we shall interchangeably use an augmented n-cubic object of $\mathcal{C}$ and an n-cubic object of $\mathcal{C}$ with an augmentation. We denote by $(Proj/\mathbb{C})$ the category of complex projective varieties.

1.10 Definition. We call a $\Box^+_n$-object of $(Proj/\mathbb{C})$ (resp. a $\Box_n$-object oof $(Proj/\mathbb{C})$) an *augmented n-cubic complex projective variety* (resp. an *n-cubic complex projective variety), or a $\Box^+_n$-complex projective variety (resp. a $\Box_n$-complex projective variety) for the abuse of language.*

1.11 Example. Let X be a complex projective variety and $\{X_r\}_{0 \leq r \leq n}$ all of irreducible components of X. For each $\alpha = (\alpha_0, \cdots, \alpha_n) \in \Box_n$ we define

$$X_\alpha := \bigcap \{X_i \mid \alpha_i = 1\}.$$

If $\alpha \leq \beta$ in $\Box_n$, then there is the natural inclusion map $X_\beta \subseteq X_\alpha$. Hence the correspondence $\alpha \in \Box_n \to X_\alpha \in (Proj/\mathbb{C})$ defines an n-cubic complex projective variety $X. : (\Box_n)^\circ \to (Proj/\mathbb{C})$. We consider X as a $\Box^+_{-1}$-complex projective variety. Then there exists naturally an augmentation $X. \to X$, which can be considered an augmented n-cubic complex projective variety (cf. Remark 1.9)

Let I be a finite ordered set. All of I-complex projective varieties, i.e., functors $I^\circ \to (Proj/\mathbb{C})$, and I-morphisms between I-complex projective varieties (cf. Definition 1.2) forms a category, which we call the category of I-complex projective varieties and denote by $(I\text{-}Proj/\mathbb{C})$(We shall mainly be concerned with the cases $I = \Box^+_n$ or $\Box_n$).

1.12 Definition. For a I-complex projective variety $X.$, a functor $Z. : (\Box^+_1)^\circ \to (I\text{-}Proj/\mathbb{C})$ is called a *2-resolution of* $X.$ if $Z.$ is defined by a cartesian square of morphisms of I-complex projective varieties as follows:

$$\begin{array}{ccc} Z_{11.} & \longrightarrow & Z_{01.} \\ \downarrow & & \downarrow \\ Z_{10.} & \longrightarrow & Z_{00.}, \end{array}$$

which satisfies the following conditions:

(i) $Z_{00.} = X.$,

(ii) $Z_{01.}$ is a smooth I-complex projective variety, i.e., a functor from I° to the category of smooth complex projective varieties,

(iii) the horizontal arrows are closed immersion of I-complex projective varieties,

(iv) f is a proper morphism between I-complex projective varieties, and

(v) f induces an isomorphism from $Z_{01\beta} - Z_{11\beta}$ to $Z_{00\beta} - Z_{10\beta}$ for any $\beta \in Ob(\square_n^+)$.

Notice that, for a finite ordered set I, a $\square_n^+$-object $X. : (\square_n^+)^\circ \to (I\text{-}Proj/\mathbb{C})$ of the category $(I\text{-}Proj/\mathbb{C})$ can naturally be identified with a $\square_n^+ \times I$-object $X.. : (\square_n^+ \times I)^\circ \to (Proj/\mathbb{C})$ of the category $(Proj/\mathbb{C})$, where $\square_n^+ \times I$ is the product finite ordered set of $\square_n^+$ and I. In the following we shall interchangeably use a $\square_n^+$-object of $(I\text{-}Proj/\mathbb{C})$ and a $\square_n^+ \times I$-object of $(Proj/\mathbb{C})$. Especially, since $\square_{n+2}^+ = \square_1^+ \times \square_n^+$, we may think of a 2-resolution $Z.$ of a $\square_n^+$-complex projective variety $X.$ as a $\square_{n+2}^+$-complex projective variety.

1.13 Remark. With the above identification, since a morphism $X. \xrightarrow{a.} X$ from a $\square_n \times I$-complex projective variety $X.$ to a $\square_{-1}^+ \times I$-complex projective variety X over the trivial functor $\square_n \times I \to \square_{-1}^+ \times I$ can be identified with a morphism from a $\square_n$-object $X.$ of I-complex projective varieties to a $\square_{-1}^+$-object X of I-complex projective varieties over the trivial functor $\square_n \to \square_{-1}^+$, the morphism $X. \xrightarrow{a.} X$ can be identified with a $\square_n^+$-object of I-complex projective varieties (cf. Remark 1.9), that is a $\square_n^+ \times I$-complex projective variety.

1.14 Definition. Let n be an integer≥ 1. Suppose we are given a sequence $\{X.^1, X.^2, \cdots, X.^n\}$ of $\square_r^+ \times I$-complex projective varieties $X.^r (1 \leq r \leq n)$ subject to the condition that the $\square_{r-1}^+ \times I$-complex projective variety $X_{00.}^{r+1}$ coincides with $X_{1.}^r$ for any r $(1 \leq r \leq n)$. Then we define, by induction on n, a $\square_n^+ \times I$-complex projective variety

$$Z. = rd(X^1_., X^2_., \cdots, X^n_.),$$

which we call *the reduction of* $\{X^1_., X^2_., \cdots, X^n_.\}$ as follows: If n=1, we define $Z. = X^1_.$. If n=2, we define $Z. = rd(X^1_., X^2_.)$ by

$$Z_{\alpha\beta} \equiv \begin{cases} X^1_{0\beta} & if \ \alpha = (0,0) \\ X^2_{\alpha\beta} & if \ \alpha \in \square_1 \end{cases}$$

for every $\beta \in \square_0^+$. If $n \geq 3$, we define

$$Z. := rd(rd(X^1_., \cdots, X^{n-1}_.), X^n_.).$$

1.15 Definition. Let X a $\square^+_{-1}$-complex projective variety, i.e., a complex projective variety, $X.$ a $\square_n$-complex projective variety, and $X. \xrightarrow{a.} X$ a morphism over the trivial functor $\square_n \to \square_{-1}$. We think of $X. \xrightarrow{a.} X$ as a $\square^+_n$-complex projective variety (cf. Remark 1.13) and we denote it by $X^+_.$. We call $X^+_.$ (resp. $X.$) an *augmented n-cubic hyperresolution of* X (resp. an *n-cubic hyperresolution of* X, or an augmented hyperresolution of X of length n (resp. a hyperresolution of X of length n) if there exist a sequence $\{X.^1, X.^2, \cdots, X.^n\}$ of $\square^+_r$-complex projective varieties $X^r_.$ $(1 \leq r \leq n)$ such that:

(i) $X^1_.$ is a two resolution of X,

(ii) for $1 \leq r < n$, $X^{r+1}_.$ is a two resolution of $X^r_{1.}$, (then the $\square^+_{r-1}$-algebraic variety $X^{r+1}_{00.}$ coinsides with $X^r_{1.}$ for every r with $1 \leq r < n$), and

(iii) $X^+_. = rd(X^1_., X^2_. \cdots, X^n_.)$

(iv) $X_{\alpha i}$ is smooth for all $(\alpha, i) \in \square_n \times I$.

1.16 Theorem.. ([7, p.14, Théorème 2.15]) *For any complex projective variety X of dimension n there exists an augmented n-cubic hyperresolution $X.^+ = \{X. \xrightarrow{a.} X\}$ of X such that*

$$dim_{\mathbb{C}} X_\alpha \leq dim X - |\alpha| + 1 \ for\ every\ \alpha \in \square_n,$$

where $|\alpha| = \sum_{i=0}^n \alpha_i$ *for* $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$.

The important fact on the cubic hyperresolution is "cohomological descent". There are two sorts of "cohomological descent"; one is of R-module sheaves (R:a commutative ring, especially R=$\mathbb{Z}, Q$, or $\mathbb{C}$)([7, p.41, Théorème 6.9]) and the other is of de Rham complexes ([7, p.61, Théorème 1.3]).

1.3 Mixed Hodge structure on the cohomology of complex projective varieties via cubic hyperresolutions

Replacing $(Proj/\mathbb{C})$ and $(I\text{-}Proj/\mathbb{C})$ by (Top) and $(I\text{-}Top)$, respectively, all notions and terminology in the previous paragraph can be transfered to the case of topological spaces. Let $a.: X. \to X$ be an augmented n-cubic topological space and let $F^{\cdot}$ be a lower bounded complex of R-module sheaves on an n-cubic topological space $X.$. We will give a concrete description of $\mathbb{R}a._*F^{\cdot} \in D^+(X,R)$. For this end we take the factorization

$$(1.9) \qquad X. \xrightarrow{a_1.} X \times \square_n \xrightarrow{a_2.} X$$

of $a.: X. \to X$ as stated in (1.6). By definition $a_{1\alpha *}F^{\cdot} = \{a_{1 \cdot *}F^\alpha\}_{\alpha \in \square_n}$, to which we associate a simple complex $s(a_{1 \cdot *}F^{\cdot})$ of R-module sheaves on X. To explain this we first give the definition of an *n-ple complex of an abelian category*. Let A be an abelian category. We denote by $C^+(A)$ the category of lower bounded complexes of A. Let n be an integer≥ 1. We denote by e_i the i-th vector of the canonical basis of $\mathbb{Z}^n$, i.e., $e_i = (0, \cdots, 1, \cdots, 0)$ (1 is at the i-th place) for $1 \leq i \leq n$.

1.17 Definition. With the notation above, an n-ple complex of A consisits of the following entities:

(i)a $\mathbb{Z}^n$-graded object $\{K^\alpha\}_{\alpha \in \mathbb{Z}^n}$ of A, and

(ii) a family $\{d_i\}_{1 \leq i \leq n}$ of differentials of $K^{\cdot}$ such that d_i is of defree e_i and they commute each other.

1.18 Definition. For $K \in n\text{-}C^+(A)$ its *associated simple complex* $s(K) \in C^+(A)$ is defined to be as follows:

$$s(K)^p := \sum_{\sum p_i = p} K^{p_1 \cdots p_n}, \quad p \in \mathbb{Z} \text{ and}$$

the defferential d of $s(K)$ is given by

$$d = \sum_{j=1}^{n} (-1)^{\varepsilon_j} d_j \quad \text{on} \quad K^{p_1 \cdots p_n},$$

where $\varepsilon_j = \sum_{i<j} p_i$.

Let $\mathcal{A}.$ be a $(\square_n^+)^\circ$-object of lower bounded complexes of R-module sheaves on a topological space, say Y, i.e., a functor $\mathcal{A}. : (\square_n^+)^\circ \to C^+(Y,R)$. We denote $\mathcal{A}.(\alpha) \in C^+(Y,R)$ by $\mathcal{A}^{\cdot}_\alpha$ for each $\alpha \in \square_n^+$. We associate to such $\mathcal{A}$ an object $K(\mathcal{A}.)$ of (n+2)-$C^+(Y,R)$, i.e., an (n+2)-ple lower bounded complex of $\mathcal{M}(Y,R)$ as follows:

$$K(\mathcal{A})^{\alpha_0 \cdots \alpha_n q} := \begin{cases} \mathcal{A}^q_\alpha & \text{if } \alpha \in \square_n^+ \\ 0 & \text{if } \alpha \in \mathbb{Z}^{n+1} - \square_n^+; \end{cases}$$

the (i+1)-th differential is the one induced by the morphism $\alpha \to \alpha + e_i$ in $\square_n^+$ for $0 \leq i \leq n$, and (n+2)-th differential is the one of objects of $C^+(Y,R)$. For the sake of simplicity we denote $s(K(\mathcal{A}.))$ by $s(\mathcal{A}.)$.

Now we return to the set-up at the beginning of this paragraph. We think of $a_{1\cdot *}F^{\cdot} = \{a_{1\alpha *}F^{\alpha}\}_{\alpha\in\square_n}$ as a $(\square_n^+)^\circ$-object of lower bounded complexes of R-module sheaves on X, defining $F^{(0,\cdots,0)} = \{0\}$ for $(0,\cdots,0)\in\square_n^+$, and form $s(a_{1\cdot *}F^{\cdot})$. Then by (1.8) we have

$$\mathbb{R}a_{2\cdot *}(a_{1\cdot *}F^{\cdot}) \cong s(a_{1\cdot *}F^{\cdot})[1]$$

in $D^+(X,R)$, where [1] stands for the shift of the degree of complexes to the left by 1, i.e., $s(a_{1\cdot *}F^{\cdot})[1]^p = s(a_{1\cdot *}F^{\cdot})^{p+1}$. Hence by (1.6)

$$\mathbb{R}a_{\cdot *}F^{\cdot} \cong s(a_{1\cdot *}F^{\cdot})[1] \tag{1.10}$$

in $D^+(X,R)$. This is an explicite description of $\mathbb{R}a_{\cdot *}F^{\cdot}$ which is needed for our arguments in the following.

For use later we will prove two facts about $(\square_n^+)^\circ$-objects $\mathcal{A}_{\cdot} = \{\mathcal{A}_\alpha\}_{\alpha\in\square_n^+}$ of complexes of R-module sheaves on a topological space Y. Since $\square_n^+ = \square_0^+ \times \square_{n-1}^+$, we may think of $\mathcal{A}_{\cdot}$ as a $(\square_0^+)^\circ$-object of $(\square_{n-1}^+)^\circ$-objects of complexes of R-module sheaves on Y. We denote it by

$$\mathcal{A}_{\cdot} = \{\mathcal{A}_{0\cdot} \xrightarrow{\delta} \mathcal{A}_{1\cdot}\},$$

where δ is the morphism between $(\square_{n-1}^+)^\circ$-objects of complexes of R-module sheaves on Y corresponding to the arrow $0\to 1$ in $\square_0^+$. δ induces a homomorphism

$$\hat{\delta} : s(\mathcal{A}_{0\cdot}) \longrightarrow s(\mathcal{A}_{1\cdot})$$

of complexes of R-module sheaves on Y. We think of this homomorphism as a $(\square_0^+)^\circ$-object of complexes of R-module sheaves on Y and form $s(s(\mathcal{A}_{0\cdot}) \to s(\mathcal{A}_{1\cdot}))$.

1.19 Proposition.. *With the notation as above, we have*

$$s(s(\mathcal{A}_{0\cdot}) \to s(\mathcal{A}_{1\cdot})) \simeq s(\mathcal{A}_{\cdot}).$$

Proof. By defintion

$$\begin{aligned} s(s(\mathcal{A}_{0\cdot}) \to s(\mathcal{A}_{1\cdot}))^p &= s(\mathcal{A}_{0\cdot})^p \oplus s(\mathcal{A}_{1\cdot})^{p-1} \\ &= (\oplus_{\alpha_1+\cdots+\alpha_n+q=p}\mathcal{A}^q_{0\alpha_1\cdots\alpha_n}) \oplus (\oplus_{\alpha'_1+\cdots+\alpha'_n+q'=p-1}\mathcal{A}^{q'}_{1\alpha'_1\cdots\alpha'_n}) \\ &= \oplus_{\alpha_0+\cdots+\alpha_n+q=p}\mathcal{A}^q_{\alpha_0\cdots\alpha_n} = s(\mathcal{A}_{\cdot})^p. \end{aligned}$$

We will use the following notation for differentials:

$$\hat{D}^{(p)} : s(s(\mathcal{A}_{0\cdot}) \to s(\mathcal{A}_{1\cdot}))^p \to s(s(\mathcal{A}_{0\cdot}) \to s(\mathcal{A}_{1\cdot}))^{p+1},$$

$$\hat{\delta}^(p) : s(\mathcal{A}_{0\cdot})^p \to s(\mathcal{A}_{1\cdot})^p,$$

(1.11) $$d_i^{(p)} : s(\mathcal{A}_{i\cdot})^p \to s(\mathcal{A}_{i\cdot}^{p+1}) \quad (i = 0, 1),$$

$$D^{(p)} : s(\mathcal{A}.)^p \to s(\mathcal{A}.)^{p+1}$$

$$\delta_j^{(\alpha_0 \cdots \alpha_n q)} : \mathcal{A}^q_{\alpha_0 \cdots \alpha_j \cdots \alpha_n} \to \mathcal{A}^q_{\alpha_0 \cdots \alpha_j + 1 \cdots \alpha_n} \quad (j = 0, \cdots, n) \quad \text{and}$$

$$d^{(q)}_{\alpha_0 \cdot \alpha_n} : \mathcal{A}^q_{\alpha_0 \cdots \alpha_n} \to \mathcal{A}^{q+1}_{\alpha_0 \cdots \alpha_n}.$$

Then by definition

$$\hat{D}^{(p)} = \hat{\delta}^{(p)} + d_0^{(p)} - d_1^{(p)} \quad \text{on} s(s(\mathcal{A}_{0\cdot} \to s(\mathcal{A}_{1\cdot}))^p$$

and.

$$\hat{D}_{|\mathcal{A}^q_{\alpha_0 \cdots \alpha_n}} =$$

$$\begin{cases} \delta_0^{(\alpha_0 \cdots \alpha_n q)} + \sum_{j=1}^n (-1)^{\alpha_1 + \cdots + \alpha_{j-1}} \delta_j^{(\alpha_0 \cdots \alpha_n q)} + (-1)^{|\alpha|} d^{(q)}_{0\alpha_1 \cdots \alpha_n} & \text{if } \alpha_0 = 0 \\ -\{\sum_{j=1}^n (-1)^{\alpha_1 + \cdots + \alpha_{j-1}} \delta_j^{(\alpha_0 \cdots \alpha_n q)} + (-1)^{|\alpha|-1} d^{(q)}_{1\alpha_1 \cdots \alpha_n}\} & \text{if } \alpha_0 = 1 \end{cases}$$

for every $(\alpha_0, \cdots, \alpha_n, q)$ with $| \alpha | + q = p$, where $| \alpha | = \alpha_0 + \cdots + \alpha_n$. Therefore, since $\delta_0^{(\alpha_0 \cdots \alpha_n q)} \equiv 0$ on $\mathcal{A}^q_{1\alpha_1 \cdots \alpha_n}$, we have

$$\hat{D}^{(p)}_{|\mathcal{A}^q_{\alpha_0 \cdots \alpha_n}}$$

$$= \sum_{j=0}^{n} (-1)^{\alpha_0 + \cdots + \alpha_{j-1}} \delta_j^{(\alpha_0 \cdots + \alpha_n q)} + (-1)^{\alpha_0 + \cdots + \alpha_n} d^{(q)}_{\alpha_0 \cdots \alpha_n}$$

$$= D^{(p)}_{|\mathcal{A}^q_{\alpha_0 \cdots \alpha_n}}.$$

Q.E.D.

Next, since $\Box_n^+ = \Box_1^+ \times \Box_{n-2}^+$, we may also think of $\mathcal{A}.$ as a $(\Box_1^+)^\circ$-object of $(\Box_{n-2}^+)^\circ$-objects of complex R-module sheaves on Y, which we denote by

$$\mathcal{A}. = \begin{array}{ccc} \mathcal{A}_{11\cdot} & \xleftarrow{\delta_0^{(01\cdot)}} & \mathcal{A}_{01\cdot} \\ \delta_1^{(10\cdot)}\uparrow & & \uparrow\delta_1^{(00\cdot)} \\ \mathcal{A}_{10\cdot} & \xleftarrow{\delta_0^{(00\cdot)}} & \mathcal{A}_{00\cdot}. \end{array}$$

Further, we denote by

$$\begin{array}{ccc} s(\mathcal{A}_{11\cdot}) & \xleftarrow{\hat\delta_0^{(01\cdot)}} & s(\mathcal{A}_{01\cdot}) \\ \hat\delta_1^{(10\cdot)}\uparrow & & \uparrow\hat\delta_1^{(00\cdot)} \\ s(\mathcal{A}_{10\cdot}) & \xleftarrow{\hat\delta_0^{(00\cdot)}} & s(\mathcal{A}_{00\cdot}) \end{array}$$

the associated $(\Box_1^+)^\circ$-object of simple complex of R-module sheaves on Y , and by

$$s\left(\begin{array}{ccc} s(\mathcal{A}_{11\cdot}) & \xleftarrow{\hat\delta_0^{(01\cdot)}} & s(\mathcal{A}_{01\cdot}) \\ \hat\delta_1^{(10\cdot)}\uparrow & & \uparrow\hat\delta_1^{(00\cdot)} \\ s(\mathcal{A}_{10\cdot}) & \xleftarrow[\hat\delta_0^{00\cdot}]{} & s(\mathcal{A}_{00\cdot}) \end{array}\right)$$

the simple complex of R-module sheaves on Y associated to this.

1.20 Proposition.. *With the notation above, we have*

$$s\left(\begin{array}{ccc} s(\mathcal{A}_{11\cdot}) & \xleftarrow{\hat\delta_0^{(01\cdot)}} & s(\mathcal{A}_{01\cdot}) \\ \hat\delta_1^{(10\cdot)}\uparrow & & \uparrow\hat\delta_1^{(00\cdot)} \\ s(\mathcal{A}_{10\cdot}) & \xleftarrow[\hat\delta_0^{(00\cdot)}]{} & s(\mathcal{A}_{00\cdot}) \end{array}\right) \simeq s(\mathcal{A}.).$$

Proof. By definition

$$s\left(\begin{array}{ccc} s(\mathcal{A}_{11\cdot}) & \longleftarrow & s(\mathcal{A}_{01\cdot} \\ \uparrow & & \uparrow \\ s(\mathcal{A}_{10\cdot}) & \longleftarrow & s(\mathcal{A}_{00\cdot}) \end{array}\right)^p$$

$$= \oplus_{|\alpha_0|+|\alpha_1|+q=p} s(\mathcal{A}_{\alpha_0\alpha_1\cdot})^q = \oplus_{|\alpha_0|+|\alpha_1|+q=p}(\oplus_{|\alpha_0|+\cdots+|\alpha_n|+r=q}\mathcal{A}^r_{\alpha_0\cdots\alpha_n})$$

$$= \oplus_{|\alpha_0|+\cdots+|\alpha_n|+r=p} \mathcal{A}^r_{\alpha_0\cdots\alpha_n} = s(\mathcal{A}.)^p.$$

We use the following notation for differentials:

$$\hat{D}^{(p)} \ : \ s\begin{pmatrix} s(\mathcal{A}_{11}\cdot) & \longleftarrow & s(\mathcal{A}_{01}\cdot) \\ \uparrow & & \uparrow \\ s(\mathcal{A}_{10}\cdot) & \longleftarrow & s(\mathcal{A}_{00}\cdot) \end{pmatrix}^p$$

$$\longrightarrow s\begin{pmatrix} s(\mathcal{A}_{11}\cdot) & \longleftarrow & s(\mathcal{A}_{01}\cdot) \\ \uparrow & & \uparrow \\ s(\mathcal{A}_{10}\cdot) & \longleftarrow & s(\mathcal{A}_{00}\cdot) \end{pmatrix}^{p+1},$$

$$\hat{\delta}_0^{(\alpha_0\alpha_1)} : s(\mathcal{A}^q_{\alpha_0\alpha_1}\cdot) \longrightarrow s(\mathcal{A}_{\alpha_0+1\alpha_1}\cdot)^q,$$

$$\hat{\delta}_1^{(\alpha_0\alpha_1)} : s(\mathcal{A}_{\alpha_0\alpha_1}\cdot)^q \longrightarrow s(\mathcal{A}_{\alpha_0\alpha_1+1}\cdot)^q,$$

$$\hat{d}^{(q)}_{\alpha_0\alpha_1} : s(\mathcal{A}^q_{\alpha_0\alpha_1}\cdot) \longrightarrow s(\mathcal{A}_{\alpha_0\alpha_1}\cdot)^{q+1},$$

and $D^{(}p)$, $\delta_j^{(\alpha_0\cdots\alpha_n q)}$, $d^{(q)}_{\alpha_0\cdots\alpha_n}$ are the same as in (1.11). Then by definition

$$\hat{D}^{(p)} = \hat{\delta}_0^{(\alpha_0\alpha_1)} + (-1)^{\alpha_0}\hat{\delta}_1^{(\alpha_0\alpha_1)} + (-1)^{\alpha_0+\alpha_1} d^{(q)}_{\alpha_0\alpha_1} \quad (1.12)$$

$$\text{on} \quad s(\mathcal{A}_{\alpha_0\alpha_1})^q.$$

Furthermore, on each $\mathcal{A}^r_{\alpha_0\cdots\alpha_n}$ with $|\alpha_0| + \cdots + |\alpha_n| + r = p$,

$$\hat{\delta}_0^{(\alpha_0\alpha_1)} = \delta_0^{(\alpha_0\cdots\alpha_n r)},$$

$$\hat{\delta}_1^{(\alpha_0\alpha_1)} = \delta_1^{(\alpha_0\cdots\alpha_n r)}, \quad \text{and}$$

$$\hat{d}^{(q)}_{\alpha_0\alpha_1} = \sum_{j=2}^{n}(-1)^{\alpha_2+\cdots+\alpha_{j-1}}\delta_j^{(\alpha_0\cdots\alpha_n r)} + (-1)^{\alpha_2+\cdots+\alpha_n} d^{(r)}_{\alpha_0\cdots\alpha_n}.$$

Therefore, by (1.12) we have

$$\begin{aligned}\hat{D}^{(p)}_{|\mathcal{A}^r_{\alpha_0\cdots\alpha_n}} &= \sum_{j=0}^{n}(-1)^{\alpha_0+\cdots+\alpha_{j-1}}\delta_j^{(\alpha_0\cdots\alpha_n r)} + (-1)^{\alpha_0+\cdots+\alpha_n} d^{(r)}_{\alpha_0\cdots\alpha_n} \\ &= D^{(p)}_{|\mathcal{A}^r_{\alpha_0\cdots\alpha_n}}.\end{aligned}$$

Q.E.D.

From now on, let X be a complex projective variety. We take a hyperresolution $X. \xrightarrow{a.} X$ of X and fix it. We denote by $D^+(X,\mathbb{Z})$ the derived category of lower bounded complexes of sheaves of $\mathbb{Z}$-modules over X. We define $K \in Ob(D^+(X,\mathbb{Z}))$ by

$$K := s(a_{1\cdot *}\mathbb{Z}_{X.}),$$

where $\mathbb{Z}_{X.}$ is the constant sheaf with value $\mathbb{Z}$ on the cubic hyperresolution $X.$ of X and $X. \xrightarrow{a_1.} X \times \square_n$ (n=the length of hyperresolution $X.$) is the factorization of $X. \xrightarrow{a.} X$ in (1.9). Notice that by the cohomological descent for $\mathbb{Z}$-module sheaves ([7, p.41, Théorème 6.9]),

$$K \simeq \mathbb{Z}_X \text{ in } D^+(X,\mathbb{Z}).$$

We define a so-called *weight filteration* W on $K_{\mathbb{Q}} = K \otimes \mathbb{Q} \in Ob(D^+(X,\mathbb{Q}))$ by

$$W_{-q}(K_{\mathbb{Q}}) := s(\sigma_{|\alpha|\geq q} a_{1\alpha *}\mathbb{Z}_{X_\alpha})$$

, where $|\alpha| = \sum_{i=0}^{n} \alpha_i$ for $\alpha = (\alpha_0, \cdots, \alpha_n) \in \square_n$. (In general, the subcomplex $\sigma_{\geq q}(K)$ of a complex K is defined as follows:

$$\sigma_{\geq q}(K)^p = \begin{cases} 0 & \text{if } p < q \\ K^p & \text{if } p \geq q \end{cases}$$

The filteration of K defined by these subcomplexes is called *stupid filteration*,cf. [4, p.37, 2.3.8].) Then $(K_{\mathbb{Q}}, W) \in Ob(D^+F(X,\mathbb{Q}))$, where $D^+F(X,\mathbb{Q})$ stands for the derived category of filtered, lower bounded complex of $\mathbb{Q}$-module sheaves over X. By the cohomological descent for de Rham complexes ([7, p.61, Théorème 1.3]), we have

$$s(a_{1\cdot *}\Omega^{\cdot}_{X.}) \simeq \mathbb{R}a_{\cdot *}\Omega^{\cdot}_{X.}$$

$$\simeq DR^{\cdot}_X \simeq \mathbb{C}_X \simeq \mathbb{Z}_X \otimes \mathbb{C}$$

in $D^+(X,\mathbb{C})$, where $\Omega^{\cdot}_{X.}$ is the de Rham complex of $X.$ and $DR^{\cdot}_X$ is the cohomological de Rham complex of X. Hence $K_{\mathbb{Q}} := K \otimes \mathbb{C}$ is quasi-isomorphic to $s(a_{1\cdot *}\Omega^{\cdot}_{X.})$. We define a so-called *Hodge filteration* F on $K_{\mathbb{C}} \simeq s(a_{1\cdot *}\Omega^{\cdot}_{X.})$ by

$$F^p(s(a_{1\cdot *}\Omega^{\cdot}_{X.})) := s(\sigma_{q\geq p} a_{1\cdot *}\Omega^q_{X.}).$$

1.21 Theorem. ([7, p.95, Proposition 1.19]) *Let X be a complex projective variety and $a. : X. \longrightarrow X$ a cubic hyperresolution of X. Then, with the same notation as above, the data :*

$\mathbb{Z}_X$, $(s(a_{1.*}\mathbb{Q}_{X.}, W)$, $\mathbb{Q}_X \simeq s(a_{1.*}\mathbb{Q}_{X.})$,
$(s(a_{1.*}\Omega^{\cdot}_{X.}), W, F)$,
$(s(a_{1.*}\mathbb{Q}_{X.}), W) \otimes \mathbb{C} \simeq (s(a_{1.*}\Omega^{\cdot}_{X.}), W)$

is a cohomological mixed Hodge complex in the sence of Deligne (For definition see [Théorie de Hodge III, (8.1.6)]). *Hence the filteration $W[\ell]$ ($W[\ell]_q := W_{q-\ell}$) on $H^\ell(X,\mathbb{Q}) \simeq H^\ell(\mathbb{R}\Gamma(X, s(a_{1.*}\mathbb{Q}_{X.})[1]))$ and the filteration on $H^\ell(X,\mathbb{C})$ $\simeq H^\ell(\mathbb{R}\Gamma(X, s(a_{1.*}\Omega^{\cdot}_{X.})[1]))(0 \le \ell \le 2\,dimX)$ defines a mixed Hodge structure.*

1.22 Remark. Thus defined mixed Hodge structure on $H^\ell(X,\mathbb{C})$ is independent of the choice of a cubic hyperresolution $X. \xrightarrow{a.} X$ and has functorial property. To prove these we need more comprehensive theory of cubic hyperresolution; that is, we have to consider not only "cubic hyperresolution" but also "k-iterated cubic hyperresolution", which is inductively defined. "1-iterated cubic hyperresolution of a complex projective variety is an ordinary "cubic hyperresolution". For $k \ge 1$, "(k+1)-iterated cubic hypereresolution" of a complex projective variety is a cubic hyperresolution of an "k-iterated cubic hyperresolution" of the comlex projective variety. For details see [7].

§2 Cubic hyperequisingular families of complex projective varieties

The purposes of this section are to give the definition of *cubic hyperequisingular families of complex projective varieties* and prove the relative version of cohomological descent of R-module sheaves (R: a commutative ring, especially R=$\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$) and de Rham complexes for these families.

2.1 Definition of cubic hyperequisingular families

2.1 Definition. By an analytic family of complex projective varieties, parametrized by a complex space M, we mean a triple $(\mathfrak{X}, \pi, M)$ satisfying the following conditions:

(i) $\pi \colon \mathfrak{X} \longrightarrow M$ is a flat surjective holomorphic map of complex spaces,and
(ii) $X_t := \pi^{-1}(t)$ is a complex projective variety for any $t \in M$.

Let $(\mathfrak{X}, \pi, M)$ and $(\mathfrak{X}', \pi', M)$ be analytic families of complex projective varieties parametrized by the same complex space M.

2.2 Definition. By a morphism (resp. an isomorphism) for $(\mathfrak{X}, \pi, M)$ to $(\mathfrak{X}', \pi', M)$ we mean a holomprphic (resp. biholomorphic) map $H \colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ such that the diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{H} & \mathfrak{X}' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{id_M} & M \end{array}$$

commutes, where id_M is the identity map on M.

We denote by $\mathcal{F}_M$ the category of analytic families of complex projective varieties, parametrized by a complex space M.

2.3 Definition. We call a $\square_n^+$-object (resp. $\square_n$-object) of $\mathcal{F}_M$ an *analytic family of augmented n-cubic (resp. n-cubic) complex projective varieties, parametrized by a complex space M.*

Let $X. \xrightarrow{b.} X$ be an augmented n-cubic complex projective variety and M a complex space. Then $X_\alpha \times M$ $(\alpha \in \square_n)$, $X \times M$, $a_\alpha := b_\alpha \times id_M : X_\alpha \times M \to X \times M$ and $\pi := Pr_M : X \times M \to M$, the projection to M consititute an analytic family of augmented n-cubic complex projective varieties , parametrized by a complex space M, which we denote by

$$X. \times M \xrightarrow{a.:=b.\times id_M} X \times M \xrightarrow{\pi:=Pr_M} M$$

and call the *product family of augmented n-cubic complex projective varieties, parametrized by a complex space M*. Let $\mathfrak{X}.^+ = \{\mathfrak{X}. \xrightarrow{a.} \mathfrak{X}\}$ be an analytic family of augmented n-cubic complex projective varieties (For notation see Remark 1.9), parametrized by a complex space M. Whenever we wish to express its parameter space M explicitly, we write

$$\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M. \tag{2.1}$$

For $t \in M$, $X_{t\alpha} := (\pi \cdot a_\alpha)^{-1}(t)$ $(\alpha \in \square_n)$, $X_t := \pi^{-1}(t)$ and $a_{t\alpha} := a_{\alpha|X_{t\alpha}} : X_{t\alpha} \to X_t$ constitute an augmented n-cubic complex projective variety, which we denote by $X_{t}. \xrightarrow{a_{t}.} X_t$ and call the fiber at $t \in M$ of an analytic family of augmented n-cubic complex projective varieties in (2.1). Similarly, for an open subset U of $\mathfrak{X}$, we form an analytic family

$$a.^{-1}(U) \xrightarrow{a.|_{a.^{-1}(U)}} U \xrightarrow{\pi} \pi(U)$$

of augmented n-cubic *analytic varieties*, parametized by a complex space $\pi(U)$. With these notions, we define an n-cubic hyperequisingular family of complex projective varieties, parametrized by a complex space as follows:

2.4 Definition. Let $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X}_0 \xrightarrow{\pi} M$ be an family of augmented n-cubic complex projective varieties, parametrized by a complex space M. We call $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X}_0 \xrightarrow{\pi} M$ an *n-cubic hyperequisingular family of complex projective varieties, parametrized by a complex space M* if it satisfies the following conditions:

(i) for any pont $t \in M$, $X_{t}. \xrightarrow{a_{t}.} X_t$ is an augmented n-cubic hyperresolution of X_t,

(ii) (analytically "local triviality") for any point $t \in \mathfrak{X}_0$, there exists an open neighborhood $\mathcal{U}$ of p in $\mathfrak{X}_0$ such that $a_\cdot^{-1}(\mathcal{U}) \xrightarrow{a_\cdot} \mathcal{U} \xrightarrow{\pi} \pi(\mathcal{U})$ is analytically isomorphic to

$$(a_\cdot^{-1}(U) \bigcap X_{\pi(p)\cdot}) \times \pi(U) \to (U \bigcap X_{\pi(p)}) \times \pi(U) \xrightarrow{Pr_{\pi(U)}} \pi(U)$$

over the identy map $id_{\pi(U)} : \pi(U) \to \pi(U)$

2.5 Proposition.. *Let* $\mathfrak{X}_\cdot \xrightarrow{a_\cdot} \mathfrak{X} \xrightarrow{\pi} M$ *be an n-cubic hyperequisingular family of complex projective varieties, parametrized by a complex manifold* M*. Then the* $\square_n$*-object* $\mathfrak{X}_\cdot \xrightarrow{\pi_\cdot} M(\pi_\cdot := \pi \cdot a_\cdot)$ *of smooth families of complex manifolds, parametrized by* M *is* C^∞ *trivial at any point of* M*; that is, for any point* $t_0 \in M$*, there exists an open neighborhood* N *of* t_0 *in* M *and a diffeomorphism* $\Phi_\cdot : (\pi_\cdot^{-1})(N) \to X_{t_0\cdot} \times N$ *of* $\square_n$*-objects of complex manifolds over the identy map* $id_N : N \to N$*. Furthermore,* $\mathfrak{X}_\cdot \xrightarrow{a_\cdot} \mathfrak{X} \xrightarrow{\pi} M$ *is topologically trivial at any point of* M*.*

Proof. Let N_1 be a coordinate neighborhood of t_0 in M with a holomorphic local coordinate system $(t_1, \cdots, t_m)$, and N a relatively compact open subset of N_1 with $\overline{N} \subset N_1$. Let $t_i = x_i + \sqrt{-1}x_{m+i}(1 \le i \le m)$ be the expression of t_i in real local coordinate functions x_i, y_i. To prove the proposition it suffices to show that for evry $\frac{\partial}{\partial x_i}$ $(1 \le i \le 2m)$ and every $\alpha \in \square_n$ there exists its liftings v_i^α to $\pi_\alpha^{-1}(N)$, i.e., a C^∞ vector field on $\pi_\alpha^{-1}(N)$ with the property

$$(d\pi_\alpha)(v_i^\alpha) = \pi_\alpha^*(\frac{\partial}{\partial x_i}),$$

subject to the requirement

$$dE_{\alpha\beta}(v_i^\beta) = E_{\alpha\beta}^*(v_i^\alpha) \tag{2.2}$$

for every pair (α, β) of elements of $Ob(\square_n)$ with $\alpha \le \beta$ in the category $\square_n$, where $E_{\alpha\beta} : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$ denotes a holomorphic maps corresponding to an arrow $\alpha \to \beta$ in $\square_n$. In fact, if such liftings $\{v_i^\alpha\}_{\alpha\in\square_n}$ exist, integrating v_i^α, we have a C^∞-trivialization of the family $\pi_\alpha : \mathfrak{X}_\alpha \to N$ along the x_i-axis in N for all $\alpha \in \square_n$ such that those trivializations commute with the maps $E_{\alpha\beta} : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$ for every pair (α, β) of elements of $Ob(\square_n)$ with $\alpha \to \beta$ in the category $\square_n$ due to the requiement (2.2). Arguing inductively on the dimension of M, we finally get the trivialization asserted in the proposition (cf. For more precise arguments we refer to Theorem 3.3 in [5]). Now we are going to prove the existence of the liftings v_i^α to $\pi_\alpha^{-1}(N)$ of $\frac{\partial}{\partial x_i}$ subject to the requirement (2.2).

We take open coverings $\mathcal{V} = \{\mathcal{V}_\lambda\}_{\lambda\in\Lambda_0}$ and $\mathcal{V}' = \{\mathcal{V}'_\lambda\}_{\lambda\in\Lambda_0}$ of $\pi^{-1}(\overline{N})$ in $\mathfrak{X}$ that satisfy the following conditions:

For every $\lambda \in \Lambda_0$,

(i) $\overline{\mathcal{V}_\lambda}$ is a compact subset of $\mathcal{V}'_\lambda$,

(ii) there exists an embedding $\varphi_\lambda : \mathcal{V}'_\lambda \to \mathbb{C}^{m_\lambda}$, and

(iii) $a.^{-1}(\mathcal{V}'_\lambda) \xrightarrow{a.} \mathcal{V}'_\lambda \xrightarrow{\pi} \pi(\mathcal{V}'_\lambda)$ is analytically trivial.

We are allowed to put the condition (iii) due to the analytically "local triviality" of the family $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$ (cf. Definition 2.4 (iii)). By this condition there exist liftings $v^\alpha_{\lambda i}$ of $\frac{\partial}{\partial x_i}$ to $a_\alpha^{-1}(\mathcal{V}'_\lambda)$ for every $\alpha \in \Box_n$ and every $\lambda \in \Lambda_0$ subject to the requirement (2.2). We take a C^∞ partition of unity $\{\rho_\lambda\}_{\lambda\in\Lambda_0}$ on $\mathfrak{X}' := \bigcup_{\lambda\in\Lambda_0} \mathcal{V}_\lambda$ subordinate to the covering $\mathcal{V} = \{\mathcal{V}_\lambda\}_{\lambda\in\Lambda_0}$, i.e., ρ_λ's are "C^∞ functions" on $\mathfrak{X}' := \bigcup_{\lambda\in\Lambda_0} \mathcal{V}_\lambda$ satisfying the following conditions:

(i) $0 \le \rho_\lambda \le 1$ for $\lambda \in \Lambda_0$,

(ii) $Supp\, \rho_\lambda \subset \mathcal{V}_\lambda$ for $\lambda \in \Lambda_0$,

(iii) $\sum_{\lambda\in\Lambda_0} \rho_\lambda \equiv 1$ on $\mathfrak{X}'$.

Notice that $\mathfrak{X}'$ is a singular space. We use here the term "C^∞ functions" in the sense of that they are locally pull-backs of C^∞ functions on $\mathbb{C}^{n_\lambda}$ via embeddings $\varphi_\lambda : \mathcal{V}'_\lambda \longrightarrow \mathbb{C}^{n_\lambda}$. The existence of C^∞-partition of unity $\{\rho_\lambda\}_{\lambda\in\Lambda_0}$ as above is guaranteed by the fact that the proof of the existence of C^∞-partition of unity subordinate to a countably indexed open covering of a C^∞-manifold is also applicable in our case (cf.[5, Chapter I, Theorem 4.6]). We define

$$v^\alpha_i := \sum_{\lambda\in\Lambda_0} a^*_\alpha(\rho_\lambda) V^\alpha_{\lambda i}$$

for $\alpha \in \Box_n$. Then we can easily check that

$$(d\pi_\alpha)(v^\alpha_i) = \pi^*_\alpha(\frac{\partial}{\partial x_i}) \quad \text{and}$$

$$(dE_{\alpha\beta})(v^\beta_i) = E^*_{\alpha\beta}(v^\alpha_i)$$

for every pair (α, β) of elements of $Ob(\Box_n)$ with $\alpha \le \beta$ in the category $\Box_n$.

Finally, we will show that the C^∞ triviality of the family $\mathfrak{X}. \xrightarrow{\pi.} M$ implies the topological triviality of the family $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$. For a fiber $X_t.$ $(t \in M)$ of the family $\mathfrak{X}. \xrightarrow{\pi.} M$, we define an equivalence relation on the topological space $\coprod_{\alpha\in\Box_n} X_{t\alpha}$ (disjoint sum) by

$$p \sim q \text{ iff } p \in X_{t\alpha}, q \in X_{t\beta} \text{ such that } \begin{cases} \alpha \le \beta & \text{and } e_{\alpha\beta}(q) = p \\ \text{or} \quad \alpha > \beta & \text{and } e_{\beta\alpha}(p) = q, \end{cases}$$

where $e_{\alpha\beta} : X_{t\beta} \longrightarrow X_{t\alpha}$ (resp. $e_{\beta\alpha} : X_{t\alpha} \longrightarrow X_{t\beta}$) is the holomorphic map corresponding to an arrow $\alpha \longrightarrow \beta$(resp. $\beta \longrightarrow \alpha$) in $\Box_n$. Then the natural map from $(\coprod_{\alpha\in\Box_n} X_{t\alpha}/ \sim)$ (the quotient topological space of $\coprod_{\alpha\in\Box_n} X_{t\alpha}$ by the equivalence relation $\sim$ above) to X give rise to a homemorphism between these spaces, because $X_t.$ is a cubic hyperresolution of X. Therefore a diffeomorphism between different fibers $X_{t\cdot}$ and $X_{t'\cdot}(t, t' \in M)$ give rise to a homeomorphism between different fibers $X_{t\cdot} \to X_t$ and $X_{t'\cdot} \to X_{t'}$ of the family $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$.

Q.E.D.

2.2 Cohomological descent of R-module sheaves for a cubic hyperequisingular family

Now we are going to prove cohomological descent of R-module sheaves for a cubic hyperequisingular family of comlex projective varieties. Let $X.$ be a $\square_n^+$-object of complex projective varieties and M a complex space. We take a two resolution of $X.$ and denote it by

$$\begin{array}{ccc} Y.' & \xrightarrow{\iota.'} & X.' \\ {\scriptstyle g.}\downarrow & & \downarrow{\scriptstyle f.} \\ Y. & \xrightarrow{\iota.} & X. \end{array} \tag{2.3}$$

We form direct product of each term in (2.2) with M. We denote it by

$$\begin{array}{ccc} Y.' \times M & \xrightarrow{I.':=\iota.'\times id_M} & X.' \times M \\ {\scriptstyle G.:=g.\times id_M}\downarrow & & \downarrow{\scriptstyle F.:=f.\times id_M} \\ Y. \times M & \xrightarrow[I.:=\iota.\times id_M]{} & X. \times M \end{array}$$

2.6 Lemma. *With the notation as above, for a R-module sheaf $\mathcal{A}$ on $X. \times M$, the simple complex associated to the $\square_1^+$−object of complexes of R-module sheaves on $X. \times M$*

$$\begin{array}{ccc} \mathbb{R}H._*H^{\cdot *}\mathcal{A} & \longleftarrow & \mathbb{R}F._*F^{\cdot *}\mathcal{A} \\ \uparrow & & \uparrow \\ I._*I.^*\mathcal{A} & \longleftarrow & \mathcal{A}, \end{array} \tag{2.4}$$

where $H. := I. \circ G. = F. \circ I.'$, is acyclic. Here we identify the derived category $D^+(X. \times M, R)$ of lower bounded complexes of R-module sheaves on $X. \times M$ with the homotopy category of bounded complexes $K^+(X. \times M, Inj\, R)$ of lower bounded complexes of injective R-module sheaves on $X. \times M$.

Proof. We proceed with the proof by induction on n.

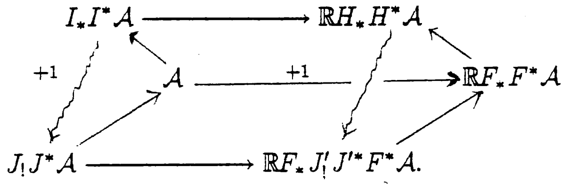
(I) The case n=1: We have short exact sequences of sheaves

$$0 \to J_!J^*\mathcal{A} \to \mathcal{A} \to I_*I^*\mathcal{A} \to 0 \tag{2.5}$$

(2.6) $$0 \to J'_! J'^* F^* \mathcal{A} \to F^* \mathcal{A} \to I'_* I'^* F^* \mathcal{A} \to 0$$

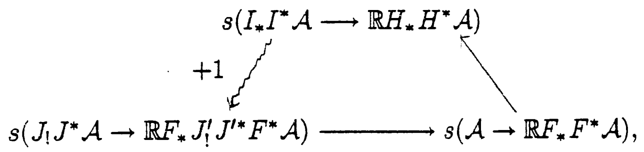
on $X \times M$ and $X' \times M$, respectively, where $J\colon (X \times M) \setminus (Y \times M)$ (*difference set*) $\to X \times M$ and $J'\colon (X' \times M) \setminus (Y' \times M) \to X' \times M$ are inclusion maps. Since F is proper, the adjunction morphism define a morphism of distinguished triangles in $D^+(X \times M, R)$

(2.7)



Hence we have a distimguished triangle in $D^+(X \times M, R)$

(2.8)



where $s(J_! J^* \mathcal{A} \to \mathbb{R}F_* J'_! J'^* F^* \mathcal{A})$ is the simple omplex associated to the $\square_0^+$-object of complexes of R-module sheaves $J_! J^* \mathcal{A} \to \mathbb{R}F_* J'_! J'^* F^* \mathcal{A}$ and so on. From (2.8) it follows a long exact cohomology sequence

$$\cdots \to H^i(s(J_! J^* \mathcal{A} \to \mathbb{R}F_* J'_! J'^* F^* \mathcal{A})) \longrightarrow H^i(s(\mathcal{A} \to \mathbb{R}F_* F^* \mathcal{A}))$$

(2.9) $$\longrightarrow H^i(s(I_* I^* \mathcal{A} \to \mathbb{R}H_* H^* \mathcal{A})) \longrightarrow H^{i+1}(s(J_! J^* \mathcal{A} \to \mathbb{R}F_* J'_! J'^* F * \mathcal{A}))$$
$$\longrightarrow \cdots$$

Since $J^{-1} \circ F \circ J' : (X' \times M) \setminus (Y' \times M) \to (X \times M) \setminus (Y \times M)$ is an analyitcal isomorphism,

$$J_! J^* \mathcal{A} \simeq \mathbb{R}F_* J'_! J'^* F^* \mathcal{A} \quad \text{in} \quad D^+(X \times M, R).$$

Hence

$$H^i(s(J_! J^* \mathcal{A} \to \mathbb{R}F_* J'_! J'^* F^* \mathcal{A})) = 0 \quad \text{for any i},$$

so by (2.9),

$$H^i(s(\mathcal{A} \to \mathbb{R}F_* F^* \mathcal{A})) \quad \cong \quad H^i(s(I_* I^* \mathcal{A} \to \mathbb{R}H_* H^* \mathcal{A}))$$

for any i. This means the morphism of complexes of R-module sheaves

$$s(\mathcal{A} \to \mathbb{R}F_* F^* \mathcal{A}) \longrightarrow s(I_* I^* \mathcal{A} \to \mathbb{R}H_* H^* \mathcal{A})$$

is a qusi-isomorphism. Hence we have

(2.10) $H^i(s(s(\mathcal{A} \to \mathbb{R}F_*F^*\mathcal{A})) \longrightarrow s(I_*I^*\mathcal{A} \to \mathbb{R}H_*H^*\mathcal{A})) = 0$ for any i.

While, by Proposition 1.19

$$s(s(\mathcal{A} \to \mathbb{R}F_*F^*\mathcal{A}) \longrightarrow s(I_*I^*\mathcal{A} \to \mathbb{R}H_*H^*\mathcal{A}))$$

is no more than the simple complex associated to the $\square_1^+$-object of complexes of R-module sheaves in (2.4). Hence we have done.

(II) The case $n \geq 2$: We denote $\square_{-1}^+$−objects corresponding to each $\alpha \in \square_n^+$ by putting α as subindexes in such a way that $\mathcal{A} = \{\mathcal{A}_\alpha\}_{\alpha \in \square_n^+}$, etc.. By the definition of a two resolution of a $\square_n^+$−object of complex projective varieties, a $\square_1^+$-object of complex projective varieties

$$\begin{array}{ccc} Y'_\alpha & \xrightarrow{\iota'_\alpha} & X\alpha' \\ {\scriptstyle g_\alpha}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ Y_\alpha & \xrightarrow[\iota_\alpha]{} & X_\alpha \end{array}$$

which comes from (2.3) is a two resolution of X_α for each $\alpha \in \square_n^+$. Hence by the result in the case of $n = 1$, the simple complex associated to a $\square_1^+$−object of comlexes of R-module sheaves on $X_\alpha \times M$

$$\begin{array}{ccc} \mathbb{R}H_{\alpha *}H_\alpha^*\mathcal{A} & \longleftarrow & \mathbb{R}F_{\alpha *}F_\alpha^*\mathcal{A} \\ \uparrow & & \uparrow \\ I_{\alpha *}I_\alpha^*\mathcal{A} & \longleftarrow & \mathcal{A}, \end{array}$$

where $H_\alpha := I_\alpha \circ G_\alpha = F_\alpha \circ I'_\alpha$, is acyclic for every $\alpha \in \square_n^+$. From this it follows that the simple complex associated to a $\square_1^+$-object of complex of R-module sheaves on $X_\cdot \times M$ in (2.4) is acyclic ([7, p.35, Proposition 5.17]).

Q.E.D.

2.7 Theorem.. *Let $\mathfrak{X}_\cdot \xrightarrow{a_\cdot} \mathfrak{X} \xrightarrow{\pi} M$ be an n-cubic ($n \geq 1$) hyperequisingular family of complex projective varieties,parametrized by a complex space M. Then, for a R-module sheaf $\mathcal{A}$ on $\mathfrak{X}$, the adjunction map*

$$\mathcal{A} \longrightarrow \mathbb{R}a_{\cdot *}a_\cdot^*\mathcal{A}$$

is an isomorphism in $D^+(\mathfrak{X}, R)$.

Proof. Let $\mathfrak{X}_\cdot \xrightarrow{a_{1\cdot}} \mathfrak{X} \times \square_n \xrightarrow{a_{2\cdot}} \mathfrak{X}$ be the factorization of $\mathfrak{X}_\cdot \xrightarrow{a_\cdot} \mathfrak{X}$ as defined in (1.6). Then by (1.10) the assertion of the theorem is equivalent to that the natural map

$$\mathcal{A} \longrightarrow s(a_{1\cdot *}a_\cdot^*\mathcal{A})[1]$$

is a quasi-isomorphism. Since this problem is of local nature with respect to $\mathfrak{X}$, by the globally topological triviality of the family $\mathfrak{X}. \xrightarrow{a.} \mathfrak{X} \xrightarrow{\pi} M$ (cf. Proposition 2.5), it suffices to prove the theorem for the case in which the family is given by

$$X. \times M \xrightarrow{a_{0.} \times id_M} X \times M \xrightarrow{Pr_M} M,$$

where $X. \xrightarrow{a_{0.}} X$ is an n-cubic hyperresolution of a complex projective variety X. In fact, we will prove the following claim by induction on n.

Claim: *Let $\{X^1_., X^2_., \cdots, X^n_.\}$ be a sequence of $\square^+_r$−complex projective varieties $X^r_.$ $(1 \leq r \leq n)$ satisfying the conditions (i) through (iii), but excluding the condition(iv) in Definition 1.15. We put $Z^{(r)}_. := rd(X^1_., \cdots, X^r_.)$ $(1 \leq r \leq n)$ and denote its natural augmentation by*

$$a^{(r)}_{0.} : Z^{(r)}_. \longrightarrow X. \tag{2.11}$$

We form direct product of each term in (2.11) with M and denote it by

$$a^{(r)}_. := a^{(r)}_{0.} \times id_M : Z^{(r)}_. \times M \longrightarrow X \times M.$$

Then, for any R-module sheave $\mathcal{A}$ on $X \times M$, $\mathbb{R}a^{(n)}_{.*} a^{(n)*}_. \mathcal{A}$ is acyclic in $D^+(X \times M, R)$.

Proof of the claim:

(I) The case $n = 1$: This is nothing but Lemma 2.6:

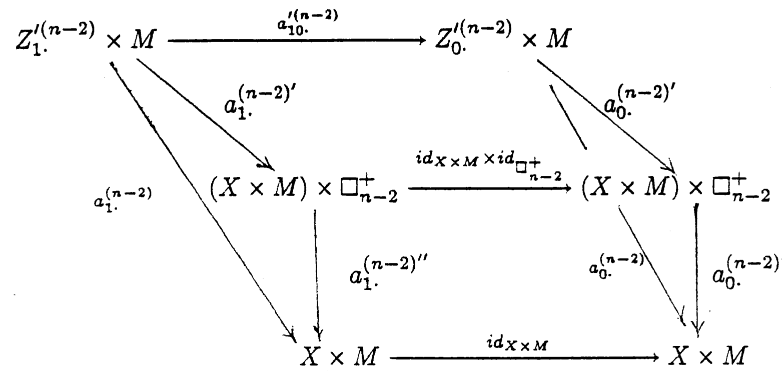
(II) The case $n \geq 2$: We assume that the assertion of the claim holds for any r with $1 \leq r \leq n-1$. We take the factorization of the natural augmentation $a^{(r)}_. : Z^{(r)}_. \times M \to X \times M$ as in (1.9)

$$\begin{array}{ccc} Z^{(r)}_. \times M & \xrightarrow{a^{(r)\prime}_.} & (X \times M) \times \square^+_r \\ & {\scriptstyle a^{(r)}_.} \searrow & \downarrow {\scriptstyle a^{(r)\prime\prime}_.} \\ & & X \times M \end{array}$$

for each r with $1 \leq r \leq n-1$. Since $Z^{(n-1)}_. \times M$ is a $\square^+_{n-1}$−object of complex projective varieties, it is represented as

$$Z^{\prime(n-2)}_{1.} \times M \xrightarrow{a^{\prime(n-2)}_{10.}} Z^{(n-2)}_{0.} \times M,$$

where $a^{(n-2)}_{10.}$ is a morphism between $\square^+_{n-2}$−objects of complex projective varieties. Let $a^{(n-2)}_{1.} : Z^{\prime(n-2)}_{1.} \times M \to X \times M$ and $a^{(n-2)}_{0.} : Z^{\prime(n-2)}_{0.} \times M \to X \times M$ be the natural augmentations. Taking the factorization of $a^{(n-2)}_{1.}$ and $a^{(n-2)}_{0.}$ as in (1.9) and putting together into one diagram, we obtain the following diagram:



For a R-module sheaf $\mathcal{A}$ on $X \times M$, by (1.10) and Proposition 1.19

$$\begin{aligned} &\mathbb{R}a_{\cdot *}^{(n-1)} a_{\cdot}^{(n-1)*}\mathcal{A} \\ &= s(s(a_{0\cdot *}^{(n-2)\prime} a_{0\cdot}^{(n-2)*}\mathcal{A}) \longrightarrow s(a_{1\cdot *}^{(n-2)\prime} a_{1\cdot}^{(n-2)*}\mathcal{A})) \end{aligned} \tag{2.12}$$

and by the induction hypothesis

$$H^i(\mathbb{R}a_{\cdot *}^{(n-1)} a_{\cdot}^{(n-1)*}\mathcal{A}) = 0 \quad \text{for any } i. \tag{2.13}$$

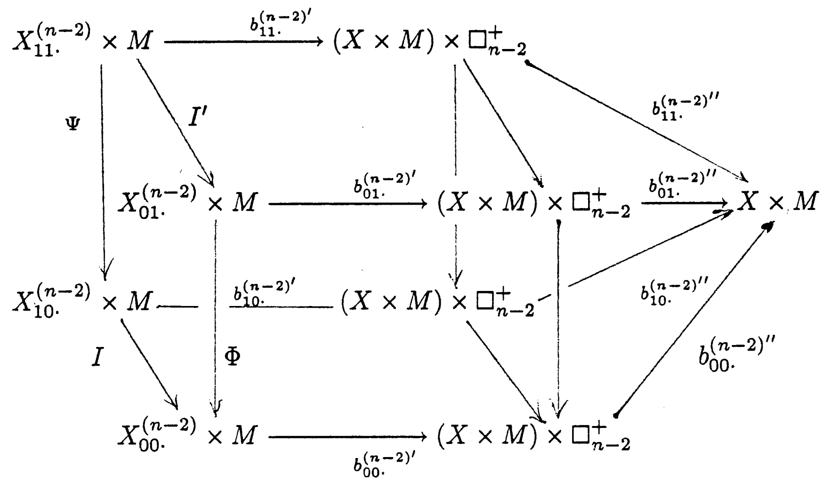
Since $X_{\cdot}^{n}$ is a 2-resolution of $Z_{1\cdot}^{\prime(n-2)}, X_{\cdot}^{n} \times M$ is represented as

$$\begin{array}{ccc} X_{11\cdot}^{(n-2)} \times M & \xrightarrow{I'} & X_{01\cdot}^{(n-2)} \times M \\ \Psi \downarrow & & \downarrow \Phi \\ X_{10\cdot}^{(n-2)} \times M & \xrightarrow{I} & X_{00\cdot}^{(n-2)} \times M \\ & & \| \\ & & Z_{1\cdot}^{\prime(n-2)} \times M, \end{array}$$

where $X_{\alpha_0\alpha_1\cdot}^{(n-2)} \times M$ $(0 \leq \alpha_0, \alpha_1 \leq 1)$ are $\square_{n-2}^{+}$-objects of complex projective varieties. Let

$$b_{\alpha_0\alpha_1\cdot}^{(n-2)} : X_{\alpha_0\alpha_1\cdot}^{(n-2)} \times M \longrightarrow X \times M$$

be the natural augmentation for each (α_0, α_1) with $0 \leq \alpha_0, \alpha_1 \leq 1$ $(b_{00\cdot}^{(n-2)} = a_{1\cdot}^{(n-2)})$. Taking the factorization of each $b_{\alpha_0\alpha_1\cdot}^{(n-2)}$ as in (1.9) and putting together into one diagram, we obtain the following:



Denoting by $b^{(n)}_{\cdot} : X^n_{\cdot} \times M \to X \times M$ the natural augmentation, for a R-module sheaf $\mathcal{A}$ on $X \times M$, we have

$$
\mathbb{R}b^{(n)}_{\cdot *}\, b^{(n)*}_{\cdot}\mathcal{A} \simeq s\begin{pmatrix} b^{(n-2)'}_{11\cdot *} b^{(n-2)*}_{11\cdot}\mathcal{A} & \longleftarrow & b^{(n-2)'}_{01\cdot *} b^{(n-2)*}_{01\cdot}\mathcal{A} \\ \uparrow & & \uparrow \\ b^{(n-1)'}_{10\cdot *} b^{(n-2)*}_{10\cdot}\mathcal{A} & \longleftarrow & b^{(n-2)'}_{00\cdot *} b^{(n-2)*}_{00\cdot}\mathcal{A} \end{pmatrix}
$$

(2.14)

$$
\simeq s\begin{pmatrix} s(b^{(n-2)'}_{11\cdot *} b^{(n-2)*}_{11\cdot}\mathcal{A}) & \longleftarrow & s(b^{(n-2)'}_{01\cdot *} b^{(n-2)*}_{01\cdot}\mathcal{A}) \\ \uparrow & & \uparrow \\ s(b^{(n-2)'}_{10\cdot *} b^{(n-2)*}_{10\cdot}\mathcal{A}) & \longleftarrow & s(b^{(n-2)'}_{00\cdot *} b^{(n-2)*}_{00\cdot}\mathcal{A}). \end{pmatrix}
$$

Here the second isomorphism is because of Proposition 1.20. Since $X^n_{\cdot} \times M$ is a 2-resolution of a $\square^+_{n-2}$-object of complex, projective varieties, by Lemma 2.6

$$
H^i(\mathbb{R}b^{(n)}_{\cdot *}\, b^{(n)*}_{\cdot}\mathcal{A}) = 0 \qquad \text{for any i.}
$$

Hence

$$
s(a^{(n-2)'}_{1\cdot *} a^{(n-2)*}_{1\cdot}\mathcal{A}) = s(b^{(n-2)'}_{00 *} b^{(n-2)*}_{00}\mathcal{A})
$$

(2.15)

$$
\simeq s\begin{pmatrix} s(b^{(n-2)'}_{11\cdot *} b^{(n-2)*}_{11\cdot}\mathcal{A}) & \longleftarrow & s(b^{(n-2)'}_{01\cdot *} b^{(n-2)*}_{01\cdot}\mathcal{A}) \\ \uparrow & & \\ s(b^{(n-2)'}_{10\cdot *} b^{(n-2)*}_{10\cdot}\mathcal{A}) & & \end{pmatrix}
$$

in $D^+(X\times M, R)$, where the last symbol stands for the simple complex associated to the $\square_1$−object of comlexes of R-module sheaves on $X \times M$. Since $Z^{(n)}_{\cdot} := rd(Z^{(n-1)}_{\cdot}, X^{(n)}_{\cdot})$, $Z^{(n)}_{\cdot} \times M$ is represented as follows:

$$Z^{(n)}_{\cdot} \times M = \begin{array}{ccc} X^{(n-2)}_{11\cdot} \times M & \xrightarrow{I'} & X^{(n-2)}_{01\cdot} \times M \\ \Psi \downarrow & & \downarrow a^{(n-2)}_{10\cdot} \cdot \Phi \\ X^{(n-2)}_{10\cdot} \times M & \xrightarrow[a^{(n-2)}_{10\cdot} \cdot I]{} & Z^{(n-2)}_{00\cdot} \times M. \end{array}$$

Hence for any R-module sheaf $\mathcal{A}$ on $X \times M$ we have

$$\mathbb{R}a^{(n)}_{\cdot *}\, a^{(n)*}_{\cdot} \mathcal{A} \simeq s \begin{pmatrix} b^{(n-2)\prime}_{11\cdot *} b^{(n-2)*}_{11\cdot} \mathcal{A} & \longleftarrow & b^{(n-2)\prime}_{01\cdot *} b^{(n-2)*}_{01\cdot} \mathcal{A} \\ \uparrow & & \uparrow \\ b^{(n-2)\prime}_{10\cdot *} b^{(n-2)*}_{10\cdot} \mathcal{A} & \longleftarrow & a^{(n-2)\prime}_{0\cdot *} a^{(n-2)*}_{0\cdot} \mathcal{A} \end{pmatrix}$$

$$\simeq s \begin{pmatrix} s(b^{(n-2)\prime}_{11\cdot *} b^{(n-2)*}_{11\cdot} \mathcal{A}) & \longleftarrow & s(b^{(n-2)\prime}_{01\cdot *} b^{(n-2)*}_{01\cdot} \mathcal{A}) \\ \uparrow & & \uparrow \\ s(b^{(n-2)\prime}_{10\cdot *} b^{(n-2)*}_{10\cdot} \mathcal{A}) & \longleftarrow & s(a^{(n-2)\prime}_{0\cdot *} a^{(n-2)*}_{0\cdot} \mathcal{A}) \end{pmatrix} \quad \text{(by Prop.1.20)}$$

$$\simeq s(s(a^{(n-2)\prime}_{0\cdot *} a^{(n-2)*}_{0\cdot} \mathcal{A} \to s(a^{(n-2)\prime}_{1\cdot *} a^{(n-2)*}_{1\cdot} \mathcal{A})) \quad \text{(by(2.15))}$$

$$\simeq \mathbb{R}a^{(n-1)}_{\cdot *}\, a^{(n-1)*}_{\cdot} \mathcal{A} \quad \text{(by (2.12))}$$

Therefore by (2.13) we have

$$H^i(\mathbb{R}a^{(n)}_{\cdot *}\, a^{(n)*}_{\cdot} \mathcal{A}) = 0 \qquad \text{for any i.}$$

This completes the proof of the claim and so we have done.

Q.E.D.

2.3 Cohomological descent of relative de Rham complexes for cubic hyperequisingular families of complex projective varieties

We are now going to define the *cohomological relative de Rham complex* $DR_{\mathfrak{X}/M} \in D^+(\mathfrak{X}, \mathbb{C})$ for an analyitc family $\mathfrak{X} \xrightarrow{\pi} M$ of complex analyitc varieties, parametrized by a complex space M. For this end we take a system of relative local embeddings $\mathcal{U} := \{(\mathcal{U}'_i, \mathcal{U}_i), \varphi_i, (\mathcal{Y}'_i, \mathcal{Y}_i, \pi_i)\}$ of $\mathfrak{X} \xrightarrow{\pi} M$ which consists of the following entities:

(i) $\{\mathcal{U}'_i\}, \{\mathcal{U}_i\}$ are open covers of $\mathfrak{X}$,

and for every i,

(ii) $\mathcal{U}_i$ is a relatively compact open subset of $\mathcal{U}'_i$,

(iii) $\mathcal{Y}'_i$ is of the forms as $\mathcal{Y}'_i = D_i \times \pi(\mathcal{U}'_i)$ where D_i is a polycylinder in a complex number space $\mathbb{C}^n$,

(iv) $\mathcal{Y}_i$ is a relatively compact open subset of $\mathcal{Y}'_i$,

(v) $\mathcal{Y}_i \xrightarrow{\pi_i} \pi(\mathcal{U}'_i)$ is a smooth family of complex manifolds, parametrized by $\pi(\mathcal{U}_i)$ such that the following diagram commutative:

$$\begin{array}{ccc} \mathcal{Y}_i & \longrightarrow & \mathcal{Y}'_i \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle Pr_{\pi(\mathcal{U}'_i)}} \\ \pi(\mathcal{U}_i) & \longrightarrow & \pi(\mathcal{U}'_i) \end{array}$$

(vi) $\varphi_i : \mathcal{U}'_i \to \mathcal{Y}'_i$ is a closed embedding over $\pi(\mathcal{U}'_i)$ such that $\varphi_i(\mathcal{U}_i) = \mathcal{Y}_i$.

For each (p+1)-tuple $(i) = \{i_0 < i_1 < \cdots < i_p\}$ we consider an open set $\mathcal{U}'_{(i)} = \mathcal{U}'_{i_0} \cap \cdots \cap \mathcal{U}'_{i_p}$ and a relative closed embedding

$$\begin{aligned} &\mathcal{U}'_{(i)} \to \mathcal{Y}'_{(i)} \\ &:= (\pi_{i_0}^{-1}(\pi(\mathcal{U}'_{(i)})) \cap \mathcal{Y}'_{i_0}) \times_{\pi(\mathcal{U}'_{(i)})} (\pi_{i_1}^{-1}(\pi(\mathcal{U}'_{(i)})) \cap \mathcal{Y}'_{i_1}) \times_{\pi(\mathcal{U}'_{(i)})} \cdots \\ &\qquad \times_{\pi(\mathcal{U}'_{(i)})} (\pi_{i_p}^{-1}(\pi(\mathcal{U}'_{(i)})) \cap \mathcal{Y}'_{i_p}) \end{aligned}$$

over $\pi(\mathcal{U}'_{(i)})$, where $\times_{\pi(\mathcal{U}'_{(i)})}$ denotes the fiber product over $\pi(\mathcal{U}'_{(i)})$; and define

$$\Omega^{\cdot}_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})\hat{|}\mathcal{U}'_{(i)}} := \varprojlim_k \Omega^{\cdot}_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})} / \mathcal{I}^k_{\mathcal{U}'_{(i)}} \cdot \Omega^{\cdot}_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}$$

where $\Omega^{\cdot}_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}$ is the relative de Rham complex of the smooth family $\mathcal{Y}'_{(i)} \xrightarrow{Pr_{\pi(\mathcal{U}'_{(i)})}} \pi(\mathcal{U}'_{(i)})$ of comlex manifolds and $\mathcal{I}_{\mathcal{U}'_{(i)}}$ is the ideal sheaf of $\mathcal{U}'_{(i)}$ in the structure sheaf $\mathcal{O}_{\mathcal{Y}'_{(i)}}$ of $\mathcal{Y}'_{(i)}$. We call $\Omega^{\cdot}_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})\hat{|}\mathcal{U}'_{(i)}}$ the completion of $\Omega^{\cdot}_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}$ along $\mathcal{U}'_{(i)}$. Then we consider a complex of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{X}$

$$C^{\cdot}_{(i)} := j_*(\Omega^{\cdot}_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})\hat{|}\mathcal{U}'_{(i)}})|_{\mathcal{U}_{(i)}},$$

where j is the inclusion of $\mathcal{U}'_{(i)}$ into $\mathfrak{X}$. Here, putting 0 outside $\mathcal{U}_{(i)}$, we consider $C^{\cdot}_{(i)}$ as a complex of sheaves on $\mathfrak{X}$. Now for any $0 \le j \le p$, let $(i') = \{i_0, \cdots, \check{i}_j, \cdots, i_p\}$ (omit i_j). Then we have natural inclusions $\mathcal{U}'_{(i)} \to \mathcal{U}'_{(i')}$, which maps $\mathcal{U}_{(i)}$ into $\mathcal{U}_{(i')}$, and $\mathcal{Y}'_{(i)} \longrightarrow \mathcal{Y}'_{(i')}$ over $\pi(\mathcal{U}'_{(i)}) \longrightarrow \pi(\mathcal{U}'_{(i')})$, which maps $\mathcal{Y}_{(i)}$ into $\mathcal{Y}_{(i')}$ over $\pi(\mathcal{U}_{(i)}) \longrightarrow \pi(\mathcal{U}_{(i')})$. Hence there is a natural map

$$\Omega^{\cdot}_{\mathcal{Y}'_{(i')}/\pi(\mathcal{U}'_{(i')})\hat{|}\mathcal{U}'_{(i')}|\mathcal{U}'_{(i)}} \longrightarrow \Omega^{\cdot}_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})\hat{|}\mathcal{U}'_{(i)}},$$

and a morphism of complexes on $\mathfrak{X}$

$$\delta_{j,(i)} : C^{\cdot}_{(i')} \longrightarrow C^{\cdot}_{(i)}$$

Notice that, by construction, for two integers $0 \leq j < k \leq p$, the corresponding four δ maps are compatible with each other. Hence we can define a double complex $\mathcal{C}(\mathcal{U})$ by

$$\mathcal{C}(\mathcal{U})^p = \prod_{|i|=p} C^{\cdot}_{(i)}$$

where $|i| = p$ for $i = (i_0, \cdots, i_p)$, and

$$\delta^{p-1} := \prod_{|i1=p} \sum_{j=0}^{p} (-1)^j \delta_{j,(i)} : \mathcal{C}(\mathcal{U})^{p-1} \longrightarrow \mathcal{C}(\mathcal{U})^p$$

We denote by $\hat{\Omega}^{\cdot}_{\mathfrak{X}/M}(\mathcal{U})$ the associated simple complex of $\mathcal{C}(\mathcal{U})$. If $\mathcal{V} = \{(\mathcal{V}'_j, \mathcal{V}_j), \psi_j, (\mathcal{Z}'_j, \mathcal{Z}_j, \pi_j)\}$ is a refinement of a system of relative local embeddings $\mathcal{U}$, then there is a natural map of double complexes $\varphi : \mathcal{C}(\mathcal{U}) \longrightarrow \mathcal{C}(\mathcal{V})$ and, as in the absolute case, we can see that the associated map $\hat{\Omega}^{\cdot}_{\mathfrak{X}/M}(\mathcal{U}) \longrightarrow \hat{\Omega}^{\cdot}_{\mathfrak{X}/M}(\mathcal{V})$ of simple complexes to φ is s qusi-isomorphism (cf. [8, p.29]). Therefore we conclude that $\hat{\Omega}^{\cdot}_{\mathfrak{X}/M}(\mathcal{U})$ defines an element of $D^+(\mathfrak{X}, \mathbb{C})$, which is independent of the choice of $\mathcal{U}$.

2.8 Definition. We call an element of $D^+(\mathfrak{X}, \mathbb{C})$ determined by the $\hat{\Omega}^{\cdot}_{\mathfrak{X}/M}(\mathcal{U})$ as above the *cohomological relative de Rham complex of* $\mathfrak{X} \xrightarrow{\pi} M$ and denote by $DR^{\cdot}_{\mathfrak{X}/M}$.

Let $\mathfrak{X}_{.} \xrightarrow{a_{.}} \mathfrak{X} \xrightarrow{\pi} M$ be an n-cubic hyperequisingular family of comlex projective varieties, parametrized by a complex space M. For each $\alpha \in \square_n$ we denote by $\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}$ the relative de Rham complex of a smooth family $\mathfrak{X}_\alpha \xrightarrow{\pi \cdot a_\alpha} M$ of complex manifolds. Then $\Omega^{\cdot}_{\mathfrak{X}_{.}/M} := \{\Omega^{\cdot}_{\mathfrak{X}_\alpha/M}\}_{\alpha \in \square_n}$ is obviously a complex of sheaves of $\mathbb{C}$-vector spaces on a $\square_n$-complex manifold $\mathfrak{X}_{.}$. The rest of this section will be devoted to proving the following theorems and a corollary.

2.9 Theorem. *(Cohomological descent of relative de Rham complexes) Under the same setting as above, tere exists naturally an isomorphism*

$$DR^{\cdot}_{\mathfrak{X}/M} \simeq \mathbb{R}a_{.*}\Omega^{\cdot}_{\mathfrak{X}_{.}/M}$$

in $D^+(\mathfrak{X}, \mathbb{C})$.

2.10 Theorem. *(Relative formal analytic Poincaré lemma) Under the same setting as above, $\hat{\Omega}^{\cdot}_{\mathfrak{X}/M}(\mathcal{U})$ yields a resolution of the sheaf $\pi^{\cdot}(\mathcal{O}_M)$ for a system of relative local embeddings $\mathcal{U} = \{(\mathcal{U}'_i, \mathcal{U}_i), \varphi_i, (\mathcal{Y}'_i, \mathcal{Y}_i, \pi_i)\}$ of $\mathfrak{X} \xrightarrow{\pi} M$, where $\pi^{\cdot}(\mathcal{O}_M)$ denotes the topological inverse of the structure sheaf of M by $\pi : \mathfrak{X} \to M$.*

2.11 Corollary. *There exist isomorphisms*

$$H^i(\mathfrak{X}, \pi^{\cdot}(\mathcal{O}_M)) \simeq H^i(\mathbb{R}\Gamma(\mathfrak{X}, s(a_{1\cdot *}\Omega^{\cdot}_{\mathfrak{X}\cdot/M}))[1])$$

$$\simeq H^i(\mathbb{R}\Gamma(\mathfrak{X}_\cdot, \Omega^{\cdot}_{\mathfrak{X}\cdot/M})[1]) \quad (1 \le i \le 2\,dim_{\mathbb{C}}\mathfrak{X}).$$

To prove this the following two theorems are essential.

2.12 Theorem. *(Mayer-Vietories sequence for relative de Rham complexes) Let $\mathfrak{Y} \xrightarrow{\pi} M$ be a flat family of analyitc varieties, parametrized by a complex space M. Suppose that $\mathfrak{Y} \xrightarrow{\pi} M$ is relatively embedded in a smooth family $\mathfrak{X} \xrightarrow{\pi'} M$ of complex manifolds, parametrized by the same complex space M, and further suppose that $\mathfrak{Y}$ is a union of two closed subvarieties $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ of $\mathfrak{X}$. Then there is a sequence of relative de Rham comlexes*

$$0 \to \Omega^{\cdot}_{\mathfrak{X}/M\hat{|}\mathfrak{Y}} \to \Omega^{\cdot}_{\mathfrak{X}/M\hat{|}\mathfrak{Y}_1} \oplus \Omega^{\cdot}_{\mathfrak{X}/M\hat{|}\mathfrak{Y}_2} \to \Omega^{\cdot}_{\mathfrak{X}/M\hat{|}\mathfrak{Y}_1\cap\mathfrak{Y}_2} \to 0,$$

which is exact on any relatively compact subset $\mathfrak{X}'$ of $\mathfrak{X}$, where $\Omega^{\cdot}_{\mathfrak{X}/M\hat{|}\mathfrak{Y}}$ is the completion of a complex of the relative de Rham complex $\Omega^{\cdot}_{\mathfrak{X}/M}$ along $\mathfrak{Y}$ and so on.

2.13 Theorem. *Let $f : X' \to X$ be a poper morphism of analytic varieties. Let Y be a closed analyitc subvariety of X, and let $Y' := f^{-1}(Y)$. Assume that f maps $X' - Y'$ isomorphically onto $X - Y$. Suppose we are given coherent sheaves $\mathcal{F}$ on X and $\mathcal{F}'$ on X', and an injective map $\mathcal{F} \to f_*\mathcal{F}'$, whose restriction to $X - Y$ is an isomorphism. Then the simple complex associated to the $\square_1^+$-object of lower bounded comlexes of sheaves of $\mathbb{C}$-vector spaces on X*

$$\begin{array}{ccc} \hat{\iota}_*\mathbb{R}\hat{F}_*\hat{\mathcal{F}}' & \longleftarrow & \mathbb{R}f_*\mathcal{F}' \\ \uparrow & & \uparrow \\ \hat{\iota}_*\hat{\mathcal{F}} & \longleftarrow & \mathcal{F} \end{array}$$

is acyclic in $D^+(X_0, \mathbb{C})$, where X_0 is any relatively compact open subset of X, ι is the closed immersion $Y \to X$ and $\hat{}$ denotes the completion along Y or Y', respectively.

The proof of Theorem 2.12 for the absolute case, i.e., $M = \{\text{one point}\}$, can be found in [8, p.89, Proposition(1.4)]. Since $\Omega^p_{\mathfrak{X}/M}$ are locally free sheaves over

$\mathcal{O}_{\mathfrak{X}}$, and since all of $\Omega^p_{\mathfrak{X}/M\uparrow\mathfrak{Y}}$, $\Omega^p_{\mathfrak{X}/M\uparrow\mathfrak{Y}_i}(i = 1,2)$, $\Omega_{\mathfrak{X}/M\uparrow\mathfrak{Y}_1\cap\mathfrak{Y}_2}$ are completions with respect to some ideal sheaves of $\mathcal{O}_{\mathfrak{X}}$, the same arguments also go well for the relative case. Hence we obtain Theorem 2.12. Theorem 2.13 is an analytic analogue of Proposition(4.3) in [8]. The key point of the proof of Proposition(4.3) in [8] is "fundamental theorem of a proper morphism" ([6, 4.1.5]), which tell us that, with the same notation as in Theorem 2.13, though under the assumption that all things are algebraic,

$$R^i\hat{f}_*\hat{\mathcal{F}}' \simeq (R^if_*\mathcal{F}')\hat{}\quad (i \geq 0),$$

where $(R^if_*\mathcal{F}')\hat{}$ is the completion of $R^if_*\mathcal{F}'$ along Y, and $R^i\hat{f}_*\hat{\mathcal{F}}'$ the i-th higher direct image sheaf of $\hat{\mathcal{F}}'$ by the morphism of formal scheames $\hat{f} : \hat{X}' \to \hat{X}$, induced by f, from the completion $\hat{X}'$ of X' along Y' to that of X along Y. Fortunately, we have analytic analogue of the "fundamental theorem of a proper morphism" due to C. Bănică and O. Stănăşă ([1, p.225, VI, Cor.4.5). Using this theorem, we can carry out the same arguments as in the proof of Proposition(4.3) in [8]. Hence we obtain Theorem 2.13.

To prove Theorem 2.9 we shall use the following theorem, which is an analytic analogue of Theorem(4.4) in [8, p.44].

2.14 Theorem. *Let $\mathfrak{X}' \xrightarrow{\pi'} M$ and $\mathfrak{X} \xrightarrow{\pi} M$ be two flat families of analytic varieties, parametrized by a complex space M. Let $\mathfrak{X}' \xrightarrow{f} \mathfrak{X}$ be a proper morphism of analytic varieties over M, $\mathfrak{Y}$ a closed subvariety of $\mathfrak{X}, \mathfrak{Y}' := f^{-1}(\mathfrak{Y})$, and $h := f_{|\mathfrak{Y}'} : \mathfrak{Y}' \to \mathfrak{Y}$ the restriction of f to $\mathfrak{Y}'$. We assume the following:*

(i) f maps $\mathfrak{X}' - \mathfrak{Y}'$ isomorphically onto $\mathfrak{X} - \mathfrak{Y}$,

(ii) there exist

(a) smooth families of complex manifolds $\mathfrak{Z}' \xrightarrow{\pi'} M$ and $\mathfrak{Z} \xrightarrow{\pi} M$, parametrized by the complex space M,

(b) closed immersions $\mathfrak{X}' \to \mathfrak{Z}'$ and $\mathfrak{X} \to \mathfrak{Z}$ over M, and

(c) a proper morphism $g : \mathfrak{Z}' \to \mathfrak{Z}$ over M

such that $g_{|\mathfrak{X}'} = f$ and g maps $\mathfrak{Z}' - g^{-1}(\mathfrak{Y})$ isomorphically onto $\mathfrak{Z} - \mathfrak{Y}$.

Then the simple complex associated to the following $\square_1^+$-object of lower bounded complexes of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{X}$

$$\begin{array}{ccc} \mathbb{R}h_*\Omega^{\cdot}_{\mathfrak{Z}'/M\uparrow\mathfrak{Y}'} & \longleftarrow & \mathbb{R}f_*\Omega^{\cdot}_{\mathfrak{Z}/M\uparrow\mathfrak{X}'} \\ \uparrow & & \uparrow \\ \Omega^{\cdot}_{\mathfrak{Z}/M\uparrow\mathfrak{Y}} & \longleftarrow & \Omega^{\cdot}_{\mathfrak{Z}/M\uparrow\mathfrak{X}} \end{array}$$

of lower bounded complexes of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{X}$ is acyclic in $D^+(\mathfrak{X}_0, \mathbb{C})$, where $\mathfrak{X}_0$ is any relatively compact open subset of $\mathfrak{X}$.

Since the proof of Theorem 2.14 is almost identical with that of the algebraic case ([8, p.44, Chapter II, Theorem(4.4)), we omit it, just mentioning that we essentially use Theorem 2.12 and Theorem 2.13 to prove it.

2.15 Proposition. *Let $\mathfrak{Y} \xrightarrow{\pi} M$ be a flat family of analytic varieties, parametrized by a complex space M, which is relatively embedded in a smooth family $\mathfrak{X} \xrightarrow{\pi'} M$ of complex manifols, parametrized by M. Suppose $\mathfrak{Y}$ is a union of finite closed subvarieties $\mathfrak{Y}_1, \cdots, \mathfrak{Y}_n (n \geq 2)$. Let $\iota : \mathfrak{Y}_. \to \mathfrak{Y}$ be the n-cubic object of analytic varieties, augmented to $\mathfrak{Y}$, which comes from the finite closed cover $\{\mathfrak{Y}_r\}_{1 \leq r \leq n}$ of Y (cf. Example 1.11). Then we have a quasi-isomorphism*

$$\Omega^{\cdot}_{\mathfrak{X}/M \hat{|} \mathfrak{Y}} \to \iota_* \Omega^{\cdot}_{\mathfrak{X}/M \hat{|} \mathfrak{Y}}$$

on any relatively compact open subset $\mathfrak{X}_0$ of $\mathfrak{X}$, where

$$\Omega^{\cdot}_{\mathfrak{X}/M \hat{|} \mathfrak{Y}.} := \{\Omega^{\cdot}_{\mathfrak{X}/M \hat{|} \mathfrak{Y}_\alpha}\}_{\alpha \in \Box_n}$$

is the complex of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{Y}$, obtained by the completions of $\Omega^{\cdot}_{\mathfrak{X}/M}$ along $\mathfrak{Y}_\alpha$ for every $\alpha \in \Box_n$.

Proof. We use induction on n. The case n=2 is nothing but Theorem 2.12. In the case $n \geq 2$ the argument is almost identical with that of Proposition 1.4 in [7, p.61], which concerns the absolute and algebraic case. Hence we omit it.

Q.E.D.

2.16 Proposition. *Let X be a complex projective variety embedded in a smooth complex projective variety Y, and let $X. \xrightarrow{a.} X$ be an n-cubic hyperresolution of X. We denote by X_h and Y_h the corresponding complex analytic varieties, and by $X_{h}. \xrightarrow{a_h.} X_h$ the corresponding n-cubic hyperreosolution of X_h in the category of complex analytic varieties. Let p be a point of X_h. We take an open neighborhood V' of p in Y_h and define $U' := V' \cap X_h$ and $U'_\alpha := a_\alpha^{-1}(U')$ for each $\alpha \in \Box_n$. We consider an n-cubic object of the product families of complex analytic varieties*

$$U'_. \times M \xrightarrow{a. \times id_M} U' \times M$$

where M is a complex space and id_M is the identy map on M. Then, for any relatively compact open neighborhood V of p, contained in V', we have a quasi-isomorphism

$$\Omega^{\cdot}_{V \times M/M \hat{|} U \times M} \to \mathbb{R}(a. \times id_M)_* \Omega^{\cdot}_{U. \times M/M} \qquad (2.16)$$

where $U := V \cap X_h$ and $U_\alpha := a_\alpha^{-1}(U)$ for $\alpha \in \Box_n$.

Proof. By the same argument used in the proof of Theorem 2.7, we can reduce the proof to the case of n=2. Hence it suffices to prove (2.16) for the folloeing $\Box_1^+$-object of complex analytic varieties:

$$\begin{array}{ccc} U_{11} \times M & \longrightarrow & U_{01} \times M \\ \downarrow & & \downarrow{\scriptstyle a_{01} \times id_M} \\ U_{10} \times M & \longrightarrow & U_{00} \times M, \\ & & \| \\ & & U \times M \subset V \times M \end{array}$$

which is a cartesian square, and where U_{01} is a smooth analytic variety, $a_{01} : U_{01} \longrightarrow U_{00}$ a proper morphism (hence so is $a_{01} \times id_M : U_{01} \times M \longrightarrow U_{00} \times M$), $U_{11} \times M \longrightarrow U_{01} \times M$ and $U_{10} \times M \longrightarrow U_{00} \times M$ are closed immersions, such that $a_{01} \times id_M : (U_{10} \times M) \setminus (U_{11} \times M) \longrightarrow (U_{00} \times M) \setminus (U_{10} \times M)$ is an isomorphism. Furthermore, using Proposition 2.15, we can reduce the proof in the case where U_{01} and U_{00} are irreducible.(For the details of this procedure we refer to the proof of Théorème 1.5 in [7, p.62]). Now we will check the proof in this case.

We write X, X', Y, Y', Z and a_{01} instead of $U_{00}, U_{01}, U_{10}, U_{11}, V$ and f, respectively. Since X, X' come from complex projective varieties, by the result of Hironaka (Elimination of points of indeterminancy of a rational mapping, [9]), there exists a commutative diagram

(2.17)
$$\begin{array}{ccccc} \overline{X'} & \xrightarrow{f_4} & \overline{X} & & \\ & {\scriptstyle f_3} \searrow & \downarrow{\scriptstyle f_2} & \searrow {\scriptstyle f_1} & \\ & & X' & \xrightarrow[f]{} & X \hookrightarrow Z \end{array}$$

such that; (i)f_1, f_2 are the composits of blowing-ups along non-singular center, (ii) $\overline{X}, \overline{X'}$ are non-singular, and (iii) f_2, f_4 are proper morphisms. Blowing up Z along the same center as X, we have the following diagram

(2.18)
$$\begin{array}{ccccc} \overline{Y} & \xrightarrow{\bar{\iota}} & \overline{X} & \longrightarrow & \overline{Z} \\ \downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} \\ Y & \xrightarrow{\iota} & X & \longrightarrow & Z \end{array}$$

where $\overline{Y} := f_1^{-1}(Y)_{red}$. Forming direct product of each term in the diagram (2.18) with M, we come to the same setting as in Theorem 2.14. Hence, by that theorem, we conclude that the simple complex associated to the following $\square_1^+$-object of lower bounded complex of sheaves of $\mathbb{C}$-vector spaces on $X \times M$

$$\begin{array}{ccc} \mathbb{R}h_{1*}\Omega^{\cdot}_{\overline{Z} \times M \hat{|} \overline{Y} \times M} & \longleftarrow & \mathbb{R}f_{1*}\Omega^{\cdot}_{\overline{Z} \times M \hat{|} \overline{X} \times M} \\ \uparrow & & \uparrow \\ \iota_* \Omega^{\cdot}_{Z \times M \hat{|} Y \times M} & \longleftarrow & \Omega^{\cdot}_{Z \times M \hat{|} X \times M}, \end{array}$$

where $h_1 := f_1 \cdot \overline{\iota}$, is acyclic in $D^+(X_0 \times M, \mathbb{C})$ for any relatively compact open subset X_0 of X. If we define $s(X \times M/Y \times M)$, $s(\overline{X} \times M/\overline{Y} \times M)$ to be the simple complexes associated to the morphisms of complexes

$$\Omega^{\cdot}_{Z\times M\upharpoonright X\times M} \to \iota_* \Omega^{\cdot}_{Z\times M\upharpoonright Y\times M} \quad \text{and}$$
$$\Omega^{\cdot}_{\overline{Z}\times M\upharpoonright \overline{X}\times M} \to \iota^* \Omega_{\overline{Z}\times M\upharpoonright \overline{Y}\times M},$$

respectively, then the above statetement is equivalent to that the morphism $f_1^* : s(X\times M/Y\times M) \to s(\overline{X}\times M/\overline{Y}\times M)$ induced by f_1 is a quasi- isomorphism on any relatively compact open subset X_0 of X. We consider the following diagram derived from the one in (2.16)

$$\begin{array}{ccccc} s(\overline{X'} \times M/\overline{Y'} \times M) & \xleftarrow{f_4^*} & s(\overline{X} \times M/\overline{Y} \times M) & & \\ & \nwarrow^{f_3^*} & \uparrow f_2^* & \nwarrow^{f_1^*} & \\ & & s(X' \times M/Y' \times M) & \xleftarrow{f^*} & s(X \times M/Y \times M) \end{array} \tag{2.19}$$

Here we should notice that since X', $\overline{X'}$ are non-singular, $s(X' \times M/Y' \times M)$, $s(\overline{X'} \times M/\overline{Y'} \times M)$ are defined as the simple complexes assciated to the morphisms of complexes

$$\Omega^{\cdot}_{X'\times M/M} \to \iota'_* \Omega^{\cdot}_{X'\times M/M\upharpoonright Y'\times M} \quad \text{and}$$
$$\Omega^{\cdot}_{\overline{X'}\times M/M} \to \overline{\iota'}_* \Omega^{\cdot}_{\overline{X}M/M\upharpoonright \overline{Y}\times M},$$

respectively, where $\iota' : Y' \to X'$, $\iota : \overline{Y'} \to \overline{X'}$ are natural inclusions. By the same reasoning as for f_1^*, we conclude that f_3^*, f_4^* are quasi-isomprphisms on any realatively compact open subsets of X, X' , respectively. Hence by the commutativity of the diagram in (2.19), we conclude that f_2^* is a quasi-isomorphism on any relatively compact open subset of X' and f^* is so. This completes the proof of the proposition.

Q.E.D.

We are now in a position to prove Theorem 2.9 and Theorem 2.10.

Proof of Theorem 2.9: By the assumption we can take a system $\mathcal{U} = \{(\mathcal{U}'_i, \mathcal{U}_i), \varphi_i, (D'_i, D_i)\}$ of relative local embeddings of $\mathfrak{X}$ which satisfies the following conditions:

For each i there exists a point $p_i \in \mathcal{U}_i$ and an embedding $e_i : X_{\pi(p_i)} \to Y_{p_i}$ of $X_{\pi(p_i)}$ (the fiber of $\mathfrak{X}$ over $\pi(p_i)$) into a smooth complex projective variety Y_{p_i} such that

(i) $a_\cdot^{-1}(\mathcal{U}_i') \xrightarrow{a_\cdot|_{a_\cdot^{-1}(\mathcal{U}_i')}} \mathcal{U}_i' \xrightarrow{\pi|_{\mathcal{U}_i'}} \pi(\mathcal{U}_i')$ is isomorphic to

$$(a_\cdot^{-1}(\mathcal{U}_i') \cap X_{\pi(p_i)}) \times \pi(\mathcal{U}_i') \xrightarrow{a_\cdot \times id_{\pi(\mathcal{U}_i')}} (\mathcal{U}_i' \cap X_{\pi(P_i)}) \times \pi(\mathcal{U}_i') \xrightarrow{Pr_{\pi(\mathcal{U}_i')}} \pi(\mathcal{U}_i')$$

(2.20) (For the notation see Definition 2.4)

(ii) D_i' is an open neighborhood of the point $e_i(p_i)$ in Y_{p_i} and

(iii) $\varphi_i(\mathcal{U}_i') = (e_i(X_{\pi(p_i)}) \cap D_i') \times \pi(\mathcal{U}_i')$.

Then by Proposition 2.16 the natural map

$$\hat{\Omega}^\cdot_{\mathcal{U}_i'/\pi(\mathcal{U}_i')} \to \mathbb{R}a_{\cdot|a_\cdot^{-1}(\mathcal{U}_i')*}\Omega_{a_\cdot^{-1}(\mathcal{U}_i')/\pi(\mathcal{U}_i')}^\cdot = \mathbb{R}a_{\cdot*}\Omega^\cdot_{\mathfrak{X}_\cdot/M|\mathcal{U}_i'}$$

is a quasi-isomorphism on every relatively compact subset $\mathcal{U}_i''$ of $\mathcal{U}_i'$ with $\mathcal{U}_i'' \supset \mathcal{U}_i$, hence

$$j_*(\Omega^\cdot_{\mathcal{U}_i'/\pi(\mathcal{U}_i')})|_{\mathcal{U}_i} \to j_*(\mathbb{R}a_{\cdot*}\Omega^\cdot_{\mathfrak{X}_\cdot/M|\mathcal{U}_i'})|_{\mathcal{U}_i}$$

is a quasi-isomorphism on $\mathfrak{X}$ for evry i, where $j : \mathcal{U}_i' \to \mathfrak{X}$ is the inclusion map. From this it follows that for any $(i) = \{i_0 < i_1 < \cdots < i_p\}$

$$C_{(i)} := j_*(\hat{\Omega}^\cdot_{\mathcal{U}_{(i)}'/\pi(\mathcal{U}_{(i)}')})|_{\mathcal{U}_i} \to D_{(i)} := j_*(\mathbb{R}a_{\cdot*}\Omega^\cdot_{\mathfrak{X}_\cdot/M|\mathcal{U}_{(i)}'})|_{\mathcal{U}_i}$$

is a quasi-isomorphism. Similarly as for $C(\mathcal{U})$, we define a double complex $D(\mathcal{U})$, using $\{D_{(i)}\}$ is nothing but $\mathbb{R}a_{\cdot*}\Omega^\cdot_{\mathfrak{X}_\cdot/M}$. Therefore we conclude that the natural map

$$\hat{\Omega}^\cdot_{\mathfrak{X}/M} \to \mathbb{R}a_{\cdot*}\Omega^\cdot_{\mathfrak{X}_\cdot/M}$$

is a quasi-isomorphism. Since any system of relative local embeddings of $\mathfrak{X}$ has its refinement satisfying the conditions (i),(ii),(iii) in (2.20) we obtain the theorem.

Proof of Theorem 2.10: Since the poblem is local, we may asuume that $\pi : \mathfrak{X} \to M$ is a product family, namely $\pi = Pr_M : \mathfrak{X} = X \times M \to M$, where X is a complex projective variety, M a complex space, and $\pi := Pr_M$ the projection to M. Furthermore we may assume that X is embedded in a smooth complex projective variety Z. We define $\mathfrak{Z} := Z \times M$ and $\pi_1 := Pr_M : \mathfrak{Z} = Z \times M \to M$ the projection to M. Under this set-up we will prove that

$$\pi^\cdot \mathcal{O}_M \to \Omega^\cdot_{\mathfrak{Z}/M\hat{|}\mathfrak{X}} \tag{2.21}$$

is a quasi-isomorphism on any relatively compact open subset of $\mathfrak{X}$. In the following we shall confuse complex algebraic objects and their associated anlytic objects, and write them by the same letters. To prove (2.21) we proceed by induction on $dim_{\mathbb{C}}X$. If $dim_{\mathbb{C}}X = 0$, this is nothing to prove. We assume that (2.21) holds for any X with $0 \leq dim_{\mathbb{C}}X < n$. By the Hironaka resolution theory ([9]) there is the following commutative diagam:

$$
\text{(2.22)} \qquad \begin{array}{ccc}
Y' & \xrightarrow{\iota'} & X' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{\iota} & X \hookrightarrow Z
\end{array}
$$

with the property $f_{|X'-Y'} : X' - Y' \to X - Y$ is an isomorphosm, where X' is a smooth complex projective variety, $f : X' \to X$ a proper morphism, Y a proper closed subvariety of X, $Y' := f^{-1}(Y)$, and ι, ι' closed immersions. Taking direct product of each term in (2.22) with M, we obtain the commutative diagram

$$
\text{(2.23)} \qquad \begin{array}{ccc}
\mathfrak{Y}' & \xrightarrow{I'} & \mathfrak{X}' \\
{\scriptstyle G}\downarrow & & \downarrow{\scriptstyle F} \\
\mathfrak{Y} & \xrightarrow{I} & \mathfrak{X} \hookrightarrow \mathfrak{Z},
\end{array}
$$

where $\mathfrak{X} := X \times M$, $\mathfrak{X}' := X' \times M$, $F := f \times id_M$, etc.. Then, by Theorem 2.14 it follows that

$$
\begin{array}{ccc}
\mathbb{R}H_*\Omega^{\cdot}_{\mathfrak{X}'/M\upharpoonright\mathfrak{Y}'} & \longleftarrow & \mathbb{R}F_*\Omega^{\cdot}_{\mathfrak{X}'/M} \\
\uparrow & & \uparrow \\
\mathbb{R}I_*\Omega^{\cdot}_{\mathfrak{Z}/M\upharpoonright\mathfrak{Y}} & \longleftarrow & \Omega^{\cdot}_{\mathfrak{Z}/M\upharpoonright\mathfrak{X}}
\end{array}
$$

is acyclic in $D^+(\mathfrak{X}_0, \mathbb{C})$ for any relatively compact open subset $\mathfrak{X}_0$ of $\mathfrak{X}$, where $H := I \circ G = F \circ I'$. Therefore we have the following long exact sequence of cohomology

$$
\begin{aligned}
&\to H^i(\mathfrak{X}_0, \Omega^{\cdot}_{\mathfrak{Z}/M\upharpoonright\mathfrak{X}}) \to H^i(\mathfrak{X}_0, \mathbb{R}I_*\Omega^{\cdot}_{\mathfrak{Z}/M\upharpoonright\mathfrak{Y}}) \oplus H^i(\mathfrak{X}_0, \mathbb{R}F_*\Omega^{\cdot}_{\mathfrak{X}'/M}) \\
\text{(2.24)} \qquad &\to H^i(\mathfrak{X}_0, \mathbb{R}H_*\Omega^{\cdot}_{\mathfrak{X}'/M\upharpoonright\mathfrak{Y}'}) \to H^{i+1}(\mathfrak{X}_0, \mathbb{R}H_*\Omega^{\cdot}_{\mathfrak{Z}/M\upharpoonright\mathfrak{X}}) \to \\
&\cdots
\end{aligned}
$$

On the other hand, applying the argument in the proof of Theorem 2.7 for the case n=1 (the length of cubic hyperresolution), we derive from (2.23) that

$$
\begin{array}{ccc}
\mathbb{R}H_*\pi^{\prime\cdot}_{|\mathfrak{Y}'}\mathcal{O}_M & \longleftarrow & \mathbb{R}F_*\pi^{\prime\cdot}\mathcal{O}_M \\
\uparrow & & \uparrow \\
\mathbb{R}I_*\pi^{\cdot}_{|\mathfrak{Y}}\mathcal{O}_M & \longleftarrow & \pi^{\cdot}\mathcal{O}_M
\end{array}
$$

is acyclic in $D^+(\mathfrak{X}_0, \mathbb{C})$ for any relatively compact open subset $\mathfrak{X}_0$ of $\mathfrak{X}$. Therefore we have the following long exact sequence of cohomology

$$\to H^i(\mathfrak{X}_0, \pi^{\cdot}\mathcal{O}_M) \to H^i(\mathfrak{X}_0, \mathbb{R}I_*\pi^{\cdot}_{|\mathfrak{Y}}\mathcal{O}_M) \oplus H^i(\mathfrak{X}_0, \mathbb{R}F_*\pi'^{\cdot}\mathcal{O}_M)$$

(2.25)

$$\to H^i(\mathfrak{X}_0, \mathbb{R}H_*\pi'^{\cdot}_{|\mathfrak{Y}'}\mathcal{O}_M) \to H^{i+1}(\mathfrak{X}_0, \pi^{\cdot}\mathcal{O}_M) \to \cdots$$

There are naturally homomorphism from (2.25) to (2.24). Among these homomorphisms,

$$H^i(\mathfrak{X}_0, \mathbb{R}I_*\pi^{\cdot}_{|\mathfrak{Y}}\mathcal{O}_M) \to H^i(\mathfrak{X}_0, \mathbb{R}I_*\Omega^{\cdot}_{\mathfrak{Z}/M|\mathfrak{Y}}),$$

$$H^i(\mathfrak{X}_0, \mathbb{R}H_*\pi'^{\cdot}_{|\mathfrak{Y}'}\mathcal{O}_M) \to H^i(\mathfrak{X}_0, \mathbb{R}H_*\Omega^{\cdot}_{\mathfrak{X}'/M|\mathfrak{Y}'})$$

are isomprphisms on $\mathfrak{X}_0$ by the induction hypothesis, and

$$H^i(\mathfrak{X}_0, \mathbb{R}F_*\pi'^{\cdot}) \to H^i(\mathfrak{X}_0, \mathbb{R}F_*\Omega^{\cdot}_{\mathfrak{X}'/M})$$

is also, because $\pi' : \mathfrak{X}' \to M$ is a smooth family ([2, p.15, 2.23.2]). Hence we conclude that

$$H^i(\mathfrak{X}_0, \pi^{\cdot}\mathcal{O}_M) \to H^i(\mathfrak{X}_0, \Omega^{\cdot}_{\mathfrak{Z}/M|\mathfrak{X}})$$

is an isomorphism on $\mathfrak{X}_0$, which means $\pi^{\cdot}\mathcal{O}_M \to \Omega^{\cdot}_{\mathfrak{Z}/M|\mathfrak{X}}$ is a quasi-isomorphism on $\mathfrak{X}_0$ as required. This completes the proof of Theorem 2.10.

Corollary 2.11 follows from Theorem 2.9 and Theorem 2.10.

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