Two classes of stochastic Dirichlet equations which admit explicit solution formulas

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Abstract

In this paper we look at stochastic Dirichlet equations of the type

$$\mathcal{A}u = (\sum_{i=1}^{m} c_i \cdot \mathrm{Exp}\{W_{\phi_x}^{(i)}\}) \diamond u - g$$
$$u|_{\partial D} = f$$

and

$$\mathrm{div}(\mathrm{Exp}\{W_{\phi_x}\} \diamond u) = \kappa \diamond u - g$$
$$u|_{\partial D} = f$$

where $\mathcal{A}$ is a uniformly elliptic second order differential operator and $\mathrm{Exp}\{W_{\phi_x}\}$,$\kappa$,$f$ and g are elements in the space $(\mathcal{S})^{-1}$ of generalized white noise distributions. With suitable conditions on κ,f and g both classes of stochastic Dirichlet equations admit unique solution formulas in the space $(\mathcal{S})^{-1}$. These are used to give explicit solution formulas to the Scrödinger and wave equation when the boundary conditions are particularly simple.

Keywords : Generalized white noise distributions, Wick product, Hermite transform.

§1 Introduction

It is well known that given a solution $u \in C(\bar{D}) \cap C^2(D)$ of the Dirichlet problem

$$\begin{aligned} \mathcal{A}u &= \kappa \cdot u - g \\ u|_{\partial D} &= f \end{aligned} \tag{1}$$

where D is an open, bounded domain, κ, g, f suitable functions and $\mathcal{A}$ a uniformly elliptic second order differential operator, then u has a stochastic representation given by

$$u(x) = E^x[f(X_{\tau_D}) \exp\{-\int_0^{\tau_D} \kappa(X_s)\, ds\} + \int_0^{\tau_D} g(X_t) \exp\{-\int_0^t \kappa(X_s)\, ds\}\, dt] \tag{2}$$

where X_s is a certain stochastic process associated with $\mathcal{A}$.
If we would like to use the Dirichlet equation for physical modeling, then it would be natural to replace $\mathcal{A}$ and/or κ by stochastic functionals. In the white noise setting, replacing κ with $\mathrm{Exp}\{W_{\phi_x}\}$ would seem to be an interesting choice, but where should we be looking for solutions? Although $\mathrm{Exp}\{W_{\phi_x}\}$ is in $\mathcal{L}^2(\mu)$, it seems clear that this would not be the case for the solution candidate (2). The next logical step would be the space $(\mathcal{S})^*$ of Hida distributions, but since, given arbitrary real constants K_1 and K_2, there always exists $x \in \mathbb{R}$ such that

$$\begin{aligned}|\mathcal{S}(\mathrm{Exp}\{-\mathrm{Exp}\{W_\phi\}\})((x + i\frac{\pi}{\|\phi\|^2})\phi)| &= |\exp\{-\exp\{x\|\phi\|^2 + i\pi\}\}| \\ &= \exp\{\exp\{x\|\phi\|^2\}\} \\ &> K_1 \exp\{K_2|x + i\frac{\pi}{\|\phi\|^2}|^2\|\phi\|^2\}\end{aligned}$$

it is clear from the characterization theorem in [HKPS] that $(\mathcal{S})^*$ is probably not the right space to look for solutions either. Fortunately, recently there have been constructed new spaces of generalized white noise distributions which will be adequate for our needs. These spaces will be described in the next section.

The methods used to solve (1) are generalizations of those used by Holden et al. in [HLØUZ] and [HLØUZ3]. In particular, theorem 3.1 may be seen as a generalization of theorem 10.2 in [HLØUZ] and theorem 4.1 generalizes theorem 3.1 in [HLØUZ3] ($\kappa \equiv 0$).

§2 Preliminaries on multidimensional white noise

There are many problems of physical nature where the need for several independent white noise sources arises. For example, given m independent positive white noise sources in a domain D, one wants to calculate the effect of these on a particle traveling in D. The result should intuitively be given by

$$\sum_{i=1}^{m} \mathrm{Exp}\{W_\phi^{(i)}\}$$

where $\{\mathrm{Exp}\{W_\phi^{(i)}\}\}_{i=1}^m$ are one dimensional independent positive white noise sources.

We will now give a short introduction of definitions and results from multidimensional Wick calculus, taken mostly from [Gj], [HLØUZ3], [HKPS] and [KLS].

In the following we will fix the parameter dimension n and space dimension m.

Let

$$\mathcal{N} := \prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n)$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^n$, and

$$\mathcal{N}^* := (\prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n))^* \approx \prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n)$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

Let $\mathcal{B} := \mathcal{B}(\mathcal{N}^*)$ denote the Borel σ-algebra on $\mathcal{N}^*$ equipped with the weak star topology and set

$$\mathcal{H} := \bigoplus_{i=1}^{m} \mathcal{L}^2(\mathbb{R}^n)$$

where $\oplus$ denotes orthogonal sum.

Since $\mathcal{N}$ is a countably Hilbert nuclear space (cf. eg.[Gj]) we get, using Minlos' theorem, a unique probability measure ν on $(\mathcal{N}^*, \mathcal{B})$ such that

$$\int_{\mathcal{N}^*} e^{i\langle\omega,\phi\rangle}\, d\nu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{\mathcal{H}}} \quad \forall \phi \in \mathcal{N}$$

where $\|\phi\|^2_{\mathcal{H}} = \sum_{i=1}^m \|\phi_i\|^2_{\mathcal{L}^2(\mathbb{R}^n)}$.

Note that if $m = 1$ then ν is usually denoted by μ.

THEOREM 2.1 [Gj] We have the following

1. $\otimes_{i=1}^m \mathcal{B}(\mathcal{S}'(\mathbb{R}^n)) = \mathcal{B}(\prod_{i=1}^m \mathcal{S}'(\mathbb{R}^n))$
2. $\nu = \times_{i=1}^m \mu$

DEFINITION 2.2 [Gj] The triple

$$(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n), \mathcal{B}, \nu)$$

is called the (m-dimensional) (n-parameter) **white noise probability space.**

For $k = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$ let

$$h_k(x) := (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}})$$

be the Hermite polynomials and

$$\xi_k(x) := \pi^{-\frac{1}{4}}((k-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{k-1}(\sqrt{2}x) \ ; \ k \geq 1$$

the Hermite functions.

It is well known that the family $\{\tilde{e}_\alpha\} \subset \mathcal{S}(\mathbb{R}^n)$ of tensor products

$$\tilde{e}_\alpha := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_n}$$

forms an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^n)$.

Give the family of all multi-indecies $\zeta = (\zeta_1, \ldots, \zeta_n)$ a fixed ordering

$$(\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(k)}, \ldots) \text{ where } \zeta^{(k)} = (\zeta_1^{(k)}, \ldots, \zeta_n^{(k)})$$

and define $\tilde{e}_k := \tilde{e}_{\zeta^{(k)}}$.

Let $\{e_k\}_{k=1}^{\infty}$ be the orthonormal basis of $\mathcal{H}$ we get from the collection

$$\{(\overbrace{0, \ldots, 0}^{i-1}, \tilde{e}_j, \overbrace{0, \ldots, 0}^{m-i}) \in \mathcal{H} \quad 1 \le i \le m, 1 \le j < \infty\}$$

and let $\gamma : \mathbb{N} \to \mathbb{N}$ be a function such that

$$e_k = (0, \ldots, 0, \tilde{e}_{\zeta(\gamma(k))}, 0, \ldots, 0).$$

Finally , let $(\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(k)}, \ldots)$ with $\beta^{(k)} = (\beta_1^{(k)}, \ldots, \beta_n^{(k)})$ be a sequence such that $\beta^{(k)} = \zeta^{(\gamma(k))}$.

If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index of non-negative integers we put

$$H_\alpha(\omega) := \prod_{i=1}^{k} h_{\alpha_i}(\langle \omega, e_i \rangle).$$

From theorem 2.1 in [HLØUZ] we know that the collection

$$\{H_\alpha(\cdot); \alpha \in \mathbb{N}_0^k; k = 0, 1, \ldots\}$$

forms an orthogonal basis for $\mathcal{L}^2(\mathcal{N}^*, \mathcal{B}, \nu)$ with $\|H_\alpha\|_{\mathcal{L}^2(\nu)} = \alpha!$ where $\alpha! = \prod_{i=1}^{k} \alpha_i!$.

This implies that any $f \in \mathcal{L}^2(\nu)$ has the unique representation

$$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

where $c_\alpha \in \mathbb{R}$ for each multi-index α and

$$\|f\|_{\mathcal{L}^2(\nu)}^2 = \sum_\alpha \alpha! c_\alpha^2.$$

DEFINITION 2.3 [Gj] The m-dimensional **white noise map** is a map

$$W : \prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n) \times \prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R}^m$$

given by

$$W^{(i)}(\phi, \omega) := \omega_i(\phi_i) \quad 1 \le i \le m$$

PROPOSITION 2.4 [Gj] The m-dimensional white noise map W satisfies the following

1. $\{W^{(i)}(\phi,\cdot)\}_{i=1}^{m}$ is a family of independent normal random variables.
2. $W^{(i)}(\phi,\cdot) \in \mathcal{L}^2(\nu)$ for $1 \leq i \leq m$.

DEFINITION 2.5 [HLØUZ3] Let $0 \leq \rho \leq 1$.

- Let $(\mathcal{S}_n^m)^\rho$, the space of **generalized white noise test functions**, consist of all
$$f = \sum_\alpha H_\alpha \in \mathcal{L}^2(\nu)$$
such that
$$\|f\|_{\rho,k}^2 := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbf{N})^{\alpha k} < \infty \quad \forall k \in \mathbb{N}$$

- Let $(\mathcal{S}_n^m)^{-\rho}$, the space of **generalized white noise distributions**, consist of all formal expansions
$$F = \sum_\alpha b_\alpha H_\alpha$$
such that
$$\sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho} (2\mathbf{N})^{-\alpha q} < \infty \text{ for some } q \in \mathbb{N}$$

where
$$(2\mathbf{N})^\alpha := \prod_{i=1}^{k} (2^n \beta_1^{(i)} \cdots \beta_n^{(i)})^{\alpha_i} \text{ if } \alpha = (\alpha_1, \ldots, \alpha_k).$$

We know that $(\mathcal{S}_n^m)^{-\rho}$ is the dual of $(\mathcal{S}_n^m)^\rho$ (when the later space has the topology given by the seminorms $\|\cdot\|_{\rho,k}$) and if $F = \sum b_\alpha H_\alpha \in (\mathcal{S}_n^m)^{-\rho}$ and $f = \sum c_\alpha H_\alpha \in (\mathcal{S}_n^m)^\rho$ then
$$\langle F, f\rangle = \sum_\alpha b_\alpha c_\alpha \alpha!.$$

It is obvious that we have the inclusions
$$(\mathcal{S}_n^m)^1 \subset (\mathcal{S}_n^m)^\rho \subset (\mathcal{S}_n^m)^{-\rho} \subset (\mathcal{S}_n^m)^{-1} \quad \rho \in [0,1]$$
and in the remaining of this paper we will consider the larger space $(\mathcal{S}_n^m)^{-1}$.

DEFINITION 2.6 [HLØUZ3] The Wick product of two elements in $(\mathcal{S}_n^m)^{-1}$ given by
$$F = \sum_\alpha a_\alpha H_\alpha \ , \ G = \sum_\beta b_\beta H_\beta$$
is defined by
$$F \diamond G = \sum_\gamma c_\gamma H_\gamma$$
where
$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta$$

LEMMA 2.7 [HLØUZ3] We have the following

1. $F, G \in (\mathcal{S}_n^m)^{-1} \Rightarrow F \diamond G \in (\mathcal{S}_n^m)^{-1}$
2. $f, g \in (\mathcal{S}_n^m)^{1} \Rightarrow f \diamond g \in (\mathcal{S}_n^m)^{1}$

DEFINITION 2.8 [HLØUZ3] Let $F = \sum b_\alpha H_\alpha$ be given. Then the Hermite transform of F,denoted by $\mathcal{H}F$, is defined to be (whenever convergent)

$$\mathcal{H}F := \sum_\alpha b_\alpha z^\alpha$$

where $z = (z_1, z_2, \cdots)$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k}$ if $\alpha = (\alpha_1, \ldots, \alpha_k)$.

LEMMA 2.9 [HLØUZ3] If $F, G \in (\mathcal{S}_n^m)^{-1}$ then

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)$$

for all z such that $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exists.

LEMMA 2.10 [HLØUZ3] Suppose $g(z_1, z_2, \cdots)$ is a bounded analytic function on $\mathbf{B}_q(\delta)$ for some $\delta > 0, q < \infty$ where

$$\mathbf{B}_q(\delta) := \{\zeta = (\zeta_1, \zeta_2, \cdots) \in \mathbb{C}_0^{\mathbb{N}}; \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2\mathbf{N})^{\alpha q} < \delta^2\}.$$

Then there exists $X \in (\mathcal{S}_n^m)^{-1}$ such that $\mathcal{H}X = g$.

LEMMA 2.11 [HLØUZ3] Suppose $X \in (\mathcal{S}_n^m)^{-1}$ and that f is an analytic function in a neighborhood of $\mathcal{H}X(0)$ in $\mathbb{C}$. Then there exists $Y \in (\mathcal{S}_n^m)^{-1}$ such that $\mathcal{H}Y = f \circ \mathcal{H}X$.

THEOREM 2.12 [KLS] Let (T, Σ, τ) be a measure space and let $\Phi : T \to (\mathcal{S}_n^m)^{-1}$ be such that there exists $q < \infty, \delta > 0$ such that

1. $\mathcal{H}\Phi_t(z) : T \to \mathbb{C}$ is measurable for all $z \in \mathbf{B}_q(\delta)$
2. there exists $C \in \mathcal{L}^1(T, \tau)$ such that $|\mathcal{H}\Phi_t(z)| \leq C(t)$ for all $z \in \mathbf{B}_q(\delta)$ and for τ-almost all t.

Then $\int_T \Phi_t \, d\tau(t)$ exists as a Bochner integral in $(\mathcal{S}_n^m)^{-1}$. In particular, $\langle \int_T \Phi_t \, d\tau(t), \phi \rangle = \int_T \langle \Phi_t, \phi \rangle \, d\tau(t)$; $\phi \in (\mathcal{S}_n^m)^{1}$.

EXAMPLE 2.13 Define the x-shift of ϕ, denoted by ϕ_x, by $\phi_x(y) := \phi(y - x)$.Then

$$\text{Exp}\{W_{\phi_x}^{(i)}\} \in (\mathcal{S}_n^m)^{-1} \quad 1 \leq i \leq m, \forall x \in \mathbb{R}^n$$

which is an immediate consequence of proposition 2.4 and lemma 2.11.

§3 The first class of stochastic Dirichlet equations

We will in this section let X_t be the solution of the stochastic integral equation

$$X_t^x = x + \int_0^t b(X_\theta^x)\,d\theta + \int_0^t \sigma(X_\theta^x)\,db_\theta \tag{3}$$

under the assumptions that

- the coefficients $b_i(x), \sigma_{i,k}(x) : \mathbb{R}^n \to \mathbb{R}$ are continuous and satisfy $\|b(x)\|_{\mathbb{R}^n}^2 + \|\sigma(x)\|_{\mathbb{R}^n}^2 \leq K(1 + \|x\|_{\mathbb{R}^n}^2)$ for all $x \in \mathbb{R}^n$, where K is a positive constant.
- the equation (3) has a weak solution $(X_t^x, b_t), (\Omega, \mathcal{F}, \hat{P}^x), \{\mathcal{F}_s\}$ for all $x \in \mathbb{R}^n$ and this solution is unique in the sense of probability law.

We will use the notation that

- $\hat{E}^x$ is expectation w.r.t. $\hat{P}^x$.
- $\tau_D = \tau_D^{X_t} := \inf\{t > 0 : X_t \in D^c\}$ is the first exit time from D for X_t.

THEOREM 3.1 Let D be an open, bounded domain in $\mathbb{R}^n$, $f \in C_b(\partial D)$, $g \in C^\alpha(\bar{D})$, $\mathcal{A}$ the differential operator on $C^2(\mathbb{R}^n)$, associated with X_t, given by

$$\mathcal{A}h(x) := \frac{1}{2}\sum_{i=1}^n \sum_{k=1}^n (\sigma\sigma^T)_{i,k}(x)\frac{\partial^2 h(x)}{\partial x_i \partial x_k} + \sum_{i=1}^n b_i(x)\frac{\partial h(x)}{\partial x_i} \;\; ; \;\; h \in C^2(D)$$

and suppose that we have the following

- $\mathcal{A}$ is uniformly elliptic in D.
- the functions $(\sigma\sigma^T)_{i,k}, b_i$ are Hölder continuous in D.
- every point $x \in \delta D$ has the exterior sphere property; i.e there exists a ball $B \ni x$ such that $\bar{B} \cap D = \emptyset, \bar{B} \cap \partial D = \{x\}$.

Then

$$u(x) = \hat{E}^x[f(X_{\tau_D})\mathrm{Exp}\{-\int_0^{\tau_D} \mathcal{U}(X_s)\,ds\} + \int_0^{\tau_D} g(X_t)\mathrm{Exp}\{-\int_0^t \mathcal{U}(X_s)\,ds\}\,dt]$$

is the unique $(\mathcal{S}_n^m)^{-1}$-valued process which solves the stochastic Dirichlet problem

$$\begin{aligned} \mathcal{A}u(x) &= \mathcal{U}(x) \diamond u(x) - g(x) \quad && x \in D \\ u(x) &= f(x) && x \in \partial D \end{aligned} \tag{4}$$

where

$$\mathcal{U}(x) = \sum_{i=1}^m c_i \cdot \mathrm{Exp}\{W_{\phi_x}^{(i)}\} \quad (c_i \in \mathbb{R}_+) \tag{5}$$

is the potential given by sources of independent positive white noise, $\hat{\mathbb{E}}^x$, $\int_0^{\tau_D} \cdot ds$ and $\int_0^t \cdot ds$ are the Bochner integrals in $(\mathcal{S}_n^m)^{-1}$.

Remark : If $u(x) \in (\mathcal{S}_n^m)^{-1}$ and $\mathcal{A}(\mathcal{H}u(x)) \in A_b(\mathbf{B}_q(\delta))$, where $A_b(\mathbf{B}_q(\delta))$ is the space of all bounded analytic functions on $\mathbf{B}_q(\delta)$, for some $q \in \mathbb{N}, \delta > 0$, we will use the convention that $\mathcal{A}u(x) := \mathcal{H}^{-1}\mathcal{A}(\mathcal{H}u(x))$.

PROOF:

We will assume that $n = m = c = 1$ for simplicity.

We must find $\delta > 0, q < \infty$ such that $\tilde{u}(x,z) := \mathcal{H}(u(x))(z) \in A_b(\mathbf{B}_q(\delta))$ solves the equation

$$\begin{aligned} \mathcal{A}\tilde{u}(x) &= \exp\{\tilde{W}_{\phi_x}\} \cdot \tilde{u}(x) - g(x) \qquad && x \in D \\ \tilde{u}(x) &= f(x) && x \in \partial D \end{aligned} \tag{6}$$

when $z \in \mathbf{B}_q(\delta)$.

Fix $\hat{q} \in \mathbb{N}$ and $\hat{\delta}$ with $0 < \hat{\delta} < \frac{\pi}{2\|\phi\|}$.

LEMMA 3.2 $\tilde{u}(x,z) \in A_b(\mathbf{B}_{\hat{q}}(\hat{\delta}))$ $\forall x \in D$.

PROOF:

Since

$$\begin{aligned} |\tilde{W}_{\phi_x}(z)|^2 &= |\sum_{k=0}^{\infty} (\phi_x, e_k) z_k|^2 \\ &\leq \sum_{k=0}^{\infty} (\phi_x, e_k)^2 \cdot \sum_{k=0}^{\infty} |z_k|^2 \\ &\leq \|\phi\|^2 \cdot \sum_{\alpha \neq 0} |z^\alpha| (2\mathbf{N})^{\alpha q} \\ &\leq \hat{\delta}^2 \|\phi\|^2 \end{aligned}$$

when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$ it follows that

$$\begin{aligned} |\tilde{u}(x,z)| \leq\ & K_f \cdot \hat{\mathbb{E}}^x[|\exp\{-\int_0^{\tau_D} \exp\{\tilde{W}_{\phi_{X_s}}\} ds\}|] \\ &+ K_g \cdot \hat{\mathbb{E}}^x[\int_0^{\tau_D} |\exp\{-\int_0^t \exp\{\tilde{W}_{\phi_{X_s}}\} ds\}| dt] \\ &= K_f \cdot \hat{\mathbb{E}}^x[|\exp\{-\int_0^{\tau_D} \exp\{\Re(\tilde{W}_{\phi_{X_s}}) + i\Im(\tilde{W}_{\phi_{X_s}})\} ds\}|] \\ &+ K_g \cdot \hat{\mathbb{E}}^x[\int_0^{\tau_D} |\exp\{-\int_0^t \exp\{\Re(\tilde{W}_{\phi_{X_s}}) + i\Im(\tilde{W}_{\phi_{X_s}})\} ds\}| dt] \end{aligned}$$

$$= K_f \cdot \hat{E}^x[|\exp\{-\int_0^{\tau_D} \exp\{\Re(\tilde{W}_{\phi_{X_s}})\}(\cos \Im(\tilde{W}_{\phi_{X_s}}) + i \sin \Im(\tilde{W}_{\phi_{X_s}}))\, ds\}|]$$
$$+ K_g \cdot \hat{E}^x[\int_0^{\tau_D} |\exp\{-\int_0^t \exp\{\Re(\tilde{W}_{\phi_{X_s}})\}(\cos \Im(\tilde{W}_{\phi_{X_s}}) + i \sin \Im(\tilde{W}_{\phi_{X_s}}))\, ds\}|\, dt]$$
$$= K_f \cdot \hat{E}^x[|\exp\{-\int_0^{\tau_D} \exp\{\Re(\tilde{W}_{\phi_{X_s}})\} \cos \Im(\tilde{W}_{\phi_{X_s}})\, ds\}|]$$
$$+ K_g \cdot \hat{E}^x[\int_0^{\tau_D} |\exp\{-\int_0^t \exp\{\Re(\tilde{W}_{\phi_{X_s}})\} \cos \Im(\tilde{W}_{\phi_{X_s}})\, ds\}|\, dt]$$
$$\leq K_f + K_g \cdot \hat{E}^x[\int_0^{\tau_D} \exp\{-t \exp\{-\hat{\delta}\|\phi\|\} \cos(\hat{\delta}\|\phi\|)\}\, dt]$$
$$\leq K_f + \frac{2K_g \cdot \exp\{\hat{\delta}\|\phi\|\}}{\cos(\hat{\delta}\|\phi\|)}$$

where

$$K_f := \sup_{x \in \partial D} |f(x)|, K_g := \sup_{x \in \bar{D}} |g(x)|.$$

■

LEMMA 3.3 $u(x)$ is well-defined as a Bochner integral in $(\mathcal{S}_n^m)^{-1}$.

PROOF:

Estimating as in lemma 3.2 , we get with

$$\Phi(x, z, \hat{\omega}) := f(X_{\tau_D}) \exp\{-\int_0^{\tau_D} \exp\{\tilde{W}_{\phi_{X_s}}\}\, ds\} + \int_0^{\tau_D} g(X_t) \exp\{-\int_0^t \exp\{\tilde{W}_{\phi_{X_s}}\}\, ds\}\, dt$$

that

$$|\Phi(x, z, \hat{\omega})| \leq K_f + \frac{2K_g \cdot \exp\{\hat{\delta}\|\phi\|\}}{\cos(\hat{\delta}\|\phi\|)}$$

whenever $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$, i.e it follows from theorem 2.12 that $u(x)$ is given as a Bochner integral in $(\mathcal{S}_n^m)^{-1}$. ■

LEMMA 3.4 $\tilde{u}(x, z)$ is the unique function which solves equation (6) when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$.

PROOF:

Since

$$\tilde{W}_{\phi_x}(z) = \sum_{k=0}^{\infty} (\phi_x, e_k) z_k$$

for all $z \in \mathbb{C}_0^{\mathbb{N}}$, it follows that $\exp\{\tilde{W}_{\phi_x}\} \geq 0\ \forall x \in D$ whenever $z \in \mathbf{B}_{\hat{q}}(\hat{\delta}) \cap \mathbb{R}_0^{\mathbb{N}}$. In this case the claim now follows from proposition 7.2 and remark 7.5 in [KS], and by expanding the natural analytic extension of $\tilde{u}(x, z)$ into real and imaginary parts, the result follows for all $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$.

■

LEMMA 3.5 $\mathcal{A}u(x)$ is well-defined as an element in $(\mathcal{S}_n^m)^{-1}$ $\forall x \in D$.

PROOF:

Since

$$\mathcal{A}\tilde{u} = \exp\{\tilde{W}_{\phi_x}\} \cdot \tilde{u} - g$$

it follows from lemma 3.2 that

$$|\mathcal{A}\tilde{u}(x,z)| \leq \exp\{\hat{\delta}\|\phi\|\} \cdot (K_f + \frac{2K_g \cdot \exp\{\hat{\delta}\|\phi\|\}}{\cos(\hat{\delta}\|\phi\|)}) + K_g$$

when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$, i.e the claim follows. ∎

The theorem now follows from the previous lemmas. ∎

REMARK 3.6 It is easy to extend theorem 3.1 into more general situations.
Let $f(x) \in (\mathcal{S}_n^m)^{-1}$ $\forall x \in \partial D$, $g(x) \in (\mathcal{S}_n^m)^{-1}$ $\forall x \in \bar{D}$ and $\mathcal{U}(x) \in (\mathcal{S}_n^m)^{-1}$ $\forall x \in \bar{D}$ be given and assume that there exits $\tilde{q} \in \mathbb{N}$ and $\tilde{\delta} > 0$ such that the following holds:

- $\mathcal{H}f(x,z) \in C(\partial D)$ when $z \in \mathbf{B}_{\tilde{q}}(\tilde{\delta})$ and $\exists K_f > 0$ (independent of x, z) such that
$$\sup_{x \in \partial D, z \in \mathbf{B}_{\tilde{q}}(\tilde{\delta})} |\mathcal{H}f(x,z)| \leq K_f$$
- $\exists \alpha(z) > 0$ such that $\mathcal{H}g(x,z) \in C^{\alpha(z)}(\bar{D})$ when $z \in \mathbf{B}_{\tilde{q}}(\tilde{\delta})$ and $\exists K_g > 0$ (independent of x, z) such that
$$\sup_{x \in \bar{D}, z \in \mathbf{B}_{\tilde{q}}(\tilde{\delta})} |\mathcal{H}g(x,z)| \leq K_g$$
- $\exists \beta(z) > 0$ such that $\mathcal{H}\mathcal{U}(x,z) \in C^{\beta(z)}(\bar{D})$ when $z \in \mathbf{B}_{\tilde{q}}(\tilde{\delta})$ and $\exists K_{\mathcal{U}}(x) > 0$ (independent of z) such that
$$\sup_{z \in \mathbf{B}_{\tilde{q}}(\tilde{\delta})} |\mathcal{H}\mathcal{U}(x,z)| \leq K_{\mathcal{U}}(x) \quad \forall x \in \bar{D}$$
- $\Re(\mathcal{H}\mathcal{U}(x,z)) \geq 0$ $\forall x \in \bar{D}, z \in \mathbf{B}_{\tilde{q}}(\tilde{\delta})$.

Then

$$u(x) = \hat{\mathbb{E}}^x[f(X_{\tau_D}) \diamond \mathrm{Exp}\{-\int_0^{\tau_D} \mathcal{U}(X_s)\, ds\} + \int_0^{\tau_D} g(X_t) \diamond \mathrm{Exp}\{-\int_0^t \mathcal{U}(X_s)\, ds\}\, dt]$$

is the unique $(\mathcal{S}_n^m)^{-1}$-valued process which solves the (modified) stochastic Dirichlet problem given by (4).

COROLLARY 3.7 (The wave equation)

Assume that the assumptions of theorem 3.1 are valid.
Then

$$\Psi(t,x) = \hat{\mathbb{E}}^x[\sinh(t) f(b_{\tau_D}) \mathrm{Exp}\{-\int_0^{\tau_D} (\mathcal{U}(b_s) \wedge c)^{\diamond -1}\, ds\}]$$

is an $(\mathcal{S}_n^m)^{-1}$-valued process which solves the stochastic wave equation

$$\begin{aligned} &\frac{\partial^2\Psi}{\partial t^2}(t,x) = (\mathcal{U}(x)\wedge c)\diamond\Delta_x\Psi(t,x) && (t,x)\in\mathbb{R}_+\times D \\ &\Psi(0,x)=0 && x\in\bar{D} \\ &\frac{\partial\Psi}{\partial t}(0,x)=f(x) && x\in\partial D \end{aligned}$$

where $c\in\mathbb{R}_+\cup\{\infty\}$ and $\mathcal{U}(x)$ is the potential given by (5).

PROOF:

It is clear that the boundary conditions are satisfied and that

$$\frac{\partial^2\Psi}{\partial t^2}(t,x)=\Psi(t,x)$$

i.e we must show that the following equation is satisfied

$$\Psi(t,x)=(\mathcal{U}(x)\wedge c)\diamond\Delta_x\Psi(t,x)$$

or equivalent

$$\Delta_x\Psi(t,x)=(\mathcal{U}(x)\wedge c)^{\diamond-1}\diamond\Psi(t,x)$$

but this is nothing but the (modified) Dirichlet equation of remark 3.6, i.e the result follows.

■

COROLLARY 3.8 (The Schrödinger equation)

Assume that the assumptions of theorem 3.1 are valid.
Then

$$\Psi(t,x)=\hat{\mathbb{E}}^x[\exp\{i\frac{\hbar}{2\bar{m}}t\}f(b_{\tau_D})\mathrm{Exp}\{-\int_0^{\tau_D}(1+\frac{2\bar{m}}{\hbar^2}\mathcal{U}(b_s))\,ds\}]$$

is an $(\mathcal{S}_n^m)_c^{-1}$-valued process which solves the stochastic Schrödinger equation

$$\begin{aligned} -\frac{\hbar^2}{2\bar{m}}\Delta_x\Psi(t,x)+\mathcal{U}(x)\diamond\Psi(t,x) &= i\hbar\frac{\partial\Psi}{\partial t}(t,x) && (t,x)\in\mathbb{R}_+\times D \\ \Psi(0,x) &= f(x) && x\in\partial D \end{aligned}$$

where $\mathcal{U}(x)$ is the potential given by (5), $i=\sqrt{-1}$ is the imaginary unit, $\hbar$ is Planck's constant divided by 2π and $\bar{m}$ is the mass of the particle in study.

PROOF:

It is clear that the boundary condition is satisfied and that

$$\frac{\partial\Psi}{\partial t}(t,x)=\frac{\hbar}{2\bar{m}}i\Psi(t,x)$$

i.e we must show that the following equation is satisfied

$$\Delta_x\Psi(t,x)=(1+\frac{2\bar{m}}{\hbar}\mathcal{U}(x))\diamond\Psi(t,x)$$

but this is again nothing but the (modified) stochastic Dirichlet equation of remark 3.6, i.e the result follows.

■

§4 The second class of stochastic Dirichlet equations

We will in this section assume that $m = 1$ and use the notation

- $\hat{E}^x$ is expectation w.r.t. the measure $\hat{P}^x$.
- $\tau_D = \tau_D^{b_t} := \inf\{t > 0 : b_t \in D^c\}$ is the first exit time from D for b_t.

where $(b_t(\hat{\omega}), \hat{P}^x)$ is a Brownian motion in $\mathbb{R}^n$.

THEOREM 4.1 Let D be an open, bounded domain in $\mathbb{R}^n$ such that every point $x \in \partial D$ has the exterior sphere property; i.e. there exists a ball $B \ni x$ such that $\bar{B} \cap D = \emptyset$, $\bar{B} \cap \partial D = \{x\}$.

Assume further that we are given functions $\bar{D} \ni x \mapsto \kappa(x) \in (\mathcal{S})^{-1}$, $\partial D \ni x \mapsto f(x) \in (\mathcal{S})^{-1}$ and $\bar{D} \ni x \mapsto g(x) \in (\mathcal{S})^{-1}$ such that

- $\exists(q_f \in \mathbb{N}, \delta_f > 0, K_f > 0)$ such that
  1. $\sup_{x \in \partial D, z \in \mathbf{B}_{q_f}(\delta_f)} |\mathcal{H}f(x,z)| \le K_f$
  2. $x \mapsto \mathcal{H}f(x,z) \in C(\partial D)$ whenever $z \in \mathbf{B}_{q_f}(\delta_f)$.
- $\exists(q_g \in \mathbb{N}, \delta_g > 0, K_g > 0)$ such that
  1. $\sup_{x \in \bar{D}, z \in \mathbf{B}_{q_g}(\delta_g)} |\mathcal{H}g(x,z)| \le K_g$
  2. $\exists(\alpha(z) > 0 \ \ \forall z \in \mathbf{B}_{q_g}(\delta_g))$ such that $x \mapsto \mathcal{H}g(x,z) \in C^{\alpha(z)}(\bar{D})$.
- $\exists(q_\kappa \in \mathbb{N}, \delta_\kappa > 0, 0 < \epsilon_\kappa < \frac{\pi}{2}, K_\kappa(x) > 0 \ \forall x \in \bar{D})$ such that
  1. $\sup_{z \in \mathbf{B}_{q_\kappa}(\delta_\kappa)} |\mathcal{H}\kappa(x,z)| \le K_\kappa(x)$
  2. $\exists(\beta(z) > 0 \ \ \forall z \in \mathbf{B}_{q_\kappa}(\delta_\kappa))$ such that $x \mapsto \mathcal{H}\kappa(x,z) \in C^{\beta(z)}(\bar{D})$.
  3. $\sup_{x \in \bar{D}, z \in \mathbf{B}_{q_\kappa}(\delta_\kappa)} |\arg(\mathcal{H}\kappa(x,z))| \le \frac{\pi}{2} - \epsilon_\kappa$.
  4. $\mathcal{H}\kappa(x,z) \ge 0 \ \ \forall x \in \bar{D}$ whenever $z \in \mathbf{B}_{q_\kappa}(\delta_\kappa) \cap \mathbb{R}_0^{\mathbb{N}}$.

Then

$$u(x) = \operatorname{Exp}\{W_{-\frac{1}{2}\phi_x}\} \diamond \hat{E}^x[f(b_{\tau_D}) \diamond \operatorname{Exp}\{-\frac{1}{2}\int_0^{\tau_D} \kappa(b_s) \diamond \operatorname{Exp}\{W_{-\phi_{b_s}}\}\,ds\} \diamond \mathcal{J}(\tau_D, x)]$$

$$+ \frac{1}{2}\operatorname{Exp}\{W_{-\frac{1}{2}\phi_x}\} \diamond \hat{E}^x[\int_0^{\tau_D} \hat{g}(b_t) \diamond \operatorname{Exp}\{-\frac{1}{2}\int_0^t \kappa(b_s) \diamond \{W_{-\phi_{b_s}}\}\,ds\} \diamond \mathcal{J}(t,x)\,dt]$$

where

$$\mathcal{J}(t,x) := \operatorname{Exp}\{\frac{1}{2}W_{\phi_{b_t}} - \frac{1}{4}\int_0^t [\frac{1}{2}(\nabla W_{\phi_x})^{\diamond 2} + \Delta W_{\phi_x}]_{x=b_s}\,ds\} \tag{7}$$

and

$$\hat{g}(x) := g(x) \diamond \operatorname{Exp}\{W_{-\phi_x}\}$$

is the unique $(\mathcal{S})^{-1}$-valued process which solves

$$\begin{aligned} \operatorname{div}(\operatorname{Exp}\{W_{\phi_x}\} \diamond \nabla u) &= \kappa(x) \diamond u(x) - g(x) \quad && x \in D \\ u(x) &= f(x) && x \in \partial D \end{aligned}$$

where $\hat{E}^x$, $\int_0^t \cdot\, ds$ and $\int_0^{\tau_D} \cdot\, ds$ are Bochner integrals in $(\mathcal{S})^{-1}$.

REMARK 4.2 : We use the convention that arg is defined to be the function $\arg : \mathbb{C} \to < -\pi, \pi]$ given by the relation $z = |z| \exp\{i \cdot \arg(z)\}$.

REMARK 4.3 : If $u(x) \in (\mathcal{S})^{-1}$ and $\operatorname{div}(\exp\{\mathcal{H}W_{\phi_x}\} \cdot \mathcal{H}u(x)) \in A_b(\mathbf{B}_q(\delta))$ for some $q \in \mathbb{N}, \delta > 0$, we will use the convention that

$$\operatorname{div}(\operatorname{Exp}\{W_{\phi_x}\} \diamond \nabla u) := \mathcal{H}^{-1}(\operatorname{div}(\exp\{\mathcal{H}W_{\phi_x}\} \cdot \mathcal{H}u(x))).$$

PROOF:

We must find $\hat{\delta} > 0$, $\hat{q} \in \mathbb{N}$ such that $\tilde{u}(x, z) := \mathcal{H}(u(x))(z) \in A_b(\mathbf{B}_{\hat{q}}(\hat{\delta}))$ solves the equation

$$\begin{aligned} \operatorname{div}(\exp\{\tilde{W}_{\phi_x}\} \cdot \nabla \tilde{u}) &= \tilde{\kappa}(x) \cdot \tilde{u}(x) - \tilde{g}(x) \quad && x \in D \\ \tilde{u}(x) &= \tilde{f}(x) && x \in \partial D \end{aligned} \tag{8}$$

when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$.

LEMMA 4.4 $\exists(\hat{\delta} > 0, \hat{q} \in \mathbb{N})$ such that $z \mapsto \tilde{u}(x, z) \in A_b(\mathbf{B}_{\hat{q}}(\hat{\delta})) \quad \forall x \in D$.

PROOF:

It is clear that

$$|\exp\{\tilde{W}_{-\frac{1}{2}\phi_x}\}| \leq \exp\{\frac{\delta}{2}\|\phi\|\}$$

and

$$|\exp\{\frac{1}{2}\tilde{W}_{\phi_{b_{\tau_D}}}\}| \leq \exp\{\frac{\delta}{2}\|\phi_{b_{\tau_D}}\|\} = \exp\{\frac{\delta}{2}\|\phi\|\}$$

when $z \in \mathbf{B}_q(\delta)$, since $\forall(q \in \mathbb{N}, \delta > 0)$

$$|\tilde{W}_{\phi_x}(z)|^2 \leq \delta^2 \|\phi\|^2.$$

Using this last estimate on

$$w(x, z) := \exp\{-\frac{1}{2} \int_0^{\tau_D} \tilde{\kappa}(b_s) \exp\{\tilde{W}_{-\phi_{b_s}}\} ds\}$$

we get

$$
\begin{aligned}
|w(x,z)| &= |\exp\{-\frac{1}{2}\int_0^{\tau_D} |\tilde{\kappa}(b_s)| \exp\{i\arg(\tilde{\kappa}(b_s))\}\exp\{\Re(\tilde{W}_{-\phi_{b_s}}) + i\Im(\tilde{W}_{-\phi_{b_s}})\}\, ds\}| \\
&= |\exp\{-\frac{1}{2}\int_0^{\tau_D} |\tilde{\kappa}(b_s)| \exp\{\Re(\tilde{W}_{-\phi_{b_s}})\}\exp\{i(\arg(\tilde{\kappa}(b_s)) + \Im(\tilde{W}_{-\phi_{b_s}}))\}\, ds\}| \\
&= \exp\{-\frac{1}{2}\int_0^{\tau_D} |\tilde{\kappa}(b_s)| \exp\{\Re(\tilde{W}_{-\phi_{b_s}})\}\cos(\arg(\tilde{\kappa}(b_s)) + \Im(\tilde{W}_{-\phi_{b_s}}))\, ds\} \\
&\leq 1
\end{aligned}
$$

whenever $q \leq \min\{q_f, q_g, q_\kappa\}$ and $0 < \delta < \min\{\delta_f, \delta_g, \delta_\kappa, \frac{\epsilon_\kappa}{\|\phi\|}\}$.

Applying these estimates on $\tilde{u}(x,z)$, still assuming $q \leq \min\{q_f, q_g, q_\kappa\}$ and $0 < \delta < \min\{\delta_f, \delta_g, \delta_\kappa, \frac{\epsilon_\kappa}{\|\phi\|}\}$, we get

$$
\begin{aligned}
|\tilde{u}(x,z)| &\leq K_f \exp\{\delta\|\phi\|\}\hat{E}^x[|\exp\{-\frac{1}{4}\int_0^{\tau_D} [\frac{1}{2}(\nabla\tilde{W}_{\phi_x})^2 + \Delta\tilde{W}_{\phi_x}]_{x=b_s}\, ds\}|] \\
&\quad + \frac{1}{2}K_g \exp\{2\delta\|\phi\|\}\hat{E}^x[\int_0^{\tau_D} |\exp\{-\frac{1}{4}\int_0^t [\frac{1}{2}(\nabla\tilde{W}_{\phi_x})^2 + \Delta\tilde{W}_{\phi_x}]_{x=b_s}\, ds\}|\, dt] \\
&\leq (K_f \exp\{\delta\|\phi\|\} + \frac{1}{2}K_g \frac{1}{c(\delta)}\exp\{2\delta\|\phi\|\})\hat{E}^x[\exp\{c(\delta)\tau_D\}]
\end{aligned}
$$

where

$$
c(\delta) := \frac{1}{4}(\frac{\delta^2}{2}\sum_{i=1}^n \|\frac{\partial\phi}{\partial y_i}\|^2 + \delta\sum_{i=1}^n \|\frac{\partial^2\phi}{\partial y_i^2}\|)
$$

since

$$
\begin{aligned}
|(\nabla\tilde{W}_{\phi_x})^2| &= |\sum_{i=1}^n (\frac{\partial}{\partial x_i}\tilde{W}_{\phi_x})^2| \\
&\leq \sum_{i=1}^n |\frac{\partial}{\partial x_i}\tilde{W}_{\phi_x}|^2 \\
&= \sum_{i=1}^n |\tilde{W}_{(\frac{\partial\phi}{\partial y_i})_x}|^2 \\
&\leq \delta^2 \sum_{i=1}^n \|\frac{\partial\phi}{\partial y_i}\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
|\Delta\tilde{W}_{\phi_x}| &= |\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\tilde{W}_{\phi_x}| \\
&\leq \sum_{i=1}^{n} |\frac{\partial^2}{\partial x_i^2}\tilde{W}_{\phi_x}| \\
&= \sum_{i=1}^{n} |\tilde{W}_{(\frac{\partial^2\phi}{\partial y_i^2})_x}| \\
&\leq \delta \sum_{i=1}^{n} \|\frac{\partial^2\phi}{\partial y_i^2}\|.
\end{aligned}
$$

We know from [DUR] that there exists $\rho > 0$ such that

$$\hat{E}^x[\exp\{\rho\tau_D\}] < \infty \quad \forall x \in D$$

i.e if we choose a $0 < \hat{\delta} < \min\{\delta_f, \delta_g, \delta_\kappa, \frac{\epsilon_\kappa}{\|\phi\|}\}$ such that

$$\frac{1}{4}(\frac{\delta^2}{2}\sum_{i=1}^{n}\|\frac{\partial\phi}{\partial y_i}\|^2 + \delta\sum_{i=1}^{n}\|\frac{\partial^2\phi}{\partial y_i^2}\|) \leq \rho$$

and a $\hat{q} \leq \min\{q_f, q_g, q_\kappa\}$, then the claim follows. ■

LEMMA 4.5 The Bochner integrals in the expression for $u(x)$ are well-defined.

PROOF:

This is obvious from the estimates in lemma 4.4. ■

LEMMA 4.6 $\tilde{u}(x,z)$ is the unique function which solves equation (8) when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$.

PROOF:

Since

$$
\begin{aligned}
\operatorname{div}(\exp\{\tilde{W}_{\phi_x}\}\cdot\nabla\tilde{u}) &= \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(\exp\{\tilde{W}_{\phi_x}\})\frac{\partial\tilde{u}}{\partial x_i} + \sum_{i=1}^{n}\exp\{\tilde{W}_{\phi_x}\}\frac{\partial^2\tilde{u}}{\partial x_i^2} \\
&= \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(\tilde{W}_{\phi_x})\exp\{\tilde{W}_{\phi_x}\}\frac{\partial\tilde{u}}{\partial x_i} + \sum_{i=1}^{n}\exp\{\tilde{W}_{\phi_x}\}\frac{\partial^2\tilde{u}}{\partial x_i^2}
\end{aligned}
$$

our problem (8) may be written as

$$
\begin{aligned}
\mathcal{A}^z\tilde{u} &= \frac{1}{2}\exp\{\tilde{W}_{-\phi_x}\}\tilde{\kappa}(x)\tilde{u} - \frac{1}{2}\exp\{\tilde{W}_{-\phi_x}\}\tilde{g}(x) \qquad x \in D \\
\tilde{u}(x) &= \tilde{f}(x) \qquad x \in \partial D
\end{aligned}
\tag{9}
$$

where $\mathcal{A}^z$ is the second order differential operator given by

$$\mathcal{A}^z h = \sum_{i=1}^{n}\frac{1}{2}\frac{\partial^2 h}{\partial x_i^2} + \sum_{i=1}^{n}\frac{1}{2}\frac{\partial}{\partial x_i}(\tilde{W}_{\phi_x})\frac{\partial h}{\partial x_i} \;;\; h \in C^2(D).$$

Assume now that $z = \zeta \in \mathbf{B}_{\hat{q}}(\hat{\delta}) \cap \mathbb{R}_0^{\mathbb{N}}$.

Then this operator is clearly uniformly elliptic in D and since the drift coefficient satisfies the linear growth condition

$$\begin{aligned}
|\,(\frac{\partial}{\partial x_i}\tilde{W}_{\phi_x})\Big|_{x=h_2} - (\frac{\partial}{\partial x_i}\tilde{W}_{\phi_x})\Big|_{x=h_1}|(\zeta) &= |\tilde{W}_{(\frac{\partial\phi}{\partial y_i})_{h_2}}(\zeta) - \tilde{W}_{(\frac{\partial\phi}{\partial y_i})_{h_1}}(\zeta)| \\
&= |\sum_{k=0}^{\infty}((\frac{\partial\phi}{\partial y_i})_{h_2} - (\frac{\partial\phi}{\partial y_i})_{h_1}, e_k)\zeta_k| \\
&\leq \sum_{k=0}^{\infty}|((\frac{\partial\phi}{\partial y_i})_{h_2} - (\frac{\partial\phi}{\partial y_i})_{h_1}, e_k)||\zeta_k| \\
&\leq (M\sum_{k=0}^{\infty}\int_{\mathbb{R}^n}|e_k(x)|\,dx|\zeta_k|)|h_2 - h_1|
\end{aligned}$$

where

$$M := \max_{1\leq i\leq n}\{\sup_{x\in\mathbb{R}^n}|\frac{\partial^2\phi}{\partial x_i^2}|\}$$

the process given by

$$dX_t = \frac{1}{2}\nabla\tilde{W}_{\phi_x}(\zeta)\,dt + db_t\ ;\ X_0 = x.$$

exists with $\mathcal{A}^{\zeta}$ as the generator.

We know that

$$\tilde{W}_{\phi_x} = \sum_{k=0}^{\infty}(\phi_x, e_k)\zeta_k$$

so it follows that $\exp\{\tilde{W}_{\phi_x}(\zeta)\} \geq 0\ \forall x \in D$ and from [KS] we know that the solution of (9) is given uniquely by

$$\begin{aligned}
\tilde{u}(x,\zeta) = \hat{\mathbb{E}}^x[\tilde{f}(X_{\tau_D^{X_t}})\exp\{-\frac{1}{2}\int_0^{\tau_D^{X_t}}\tilde{\kappa}(X_s)\exp\{\tilde{W}_{-\phi_{X_s}}\}\,ds\}] \\
+\frac{1}{2}\hat{\mathbb{E}}^x[\int_0^{\tau_D^{X_t}}\tilde{g}(X_t)\exp\{\tilde{W}_{-\phi_{X_t}}\}\exp\{-\int_0^t\tilde{\kappa}(X_s)\exp\{\tilde{W}_{-\phi_{X_s}}\}\,ds\}\,dt].
\end{aligned}$$

By a change of measure this may be written as

$$\begin{aligned}
\tilde{u}(x,\zeta) = \hat{\mathbb{E}}^x[\tilde{f}(b_{\tau_D})\exp\{-\frac{1}{2}\int_0^{\tau_D}\tilde{\kappa}(b_s)\exp\{\tilde{W}_{-\phi_{b_s}}\}\,ds\}\mathcal{M}(\tau_D)] \\
+\frac{1}{2}\hat{\mathbb{E}}^x[\int_0^{\tau_D}\tilde{g}(b_t)\exp\{\tilde{W}_{-\phi_{b_t}}\}\exp\{-\int_0^t\tilde{\kappa}(b_s)\exp\{\tilde{W}_{-\phi_{b_s}}\}\,ds\}\mathcal{M}(t)\,dt]
\end{aligned}$$

where

$$\mathcal{M}(t) := \exp\{\frac{1}{2}\int_0^t (\nabla \tilde{W}_{\phi_x})_{x=b_s}\, db_s - \frac{1}{8}\int_0^t (\nabla \tilde{W}_{\phi_x})^2_{x=b_s}\, ds\} \tag{10}$$

and by applying the Ito-formula we know that

$$\frac{1}{2}\int_0^t [\nabla \tilde{W}_{\phi_x}]_{x=b_s}\, db_s = -\frac{1}{4}\int_0^t [\Delta \tilde{W}_{\phi_x}]_{x=b_s}\, ds + \frac{1}{2}\tilde{W}_{\phi_{b_t}} - \frac{1}{2}\tilde{W}_{\phi_x}$$

so finally, by substituting this expression into (10), we obtain equation (8).

This expression is easily seen to have an analytic extension to all $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$ and by applying the generator of b_t on both the real and imaginary part of $\tilde{u}(x,z)$ we see that equation (8) also holds in this case. ∎

LEMMA 4.7 The differential operator $\operatorname{div}(\operatorname{Exp}\{W_{\phi_x}\} \diamond \nabla u)$ is well-defined as an element in $(\mathcal{S})^{-1} \quad \forall x \in D$.

PROOF:

This is an obvious consequence of lemma 4.4 since we have shown that

$$\operatorname{div}(\exp\{\tilde{W}_{\phi_x}\} \diamond \nabla \tilde{u}) = \tilde{\kappa}(x)\tilde{u}(x) - \tilde{g}(x)$$

in lemma 4.6. ∎

The theorem now follows from the previous lemmas. ∎

COROLLARY 4.8 (The wave equation in an isotropic stochastic medium)

Assume that the assumptions of theorem 4.1 are valid.

Then

$$\Psi(t,x) = \operatorname{Exp}\{W_{-\frac{1}{2}\phi_x}\} \diamond \hat{E}^x[\sinh(t) f(b_{\tau_D}) \diamond \operatorname{Exp}\{-\frac{1}{2}\int_0^{\tau_D} \operatorname{Exp}\{W_{-\phi_{b_s}}\}\, ds\} \diamond \mathcal{J}(\tau_D, x)]$$

where $\mathcal{J}(t,x)$ is given in (7) is an $(\mathcal{S})^{-1}$-valued process which solves the wave equation in an isotropic medium given by

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial t^2}(t,x) &= \operatorname{div}(\operatorname{Exp}\{W_{\phi_x}\} \diamond \nabla_x \Psi) && x \in D \\ \Psi(0,x) &= 0 && x \in \bar{D} \\ \frac{\partial \Psi}{\partial t}(0,x) &= f(x) && x \in \partial D. \end{aligned}$$

PROOF:

It is clear that the boundary conditions are satisfied and that

$$\frac{\partial^2 \Psi}{\partial t^2}(t,x) = \Psi(t,x)$$

i.e we must show that

$$\Psi(t,x) = \mathrm{div}(\mathrm{Exp}\{W_{\phi_x}\} \diamond \nabla_x \Psi)$$

but this is nothing but the Dirichlet equation of theorem 4.1. ■

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