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A VERIFICATION THEOREM FOR COMBINED STOCHASTIC CONTROL AND IMPULSE CONTROL

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Abstract

We formulate a general combined stochastic control and impulse control (sequential optimal stopping) problem. A sufficient condition for a function and a strategy to be a solution is given. The condition involves a quasivariational version of the Hamilton-Jacobi-Bellman (HJB) equation.

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1. Introduction

Suppose that a person has M different – more or less risky – assets, in which he can invest his money and that at any time he can – at a certain transaction cost – change his portfolio, i.e. transfer some of the money from any of the investments to any of the others. In addition he has at any time t the choice of a consumption rate $c_t \geq 0$. The goal is to maximize the expected total utility of this consumption up to the first time either t either reaches the future value T or the fortune X_t reaches the value 0 (indicating that the person is bancrupt). What consumption rate should he choose, at what times should he transfer money, how much money and from where to where, in order to achieve this maximum?

This is an example of a " combined stochastic control" – problem: The person has to find an optimal continuous time control of the stochastic system as well as optimal times (and corresponding optimal amounts) of interventions into the system. In other words, the problem is a combination of a stochastic control and an impulse control (sequential optimal stopping) problem. We call such controls *combined stochastic controls* for short.

The purpose of this paper is to prove a sufficient condition (a verification theorem) in terms of quasi-variational inequalities for such a combined stochastic control to be optimal and for a given smooth function to coincide with the corresponding maximal expected utility (the *value function*). The result – and the theory – constitutes a synthesis of the theory of stochastic control and the theory of impulse control. Our result may be regarded as a quasi-variational extension of the familiar Hamilton-Jacobi-Bellman equation from classical stochastic control theory.

There is a rich literature on optimal stopping and stochastic control separately. Impulse control (or sequential optimal stopping) is discussed in detail in [BL]. A slightly extended version with application to optimal starting and stopping of sections of an economy under uncertainty is presented in [BØ1]. A special case of a combined stochastic control problem for piecewise deterministic processes is considered in [DF]. In this paper it is proved that the value function is the unique viscosity solution of the corresponding variational inequalities. The quasi-variational inequalities associated to combined stochastic control are studied in [P1], [P2] in a Sobolev space setting. The conditions assumed there, however, are too strong for many applications.

2. Combined stochastic control

Suppose that - if there are no interventions - the state X_t at time t of the system we consider satisfies an Ito stochastic differential equation of the form

$$dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)dB_t \quad ; \; X_0 = x \in \mathbf{R}^n \tag{2.1}$$

Here $b : \mathbf{R}^{n+k} \to \mathbf{R}^n$ and $\sigma : \mathbf{R}^{n+k} \to \mathbf{R}^{n\times m}$ are given Lipschitz continuous functions, $B_t = B_t(\omega)$; $t \geq 0$, $\omega \in \Omega$, is Brownian motion in $\mathbf{R}^m$ with filtration $\mathcal{F}_t$ and probability law P^y when starting at y, and u_t is an $\mathcal{F}_t$-adapted stochastic process with values in a given set $U \subset \mathbf{R}^k$. The process $u_t = u_t(\omega)$ is our *stochastic control*, whose value we are free to choose (within U) at any time t and for any $\omega \in \Omega$. Here we will only consider *Markov* controls, i.e. controls of the form $u_t(\omega) = u(X_t(\omega))$ where $u : \mathbf{R}^n \to U$ is a measurable (deterministic) function. (One can show that in general one can obtain just as good performance by Markov controls as one can by using the (larger) class of $\mathcal{F}_t$-adapted controls. See e.g. [Ø], Ch. XI.)

Let $\mathcal{U}$ denote the set of Markov controls $u : \mathbf{R}^n \to U$ such that (2.1) has a unique weak solution $X_t = X_t^u$.

Suppose that we at state $y \in \mathbf{R}^n$ decide to intervene and give the system an *impulse* $\zeta \in Z \in \mathbf{R}^l$, where Z is a given set (the set of admissable impulse values). Suppose that the result of giving this impulse is that the state of the system jumps immediately from y to a new state $\gamma(y, \zeta)$, where $\gamma : \mathbf{R}^n \times Z \to \mathbf{R}^n$ is a given function.

An *impulse control* for this system is a double (possibly finite) sequence

$$v = (\tau_1, \tau_2, \cdots, \tau_k, \cdots; \zeta_1, \zeta_2, \cdots, \zeta_k \cdots)_{k \le N} \quad (N \le \infty) \tag{2.2}$$

where $\tau_1 \le \tau_2 \le \cdots$ are $\mathcal{F}_t$-stopping times and $\zeta_1, \zeta_2, \cdots$ are the impulses, $\zeta_k \in Z$. We interpret $\tau_1, \tau_2, \cdots$ as the *intervention times*, i.e. the times when we decide to intervene and give the system the impulses $\zeta_1, \zeta_2, \cdots$, respectively. Let $\mathcal{V}$ denote the set of all impulse controls.

If $u \in \mathcal{U}$ is a stochastic control and $v \in \mathcal{V}$ is an impulse control, we call the pair $w = (u, v) \in \mathcal{U} \times \mathcal{V}$ a *combined stochastic control.*

If $w = (u, v) \in \mathcal{U} \times \mathcal{V}$, with $v = (\tau_1, \tau_2, \cdots; \zeta_1, \zeta_2, \cdots)$, is applied to the system $\{X_t\}$, it gets the value $\{X_t^w\}$, which inductively can be described as follows:

$$\begin{cases} X_t^w = X_{\tau_{k-1}}^w + \int\limits_{\tau_{k-1}}^{t} b(X_s^w, u_s)ds + \int\limits_{\tau_{k-1}}^{t} \sigma(X_s^w, u_s)dB_s; & \tau_{k-1} \le t < \tau_k < T^* \\ X_{\tau_k}^w = \gamma(X_{\tau_k^-}^w, \zeta_k)\ ; \quad k = 1, 2, \ldots\ , & \tau_k < T^* \end{cases} \tag{2.3}$$

where we put $\tau_0 = 0$. Here $T^* = T^*(\omega)$ is the *explosion time* of the process X_t^w, defined by

$$T^*(\omega) = \lim_{R \to \infty} (\inf\{t > 0; |X_t^w(\omega)| \ge R\}) \quad (\le \infty).$$

Let $Q^x = Q^{x,w}$ denote the law of the stochastic process $X_t^{x,w}$ starting at $X_0^{x,w} = x$.

We now describe the *performance criterion* for our system:

Let $S \subset \mathbf{R}^n$ be a fixed domain and define

$$T = T_S^{x,w}(\omega) = \inf\{t \in (0, T^*(\omega)); X_t^{x,w}(\omega) \notin S\}. \tag{2.4}$$

(If $X_t^{x,w}(\omega) \in S$ for all $t \in (0, T^*(\omega))$ we set $T(\omega) = T^*(\omega)$.)

We can think of S as our "universe", in the sense that we are only interested in the system up to the first exit time from S.

DEFINITION 2.1. The space $\mathcal{W}$ of *admissible combined stochastic controls* consists of those combined stochastic controls $w = (u, v) \in \mathcal{U} \times \mathcal{V}$ such that
$v = (\tau_1, \tau_2, \cdots; \zeta_1, \zeta_2, \cdots)$ with $\{\tau_k\}$ satisfying

$$\tau_k \uparrow T \quad \text{a.s. } Q^{x,w}\text{ , for all } x \in \mathbf{R}^n. \tag{2.5}$$

Suppose that the *profit/utility rate* when the system is in state y and the control value is α is $f(y, \alpha)$, where $f : S \times U \to \mathbf{R}$ is a given function. Let $g : \partial S \to \mathbf{R}$ be a *bequest* function (∂S denotes the boundary of S).

Suppose the *profit/utility* of performing an intervention with impulse ζ when the system is in state y is $K(y, \zeta)$, where $K : \overline{S} \times Z \to \mathbf{R}$ is a given function ($\overline{S}$ denotes the closure of S).

Then the *performance* or *total expected profit/utility* obtained when applying the combined stochastic control $w = (u, v) \in \mathcal{W}$, with $v = (\tau_1, \tau_2, \cdots; \zeta_1, \zeta_2, \cdots)$, is defined by

$$J^w(x) = E^x\Big[\int_0^T f(X_s^w, u_s)ds + g(X_T^w) \cdot \chi_{\{T<T^*\}} + \sum_{k=1}^N K(X_{\tau_k^-}, \zeta_k)\Big] \tag{2.6}$$

where E^x is the expectation w.r.t. Q^x.

The *combined stochastic control problem* is the following: Find the *value function* Φ defined by

$$\Phi(x) = \sup_{w \in \mathcal{W}} J^w(x) \; ; \; x \in \overline{S} \tag{2.7}$$

and find an *optimal admissible combined stochastic control* $w^* = w^*(x) = (u^*, v^*) \in \mathcal{W}$ such that

$$\Phi(x) = J^{w^*}(x); \; x \in \overline{S}. \tag{2.8}$$

We assume that

$$E^x\Big[\int_0^T |f(X_t^w, u_t)|dt\Big] < \infty \tag{2.9}$$

for all $x \in S, w \in \mathcal{W}$ and that

$$g : \partial S \to \mathbf{R} \quad \text{is bounded and measurable.} \tag{2.10}$$

REMARK 2.2. Note that the impulse control part $v \in \mathcal{V}$ of our combined stochastic control is more general than what is usually considered in the literature, e.g. in [BL], because we do *not* assume that the intervention profit $K(y, \zeta)$ is non-zero. (See, however, Remark 3.2.)

3. Quasi-variational HJB inequalities

In this section we state and prove sufficient quasi-variational inequalities for the problem (2.7), (2.8). These inequalities contain as special cases both the quasi-variational inequalities for impulse control (see [BL], [BØ2]) and the Hamilton-Jacobi-Bellman (HJB) equation for stochastic control (see e.g. [Ø], Ch. XI).

First we introduce some notation and terminology. Let A^u be the generator of the Ito diffusion X_t^u obtained by applying the Markov control u and no interventions. Then A^u coincides on $C_0^2(\mathbf{R}^n)$ (the twice continuously differentiable functions on $\mathbf{R}^n$ with compact support) with the partial differential operator L^u defined by

$$L^u\phi(x) = \sum_{i=1}^{n} b_i(x,u)\frac{\partial \phi}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(x,u)\frac{\partial^2\phi}{\partial x_i \partial x_j}, \tag{3.1}$$

for all functions $\phi : \mathbf{R}^n \to \mathbf{R}$ for which the derivatives involved exist at x.

The *Green measure* $G(x,\cdot)$ of X_t^w in a domain $V \subset \mathbf{R}^n$ (w.r.t. the starting point $x \in V$) is defined by

$$G(x,H) = G_V^{X^w}(x,H) = E^x[\int_0^{\tau_V} \chi_H(X_t^w)dt] \ ; \ H \subset V \quad \text{Borel sets} \tag{3.2}$$

where

$$\tau_V = \inf\{t > 0;\ X_t^w \notin V\}.$$

A continuous function $\phi : \overline{S} \to \mathbf{R}$ is called *stochastically* C^2 w.r.t. $X_t = X_t^u$ if $L^u\phi(y)$ exists for a.a.y w.r.t the Green measure $G_S^{X^u}(x,\cdot)$ and the *generalized Dynkin formula* holds for ϕ, i.e.

$$E^x[\phi(X_{\tau'})] = E^x[\phi(X_\tau)] + E^x[\int_\tau^{\tau'} L^{u(X_t)}\phi(X_t)dt] \tag{3.3}$$

for all bounded stopping times τ, τ' with $\tau \le \tau' \le \inf\{t > 0; |X_t^u| \ge R\}$ for some $R < \infty$.

Remark. This concept was introduced in [BØ1]. There it was proved that, under some conditions, a function ϕ which is C^1 everywhere and C^2 outside a 'thin' set (in a Green measure sense) is stochastically C^2. For details see [BØ1].

Let $\mathcal{H}$ denote the space of all measurable functions $h : \mathbf{R}^n \to \mathbf{R}$. Define the *intervention operator* (or *switching* operator) $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{M}h(x) = \sup_{\zeta \in Z}\{h(\gamma(x,\zeta)) + K(x,\zeta)\} \ ; \ h \in \mathcal{H}, x \in \mathbf{R}^n. \tag{3.4}$$

Suppose that for each $x \in \mathbf{R}^n$ there exists at least one $\hat{\zeta} = \hat{\zeta}(x) \in Z$ such that the supremum in (3.4) is attained and that a *measurable selection* $\hat{\zeta} = \mathcal{R}_h(x)$ of such maximum points $\hat{\zeta}$ exists. Then we have

$$\mathcal{M}h(x) = h(\gamma(x,\mathcal{R}_h(x))) + K(x,\mathcal{R}_h(x));\ x \in \mathbf{R}^n. \tag{3.5}$$

We now formulate the quasi-variational HJB inequalities. They provide *sufficient* conditions that a given function ϕ and a combined stochastic control $w \in \mathcal{W}$ actually satisfies $\phi = \Phi$ and $w = w^*$ in (2.7), (2.8).

THEOREM 3.1 (Sufficient quasi-variational HJB inequalities)

a) Suppose we can find a continuous function $\phi : \overline{S} \to \mathbf{R}$ such that

(3.6) ϕ is stochastically C^2 w.r.t. X_t^w in S, for all $w \in \mathcal{W}$ without interventions

(3.7) $L^\alpha \phi(x) + f(x,\alpha) \le 0$ for a.a. $x \in S$ with respect to $G(y,\cdot)$, for all $y \in S$ and all $\alpha \in U$

(3.8) $\phi \ge \mathcal{M}\phi$ on $\overline{S}$

(3.9) $\phi(X_t) \to g(X_T) \cdot \chi_{\{T<T^*\}}$ as $t \to T$ a.s. $Q^{x,w}$ for all $x \in \mathcal{S}, w \in \mathcal{W}$, and

(3.10) $\{\phi(X_\tau)\}_{\tau \le T}$ is uniformly integrable w.r.t. $Q^{x,w}$, for all $x \in S, w \in \mathcal{W}$.

Then

$$\phi(x) \ge J^w(x) \quad \text{for all} \quad w \in \mathcal{W}, x \in S. \tag{3.11}$$

b) Define

$$D = \{x;\ \phi(x) > \mathcal{M}\phi(x)\}. \tag{3.12}$$

Suppose that, in addition to (3.6)-(3.10) above, there exists a function $\hat{u} : D \to U$ such that

$$L^{\hat{u}(x)}\phi(x) + f(x,\hat{u}(x)) = 0 \quad \text{for all} \quad x \in D \tag{3.13}$$

Define the impulse control

$$\hat{v} := (\hat{\tau}_1, \hat{\tau}_2, \cdots; \hat{\zeta}_1, \hat{\zeta}_2, \cdots)$$

inductively as follows: $\hat{\tau}_0 = 0$ and

$$\hat{\tau}_{k+1} = \inf\{t \in (\hat{\tau}_k, T^*); X_t^{(k)} \notin D\} \wedge T, \tag{3.14}$$

$$\hat{\zeta}_{k+1} = \mathcal{R}_\phi(X^{(k)}_{\hat{\tau}^-_{k+1}}) \quad \text{if} \quad \hat{\tau}_{k+1} < T, \tag{3.15}$$

where $X_t^{(k)}$ is the result of applying the combined stochastic control

$$\hat{w}_k := (\hat{u}, (\hat{\tau}_1, \cdots, \hat{\tau}_k; \hat{\zeta}_1, \cdots, \hat{\zeta}_k)).$$

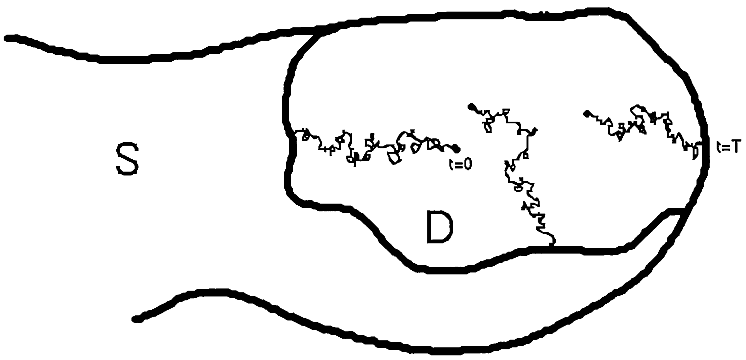
Put $\widehat{w} = (\widehat{u}, \widehat{v})$ and suppose

$$\lim_{k\to\infty} \hat{\tau}_k = T \quad \text{a.s. } Q^{x,\widehat{w}} \text{ for all } x \in \mathbf{R}^n. \tag{3.16}$$

Then

$$\phi(x) = \Phi(x) \tag{3.17}$$

and the combined stochastic control $w^* = \widehat{w} \in \mathcal{W}$ is optimal.



Proof. a) Assume that ϕ satisfies (3.6)-(3.10). Choose $w = (u, v) \in \mathcal{W}$, where

$$v = (\tau_1, \tau_2, \cdots; \zeta_1, \zeta_2, \cdots).$$

For $R > 0$ put

$$T_R = R \wedge \inf\{t > 0; |X_t^w| \geq R\}$$

and set

$$\theta_k = \theta_k^{(R)} = \tau_k \wedge T_R, X_t = X_t^w.$$

Then by the generalized Dynkin formula and (3.7)

$$\begin{aligned} E^x[\phi(X_{\theta_{k+1}^-})] &= E^x[\phi(X_{\theta_k})] + E^x[\int_{\theta_k}^{\theta_{k+1}} A^{u(X_t)}\phi(X_t)dt] \\ &\leq E^x[\phi(X_{\theta_k})] - E^x[\int_{\theta_k}^{\theta_{k+1}} f(X_t, u(X_t))dt]. \end{aligned} \tag{3.18}$$

Or,

$$E^x[\phi(X_{\theta_k})] - E^x[\phi(X_{\theta_{k+1}^-})] \geq E^x[\int_{\theta_k}^{\theta_{k+1}} f(X_t, u(X_t))dt].$$

Letting $R \to \infty$ we obtain

$$E^x[\phi(X_{\tau_k})] - E^x[\phi(X_{\tau_{k+1}^-})] \geq E^x[\int_{\tau_k}^{\tau_{k+1}} f(X_t, u(X_t))dt]. \tag{3.19}$$

Summing from $k = 0$ to $k = m$ gives

$$\begin{aligned} &\phi(x) + \sum_{k=1}^{m} E^x[\phi(X_{\tau_k}) - \phi(X_{\tau_k^-})] - E^x[\phi(X_{\tau_{m+1}^-})] \\ &\geq E^x[\int_0^{\tau_{m+1}} f(X_t, u(X_t))dt]. \end{aligned} \tag{3.20}$$

Now

$$\phi(X_{\tau_k}) = \phi(\gamma(X_{\tau_k^-}, \zeta_k)) \le \mathcal{M}\phi(X_{\tau_k^-}) - K(X_{\tau_k^-}, \zeta_k) \quad \text{if} \quad \tau_k < T$$
$$\phi(X_{\tau_k}) = \phi(X_{\tau_k^-}) = g(X_T) \cdot \chi_{\{T<T^*\}} \quad \text{if} \quad \tau_k = T, \tag{3.21}$$

and therefore

$$\phi(x) + \sum_{k=1}^{m} E^x[(\mathcal{M}\phi(X_{\tau_k^-}) - \phi(X_{\tau_k^-})) \cdot \chi_{\tau_k<T}]$$
$$\ge E^x[\int_0^{\tau_{m+1}} f(X_t, u(X_t))dt + \phi(X_{\tau_{m+1}^-}) + \sum_{k=1}^{m} K(X_{\tau_k^-}, \zeta_k)] \tag{3.22}$$

By (3.8)

$$\mathcal{M}_\phi(X_{\tau_k^-}) - \phi(X_{\tau_k^-}) \le 0 \tag{3.23}$$

and hence

$$\phi(x) \ge E^x[\int_0^{\tau_{m+1}} f(X_t, u(X_t))dt + \phi(X_{\tau_{m+1}^-}) + \sum_{k=1}^{m} K(X_{\tau_k^-}, \zeta_k)]. \tag{3.24}$$

Letting $m \to N$ ($N \le \infty$) we get, using (2.5), (2.9), (3.9) and (3.10),

$$\phi(x) \ge E^x[\int_0^{T} f(X_t, u(X_t))dt + g(X_T) \cdot \chi_{\{T<T^*\}} + \sum_{k=1}^{N} K(X_{\tau_k^-}, \zeta_k)]. \tag{3.25}$$

Hence $\phi(x) \ge J^w(x)$ as claimed in (3.11).

b) Next, assume that (3.13) also holds. Define $\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \cdots; \hat{\zeta}_1, \hat{\zeta}_2, \cdots)$ by the formulas (3.14) and (3.15) and put $\hat{w} = (\hat{u}, \hat{v})$. Then repeat the argument in part a) for $w = \hat{w}$: We see that in all the inequalities (3.18) - (3.24) we now get *equalities*. So we conclude that

$$\phi(x) = E^x[\int_0^{\hat{\tau}_{m+1}} f(X_t, \hat{u}(X_t))dt + \phi(X_{\hat{\tau}_{m+1}^-}) + \sum_{k=1}^{m} K(X_{\hat{\tau}_k^-}, \hat{\zeta}_k)] \quad \text{for all } m. \tag{3.26}$$

By (3.16) we have that $\hat{w} = (\hat{u}, \hat{v}) \in \mathcal{W}$ and by letting $m \to \infty$ in (3.26) we get

$$\phi(x) = J^{\hat{w}}(x). \tag{3.27}$$

Combining this with (3.11) we obtain

$$\phi(x) \ge \sup_{w \in \mathcal{W}} J^w(x) \ge J^{\hat{w}}(x) = \phi(x)$$

Hence $\phi = \Phi$ and $w^* = \widehat{w}$ is optimal. □

REMARK 3.2. Condition (3.16) in Theorem 3.1 will hold automatically if the intervention profit $K(y,\zeta)$ satisfies the following condition:

(3.28) For all $x_0 \in \overline{S}$ there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |K(x,\zeta)| \geq \delta$ for all $\zeta \in Z$.

(In other words, $K(x,\zeta)$ is locally bounded away from 0.)

To see this assume that (3.28) holds and define

$$\hat{\tau} = \lim_{k\to\infty} \hat{\tau}_k.$$

Then $\hat{\tau} \leq T$. Let $\Omega_0 = \{\omega; \hat{\tau}(\omega) < T\}$. For $\omega \in \Omega_0$ we have that

$$\lim_{m\to\infty} X_{\hat{\tau}_m^-}(\omega) = X_{\hat{\tau}^-}(\omega) \in S$$

and so there exists by (3.28) $\delta(\omega) > 0$ and $k_0(\omega) < \infty$ such that

$$|K(X_{\hat{\tau}_k^-}(\omega), \hat{\zeta}_k)| \geq \delta(\omega) \quad \text{for all} \quad k \geq k_0(\omega).$$

Therefore we get a divergent series on the right hand side of (3.26), for all $\omega \in \Omega_0$. Hence $Q^x[\Omega_0] = 0$ and (3.16) is proved.

REMARK 3.3. In [MØ] Theorem 3.1 is applied to the problem of controlling optimally the currency exchange rate under uncertainty.

In [W] the impulse control part of Theorem 3.1 is applied to the stochastic rotation problem in forestry.

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