ON FINITELY TOPOLOGICALLY DETERMINED MAP–GERMS

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§0. Introduction

Let f, $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two C^∞ map-germs. We say that f and g are C^0-*equivalent* if there exist homeomorphism-germs h and l of $(\mathbb{R}^n, 0)$ and $(\mathbb{R}^p, 0)$ respectively such that $g = l \circ f \circ h^{-1}$. Let k be a positive integer. We say that a germ f is k-C^0-*determined* if every germ g with $j^k g(0) = j^k f(0)$ is C^0-equivalent to f. Moreover, we say that f is *finitely topologically determined* if f is k-C^0-determined for some finite k.

In this article we are going to prove a theorem giving a sufficient condition for a germ to be finitely topologically determined. To be able to formulate this theorem we need to introduce some notation.

Let N and P be two C^∞ manifolds. Consider the jet bundle $J^k(N, P)$ with fiber $J^k(n, p)$. Let $z \in J^k(n, p)$ and let f be such that $z = j^k f(0)$. Define

$$\chi(f) = \dim_{\mathbb{R}} \frac{\theta(f)}{tf(\theta(n)) + f^*(m_p)\theta(f)}.$$

Whether $\chi(f) < k$ depends only on z, not on f (see [8]). We can therefore define the set

$$W^k = W^k(n, p) = \{z \in J^k(n, p) | \chi(f) \geq k \text{ for some representative } f \text{ of } z\}.$$

W^k is semialgebraic and invariant under the contact group $\mathcal{K}$. Let $W^k(N, P)$ be the subbundle of $J^k(N, P)$ with fiber $W^k(n, p)$. In [8], Mather constructs a finite Whitney (b)-regular stratification $\mathcal{S}^k(n, p)$ of $J^k(n, p) - W^k(n, p)$ such that all strata are semialgebraic and $\mathcal{K}$-invariant, having the property that if $\mathcal{S}^k(N, P)$ denotes the corresponding stratification of $J^k(N, P) - W^k(N, P)$ and $f \in C^\infty(N, P)$ is a C^∞ map such that $j^k f$ is multitransverse to $\mathcal{S}^k(N, P)$, $j^k f(N) \cap W^k(N, P) = \varnothing$ and N is compact (or f is proper), then f is topologically stable.

Now consider the set $\mathbb{R}^p \times J^k(n, p)$ and, changing notation slightly, define for each $f : \mathbb{R}^n \to \mathbb{R}^p$, $j^k f : \mathbb{R}^n \to \mathbb{R}^p \times J^k(n, p)$ as in [2], namely $j^k f(x_0) = (f(x_0), z)$ where z is the k-jet at 0 of $f(x + x_0) - f(x_0)$. Also we let $(j^k f)^m : (\mathbb{R}^n)^m \to (\mathbb{R}^p \times J^k(n, p))^m$ denote the induced map on m-fold Cartesian products. Let $\mathcal{S}^k$ denote the product stratification $\mathbb{R}^p \times \mathcal{S}^k(n, p)$ of $\mathbb{R}^p \times (J^k(n, p) - W^k(n, p))$. Let $\pi_k : J^{k+1}(n, p) \to J^k(n, p)$ be the projection.

Consider the subset

$$NT = NT^{(k,m)} = NT^{(k,m)}(n, p) \subset (\mathbb{R}^p \times J^{k+1}(n, p))^m$$

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consisting of those jets $z = ((y_1, z_1), \dots, (y_m, z_m))$ such that $y_1 = \dots = y_m$ and either

(i) $\pi_k(z_i) \in W^k(n,p)$ for some i

or

(iia) $\pi_k(z_i) \in J^k(n,p) - W^k(n,p)$ for all i

and z satisfies the non-transversality condition (iib) explained below (for brevity we introduce the notation $(y_i, z_i)_{i=1}^m$ for $((y_1, z_1), \dots, (y_m, z_m))$.

Let $z_i = j^{k+1} f_i(0)$ for some maps $f_i : \mathbb{R}^n \to \mathbb{R}^p$, $i = 1, \dots, m$. Take m copies of $\mathbb{R}^n$ and form the disjoint topological union $\sum_{i=1}^m \mathbb{R}^n$. The f_i's then induce a map $\tilde{f} : \sum_{i=1}^m \mathbb{R}^n \to \mathbb{R}^p$. Let $S \subset \sum_{i=1}^m \mathbb{R}^n$ be the subset of m points consisting of 0 in each topological component. Consider the condition:

(iib) $j^k \tilde{f}$ is not multitransverse to $\mathcal{S}^k$ at S.

Consider $f \in C^\infty(U, \mathbb{R}^p)$, where U is an open neighborhood of 0 in $\mathbb{R}^n$. Define

$$D_m(f) = \{(x_1, \dots, x_m) \in (U)^m |\ x_i = x_j \in \Sigma(f) \text{ for some } i \neq j \text{ or } x_i = 0 \text{ for some } i\}.$$

Here $\Sigma(f)$ denotes the set of points in U where the derivative map Df is not surjective.

Now the main result of this article is the following theorem.

(0.1) Theorem. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a map-germ. Suppose there exists a representative $f : U \to \mathbb{R}^p$ and, for $m = 1, \dots, p+1$, positive constants C_m and α_m such that for some positive integer k and for any $x \in U^m$*

$$d((j^{k+1} f)^m(x), NT^{(k,m)}(n,p)) \geq C_m d(x, D_m(f))^{\alpha_m};$$

then f is finitely topologically determined.

In the above Theorem, $d(\cdot, \cdot)$ denotes the Euclidean distance. Moreover, note that we have used the same symbol to denote a germ and a representative of the germ. We will continue to do so throughout the article.

Remark. It follows from our definitions above that if $((j^{k+1} f)^m)^{-1}(NT^{(k,m)}) \subset D_m(f)$ for $m = 1, \dots, p+1$ then $j^k f$ hits and is multitransverse to the canonical stratification outside 0. Our theorem therefore says that if $j^k f$ is multitransverse to the canonical stratification on some punctured neighborhood of 0, and becomes non-transverse, when we move towards 0, at a rate controlled by some Łojasiewicz inequalities, then f is finitely topologically determined.

The proof of (0.1) will occupy the rest of this article. In the proof the key tool will be a theorem appearing in [2]. To formulate this theorem we must introduce some notation.

Consider the set $\text{Uns}(m) \subset (\mathbb{R}^p \times \mathbb{R}^c \times J^{p+c+1}(n+c, p+c))^m$ consisting of unstable m-tuples of C^∞ $(p+c+1)$-jets. Let $F : (\mathbb{R}^n \times \mathbb{R}^c, 0) \to (\mathbb{R}^p \times \mathbb{R}^c, 0)$ be a C^∞ map-germ of the form $F(x,u) = (f(x,u), u)$. Consider the condition (e$'$) defined below.

(e$'$) There exist a positive constant α and a representative $F : U \to \mathbb{R}^p \times \mathbb{R}^c$ such that

$$d((j^{p+c+1} F)^m(x, 0), \text{Uns}(m)) \geq d((x,0), D_m(F))^\alpha$$

for any $(x, 0) \in (U \cap (\mathbb{R}^n \times \{0\}))^m$ and $m = 1, \dots, p+c+1$.

Then, the following theorem appears in [2].

(0.2) Theorem. *Let F be as described above and assume that F satisfies condition (e$'$). Then there exist a constant $\lambda > 0$ and a function $\gamma : \mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}$, with $\gamma(k) < \infty$ for $k < \infty$ and $\gamma(\infty) = \infty$, having the following properties:*

Let $G : (\mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}, 0 \times I) \to (\mathbb{R}^p \times \mathbb{R}^c \times \mathbb{R}, 0 \times I)$ be a germ of an unfolding of F of the form $G(x,u,t) = (g(x,u,t),u,t) = (g_t(x,u),t)$ with $I = [0,\ 1]$, $g_0 = F$ and $j^{\gamma(k)}g_t(0) = j^{\gamma(k)}F(0)$ for some k. Let $H(\lambda)$ be the germ of the "hornshaped" neighborhood given by

$$H(\lambda) = \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^c |\ \|u\| < \|x\|^{\lambda}\}.$$

Then there exist a representatives and $G : U \times O \to V \times O$ and $H(\lambda)$ (of the germs G and $H(\lambda)$) with $H(\lambda) \subset U$, where U and V are neighborhoods of 0 in $\mathbb{R}^n \times \mathbb{R}^c$ and $\mathbb{R}^p \times \mathbb{R}^c$, and O is a neighborhood of I in $\mathbb{R}$, together with level-preserving maps $L : H(\lambda) \times O \to U \times O$ and $K : V \times O \to \mathbb{R}^{p+c} \times O$ satisfying the following conditions:

(1) The germ of $g_0 = F$ is multistable at each finite set of points in $H(\lambda)$.
(2) $K \circ G \circ L = F \times Id_{\mathbb{R}}$ on $H(\lambda) \times O$.
(3) L and $K|(V - \{0\}) \times O$ are C^∞ diffeomorphisms onto their respective images and the conditions

(i) $$\|Id_{\mathbb{R}^{n+c}\times\mathbb{R}} - L\|^*_{k,(x,u,t)} = o(\|(x,u)\|^k) \text{ and}$$
(ii) $$\|Id_{\mathbb{R}^{p+c}\times\mathbb{R}} - K\|^*_{k,(y,u,t)} = o(\|(y,u)\|^k)$$

hold where $(x,u,t) \in H(\lambda) \times O$ and $(y,u,t) \in V \times O$ respectively.
Furthermore, in the case $k = \infty$, (i) *and* (ii) *can be replaced by the conditions*

(i) $$\|Id_{\mathbb{R}^{n+c}\times\mathbb{R}} - L\|^*_{m,(x,u,t)} = o(\|(x,u)\|^m) \text{ and}$$
(ii) $$\|Id_{\mathbb{R}^{p+c}\times\mathbb{R}} - K\|^*_{m,(y,u,t)} = o(\|(y,u)\|^m)$$

for any m where $(x,u,t) \in H(\lambda) \times O$ and $(y,u,t) \in V \times O$ respectively. (Here, the norms $\|\ldots\|^*_{\ldots,}$ are defined in [2]*. They are equivalent with those defined in* [5]*.)*

We finish this section with a brief sketch of the proof of (0.1). As is well known, a finitely $\mathcal{K}$-determined map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ admits a stable unfolding $F : (\mathbb{R}^{n+c}, 0) \to (\mathbb{R}^{p+c}, 0)$. Mather's theory (see [8]) produces a Whitney stratification $\mathcal{S}_2$ of $(\mathbb{R}^{p+c}, 0)$ such that, letting j denote the natural embedding of $\mathbb{R}^p$ into $\mathbb{R}^{p+c}$, then $j \pitchfork \mathcal{S}_2$ implies that f is topologically stable.

Let f be a germ satisfying the hypothesis of (0.1). While f is not finitely $\mathcal{K}$-determined, we construct in §1 an unfolding F of f satisfying condition (e$'$) and therefore the conclusion of (0.2).

In §2 we consider the unfolding F of f constructed in §1. Let k_0 be a positive integer. Since F satisfies the conclusion of (0.2), we can obtain a commutative diagram

$$(*) \qquad \begin{array}{ccc} H(\lambda) & \xrightarrow{\ l\ } & U \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle R} \\ \mathbb{R}^p \times \mathbb{R}^c & \xleftarrow{\ k\ } & V, \end{array}$$

where $R = j^{\gamma(k_0)}F(0)$ and l and $k|V - \{0\}$ are C^∞-diffeomorphisms onto their respective images. Using (1) and (i) of (0.2), we can show that there exists λ' such that $l(H(\lambda)) \supset \overline{H(\lambda')} - \{0\}$ and such that if $C = R(\Sigma(R) \cap (\overline{H(\lambda')} - H(\lambda')))$ and $K = R^{-1}(C)$, then $R|H(\lambda') - K : H(\lambda') - K \to \mathbb{R}^p \times \mathbb{R}^c - C$ is infinitesimally stable. We can consequently use the results of Mather in [8], and we get induced stratifications $\mathcal{S}_i(R)$, $i = 1, 2, 3$, where $\mathcal{S}_i(R)$, $i = 1, 3$, are stratifications of $H(\lambda') - K$ and $\mathcal{S}_2(R)$ is a stratification of $\mathbb{R}^p \times \mathbb{R}^c - C$. Moreover, since F is an unfolding of f, we can use $(*)$ to construct another unfolding

$$(**) \qquad \begin{array}{ccc} O & \xrightarrow{\tilde{i}} & \mathbb{R}^{n+c} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle R} \\ W & \xrightarrow{\tilde{j}} & \mathbb{R}^{p+c}. \end{array}$$

We end §2 by formulating an important lemma (Lemma (2.7)), concluding that for each $\beta > 0$, $\tilde{i}|O - H(f^{-1}(0), \beta)$ (where $H(f^{-1}(0), \beta) = \{x | d(x, f^{-1}(0)) \leq \|x\|^\beta\}$) and $\tilde{j}$ satisfy certain Łojasiewicz inequalities implying that these maps are transverse to $\mathcal{S}_3(R)$ and $\mathcal{S}_2(R)$ respectively.

In §3 we prove some general metric properties of semialgebraic stratifications. The results here are used in §4 and §5.

In §4 we prove Lemma (2.7). In fact, we consider a one-parameter unfolding $g(x, t) = (g_t(x), t) = (f(x) + th(x), t)$ with $j^s h(0) = 0$. If s is large we can construct a one-parameter version of $(**)$,

$$(***) \qquad \begin{array}{ccc} \bigcup_{t \in I} (O - g_t^{-1}(0)) \times \{t\} & \xrightarrow{(i_t, t)} & (H(\lambda') - K) \times I \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle R \times id} \\ (W - \{0\}) \times I & \xrightarrow{(j_t, t)} & (\mathbb{R}^{p+c} - C) \times I, \end{array}$$

and we prove a one-parameter version of (2.7) for $(***)$.

In §5 we consider $(***)$. Since we have proved a one-parameter version of (2.7) for $(***)$, we know that $(i_t, t)| \bigcup_{t \in I} (O - H(g_t^{-1}(0), \beta)) \times \{t\}$ and (j_t, t) are transverse to $\mathcal{S}_3(R) \times I$ and $\mathcal{S}_2(R) \times I$ respectively. We can therefore take the pullback of these stratifications via (i_t, t) and (j_t, t) respectively. If s and β are large, we can also show that $g| \bigcup_{t \in I} H(g_t^{-1}(0), \beta)) \times \{t\} - (\{0\} \times I)$ is a submersion, and we use this to construct a stratification of $\bigcup_{t \in I} H(g_t^{-1}(0), \beta)) \times \{t\} - (\{0\} \times I)$, which fits together with the pullback of $\mathcal{S}_2(R)$ via (i_t, t). We then add $\{0\} \times I$ as strata in source and target, and obtain stratifications of $O \times I$ and $W \times I$. Now, using the results of §2, §3 and §4, we prove that these stratifications are Whitney regular, and that g is a stratified map. So by the second isotopy lemma each level g_t is topological equivalent with f. Thus if s is sufficiently large, f and $f + h$ are topologically equivalent, hence f topologically s-determined, proving (0.1).

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§1. Stable unfoldings on hornshaped neighborhoods

In this section we will construct a stable unfolding on a hornshaped neighborhood of the punctured domain of a map. The rate at which the unfolding becomes unstable will be controlled by a Łojasiewicz inequality.

Now assume U is a neighborhood of 0 in $\mathbb{R}^n$. Let $N = U - \{0\}$. The maps $f : U \to \mathbb{R}^p$ for which we will construct stable unfoldings will be those which satisfy

$$(1.1.1) \qquad j^k f(N) \cap W^k(U, \mathbb{R}^p) = \varnothing \text{ for some fixed } k$$

and, letting Σ_y denote $f^{-1}(y) \cap \Sigma(f)$ for $y \neq 0$ and Σ_0 denote $f^{-1}(0) \cap \Sigma(f) \cap N$,

$$(1.1.2)) \qquad \#\Sigma_y \leq s \text{ for some fixed } s.$$

It is proven in [6] that (1.1.1) implies:

$$(1.2) \qquad tf_x\theta_x + f^* m_{f(x)}\theta(f_x) \supset m_x^{k-1}\theta(f_x), \forall x \in N.$$

Fix a positive integer d. Let g_i, $i = 1, \ldots, c = c(d)$, be the collection of all polynomial maps $g_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ in which only one component is nonzero, and that component is a monomial of degree $\leq d$ with coefficient 1. Define $F : U \times \mathbb{R}^c \to \mathbb{R}^p \times \mathbb{R}^c$ by

$$(1.3) \qquad F(x, u) = (f(x) + \sum_{i=1}^{c} u_i g_i(x), u).$$

We wish to choose d so that, if f satisfies (1.1), then F will be a stable germ at each Σ_y, $y \in \mathbb{R}^p$.

Let S be a finite set in N. Let f_S (respectively, $(g_i)_S$) denote the germ of F (respectively g_i) at S. Then we have:

(1.4) Proposition. *F is stable at S if, and only if, the images of $(g_i)_S$, $i = 1, \ldots, c$, span*

$$\frac{\theta(f_S)}{tf_S(\theta_S) + (f_S^* m_{f(S)})\theta(f_S) + \sum_{j=1}^{p} \mathbb{R}\frac{\partial}{\partial y_i} \circ f}.$$

(Note that in (1.4) we have identified each $(g_i)_S$ with a vector field in $\theta(f_S)$ in an obvious manner.)

Proof. Alter the proof of Theorem 2.1 (p. 200) and Proposition 3.1 (p. 204) of [4] by replacing 0 in $\mathbb{R}^s$ (respectively, $\mathbb{R}^t$) by $S \subset \mathbb{R}^n$ (respectively, $f(S) \subset \mathbb{R}^p$). □

If f satisfies (1.2) then, for any finite $S \subset N$,

$$tf_S(\theta_S) + (f_S^* m_{f(S)})\theta(f_S) \supset (m_S^{k-1})\theta(f_S).$$

Proposition (1.4) implies that F will be stable at S if $(g_i)_S$, $i = 1, \ldots, c$, span a complement of $(m_S^{k-1})\theta(f_S)$ in $(m_S)\theta(f_S)$.

We now have the following proposition:

(1.5) Proposition. *Suppose $S = \{x_1, \ldots, x_s\} \subset \mathbb{R}^n$ and z_i is a k-jet at $x_i, i = 1, \ldots, s$, $k \geq 2$. Let $d = sk$. Then there is a polynomial map g of degree $\leq d$ such that $j^k g(x_i) = z_i$ for $i = 1, \ldots, s$.*

Proof. This is a special case of [13, bottom of p. 153], also see [3, Lemma 4.1].□

Thus if f satisfies (1.2) and F is defined as in (1.3), with $d = s(k-2)$, then F_S is stable for each $S \in N$ with $\#S \leq s$. It then immediately follows that:

(1.6) Proposition. *If f satisfies (1.1) and F is defined as in (1.3), then F_{Σ_y} is stable for all y in $\mathbb{R}^p$.*

Next we will control the rate at which f becomes unstable. For any $f : \mathbb{R}^n \to \mathbb{R}^p$ define $j^r f : \mathbb{R}^n \to \mathbb{R}^p \times J^r(n,p)$ as in §0. If z is an r-jet in $J^r(n,p)$ given by $j^r f(x) = (f(x), z)$, then we also let z denote the r-jet in $J^r(n+c, p+c)$ given by $j^r(f(x), u) = ((f(x), u), z)$. For $l = 1, \ldots s$, define $\Phi : (\mathbb{R}^n \times \mathbb{R}^p \times J^r(n,p))^l \to (\mathbb{R}^{p+c} \times J^r(n+c, p+c))^l$ by $\Phi = (\Phi_1, \ldots, \Phi_l)$, where

$$\Phi^j((x_1, y_1, z_1), \ldots, (x_s, y_s, z_s)) = ((y_j, 0), z_j) + j^r(\sum_{i=1}^{c} u_i g_i(x))(x_j, 0),$$

(g_i and c are as in (1.3)). If $f : U \to \mathbb{R}^p$ is any mapping and F is as in (1.3), then

$$\Phi \circ ((x_1, j^r f(x_1)), \ldots, (x_l, j^r f(x_l))) = (j^r F(x_1, 0), \ldots, j^r F(x_l, 0))$$

(here, $x_1, \ldots, x_l$ are not necessarily distinct). If the critical points of f among $x_1, \ldots, x_l$ are distinct, $j^k f(x_i) \notin \mathbb{R}^p \times W^k(n,p)$, $r \geq k \geq 2$, and $d = s(k-2)$, then F is multistable at $\{(x_1, 0), \ldots, (x_l, 0)\}$. Now assume that $r \geq p + c + 1$. Let $\mathrm{Uns} = \mathrm{Uns}(r, l)$ denote the set of unstable multijets in $(\mathbb{R}^{p+c} \times J^r(n+c, p+c))^l$ (see [2] for a precise definition). Thus it follows from above that

$$\Phi^{-1}(\mathrm{Uns}) \subset W^{r,k,l} \cup \Delta^l_\Sigma,$$

where $W^{r,k,l}$ consists of those points $(x_i, y_i, z_i)_{i=1}^l$ with $y_1 = \cdots = y_l$ and $z_j \in (\Pi^r_k)^{-1}(W^k(n,p))$ for some j, and Δ^l_Σ are points $(x_i, y_i, z_i)_{i=1}^l$ with $y_1 = \cdots = y_l$, $x_i = x_j$ and z_i, z_j both critical jets for at least two indices i and j. (Here, Π^r_k is the projection of $J^r(n,p)$ onto $J^k(n,p)$.) For any set X and any natural number r, we let ΔX denote the diagonal $\{(x, \ldots, x) : x \in X\} \subset X^r$, where the r will always be clear by context.

Finally, we assume

$$d(j^k f(x), \mathbb{R}^p \times W^k(n,p)) \geq C\|x\|^\alpha, \tag{1.7.1}$$

$$d(j^1 f(x), \mathbb{R}^p \times \Sigma(n,p)) \geq C d(x, \Sigma(f))^\alpha \tag{1.7.2}$$

and for $x = (x_1, \ldots, x_{s+1}) \in U^{s+1}$, $s \geq 1$,

$$d((j^1 f)^{s+1}(x), \Delta\mathbb{R}^p \times (\Sigma(n,p))^{s+1}) \geq C d(x, D_{s+1}(f))^\alpha \tag{1.7.3}$$

for some positive constants C and α.

Let $x = (x_1, \ldots, x_l)$, $l \leq s$. From (1.7.1) it follows that $d((x, (j^r f)^l(x)), W^{r,k,l})$ satisfies a Łojasiewicz inequality with respect to $\min_i\{\|x_i\|\}$, a fortiori with respect to $d(x, D_l(f))$.

To realize that (1.7.2) implies that $d((x, (j^r f)^l(x)), \Delta^l_\Sigma)$ satisfies a Łojasiewicz inequality with respect to $d(x, D_l(f))$ is somewhat more involved. To this end, let $\bar{z} = (\bar{x}_m, \bar{y}_m, \bar{z}_m)^l_{m=1}$ be a point in Δ^l_Σ such that

$$d((x, (j^r f)^l(x)), \Delta^l_\Sigma) = \|(x, (j^r f)^l(x)) - \bar{z}\|.$$

Let i and j be indices such that $\bar{x}_i = \bar{x}_j = \bar{x}$ and $\bar{z}_i$ and $\bar{z}_j$ are both critical jets. Assume first that $\|x_i - x_j\| \geq d((x_i, x_j), \Delta\Sigma(f))^2$. Then

$$\|(x, (j^r f)^l(x)) - \bar{z}\| \geq \|(x_i, x_j) - (\bar{x}, \bar{x})\| \geq \frac{1}{\sqrt{2}}\|x_i - x_j\| \geq \frac{1}{\sqrt{2}} d((x_i, x_j), \Delta\Sigma(f))^2,$$

showing that we have satisfied a Łojasiewicz inequality in this case.

Assume now $\|x_i - x_j\| < d((x_i, x_j), \Delta\Sigma(f))^2$. Let $\hat{x}_i$ and $\hat{x}_j$ be points in $\Sigma(f)$ such that $d(x_i, \Sigma(f)) = \|x_i - \hat{x}_i\|$ and $d(x_j, \Sigma(f)) = \|x_j - \hat{x}_j\|$. Then

$$\begin{aligned}
&\|x_i - \hat{x}_i\| + \|x_j - \hat{x}_j\| \geq \|x_j - \hat{x}_i\| + \|x_j - \hat{x}_j\| - \|x_i - x_j\| \geq \\
&\|x_j - \hat{x}_j\| + \|x_j - \hat{x}_j\| - \|x_i - x_j\| \geq \|x_i - \hat{x}_j\| + \|x_j - \hat{x}_j\| - 2\|x_i - x_j\| \geq \\
&d((x_i, x_j), \Delta\Sigma(f)) - 2d((x_i, x_j), \Delta\Sigma(f))^2 \geq d((x_i, x_j), \Delta\Sigma(f))^2 \geq d(x, D_l(f))^2.
\end{aligned}$$

(1.7.2) now implies that

$$\|(x, (j^r f)^l(x)) - \bar{z}\| \geq \|(j^r f)^l(x) - (\bar{y}_m, \bar{z}_m)^l_{m=1}\| \geq C(\|x_i - \hat{x}_i\|^\alpha + \|x_j - \hat{x}_j\|^\alpha),$$

and this together with the inequality $\|x_i - \hat{x}_i\| + \|x_j - \hat{x}_j\| \geq d(x, D_l(f))^2$ give us that $d((x, (j^r f)^l(x)), \Delta^l_\Sigma)$ satisfies a Łojasiewicz inequality with respect to $d(x, D_l(f))$ in this case too.

From all this we see that $d((x, (j^r f)^l(x)), W^{r,k,l} \cup \Delta^l_\Sigma)$ also satisfies a Łojasiewicz inequality with respect to $d(x, D_l(f))$.

We now have the following

(1.8) Lemma. *Suppose Ψ is a polynomial map from $\mathbb{R}^s$ to $\mathbb{R}^t$, $A \subset \mathbb{R}^s$, $B \subset \mathbb{R}^t$ are closed sets, B is semialgebraic and $\Psi^{-1}(B) \subset A$. For each $P \in \mathbb{R}^s$, there exist positive constants C and α such that $d(\Psi(x), B) \geq Cd(x, A)^\alpha$ on some neighborhood of P.*

Proof. We may suppose that $\Psi(P) \in B$, otherwise the inequality is obvious. Let $V = \{(x, \Psi(x))\}$ and $W = \mathbb{R}^s \times B$. Clearly V is an algebraic and W is a semialgebraic subset of $\mathbb{R}^s \times \mathbb{R}^t$ and therefore they are regularly situated. So there exist α, $C > 0$ such that in a compact neighborhood of P we have that $d((x, \Psi(x)), W) \geq Cd((x, \Psi(x)), V \cap W)^\alpha$. Let $(y, b) \in V \cap W$ be such that $d((x, \Psi(x)), V \cap W) = \|(x, \Psi(x)) - (y, b)\|$. Since $b \in B$ and $\Psi(y) = b$, we have that $y \in \Psi^{-1}(B)$ and thus $d(\Psi(x), B) = d((x, \Psi(x)), W) \geq C\|(x, \Psi(x)) - (y, b)\|^\alpha \geq C\|x - y\|^\alpha \geq Cd(x, \Psi^{-1}(B))^\alpha \geq Cd(x, A)^\alpha$. □

Now we wish to see that (1.1) and (1.7) imply that F satisfies the hypothesis of (0.2). To this end, apply (1.8) to $\Psi = \Phi$, $B = \mathrm{Uns}$, $A = W^{r,k,l} \cup \Delta^l_\Sigma$. We then get that $d((j^rF)^l(x,0), \mathrm{Uns}(r,l))$ is Łojasiewicz with respect to $d(x, D_l(f))$ and a fortiori with respect to $d((x,0), D_l(F))$, for $l = 1, \dots, s$.

We need that $d((j^rF)^l(x,0), \mathrm{Uns}(r,l))$ is Łojasiewicz with respect to $d((x,0), D_l(F))$ for $l = s+1, \dots, p+c+1$ as well. Consider the map $\psi : J^1(n,p) \to J^1(n+c, p+c)$ defined by

$$\psi(A) = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

If we let a_{ij} denote the standard coordinates in $J^1(n+c, p+c)$, then clearly $\mathrm{im}\, D\psi$ contains the vectors $\frac{\partial}{\partial a_{ij}}$, $1 \le i \le p$, $1 \le j \le n$. Suppose B is in $\mathrm{im}\,\psi$ and has rank r, and let Σ_r denote the manifold of 1-jets of rank r. It is easy to see that $T_B\Sigma_r$ contains the vectors $\frac{\partial}{\partial a_{ij}}$, $i > p$ or $j > n$. Thus ψ is transverse to the stratification of $\Sigma(n+c, p+c)$ by rank. It follows from the main theorem of [9] that

$$d(DF(y,0), \Sigma(n+c, p+c)) \ge C d(Df(y), \Sigma(n,p)).$$

(A more general result with a simpler proof can be found as Proposition 2.16 of [12]). Putting this and (1.7.3) together, we get that $d((j^1F)^{s+1}(x,0), \Delta\mathbb{R}^{p+c} \times (\Sigma(n+c, p+c))^{s+1})$ satisfies a Łojasiewicz inequality with respect to $d(x, D_{s+1}(f))$, a fortiori with respect to $d((x,0), D_{s+1}(F))$, and it follows that the corresponding inequality also holds for all $l > s+1$. Now $\mathrm{Uns}(r,l)$ is a subset of $\Delta(\mathbb{R}^{p+c})^l \times (\Sigma(n+c, p+c)^l)$; we therefore get that $d((j^rF)^l(x,0), \mathrm{Uns}(r,l))$ is Łojasiewicz with respect to $d((x,0), D_l(F))$ for all $l > s$. Since stable maps from $\mathbb{R}^{n+c}$ to $\mathbb{R}^{p+c}$ are $(p+c+1)$-determined, we have that $d((j^rF)^l(x,0), \mathrm{Uns}(r,l)) = d((j^{p+c+1}F)^l(x,0), \mathrm{Uns}(p+c+1, l))$, and it follows from above that the hypothesis of (0.2) is satisfied. We can consequently apply the conclusion of (0.2) for the germ F and the subset $\mathbb{R}^n$.

(1.9) Examples.

(1). Let $f(x,y) = (x, y^3 + \alpha(x)y)$, where $\alpha(x)$ is infinitely flat at 0 but positive at all $x \ne 0$. Then f satisfies (1.1.1) with $k = 3$ and (1.1.2) with $s = 0$, so $d = s(k-2) = 0$, and f satisfies (1.7.1) (since $s = 0$, (1.7.3) doesn't apply). However (1.7.2) fails: if $x_n \to 0$, then $d(j^1 f(x_n, 0), \Sigma(2,2))$ is flat along $d((x_n, 0), \Sigma(f)) = |x_n|$. We get the unfolding F by unfolding with constant polynomials in each component (in fact, these aren't necessary, we could take $F = f$; since we haven't shown this however, we will stick with the unfolding by constants; consequently, $p+c+1 = 5$). Now if we construct an unfolding of $g(x,y) = (x, y^3)$ by unfolding by the same terms, then we get an unfolding G. So j^5G is unstable and infinitely close to j^5F at the points $(x_n, 0, 0, 0)$. F will consequently not satisfy the hypothesis of (0.2). This shows the need for condition (1.7.2).

(2). If we compose the fold $(x,y) \to (x, y^2)$ with the complex function $z \to z^2$, we obtain a map g from the plane to the plane which has fold points along the x-axis intersecting nontransversally in the target. So the composed map is unstable at the pairs of points $\{(x,0), (-x,0)\}$. Now we make a flat perturbation of the map z^2 such that the perturbed map is one-to-one on the halfplane $y \ge 0$ and the images of the positive and negative x-axes become two non-intersecting curves which are "flatly" tangent at the origin. Compose the fold with this map instead. Then we have obtained a map f which outside 0 has only non-intersecting fold singularities on the x-axis, but $(j^2)^2$ of this map is infinitely close to a halfline of unstable multijets on the x-axis. However f satisfies (1.1.1) with $k = 2$ and (1.1.2) with $s = 1$, so $d = s(k-2) = 0$ and we get the unfolding F by unfolding with constant polynomials in each component (in fact, these aren't necessary; we could take $F = f$). Now if we construct an unfolding of g by unfolding by the same terms and let u_1, u_2 denote the unfolding parameters, then we get an unfolding G which has nontransverse intersecting fold points at $\{(x,0,0,0), (-x,0,0,0)\}$. So $(j^5)^2G$ is unstable and infinitely close to $(j^5)^2F$ at $\{(x,0,0,0), (-x,0,0,0)\}$. F will consequently not satisfy the hypothesis of (0.2). This shows the need for condition (1.7.3).

§2. Beginning the proof of (0.1)

We start this section with the following lemma.

(2.1) Lemma. *Assume that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ satisfies the hypothesis of (0.1), then f also satisfies the conditions (1.1) and (1.7) with $s = p+1$.*

Proof. Since $\mathbb{R}^p \times W^k(n,p) \subset NT^{(k,1)}$ and $D_1(f) = \{0\}$, (1.1.1) and (1.7.1) follow immediately. Recall from the construction of the canonical stratification in [8], that the set of critical jets in $J^k(n,p) - W^k(n,p)$ is a union of strata in $S^k(n,p)$. Therefore a map $f : \mathbb{R}^n \to \mathbb{R}^p$ cannot be multitransverse to the canonical stratification at a set consisting of $p+1$ singular points with a common image. It follows that $\Delta\mathbb{R}^p \times (\Sigma(n,p))^{p+1} \subset NT^{(k,p+1)}$ and (1.1.2) and (1.7.3) will follow from the hypothesis of (0.1) with $s = p$. The proof of (1.7.2) is more involved. To this end, assume that there exists a sequence $\{x_n\}$, $x_n \to 0$ such that $d(j^1 f(x_n), \mathbb{R}^p \times \Sigma(n,p))$ is flat along $d(x_n, \Sigma(f))$ (we say that a sequence $y_n \to 0$ is flat along another sequence $z_n \to 0$ if $|y_n| = o(|z_n|^m)$ for every $m \in \mathbb{N}$). Define a finite product stratification of $\mathbb{R}^p \times \Sigma(n,p) \subset J^k(n,p)$ trivial in the $\mathbb{R}^p$ direction, by letting the strata in $\Sigma(n,p) - W^k(n,p)$ be strata in $S^k(n,p)$ and choosing some finite (arbitrary) stratification of the algebraic set $W^k(n,p)$. Using [15, Lemma 2.2], we can find a map $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $j^\infty g(0) = j^\infty f(0)$, and a sequence $y_n \to 0$ such that $j^k g$ is not transverse to the stratification of $\mathbb{R}^p \times \Sigma(n,p)$ described above at the sequence $\{y_n\}$. Since $\|j^k f(x) - j^k g(x)\|$ is flat along $\|x\|$, it follows from (1.7.1) that $j^k g(y_n)$ cannot hit $W^k(n,p)$. So, $j^k g$ hits S^k non-transversally at y_n. On the other hand, since $d(j^k f(x), NT^{(k,1)}) \geq C_1 \|x\|^{\alpha_1}$, and $\|j^k f(x) - j^k g(x)\|$ is flat along $\|x\|$, we must have $d(j^k g(x), NT^{(k,1)}) \geq \frac{C_1}{2} \|x\|^{\alpha_1}$, and $j^k g$ must hit S^k transversally outside 0. We have thus obtained a contradiction. □

From (2.1) and the results of §1, it follows that we can construct an unfolding

$$F : \mathbb{R}^n \times \mathbb{R}^c \to \mathbb{R}^p \times \mathbb{R}^c$$

of f such that this unfolding satisfies the hypothesis of (0.2) (with respect to the subspace $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+c}$).

Let k_0 be a positive integer. Let $R = j^{\gamma(k_0)} F(0)$ and let G be defined by $G(x,u,t) = (F(x,u) + t(R-F)(x,u), t)$. We can then apply the conclusion of (0.2) to the unfolding

G. Let l and k be the mappings we get by restricting the maps L and K of (0.2) to the level $t = 1$. By (2) of (0.2) we then obtain a commutative diagram

$$\begin{array}{ccc} H(\lambda) & \xrightarrow{\ l\ } & U \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle R} \\ \mathbb{R}^p \times \mathbb{R}^c & \xleftarrow{\ k\ } & V. \end{array} \tag{2.2}$$

Here U is a neighborhood of 0 in $\mathbb{R}^n \times \mathbb{R}^c$ such that $l(H(\lambda)) \subset U$ and $R(U) \subset V$. Now, if $k_0 \geq \lambda$, it will follow from (3)(i) of (0.2) that $l(H(\lambda))$ contains a representative of the hornshaped neighborhood $H(\lambda')$, $\lambda' > \lambda$. We may assume that λ' is an integer which may be chosen independent of our particular choice of k_0, and that the representative is given by

$$H(\lambda') = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^c \mid \|u\| < \|x\|^{\lambda'} \text{ and } \|(x, u)\| < \epsilon\}$$

for some ϵ. From (0.2)(1) and (2.2), it follows that R becomes multistable at finite sets of points in $H(\lambda')$. In fact, it is not hard to see that λ' and ϵ can be chosen such that R is multistable at finite sets of points in $\overline{H(\lambda')} - \{0\}$ and that $R^{-1}(0) \cap \Sigma(R) \cap \overline{H(\lambda')} = \{0\}$.

Let C denote the set $R(\Sigma(R) \cap (\overline{H(\lambda')} - H(\lambda')))$. It follows from the Tarski-Seidenberg Theorem that C and consequently $\mathbb{R}^p \times \mathbb{R}^c - C$ are semialgebraic sets. Assuming that 0 is a critical point, we get that $0 \in C$. Let $K = R^{-1}(C)$. Then $H(\lambda') - K$ is a semialgebraic set in $\mathbb{R}^n \times \mathbb{R}^c$. We now have the following lemma

(2.3) Lemma. *$R|H(\lambda') - K : H(\lambda') - K \to \mathbb{R}^p \times \mathbb{R}^c - C$ is an infinitesimally stable mapping.*

Proof. It is obvious from our definitions that the map in (2.3) restricted to its critical set is a proper map. Since R is multistable at finite subsets of $H(\lambda') - K$, the conclusion of (2.3) follows from [7, Proposition 5.1] □.

Since $R|H(\lambda') - K : H(\lambda') - K \to \mathbb{R}^p \times \mathbb{R}^c - C$ is an infinitesimally stable mapping, we can use the results of [8] to stratify this map.

In §10 of Mather's article, Mather constructs a collection S_1 of infinitesimally stable mapping classes. If $f : N \to P$ is an infinitesimally stable mapping between manifolds N and P, S_1 induces three Whitney stratifications $\mathcal{S}_i(f)$, $i = 1, 2, 3$. The properties of these stratifications are described in [8]. Applying this to the map of Lemma (2.3), we get induced three stratifications which we will denote by $\mathcal{S}_i(R)$, $i = 1, 2, 3$. $\mathcal{S}_i(R)$, $i = 1, 3$ are stratifications of $H(\lambda') - K$ and $\mathcal{S}_2(R)$ is a stratification of $\mathbb{R}^p \times \mathbb{R}^c - C$. These stratifications will be an essential ingredient of the proof of (0.2). However, before we can make any use of these stratifications we need to prove the following technical lemma:

(2.4) Lemma. *Let $k_0, R, \lambda', \epsilon$ and C be as described above. Then, for each choice of k_0, λ' and ϵ, we can choose ϵ' such that if*

$$H(2\lambda') = \{(y, u) \in \mathbb{R}^p \times \mathbb{R}^c |\ \|u\| < \|y\|^{2\lambda'} \text{ and } \|(y, u)\| < \epsilon'\},$$

then $H(2\lambda') \subset \mathbb{R}^p \times \mathbb{R}^c - C$.

Proof. Recall that

$$\begin{aligned} C = R(\Sigma(R) \cap (\overline{H(\lambda')} - H(\lambda'))) = \\ R(\Sigma(R) \cap \{(x,u) |\ \|u\| = \|x\|^{\lambda'} \text{ and } \ \|(x,u)\| < \epsilon\}) \\ \cup R(\Sigma(R) \cap \{(x,u) |\ \|u\| \leq \|x\|^{\lambda'} \text{ and } \ \|(x,u)\| = \epsilon\}). \end{aligned}$$

Let $M = \max\{\sup_{\|(x,u)\| \leq 1} \|DR(x,u)\|, 1\}$, where $\|DR(x,u)\|$ is the operator norm. We may assume that ϵ occurring in the definition of the representative $H(\lambda')$ also is chosen such that $\epsilon \leq 1/M^2 \leq 1$; hence $\|DR(x,u)\| \leq M$ when $\|(x,u)\| \leq \epsilon$. Now, if $\|(x,u)\| \leq \epsilon$, we get that $\|x\| \leq 1/M^2$ and hence $(y,u) = R(x,u) = (R_u(x), u)$ satisfies $\|y\| \leq M\|x\| \leq \|x\|^{1/2}$. So $\|y\|^2 \leq \|x\|$. So if $\|u\| = \|x\|^{\lambda'}$, we get that $\|u\| \geq \|y\|^{2\lambda'}$ and consequently that

$$H(2\lambda') \subset \sim R(\Sigma(R) \cap \{(x,u) |\ \|u\| = \|x\|^{\lambda'} \text{ and } \ \|(x,u)\| < \epsilon\})$$

($\sim$ denotes complement). Now, recall that we had $\Sigma(R) \cap \overline{H(\lambda')} \cap R^{-1}(0) = \{0\}$. So

$$\Sigma(R) \cap \overline{H(\lambda')} \cap R^{-1}(0) \cap \{(x,u) \mid \|(x,u)\| = \epsilon\} = \varnothing.$$

The compact set $R(\Sigma(R) \cap \overline{H(\lambda')} \cap \{(x,u) \mid \|(x,u)\| = \epsilon\})$ must therefore avoid 0 in $\mathbb{R}^p \times \mathbb{R}^c$, and we can consequently find $\epsilon' > 0$ such that $R(\Sigma(R) \cap \overline{H(\lambda')} \cap \{(x,u) \mid \|(x,u)\| = \epsilon\})$ does not intersect $\{(y,u) | \|(y,u)\| < \epsilon'\}$. So with this ϵ' we also get that

$$H(2\lambda') \subset \sim R(\Sigma(R) \cap \{(x,u) |\ \|u\| \leq \|x\|^{\lambda'} \text{ and } \ \|(x,u)\| = \epsilon\}),$$

and we obtain the conclusion of the lemma. □

Let $i : \mathbb{R}^n \to \mathbb{R}^{n+c} \times \mathbb{R}$ and $j : \mathbb{R}^p \to \mathbb{R}^{p+c} \times \mathbb{R}$ be the standard imbeddings. Since k of (2.2) is a C^{k_0} diffeomorphism which is C^∞ outside 0, we can, restricting to neighborhoods O and W of 0 in $\mathbb{R}^n$ and $\mathbb{R}^p$, compose i and j with l and k^{-1} respectively, and we obtain a commutative diagram

(2.5)
$$\begin{array}{ccc} O & \xrightarrow{\tilde{i}} & \mathbb{R}^{n+c} \\ \downarrow f & & \downarrow R \\ W & \xrightarrow{\tilde{j}} & \mathbb{R}^{p+c} \end{array}$$

where $\tilde{i} = l \circ i$ and $\tilde{j} = k^{-1} \circ j$. It follows from (0.2) (3) (i) and (ii) that $\tilde{i}$ and $\tilde{j}$ will be close to i and j if k_0 is chosen large. From this, it follows that W can be chosen such that $\tilde{j}(W - \{0\}) \subset H(2\lambda')$ and it follows from (2.4) and (2.5) that O can be chosen such that

$\tilde{i}(O - f^{-1}(0)) \subset H(\lambda') - K$ (provided k_0 is sufficiently large). We therefore obtain the following commutative diagram

$$\begin{array}{ccc} O - f^{-1}(0) & \xrightarrow{\tilde{i}} & H(\lambda') - K \\ \Big\downarrow f & & \Big\downarrow R \\ W - \{0\} & \xrightarrow{\tilde{j}} & \mathbb{R}^{p+c} - C\,. \end{array} \tag{2.6}$$

Here, all the maps are C^∞.

Now, consider the sets $B(i)(R)$, $i = 2, 3$, defined below:

$$B(i) = \{(z, H) | H \text{ is not transverse to } T_z X \text{ where } X \text{ is the stratum in } \mathcal{S}_i(R) \text{ passing through } z\},$$

where $(z, H) \in (\mathbb{R}^{p+c} - C) \times L(\mathbb{R}^p, \mathbb{R}^{p+c})$ when $i = 2$ and $(z, H) \in (H(\lambda') - K) \times L(\mathbb{R}^n, \mathbb{R}^{n+c})$ when $i = 3$. The following technical lemma is crucial in the proof of (0.1).

(2.7) Lemma. *Let $\beta > 0$. Then there exists $\alpha = \alpha(\beta) > 0$ and neighborhoods O and W as in (2.6) such that condition* (a) *and* (b) *below hold.*

(a)

$$(i = 3) \quad d((\tilde{i}(x), D\tilde{i}(x)), B(3)) \geq \|x\|^\alpha \text{ when } x \in O \cap \sim \{x \,|\, d(x, f^{-1}(0)) \leq \|x\|^\beta\} \text{ and}$$

$$(i = 3) \quad d((\tilde{j}(y), D\tilde{j}(y)), B(2)) \geq \|y\|^\alpha \text{ when } y \in W - \{0\}.$$

(b) Let $\mathcal{L}$ be a relatively closed set in $H(\lambda') - K$ or $\mathbb{R}^{p+c} - C$ whose closure in $\mathbb{R}^{n+c}$ or $\mathbb{R}^{p+c}$ contains 0, and assume that $\mathcal{L}$ is a union either of $\mathcal{S}_3(R)$ or $\mathcal{S}_2(R)$ strata. Then, in the case $\mathcal{L} \subset H(\lambda') - K$, we have that

$$d(\tilde{i}(x), \mathcal{L}) \geq \|x\|^\alpha d(\tilde{i}(x), \operatorname{im} \tilde{i} \cap \overline{\mathcal{L}}),$$

for $x \in O \cap \sim \{x | d(x, f^{-1}(0)) \leq \|x\|^\beta\}$, and if $\mathcal{L} \subset \mathbb{R}^{p+c} - C$, we have that

$$d(\tilde{j}(y), \mathcal{L}) \geq \|y\|^\alpha d(\tilde{j}(y), \operatorname{im} \tilde{j} \cap \overline{\mathcal{L}}),$$

for $y \in W - \{0\}$.

The proof of (2.7) is postponed to §4. To be able to prove (2.7), we will however need some results concerning metric properties of semialgebraic stratifications which will be given in the following section.

§3. Some metric properties of semialgebraic stratifications

Let V, W be linear subspaces of a common Euclidean space. We define the distance, $d(V,W)$ between V and W by

$$d(V,W) = \sup_{\substack{v\in V\\ \|v\|=1}} \inf_{w\in W} \|v-w\|.$$

(Note that in general, $d(\cdot,\cdot)$ is not symmetric, since $V \subset W$ implies $d(V,W)=0 \neq d(W,V)$; on the other hand $d(\cdot,\cdot)$ restricts to a metric on Grassmannians (see e.g. [14]).) We start with the following lemma.

(3.1) Lemma. *Let A be a semialgebraic set $\mathbb{R}^N$ and let $\mathcal{S}$ be a Whitney (a)-regular stratification of A with semialgebraic strata. Let x_0 be a point in A. Then there exist a neighborhood U of x_0 and positive constants α and β such that the following hold. Assume that $x \in U \cap X$ and $y \in U \cap Y$, where X and Y are strata in $\mathcal{S}$ with $X \subset \overline{Y}$, and that $\|x-y\| \leq d(x, \overline{X}-X)^\alpha$. Then $d(T_xX, T_yY)^\beta \leq \|x-y\|$.*

To prove (3.1) we need the following sublemma.

(3.2) Sublemma. *Let $A, \mathcal{S}$ and x_0 be as in (3.1). Then there exist a neighborhood U of x_0 and positive constants α and β such that the following hold. Assume that x and $\bar{x}$ are in $U \cap X$, where X is some stratum in $\mathcal{S}$, and that $\|x-\bar{x}\| \leq d(x,\overline{X}-X)^\alpha$. Then $d(T_xX, T_{\bar{x}}X)^\beta \leq \|x-\bar{x}\|$.*

Proof of (3.2). Suppose the conclusion of (3.2) is wrong. Then there exist sequences $\{x_n\}$ and $\{\bar{x}_n\}$ tending to x_0 such that $\|x_n-\bar{x}_n\|$ is flat along $d(x_n,\overline{X}-X)$ and also along $d(T_{x_n}X, T_{\bar{x}_n}X)$. Using compactness of the Grassmannian, we may suppose that $T_{x_n}X \to F$ and $T_{\bar{x}_n}X \to H$ as $n\to\infty$, where F and H are in the Grassmannian G of $\dim X$ dimensional subspaces in $\mathbb{R}^N$. Consider the sets

$$B = \overline{\{(x,\bar{x},L,T_{\bar{x}}X) | (x,\bar{x}) \in X\times X, L\in G\}}$$

and

$$C = \overline{\{(x,x,T_xX,L) | x\in X, L \in G\}}.$$

Clearly $(x_0,x_0,F,H) \in B\cap C$. Since B and C are semialgebraic sets and therefore regularly situated, there exists a (compact) neighborhood U of (x_0,x_0,F,H) and positive constants M,γ such that $d((x,\bar{x},L,\bar{L}), C\cap U) \geq M d((x,\bar{x},L,\bar{L}), B\cap C\cap U)^\gamma$ for $(x,\bar{x},L,\bar{L}) \in B\cap U$. Now, if n is sufficiently large, we have that $(x_n,\bar{x}_n,T_{x_n}X,T_{\bar{x}_n}X) \in B\cap U$ and also that $(x_n,x_n,T_{x_n}X,T_{\bar{x}_n}X) \in C\cap U$. So we get that

$$(*) \qquad \|x_n-\bar{x}_n\| \geq M d((x_n,\bar{x}_n,T_{x_n}X,T_{\bar{x}_n}X), B\cap C\cap U)^\gamma.$$

Let $(\tilde{x}_n,\tilde{x}_n,F_n,H_n) \in B\cap C\cap U$ be such that

$$d((x_n,\bar{x}_n,T_{x_n}X,T_{\bar{x}_n}X), B\cap C\cap U) = d((x_n,\bar{x}_n,T_{x_n}X,T_{\bar{x}_n}X),(\tilde{x}_n,\tilde{x}_n,F_n,H_n)).$$

Then it follows from $(*)$ that $\|x_n-\bar{x}_n\| \geq M\|x_n-\tilde{x}_n\|^\gamma$, and since $\|x_n-\bar{x}_n\|$ is flat along $d(x_n,\overline{X}-X)$, $\|x_n-\tilde{x}_n\|$ is also flat along $d(x_n,\overline{X}-X)$. We must therefore have that $\tilde{x}_n \in X$ for sufficiently large n. So we must have $F_n = T_{\tilde{x}_n}X = H_n$. From (*) we also deduce that $\|x_n - \tilde{x}_n\| \geq Md(T_{x_n}X, T_{\tilde{x}_n}X)^\gamma$. This proves that $\|x_n - \bar{x}_n\|$ is not flat along $d(T_{x_n}X, T_{\tilde{x}_n}X)$. A similar argument shows that $\|x_n - \bar{x}_n\|$ is not flat along $d(T_{\bar{x}_n}X, T_{\tilde{x}_n}X)$ either. Now it follows from the triangle inequality that $\|x_n - \bar{x}_n\|$ is not flat along $d(T_{x_n}X, T_{\bar{x}_n}X)$ which gives us a contradiction. □

Proof of (3.1). Suppose the conclusion of (3.1) is wrong. Then there exists a sequence $(x_n, y_n) \to (x_0, x_0)$, $(x_n, y_n) \in X \times Y$ such that $\|x_n - y_n\|$ is flat along $d(x_n, \overline{X} - X)$ and also along $d(T_{x_n}X, T_{y_n}Y)$. Let us suppose that $T_{x_n}X \to F$ and $T_{y_n}Y \to H$ in the appropriate Grassmannians. Let $B = \overline{\{(y, T_yY) | y \in Y\}}$ and $C = \overline{\{(x, L) | x \in X, L \in G\}}$ where G is the Grassmannian of $\dim Y$ planes in $\mathbb{R}^N$. Since B and C are semialgebraic, we have that B and C are regularly situated. So there exist $M, \gamma > 0$, and a neighborhood V of $(x_0, H) \in B \cap C$ such that

$$(1) \qquad d((y, L), C \cap V) \geq Md((y, L), B \cap C \cap V)^\gamma$$

for each $(y, L) \in B \cap V$. Then for sufficiently large n, we have that $(y_n, T_{y_n}Y) \in B \cap V$ and $(x_n, T_{y_n}Y) \in C \cap V$. Let $(\bar{x}_n, L_n)$ be a point in $B \cap C \cap V$ such that

$$d((y_n, T_{y_n}Y), B \cap C \cap V) = d((y_n, T_{y_n}Y), (\bar{x}_n, L_n)).$$

It follows from (1) that

$$d((y_n, T_{y_n}Y), (x_n, T_{y_n}Y)) \geq Md((y_n, T_{y_n}Y), (\bar{x}_n, L_n))^\gamma.$$

Taking components of this inequality, we get that

$$(2) \qquad \|y_n - x_n\| \geq M\|y_n - \bar{x}_n\|^\gamma$$

and

$$(3) \qquad \|y_n - x_n\| \geq Md(T_{y_n}Y, L_n)^\gamma.$$

Since we have supposed that $\|y_n - x_n\|$ is flat along $d(x_n, \overline{X} - X)$, (2) proves that $\|y_n - \bar{x}_n\|$ is also flat along $d(x_n, \overline{X} - X)$. So, using the triangle inequality $\|x_n - \bar{x}_n\|$ is flat along $d(x_n, \overline{X} - X)$. So, for large n, $\bar{x}_n$ is a point in X not in $\overline{X} - X$. From the definition of B it follows that each L_n is a limit of tangent spaces for a sequence of points in Y tending to $\bar{x}_n$. It follows from Whitney condition (a), that $T_{\bar{x}_n}X \subset L_n$. Since $\|x_n - \bar{x}_n\|$ is flat along $d(x_n, \overline{X} - X)$, it follows from (3.2) above that $\|x_n - \bar{x}_n\|$ is not flat along $d(T_{x_n}X, T_{\bar{x}_n}X)$. Now, since $T_{\bar{x}_n}X \subset L_n$, we get that $d(T_{\bar{x}_n}X, T_{y_n}Y) \leq d(L_n, T_{y_n}Y)$, and this together with (3) implies $\|x_n - y_n\|$ is not flat along $d(T_{\bar{x}_n}X, T_{y_n}Y)$. On the other hand, using (2) and the triangle inequality, we get that $\|x_n - y_n\|$ is not flat along $\|x_n - \bar{x}_n\|$, and we deduce that $\|x_n - y_n\|$ is not flat along $d(T_{x_n}X, T_{\bar{x}_n}X)$ either. Using the definition of $d(\cdot, \cdot)$, it is not hard to deduce that

$$d(T_{x_n}X, T_{y_n}Y) \leq d(T_{x_n}X, T_{\bar{x}_n}X) + d(T_{\bar{x}_n}X, T_{y_n}Y).$$

Therefore, we get, putting everything together, that $\|x_n - y_n\|$ is not flat with respect to $d(T_{x_n}X, T_{y_n}Y)$. This gives us the desired contradiction. □

The next lemma is in the same spirit as (3.1) and (3.2).

(3.3) Lemma. *Let $X \subset \mathbb{R}^N$ be a semialgebraic set and a C^∞ manifold. Let $R : \mathbb{R}^N \to \mathbb{R}^P$ be a polynomial map such that $R|X$ is an immersion. Let $x_0 \in \overline{X} - X$. Then there exist a neighborhood U of x_0 and a constant $\alpha > 0$, such that*

$$\|DR(x)(v)\| \geq d(x, \overline{X} - X)^\alpha$$

holds for any $x \in U \cap X$ and any unit vector $v \in T_x X$.

Proof. To obtain a contradiction, assume that there exist a sequence of points $\{x_n\} \subset X$, $x_n \to x_0$, and a sequence of unit vectors $\{v_n\}$, $v_n \in T_{x_n}X$ such that $\|DR(x_n)(v_n)\|$ is flat along $d(x_n, \overline{X} - X)$. We may assume that $v_n \to v_0$ for some unit vector $v_0 \in \mathbb{R}^N$. Consider the sets A and B defined by

$$A = \overline{\{(x, v, DR(x)(v)) | (x, v) \in X \times T_x X, \|v\| = 1\}},$$
$$B = \overline{\{(x, v, 0) | (x, v) \in X \times T_x X, \|v\| = 1\}}.$$

By standard arguments, these sets are semialgebraic. Moreover, by the assumptions we have made, $(x_0, v_0, 0) \in A \cap B$. Now we have that

$$\|DR(x_n)(v_n)\| = \|(x_n, v_n, DR(x_n)(v_n) - (x_n, v_n, 0)\| \geq d((x_n, v_n, DR(x_n)(v_n)), B).$$

For each n, let $(\tilde{x}_n, \tilde{v}_n, 0) \in A \cap B$ be a point minimizing $d((x_n, v_n, DR(x_n)(v_n)), A \cap B)$. Since $(\tilde{x}_n, \tilde{v}_n, 0) \in A$ and $R|X$ is an immersion, we must have $\tilde{x}_n \in \overline{X} - X$. It is also clear that $(\tilde{x}_n, \tilde{v}_n, 0) \to (x_0, v_0, 0)$. Since semialgebraic sets are regularly situated, we can pick a neighborhood V of $(x_0, v_0, 0)$ and constants $M, \gamma > 0$ such that

$$d((x, v, DR(x)(v)), B) \geq Md((x, v, DR(x)(v)), A \cap B)^\gamma$$

for $(x, v, DR(x)(v)) \in A \cap V$. In particular, we get that

$$\|DR(x_n)(v_n)\| \geq M\|(x_n, v_n, DR(x_n)(v_n)) - (\tilde{x}_n, \tilde{v}_n, 0)\|^\gamma \geq M\|x_n - \tilde{x}_n\|^\gamma \geq d(x_n, \overline{X} - X)^\gamma.$$

We have therefore obtained a contradiction. □

We close this section with another lemma which will be used in §4.

(3.4) Lemma. *Let $R : \mathbb{R}^N \to \mathbb{R}^P$ be a polynomial map where $N > P$. Let $x_0 \in \Sigma(R)$. Then there exist constants $\alpha, \beta > 0$, and a neighborhood U of x_0 such that the following conditions are satisfied.*

(a) Let $x, y \in U \cap (\mathbb{R}^N - \Sigma(R))$. Assume that $\|x - y\| < d(x, \Sigma(R))^\alpha$. Then

$$d(\ker DR(x), \ker DR(y))^\beta \leq \|x - y\|.$$

(b) With the same assumptions as in (a), we also have that

$$d((\ker DR(x))^\perp, (\ker DR(y))^\perp)^\beta \leq \|x - y\|.$$

(c) For any unit vector $v \in (\ker DR(x))^\perp$ we have that $\|DR(x)(v)\| > d(x, \Sigma(R))^\beta$.

Proof. (a) and (b) are proven in the same manner as (3.2), replacing X by $\mathbb{R}^N - \Sigma(R)$ and $T_x X$ by $\ker DR(x)$ or $(\ker DR(x))^\perp$ respectively. (Actually $d(V, W) = d(V^\perp, W^\perp)$, so (*b*) is equivalent to (*a*)).

(c) is proven in the same manner as (3.3), replacing X by $\mathbb{R}^N - \Sigma(R)$ and $T_x X$ by $(\ker DR(x))^\perp$. □

§4. Proof of (2.7)

We start this section with a technical lemma.

(4.1) Lemma. *Let $R, H(\lambda'), C, K$ and $\mathcal{S}_i(R)$ be as described in §2. There exist constants $\alpha, \tilde{\lambda} > 0$, and a representative $H(\tilde{\lambda})$ of the set germ*

$$H(\tilde{\lambda}) = \{(y,u) \in \mathbb{R}^{p+c} | \|u\| < \|y\|^{\tilde{\lambda}}\}$$

such that the following conditions are satisfied: $H(\tilde{\lambda}) \cap C = \varnothing$, and if

$$(y,u) \in R(\Sigma(R) \cap H(\lambda')) \cap H(\tilde{\lambda})$$

and

$$\{(x_1,u),(x_2,u)\} \subset R^{-1}(y,u) \cap \Sigma(R) \cap H(\lambda')$$

with $x_1 \neq x_2$, then

$$\|x_1 - x_2\| \geq d((y,u), \overline{Y} - Y)^{\alpha},$$

where Y is the connected component of the $\mathcal{S}_2(R)$ stratum in $\mathbb{R}^{p+c} - C$ that (y,u) belongs to, and $\overline{Y}$ denotes the closure of Y in $\mathbb{R}^{p+c}$.

Proof. The assertion that $H(\tilde{\lambda}) \cap C = \varnothing$ follows from (2.4), taking $\tilde{\lambda} > 2\lambda'$. Let us therefore assume this and prove the remaining part of the conclusion of (4.1).

Since the stratification $\mathcal{S}_3(R)$ of $H(\lambda') - K$ is finite with semialgebraic strata, we can find a neighborhood U of 0 in $\mathbb{R}^{n+c}$ such that if X is a connected component of a critical-point stratum in $\mathcal{S}_3(R)$ with $X \cap U \neq \varnothing$, then $0 \in \overline{X}$. Let X_1 and X_2 be two (not necessarily distinct) such components. Assume that $\#\{R^{-1}(R(z)) \cap (X_1 \cup X_2)\} > 1$ for some $z \in X_1 \cup X_2$. Then it follows from the way $\mathcal{S}_3(R)$ is constructed (see [8]) that $\#\{R^{-1}(R(z)) \cap (X_1 \cup X_2)\} > 1$ for any $z \in X_1 \cup X_2$.

Let V_1 and V_2 be the semialgebraic sets in $(\mathbb{R}^{n+c})^2$ defined by

$$V_1 = \overline{\{(z_1,z_2) \in (X_1 \cup X_2) \times (X_1 \cup X_2) | R(z_1) = R(z_2)\} - \Delta(X_1 \cup X_2)}$$

and $V_2 = \overline{\Delta(X_1 \cup X_2)}$. Since we have assumed $0 \in \overline{X_i}$ for $i = 1,2$, we must have that $(0,0) \in V_1 \cap V_2$. Since V_1 and V_2 are regularly situated, we can find a neighborhood $U_1 \subset U$ of 0 and a constant $\alpha_1 > 0$ such that

$$d((z_1,z_2), V_2) \geq d((z_1,z_2), V_1 \cap V_2)^{\alpha_1}, \tag{4.1.1}$$

for every $(z_1,z_2) \in U_1 \cap V_1$. Since we have only finitely many strata, we can suppose that U_1 and α_1 are chosen such that (4.1.1) holds for any such pair of strata $X_i, i = 1,2$.

Recall from §2, that we have chosen $H(\lambda')$ such that $R^{-1}(0) \cap \Sigma(R) \cap \overline{H(\lambda')} = \{0\}$. It follows that $R(\Sigma(R) \cap H(\lambda') \cap (\sim U_1))$ is bounded away from 0. We can therefore also choose our representative $H(\tilde{\lambda})$ such that $R^{-1}(H(\tilde{\lambda})) \cap \Sigma(R) \cap H(\lambda') \subset U_1$.

Now, let $(y,u), (x_1,u), (x_2,u)$ and Y be as described in the hypothesis of this lemma. Since $H(\tilde{\lambda}) \cap C = \varnothing$, we have $\{(x_1,u),(x_2,u)\} \subset H(\lambda') - K$, and thus we can let $X_i, i = 1,2$ denote the connected components of the strata in $\mathcal{S}_3(R)$ such that $(x_i,u) \in X_i$. From the construction of the stratifications $\mathcal{S}_i(R)$ given in [8], it follows that $R|X_i$, $i = 1,2$ are immersions, and that $\dim X_i = \dim Y$, $i = 1,2$. So the latter implies that no one of the two components X_i can be contained in the closure of the other unless actually $X_1 = X_2$. Let V_j, $j = 1,2$, be the semialgebraic sets we constructed above from the pair X_i, $i = 1,2$. Let $(z,z) \in V_1 \cap V_2$ be such that

$$d(((x_1,u),(x_2,u)), V_1 \cap V_2) = \|((x_1,u),(x_2,u)) - (z,z)\|.$$

Since z is the limit of a sequence of double points from $X_1 \cup X_2$, and each $R|X_i$ is an immersion, it follows that either $z \in \overline{X_1} - X_1$ or $z \in \overline{X_2} - X_2$. Now, either $z \in K$ or z belongs to a stratum in $\mathcal{S}_3(R)$ of dimension strictly less than $\dim X_i$. In the latter case this stratum is mapped to a stratum in $\mathcal{S}_2(R)$ with dimension strictly less than $\dim Y$. So, in any case we must have $R(z) \in \overline{Y} - Y$. Since $R^{-1}(y,u) \cap \Sigma(R) \cap H(\lambda') \subset U_1$, we get from (4.1.1) that

$$\text{(4.1.2)}\quad \|((x_1,u),(x_2,u)) - ((x_2,u),(x_2,u))\| = \|x_1 - x_2\| \geq \|((x_1,u),(x_2,u)) - (z,z)\|^{\alpha_1}.$$

Since the derivative of R is bounded, it is clear that $H(\lambda')$ can be chosen such that $\|z_1 - z_2\| \geq \|R(z_1) - R(z_2)\|^2$ for any $z_1, z_2 \in \overline{H(\lambda')}$. This and (4.1.2) imply that

$$\text{(4.1.3)}\qquad \|x_1 - x_2\| \geq \|(y,u) - R(z)\|^{2\alpha_1} \geq d((y,u), \overline{Y} - Y)^{2\alpha_1},$$

and (4.1) follows choosing $\alpha = 2\alpha_1$. □

Recall that in the diagram (2.6) we assumed that k_0 was chosen so large that

$$\tilde{\jmath}(W - \{0\}) \subset H(2\lambda') \subset \mathbb{R}^{p+c} - C.$$

Since it follows from the proof of (4.1) that $\tilde{\lambda}$ is independent of the particular k_0 (because λ' is independent of k_0), it is clear that k_0 can be chosen such that this inclusion also is valid if $2\lambda'$ is replaced by the exponent $\tilde{\lambda}$ of (4.1). Since C avoids $H(\tilde{\lambda})$ we may also assume that k_0 is chosen so large that $y \in W - \{0\}$ implies that $d(\tilde{\jmath}(y), C) > \|y\|^\delta$ for some constant $\delta > 0$. With these additional assumptions on $\tilde{\jmath}$ we will now prove part (a) of (2.7) in the case $i = 2$. To obtain a contradiction assume that (a) of (2.7) does not hold for any $\alpha > 0$. We can then find two sequences $\{y_n\} \subset \mathbb{R}^p$ and $\{((\tilde{y}_n, \tilde{u}_n), H_n)\} \subset B(2)$ such that $d((\tilde{\jmath}(y_n), D\tilde{\jmath}(y_n)), ((\tilde{y}_n, \tilde{u}_n), H_n))$ is flat along $\|y_n\|$. Since our stratification is finite, we may assume that the sequence $\{(\tilde{y}_n, \tilde{u}_n)\}$ is contained in a connected component of a single $\mathcal{S}_2(R)$ stratum Y, and we may suppose that the dimension of this stratum is chosen as small as possible. Now we will first prove the following claim.

(4.2) Claim. *There exists $\alpha > 0$ such that $d((\tilde{y}_n, \tilde{u}_n), \overline{Y} - Y) \geq \|y_n\|^\alpha$ for sufficiently large n.*

Proof of (4.2). Suppose that (4.2) does not hold. Then there exists a sequence $\{(\hat{y}_n, \hat{u}_n)\}$ in $\overline{Y} - Y$ such that $\|(\tilde{y}_n, \tilde{u}_n) - (\hat{y}_n, \hat{u}_n)\|$ is flat along $\|y_n\|$. From the assumptions on $\tilde{\jmath}$ it is clear that $(\hat{y}_n, \hat{u}_n)$ does not belong to C for n large. We may therefore assume that $(\hat{y}_n, \hat{u}_n)$ belongs to a connected component of an $\mathcal{S}_2(R)$-stratum in $\overline{Y} - Y$ chosen of dimension as small as possible. Denote this component by Y'. Since the dimension of Y' is minimal, $d((\hat{y}_n, \hat{u}_n), \overline{Y'} - Y')$ cannot be flat along $\|y_n\|$, and $\|(\tilde{y}_n, \tilde{u}_n) - (\hat{y}_n, \hat{u}_n)\|$ is consequently flat along $d((\hat{y}_n, \hat{u}_n), \overline{Y'} - Y')$. Since $\mathcal{S}_2(R)$ is a Whitney regular stratification it follows from (3.1) that $d(T_{(\hat{y}_n, \hat{u}_n)}Y', T_{(\tilde{y}_n, \tilde{u}_n)}Y)$ is flat with respect to $\|y_n\|$. Let V_n be the image of $T_{(\hat{y}_n, \hat{u}_n)}Y'$ under the orthogonal projection onto $T_{(\tilde{y}_n, \tilde{u}_n)}Y$. Since we have that $((\tilde{y}_n, \tilde{u}_n), H_n) \in B(2)$, we have that H_n is not transverse to $T_{(\tilde{y}_n, \tilde{u}_n)}Y$ and therefore not to V_n either. Since $d(T_{(\hat{y}_n, \hat{u}_n)}Y', V_n) = d(T_{(\hat{y}_n, \hat{u}_n)}Y', T_{(\tilde{y}_n, \tilde{u}_n)}Y)$ is flat along $\|y_n\|$, it is not hard to see that we can find linear maps $\hat{H}_n$ not transverse to $T_{(\hat{y}_n, \hat{u}_n)}Y'$, such that $d(\hat{H}_n, H_n)$ is flat along $\|y_n\|$. We therefore have that $((\hat{y}_n, \hat{u}_n), \hat{H}_n) \in B(2)$. From everything above it follows that $d((\tilde{j}(y_n), D\tilde{j}(y_n)), ((\hat{y}_n, \hat{u}_n), \hat{H}_n))$ is flat along $\|y_n\|$. Since $(\hat{y}_n, \hat{u}_n) \in Y' \subset \overline{Y} - Y$ and consequently that $\dim Y' < \dim Y$, we have obtained a contradiction to the assumptions that $\{(\tilde{y}_n, \tilde{u}_n)\}$ was chosen from a stratum with as small dimension as possible. This proves (4.2). □

To proceed with the proof of (2.7) (a) in the case $i = 2$, we let, for each n,

$$R^{-1}(\tilde{y}_n, \tilde{u}_n) \cap \Sigma(R) \cap H(\lambda') = \{\tilde{z}_n^1, \ldots, \tilde{z}_n^l\}.$$

Since we have assumed that all $(\tilde{y}_n, \tilde{u}_n)$ belongs to the same $\mathcal{S}_2(R)$ stratum, l is independent of n. Let $\tilde{z}_n = (\tilde{z}_n^1, \ldots, \tilde{z}_n^l) \in H(\lambda')^l$. We now have the following.

(4.3) Claim. *There exists $\alpha > 0$ such that for n sufficiently large, the following two conditions are satisfied:*
(i) $\|\tilde{z}_n^i - \tilde{z}_n^j\| \geq \|\tilde{z}_n\|^\alpha$ for $i \neq j, 1 \geq i, j \geq l$;
(ii) $\|\tilde{z}_n^i\| \geq \|\tilde{z}_n\|^\alpha$ for $1 \geq i \geq l$.

Proof of (4.3). Since we know from (4.2) that $d((\tilde{y}_n, \tilde{u}_n), \overline{Y} - Y)$ is not flat along $\|y_n\|$, and we also, by choosing k_0 large, may assume that $(\tilde{y}_n, \tilde{u}_n) \in H(\tilde{\lambda})$ (this holds since $\tilde{j}$ is close to the standard imbedding j and $\|\tilde{j}(y_n) - (\tilde{y}_n, \tilde{u}_n)\|$ is flat along $\|y_n\|$), we can apply (4.1) to conclude that each $\|\tilde{z}_n^i - \tilde{z}_n^j\|$ is not flat along $\|y_n\|$. Since $\tilde{j}$ is close to j, $\|\tilde{j}(y_n)\| = O(\|y_n\|)$ and $\|y_n\| = O(\|\tilde{j}(y_n)\|)$, so $\|\tilde{z}_n^i - \tilde{z}_n^j\|$ is not flat along $\|(\tilde{y}_n, \tilde{u}_n)\|$ either. Since R is a polynomial which only vanishes at 0 on the compact set $\Sigma(R) \cap \overline{H}(\lambda')$, $\|(\tilde{y}_n, \tilde{u}_n)\|$ is not flat along any of $\|\tilde{z}_n^i\|$ and therefore is not flat along $\|\tilde{z}_n\|$ either. Since the derivative of R is bounded, we also get that each $\|\tilde{z}_n^i\|$ is not flat along $\|(\tilde{y}_n, \tilde{u}_n)\|$, and putting these facts together we get (i) and (ii). □

Let us finish the proof of (2.7) (a), case $i = 2$. From (2.2) and the fact that $l(H(\lambda)) \supset H(\lambda')$, we can construct a diagram

$$\begin{array}{ccccc} O - f^{-1}(0) & \xrightarrow{i} & l^{-1}(H(\lambda') - K) & \xrightarrow{l} & H(\lambda') - K \\ \downarrow f & & \downarrow F & & \downarrow R \\ W - \{0\} & \xrightarrow{j} & k^{-1}(V - C) & \xleftarrow{k} & V - C. \end{array} \tag{4.4}$$

From (2.3) and (4.4), it follows that $F|l^{-1}(H(\lambda') - K)$ is infinitesimally stable. So we can construct stratifications $\mathcal{S}_i(F)$, $i = 2, 3$, of $F|l^{-1}(H(\lambda') - K)$. From [8] it follows that the strata of $\mathcal{S}_i(R)$ and $\mathcal{S}_i(F)$ are mapped to each other via l^{-1} and k and vice versa. We can also construct bad sets $B(i)(F)$, $i = 2,3$, depending on $\mathcal{S}_i(F)$ and the imbeddings i and j. It is clear that the map $(z,H) \longrightarrow (l^{-1}(z), Dl^{-1}(z)\circ H)$ (respectively, $(z,H) \longrightarrow (k(z), Dk(z)\circ H)$) maps $B(i)(R)$, $i=3$ (respectively, $i=2$), to $B(i)(F)$, $i=3$ (respectively, $i=2$).

Recall that we had that

$$d((\tilde{j}(y_n), D\tilde{j}(y_n)), ((\tilde{y}_n, \tilde{u}_n), H_n)) \text{ is flat along } \|y_n\|.$$

Since $k|U-C$ is C^∞ and close to the identity, it follows that

$$d((j(y_n), Dj(y_n)), (k(\tilde{y}_n, \tilde{u}_n), Dk(\tilde{y}_n, \tilde{u}_n)\circ H_n)) \text{ is flat along } \|y_n\|.$$

Let $Dk(\tilde{y}_n, \tilde{u}_n)\circ H_n = H'_n$ and $k(\tilde{y}_n, \tilde{u}_n) = (y'_n, u'_n)$. Using [15, Lemma 2.2] we can construct a C^∞ imbedding $\hat{j} : W \to \mathbb{R}^{p+c}$ with the same ∞-jet at 0 as j such that $(\hat{j}(y_n), D\hat{j}(y_n)) = ((y'_n, u'_n), H'_n)$ for infinitely many n's. Note that since $((y'_n, u'_n), H'_n) \in B_2(F)$, $\hat{j}$ is not transverse to $\mathcal{S}_2(F)$ at $\{(y_n)\}$. Since $F \pitchfork j$ and $j^1\hat{j}(0) = j^1 j(0)$, $F \pitchfork \hat{j}$ at 0, and hence close to 0.

Let $j_t = (1-t)j + t(\hat{j})$, $t \in [0,1]$. It is easy to see that there are submersion germs ϕ_t with $\phi_t^{-1}(0) = \operatorname{im} j_t$ and $j^\infty\phi_t(0) = j^\infty\phi_0(0)$. Then $\Phi_t = \phi_t \circ F$ is a family of submersions with $j^\infty\Phi_t(0) = j^\infty\Phi_0(0)$; therefore there are diffeomorphism germs ψ_t such that $j^\infty\psi_t(0) = I$ and $\Phi_t\circ\psi_t = \Phi_0$. Therefore we have a diffeomorphism germ $\hat{i} = \psi_1|F^{-1}(\operatorname{im} j)$, $\hat{i} : F^{-1}(\operatorname{im} j) = (\mathbb{R}^n, 0) \to F^{-1}(\operatorname{im}\hat{j})$ with the same ∞-jet as i at 0. Put $g = \hat{j}^{-1}\circ F\circ\hat{i}$. Then $j^\infty g(0) = j^\infty f(0)$. From arguments similar to those given at pp. 173–174 in [8], the fact that $\hat{j}$ is not transverse to $\mathcal{S}_2(F)$ at $\{y_n\}$ implies that g is not multitransverse to the canonical stratification at $g^{-1}(y_n) \subset \Sigma(F) = \{x_n^1, \dots, x_n^l\}$. Since the canonical stratification restricts to a stratification of the singular k-jets in $J^k(\mathbb{R}^n, \mathbb{R}^p) - W^k(\mathbb{R}^n, \mathbb{R}^p)$ and each intersection of $p+1$ singular image points are automatically nontransverse, we may (by taking a subset if necessary) assume that $l \le p+1$. Let $\{\tilde{z}_n^j\}$ be the points which occur in (4.3). We can order our points such that we have $x_n^j = \hat{i}^{-1}\circ l^{-1}(\tilde{z}_n^j)$. Since l is a C^∞ diffeomorphism close to the identity and $\hat{i}$ is close to i, $\{x_n^j\}$ must satisfy inequalities similar to those occurring in the conclusion of (4.3). These inequalities will imply that $d(x_n, D_l(f))$ is not flat along $\|x_n\|$, where $x_n = (x_n^1, \dots, x_n^l)$. The inequality given in the hypothesis of (0.1) then show that $d((j^{k+1}f)^l(x_n), NT^{(k,l)})$ is not flat along $\|x_n\|$. On the other hand $j^\infty g(0) = j^\infty f(0)$ implies that $\|(j^{k+1}f)^l(x_n) - (j^{k+1}f)^l(x_n)\|$ is flat along $\|x_n\|$ which implies that $d((j^{k+1}g)^l(x_n), NT^{(k,l)})$ is not flat along $\|x_n\|$ either, and since $\|x_n\| \neq 0$, g must be multitranverse to the canonical stratification at $\{x_n^1, \dots, x_n^l\} = g^{-1}(y_n) \cap \Sigma(g)$. This gives us a contradiction and proves that (2.7)(a) holds when $i=2$.

Next, we will prove that (2.7)(a) holds with $i=3$. We start with some lemmas.

(4.5) Lemma. *Assume that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ satisfies the condition given in the hypothesis of (0.1). Then there exists $C > 0$ and $\alpha > 0$ such that the inequality*

$$\|f(x)\| + |\det Df(x)Df(x)^T| \ge C\|x\|^\alpha$$

holds in some neighborhood of 0.

Proof. Assume that no such C and α exist. Then we can find a sequence $x_n \to 0$ such that $\|f(x)\| + |\det Df(x)Df(x)^T|$ is flat along $\|x_n\|$. Let $\Sigma \subset \mathbb{R}^p \times J^1(n,p)$ be the set consisting of pairs $(0,z)$ with z a singular jet. Then it is easy to see that the assumption above is equivalent to assuming that the distance $d((f(x_n), j^1 f(x_n)), \Sigma)$ is flat with respect to $\|x_n\|$. Using [15, Lemma 2.2], we can find a map g with $j^\infty g(0) = j^\infty f(0)$ such that g has singular zeros along a subsequence of $\{x_n\}$. This subsequence, which we also denote by $\{x_n\}$, can be chosen such that any $(p+1)$-tuple $(x_n, x_{n+1}, \dots, x_{n+p})$ satisfies $\|x_{n+i}\| \geq \|(x_n, \dots, x_{n+p})\|^\alpha$ and $\|x_{n+i}\| \geq \|x_{n+i} - x_{n+j}\|^\alpha$ for some $\alpha > 0$ (independent of n) and any i, j, $i \neq j$. Now the hypothesis of (0.1) and an argument similar to the one given in the end of the proof of (2.7)(a), i=2, will give us that g is multitransverse to the canonical stratification at the pointset $\{x_n, \dots, x_{n+p}\}$, which contradicts the fact that this pointset consists of $p+1$ singular points with the same image. Hence the conclusion of (4.5) follows. □

The next lemma is another technical lemma in the same spirit.

4.6 Lemma. *Let f be as in (4.5). Let $\beta > 0$, and let $H(f^{-1}(0), \beta) = \{x | d(x, f^{-1}(0)) \leq \|x\|^\beta\}$. Then there exist $\alpha = \alpha(\beta) > 0$ and a neighborhood U such that $\|f(x)\| \geq \|x\|^\alpha$ for $x \in \sim H(f^{-1}(0), \beta) \cap U$.*

Proof. If (4.6) is false, then there exists a sequence $x_n \to 0$, $x_n \in \sim H(f^{-1}(0), \beta)$, such that $\|f(x_n)\|$ is flat along $\|x_n\|$. Using [15, Lemma 2.2] again, we can construct a map-germ $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $j^\infty f(0) = j^\infty g(0)$ and with zeroes along a subsequence of $\{x_n\}$. It follows from (4.5) and the results in [1] that f and g are $\mathcal{K}$-equivalent with a conjugating contact-diffeomorphism having the same ∞-jet as the identity. Therefore there exists a diffeomorphism with the same ∞-jet as the identity moving the points of a subsequence of $\{x_n\}$ to points in $f^{-1}(0)$. An easy estimate will however show that it is impossible for a germ with the same ∞-jet as the identity to move points outside $H(f^{-1}(0), \beta)$ to points in $f^{-1}(0)$, and we obtain a contradiction. □

Let us proceed with the proof of (2.7)(a) in the case $i = 3$.
Recall that we could assume $d(\tilde{j}(y), C) \geq \|y\|^\delta$ for some $\delta > 0$. Now since the diagram (2.6) is commutative and the derivative of R is bounded, it follows from (4.6) that for each constant $\beta > 0$ there exists another constant $\gamma = \gamma(\beta)$ such that we have $d(\tilde{i}(x), K) \geq \|x\|^\gamma$ for each $x \in \sim H(f^{-1}(0), \beta)$. Let $\beta > 0$ and assume, to obtain a contradiction, that we can find sequences $\{x_n\} \subset \sim H(f^{-1}(0), \beta)$ and $((\tilde{x}_n, \tilde{u}_n), H_n) \in B(3)$ such that $d((\tilde{i}(x_n), D\tilde{i}(x_n)), ((\tilde{x}_n, \tilde{u}_n), H_n))$ is flat along $\|x_n\|$. As in the case $i = 2$, we may assume that the sequence $\{(\tilde{x}_n, \tilde{u}_n)\}$ is chosen from a connected component X of an $\mathcal{S}_3(R)$ stratum of as small dimension as possible. Exactly as in (4.2) we can prove that there exists $\alpha = \alpha(\beta)$ such that in some neighborhood of 0 we have

$$d((\tilde{x}_n, \tilde{u}_n), \overline{X} - X) \geq \|x_n\|^\alpha. \tag{4.7}$$

Now, two cases may arise:
(1) $X \subset \Sigma(R)$;

(2) $X \subset\sim \Sigma(R)$.

We will first consider case (1). In this case we know that $R|X$ is an immersion. Let $v_n^1, \dots, v_n^l$ be an orthogonal set of unit vectors in $\mathbb{R}^n$ such that the $H_n(v_n^i)$ form a basis for $\operatorname{im} H_n \cap T_{(\tilde{x}_n, \tilde{u}_n)}X$. Then $\|D\tilde{i}(x_n)(v_n^i) - H_n(v_n^i)\|$ is flat along $\|x_n\|$ for each i and, since $\|\tilde{i}(x_n) - (\tilde{x}_n, \tilde{u}_n)\|$ is flat along $\|x_n\|$, we must also have that

$$\|DR(\tilde{i}(x_n)) \circ D\tilde{i}(x_n)(v_n^i) - DR(\tilde{x}_n, \tilde{u}_n) \circ H_n(v_n^i)\|$$

is flat along $\|x_n\|$. Now, since H_n is close to the standard imbedding $i : \mathbb{R}^n \to \mathbb{R}^{n+c}$, we must have $\|H_n(v_n^i)\| > \frac{1}{2}$ for each i. Using this, (3.3) and (4.7), we get that $\|DR(\tilde{x}_n, \tilde{u}_n) \circ H_n(v_n^i)\|$ is not flat along $\|x_n\|$. From these arguments it also follows that the distance from any of the vectors $DR(\tilde{x}_n, \tilde{u}_n) \circ H_n(v_n^i)$ to the subspace spanned by the others cannot be flat along $\|x_n\|$ (otherwise we could construct a unit length v_n in the span of $v_n^1, \dots, v_n^l$ such that $\|DR(\tilde{x}_n, \tilde{u}_n) \circ H_n(v_n)\|$ is flat along $\|x_n\|$). It also follows that the same statement must be true for the set of vectors $\{DR(\tilde{i}(x_n)) \circ D\tilde{i}(x_n)(v_n^i)\}$. This set of vectors are equal to the set $\{D\tilde{j}(f(x_n)) \circ Df(x_n)(v_n^i)\}$. Now, we may assume (by passing to a subsequence if necessary) that l is independent of n. So for each n, $\{Df(x_n)(v_n^i)\}$ are l linearly independent vectors, and since $\tilde{j}$ is close to the identity neither the norms nor the distance from each of these vectors to the subspace spanned by the others can be flat along $\|x_n\|$.

Now, let us define a linear map $K_n : \mathbb{R}^p \to \mathbb{R}^{p+c}$ by mapping each $Df(x_n)(v_n^i)$ to $DR(\tilde{x}_n, \tilde{u}_n) \circ H_n(v_n^i)$ and letting the restriction of K_n to the orthogonal compliment of $\{Df(x_n)(v_n^i)\}$ be equal to the restriction of $D\tilde{j}(f(x_n))$ to this compliment. From the construction of K_n, it follows that $\|K_n - D\tilde{j}(f(x_n))\|$ is flat along $\|x_n\|$ and K_n is not transverse to $T_{R(\tilde{x}_n, \tilde{u}_n)}Y$, where Y is the stratum in $\mathcal{S}_2(R)$ such that $R(X) \subset Y$. Moreover, it is also clear that $\|\tilde{j}(f(x_n)) - R(\tilde{x}_n, \tilde{u}_n)\|$ is flat along $\|x_n\|$. So, putting $y_n = f(x_n)$ and $(\tilde{y}_n, \tilde{u}_n) = R(\tilde{x}_n, \tilde{u}_n)$, we have produced a sequence $\{y_n\} \in \mathbb{R}^p$ and $((\tilde{y}_n, \tilde{u}_n), K_n) \in B(2)$ such that $d((\tilde{j}(y_n), D\tilde{j}(y_n)), ((\tilde{y}_n, \tilde{u}_n), K_n))$ is flat along $\|x_n\|$. On the other hand, since $x_n \in\sim H(f^{-1}(0), \beta)$, it follows from (4.6) that the distance also is flat along $\|y_n\|$, and we have therefore obtained a contradiction to what we already have proved in the case $i = 2$.

We will now consider case (2), $X \subset\sim \Sigma(R)$. We start by proving the following statement.

(4.8). *There exists $\alpha > 0$ such that $d((\tilde{x}_n, \tilde{u}_n), \Sigma(R)) > \|\tilde{x}_n\|^\alpha$ for n sufficiently large.*

Proof. Assume that (4.8) is false. Then $d((\tilde{x}_n, \tilde{u}_n), \Sigma(R))$ is flat along $\|\tilde{x}_n\|$. Recall that we earlier in this section have pointed out that $d(\tilde{j}(y), C) > \|y\|^\delta$ for some δ. Then it follows from (2.6), (4.6) and the assumption that $\|\tilde{i}(x_n) - (\tilde{x}_n, \tilde{u}_n)\|$ is flat along $\|x_n\|$ that $d((\tilde{x}_n, \tilde{u}_n), K)$ is not flat along $\|\tilde{x}_n\|$. So points in $\Sigma(R)$ minimizing $d((\tilde{x}_n, \tilde{u}_n), \Sigma(R))$ cannot be points in K. We can therefore find a connected component of an $\mathcal{S}_3(R)$ stratum $Y \subset \Sigma(R)$ such that $d((\tilde{x}_n, \tilde{u}_n), Y)$ is flat along $\|\tilde{x}_n\|$. Since $X \subset\sim \Sigma(R)$ we cannot have $X \subset \overline{Y}$, so $\overline{X} \cap \overline{Y} \subset \overline{X} - X$. Now since $\overline{X}$ and $\overline{Y}$ are regularly situated, we must have that $d((\tilde{x}_n, \tilde{u}_n), \overline{X} \cap \overline{Y})$ also is flat along $\|\tilde{x}_n\|$. Since $\|\tilde{x}_n - x_n\|$ is flat along $\|x_n\|$, we thus get that $d((\tilde{x}_n, \tilde{u}_n), \overline{X} \cap \overline{Y})$ is flat along $\|x_n\|$. Since $d((\tilde{x}_n, \tilde{u}_n), \overline{X} \cap \overline{Y}) \geq d((\tilde{x}_n, \tilde{u}_n), \overline{X} - X)$ this will however contradict what we already have proved in (4.7). □

Since $\operatorname{im} H_n$ is not transverse to $T_{(\tilde{x}_n,\tilde{u}_n)}X$, we must have

$$\operatorname{codim}(\operatorname{im} H_n \cap T_{(\tilde{x}_n,\tilde{u}_n)}X, T_{(\tilde{x}_n,\tilde{u}_n)}X) < \operatorname{codim} \operatorname{im} H_n = c.$$

From the construction of our stratification (again see [8]) we have that $\ker DR(\tilde{x}_n,\tilde{u}_n) \subset T_{(\tilde{x}_n,\tilde{u}_n)}X$. From the construction of diagram (2.6) we also have that $\ker DR(\tilde{i}(x_n)) \subset \operatorname{im} D\tilde{i}(x_n)$. Since $\|H_n - D\tilde{i}(x_n)\|$ is flat along $\|x_n\|$, it is not hard to see that we can find a perturbation of each H_n which also is flat along $\|x_n\|$ such that each of the perturbed maps contains $\ker DR(\tilde{i}(x_n))$ and remains non-transverse to $T_{(\tilde{x}_n,\tilde{u}_n)}X$. Let us replace the original H_n with these perturbed maps, which we also denote by H_n. It is then clear that the inclusion $(\ker DR(\tilde{x}_n,\tilde{u}_n))^\perp \cap T_{(\tilde{x}_n,\tilde{u}_n)}X \longrightarrow T_{(\tilde{x}_n,\tilde{u}_n)}X$ is transverse to $\operatorname{im} H_n \cap T_{(\tilde{x}_n,\tilde{u}_n)}X$ (in $T_{(\tilde{x}_n,\tilde{u}_n)}X$). Put

$$\tilde{H}_n = \operatorname{im} H_n \cap T_{(\tilde{x}_n,\tilde{u}_n)}X \cap \ker DR(\tilde{x}_n,\tilde{u}_n)^\perp.$$

We now have that

$$\begin{aligned}&\operatorname{codim}(\tilde{H}_n, T_{(\tilde{x}_n,\tilde{u}_n)}X \cap \ker DR(\tilde{x}_n,\tilde{u}_n)^\perp) = \\ &\operatorname{codim}(\operatorname{im} H_n \cap T_{(\tilde{x}_n,\tilde{u}_n)}X, T_{(\tilde{x}_n,\tilde{u}_n)}X) < c.\end{aligned}$$

Let $l = \dim \tilde{H}_n$. Pick l orthogonal unit vectors $v_n^1, \ldots, v_n^l$ such that $\{H(v_n^i)\}$ span $\tilde{H}_n$. We can now argue exactly as in case (1), but use (3.4)(c) and (4.8) instead of (3.3) and (4.7). We will then obtain a sequence of points in $B(2)$ whose distance to the sequence $(\tilde{j}(f(x_n)), D\tilde{j}(f(x_n)))$ is flat along $\|x_n\|$. As in case (1), this will contradict (2.7)(a), $i = 2$. This completes the proof of case 2 (2.7)(a), $i = 3$. □

We will now prove (2.7)(b); in fact we will prove a slightly stronger statement. To be able to formulate this statement, we will consider the following situation.

Let h be a map with $j^s h(0) = 0$. Consider the 1-parameter unfolding defined by $g(x,t) = (f(x) + th(x), t)$. Let G be the corresponding 1-parameter unfolding of F defined by $G(x,u,t) = (F(x,u),t) + (th(x),0,0)$. Then the t-levels of G have the same s-jet as F at 0. We can also consider G as an unfolding of g. So, if $s > \gamma(k_0)$ where γ is given in (0.2), we get a diagram of maps (where L and K are as in (0.2)).

$$\text{(4.9)} \qquad \begin{array}{ccccc} O \times I & \xrightarrow{i \times id} & \mathbb{R}^{n+c} \times I & \xleftarrow{L} & H(\lambda) \times I \\ \downarrow g & & \downarrow G & & \downarrow F \times id \\ W \times I & \xrightarrow{j \times id} & V \times I & \xrightarrow{K} & \mathbb{R}^{p+c} \times I. \end{array}$$

Here $I = [0,1]$ and i and j are the standard imbeddings, and O and W are adjusted so that $j(W) \subset V$ and $g(O) \subset W$. Consider the diagram given in (2.2). Recall that l and k were close to the identity. Also the t-levels in L and K are close to the identity and become closer as s is chosen greater. Then it is not hard to see that everything can be adjusted such that L^{-1} is defined in a neighborhood of the image of $i \times id$ and that L^{-1} maps this neighborhood into $H(\lambda) \times I$ and we can consequently define the composition $(l \times id) \circ L^{-1} \circ (i \times id)$. We may also assume that $(k \times id)^{-1}$ is defined in a neighborhood of $\operatorname{im} K \circ (j \times id)$ and we can therefore define $(k \times id)^{-1} \circ K \circ (j \times id)$. If k_0 and s are sufficiently large $(l\times id)\circ L^{-1}\circ(i\times id)$ and $(k\times id)^{-1}\circ K\circ(j\times id)$ will map $\bigcup_{t\in I}(O-g_t^{-1}(0))\times\{t\}$ and $(W-\{0\})\times I$ into $(H(\lambda')-K)\times I$ and $(\mathbb{R}^{p+c}-C)\times I$, respectively. (This follows from (2.4) since our maps become closer to the standard imbeddings when s and k_0 become greater.) Denoting the t-levels of the two compositions above by i_t and j_t respectively, we can therefore adjust the situation such that we can put (2.2) and (4.9) together to yield a commutative diagram

$$\text{(4.10)}\qquad \begin{array}{ccc} \bigcup_{t\in I}(O-g_t^{-1}(0))\times\{t\} & \xrightarrow{(i_t,t)} & (H(\lambda')-K)\times I \\ \big\downarrow g & & \big\downarrow R\times id \\ (W-\{0\})\times I & \xrightarrow{(j_t,t)} & (\mathbb{R}^{p+c}-C)\times I\,. \end{array}$$

Now we have the following:

(4.11) Lemma. *For each $\beta>0$ there exist $\alpha=\alpha(\beta)>0$ and $s=s(\beta)$ such that the following statements hold:*
(i) $d((i_t(x),Di_t(x)),B(3))\geq\|x\|^\alpha$ for $x\in\sim H(g_t^{-1}(0),\beta)$;
(ii) $d((j_t(y),Dj_t(y)),B(2))\geq\|y\|^\alpha$.
Moreover, let $\mathcal{L}$ be defined as in (2.7)(b); then we also have that:
(iii)

$$d(i_t(x),\overline{\mathcal{L}})\geq\|x\|^\alpha d(i_t(x),\operatorname{im} i_t\cap\overline{\mathcal{L}})$$

when $x\in O\cap\sim\{x|d(x,g_t^{-1}(0)\leq\|x\|^\beta\}$ and $\mathcal{L}\subset H(\lambda')$;

(iv) if $y\in W-\{0\}$ and $\mathcal{L}\subset\mathbb{R}^{p+c}-C$, we have that

$$d(j_t(y),\overline{\mathcal{L}})\geq\|y\|^\alpha d(j_t(y),\operatorname{im}\tilde{j}_t\cap\overline{\mathcal{L}})\text{ for }y\in W-\{0\}.$$

Note that from the proof of (0.2) in [2], it follows that the 0–levels of L and K are equal to the identities. It therefore follows that $i_0=\tilde{i}$ and $j_0=\tilde{j}$ so (2.7)(b) follows from (iii) and (iv) restricting to the level $t=0$. (i) and (ii) is a corresponding generalization of (2.7)(a) which will be needed in the proof of (iii) and (iv). We therefore start with the proof of (i) and (ii).

Proof of (4.11) (i) and (ii). From the characterizations of ∞-$\mathcal{K}$-determinacy given in [1], it will follow from (4.5) that f is ∞-$\mathcal{K}$-determined and therefore also finitely $\mathcal{K}^k$-determined for any finite k. So given k there exists $s=s(k)$ such that each g_t and f are $\mathcal{K}^k$–equivalent and $g_t^{-1}(0)$ and $f^{-1}(0)$ are therefore C^k diffeomorphic. From [1] it also follows that s can be chosen such that the diffeomorphism between these zero sets has the same k-jet as the identity. This implies that $g_t^{-1}(0)$ is close to $f^{-1}(0)$. Therefore, given $\beta>0$, it is not hard to see that there exists an $s=s(\beta)$ such that, for this choice of s, we obtain $H(f^{-1}(0),2\beta)\subset H(g_t^{-1}(0),\beta)$ for each $t\in I$. Now, let $\tilde{\alpha}$ be the exponent such that (2.7)(a) holds when β is replaced by 2β. From our definitions of i_t and j_t , it follows that if s is chosen perhaps even greater then we can obtain that $\|\tilde{i}-i_t\|=o(\|x\|^{\tilde{\alpha}})$, $\|\tilde{j}(y)-j_t(y)\|=o(\|y\|^{\tilde{\alpha}})$, and similar statements also hold for the derivatives of these maps. So, since $x \in \sim H(g_t^{-1}(0), \beta) \Rightarrow x \in \sim H(f^{-1}(0), 2\beta)$, we can prove (4.11)(i) and (ii), using the conclusion of (2.7)(a) together with an estimate of the distance between (i_t, Di_t) and $(\tilde{i}, D\tilde{i})$ and (j_t, Dj_t) and $(\tilde{j}, D\tilde{j})$ respectively, and we will then see that (i) and (ii) hold with $\alpha = \tilde{\alpha} + 1$.

Proof of (4.11) (iii) and (iv). We only prove (iv) since the proof of (iii) is similar. Assume that s is chosen such that (4.11) (ii) holds for some α, but (4.11) (iv) fails for any exponent. We can then find a set $\mathcal{L}$, a sequence of functions h_n with $j^s h_n(0) = 0$ and a sequence $\{y_n, t_n\} \subset \mathbb{R}^p \times I$ such that $(y_n, t_n) \to (0, t_0)$ for some $t_0 \in I$,

$$d((j_{t_n}(y_n), Dj_{t_n}(y_n)), B(2)) \geq \|y_n\|^\alpha$$

(where j_{t_n} is the imbedding we get by applying the construction (4.10) to the map $h = h_n$) and

$$d(j_{t_n}(y_n), \overline{\mathcal{L}}) \leq \|y_n\|^n d(j_{t_n}(y_n), \operatorname{im} j_{t_n} \cap \overline{\mathcal{L}}).$$

Let D_n be the closed disk in $\mathbb{R}^p$ with center y_n and radius $r_n = \frac{1}{2} d(j_{t_n}(y_n), \operatorname{im} j_{t_n} \cap \overline{\mathcal{L}})$. Then since j_{t_n} is close to $\tilde{j}$ which again is close to the standard imbedding, we must have $j_{t_n}(D_n) \cap \overline{\mathcal{L}} = \varnothing$. Furthermore, $d(j_{t_n}(D_n), \overline{\mathcal{L}})$ is flat along $\|y_n\|$. We can now use a perturbation technique similar to the one used in the proof of [15, Lemma 2.3] and for each n construct another imbedding $\tilde{j}_{t_n}$ which agrees with j_{t_n} outside the interior of D_n and which satisfies

$$\|j_{t_n} - \tilde{j}_{t_n}\| = O(d(j_{t_n}(y_n), \overline{\mathcal{L}})) \text{ and } \|Dj_{t_n} - D\tilde{j}_{t_n}\| = O(d(j_{t_n}(y_n), \overline{\mathcal{L}})/r_n),$$

and which has the property that there exists a point $\tilde{y}_n \in \mathring{D}$ such that $\tilde{j}_{t_n}$ is not transverse to some $\mathcal{S}_2(R)$ stratum in $\mathcal{L}$ at $\tilde{y}_n$. So $(\tilde{j}_{t_n}(\tilde{y}_n), D\tilde{j}_{t_n}(\tilde{y}_n)) \in B(2)$. On the other hand, combining inequalities given above it is clear that $d((\tilde{j}_{t_n}(\tilde{y}_n), D\tilde{j}_{t_n}(\tilde{y}_n)), (j_{t_n}(\tilde{y}_n), Dj_{t_n}(\tilde{y}_n)))$ is also flat along $\|y_n\|$. Since $0 \in \overline{\mathcal{L}}$ we must have that $r_n \leq \frac{1}{2}\|y_n\|$ and consequently that $\|y_n\| = O(\|\tilde{y}_n\|)$. Therefore $d((\tilde{j}_{t_n}(\tilde{y}_n), D\tilde{j}_{t_n}(\tilde{y}_n)), (j_{t_n}(\tilde{y}_n), Dj_{t_n}(\tilde{y}_n)))$ is also flat along $\|\tilde{y}_n\|$. Since $(\tilde{j}_{t_n}(\tilde{y}_n), D\tilde{j}_{t_n}(\tilde{y}_n)) \in B(2)$, this contradicts what we already have proved in (ii). This completes the proof of (2.7). $\square$

§5. Completing the proof of (0.1)

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ satisfy the hypothesis of (0.1). Let k_0 be a positive integer, let F be the unfolding constructed in §1 and let $R = j^{\gamma(k_0)}F(0)$ be as in §2. Then we showed in §4 that, given $\beta > 0$, we can find $s = s(\beta, k_0)$ such that for each h with $j^s h(0) = 0$, $s \geq s(\beta, k_0)$, we have an unfolding of the map $g(x,t) = (f(x) + th(x), t) = (g_t(x), t)$ of the form given by the diagram

$$\text{(5.1)} \qquad \begin{array}{ccc} \bigcup_{t \in I} (O - H(g_t{}^{-1}(0), \beta)) \times \{t\} & \xrightarrow{(i_t, t)} & (H(\lambda') - K) \times I \\ \Big\downarrow g & & \Big\downarrow R \times id \\ (W - \{0\}) \times I & \xrightarrow{(j_t, t)} & (\mathbb{R}^{p+c} - C) \times I, \end{array}$$

where $H(\lambda'), K, C, i_t, j_t$ and $H(g_t^{-1}(0), \beta)$ are explained in §2 and §4, and i_t and j_t satisfy the conclusion of (4.11). ¿From now on we abbreviate $\bigcup_{t \in I}$ by $\cup_t$.

Now, we wish to pull back the stratifications $\mathcal{S}_2(R) \times I$ and $\mathcal{S}_3(R) \times I$ via (j_t, t) and (i_t, t), and then add $\{0\} \times I$ in the target and also stratify $\cup_t H(g_t^{-1}(0), \beta) \times \{t\}$ in the source such that we obtain Whitney regular stratifications of $O \times I$ and $W \times I$, and such that g is a stratified map with respect to these stratifications. Then Thom's second isotopy lemma will show that f is s-C^0-determined, proving (0.1).

To be able to prove Whitney regularity, we will compare the pullback-stratifications with some semi-algebraic stratifications we get induced in the same manner. To this end, put $h = j^s f(0) - f$, let i_1 and j_1 denote the 1-levels of the maps i_t and j_t of (5.1) with this h, and let $j^s f(0) = q_s$. Then we obtain a diagram

$$\begin{array}{ccc} O - H(q_s{}^{-1}(0), \beta) & \xrightarrow{i_1} & H(\lambda') - K \\ \big\downarrow q_s & & \big\downarrow R \\ W - \{0\} & \xrightarrow{j_1} & \mathbb{R}^{p+c} - C\,. \end{array} \tag{5.2}$$

We now have the following lemma:

(5.3) Lemma. *For each k_0 there exists $s(k_0)$ such that, if $s \geq s(k_0)$, then the pullbacks of $\mathcal{S}_2(R)$ and $\mathcal{S}_3(R)$ by i_1 and j_1 are Whitney regular stratifications such that the strata in these pullback-stratifications also are Whitney regular with respect to 0.*

Proof. From the constructions in §2 and §4 we have that (5.2) was obtained from putting together (2.2) and the 1-level of (4.9). So if s is chosen greater than the maximal degree of the unfolding terms of F, (5.2) is obtained from the following type of diagram:

$$\begin{array}{ccccccc} O & \xrightarrow{i} & \mathbb{R}^{n+c} & \longleftarrow & H(\lambda) & \longrightarrow & U \\ \downarrow q_s & & \downarrow j^s F(0) & & \downarrow F & & \downarrow R \\ W & \xrightarrow[j]{} & V & \longrightarrow & \mathbb{R}^{p+c} & \longleftarrow & V\,. \end{array} \tag{5.4}$$

Here i and j are the standard imbeddings and the other horizontal maps are germs of diffeomorphisms which approximate the identities as well as we want when $s, k_0 \to \infty$.

Recall that the image of $H(\lambda)$ contains $H(\lambda')$. Since our maps are close to the identities it is not hard to see that we can find $\lambda'' > \lambda'$ such that we can obtain a diagram

$$\begin{array}{ccccccc} O - H(q_s^{-1}(0), \beta) & \xrightarrow{i} & H(\lambda'') & \longrightarrow & \operatorname{im} H(\lambda'') & \subset & H(\lambda') \\ \downarrow q_s & & \downarrow j^s F(0) & & \downarrow R & & \downarrow R \\ W & \xrightarrow[j]{} & V & \longrightarrow & \operatorname{im} V & \subset & \mathbb{R}^{p+c}\,, \end{array} \tag{5.5}$$

where i_1 and j_1 of (5.2) are respectively the compositions of the upper horizontal maps and lower horizontal maps of (5.5). As we did for R, we can also stratify $j^s F(0)$ by throwing away bad sets $C = C(j^s F(0), \lambda'')$ and $K = K(j^s F(0), \lambda'')$, and we obtain a stratified map $j^sF(0) : (H(\lambda'') - K) \to \mathbb{R}^{p+c} - C$. Pulling back these stratifications by i and j, we obtain a semialgebraic partition of $O - H(q_s^{-1}(0), \beta)$ and $W - \{0\}$. If we can see that the pullbacks of these stratifications are the same as what we get by pulling back the stratifications $\mathcal{S}_3(R)$ and $\mathcal{S}_2(R)$ by i_1 and j_1, we obtain that this is a semialgebraic Whitney regular stratification (because it follows from (4.11) that i_1 and j_1 are transverse to $\mathcal{S}_3(R)$ and $\mathcal{S}_2(R)$, so the pullback stratifications are therefore regular; note that i_1 and j_1 are not necessarily algebraic, which is the reason for this roundabout argument for the semialgebricity of the pullback stratifications), and because the strata are semialgebraic they must be Whitney regular to 0 as well. So (5.3) will follow from this assumption.

We must therefore prove that the pullbacks of the source and target stratifications of $j^sF(0)$ and R coincide. Now since the canonical stratification is $\mathcal{A}$-invariant, it is clear from the commutative diagram (5.5) that the strata of the source and target stratifications of $j^sF(0)$ will map to the strata of the corresponding stratifications of $R|\ \operatorname{im} H(\lambda'')$. So the pullbacks of these stratifications to $O - H(q_s^{-1}(0), \beta)$ and $W - \{0\}$ will coincide. The problem, however, is that the stratifications of $R|\ \operatorname{im} H(\lambda'')$ are not necessarily the restriction of the stratifications $\mathcal{S}_i(R)$ to the smaller neighborhoods where the stratifications of $R|\ \operatorname{im} H(\lambda'')$ are defined, because $R|H(\lambda')$ may contain more points in a critical fiber than perhaps $R|\ \operatorname{im} H(\lambda'')$ does.

For our purpose, however, it suffices to see that there exist neighborhoods around $i_1(O - H(q_s^{-1}(0), \beta))$ and $j_1(W - \{0\})$ such that the stratifications of the two restrictions of R coincide here. To see this, note that since the map $H(\lambda'') \to \operatorname{im} H(\lambda'')$ of (5.5) is close to the identity in the sense of (0.2), we can, if s and k are sufficiently large, assume that $\operatorname{im} H(\lambda'')$ contains another hornshaped neighborhood, say $H(\tilde{\lambda})$, where $\tilde{\lambda} > \lambda'' > \lambda'$. Arguing as in (2.4), we will find that the hornshaped neighborhood $H(2\tilde{\lambda})$ in the target will not contain any R-critical values from $\overline{\operatorname{im} H(\lambda'')} - \operatorname{im} H(\lambda'')$, and since $2\tilde{\lambda} > 2\lambda'$, it follows from (2.3) that $H(2\tilde{\lambda})$ will not contain any R-critical values from $\overline{H(\lambda')} - H(\lambda')$ either. Moreover, since R is bounded and u-level preserving, it is not hard to see that $R^{-1}(H(2\tilde{\lambda})) \subset H(\tilde{\lambda}) \subset \operatorname{im} H(\lambda'')$. From this it follows that the target-stratification we get from stratifying $R| \operatorname{im} H(\lambda'')$ will coincide with $\mathcal{S}_2(R)$ on the neighborhood $H(2\tilde{\lambda})$.

Moreover, note that since the hornshaped neighborhood $H(\lambda)$ of (0.2) can be kept fixed if we change s and k_0, and all our mappings becomes closer to their respective identities if we let s and k_0 tend to ∞, we do not have to alter our choice of λ', λ'' and $\tilde{\lambda}$, if we change our maps by choosing s and k_0 larger. We can therefore conclude from (0.2) that if s and k_0 are chosen sufficiently large, j_1 becomes so close to our standard imbedding that we must have $\operatorname{im} j_1 \subset H(2\tilde{\lambda})$. We must therefore have that the two pullback stratifications of $W - \{0\}$ coincide. From the way the $\mathcal{S}_3$ stratifications are constructed, and from the fact that $\operatorname{im} i_1(O - q_s^{-1}(0)) \subset R^{-1}(\operatorname{im} j_1) \subset R^{-1}H(2\tilde{\lambda})$, the same must be the case for the two pullback stratifications in the source. We have therefore proved (5.3). □

Our next goal is to prove that, by pulling back the stratification $\mathcal{S}_i(R) \times I$ via the imbeddings (i_t, t) and (j_t, t) of (5.1) and then adding $\{0\} \times I$ as a stratum in source and target, we obtain Whitney regular stratifications. We will here make use of the following technical lemma.

(5.6) Lemma. *Let G_q^N and G_r^N denote the Grassmannians of q-planes and r-planes in*

$\mathbb{R}^N$. *For* $W \in G_q^N$ *and* $V \in G_r^N$, *we denote by* $D(V, W)$ *the exterior angle, that is the distance from* V *to the set of* r*-dimensional planes not transverse to* W.
(a) *Then there exist* $K > 0$ *and* $\epsilon > 0$ *such that, for any* $W \in G_q^N$ *and any two* $V, V' \in G_r^N$ *with* $d(V', V) < \frac{1}{2}D(V, W)$,

$$d(V' \cap W, V \cap W) \leq Kd(V', V)/D(V, W)^\epsilon$$

(*where* $d(\cdot, \cdot)$ *is defined in* §3).
(b) *For any* $W \in G_q^N$ *and* $V \in G_r^N$, $D(V, W) = D(W, V)$.

Proof. It is enough to prove (a) with fixed W. So fix W and denote by Z the set of planes in G_r^N not transverse to W. Put $U = G_r^N - Z$. Then the map $g : U \to G_{r+q-N}^N$ given by $g(V) = V \cap W$ is a restriction of a rational map. We can therefore find K', ϵ' such that $d(g(V), g(V')) \leq (K'/(D(V', W)D(V, W))^{\epsilon'})d(V', V)$. If $d(V', V) < \frac{1}{2}D(V, W)$, then $D(V', W) \geq \frac{1}{2}D(V, W)$ so (a) follows.

Since $D(V, W) = 0$ if, and only if, $V \not\pitchfork W$ if, and only if, $D(W, V) = 0$, we can without loss of generality assume that $V \pitchfork W$. Let $K = V \cap W$ and let $V_1 = V \cap K^\perp$ and $W_1 = W \cap K^\perp$. For any nonzero vectors v and w, let $S(v, w)$ denote the sine of the angle between them, and let $\langle v \rangle$ denote the line spanned by v. Then (*b*) follows immediately from the following formula:

$$D(V, W) = \inf_{\substack{v \in V_1 - \{0\} \\ w \in W_1 - \{0\}}} S(v, w) = (*).$$

Suppose $V' \in Z$. Then $\dim V' \cap W > \dim K$, hence there exists $w_0 \neq 0$, $w_0 \in V' \cap W \cap K^\perp$. We may assume that V' was chosen so that

$$\begin{aligned} D(V, W) = d(V, V') = d(V', V) &= \sup_{w \in V' - \{0\}} \inf_{v \in V - \{0\}} S(v, w) \\ &\geq \inf_{v \in V - \{0\}} S(v, w_0) = S(\pi_V w_0, w_0). \end{aligned}$$

Since $w_0 \in K^\perp \supset V^\perp$, $\pi_V w_0 \in V_1$. Thus $D(V, W) \geq (*)$.

For each $w \notin V^\perp$, let $v = \pi_V w$. Let $V_w = (\langle v \rangle^\perp \cap V) + \langle w \rangle$. Then $d(V, V_w) = S(v, w)$. If $w \in W_1 - V^\perp$, then $V_w \in Z$. Thus if $W_1 \neq V^\perp$, then

$$D(V, W) \leq \inf_{w \in W_1 - V^\perp} d(V, V_w) = \inf_{w \in W_1 - V^\perp} S(\pi_V w, w) = \inf_{\substack{v \in V_1 - \{0\} \\ w \in W_1 - V^\perp}} S(v, w) = (*),$$

since $\inf_{v \in V_1 - \{0\}} S(v, w) = S(\pi_V w, w)$ and $W_1 - V^\perp$ is dense in $W_1 - \{0\}$. On the other hand, if $W_1 = V^\perp$, then $D(V, W) = (*) = 1$. □

We will now return to the situation described in diagram (5.1). Let $\mathcal{S}_i(g)$, $i = 2, 3$, denote the stratifications we obtain by pulling $\mathcal{S}_i(R) \times I$, $i = 2, 3$, back to $\cup_t(O - H({g_t}^{-1}(0), \beta)) \times \{t\}$ and $(W - \{0\}) \times I$ respectively. Using (4.11) it follows immediately that these stratifications are Whitney regular. We want, however, to see that it is also possible to choose s such that the strata in $\mathcal{S}_i(g)$ are Whitney regular over $\{0\} \times I$ in source and target. To prove this, we will prove that they are (a) and (b′) regular since this is equivalent to (b)-regularity (see Lemma 1 of [11]). We will prove this only for $\mathcal{S}_2(g)$ since the proof in the case involving $\mathcal{S}_3(g)$ is similar. The regularity of $\mathcal{S}_2(g)$ along $\{0\} \times I$ follows directly from the lemma below.

(5.7) Lemma. *For each k_0 and each non-negative integer i we can find an integer $s(i, k_0)$ and a positive constant $\alpha_i = \alpha_i(i, k_0)$ such that, if h in (5.1) satisfies $j^s h(0) = 0$ with $s \geq s(i, k_0)$, then the Whitney conditions* (a) *and* (b′) *hold along $\{0\} \times I$ when we consider sequences of points tending to points in $\{0\} \times I$ and the points in such sequences are chosen from the sets $N_i(\alpha_i)$ defined by*

$$N_i(\alpha_i) = \{(y, t) | d(J(y, t), F_i) \leq \|y\|^{\alpha_i}\},$$

where F_i denotes the closure of the union of all $\mathcal{S}_2(R) \times I$ strata of $\dim \leq i$, *and $J(y, t) = (j_t(y), t)$.*

Proof. The proof uses induction on i. Let i_0 be the minimal dimension of strata in $\mathcal{S}_2(R) \times I$ intersecting $J((W - \{0\}) \times I)$. We will first prove the statement in (5.7) for all $i < i_0$. Using (4.11)(iv), we can choose s such that we obtain

$$d(J(y, t), F_{i_0-1}) \geq \|y\|^{\alpha} d(J(y, t), \operatorname{im} J \cap F_{i_0-1}).$$

By assumption, $\operatorname{im} J \cap F_{i_0-1} = \{0\} \times I$. So, since $j_t(y)$ is close to the standard imbedding, we must have that $d(J(y, t), F_{i_0-1}) \geq \frac{1}{2}\|y\|^{\alpha+1}$. This shows that $N_i(\alpha + 2) = \{0\} \times I$ for all $i < i_0$, and the conclusion of (5.7) holds trivially for these i's.

Now let $i \geq i_0$ and assume by induction that the statement of (5.7) holds for all $j < i$. Let Y_i denote the union of all strata in $\mathcal{S}_2(R) \times I$ of dimension i. We must consider the following two cases:
(1) $J((W - \{0\}) \times I) \cap Y_i = \varnothing$,
(2) $J((W - \{0\}) \times I) \cap Y_i \neq \varnothing$,
where $J((W - \{0\}) \times I) \cap Y_i$ is considered as a set-germ at $\{0\} \times I$.

In case (1) we must have $\operatorname{im} J \cap F_i \subset F_{i-1}$. Again, by (4.11)(iv), we can find α such that

$$d(J(y, t), F_i) \geq \|y\|^{\alpha} d(J(y, t), F_{i-1}).$$

If $(y, t) \notin N_{i-1}(\alpha_{i-1})$, we therefore have that

$$d(J(y, t), F_i) \geq \|y\|^{\alpha+\alpha_{i-1}}.$$

This implies that $N_i(\alpha + \alpha_{i-1}) \subset N_{i-1}(\alpha_{i-1})$, and we can choose $\alpha_i = \alpha_{i-1} + \alpha$ in this case.

Now, consider case (2). Since we assume that $\operatorname{im} J \cap Y_i \neq \varnothing$, we can pick a sequence $J((y_n, t_n)) = (\tilde{y}_n, t_n) \in \operatorname{im} J \cap Y_i$ such that $(\tilde{y}_n, t_n) \to (0, t_0)$ for some $t_0 \in I$. First, we wish to prove that when s is sufficiently large, then Whitney conditions (a) and (b′) hold when we pass to limits along such sequences. Since we already know this on $N_{i-1}(\alpha_{i-1})$, we assume that $(y_n, t_n) \in \sim N_{i-1}(\alpha_{i-1})$. Now consider the map $j_1 \times id$ of the diagram (5.2). Put $(j_1(y_n), t_n) = (\hat{y}_n, t_n)$. Since the levels of g_t have s-jet equal q_s, J and $j_1 \times id$ must be close. In fact from the construction of the diagrams (5.1), (5.2) and the estimates (ii) of (0.2), it follows that, given k, it is possible to find s such that the k-jets of J and $j_1 \times id$ agree on $\{0\} \times I$. This will imply that $\|(\hat{y}_n, t_n) - (\tilde{y}_n, t_n)\| = o(\|y_n\|^k)$. We will determine below how large we want to choose this k. Clearly, we may assume that the two sequences $(\tilde{y}_n, t_n)$ and $(\hat{y}_n, t_n)$ belong to two connected components of strata in $\mathcal{S}_2(R) \times I$, which we will

denote by Y and $\hat{Y}$, respectively. From above it follows that $d((\tilde{y}_n, t_n), \hat{Y}) = o(\|y_n\|^k)$. On the other hand, since Y and $\hat{Y}$ are semialgebraic and therefore regularly situated, we must have that $d((\tilde{y}_n, t_n), \overline{\hat{Y}}) \geq d((\tilde{y}_n, t_n), \overline{\hat{Y}} \cap \overline{Y})^{\bar{\alpha}}$ for some $\bar{\alpha}$. (Note that since our stratification is finite, we can assume that $\bar{\alpha}$ actually is independent of the particular strata $\hat{Y}$ and Y.) From this we get that $d((\tilde{y}, t_n), \overline{\hat{Y}} \cap \overline{Y}) = o(\|y_n\|^{k/\bar{\alpha}})$. Now, since $\overline{Y} - Y \subset F_{i-1}$ and we assumed that $(y_n, t_n) \in \sim N_{i-1}(\alpha_{i-1})$, we have that $d((\tilde{y}_n, t_n), \overline{Y} - Y)) > \|y_n\|^{\alpha_{i-1}}$. So if we have $k > \bar{\alpha}\alpha_{i-1}$ and s is chosen according to this, we will get from above that $d((\tilde{y}_n, t_n), \overline{\hat{Y}} \cap \overline{Y}) = o(\|y_n\|^{\alpha_{i-1}})$, and therefore $\overline{\hat{Y}} \cap Y$ must contain points in Y. We must therefore either have
(A) $Y = \hat{Y}$ or
(B) $Y \subset \overline{\hat{Y}} - \hat{Y}$.

We will first consider case (A). In this case $(\hat{y}_n, t_n) \in Y$. From above we have that $\|\hat{y}_n - \tilde{y}_n\| = o(\|y_n\|^k) < d((\tilde{y}_n, t_n), \overline{Y} - Y)^{k/\alpha_{i-1}}$. Now we will apply (3.2) to the stratification $\mathcal{S}_2(R) \times I$ (including $\{0\} \times I$ as a stratum in this stratification), and to $x_0 = (0, t_0)$; let α and β denote the exponents occurring in (3.2). Then, if $k > \alpha\alpha_{i-1}$ and s is chosen according to this, we get that

$$d(T_{(\tilde{y}_n, t_n)}Y, T_{(\hat{y}_n, t_n)}Y) \leq \|\tilde{y}_n - \hat{y}_n\|^{1/\beta} = o(\|y_n\|^{k/\beta}).$$

Since it follows from (4.11) that J and $j_1 \times id$ are transverse to $\mathcal{S}_2(R) \times I$ outside $\{0\} \times I$, we can use (5.6) to get

$$(*) \quad \begin{aligned} & d(T_{(\tilde{y}_n, t_n)}Y \cap T_{(\tilde{y}_n, t_n)}(\text{im } J), T_{(\hat{y}_n, t_n)}Y \cap T_{(\hat{y}_n, t_n)}(\text{im } j_1 \times id)) \\ \leq & d(T_{(\tilde{y}_n, t_n)}Y \cap T_{(\tilde{y}_n, t_n)}(\text{im } J), T_{(\hat{y}_n, t_n)}Y \cap T_{(\tilde{y}_n, t_n)}(\text{im } J)) \\ + & d(T_{(\hat{y}_n, t_n)}Y \cap T_{(\tilde{y}_n, t_n)}(\text{im } J), T_{(\hat{y}_n, t_n)}Y \cap T_{(\hat{y}_n, t_n)}(\text{im } j_1 \times id)) \\ \leq & K\left(\frac{d(T_{(\tilde{y}_n, t_n)}Y, T_{(\hat{y}_n, t_n)}Y)}{(D(T_{(\tilde{y}_n, t_n)}Y, T_{(\tilde{y}_n, t_n)}(\text{im } J)))^{\epsilon}} + \frac{d(T_{(\tilde{y}_n, t_n)}(\text{im } J), T_{(\hat{y}_n, t_n)}(\text{im } j_1 \times id))}{(D(T_{(\hat{y}_n, t_n)}(\text{im } j_1 \times id), T_{(\hat{y}_n, t_n)}Y)))^{\epsilon}}\right) \\ = & \frac{o(\|y_n\|^{k/\beta})}{\|y_n\|^{\alpha\epsilon}} + \frac{o(\|y_n\|^{k-1})}{\|y_n\|^{\alpha\epsilon}}, \end{aligned}$$

where this time α is the α of (4.11) and we have used that the k-jets of $j_1 \times id$ and J agree on $\{0\} \times I$. Now if $k \geq \max\{\beta\alpha\epsilon, \alpha\epsilon + 1\}$ and s is chosen according to this, then $(*)$ tends to 0 when n tends to ∞. This will prove that $T_{(\tilde{y}_n, t_n)}(Y \cap \text{im } J)$ and $T_{(\hat{y}_n, t_n)}(Y \cap \text{im } (j_1 \times id))$ tend to the same limit. On the other hand since J and $j_1 \times id$ have the same 1-jet as the standard imbedding, $T_{(y_n, t_n)}(J^{-1}Y)$ and $T_{(y_n, t_n)}((j_1 \times id)^{-1}Y)$ must also tend to this limit. Since we already have proved that $j_1^{-1}(\mathcal{S}_2(R)) \times I$ is Whitney regulars along $\{0\} \times I$ (this follows from (5.3)), it follows that $J^{-1}(\mathcal{S}_2(R) \times I)$ is Whitney (a)-regular when we tend to $(0, t_0)$ along the sequence (y_n, t_n). Also since $j_1^{-1}(\mathcal{S}_2(R)) \times I$ is (b$'$)-regular along $\{0\} \times I$ and $\lim_{n \to \infty} T_{(y_n, t_n)}(J^{-1}(Y)) = \lim_{n \to \infty} T_{(y_n, t_n)}((j_1 \times id)^{-1}(Y))$, it follows that $\lim_{n \to \infty} \frac{y_n}{\|y_n\|} \in \lim_{n \to \infty} T_{(y_n, t_n)}(J^{-1}(Y))$. So we get (b$'$)-regularity along (y_n, t_n) as well.

Next, we must consider case (B), in which $Y \subset \overline{\hat{Y}} - \hat{Y}$. We have from (4.11) that

$$d((\hat{y}_n, t_n), \overline{Y}) \geq \|y_n\|^{\alpha} d((\hat{y}_n, t_n), \text{im}(j_1 \times id) \cap \overline{Y}),$$

and since $d((\hat{y}_n, t_n), \overline{Y}) = o(\|y_n\|^k)$, we get that $d((\hat{y}_n, t_n), \mathrm{im}(j_1 \times id) \cap \overline{Y}) = o(\|y_n\|^{k-\alpha})$. We can therefore find points $(\bar{y}_n, t_n) \in \mathrm{im}(j_1 \times id) \cap Y$ such that $\|\hat{y}_n - \bar{y}_n\| < \|y_n\|^{k-\alpha}$. So we get $\|\tilde{y}_n - \bar{y}_n\| < \|y_n\|^{k-\alpha} + \|y_n\|^k < \|y_n\|^{k-\alpha-1}$. Now, choosing k and therefore s sufficiently large, we can get $\|\tilde{y}_n - \bar{y}_n\|$ as small as we want, and since $(\bar{y}_n, t_n) \in Y$, we can repeat the arguments from case (A) over again to prove (a)-regularity. The case of (b′)-regularity differs slightly from what we had in (A), since now $(\bar{y}_n, t_n)$ is not the $j_1 \times id$ image of the point (y_n, t_n), but of some other point say, (y'_n, t_n). However, since $j_1 \times id$ and J both are close to the standard imbedding and $\|\tilde{y}_n - \bar{y}_n\|$ is small, an easy estimate shows that

$$\lim_{n\to\infty} \frac{y_n}{\|y_n\|} = \lim_{n\to\infty} \frac{\tilde{y}_n}{\|\tilde{y}_n\|} = \lim_{n\to\infty} \frac{\bar{y}_n}{\|\bar{y}_n\|} = \lim_{n\to\infty} \frac{y'_n}{\|y'_n\|}.$$

So (b′) follows from applying (5.3) in this case too.

To sum up, we have now proved Whitney regularity along sequences in $J^{-1}(Y_i) \cup N_{i-1}(\alpha_{i-1})$. Now consider the set

$$\tilde{Y}_i = \{(y,t) \in Y_i |\ d((y,t), F_{i-1}) > \frac{1}{2}\|y\|^{\alpha_{i-1}}\},$$

and let

$$\tilde{N}_i(\gamma) = \{(y,t)|\ d(J(y,t), \overline{\tilde{Y}_i}) < \|y\|^\gamma\}.$$

We wish to see that if γ is sufficiently large, then we also have Whitney regularity along $\tilde{N}_i(\gamma)$.

To this end, let $\gamma > 0$ and consider a sequence $(y_n, t_n) \in \tilde{N}_i(\gamma)$. Let $J(y_n, t_n) = (\hat{y}_n, t_n)$ and assume that $(\hat{y}_n, t_n)$ belongs to a connected component $\hat{Y}$ of a stratum in $\mathcal{S}_2(R) \times I$. Then we can find points $(\tilde{y}_n, t_n)$ in $\tilde{Y}_i$ such that $\|\hat{y}_n - \tilde{y}_n\| < \|y_n\|^\gamma$. Assume that $\tilde{y}_n$ belongs to a corresponding component Y. Repeating an argument given above, we find that if γ is chosen sufficiently large then either (A) $Y = \hat{Y}$, or (B) $Y \subset \overline{\hat{Y}} - \hat{Y}$. In case (A) we have already proven Whitney regularity along (y_n, t_n) (since then $Y = \hat{Y} \subset Y_i$). We therefore only need to consider case (B). By (4.11)(iv) we have

$$\|y_n\|^\gamma > d((\hat{y_n}, t_n), \overline{Y}) \geq \|y_n\|^\alpha d((\hat{y_n}, t_n), \mathrm{im}\, J \cap \overline{Y}).$$

So $d((\hat{y_n}, t_n), \mathrm{im}\, J \cap \overline{Y}) < \|y_n\|^{\gamma-\alpha}$. We can therefore find points $(y'_n, t'_n) \in \mathrm{im}\, J \cap \overline{Y}$ such that $\|(\hat{y}_n, t_n) - (y'_n, t'_n)\| < \|y_n\|^{\gamma-\alpha}$. Since $(\tilde{y}_n, t_n) \in \tilde{Y}_i$, we have that

$$d((\tilde{y}_n, t_n), \overline{Y} - Y) > \frac{1}{2}\|\tilde{y}_n\|^{\alpha_{i-1}}.$$

So

$$d((y'_n, t'_n), \overline{Y} - Y) > \frac{1}{2}\|\tilde{y}_n\|^{\alpha_{i-1}} - \|y_n\|^\gamma - \|y_n\|^{\gamma-\alpha} > \|y_n\|^{\alpha_{i-1}+1}$$

if γ is sufficiently large. (In the above we have used that our various inequalities imply that $\|y_n\| = O(\|\tilde{y}_n\|)$.)

Now let $\tilde{\alpha}$ and $\tilde{\beta}$ be the constants we get from applying (3.1) to the stratification $\mathcal{S}_2(R) \times I$. If $\gamma > \tilde{\alpha}(\alpha_{i-1} + 1) + \alpha$, we have that

$$\|(\hat{y}_n, t_n) - (y'_n, t'_n)\| \leq d((y'_n, t'_n), \overline{Y} - Y)^{\tilde{\alpha}},$$

and we get from (3.1) that $d(T_{(y'_n,t'_n)}Y, T_{(\hat{y_n},t_n)}\hat{Y}) \leq \|y_n\|^{\frac{\gamma-\alpha}{\beta}}$. Let H_n be a $\dim Y$ dimensional subspace of $T_{(\hat{y}_n,t_n)}\hat{Y}$ such that

$$d(T_{(y'_n,t'_n)}Y, T_{(\hat{y}_n,t_n)}Y) = d(T_{(y'_n,t'_n)}Y, H_n).$$

We can now use the estimate $d(T_{(y'_n,t'_n)}Y, H_n) \leq \|y_n\|^{\gamma/(\alpha\tilde{\beta})}$, together with (4.11)(ii) and (3.6), to estimate

$$d(T_{y'_n}Y \cap T_{(y'_n,t'_n)}(\text{im } J), H_n \cap T_{(\hat{y}_n,t_n)}(\text{im } J));$$

this estimate shows that $T_{J^{-1}(y'_n,t'_n)}(J^{-1}Y)$ and $(DJ(y_n,t_n))^{-1}(H_n)$ tend to the same limit as n tends to ∞ provided γ is chosen large enough. Using this, the estimate $\|(\hat{y}_n,t_n) - (y'_n,t'_n)\| \leq \|y_n\|^{\gamma-\alpha}$, the fact that $J^{-1}(y'_n,t'_n) \in J^{-1}(Y) \subset J^{-1}(Y_i)$ and finally that we already have proven Whitney regularity for sequences in this set, it follows that we have Whitney (a)- and (b′)-regularity along (y_n,t_n). (Here we leave out further details since we have given details in the similar case in which we considered sequences in $J^{-1}(Y_i)$.) We have therefore proven that if γ is sufficiently large we have Whitney regularity along sequences in $\tilde{N}_i(\gamma) \cup N_{i-1}(\alpha_{i-1})$.

Finally we need to see that given γ there is a constant α_i such that $N_i(\alpha_i) \subset \tilde{N}_i(\gamma) \cup N_{i-1}(\alpha_{i-1})$. To this end, consider $(\tilde{y},\tilde{t}) \in Y_i$ such that $d((\tilde{y},\tilde{t}), F_{i-1}) \leq \frac{1}{2}\|\tilde{y}\|^{\alpha_{i-1}}$. We must find α_i such that $\|J(y,t) - (\tilde{y},\tilde{t})\| \leq \|y\|^{\alpha_i}$ implies that $(y,t) \in N_{i-1}(\alpha_{i-1})$.

Let $(y',\tilde{t}) \in F_{i-1}$ be such that $\|(\tilde{y},\tilde{t}) - (y',\tilde{t})\| \leq \frac{1}{2}\|\tilde{y}\|^{\alpha_{i-1}}$. Then $\|J(y,t) - (y',\tilde{t})\| \leq \|y\|^{\alpha_i} + \frac{1}{2}\|\tilde{y}\|^{\alpha_{i-1}}$. If all our exponents are greater than some lower bound, we must have $O(\|y\|) = O(\|y'\|) = O(\|\tilde{y}\|)$ (since J is close to the standard imbedding). So $\alpha_i > \alpha_{i-1}+1$ implies $\|J(y,t) - (y',\tilde{t})\| < \|y\|^{\alpha_{i-1}}$(provided y is sufficiently close to 0). Thus, if $\alpha_i = \max(\gamma, \alpha_{i-1}+1)$, we obtain $N_i(\alpha_i) \subset \tilde{N}_i(\gamma) \cup N_{i-1}(\alpha_{i-1})$, and we have completed the induction step and thus proven (5.7). □

(5.8) Remark. In the proof given above there are several possibilities at each step, and it may seem that our choice of $s(i,k_0)$ and $\alpha(i,k_0)$ will depend on the case which actually occurs. Note, however, that when we keep k_0 fixed, the stratification $\mathcal{S}_2(R)$ will be kept fixed. Also note that the exponent α of (4.11)(ii) and (iv) is not dependent of our unfolding g if s is sufficiently large. From our proof we see that in all cases our integers s and exponents α_i are constructed from this α together with exponents occurring in inequalities expressing various metric properties of the stratification $\mathcal{S}_2(R)$. Since these constants are not affected when we change g by choosing s somewhat larger, there will always be only finitely many choices for α_i, and we can find a constant working in all cases by choosing the greatlargeest one.

Arguing as in the proof of (5.7), we can also prove that the the strata in the stratification $\mathcal{S}_3(g)$ of $\cup_t(O - H(g_t{}^{-1}(0),\beta)) \times \{t\}$ are regular along $\{0\} \times I$, using (4.11)(i) and (iii). Note however that the integer s and exponent α here are dependent on β. So the lower bound for the integer s which gives us Whitney regularity along $\{0\} \times I$ will here also depend on β. In the proof we will also apply the conclusion of (5.3) giving Whitney regularity with respect to 0 for the pullback in the source. Here we must however take some extra care, namely, given β, we must choose s so large that $H(q_s^{-1}(0), 2\beta) \subset H(g_t^{-1}(0), \beta)$. We will then see that the sequences in the source corresponding to (y_n, t_n) and (y'_n, t_n) in the proof of (5.7) will be contained in the compliment of $H(q_s^{-1}(0), 4\beta)$. We can then apply the conclusion of (5.3) with s chosen according to the constant 4β and proceed as in the proof of (5.7).

If we only consider Whitney condition (a), we can give a much simpler proof that $\mathcal{S}_2(g)$ is (a)-regular along $\{0\} \times I$. Actually we can prove a stronger statement: if s is sufficiently large, then $\mathcal{S}_2(g)$ will be close to a product stratification in the sense described in the lemma below.

(5.9) Lemma. *Given $k > 0$ there exists s such that, if $j^s g_t(0) = j^s f(0)$ for all t, then for each stratum Y in $\mathcal{S}_2(g)$ and point $(y,t) \in S_2(g)$, $T_{(y,t)}Y$ contains a vector $v = v_t(y)$ such that $v = \begin{pmatrix} v_1 \\ 1 \end{pmatrix}$ with $v_1(y,t) = o(\|y\|^k)$.*

Proof. Let Y be an $\mathcal{S}_2(g)$ stratum. Let $\tilde{Y}$ be the $\mathcal{S}_2(R) \times I$ stratum such that $Y = J^{-1}(\tilde{Y})$. Let $(y_0, t_0) \in Y$ and let $\tilde{y}_0 = j_{t_0}(y_0)$. Let t_n be a sequence of points such that $t_n \to t_0$. Let $\tilde{y}_n = j_{t_n}(y_0)$. From the estimate in (0.2) and the construction of diagram (5.1), we see that the imbedding $J(y,t) = \begin{pmatrix} j_t(y) \\ t \end{pmatrix}$ has the property that the levels $j_t(y)$ get closer to $j_0(y)$ (in the sense that they will share the same high order jet at 0) as s becomes larger. We can consequently choose s such that j_t has the form $j_0(y) + t\tilde{j}_t(y)$ with $\|\tilde{j}_t(y)\| < \|y\|^k$ (when y is close to 0), and we may also assume that the derivative of $\tilde{j}_t(y)$ (in y and t) satisfies the same estimate. So a Taylor expansion of $t\tilde{j}_t(y)$ shows that $\|\tilde{y}_n - \tilde{y}_0\| \le |t_n - t_0| \|y_0\|^k$. Since $(\tilde{y}_0, t_n) \in \tilde{Y}$, we must have that

$$d(\tilde{y}_n, \tilde{Y} \cap (\mathbb{R}^{p+c} \times \{t_n\})) \le |t_n - t_0| \|y_0\|^k.$$

On the other hand (4.11) implies that

$$d(\tilde{y}_n, \overline{\tilde{Y} \cap (\mathbb{R}^{p+c} \times \{t_n\})}) \ge \|y_0\|^\alpha d(\tilde{y}_n, (\text{im } j_{t_n}) \cap \overline{\tilde{Y} \cap (\mathbb{R}^{p+c} \times \{t_n\})}).$$

So

$$d(\tilde{y}_n, (\text{im } j_{t_n}) \cap \overline{\tilde{Y} \cap (\mathbb{R}^{p+c} \times \{t_n\})}) \le |t_n - t_0| \|y_0\|^{k-\alpha}.$$

We can therefore find a sequence of points (indexed by n) from $(\text{im } j_{t_n}) \cap \overline{\tilde{Y} \cap (\mathbb{R}^{p+c} \times \{t_n\})}$ whose distance to $\tilde{y}_n$ is no more than $\|t_n - t_0\| \|y_0\|^{k-\alpha}$. If k is large this sequence cannot be contained in $\{0\} \times I$ and, since j_{t_n} is transverse to $\mathcal{S}_2(R)$ we can actually choose this sequence from $(\text{im } j_{t_n}) \cap \tilde{Y} \cap (\mathbb{R}^{p+c} \times \{t_n\})$. Let us denote this sequence by $\hat{y}_n$. So we have

$$\|\hat{y}_n - \tilde{y}_0\| \le \|t_n - t_0\|(\|y_0\|^k + \|y_0\|^{k-\alpha}),$$

which implies $\hat{y}_n \to \tilde{y}_0$.

So, since $(\text{im } J) \cap \tilde{Y}$ is smooth at $(\tilde{y}_0, t_0)$,

$$\frac{(\hat{y}_n, t_n) - (\tilde{y}_0, t_0)}{\|(\hat{y}_n, t_n) - (\tilde{y}_0, t_0)\|}$$

must tend to a unit vector $\hat{v}(y_0, t_0)$ in $T_{(\tilde{y}_0,t_0)}((\text{im } J) \cap \tilde{Y})$ and, since

$$\|\hat{y}_n - \tilde{y}_0\| \leq \|t_n - t_0\|(\|y_0\|^k + \|y_0\|^{k-\alpha}),$$

this unit vector must have the form $\begin{pmatrix} O(\|y_0\|)^{k-\alpha} \\ 1 \end{pmatrix}$. Now,

$$DJ = \begin{pmatrix} D_x j_t & h \\ 0 & 1 \end{pmatrix}$$

with $h = o(\|y_0\|^s)$ and $D_x j_t = O(\|y_0\|)$. It is therefore clear that the vector $v(y_0, t_0)$ which is mapped by DJ to $\hat{v}(y_0, t_0)$ also must have the form $\begin{pmatrix} O(\|y_0\|)^{k-\alpha} \\ 1 \end{pmatrix}$. Since we can choose k as large as we want, we have proved (5.9). □

Lemma (4.5) together with an easy estimate will show that β and s can be chosen such that

$$|\det Dg_t(x)Dg_t(x)^T| \geq \frac{C}{2}\|x\|^\alpha \text{ on } H(g_t^{-1}(0), \beta), \text{ for all } t \in I.$$

So we may assume that g is a submersion on $\cup_t(H(g_t^{-1}(0), \beta) \times \{t\} - (\{0\} \times I)$. We therefore get a stratification of $\cup_t H(g_t^{-1}(0), \beta) \times \{t\}$ by pulling $\mathcal{S}_2(g)$ back to $\cup_t(H(g_t^{-1}(0), \beta) - g_t^{-1}(0)) \times \{t\}$, and then inserting $\cup_t(g_t^{-1}(0) - \{0\}) \times \{t\}$ and $\{0\} \times I$ as two additional strata. It follows from the commutative diagram (4.10) that these fit together with the stratification $\mathcal{S}_3(g)$ defined outside $\cup_t H(g_t^{-1}(0), \beta) \times \{t\}$. We can therefore put these two stratifications together to form a stratification on the neighborhood $O \times I$ in the source. Let this stratification also be denoted by $\mathcal{S}_3(g)$. We now have the following lemma.

(5.10) Lemma. *If s and β are chosen sufficiently large, then the restriction of $\mathcal{S}_3(g)$ to $\cup_t H(g_t^{-1}(0), \beta) \times \{0\}$ is (a)- and (b')-regular.*

Proof. As we pointed out above we may assume that β and s are chosen such that

$$|\det Dg_t(x)Dg_t(x)^T| \geq \frac{C}{2}\|x\|^\alpha \text{ on } H(g_t^{-1}(0), \beta).$$

Since g is a submersion on $\cup_t H(g_t^{-1}(0), \beta) \times \{t\} - (\{0\} \times I)$, Whitney regularity holds trivially here (since $\mathcal{S}_2(g)$ is Whitney regular by (5.7)).

To prove Whitney regularity along $\{0\} \times I$, consider the equation

$$\frac{\partial}{\partial t} = Dg(\zeta).$$

From the estimate $|\det Dg_t(x)Dg_t(x)^T| \geq \frac{C}{2}\|x\|^\alpha$ we find that this equation can be solved on $\cup_t H(g_t^{-1}(0), \beta) \times \{t\} - (\{0\} \times I)$ by putting

$$\zeta = \begin{pmatrix} Dg_t^T(Dg_t Dg_t^T)^{-1}(-h) \\ 1 \end{pmatrix}.$$

(Recall that g_t is of the form $f+th$). From above we also get that $\zeta(x,t) = \begin{pmatrix} o(\|x\|^{s-\alpha}) \\ 1 \end{pmatrix}$. Now we have that $\zeta \in T_{(x,t)}(\cup_t g_t^{-1}(0) \times \{t\})$ when $x \in g_t^{-1}(0)$ and, since $\zeta(x,t) \to \frac{\partial}{\partial t}$ as x tends to 0, we get (a)-regularity for the pair $(\cup_t g_t^{-1}(0) \times \{t\} - (\{0\} \times I), \{0\} \times I)$.

Now consider another stratum in $\mathcal{S}_3(g)$ intersecting $\cup_t H(g_t^{-1}(0), \beta) \times \{t\}$. Let (x,t) be a point in this stratum and suppose $g(x,t) = (y,t) \in Y$, $Y \in \mathcal{S}_2(g)$. From (5.9) we can find a vector $v = v_t(y)$ in $T_{(y,t)}Y$ such that $v = \begin{pmatrix} v_1 \\ 1 \end{pmatrix}$ with $\|v_1\| = o(\|y\|^k)$. Now putting

$$\eta(x,t) = \begin{pmatrix} Dg_t^T(Dg_t(Dg_t)^T)^{-1}(v_1 - h) \\ 1 \end{pmatrix},$$

we get that $Dg(x,t)(\eta(x,t)) = v_t(y)$. We also see that the $\mathbb{R}^n$-component of $\eta(x,t)$ is of the form $\frac{o(\|y\|^k)+o(\|x\|^s)}{\|x\|^\alpha}$. Now $y = g_t(x) = O(\|x\|)$. It is therefore clear that with k and s sufficiently large, we must have $\eta(x,t) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $x \to 0$. So we have (a)-regularity in this case too.

Next we prove that the stratification also is (b′)-regular on $\cup_t H(g_t^{-1}(0), \beta) \times \{t\}$. Consider the vector field ζ constructed above with the property that $Dg(x,t)(\zeta(x,t)) = \frac{\partial}{\partial t}$. Let $\zeta_1(x,t) = Dg_t^T(Dg_t Dg_t^T)^{-1}(-h)$, the $\mathbb{R}^n$-component of ζ. From the inequality $|\det Dg_t(x)Dg_t(x)^T| \geq \frac{C}{2}\|x\|^\alpha$ it follows that, given k, we can choose $s = s(k)$ such that $\zeta_1(x,t)$ is C^k on $\cup_t H(g_t^{-1}(0), \beta/2) \times \{t\} - (\{0\} \times I)$, with $\|j^k\zeta_1(x,t)\| = o(\|x\|^k)$ on this set.

From (4.5) it also follows that f is ∞-$\mathcal{K}$-determined and finitely $\mathcal{K}^k$-determined. It then follows from the results in [1] that, if s is sufficiently large, there exists a one-parameter family of C^k-diffeomorphisms $\{\phi_t\}$ with $j^k(\phi_t - id_{\mathbb{R}^n})(0) = 0$, such that $\phi_t(g_t^{-1}(0)) = f^{-1}(0)$. Given t and t' we thus get that $\phi_{t'}^{-1} \circ \phi_t(g_t^{-1}(0)) = g_{t'}^{-1}(0)$. Now if $x \in g_t^{-1}(0)$ we may then assume that k and $s = s(k)$ are chosen so large compared to β that with $x' \in \phi_{t'}^{-1} \circ \phi_t(x)$ we have

$$\|x' - x\| = \|(\phi_{t'}^{-1} \circ \phi_t - id_{\mathbb{R}^n})(x)\| \leq \min(\|x'\|, \|x\|)^\beta.$$

Now we choose representatives of $\cup_t H(g_t^{-1}(0), \beta) \times \{t\}$ and $\cup_t H(g_t^{-1}(0), \frac{\beta}{2}) \times \{t\}$ such that

$$K = \overline{\cup_t H(g_t^{-1}(0), \beta) \times \{t\}} \subset \cup_t H(g_t^{-1}(0), \frac{\beta}{2}) \times \{t\}.$$

Define a Taylor field Q of order 1 on K by letting $Q(x,t) = j^1\zeta_1(x,t)$ on $(K - \{0\}) \times I$ and $Q(0,t) = 0$. We wish to see that k and s can be chosen such that this is a Whitney field. Let us use the notation of [10] concerning Whitney fields.

Consider first points $p = (x,t)$ and $q = (0,t')$. Then

$$|(R_q^1 Q)^m(p)| \leq \|j^1\zeta_1(x,t)\| = o(\|(x,t)\|^k) = o(\|(x,t)\|)$$

if $k \geq 1$, $|m| = 0, 1$. Likewise

$$|(R_p^1 Q)^m(q)| = o(\|(x,t)\|)$$

if $|m| = 0, 1$.

Next assume $p = (x, t)$, $q = (x', t')$ and that $\|p - q\| \geq \max(\|x\|, \|x'\|)^\beta$. Then, for some $C > 0$,

$$\begin{aligned}\|(R^1_q Q)^m(p)\| &\leq C(\|j^1\zeta_1(p)\| + \|j^1\zeta_1(q)\|) \leq C(\|x\|^k + \|x'\|^k)\\ &\leq 2C\max(\|x\|, \|x'\|) \leq 2C\|p - q\|^{k/\beta}.\end{aligned}$$

So with k chosen such that $k > \beta + 1$ we get $|(R^1_q Q)^m(p)| = o(\|p - q\|)$.

Finally consider two points $p = (x, t)$ and $q = (x', t')$ with $\|p - q\| < \max(\|x\|, \|x'\|)^\beta$. Let $\bar{p} = (\bar{x}, t)$ be a point in $g_t^{-1}(0)$ such that

$$\|p - \bar{p}\| = \|x - \bar{x}\| < \|x\|^\beta.$$

Let $\bar{x}' = \phi_{t'}^{-1} \circ \phi_t(\bar{x})$ and put $\bar{q} = (\bar{x}', t') \in g_{t'}^{-1}(0)$. As above we have

$$\|\bar{x}' - \bar{x}\| \leq \min(\|\bar{x}'\|, \|\bar{x}\|)^\beta \leq \|\bar{x}\|^\beta.$$

If β is large and we are close to $\{0\} \times I$, then

$$\|x\|^\beta < 2\|x'\|^\beta \text{ and } \|\bar{x}\|^\beta \leq 2\|x\|^\beta < 4\|x'\|^\beta.$$

Thus,

$$\|x - x'\| \leq \|p - q\| < 2\|x'\|^\beta.$$

Putting this all together, we get

$$\begin{aligned}\|q - \bar{q}\| = \|x' - \bar{x}'\| &\leq \|x - x'\| + \|x - \bar{x}\| + \|\bar{x}' - \bar{x}\|\\ &< 2\|x'\|^\beta + 2\|x'\|^\beta + 4\|x'\|^\beta = 8\|x'\|^\beta < \|x'\|^{\beta/2}\end{aligned}$$

if we are close to $\{0\} \times I$. Although we have supposed that $q \in K \subset \cup_t H(g_t^{-1}(0), \beta/2) \times \{t\}$, the argument shows that the ball with center p and radius $\max(\|x\|, \|x'\|)^\beta$ is contained in $\cup_t H(g_t^{-1}(0), \beta/2) \times \{t\}$. We may therefore suppose that the line segment between q and p is contained in $\cup_t H(g_t^{-1}(0), \beta/2) \times \{t\}$. If β is large, this line segment does not intersect $\{0\} \times I$.

Now, since ζ_1 is C^k on $\cup_t H(g_t^{-1}(0), \beta/2) \times \{t\} - (\{0\} \times I)$, and we may assume that $k \geq 2$ and that derivatives of ζ of order ≤ 2 are bounded, we get from Taylor's formula with remainder that

$$|(R^1_p Q)^m(q)| \leq C\|p - q\|^{2-|m|} = o(\|p - q\|^{1-|m|}).$$

Putting all the estimates for R together, we see that Q is a C^1 Whitney field on K.

By Whitney's Extension Theorem, extend Q to a C^1 function $\tilde{\zeta}_1(x, t)$ on $\mathbb{R}^n \times I$ which actually is C^∞ on the interior of $\cup_t H(g_t^{-1}(0), \beta) \times \{t\}$. Put $\tilde{\zeta}(x, t) = \begin{pmatrix} \tilde{\zeta}_1(x, t) \\ 1 \end{pmatrix}$. Integrating this vector field, we get a one parameter family of C^1 diffeomorphisms $\{h_t\}$ of $\mathbb{R}^n$

which trivialize the family g_t inside $\cup_t H(g_t^{-1}(0),\beta)\times\{t\}$ (this follows since ζ is mapped on $\frac{\partial}{\partial t}$ in this neighborhood). From our construction it also follows that each h_t has the same 1-jet at 0 as the identity; from this it is not hard to see that everything can be adjusted such that $H(f^{-1}(0),\beta)=H(g_0^{-1}(0),\beta)$ contains another hornshaped neighborhood $H(f^{-1}(0),\beta')$ such that for each t we have that $h_t(H(f^{-1}(0),\beta'))\subset H(g_t^{-1}(0),\beta)$.

Now, put $\tilde{f}=j^s f(0)-f$ and consider the map $\tilde{g}(x,t)=(f(x)+t\tilde{f}(x),t)$. Applying the above to this unfolding $\tilde{g}$ of f, we get a family of diffeomorphisms $\tilde{h}_t$ with properties described above. Letting $q_s=j^s f(0)$ and $\tilde{h}=\tilde{h}_1$, we get a diagram of mappings

$$\begin{array}{ccccc} H(q_s^{-1}(0),\beta) & \overset{\tilde{h}}{\leftarrow} & H(f^{-1}(0),\beta') & \overset{h_t}{\rightarrow} & H(g_t^{-1}(0),\beta) \\ \downarrow q_s & & \downarrow f & & \downarrow g_t \\ \mathbb{R}^p & \overset{id}{\leftarrow} & \mathbb{R}^p & \overset{id}{\rightarrow} & \mathbb{R}^p\,. \end{array} \tag{5.11}$$

Now since $f^{-1}(0)$ is mapped by h_t to $g_t^{-1}(0)$ and each h_t has the same 1-jet as the identity, we can find $\tilde{\beta}$ such that $H(g_t^{-1}(0),\tilde{\beta})\subset h_t(H(f^{-1}(0),\beta')$. From all this we get a diagram

$$\begin{array}{ccc} H(q_s^{-1}(0),\beta)\times I & \overset{(\tilde{h}\circ h_t^{-1})\times id_I}{\longleftarrow} & \cup_t H(g_t^{-1}(0),\tilde{\beta})\times\{t\} \\ \downarrow q_s\times id & & \downarrow g \\ \mathbb{R}^p\times I & \overset{id}{\rightarrow} & \mathbb{R}^p\times I\,. \end{array} \tag{5.12}$$

Since q_s is a polynomial map, the pair of strata $((q_s^{-1}(0)-\{0\})\times I,\{0\}\times I)$ is (b′)-regular. Since the pair $(\cup_t(g_t^{-1}(0)-\{0\})\times\{t\},\{0\}\times I)$ is mapped to that pair of strata by the C^1-diffeomorphism $(\tilde{h}\circ h_t^{-1})\times id_I$, the latter pair of strata is also (b′)-regular.

Now we want to prove (b′)-regularity when we consider sequences from other strata in $\cup_t H(g_t^{-1}(0),\tilde{\beta})\times\{t\}$. To this end, consider a point

$$(x,t)\in(H(q_s^{-1}(0),\beta)-\{0\})\times I.$$

Since we have seen that

$$\det(Dq_s(x)Dq_s(x)^T)\geq\frac{C}{2}\|x\|^\alpha \text{ on } H(q_s^{-1}(0),\beta),$$

$q_s\times id_I$ is a submersion on the smaller set $(H(q_s^{-1}(0),\beta)-\{0\})\times I$. Let $x'\in q_s^{-1}(0)$ be such that $\|x-x'\|$ is minimal, hence $\|x-x'\|\leq\|x\|^\beta$. Let α' and β' denote the α and β of Lemma 3.4(a), applied with $R=q_s$ and $x_0=0$. Using $\|x-x'\|\leq\|x\|^\beta$, the inequality $\det Dq_s(x)Dq_s(x)^T\geq\frac{C}{2}\|x\|^\alpha$ and the fact that the zero set of $\det(Dq_s(x)Dq_s(x)^T)$ is $\Sigma(q_s)$, an easy estimate will show that β can be chosen such that $\|x-x'\|\leq d(x,\Sigma(q_s))^{\alpha'}$. From (3.4) we get that

$$d(\ker Dq_s(x),\ker Dq_s(x'))\leq\|x\|^{\beta/\beta'}.$$

So, to prove (b′) for $S_3(g)$ inside $\cup_t H(g_t^{-1}(0),\tilde{\beta})\times\{t\}$, let Y be a stratum in this stratification, and let (x_n,t_n) be a sequence of points in $Y\cap(\cup_t H(g_t^{-1}(0),\tilde{\beta})\times\{t\})$ converging to a point $(0,t_0)$. We must prove that

$$\lim_{n\to\infty}T_{(x_n,t_n)}Y\supset\lim_{n\to\infty}\frac{(x_n,0)}{\|x_n\|}.$$

Let

$$(\tilde{x}_n, t_n) = (\tilde{h} \circ h_{t_n}^{-1}(x_n), t_n).$$

Let $\hat{x}_n$ be the point in $q_s^{-1}(0)$ closest to $\tilde{x}_n$. Then $\|\hat{x}_n - \tilde{x}_n\| < \|\tilde{x}_n\|^\beta$ and it is easy to see that $\lim_{n\to\infty} \frac{\hat{x}_n}{\|\hat{x}_n\|} = \lim_{n\to\infty} \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$ (provided the limit exists). Denote this limit by $\hat{v}$. Since we know that the pair $(q_s^{-1}(0) - \{0\}, \{0\})$ is Whitney regular, $\hat{v} \subset \lim_{n\to\infty} T_{\hat{x}_n} q_s^{-1}(0)$. On the other hand, since

$$T_{\hat{x}_n} q_s^{-1}(0) = \ker Dq_s(\hat{x}_n)$$

and

$$d(\ker Dq_s(\hat{x}_n), \ker Dq_s(\tilde{x}_n)) \le \|\tilde{x}_n\|^{\beta/\beta'}$$

(recall that, here, β' is the exponent of (3.4)), we also get that β can be chosen such that

$$\lim_{n\to\infty} \ker Dq_s(\hat{x}_n) = \lim_{n\to\infty} \ker Dq_s(\tilde{x}_n) \supset \hat{v}.$$

Now $\tilde{x}_n = \tilde{h} \circ h_{t_n}^{-1}(x_n)$ and $\tilde{h} \circ h_{t_n}^{-1}$ has the same 1-jet as the identity at 0. We must therefore also have $\lim_{n\to\infty} \frac{x_n}{\|x_n\|} = \hat{v}$. On the other hand,

$$\ker Dg_{t_n}(x_n) = D(h_{t_n} \circ \tilde{h}^{-1})(\tilde{x}_n) \ker Dq_s(\tilde{x}_n)$$

and, since $h_{t_n} \circ \tilde{h}^{-1}$ also has the same 1-jet as the identity, we must have that

$$\lim_{n\to\infty} \ker Dg_{t_n}(x_n) = \lim_{n\to\infty} \ker Dq_s(\tilde{x}_n).$$

We therefore get from above that

$$\lim_{n\to\infty} \frac{x_n}{\|x_n\|} \subset \lim_{n\to\infty} \ker Dg_{t_n}(x_n).$$

Now since Y is the pullback through a submersion of a stratum in the target, $T_{(x_n,t_n)}Y$ must contain $\ker Dg_{t_n}(x_n)$ and we consequently get

$$\lim_{n\to\infty} \frac{x_n}{\|x_n\|} \subset \lim_{n\to\infty} T_{(x_n,t_n)}Y,$$

proving that (b$'$) holds inside $\cup_t H(g_t^{-1}(0), \tilde{\beta}) \times \{t\}$. Letting β' be the β in the statement of (5.10), we have proved (a) and (b$'$) hold inside $\cup_t H(g_t^{-1}(0), \beta) \times \{t\}$. This completes the proof of 5.10. □

We have now established that if s is chosen sufficiently large our stratification in the target is both (a) and (b$'$). The stratification in the source is also (a)- and (b$'$)-regular both inside (by (5.10)) and outside (by (5.7) and the comments which follow (5.7)) a neighborhood $\cup_t H(g_t^{-1}(0), \beta) \times \{t\}$, and consequently is (b)-regular in a neighborhood of $\{0\} \times I$. From the construction of the stratification it also follows that the map $g : \mathbb{R}^n \times I \to \mathbb{R}^p \times I$ is a stratified map if we restrict to a sufficiently small neighborhood of $\{0\} \times I$. The family g_t is therefore topological trivial by the second isotopy lemma, and we have consequently completed the proof of (0.1).

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