SPINORS IN THE MINKOWSKI SPACE

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March 23, 1995

ABSTRACT. We propose a realization of spinors in the Minkowski space as exterior forms of a special type. It is well known that the symbol of the operator $d + \delta$ determines a Clifford module structure in the space of exterior forms. This module splits into a sum of two simple ones called a *spinor space* and a *dual spinor space*. The decomposition is determined by two projectors. We suggest a description of these projectors in terms of hyperbolic planes. This description produces two algebraic structures into the spinor space: (1) the canonically determined complex structure, and (2) a noncanonically determined quaternion structure. Tyhe choise of the latter structure depends on a basis in the elliptic plane orthogonal to the hyperbolic one. In this case the Clifford algebra is realized as 2×2 quaternion matrices algebra.

1. INTRODUCTION

This article arises from the attempt to understand a sense of Dirac equation solutions. From the formal point of view Dirac equation solutions are spinor fields. But such an approach gives no indications on a measurement of a such field. In this article we give realisation of spinors in the Minkowski space as a special type of exterior forms. One obtains two corollaries from this description. First, spinor fields can be measured like electromagnetic fields. And, second, a spinor structure is given by an measurable object which is an exterior decomposable 2-form of the unit length.

Recall that when introducing spinors the main Dirac's idea was to compute a square root of the wave operator [[1][2]]. In differential geometry this problem resolves by means of the operator $d + \delta$. But $d + \delta$ is not the Dirac operator, because it is reducible. Namely, the algebra of differential forms can be represented as a sum of two submodules that invariant with respect to differential operator $d + \delta$ with possible addition of a zero order terms.

Considering symbols of these operators one obtains the following problem in linear algebra which is solved in this article. The symbol of operator $d + \delta$ determines

1991 *Mathematics Subject Classification.* 35A30,58C27,58C28,58G17.

Key words and phrases. Minkowski space, Clifford Algebra, Spinors, hyperbolic plane.

a Clifford module structure in the exterior forms space. This module splits into a sum of two simple ones called a *spinor space* and a *dual spinor space.* We show that projectors corresponding to this decomposition are determined by hyperbolic planes. Moreover, the spinor space is equipped with a canonical complex structure and a noncanonically determined quaternion structure. The choice of the latter one depends on a basis in the elliptic plane orthogonal to the hyperbolic one. The Clifford algebra is realized as the 2×2 quaternion matrices algebra.

In conclusion note that the Dirac operator naturally arises as an operator satisfying the De Broglie principle. It means that singularities of solutions of corresponding equations move as material points. This remark together with the observation that the Schrödinger operator is the transport operator for special type singular solutions of the Dirac equation as well as the fact that spinors can be realized as special differential forms allow us to indicate geometrical sense of Ψ−function as a section of a quaternion fibre bundle.[1]

2. An Exterior Algebra

In this section we give a brief review of structures into the exterior algebra over vector space equipped with (pseudo)metric.

Let E be a vector space over $\mathbb{R}$, $\dim E = n$, and $E^* = \mathrm{Hom}(E, \mathbb{R})$ be the dual space.

Denote by $\Lambda^i(E^*)$ the space of $i-$ dimensional skew symmetric forms on the space E, and by

$$\Lambda^\bullet(E^*) = \bigoplus_{i=1}^{n} \Lambda^i(E^*)$$

the graded exterior algebra of the space E^*.

This algebra is a direct sum of the subalgebra

$$\Lambda^{\mathrm{ev}}(E^*) \subset \Lambda^\bullet(E^*)$$

where

$$\Lambda^{\mathrm{ev}}(E^*) = \bigoplus_{k \equiv 0 \bmod 2} \Lambda^k(E^*),$$

and the $\Lambda^{\mathrm{ev}}(E^*)$−module

$$\Lambda^{\mathrm{od}}(E^*) \subset \Lambda^\bullet(E^*)$$

[1]The research described in this publications was made possible in part by Grant N2F00 from the International Science Foundation

where

$$\Lambda^{\text{od}}(E^*) = \bigoplus_{k\equiv 1 \bmod 2} \Lambda^k(E^*).$$

There are two natural operators in the algebra $\Lambda^\bullet(E^*)$:
1°. operator of *exterior multiplication:*

$$\begin{aligned} e_\theta : \ \Lambda^i(E^*) &\longrightarrow \Lambda^{i+1}(E^*), \\ \omega &\longmapsto \theta \wedge \omega \end{aligned}$$

by a covector $\theta \in E^*$, and
2°. operator of *interior multiplication:*

$$\begin{aligned} i_X : \ \Lambda^i(E^*) &\longrightarrow \Lambda^{i-1}(E^*), \\ \omega &\longmapsto i_X\omega \end{aligned}$$

by a vector $X \in E$.
We assumed here that

$$(i_X\omega)(X_1, \cdots, X_{i-1}) = \omega(X, X_1, \cdots, X_{i-1})$$

for all vectors $X_1, \cdots, X_{i-1} \in E$.
Operators e and i satisfy the following well known relations:

$$\begin{aligned} e_{\theta_1} \circ e_{\theta_2} + e_{\theta_2} \circ e_{\theta_1} &= 0, \\ i_{X_1} \circ i_{X_2} + i_{X_2} \circ i_{X_1} &= 0, \\ i_X \circ e_\theta + e_\theta \circ i_X &= e_{i_X(\theta)}, \end{aligned}$$

for all $X \in E, \theta \in E^*$.
Therefore, we have the following representations of the graded algebra $\Lambda^\bullet(E^*)$
(1)

$$\begin{aligned} e : \ \Lambda^\bullet(E^*) &\longrightarrow \text{End}(\Lambda^\bullet(E^*)), \\ \theta &\longmapsto e_\theta \end{aligned}$$

where e_θ has the form

$$e_\theta = e_{\theta_1} \circ \cdots \circ e_{\theta_k},$$

on decomposable elements $\theta = \theta_1 \wedge \cdots \wedge \theta_k, \ \theta_j \in E^*, \ j = 1, \cdots, k$ and

(2)

$$\begin{array}{rcl} i: \ \Lambda^\bullet(E^*) & \longrightarrow & \mathrm{End}(\Lambda^\bullet(E^*), \\ X & \longmapsto & i_X \end{array}$$

where the operator i_X has the form

$$i_X = i_{X_1} \circ \cdots \circ i_{X_k},$$

on decomposable elements $X = X_1 \wedge \cdots \wedge X_k, \ \ X_j \in E, j = 1, \cdots, k$.

Denote by $\langle , \rangle$ the natural pairing between spaces $\Lambda^k(E)$ and $\Lambda^k(E^*)$:

$$\begin{array}{rcl} \langle , \rangle : \ \Lambda^k(E) \times \Lambda^k(E^*) & \longrightarrow & \mathbb{R}, \\ (a, b^*) & \longmapsto & i_a b^*. \end{array}$$

Let g be (pseudo)metric of the signature $(n - l, l)$ on the space E.

In a usual way we will identify the metric g with the symmetric operator:

$$A_g : E \longrightarrow E^*$$

where $\langle A_g(X), Y\rangle = g\,(X, Y)$, for all $X, Y \in E$.

Presenting the metric g (or the operator A_g) allows us to determine in a natural way the metric g^* on the dual space E^*:

$$A_g^* = A_g^{-1} : E^* \longrightarrow E,$$

as well as metrics g_k and g_k^* on the spaces $\Lambda^k(E)$ and $\Lambda^k(E^*)$ respectively.

Namely,

$$A_{g_k} = \Lambda^k(A_g), A_{g_k^*} = \Lambda^k\left(A_g^{-1}\right).$$

In the sequel, to simplify our notations, we will omit indices k and $*$ and denote metrics g_k and g_k^* by g.

Denote by $\hat{\theta} \in \Lambda^\bullet(E)$ and $\hat{X} \in \Lambda^\bullet(E^*)$ elements $A_g(\theta)$ and $A_g^{-1}(X)$, for elements $\theta \in \Lambda^\bullet(E^*)$ and $X \in \Lambda^\bullet(E)$ respectively.

In these notations, one obtains the following relations:

$$g(\theta_1, \theta_2) = \langle A_g(\theta_1), \theta_2\rangle = i_{\hat{\theta}_1}\theta_2.$$

Hence, the following relations between operators e, i and the metric g :

$$i_{\hat{\theta}_1} \circ e_{\theta_2} + e_{\theta_2} \circ i_{\hat{\theta}_1} = g(\theta_1, \theta_2)\,\mathbf{1} \quad (1.1)$$

holds, for any $\theta_1, \theta_2 \in E^*$.

Note also that the following equality is valid

$$\operatorname{sign}(g(\theta,\theta)) = (-1)^{\nu},$$

for any nonvanishing form $\theta \in \Lambda^n(E^*)$. Here $\nu = \left[\frac{n}{2}\right] + l$.

3. Clifford Modules

Assume that the space E is oriented and choose a volume form $\Omega_g \in \Lambda^n(E^*)$ such that

$$g(\Omega_g, \Omega_g) = (-1)^{\nu}.$$

The volume form Ω_g determines the Hodge operator:

$$\begin{array}{cccc} \# : & \Lambda^k(E^*) & \longrightarrow & \Lambda^{n-k}(E^*), \\ & \theta & \longmapsto & i_{\hat{\theta}}\Omega_g, \\ & k = 1,\cdots,n. & k = 1,\cdots,n. & k = 1,\cdots,n. \end{array}$$

In other words one has the following formulae

$$\#(\theta_1 \wedge \cdots \theta_k) = i_{\hat{\theta}_1} \circ \cdots \circ i_{\hat{\theta}_k}(\Omega_g), \quad (2.1)$$

for the action of the operator $\#$ on decomposable forms.
Here $\theta_1, \cdots, \theta_k \in E^*$.

Proposition 3.1. *Let g be a metric of the signature $(n-l, l)$ on an oriented vector space E, and $\Omega_g \in \Lambda^n(E^*)$ be a volume form such, that $g(\Omega_g, \Omega_g) = (-1)^{\nu}, \nu = \left[\frac{n}{2}\right] + l$. Then the Hodge operator $\#$ satisfies following conditions:*

(1) $\#^2 = (-1)^{\nu}\mathbf{1}$,
(2) $\#\Omega_g = (-1)^{\nu}, \#1 = \Omega_g$,
(3) $\# \circ e_\theta = i_{\hat{\theta}} \circ \#$,
for all $\theta \in \Lambda^{\bullet}(E^*)$.
(4) $g(\theta_1, \theta_2) = \#^{-1} \circ e_{\theta_1} \circ \#(\theta_2)$,
for all $\theta_1, \theta_2 \in \Lambda^k(E^*)$.

From now on we assume that $\nu \equiv 1 \bmod 2$.
In this case the Hodge operator $\#$ determines the complex structure on the space $\Lambda^{\bullet}(E^*)$.
Consider morphisms:

$$\begin{array}{cccc} \sigma^+ : & E^* & \longrightarrow & \operatorname{End}(\Lambda^{\bullet}(E^*)), \\ & \theta & \longmapsto & e_\theta + i_{\hat{\theta}} \end{array}$$

and

$$\begin{array}{rccl}\sigma^{-}: & E^{*} & \longrightarrow & \operatorname{End}\left(\Lambda^{\bullet}\left(E^{*}\right)\right), \\ & \theta & \longmapsto & e_{\theta}-i_{\hat{\theta}}.\end{array}$$

Let $\sigma^{+}(\theta)=\sigma_{\theta}^{+}, \sigma^{-}(\theta)=\sigma_{\theta}^{-}$.
Then one has

$$\sigma_{\theta_1}^{+} \circ \sigma_{\theta_2}^{+}+\sigma_{\theta_2}^{+} \circ \sigma_{\theta_1}^{+}=2 g\left(\theta_1, \theta_2\right) \mathbf{1}, \qquad (3.1)$$

and

$$\sigma_{\theta_1}^{-} \circ \sigma_{\theta_2}^{-}+\sigma_{\theta_2}^{-} \circ \sigma_{\theta_1}^{-}=-2 g\left(\theta_1, \theta_2\right) \mathbf{1}. \qquad (3.2)$$

for any covectors $\theta_1, \theta_2 \in E^{*}$.

Due to these relations the mapping σ^{+} defines a representation $C\left(\sigma^{+}\right)$ of the Clifford algebra $C(E, g)$ and the mapping σ^{-} determines the representation $C\left(\sigma^{-}\right)$ of the Clifford algebra $C(E,-g)$.

Proposition 3.2. *The mappings σ^{+} and σ^{-} anticommute. That is,*

$$\sigma_{\theta_1}^{+} \circ \sigma_{\theta_2}^{-}+\sigma_{\theta_2}^{-} \circ \sigma_{\theta_1}^{+}=0$$

for any covectors $\theta_1, \theta_2 \in E^{*}$.

Proposition 3.3. *For any covector $\theta \in E^{*}$ the mappings σ_{θ}^{+} and σ_{θ}^{-} are linear and antilinear ones with respect to the complex structure is given by the operator $\#$:*

$$\begin{aligned} \# \circ \sigma_{\theta}^{+} &= \sigma_{\theta}^{+} \circ \#, \\ \# \circ \sigma_{\theta}^{-} &= -\sigma_{\theta}^{-} \circ \#. \end{aligned}$$

4. Clifford Modules over Minkowski Space

From here on we assume that (E, g) is the 4-dimensional Minkowski space with a metric g of the signature $(1,3)$.

Recall that a spinor space is, by definition, an irreducible module over the Clifford algebra $C(E, g)$. Using representation $C\left(\sigma^{+}\right)$, one can realize this module as a submodule of the module $\Lambda^{\bullet}\left(E^{*}\right)$.

We will look for this submodule in a form $P \Lambda^{\bullet}\left(E^{*}\right)$ where $P \in C^{0}\left(\sigma^{-}\right)$ is a projector. Here we denoted by $C^{0}\left(\sigma^{-}\right)$ the image of the even subalgebra of the Clifford algebra $C(E,-g)$ under the mapping σ^{-}.

These projectors can be described in terms of hyperbolic planes.

Definition 4.1. *An oriented 2-dimensional plane $E_h \subset E$ is called* hyperbolic, *if the restriction of the metric g on this plane has the signature $(1,1)$.*

Any hyperbolic plane E_h is uniquely determined by two isotropic directions that are intersection of the plane E_h with the cone of isotropic vectors.

Let v_1, v_2 be isotropic vectors forming a basis in E_h.

Choose vectors v_1, v_2 in such a way, that

$$g(v_1, v_2) = -1.$$

Note that the choice of such vectors is determined up to a scale factor $v_1 \longmapsto t \cdot v_1,\ v_2 \longmapsto t^{-1} \times v_2$ where $t \in \mathbb{R} \setminus 0$.

By virtue of this remark the exterior 2-form $\omega = \hat{v}_1 \wedge \hat{v}_2$ is determined by the hyperbolic plane E_h uniquely.

Therefore, one obtains the following statement.

Proposition 4.1. *There exists a one-to-one correspondence between hyperbolic planes and exterior 2-forms $\omega \in \Lambda^2(E^*)$ such that*

(1) $\omega \wedge \omega = 0$, *and*
(2) $\omega \wedge \#\omega = \Omega_g$.

Proposition 4.2. *Any projector $\mathrm{P} \in C^0(\sigma^-)$ is a uniquely determined by some hyperbolic plane $E_h \subset E$ and, conversely, any hyperbolic plane E_h, determined by an exterior 2-form $\omega \in \Lambda^2(E^*)$, corresponds to the projector*

$$\mathrm{P}_\omega = \frac{1 + S_\omega^-}{2}$$

where $\omega = \theta_1 \wedge \theta_2, \theta_1 = \hat{v}_1, \theta_2 = \hat{v}_2$, and

$$S_\omega^- = \frac{1}{2} \times \left[\sigma_{\theta_1}^- \circ \sigma_{\theta_2}^- - \sigma_{\theta_2}^- \circ \sigma_{\theta_1}^-\right].$$

Remark 4.1. *It is easy to check that*

$$P_\omega = \frac{1}{2}\sigma_{\theta_1} \circ \sigma_{\theta_2}.$$

To prove proposition 4.2 we use the following statements.

Lemma 4.3. *Operators $\sigma_\alpha^- \circ \sigma_\beta^-$ corresponding to pairs of orthonormal covectors $\alpha, \beta \in E^*$, are traceless:*

$$tr_{\mathbb{C}}\left(\sigma_\alpha^- \circ \sigma_\beta^-\right) = 0.$$

as linear operators in the complex structure defined by the operator $\#$.

Proof. The spectrum of operator σ_β^- coincides with ± 1 or $\pm i$ depending on covector β. It is enough to note that the operator σ_α^- transposes proper subspases $E(\lambda)$ of the operator σ_β^- :

$$\sigma_\alpha^- : E(\lambda) \longrightarrow E(-\lambda).$$

Lemma 4.4. *Introduce the operator*

$$a_2 = \sum_{i<j} a_{ij}\, \sigma_i^- \circ \sigma_j^-,$$

where $\sigma_i^- = \sigma_{\theta_i}^-$ *for some orthonormal basis* $\theta_1, \cdots, \theta_4$ *of the space* E^*. *Then*

$$a_2^2 = \mathrm{Pf}\,[\nu]\,\gamma + g(\nu,\nu), \quad (4.1)$$

where $\gamma = \sigma_1^- \circ \sigma_2^- \circ \sigma_3^- \circ \sigma_4^-, \nu = \sum_{i<j} a_{ij}\theta_i \wedge \theta_j$ *and* Pf $[\nu]$ *is the* Pfaffian *of 2-form* ν.

Proof. (Proposition 4.2) Represent the projector P in the form

$$P = a_0 + a_2 + b\gamma,$$

where $a_0, b \in \mathbf{R}$.
Then, by using lemma 4.1, one obtains

$$tr_{\mathbf{C}}\, P = 8\,(a_0 + ib).$$

Since P is a projector, $tr_{\mathbf{C}} P$ is a natural number. Hence $b = 0$. Using lemma 4.2 and the equality $P^2 = P$,one gets

$$a_0^2 + 2a_0a_2 + g(\nu,\nu) + \mathrm{Pf}(\nu)\,\gamma = a_0 + a_2.$$

Therefore,

$$\mathrm{Pf}(\nu) = 0, a_0 = \frac{1}{2}, g(\nu,\nu) = \frac{1}{4}.$$

Introducing an exterior 2-form $\omega = 2\nu$, we obtain the result.

Remark 4.2. *A change of the plane* E_h *orientation leads to the change* $\omega \longrightarrow \bar{\omega} = -\omega$. *In addition, the projector* $\mathrm{P}_{\bar{\omega}}$ *is a complementary to* P_ω :

$$\mathrm{P}_\omega + \mathrm{P}_{\bar{\omega}} = 1, \ \textit{and} \ \ \mathrm{P}_\omega \cdot \ \mathrm{P}_{\bar{\omega}} = 0.$$

Hyperbolic plane $E_h \subset E$ determines the following a direct sum decomposition of the space E

$$E = E_h \oplus E_e$$

where 2-dimensional (*elliptic*) plane E_e is the orthogonal complement of the plane E_h in E.

This decomposition leads to a decomposition of the exterior algebra

$$\Lambda^\bullet(E^*) = \sum_i \Lambda^i(E_h^*) \otimes \Lambda^{\bullet - i}(E_e^*).$$

In other words, any exterior form $\alpha \in \Lambda^k(E^*)$ can be represented in the form:

$$\alpha = \alpha_0 + \theta_1 \wedge \alpha_1 + \theta_2 \wedge \alpha_2 + \theta_1 \wedge \theta_2 \wedge \alpha_3$$

where

$$\alpha_0 \in \Lambda^k(E_e^*),\ \alpha_1, \alpha_2 \in \Lambda^{k-1}(E_e^*),\ \alpha_3 \in \Lambda^{k-2}(E_e^*).$$

We identified here forms $\beta \in \Lambda^\bullet(E_e^*)$ with exterior forms on E that satisfied to the following conditions:

$$i_{\hat{\theta}_j}\beta = 0,$$

for all $j = 1, 2$.

Proposition 4.5.

$$\text{Im P}_\omega = (1 + e_\omega)\Lambda^\bullet(E_e^*) + e_{\theta_1}\Lambda^\bullet(E_e^*)$$

In other words, any element $x \in \text{Im } P_\omega$ can be represented as a pair (x_0, x_1) where $x_0, x_1 \in \Lambda^\bullet(E_e^*)$, and

$$x = (1 + e_\omega)x_0 + e_{\theta_1}x_1.$$

Proposition 4.6. *The operator $\sigma_{\theta_2}^-$ determines an isomorphism*

$$\sigma_{\theta_2}^- : \text{Im } P_\omega \longrightarrow \text{Im } P_{\bar{\omega}}.$$

Proof. By straightforward computations one gets

$$\sigma_{\theta_2}^{-}\left[(1+e_\omega)x_0 + e_{\theta_1}x_1\right] = 2e_{\theta_2}x_0 + (1+e_\omega)x_1.$$

Proposition 4.7.

$$\text{Ker } \mathrm{P}_\omega = \operatorname{Ker}\sigma_{\theta_2}^{-}.$$

Proof. The statement of the proposition follows from the equality:

$$\sigma_{\theta_2}^{-}\, P_\omega = \sigma_{\theta_2}.$$

Corollary 4.8.

$$\text{Im } \mathrm{P}_\omega \simeq \operatorname{Coker}\sigma_{\theta_2}^{-}$$

Proposition 4.9. *Isotropic direction θ_2 determines a pair of projectors $P_\omega, P_{\bar\omega}$ such that $P_\omega + P_{\bar\omega} = 1$. That is, the direction determines a non oriented hyperbolic plane $E_h \subset E$.*

5. Spinors

The restriction of the metric g onto the elliptic plane E_e is a negative definite metric g_e.

Therefore, operators σ_θ^+ where $\theta \in E_e^*$ induce an action of the Clifford algebra $C(E_e, g_e)$ (isomorphic to the division ring of quaternions) on the exterior algebra $\Lambda^\bullet(E_e^*)$.

In addition, the Hodge operator $\#$ induces the operator

$$\#_e : \Lambda^i(E_e^*) \longrightarrow \Lambda^{2-i}(E_e^*)$$

for all $i = 1, 2$, and

$$\#_e^2 = -\mathbf{1}.$$

Hence, the space $\Lambda^\bullet(E_e^*)$ is a quaternion and complex space simultaneously.

Note also, that the operator $\#$ admits the restriction onto the subspace Im P_ω and acts on in the following way:

$$\# : (x_0, x_1) \longrightarrow (\#_e x_0, -\#_e x_1).$$

Let's show that the module Im P_ω is a simple.

Consider a cyclic vector (x, y) in some irreducible σ^+- module. Then one gets all covectors $(ax+by, cx+dy)$, for any real numbers a, b, c, d by acting σ_θ^+,for all $\theta \in E_h^*$, and considering all linear combinations.

Consider now the subspace spanned in $\Lambda^\bullet(E_e^*)$ by covectors x and y. Acting by elements $\sigma_\theta^+, \theta \in E_e^*$, we obtain the whole space $\Lambda^\bullet(E_e^*)$. Thus the module under consideration coincides with the whole image Im P_ω.

Denote this module by S_ω. We call elements of the module S_ω *spinors* with the orientation ω, while elements of the module $S_{\bar\omega}$ are called *dual spinors* with the orientation ω.

Note that there exists an isomorphism between spinors S_ω and dual spinors $S_{\bar\omega}$ with the orientation ω (Proposition 5.2) and the exterior algebra $\Lambda^\bullet(E^*)$ decomposes now into the direct sum

$$\Lambda^\bullet(E^*) = S_\omega \oplus S_{\bar\omega}.$$

Note also, that the module S_ω, in addition to the complex $\#$, carries a module structure over the Clifford algebra $C(E_e, g_e)$. The latter algebra is isomorphic (but noncanonically) to the quaternion algebra.

Moreover, the complex structure $\#$ is induced by the Clifford multiplication on the element $\#\omega$.

In other words, we can consider the module S_ω as a module over algebra $C(E_e, g_e)$, while the Clifford algebra $C(E, g)$ under the representation σ^+ is mapped onto the 2×2 matrices algebra over algebra $C(E_e, g_e)$.

Namely,
$\sigma_{\theta_1}^+ : (x, y) \longmapsto (0, x)$,
$\sigma_{\theta_2}^+ : (x, y) \longmapsto (-2y, 0)$,
$\sigma_{\theta_e}^+ : (x, y) \longmapsto (\sigma_{\theta_e}^+ x, -\sigma_{\theta_e}^+ y), \theta_e \in E_e^*$.

Therefore, the following theorem is proved.

Theorem 5.1. *Let (E, g) be the 4-dimensional Minkowski space with a metric g of signature $(1, 3)$.*

Then the following statements hold.

$1°$. *The spinor structure on the space E is determined by a hyperbolic plane $E_h \subset E$, or by the exterior 2-form $\omega \in \Lambda^2(E^*)$ such that*

a. $\omega \wedge \omega = 0$,

b. $g(\omega, \omega) = 1$.

The hyperbolic plane E_h uniquely determined by two isotropic directions v_1, v_2, such that $g(v_1, v_2) = -1$.

The form ω (one of two possible orientations of the plane E_h) coincides up to sign with the form $\theta_1 \wedge \theta_2, \theta_1 = \bar v_1, \theta_2 = \bar v_2$. The pair of forms $(\omega, -\omega)$ corresponds to a non oriented plane determined by an isotropic direction.

2°. *The spinor space S_ω corresponding to the hyperbolic plane $E_h \subset E$ with the orientation ω is isomorphic to the space* $\operatorname{Coker} \sigma^-_{\theta_2}$ *or the space*

$$(1 + e_\omega)\Lambda^\bullet(E_e^*) + e_{\theta_1}\Lambda^\bullet(E_e^*)$$

where E_e is the orthogonal complement to the plane E_h.

3°. *There exists an isomorphism between the spinor space S_ω and the dual spinor space $S_{\bar\omega}, \bar\omega = -\omega$. This isomorphism is given by the mapping $\sigma^-_{\theta_2} : S_\omega \longrightarrow S_{\bar\omega}$.*

4°. *The exterior algebra $\Lambda^\bullet(E^*)$ splits into the direct sum:*

$$\Lambda^\bullet(E^*) = S_\omega \oplus S_{\bar\omega}.$$

5°. *The spinor space S_ω is a module over the Clifford algebra $C(E_e, g_e)$ where g_e is the restriction of the metric g onto the elliptic plane E_e. This algebra possesses the canonically determined complex structure # and is noncanonically isomorphic to the division ring of quaternions.*

6°. *The representation σ^+ of the Clifford algebra $C(E, g)$ in the spinor space S_ω is irreducible and defines the isomorphism of the algebra $C(E, g)$ with the 2×2 matrices algebra over the division ring of quaternions.*

6. Miscellany

In conclusion we describe the kernel of the operator

$$\sigma_\theta^+ : S_\omega \longrightarrow S_\omega.$$

It is easily seen that the nontriviality kernel condition implies isotropy of the covector θ. Isotropy of the covector θ implies in turn an isomorphism

$$\sigma_\theta^+ : \operatorname{Coker} \sigma_\theta^+ \longrightarrow \operatorname{Im} \sigma_\theta^+ \simeq \operatorname{Ker} \sigma_\theta^+.$$

Using the decomposition $E = E_h \oplus E_e$, it is possible to represent any covector θ in the form

$$\theta = \theta_e + \lambda_1\theta_1 + \lambda_2\theta_2,$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, and $\theta_e \in E_e^*$.

In terms of this decomposition the isotropy condition takes the form:

$$2\lambda_1\lambda_2 - g(\theta_e, \theta_e) = 0$$

while the action of the operator σ_θ^+ on an element $(x, y) \in S_\omega$ is given by

$$\sigma_\theta^+ : (x, y) \longmapsto (-2\lambda_2 y + \sigma_{\theta_e}^+ x, \lambda_1 x - \sigma_{\theta_e}^+ y). \quad (7.1)$$

We set:

$$\lambda_\theta = \mid g(\theta_e, \theta_e) \mid^{\frac{1}{2}},$$
$$\#_\theta = \mid g(\theta_e, \theta_e) \mid^{-\frac{1}{2}} \bullet \sigma_{\theta_e}^+,$$

if $\theta_e \neq 0$.

Remark that the operator $\#_\theta$ defines a complex structure in the module $\Lambda^\bullet(E_e^*)$. In this situation we define the transformation

$$\Xi^\theta : E_e^* \longrightarrow \mathrm{End}\,(\Lambda^\bullet(E_e^*))$$

as a composition of two transformations:

1) reflection in the plane E_e^* with respect to the line orthogonal to the covector θ_e, and

2) the transformation σ^+.

From the definition of the transformation Ξ^θ it follows that the following relations is valid

$$\sigma_\lambda^+ \circ \Xi_\mu^\theta = \sigma_\mu^+ \circ \sigma_\lambda^+,$$

for any elements $\mu, \lambda \in E_e^*$.

Therefore, transformation Ξ^θ determines a $C(E_e, g_e)-$ module structure in the space $\mathrm{Ker}\,\sigma_\theta^+$.

Keeping in mind the description of the space $\mathrm{Ker}\,\sigma_\theta^+$, denote by $S_{\langle\omega|1\rangle}$ the $C(E_e, g_e)-$ module $\mathrm{Ker}\,\sigma_{\theta_1}^+ \mid_{S_\omega}$. The module $S_{\langle\omega|1\rangle}$ is generated by spinors of the form $(0, y)$, where $y \in \Lambda^\bullet(E_e^*)$.

Denote also by $S_{\langle\omega|2\rangle}$ the $C(E_e, g_e)-$ module $\mathrm{Ker}\,\sigma_{\theta_2}^+ \mid_{S_\omega}$ generated by spinors of the type $(x, 0)$ where $x \in \Lambda^\bullet(E_e^*)$.

These remarks prove the following

Proposition 6.1. *Let θ be an isotropic covector in the space (E^*, g). Then,*

1) *there exist an isomorphism*

$$\mathrm{Coker}\,\sigma_\theta^+ \longrightarrow \mathrm{Im}\,\sigma_\theta^+ \simeq \mathrm{Ker}\,\sigma_\theta^+,$$

2) $\mathrm{Ker}\,\sigma_\theta^+$ *is a module over the Clifford algebra $C(E, -g)$ with respect to the representation $C(\sigma^-)$,*

3) *the kernel of the operator*

$$\sigma_\theta^+ : S_\theta \longrightarrow S_\theta$$

is nontrivial and have the following description:

a. if $\theta_e = 0$, *then the subspace* $\operatorname{Ker}\sigma_\theta^+$ *coincides with one of* $C(E_e, g_e)-$ *modules* $S_{\langle\omega|i\rangle}, i = 1, 2,$

b. if $\theta_e \neq 0$, *then the subspace* $\operatorname{Ker}\sigma_\theta^+$ *is generated by spinors of the type*

$$(\lambda_1 \#_e x, \lambda_2 x),$$

where $x \in \Lambda^\bullet(E_e^*), \lambda_1, \lambda_2 \in \mathbb{R}$.

This space has an additional module structure over the Clifford algebra $C(E_e, g_e)$ *with respect to the representation* Ξ^θ.

Proof. Prove the statement 1°. Note that for any isotropic covector θ the embedding $\operatorname{Im}\sigma_\theta^+ \subset \operatorname{Ker}\sigma_\theta^+$ is obvious. To prove the embedding $\operatorname{Ker}\sigma_\theta^+ \subset \operatorname{Im}\sigma_\theta^+$, we consider a covector $\theta_0 \subset E^*$ such that $g(\theta_0, \theta_0) = -\frac{1}{2}$. If $\varsigma \in \operatorname{Ker}\sigma_\theta^+$, then using the formula (3.1), one gets

$$0 = \sigma_{\theta_0}^+ \circ \sigma_\theta^+ \varsigma = \left[\sigma_{\theta_0}^+, \sigma_\theta^+\right] \varsigma = -\varsigma + \sigma_\theta^+ \circ \sigma_{\theta_0}^+ \varsigma.$$

Therefore, $\varsigma \in \operatorname{Im}\sigma_\theta^+$.

Statement 2° follows from the anticommutativity of reprentations σ^+ and σ^- (proposition 3.1).

Statement 3° follows from description (7.1) of the operator σ_θ^+ action, $\theta \in E^*$ and from the definition of the representation Ξ^θ.

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