An equation modeling Poisson distributed pollution in a stochastic medium
– A white noise approach –

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Abstract

In this paper we look at a model for pollution given by the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\eta^2 \Delta u + \vec{W}_{\phi_x} \diamond \nabla u - \kappa \diamond u + \sum_{j=1}^{l} \psi_j(t,x) P^{(j)}_{t/\mu_j}$$

where α, η and $\{\mu_j\}_{j=1}^{l}$ are constants, $\vec{W}_{\phi_x}$ is a Gaussian white noise vector, $\{P^{(j)}_{t/\mu_j}\}_{j=1}^{l}$ are Poisson white noises, κ is a generalized white noise distribution and $\{\psi_j\}_{j=1}^{l}$ are suitable real functions. This equation will be studied on $\mathbb{R}^n$ and also on any bounded domain. We will show that this equation has a unique solution given by an explicit solution formula.

Keywords: Gaussian and Poisson generalized white noise distributions, Wick product, Hermite transform.

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§1 Introduction

We will consider a stochastic model based on the PDE

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\eta^2 \Delta u(t,x) - V \cdot \nabla u(t,x) - \alpha u(t,x) + \xi(t,x) \qquad (1)$$

where $\frac{1}{2}\eta^2$ is the dispersion coefficient, V is the water velocity, α is the leakage rate, $\xi(t,x)$ is the rate of increase of the chemical concentration at (t,x) and $u(t,x)$ is the chemical concentration at time t on location x. This equation was studied in the paper of Kallianpur et al. [KX] where the rate of increase $\xi(t,x)$ was supposed to be a Poisson random variable. We will in addition consider the case where the drift vector V is modeled as an n-dimensional Gaussian white noise (with independent components) and α is a generalized white noise distribution. It will be convenient to work in the space $(\mathcal{S})^{-1}$ of generalized white noise distributions, since this space allows explicit solution formulas for a wide range of possible choices for random ξ, V and α's. We will emphasis on the possibility of transforming the equation into a Gaussian equation. After solving the Gaussian equation we are able to transform the Gaussian solution into the solution of the mixed case.

The method used to solve the Gaussian stochastic version of equation (1) is the same as the one used by Holden et al. in [HLØUZ3]. Several other SPDE's are solved in a similarly fashion:

- The transport equation ([GjHØUZ]).
- The pressure equation for fluid flow ([HLØUZ3]).
- The Dirichlet equation ([Gj2]).
- The Burgers equation ([HLØUZ2]).
- The Schrödinger equation ([HLØUZ]).

For more examples and background on white noise, the author would recommend [Ø3].

§2 Preliminaries on multidimensional white noise

We will start by giving a short introduction of definitions and results from multidimensional Gaussian Wick calculus, taken mostly from [Gj], [HLØUZ3], [HKPS] and [KLS].

In the following we will fix the parameter dimension n and space dimension m.

Let

$$\mathcal{N} := \prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n)$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^n$, and

$$\mathcal{N}^* := (\prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n))^* \approx \prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n)$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

Let $\mathcal{B} := \mathcal{B}(\mathcal{N}^*)$ denote the Borel σ-algebra on $\mathcal{N}^*$ equipped with the weak star topology and set

$$\mathcal{H} := \bigoplus_{i=1}^{m} \mathcal{L}^2(\mathbb{R}^n)$$

where $\oplus$ denotes orthogonal sum.

Since $\mathcal{N}$ is a countably Hilbert nuclear space (cf. eg.[Gj]) we get, using Minlos' theorem, a unique probability measure ν_G on $(\mathcal{N}^*, \mathcal{B})$ such that

$$\int_{\mathcal{N}^*} e^{i\langle\omega,\phi\rangle}\, d\nu_G(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{\mathcal{H}}} \quad \forall\phi \in \mathcal{N}$$

where $\|\phi\|^2_{\mathcal{H}} = \sum_{i=1}^{m} \|\phi_i\|^2_{\mathcal{L}^2(\mathbb{R}^n)}$.

Note that if $m = 1$ then ν_G is usually denoted by μ_G.

THEOREM 2.1 [Gj] We have the following

1. $\otimes_{i=1}^{m}\mathcal{B}(\mathcal{S}'(\mathbb{R}^n)) = \mathcal{B}(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n))$
2. $\nu_G = \times_{i=1}^{m} \mu_G$

DEFINITION 2.2 [Gj] The triple

$$(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n), \mathcal{B}, \nu_G)$$

is called the (m-dimensional) (n-parameter) **white noise probability space.**

For $k = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$ let

$$h_k(x) := (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}})$$

be the Hermite polynomials and

$$\xi_k(x) := \pi^{-\frac{1}{4}}((k-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{k-1}(\sqrt{2}x) \ ; \ k \geq 1$$

the Hermite functions.

It is well known that the family $\{\tilde{e}_\alpha\} \subset \mathcal{S}(\mathbb{R}^n)$ of tensor products

$$\tilde{e}_\alpha := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_n}$$

forms an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^n)$.

Give the family of all multi-indecies $\zeta = (\zeta_1, \dots, \zeta_n)$ a fixed ordering

$$(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(k)}, \dots) \text{ where } \zeta^{(k)} = (\zeta_1^{(k)}, \dots, \zeta_n^{(k)})$$

and define $\tilde{e}_k := \tilde{e}_{\zeta^{(k)}}$.

Let $\{e_k\}_{k=1}^{\infty}$ be the orthonormal basis of $\mathcal{H}$ we get from the collection

$$\{(\overbrace{0, \dots, 0}^{i-1}, \tilde{e}_j, \overbrace{0, \dots, 0}^{m-i}) \in \mathcal{H} \;\; 1 \le i \le m, 1 \le j < \infty\}$$

and let $\gamma : \mathbb{N} \longrightarrow \mathbb{N}$ be a function such that

$$e_k = (0, \dots, 0, \tilde{e}_{\zeta^{(\gamma(k))}}, 0, \dots, 0).$$

Finally , let $(\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(k)}, \dots)$ with $\beta^{(k)} = (\beta_1^{(k)}, \dots, \beta_n^{(k)})$ be a sequence such that $\beta^{(k)} = \zeta^{(\gamma(k))}$.

If $\alpha = (\alpha_1, \dots, \alpha_k)$ is a multi-index of non-negative integers we put

$$H_\alpha(\omega) := \prod_{i=1}^{k} h_{\alpha_i}(\langle \omega, e_i \rangle).$$

From theorem 2.1 in [HLØUZ] we know that the collection

$$\{H_\alpha(\cdot); \alpha \in \mathbb{N}_0^k; k = 0, 1, \dots\}$$

forms an orthogonal basis for $\mathcal{L}^2(\mathcal{N}^*, \mathcal{B}, \nu_G)$ with $||H_\alpha||_{\mathcal{L}^2(\nu_G)} = \alpha!$ where $\alpha! = \prod_{i=1}^{k} \alpha_i!$.

This implies that any $f \in \mathcal{L}^2(\nu_G)$ has the unique representation

$$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

where $c_\alpha \in \mathbb{R}$ for each multi-index α and

$$||f||^2_{\mathcal{L}^2(\nu_G)} = \sum_\alpha \alpha! c_\alpha^2.$$

DEFINITION 2.3 [Gj] The m-dimensional **white noise map** is a map

$$W : \prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n) \times \prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R}^m$$

given by

$$W^{(i)}(\phi, \omega) := \omega_i(\phi_i) \quad 1 \le i \le m$$

PROPOSITION 2.4 [Gj] The m-dimensional white noise map W satisfies the following

1. $\{W^{(i)}(\phi, \cdot)\}_{i=1}^{m}$ is a family of independent normal random variables.
2. $W^{(i)}(\phi, \cdot) \in \mathcal{L}^2(\nu_G)$ for $1 \le i \le m$.

DEFINITION 2.5 [HLØUZ3] Let $0 \le \rho \le 1$.

- Let $(\mathcal{S}_n^m)^\rho$, the space of **generalized white noise test functions**, consist of all

  $$f = \sum_\alpha H_\alpha \in \mathcal{L}^2(\nu_G)$$

  such that

  $$\|f\|_{\rho,k}^2 := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{\alpha k} < \infty \quad \forall k \in \mathbb{N}$$

- Let $(\mathcal{S}_n^m)_G^{-\rho}$, the space of **generalized white noise distributions**, consist of all formal expansions

  $$F = \sum_\alpha b_\alpha H_\alpha$$

  such that

  $$\sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-\alpha q} < \infty \text{ for some } q \in \mathbb{N}$$

where

$$(2\mathbb{N})^\alpha := \prod_{i=1}^{k} (2^n \beta_1^{(i)} \cdots \beta_n^{(i)})^{\alpha_i} \text{ if } \alpha = (\alpha_1, \dots, \alpha_k).$$

We know that $(\mathcal{S}_n^m)_G^{-\rho}$ is the dual of $(\mathcal{S}_n^m)_G^\rho$ (when the later space has the topology given by the seminorms $\|\cdot\|_{\rho,k}$) and if $F = \sum b_\alpha H_\alpha \in (\mathcal{S}_n^m)_G^{-\rho}$ and $f = \sum c_\alpha H_\alpha \in (\mathcal{S}_n^m)_G^\rho$ then

$$\langle F, f \rangle = \sum_\alpha b_\alpha c_\alpha \alpha!.$$

It is obvious that we have the inclusions

$$(\mathcal{S}_n^m)_G^1 \subset (\mathcal{S}_n^m)_G^\rho \subset (\mathcal{S}_n^m)_G^{-\rho} \subset (\mathcal{S}_n^m)_G^{-1} \quad \rho \in [0, 1]$$

and in the remaining of this paper we will consider the larger space $(\mathcal{S}_n^m)_G^{-1}$.

DEFINITION 2.6 [HLØUZ3] The Wick product of two elements in $(\mathcal{S}_n^m)_G^{-1}$ given by

$$F = \sum_\alpha a_\alpha H_\alpha \ , \quad G = \sum_\beta b_\beta H_\beta$$

is defined by

$$F \diamond G = \sum_\gamma c_\gamma H_\gamma$$

where

$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta$$

LEMMA 2.7 [HLØUZ3] We have the following

1. $F, G \in (\mathcal{S}_n^m)_G^{-1} \Rightarrow F \diamond G \in (\mathcal{S}_n^m)_G^{-1}$
2. $f, g \in (\mathcal{S}_n^m)_G^{1} \Rightarrow f \diamond g \in (\mathcal{S}_n^m)_G^{1}$

DEFINITION 2.8 [HLØUZ3] Let $F = \sum b_\alpha H_\alpha$ be given. Then the Hermite transform of F, denoted by $\mathcal{H}F$, is defined to be (whenever convergent)

$$\mathcal{H}F := \sum_\alpha b_\alpha z^\alpha$$

where $z = (z_1, z_2, \cdots)$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k}$ if $\alpha = (\alpha_1, \ldots, \alpha_k)$.

LEMMA 2.9 [HLØUZ3] If $F, G \in (\mathcal{S}_n^m)_G^{-1}$ then

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)$$

for all z such that $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exists.

LEMMA 2.10 [HLØUZ3] Suppose $g(z_1, z_2, \cdots)$ is a bounded analytic function on $\mathbf{B}_q(\delta)$ for some $\delta > 0, q < \infty$ where

$$\mathbf{B}_q(\delta) := \{\zeta = (\zeta_1, \zeta_2, \cdots) \in \mathbb{C}_0^{\mathbb{N}}; \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2\mathbf{N})^{\alpha q} < \delta^2\}.$$

Then there exists $X \in (\mathcal{S}_n^m)_G^{-1}$ such that $\mathcal{H}X = g$.

LEMMA 2.11 [HLØUZ3] Suppose $X \in (\mathcal{S}_n^m)_G^{-1}$ and that f is an analytic function in a neighborhood of $\mathcal{H}X(0)$ in $\mathbb{C}$. Then there exists $Y \in (\mathcal{S}_n^m)_G^{-1}$ such that $\mathcal{H}Y = f \circ \mathcal{H}X$.

THEOREM 2.12 [KLS] Let (T, Σ, τ) be a measure space and let $\Phi : T \to (\mathcal{S}_n^m)_G^{-1}$ be such that there exists $q < \infty, \delta > 0$ such that

1. $\mathcal{H}\Phi_t(z) : T \to \mathbb{C}$ is measurable for all $z \in \mathbf{B}_q(\delta)$
2. there exists $C \in \mathcal{L}^1(T,\tau)$ such that $|\mathcal{H}\Phi_t(z)| \leq C(t)$ for all $z \in \mathbf{B}_q(\delta)$ and for τ-almost all t.

Then $\int_T \Phi_t \, d\tau(t)$ exists as a Bochner integral in $(\mathcal{S}_n^m)_G^{-1}$. In particular, $\langle \int_T \Phi_t \, d\tau(t), \phi \rangle = \int_T \langle \Phi_t, \phi \rangle \, d\tau(t) \;;\; \phi \in (\mathcal{S}_n^m)_G^1$.

EXAMPLE 2.13 Define the x-shift of ϕ, denoted by ϕ_x, by $\phi_x(y) := \phi(y-x)$. Then

$$\mathrm{Exp}\{W_{\phi_x}^{(i)}\} \in (\mathcal{S}_n^m)_G^{-1} \quad 1 \leq i \leq m, \forall x \in \mathbb{R}^n$$

which is an immediate consequence of proposition 2.4 and lemma 2.11.

Instead of developing the Poisson analysis as we did with the Gaussian analysis, we only mention the existence of a measure ν_P on $\prod_{j=1}^{l} \mathcal{S}'(\mathbb{R}^n)$ together with independent random variables $\{P_t^{(j)}\}_{j=1}^{l}$ which are all Poisson distributed. Moreover, we also know from [BGj] that

$$\mathcal{L}^2(\prod_{i=1}^{l} \mathcal{S}'(\mathbb{R}^n), \nu_G) \overset{U}{\approx} \mathcal{L}^2(\prod_{i=1}^{l} \mathcal{S}'(\mathbb{R}^n), \nu_P)$$

where U is a unitary mapping with $U(B_t^{(j)}) = P_t^{(j)} - t$, $\{B_t^{(j)}\}_{j=1}^{l}$ being standard independent Brownian motions. In particular,

$$\begin{aligned}
\mathcal{L}^2(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n) \times \prod_{i=1}^{l} \mathcal{S}'(\mathbb{R}^n), \nu_G \times \nu_P) &\approx \mathcal{L}^2(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n), \nu_G) \otimes \mathcal{L}^2(\prod_{i=1}^{l} \mathcal{S}'(\mathbb{R}^n), \nu_P) \\
&\approx \mathcal{L}^2(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n), \nu_G) \otimes \mathcal{L}^2(\prod_{i=1}^{l} \mathcal{S}'(\mathbb{R}^n), \nu_G) \\
&\approx \mathcal{L}^2(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n) \times \prod_{i=1}^{l} \mathcal{S}'(\mathbb{R}^n), \nu_G \times \nu_G)
\end{aligned}$$

where the composed unitary mapping is denoted by V. Note that the last space is a Gaussian space ($\mathcal{N} = \prod_{i=1}^{m+l} \mathcal{S}'(\mathbb{R}^n)$). By extending the unitary mapping V, as explained in [BGj], we may define the combined Gaussian and Poisson distribution space, denoted by $(\mathcal{S}_n^{m,l})_{G,P}^{-1} := V^{-1}((\mathcal{S}_n^{m+l})_G^{-1})$.

§3 The pollution model in $\mathbb{R}^n$

We will in this and the next section assume that $(b_s^{(t,x)}(\omega), \hat{P}^{t,x})$ is a Brownian motion starting at location $x \in \mathbb{R}^n$ at time t, and use the notation

- $\hat{E}^{t,x}$ is expectation w.r.t. the measure $\hat{P}^{t,x}$.
- $C^2(\mathbb{R}^n)$ are the functions on $\mathbb{R}^n$ with continuous derivatives up to order 2.
- $C_0^2(\mathbb{R}^n)$ are the functions on $\mathbb{R}^n$ with compact support and continuous derivatives up to order 2.
- $C_0^2([0,T]\times\mathbb{R}^n)$ are the functions on $[0,T]\times\mathbb{R}^n$ with compact support, t-continuity and continuous x-derivatives up to order 2.

We are now ready to state our main result of this section:

THEOREM 3.1 Let $T > 0$ be given and assume furthermore that we are given functions $\mathbb{R}^n \ni x \mapsto f(x) \in (\mathcal{S}_n^{n,l})_{G,P}^{-1}$, $[0,T]\times\mathbb{R}^n \ni (t,x) \mapsto \kappa(t,x) \in (\mathcal{S}_n^{n,l})_{G,P}^{-1}$, $\psi_j \in C_0^2([0,T]\times\mathbb{R}^n)$ and constants $\mu_j > 0$ $(1 \le j \le l)$ such that

- $\exists(q_f \in \mathbb{N}, \delta_f > 0, K_f > 0)$ such that
  1. $\sup_{x\in\mathbb{R}^n, z\in\mathbf{B}_{q_f}(\delta_f)} |\mathcal{H}f(x,z)| \le K_f$.
  2. $x \mapsto \mathcal{H}f(x,z) \in C_0^2(\mathbb{R}^n)$ whenever $z \in \mathbf{B}_{q_f}(\delta_f)$.
- $\exists(q_\kappa \in \mathbb{N}, \delta_\kappa > 0, K_\kappa > 0)$ such that
  1. $\sup_{(t,x)\in[0,T]\times\mathbb{R}^n, z\in\mathbf{B}_{q_\kappa}(\delta_\kappa)} |\mathcal{H}\kappa(t,x,z)| \le K_\kappa$.
  2. $x \mapsto \mathcal{H}\kappa(t,x,z) \in C_0^2(\mathbb{R}^n)$ whenever $t \in [0,T], z \in \mathbf{B}_{q_\kappa}(\delta_\kappa)$.
  3. $\exists(\beta(z) > 0 \ \forall z \in \mathbf{B}_{q_\kappa}(\delta_\kappa))$ such that $(t,x) \mapsto \mathcal{H}\kappa(t,x,z)$ is uniformly Hölder continuous (exponent $\beta(z)$) in (t,x) on compact subsets of $[0,T]\times\mathbb{R}^n$.
  4. $\mathcal{H}\kappa(t,x,z) \ge 0$ whenever $z \in \mathbf{B}_{q_\kappa}(\delta_\kappa) \cap \mathbb{R}_0^N$.

Then

$$u(t,x) = \hat{E}^{T-t,x}[f(b_T) \diamond \mathrm{Exp}\{-\int_{T-t}^{T} \kappa(T-\theta, b_\theta)\,d\theta\} \diamond \mathcal{J}_{t,T}] \tag{2}$$

$$+ \hat{E}^{T-t,x}[\int_{T-t}^{T} g(T-s, b_s) \diamond \mathrm{Exp}\{-\int_{T-t}^{s} \kappa(T-\theta, b_\theta)\,d\theta\}\,ds \diamond \mathcal{J}_{t,T}]$$

where

$$\mathcal{J}_{t,T} := \mathrm{Exp}\{\sum_{i=1}^{n} \eta^{-1} \int_{T-t}^{T} [W_{\phi_y}^{(i)}]_{y=\eta b_s}\,db_s^i - \frac{1}{2}\sum_{i=1}^{n} \eta^{-2} \int_{T-t}^{T} [W_{\phi_y}^{(i)}]_{y=\eta b_s}^{\diamond 2}\,ds\} \tag{3}$$

and

$$g(t,x) := \sum_{j=1}^{l} \psi_j(t,x) P^{(j)}_{t/\mu_j}$$

is the unique $(\mathcal{S}^{n,l}_n)^{-1}_{G,P}$-valued process which solves

$$\frac{\partial u}{\partial t} = \frac{1}{2}\eta^2 \Delta u + \vec{W}_{\phi_x} \diamond \nabla u - \kappa(t,x) \diamond u + \sum_{j=1}^{l} \psi_j(t,x) P^{(j)}_{t/\mu_j} \qquad (t,x) \in [0,T] \times \mathbb{R}^n \quad (4)$$

$$u(0,x) = f(x) \qquad x \in \mathbb{R}^n \quad (5)$$

where $\hat{E}$ and $\int \cdot\, ds$ are Bochner integrals in $(\mathcal{S}^{n,l}_n)^{-1}_{G,P}$.

REMARK 3.2 If $u(t,x) \in (\mathcal{S}^{n,l}_n)^{-1}_{G,P}$ and $\mathcal{A}(\mathcal{H}u(t,x)) \in A_b(\mathbf{B}_q(\delta))$ for some $q \in \mathbb{N}, \delta > 0$, where $A_b(\mathbf{B}_q(\delta))$ is the space of all bounded analytic functions on $\mathbf{B}_q(\delta)$ and $\mathcal{A} := \frac{\partial}{\partial t} - \frac{1}{2}\eta^2\Delta - \vec{W}_\phi \diamond \nabla$, we will use the convention that $\mathcal{A}u(t,x) := \mathcal{H}^{-1}\{\mathcal{H}\mathcal{A}(\mathcal{H}u(t,x))\}$.

PROOF:

We will now adapt the all Gaussian point of view as explained in the end of section 2. We will therefore replace g by

$$g(t,x) := \sum_{j=1}^{l} \psi_j(t,x)(B^{(j)}_{t/\mu_j} + t/\mu_j) \qquad (6)$$

Note that given $\delta_g > 0$ and $q_g \in \mathbb{N}$, using definition 2.8 and lemma 2.10, we have

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n, z\in \mathbf{B}_{q_g}(\delta_g)} |\mathcal{H}g(t,x,z)| \le K_g$$

for some constant $K_g > 0$.

To solve equation (4), we must find $\hat{q} \in \mathbb{N}$ and $\hat{\delta} > 0$ such that $\tilde{u}(t,x,z) := \mathcal{H}(u(t,x))(z) \in A_b(\mathbf{B}_{\hat{q}}(\hat{\delta}))$ solves the equation

$$\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2}\eta^2 \Delta \tilde{u} + \tilde{\vec{W}}_{\phi_x} \cdot \nabla \tilde{u} - \tilde{\kappa}(t,x) \cdot \tilde{u} + \tilde{g}(t,x) \qquad (t,x) \in [0,T] \times \mathbb{R}^n \quad (7)$$

$$\tilde{u}(0,x) = \tilde{f}(x) \qquad x \in \mathbb{R}^n \quad (8)$$

when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$.

LEMMA 3.3 $\exists(\hat{\delta} > 0, \hat{q} \in \mathbb{N})$ such that $z \mapsto \tilde{u}(t,x,z) \in A_b(\mathbf{B}_{\hat{q}}(\hat{\delta})) \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n$.

PROOF:

By taking absolute values in (2), we get

$$|\tilde{u}(t,x,z)| \leq K_f e^{TK_\kappa} + TK_g e^{TK_\kappa}$$

whenever $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$ where $\hat{q} \geq \max\{q_f, q_g, q_\kappa\}$ and $0 < \hat{\delta} \leq \min\{\delta_f, \delta_g, \delta_\kappa\}$. Note that we have used the equality

$$\hat{E}^{T-t,x}[|\mathcal{J}_{t,T}|] = \hat{E}^{T-t,x}[\exp\{\sum_{i=1}^{n} \eta^{-1} \int_{T-t}^{T} \Re[\tilde{W}^{(i)}_{\phi_y}]_{y=\eta b_s}\, db^i_s - \frac{1}{2}\sum_{i=1}^{n} \eta^{-2} \int_{T-t}^{T} \Re[\tilde{W}^{(i)}_{\phi_y}]^2_{y=\eta b_s}\, ds\}]$$
$$\equiv 1.$$

which follows by using [Ø, Corollary 8.23]. ■

LEMMA 3.4 The Bochner integrals in the expression for $u(t,x)$ are well-defined.

PROOF:

This is obvious from the estimates in lemma 3.3 ■

LEMMA 3.5 $\mathcal{A}u(t,x)$ is well-defined as an element in $(\mathcal{S}^{n,1}_n)^{-1}_{G,P}$ $\forall (t,x) \in [0,T] \times \mathbb{R}^n$.

PROOF:

Since

$$\mathcal{A}\tilde{u} = -\tilde{\kappa} \cdot \tilde{u} + \tilde{g}$$

it follows from lemma 3.3 that

$$|\mathcal{A}\tilde{u}(t,x,z)| \leq K_\kappa(K_f e^{TK_\kappa} + TK_g e^{TK_\kappa}) + K_g$$

when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$, i.e. the claim follows. ■

LEMMA 3.6 $\tilde{u}(t,x,z)$ is the unique function which solves equation (7) when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$.

PROOF:

Equation (7) may be written as

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{\kappa}\tilde{u} = \mathcal{A}^{\xi}\tilde{u} + \tilde{g} \quad (t,x) \in [0,T] \times \mathbb{R}^n \tag{9}$$

$$\tilde{u}(0,x) = \tilde{f} \qquad x \in \mathbb{R}^n \tag{10}$$

where $\mathcal{A}^{\xi}$ is the second order differential operator given by

$$\mathcal{A}^{\xi} = \sum_{i=1}^{n} \frac{1}{2}\eta^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n} \tilde{W}^{(i)}_{\phi_x}(\xi) \frac{\partial}{\partial x_i}.$$

Assume now that $\xi \in \mathbf{B}_{\hat{q}}(\hat{\delta}) \cap \mathbb{R}_0^{\mathbb{N}}$.

The operator $\mathcal{A}^{\xi}$ is clearly uniformly elliptic with drift term which satisfies the linear growth condition

$$
\begin{aligned}
|\tilde{W}^{(i)}_{\phi_x} - \tilde{W}^{(i)}_{\phi_y}|(\xi) &= |\sum_{k=0}^{\infty} (\phi_x - \phi_y, e_k)\xi_k| \\
&\leq \sum_{k=0}^{\infty} |(\phi_x - \phi_y, e_k)||\xi_k| \\
&\leq (M \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} |e_k(x)|\, dx |\xi_k|)|x-y|
\end{aligned}
$$

where

$$
M := \max_{1 \leq i \leq n} \{ \sup_{x \in \mathbb{R}^n} |\frac{\partial \phi}{\partial x_i}| \} < \infty.
$$

It follows by standard results that the stochastic process

$$
dX_t^{\xi} = \tilde{W}_{\phi_{X_t^{\xi}}}(\xi)\, dt + db_t \ ; \ X_0^{\xi} = x
$$

exists with $\mathcal{A}^{\xi}$ as generator.

The solution of (9) is given by the Feynman-Kac formula [KS, Theorem 5.7.6]

$$
\begin{aligned}
\tilde{u}(t,x,\xi) = \hat{E}^{T-t,x}[\tilde{f}(X_T^{\xi}) \exp\{-\int_{T-t}^{T} \tilde{\kappa}(T-\theta, X_\theta^{\xi})\, d\theta\}] \\
+ \hat{E}^{T-t,x}[\int_{T-t}^{T} \tilde{g}(T-s, X_s^{\xi}) \exp\{-\int_{T-t}^{s} \tilde{\kappa}(T-\theta, X_\theta^{\xi})\, d\theta\}\, ds]
\end{aligned}
$$

and by a change of measure this may be written as

$$
\begin{aligned}
\tilde{u}(t,x,\xi) = \hat{E}^{T-t,x}[\tilde{f}(b_T) \exp\{-\int_{T-t}^{T} \tilde{\kappa}(T-\theta, b_\theta)\, d\theta\} \mathcal{M}_{t,T}] \\
+ \hat{E}^{T-t,x}[\int_{T-t}^{T} \tilde{g}(T-s, b_s) \exp\{-\int_{T-t}^{s} \tilde{\kappa}(T-\theta, b_\theta)\, d\theta\}\, ds \mathcal{M}_{t,T}]
\end{aligned}
$$

where

$$
\mathcal{M}_{t,T} := \exp\{\sum_{i=1}^{n} \eta^{-1} \int_{T-t}^{T} [\tilde{W}^{(i)}_{\phi_y}]_{y=\eta b_s}\, db_s^i - \frac{1}{2} \sum_{i=1}^{n} \eta^{-2} \int_{T-t}^{T} [\tilde{W}^{(i)}_{\phi_y}]^2_{y=\eta b_s}\, ds\}.
$$

This expression is easily seen to have an analytic extension for all $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$ and this completes the proof. ■

The theorem now follows from the previous lemmas. ■

§4 The pollution model in a bounded domain

THEOREM 4.1 Let $T > 0$ be given and suppose $D \subset \mathbb{R}^n$ is a bounded domain such that every point on the boundary of δD has the exterior sphere property; i.e. there exists a ball $B \ni x$ such that $\bar{B} \cap D = \emptyset$, $\bar{B} \cap \delta D = \{x\}$.

Assume furthermore that we are given functions $[0,T] \times \partial D \ni (t,x) \mapsto h(t,x) \in (\mathcal{S}_n^{n,l})_{G,P}^{-1}$, $D \ni x \mapsto \phi(x) \in (\mathcal{S}_n^{n,l})_{G,P}^{-1}$ and $[0,T] \times D \ni (t,x) \mapsto \kappa(t,x) \in (\mathcal{S}_n^{n,l})_{G,P}^{-1}$, $\psi_j \in C_0^2([0,T] \times D)$ and constants $\mu_j > 0$ $(1 \leq j \leq l)$ such that

- $\exists(q_h \in \mathbb{N}, \delta_h > 0, K_h > 0)$ such that
  1. $\sup_{(t,x)\in[0,T]\times\partial D, z\in\mathbf{B}_{q_h}(\delta_h)} |\mathcal{H}h(t,x,z)| \leq K_h$.
  2. $x \mapsto \mathcal{H}h(t,x,z) \in C^2([0,T] \times \partial D)$ whenever $t \in [0,T], z \in \mathbf{B}_{q_h}(\delta_h)$.
- $\exists(q_\phi \in \mathbb{N}, \delta_\phi > 0, K_\phi > 0)$ such that
  1. $\sup_{x\in D, z\in\mathbf{B}_{q_\phi}(\delta_\phi)} |\mathcal{H}\phi(x,z)| \leq K_\phi$.
  2. $x \mapsto \mathcal{H}\phi(x,z) \in C^2(D)$ whenever $z \in \mathbf{B}_{q_\phi}(\delta_\phi)$.
- $\exists(q_\kappa \in \mathbb{N}, \delta_\kappa > 0, K_\kappa > 0)$ such that
  1. $\sup_{(t,x)\in[0,T]\times D, z\in\mathbf{B}_{q_\kappa}(\delta_\kappa)} |\mathcal{H}\kappa(t,x,z)| \leq K_\kappa$.
  2. $x \mapsto \mathcal{H}\kappa(t,x,z) \in C^2(D)$ whenever $t \in [0,T], z \in \mathbf{B}_{q_\kappa}(\delta_\kappa)$.
  3. $\exists(\beta(z) > 0 \ \forall z \in \mathbf{B}_{q_\kappa}(\delta_\kappa))$ such that $(t,x) \mapsto \mathcal{H}\kappa(t,x,z)$ is uniformly Hölder continuous (exponent $\beta(z)$) in (t,x) in compact subsets of $[0,T] \times D$.
  4. $\mathcal{H}\kappa(t,x,z) \geq 0$ whenever $z \in \mathbf{B}_{q_\kappa}(\delta_\kappa) \cap \mathbb{R}_0^N$.
- $h(0,x) = \phi(x)$ $\forall x \in \partial D$

Then

$$u(t,x) = \hat{E}^{T-t,x}[h(T-\tau, b_\tau) \diamond \mathrm{Exp}\{-\int_{T-t}^{\tau} \kappa(T-s, b_s)\, ds\}\chi_{\tau<T} \diamond \mathcal{J}_{t,T}]$$

$$+ \hat{E}^{T-t,x}[\phi(b_T) \diamond \mathrm{Exp}\{-\int_{T-t}^{\tau} \kappa(T-s, b_s)\, ds\}\chi_{\tau=T} \diamond \mathcal{J}_{t,T}]$$

$$+ \hat{E}^{T-t,x}[\int_{T-t}^{\tau} g(T-s, b_s) \diamond \mathrm{Exp}\{-\int_{T-t}^{s} \kappa(T-\theta, b_\theta)\, d\theta\}\, ds \diamond \mathcal{J}_{t,T}]$$

where $\mathcal{J}_{t,T}$ is given by (3), g by (6) and τ is the first time $\lambda \in [t,T]$ that b_s leaves D if such a time exists and $\tau := T$ otherwise, is the unique $(\mathcal{S}_n^{n,l})_{G,P}^{-1}$-valued process which solves

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\eta^2 \Delta u + \vec{W}_{\phi_x} \diamond \nabla u - \kappa(t,x) \diamond u + \sum_{j=1}^{l} \psi_j(t,x) P^{(j)}_{t/\mu_j} && (t,x) \in [0,T] \times D \\ u(0,x) &= \phi(x) && x \in D \\ u(t,x) &= h(t,x) && (t,x) \in [0,T] \times \delta D \end{aligned}$$

where $\hat{E}$ and $\int \cdot \, ds$ are Bochner integrals in $(\mathcal{S}_n^{n,l})_{G,P}^{-1}$.

PROOF:

This follows, since $\tau \leq T$, as in the proof of theorem 3.1, but instead of using the Feynman-Kac formula, we use [F2, Theorem 5.2]. ■

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