Strategic investment behaviour under uncertainty: A stylized model

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Abstract

A stylized model is considered: Two firms have one irreversible investment opportunity each. The firm which invests first can enjoy being the sole producer in a market with uncertain demand, until the other firm invests. A nontrivial game arises if both firms have incentive to exploit that advantage.

Equilibrium investment strategies are found. If the equilibrium strategies are played and the firms have equal unit cost, the smaller of the firms will invest first. The equilibrium strategies might lead to earlier investment than in the monopoly case.

1 Introduction

The classical attempt to explain investment behaviour is the *present value principle* (See for instance Copeland and Weston for an overview [3]). Intuitively, the present value principle is not an optimal investment rule except for the unreal case of full certainty. When the expected reward from investing is zero, one would be at least as well off not investing at all, and it therefore seems reasonable to demand an expected reward higher than zero. In other words, the value of having the *option* to invest later is strictly positive if there is any possibility that conditions might turn out more favourable in the future.

These and related ideas are discussed in the literature on *real options*, which treats the problem of how to take uncertain future price development into account when making investment decisions. The main body of the literature considers the investment problem of a monopoly firm (for instance McDonald and Siegel [8] or Kobila [7]). A broader discussion may be found in Dixit and Pindyck [4]. According to the analysis conducted in most of these papers, investors should postpone exercising their options to invest until expected reward has reached a level which exceeds the option value of their investment possibility. The real option approach seems to be more consistent with empirical observation than the classical attempts to explain investment behaviour, although thorough statistical analysis of the matter has been difficult because of lack of suitable data (see Abel and Blanchard [1]).

A firm waiting for a high expected reward, must in many cases be prepared for the possibility that competitors might be tempted by the profit opportunities, and choose to enter the market at some time (slightly) before the time at which the waiting firm intends to invest. This will alter the profit outlook for the firm. Concequently, if entry is possible, the investor has to think strategically.

The purpose of this paper is to investigate strategic behaviour in this context. The simplest possible case of only two firms each owning one exlusive investment possibility is considered through a stylized model. The analysis will reveal that an equilibrium solution is that the firm which has the lowest point of indifference between investing first and secondly invests first, and does so at or before the competitor's corresponding point of indifference. In some cases, the problem reduces to a triviality since one of the firms will always want to invest first, and the other will always want to wait. If unit cost is equal, the smaller firm invests first, and the first investment is not undertaken at a time later than the time a monopoly firm owning both projects would choose. It might very well be undertaken at some earlier point in time.

Since the recent development in real option theory is formulated in continuous time, it is natural to look to continous game theory for mathematical tools. A discussion of zero sum stochasic games viewed as generalized stopping problems may be found in Avner Friedman's book on stochastic differential equations [6]. Nash equilibria, however, which are the relevant solution concept in most economic applications such as the one at hand, are not tackled in his paper.

Dutta and Rustichini [5] discuss non zero sum symmetric stochastic games. Their results are too restrictive for the present situation, since we want to consider cases where the firms might have different payoff functions due to differences in the investment possibilities. A simplification of the forthcoming model to the case that the investment possibilities are identical has been considered in an unpublished paper by Smets [10], cited in Dixit and Pindyck's chapter on industry equilibrium. Smets builds his analysis on Dutta and Rustichini.

2 The game model

Let Q_t be the solution to the stochastic differential equation

$$dQ_t = \alpha Q_t dt + \beta Q_t dB_t,\ Q_0 = q > 0. \tag{1}$$

Consider the following investment game involving two investors A and B, where $i = A, B$, $j = A, B$ and $i \neq j$:

- The set of strategies are taken to be all pairs of stopping times (τ_i^1, τ_i^2).
- Each pair of strategies $[(\tau_A^1, \tau_A^2), (\tau_B^1, \tau_B^2)]$ has *investment times* given by

$$\tau_A = \begin{cases} \tau_A^1 & \text{if } \tau_A^1 \leq \tau_B^1 \\ \tau_A^2 & \text{if } \tau_A^1 > \tau_B^1, \end{cases} \tag{2}$$

  and

$$\tau_B = \begin{cases} \tau_B^1 & \text{if } \tau_B^1 < \tau_A^1 \\ \tau_B^2 & \text{if } \tau_B^1 \geq \tau_A^1. \end{cases} \tag{3}$$

- Player i's expected payoff is given by

$$J_i^q[(\tau_A^1, \tau_A^2), (\tau_B^1, \tau_B^2)] = E^q \left[\int_{\tau_i}^{\infty} \frac{Q_t}{S_t} s_i e^{-rt} dt - C_i e^{-r\tau_i} \right], \tag{4}$$

  where

  - r is the discount factor
  - $S_t = s_A \chi_{t \geq \tau_A}(t) + s_B \chi_{t \geq \tau_B}(t)$
  - s_i is the constant production capacity resulting from player i's investment
  - C_i denotes the cost of player i's investment

For a strategy (τ_i^1, τ_i^2), the first entry is interpreted as the stopping time at which investor i plans to invest if he is first, and the second entry the stopping time he otherwise plans to choose. In order to simplify notation, it is tempting to specify player i's strategy space to be only all stopping times τ_i and thereby neglect his choice of stopping time in case he is not first. One might reason that once one of the players has invested, the game is resolved, and the choice of stopping time in case one is not first to invest is therefore superflous. A priori, that is too restrictive, since player j's choice τ_j^2 normally will influence player i's choice τ_i^1. On the other hand, once one of the players has moved, the other player can do nothing better than to optimize his behaviour given that he cannot be first. In fact, we shall be interested in eliminating equilibria which involve strategies which do not optimize behaviour out of equilibrium.

For a given strategy pair, the investment times indicate when each of the investors will invest if they stick to their strategies. Note that the game is specified so that simultaneous action cannot happen. If $\tau_i^1 = \tau_i^2$, we postulate that only one of the players can invest, and we label the investors such that it is investor A who invests first in case of a tie. This way to tackle the problem of ties is the same as in Friedman [6].

The fraction $\frac{Q_t}{S_t}$ is interpreted as the market price at time t. The process Q_t therefore indicates market demand level. The payoff is in other words the integral of the discounted cash flow, minus the discounted cost which occurs only at the time when the investment is undertaken. We shall suppose that $\alpha < r$.

We shall use the following solution concept:

Definition A *stochastic game Nash equilibrium* (at q) is a strategy pair

$$[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)] \tag{5}$$

such that

$$J_A^q[(\tau_A^1, \tau_A^2), ({\tau_B^1}^*, {\tau_B^2}^*)] \leq J_A^q[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)] \tag{6}$$

for all strategies (τ_A^1, τ_A^2), and

$$J_B^q[({\tau_A^1}^*, {\tau_A^2}^*), (\tau_B^1, \tau_B^2)] \leq J_B^q[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)] \tag{7}$$

for all strategies (τ_B^1, τ_B^2).

A Nash equilibrium is nothing but a strategy combination which none of the players has incentive to deviate from, given the competitor's strategy. As pointed out above, there might be many Nash equilibria which are not credible in the sense that they depend on one of the players having a strategy which implies irrational behaviour in out of equilibrium play. In other words, there are equilibria which depend on one of the players threatening to play in a way which would be counter to his own interest, given the opportunity to play in the way he threats. In our case we shall require so called *subgame perfection* in order to rule out such equilibria. More presicely, if for some stochastic game Nash equilibrium $[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)]$, investment times for player A and B respectively are ${\tau_A^1}^*$ and ${\tau_B^2}^*$, then we require ${\tau_B^1}^*$ to be optimal for player B conditioned that player A plays ${\tau_A^2}^*$, and ${\tau_A^2}^*$ to be optimal for player A conditioned that player B plays ${\tau_B^1}^*$. In other words, if A plays ${\tau_A^2}^*$ which would be an out of equilibrium action, then B can gain nothing from changing strategy. Similarly, if B plays ${\tau_B^1}^*$ which also would be an out of equilibrium action, then A can gain nothing from changing strategy.

Definition Let

$$f_i(q) = \sup_{\sigma} E^q \left[\int_{\sigma}^{\infty} \frac{Q_t}{s_i + s_j} s_i e^{-rt} dt - C_i e^{-r\sigma} \right], \tag{8}$$

and let the optimal stopping time that solves the maximization problem be denoted by τ_i^2. Furthermore, let

$$l_i(q) = E^q \left[\int_0^{\infty} \frac{Q_t}{s_i + s_j \chi_{t \geq \tau_j^2}(t)} s_i e^{-rt} dt - C_i \right]. \tag{9}$$

We shall call $l_i(q)$ the *value of investing first at* q, or *leader's payoff at* q. The function $f_i(q)$ is called the *value at* q *given that the opponent has already invested* or *follower's payoff at* q. Note that subgame perfection is implicitly assumed in the definition of follower's payoff. The functions $l_i(q)$ and $f_i(q)$ will play a central role in characterizing subgame perfect stochastic game Nash equilibria to the game.

In appendix 1 it is shown that

$$f_i(q) = \begin{cases} \frac{C_i}{\gamma-1}\left(\frac{q}{q_i^2}\right)^\gamma & \text{if } q < q_i^2 \\ \frac{s_i}{(s_i+s_j)(r-\alpha)}q - C_i & \text{if } q \geq q_i^2, \end{cases} \tag{10}$$

where

$$q_i^2 = \frac{\gamma}{\gamma-1}(r-\alpha)\left(\frac{s_i+s_j}{s_i}\right)C_i, \tag{11}$$

and

$$\gamma = \frac{\frac{\beta^2}{2} - \alpha + \sqrt{\left(\frac{\beta^2}{2}-\alpha\right)^2 + 2\beta^2 r}}{\beta^2} > 1. \tag{12}$$

Furthermore,

$$l_i(q) = \begin{cases} \frac{q}{r-\alpha} - C_j\frac{\gamma}{\gamma-1}\left(\frac{q}{q_j^2}\right)^\gamma - C_i & \text{if } q < q_j^2 \\ \frac{s_i}{(s_i+s_j)(r-\alpha)}q - C_i & \text{if } q \geq q_j^2 \end{cases} \tag{13}$$

Finally, if it is predetermined that firm i shall invest first, and that firm j will invest at $q \geq q_j^2$, then if $l_i(q_i^1) > 0$, the unique optimal trigger level of firm i is given by

$$q_i^1 = \frac{\gamma}{\gamma-1}(r-\alpha)C_i. \tag{14}$$

3 Analysis of the game

Recall that $l_i(q)$ is the value of investing first at the starting point q, while $f_i(q)$ is the value of i's investment possibility at q, given that player j has already invested. The number q_i^1 is the leader's optimal trigger level if it is predetermined that he shall be the leader.

Definition Let

$$\bar{q}_i = \min\{q : l_i(q) = f_i(q)\}. \tag{15}$$

Examining the functions $l_i(q)$ and $f_i(q)$, it is clear that there is a least point $\bar{q}_i$ of intersection, that is, there is a point where player i is indifferent between investing first and secondly. For starting points q less than $\bar{q}_i$, it is a disadvantage to invest first, while for starting points q bigger than $\bar{q}_i$, it is advantageous to invest first.

Observe the following: If $[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)]$ is a Nash equilibrium at some starting point $\tilde{q}$ and $\tilde{q} < {q_A^1}^* \wedge {q_B^1}^*$, then $[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)]$ is a Nash equilibrium at all $q \leq {q_A^1}^* \wedge {q_B^1}^*$. Therefore it is sufficient to examine whether $[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)]$ is a Nash equilibrium at ${q_A^1}^* \wedge {q_B^1}^*$ to establish whether $[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)]$ is a Nash equilibrium at $q \in (0, {q_A^1}^* \wedge {q_B^1}^*]$. But since

$$J_A^q[(0, \tau_A^2), (\tau_B^1, \tau_B^2)] = l_A(q) \tag{16}$$

if $\tau_B^1 \geq \tau_q$ and

$$J_A^q[(\tau_A^1, \tau_A^2), (0, \tau_B^2)] = f_A(q) \tag{17}$$

if $\tau_A^1 > \tau_q$ and τ_A^2 and τ_B^2 are optimally chosen, and likewise for $l_B(q)$ and $f_B(q)$, this can be done comparing the functions $l_i(q)$ and $f_i(q)$.

Theorem 3.1 *Choose labels A and B such that $\bar{q}_B \geq \bar{q}_A$. For all starting points $q \in (0, q_A^1 \wedge \bar{q}_B]$ there exists a unique subgame perfect stochastic game Nash equilibrium*

$$[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)]$$

with investment times ${\tau_A^1}^*$ and ${\tau_B^2}^*$ satisfying ${\tau_A^1}^* = \tau_A^1 \wedge \bar{\tau}_B$ and ${\tau_B^2}^* = \tau_B^2$.

Proof We shall use that the stopping time τ_A^1 which player would choose if he were sure to invest first, is given by a trigger level q_A^1 such that $\tau_A^1 = \tau_{q_A^1}$ (see proposition 6.3).

First of all, it cannot be optimal for player B to choose ${\tau_B^1}^* < \bar{\tau}_B$. Therefore a subgame perfect equilibrium satisfies ${\tau_B^1}^* \geq \bar{\tau}_B$. Let B's strategy be $(\bar{\tau}_B, \tau_B^2)$. Since $\bar{q}_A \leq \bar{q}_B$, A's best response is investing at $\bar{\tau}_B$ or at some time before $\bar{\tau}_B$ and preferably at τ_A^1. Suppose that $q_A^1 > \bar{q}_B$. Then since the value at the starting point q of investing first at $\tau_{\tilde{q}}$,

$$l_i(\tilde{q}) \left(\frac{q}{\tilde{q}}\right)^\gamma, \tag{18}$$

is increasing in $\tilde{q}$ for all $\tilde{q} \leq \bar{q}_B$ (see the proof of Proposition 6.3), A's best response is to invest at $\bar{\tau}_B$.

Suppose that B's strategy is $(\tilde{\tau}_B, \tau_B^2)$ where $\bar{q}_B < \tilde{q}_B \leq q_B^2$ and let $q_A^1 \wedge \bar{q}_B = \bar{q}_B$. Then by the same argument as above, A's best response is ${\tau_A^1}^* = \tau_A^1 \wedge \tilde{\tau}_B$. But then by the same argument, B's best response is some ${\tau_B^1}^*$ satisfying ${q_B^1}^* \in (\bar{q}_B, q_A^1 \wedge \tilde{q}_B)$ which implies that $(\tilde{\tau}_B, \tau_B^2)$ cannot be an optimal strategy for B if $q_A^1 > \bar{q}_B$.

Suppose that A's strategy is $(\tau_A^1 \wedge \bar{\tau}_B, \tilde{\tau}_A)$, where $\tilde{q}_A$ satisfies $\tilde{q}_A \geq q_A^1 \wedge \bar{q}_B$. Then obviously it cannot be an equilibrium strategy for B to invest before A, and B's best response is τ_B^2. $\square$

3.1 High starting point levels

The main theorem states a possible solution if the starting point q satisfies $q \in (0, q_A^1 \wedge \bar{q}_B]$, that is, the game is entered at some starting point which allows for the equilibrium solution stated above. The question is, what if $q > q_A^1 \wedge \bar{q}_B$? In order to answer this, we need to examine the geometry of the functions $l_i(q)$ and $f_i(q)$ more closely.

Lemma 3.2 *Suppose that $q_j^2 > q_i^2$. Then there is exactly one point, $\bar{q}_i < q_j^2$ such that*

$$l_i(q) \begin{cases} < f_i(q) & \text{if } q \in (0, \bar{q}_i) \\ = f_i(q) & \text{if } q = \bar{q}_i \text{ or } q \geq q_j^2 \\ > f_i(q) & \text{if } q \in (\bar{q}_i, q_j^2). \end{cases} \tag{19}$$

Further, we have that l_j equals f_j in exactly two different points $\bar{q}_i$ and $\hat{q}$, satisfying $\bar{q}_i < \hat{q} < q_j^2$, or l_j equals f_j in fewer than two points. In case both $\bar{q}_i$ and $\hat{q}$ exist, then

$$l_j(q) \begin{cases} < f_j(q) & \text{if } q \in (0, \bar{q}_i) \text{ or } q \in (\hat{q}, q_j^2) \\ = f_j(q) & \text{if } q = \bar{q}_i \text{ or } q = \hat{q} \text{ or } q \geq q_j^2 \\ > f_j(q) & \text{if } q \in (\bar{q}_i, \hat{q}), \end{cases} \tag{20}$$

otherwise $l_j(q) \leq f_j(q)$ for all positive q.

Proof The lemma follows directly from considering the expressions for the functions $l_i(q)$ and $f_i(q)$. □

There are two cases: Either l_j equals f_j in exactly two different points $\bar{q}_j$ and $\hat{q}$, $\bar{q}_j < \hat{q} < q_j^2$, or l_j equals f_j in no points or only in one tangency point. In the last case, the game is of course trivial since player j has no incentive to invest first.

In light of the lemma above, it is clear that $0 < \bar{q}_i < q_i^1$. This follows from the fact that if $l_i(q)$ and $f_i(q)$ meet in one tangency point, that point is q_i^1.

It is natural to ask whether $q_j^2 > q_i^2$ implies that $\bar{q}_i \leq \bar{q}_j$. That may be concluded if the additional condition $C_i < \gamma C_j$ is fulfilled (see theorem 3.6 and lemma 7.1).

Theorem 3.3 *Let $\tilde{q}$ be the point $q_A^1 \wedge \bar{q}_B$ found in theorem 3.1. Now choose the labels A and B such that $q_A^2 \leq q_B^2$ and let l_B equal f_B in exactly two different points $\bar{q}_B$ and $\hat{q}$. If the starting point q satisfies $\tilde{q} < q$ then all subgame perfect stochastic game Nash equilibria at q,*

$$[(\tau_A^{1*}, \tau_A^{2*}), (\tau_B^{1*}, \tau_B^{2*})],$$

have investment times τ_A^{1*} and τ_B^{2*} satisfying $\tau_A^{1*} = \tau_q$ and $\tau_B^{2*} = \tau_q \vee \tau_B^2$.

Proof There are three cases to be considered:

- If the starting point q satisfies $\tilde{q} < q < \hat{q}$, the result follows by the same type of argument as in the proof of Theorem 3.1.
- Suppose that the starting point q satisfies $\hat{q} \leq q < q_B^2$. First, B has no incentive to invest in the region $\hat{q} < q < q_B^2$. Let B's strategy be $(\tau_{\hat{q}} \wedge \tau_B^2, \tau_B^2)$. If $q \geq q_A^1$, it follows from the argument in the proof of proposition 6.3 that the best response from A is to invest immediately. If $q < q_A^1$, A must weigh the value of waiting against the reward of investing right away. But since the reward function is concave and the value of waiting is a convex function combining two points on the reward function, the reward from investing right away is strictly higher than the value of waiting. Therefore, A's best response to B's strategy is to invest at q. Obviously, firm A's best response if firm B invests first at q is τ_A^2 if $q < q_A^2$, else τ_q. If A's strategy is $(\tau_q, \tau_A^2$ if $q < q_A^2$ else $\tau_q)$, it is easily seen that the strategy $(\tau_{\hat{q}} \wedge \tau_B^2, \tau_B^2)$ is optimal for player B.
- If the starting point q satisfies $q \geq q_B^2$ the result is trivial.

□

Note that if the game is entered at demand levels exeeding $q_A^1 \wedge \bar{q}_B$, where the labels A and B are chosen as in theorem 3.1, it is no longer the least points of indifference which is relevant to which firm will invest first, but rather which firm has the least trigger level in case it is second to invest.

3.2 The trivial case of no conflicting interests

It is of interest to single out the case where one of the firms has no incentive to invest first, since the game then reduces to a triviality. Intuitively, that might happen if one of the firms has so high cost relative to production capacity that very high demand levels are needed in order that the investment be profitable.

Proposition 3.4 *The inequality $l_i(q) \leq f_i(q)$ holds for all nonnegative q if and only if*

$$\left(\frac{s_j}{s_i} + 1\right)^{\gamma} - \gamma \frac{C_j}{C_i} \left(\frac{C_i s_j}{C_j s_i}\right)^{\gamma} \leq 1. \tag{21}$$

Proof The proof may be found in appendix 2. □

We may conclude that if the fraction $\frac{C_i}{C_j}$ is sufficiently large, or $\frac{C_i}{C_j} > 1$ and $\frac{s_j}{s_i}$ is sufficiently large, the expression is less than or equal to 1. Put differently, if one of the firms has sufficiently higher cost than the other firm, or higher cost and sufficiently low production capacity relative to the competitor, the firm which has the higher cost has no incentive to invest first. In other words:

Proposition 3.5 *Let $l_i \leq f_i$ for all $q > 0$. Then for all starting points $q > 0$ there is a unique subgame perfect stochastic game Nash equilibrium*

$$[({\tau_A^1}^*, {\tau_A^2}^*), ({\tau_B^1}^*, {\tau_B^2}^*)]$$

which has investment times ${\tau_A^1}^*$ and ${\tau_B^2}^*$ satisfying ${\tau_A^1}^* = \tau_A^1 \vee q$ and ${\tau_B^2}^* = \tau_B^2 \vee q$.

3.3 Comparison of the characteristics of the firm which will invest first to the characteristics of the other firm

It is natural to ask what characterizes the firm which has the lowest point of indifference between being first and second to invest and thereby, by theorem 3.1 which firm will invest first, in terms of cost and production capacity compared to the other firm.

Theorem 3.6 *If $C_i < \gamma C_j$ and $\frac{C_i}{C_j} < \frac{s_i}{s_j}$, then $\bar{q}_j > \bar{q}_i$, and by theorem 3.1 firm i will invest first.*

Proof The proof may be found in appendix 2. □

Intuitively, the greater firm will plan its investment more independently of the smaller firm, since the smaller production capacity will have lesser impact on the product price than the bigger. If the difference in size is sufficient, it will be relatively profitable for the little firm to be alone in the market, while the bigger firm will plan its investment almost as if the smaller firm did not exist. Therefore, one would suspect that the smaller firm will invest first. On the other hand, if the firms are more equally sized, both will have a strong interest in being first. Intuition suggests that in this case the bigger firm might be able to exploit lower cost-production capacity ratio to invest profitably at lower prices than the smaller, and thus manage to be first. Theorem 3.6 implies that both views may be correct.

3.4 The case of equal unit cost

It turns out to be possible to deduce sharper results if unit costs are equal:

Theorem 3.7 *Suppose that $\frac{C_A}{C_B} = \frac{s_A}{s_B}$. Then $C_B > C_A$ if and only if $\bar{q}_B > \bar{q}_A$, and by theorem 3.1 firm A will invest first.*

Proof The proof may be found in appendix 2. □

In other words, if the firms have identical unit costs, the intuition that the greater firm will plan its investment more independently of the smaller firm than vice versa holds. The smaller production capacity has, when realized, lesser impact on the product price than the bigger. The smaller firm invests first, and thus achieves an extraordinary profit.

3.4.1 The firms are equal

If the firms are equal, we may write $l_i = l_j$, $f_i = f_j$, $\bar{q}_i = \bar{q}_j = \bar{q}$. In this case $q_A^1 \wedge \bar{q}_B = q_B^1 \wedge \bar{q}_A = \bar{q}$. Since the firms are indistinguishable, it is impossible to say which firm will invest first. If $q = \bar{q}$, reward from investing first at q is equal to investing secondly at q^2. The prediction is in other words that the firms will have equal expected profit unless q already exeeds $\bar{q}$ when the players enter the game.

4 Comparison with the monopoly case

In this section, investment times chosen by the participants in the game is compared to the investment times chosen by a monopolist owning both investment projects. The case when the two investment projects in question have equal unit cost, but are not equal, is considered. We shall se that competition might lead to earlier investments and thereby lower product prices.

Theorem 4.1 *Suppose that $\frac{C_A}{C_B} = \frac{s_A}{s_B}$ and that $C_A < C_B$. Suppose also that a monopoly firm owns both investment projects A and B. Then the monopoly firm will exercise the smaller investment project, A, first at τ_A^1, and the biggest investment project, B, at $\tau_A^2 = \tau_B^2$.*

Proof It follows from the discussion in chapter 1 that the monopoly firm will invest at either τ_A^1 or at τ_B^1.

Since $\frac{C_A}{C_B} = \frac{s_A}{s_B}$, then by lemma 7.1, $q_B^2 = q_A^2 = q^2$, and the monopoly firm will realize the second investent at τ_{q^2} no matter which investment project will be realized first. The project that will yield most profit before time τ_{q^2} will therefore be realized first. Profits from the two projects before time τ_{q^2} is given by

$$E^q \int_{\tau_A^1}^{\tau^2} \frac{Q_t}{s_A} s_A e^{-rt} dt \text{ and } E^q \int_{\tau_B^1}^{\tau^2} \frac{Q_t}{s_B} s_B e^{-rt} dt. \tag{22}$$

Since the integrand is the same in the two expressions, clearly the investment that is expected to be undertaken at the earliest time will be most profitable. But if $\frac{C_A}{C_B} = \frac{s_A}{s_B}$, then by theorem 3.7 and theorem 6.3,

$$C_A < C_B \text{ if and only if } \bar{q}_A < \bar{q}_B \text{ if and only if } q_A^1 \wedge q_B^1 = q_A^1.$$

□

5 Conclusion

The analysis has revealed that within the frames of the stated stylized model, it is the firm which has the lowest point of indifference between investing first and waiting to be second which invests first.

It is tempting to try to single out the underlying principle guiding optimal timing behaviour more generally in order to give investors some rule of thumb for making investment decisions in practice, and social planners some idea of what such strategic decisions might be. The analysis suggests the following principle the validity of which might be a topic for further research:

> *Act at or before the time at which the competititor is indifferent between acting first and secondly, if you thereby can achieve higher payoff than if you act secondly.*

The underlying intuition is the following: Both players will want to invest first at the point which would be optimal if the competitor were certain to invest secondly. They will do so unless the competitor in that case has an incentive to invest first. To prevent that the competitor does so, the player must invest at the (first) point where the competitor is indifferent between investing first and secondly, or before that point. He will of course not do so unless his own (first) point of indifference comes before the competitor's point of indifference.

In the case that unit cost is equal for the two investment projects, and the investors play the equilibrium strategy, we have the following interesting results: A monopoly firm owning both investment projects will exercise the two projects in the same order as would result in the competitive case. Furthermore, a competitive solution sometimes, but not always, implies that the first investment will be undertaken at an earlier time than in the monopolist case.

6 Appendix 1: Some stopping time and boundary value calculations

We solve the problem of finding $l_i(q)$ and $f_i(q)$ backwards, by first calculating the value of the firm which invests secondly, and thereafter the reward of being first, using techniques from optimal stopping theory and the theory of stochastic boundary value problems (see for instance Øksendal [9]).

6.1 Follower's payoff

Proposition 6.1 *The value at q of waiting to be second to invest, $f_i(q)$, is given by*

$$f_i(q) = \begin{cases} \frac{C_i}{\gamma-1}\left(\frac{q}{q_i^2}\right)^{\gamma} & \text{if } q < q_i^2 \\ \frac{s_i}{(s_i+s_j)(r-\alpha)}q - C_i & \text{if } q \geq q_i^2, \end{cases}$$

where

$$q_i^2 = \frac{\gamma}{\gamma-1}(r-\alpha)\left(\frac{s_i+s_j}{s_i}\right)C_i,$$

and

$$\gamma = \frac{\frac{\beta^2}{2} - \alpha + \sqrt{\left(\frac{\beta^2}{2} - \alpha\right)^2 + 2\beta^2 r}}{\beta^2} > 1.$$

Proof The reward from investing secondly at q is given by

$$h_i(q) =$$

$$E^q\left[\int_0^\infty e^{-ru} p_u s_i du - C_i\right] = \frac{s_i}{s_i + s_j}\int_0^\infty e^{-ru} E^q[Q_u] du - C_i = K_i\left[q - \tilde{C}_i\right], \tag{23}$$

where

$$\tilde{C}_i = \frac{(s_i + s_j)(r - \alpha)}{s_i} C_i \text{ and } K_i = \frac{s_i}{(s_i + s_j)(r - \alpha)}. \tag{24}$$

We need the optimal stopping time τ_i^2 and the expression for $f_i(q)$ such that the equations

$$f_i(q) = E^q\left[e^{-r\tau_i^2} h_i^{Q_{\tau_i^2}}\right] = K_i\left(\sup_\tau E^q\left[\left(Q_\tau - \tilde{C}_i\right) e^{-r\tau}\right]\right) \tag{25}$$

hold. A solution to this problem may be found for instance in chapter 5 of Dixit and Pindyck [4]. Using their results, we get that $Q_{\tau_i^2} = q_i^2$ which is the level that triggers investment is given by

$$q_i^2 = \frac{\gamma}{\gamma - 1}\tilde{C}_i, \tag{26}$$

and

$$f_i(q) = \begin{cases} K_i(q_i^2 - \tilde{C}_i)\left(\frac{q}{q_i^2}\right)^\gamma & \text{if } q < q_i^2 \\ K_i[q - \tilde{C}_i] & \text{if } q \geq q_i^2, \end{cases} \tag{27}$$

which simplifies to the formulas asserted in the theorem. □

6.2 Leader's payoff

Since the trigger level of the second firm q_j^2 is known, we are able to compute the expected present value of the leader's cash flow.

Proposition 6.2 *The value of investing first at q, $l_i(q)$, is given by*

$$l_i(q) = \begin{cases} \frac{q}{r-\alpha} - C_j\frac{\gamma}{\gamma-1}\left(\frac{q}{q_j^2}\right)^\gamma - C_i & \text{if } q < q_j^2 \\ \frac{s_i}{(s_i+s_j)(r-\alpha)} q - C_i & \text{if } q \geq q_j^2 \end{cases}$$

Proof Suppose $q \geq q_j^2$. Then the second firm will invest immediately once the first firm has invested, and the reward of the first firm is qualitatively equal to the reward of the second firm. In other words,

$$l_i(q) = \frac{s_i}{(s_i + s_j)(r - \alpha)} q - C_i \tag{28}$$

for all $q \geq q_j^2$.

Suppose $q < q_j^2$. Then the second firm will postpone its investment, and the reward $l_i(q)$ of the firm that invests first is the sum of the reward at the stopping time τ_j^2 when the second firm invests, and the value of the production in the period untill the stopping time τ_j^2 is reached. Therefore

$$\begin{aligned} l_i(q) &= E^q \int_0^\infty \frac{Q_t}{S_t} s_i e^{-rt} dt - C_i & (29)\\ &= E^q \left[\int_0^{\tau_j^2} \frac{Q_t}{s_i} s_i e^{-rt} dt + \int_{\tau_j^2}^\infty \frac{Q_t}{s_i + s_j} s_i e^{-rt} dt \right] - C_i & (30)\\ &= E^q \left[\int_0^{\tau_j^2} Q_t e^{-rt} dt + \int_{\tau_j^2}^\infty \left(Q_t - \frac{Q_t}{s_i + s_j} s_j \right) e^{-rt} dt \right] - C_i & (31)\\ &= E^q \left[\int_0^\infty Q_t e^{-rt} dt - \frac{s_j}{s_i + s_j} \int_{\tau_j^2}^\infty Q_t e^{-rt} dt \right] - C_i & (32)\\ &= \frac{q}{r - \alpha} - \frac{s_j}{s_i + s_j} E^q \left[\int_{\tau_j^2}^\infty Q_t e^{-rt} dt \right] - C_i, & (33) \end{aligned}$$

where

$$E^q \left[\int_{\tau_j^2}^\infty Q_t e^{-rt} dt \right] = E^q e^{-r\tau_j^2} E^{Q_{\tau_j^2} = q_j^2} \left[\int_0^\infty Q_t e^{-rt} dt \right] = \frac{q_j^2}{r - \alpha} E^q \left[e^{-r\tau_j^2} \right]. \tag{34}$$

Let

$$\phi(q) = E^q \left[e^{-r\tau_j} \right], \text{ and } \psi(t, q) = e^{-rt} \phi(q). \tag{35}$$

The theory on boundary value problems, for instance in Øksendal [9], gives that the differential equation

$$\frac{\partial \psi}{\partial t} + \alpha q \frac{\partial \psi}{\partial q} + \frac{1}{2} \beta^2 q^2 \frac{\partial^2 \psi}{\partial q^2} = 0 \tag{36}$$

must be satisfied for all $q \leq q_i^2$. This equation can be written

$$e^{-rt} \left(\frac{\beta^2}{2} q^2 \phi^{''}(q) + \alpha q \phi^{'}(q) - r\phi(q) \right) = 0. \tag{37}$$

Solving this equation and applying the boundary conditions $\lim_{q \downarrow 0} \phi(q) < \infty$, and $\phi(q_j^2) = E^{q_j^2}\left[e^{-r\tau_j^2}\right] = 1$, gives

$$\phi(q) = \left(\frac{q}{q_j^2}\right)^{\gamma} \tag{38}$$

for $q \leq q_j^2$, and it follows that $l_i(q)$ is equal to the asserted expression. □

6.3 Optimal trigger level of the firm which invests first

Once the leader's reward-functionis determined, the problem of finding his optimal investment time is reduced to an ordinary optimal stopping problem.

Proposition 6.3 *Let it be predetermined that firm i shall invest first, and that firm j will invest at $q \geq q_j^2$. If $l_i(q_i^1) > 0$, then the unique optimal trigger level of firm i is given by*

$$q_i^1 = \frac{\gamma}{\gamma - 1}(r - \alpha)C_i.$$

Proof. Like in proof of Proposition 6.1, the problem is to find the optimal stopping time τ_i^1 such that the equation

$$l_i^*(q) = E^q\left[e^{-r\tau_i^1} l_i(Q_{\tau_i^1})\right] = \sup_{\tau} E^q\left[e^{-r\tau} l_i(Q_\tau)\right] \tag{39}$$

holds.

Let $\phi(q) = l_i^*(q)$, and suppose that q_i^1 is a fixed boundary. Then the same reasoning as in the proof of Proposition 6.2, gives that the differential equation

$$e^{-rt}\left(\frac{\beta^2}{2}q^2\phi^{''}(q) + \alpha q\phi^{'}(q) - r\phi(q)\right) = 0 \tag{40}$$

must be satisfied for all $q \leq q_i^1$. Solving this differential equation and applying the boundary conditions $\lim_{q \downarrow 0} \phi(q) < \infty$ and $\phi(q_i^1) = l_i(q_i^1)$, gives that

$$\phi(q) = l_i^*(q) = l_i(q_i^1)\left(\frac{q}{q_i^1}\right)^{\gamma} \tag{41}$$

for all $q \leq q_i^1$. The derivative with respect to q_i^1 of this expression is zero at

$$q_i^1 = \frac{\gamma}{\gamma - 1}(r - \alpha)C_i. \tag{42}$$

According to Brekke and Øksendal [2], this (high contact) is a sufficient condition for optimality in the continuation region $0 < q \leq q_i^1$, and we conclude that $Q_{\tau_i^1} = q_i^1$.

If $q \geq q_j^2$, the dominant strategy for firm i is to invest immediately. Therefore, if $q_i^1 < q < q_j^2$, waiting can only decrease i's profit since

$$l_i(\tilde{q})\left(\frac{q}{\tilde{q}}\right)^\gamma \tag{43}$$

is decreasing in $\tilde{q} > q_i^1$. Concequently, there are no other continuation regions. Finally, if $l_i(q_i^1) \leq 0$, the optimal stopping time of i is τ_i^2. □

Remark The proof of Proposition 6.3 gives that

$$l_i^*(q) = \begin{cases} l_i(q_i^1)\left(\frac{q}{q_i^1}\right)^\gamma & \text{if } q < q_i^1 \\ l_i(q) & \text{if } q \geq q_i^1. \end{cases} \tag{44}$$

7 Appendix 2: Some technical proofs

Proof of proposition 3.4: *The inequality $l_i(q) \leq f_i(q)$ holds for all nonnegative q if and only if*

$$\left(\frac{s_j}{s_i}+1\right)^\gamma - \gamma\frac{C_j}{C_i}\left(\frac{C_i s_j}{C_j s_i}\right)^\gamma \leq 1.$$

Proof If the curves $f_i(q)$ and $l_i(q)$ meet in one point, then that point is a tangency point. In other words, there is high contact in that point, which by the proof of Proposition 6.3 implies that the point is equal to the optimal trigger level q_i^1, and $l_i(q_i^1) = f_i(q_i^1)$. Therefore, $l_i \leq f_i$ for all nonnegative q if and only if $l_i(q_i^1) \leq f_i(q_i^1)$, which holds if and only if $\frac{l_i(q_i^1)}{f_i(q_i^1)} \leq 1$. But

$$\frac{l_i(q_i^1)}{f_i(q_i^1)} = \frac{C_i\left(\frac{\gamma}{\gamma-1}-1\right) - \frac{\gamma}{\gamma-1}C_j\left(\frac{C_i}{C_j}\frac{s_j}{s_i+s_j}\right)^\gamma}{C_i\left(\frac{\gamma}{\gamma-1}-1\right)\left(\frac{s_i}{s_i+s_j}\right)^\gamma} = \left(\frac{s_j}{s_i}+1\right)^\gamma - \gamma\frac{C_j}{C_i}\left(\frac{C_i s_j}{C_j s_i}\right)^\gamma. \tag{45}$$

□

Proof of theorem 3.6: *If $C_i < \gamma C_j$ and $\frac{C_i}{C_j} < \frac{s_i}{s_j}$, then $\bar{q}_j > \bar{q}_i$, and by theorem 3.1 firm i will invest first.*

Proof The condition $f_i(q) = l_i(q)$ may be written

$$LS_i(q) = \frac{q}{r-\alpha} - C_j\frac{\gamma}{\gamma-1}\left(\frac{q}{q_j^2}\right)^\gamma - C_i = C_i\left(\frac{\gamma}{\gamma-1}-1\right)\left(\frac{q}{q_i^2}\right)^\gamma = RS_i(q). \tag{46}$$

Equality is obtained for $q = \bar{q}_i$. If $\bar{q}_i < q < \tilde{q}$, where $\tilde{q} = \hat{q}$ or $\tilde{q} = q_i^2$, $LS_i(q) > RS_i(q)$. If $q < \bar{q}_i$, then $LS_i(q) < RS_i(q)$. Observe that $C_i < \gamma C_j$ implies that $C_j \frac{\gamma}{\gamma-1} > C_i \left(\frac{\gamma}{\gamma-1} - 1\right)$, and $\frac{C_i}{s_i} < \frac{C_j}{s_j}$ implies that $q_i^2 > q_j^2$.

The suppósition that $\bar{q}_i \geq \bar{q}_j$ together with $C_i < \gamma C_j$ and $C_i < \frac{s_i}{s_j} C_j$ leads to the conclusion that $LS_j(\bar{q}_j) < RS_j(\bar{q}_j)$ which is a contradiction since $LS_j(q) = RS_j(q)$ is fulfilled for $q = \bar{q}_j$. □

Proof of theorem 3.7: *Suppose that $\frac{C_A}{C_B} = \frac{s_A}{s_B}$. Then $C_B > C_A$ if and only if $\bar{q}_B > \bar{q}_A$, and by theorem 3.1 firm A will invest first.*

We shall need the following lemma:

Lemma 7.1 *$q_i^2 = q_j^2$ if and only if $\frac{C_i}{C_j} = \frac{s_i}{s_j}$.*

Proof of lemma: $q_i^2 = q_j^2$ if and only if $\frac{\gamma}{\gamma-1}(r-\alpha)\left(\frac{s_i+s_j}{s_i}\right) C_i = \frac{\gamma}{\gamma-1}(r-\alpha)\left(\frac{s_i+s_j}{s_j}\right) C_j$ if and only if $\frac{C_i}{C_j} = \frac{s_i}{s_j}$. □

As a concequence, we may write $q_i^2 = q_j^2 = q^2$ if $\frac{C_i}{C_j} = \frac{s_i}{s_j}$. By symmetry, $l_i = f_i$ at q^2.

Proof of theorem: The point $\bar{q}_B$ satisfies the equation $f_B(q) = l_B(q)$ which is equivalent to

$$C_A \frac{\gamma}{\gamma-1}\left(\frac{q}{q_A^2}\right)^{\gamma} = \frac{q}{r-\alpha} - C_B\left(\frac{\gamma}{\gamma-1} - 1\right)\left(\frac{q}{q_B^2}\right)^{\gamma} - C_B. \tag{47}$$

Since $q^2 = q_A^2 = q_B^2$, we may write

$$LS(q) = \frac{q}{r-\alpha} - C_B\left(1 - \left(\frac{q}{q^2}\right)^{\gamma}\right) = \frac{\gamma}{\gamma-1}(C_B + C_A)\left(\frac{q}{q^2}\right)^{\gamma} = RS(q). \tag{48}$$

Since $\frac{C_B}{C_A} = \frac{s_B}{s_A}$,

$$RS'(q)_{q=q^2} = \frac{\partial}{\partial q}\left[\frac{\gamma}{\gamma-1}(C_B + C_A)\left(\frac{q}{q^2}\right)^{\gamma}\right]_{q=q^2} = \frac{\gamma}{r-\alpha}, \tag{49}$$

and

$$LS'(q)_{q=q^2} = \frac{\partial}{\partial q}\left[\frac{q}{r-\alpha} - C_B\left(1 - \left(\frac{q}{q^2}\right)^{\gamma}\right)\right]_{q=q^2} = \left(\frac{1}{r-\alpha}\right)\frac{\gamma C_B + C_A}{C_A + C_B}. \tag{50}$$

We conclude that

$$RS'(q)_{q=q^2} > LS'(q)_{q=q^2}, \tag{51}$$

since if $RS'(q)_{q=q^2} \leq LS'(q)_{q=q^2}$, then $\gamma \leq 1$, which is a contradiction.

Recall that $RS = LS$ in two points, q^2 and $\bar{q}_B$. Both RS and LS are continuous functions in q with continuous first derivatives. Therefore, since $RS'(q)_{q=q^2} > LS'(q)_{q=q^2}$, we conclude that $RS'(q)_{q=\bar{q}_B} < LS'(q)_{q=\bar{q}_B}$.

Equation 48 above holds for $q = \bar{q}_A$, when C_B is replaced with C_A and C_A is replaced with C_B. Suppose that $C_B > C_A$ and let $\bar{q}_B \leq \bar{q}_A$. Then

$$\frac{\bar{q}_A}{r-\alpha} - C_A\left(1-\left(\frac{\bar{q}_A}{q^2}\right)^{\gamma}\right) > \frac{\gamma}{\gamma-1}(C_B + C_A)\left(\frac{\bar{q}_A}{q^2}\right)^{\gamma}, \tag{52}$$

which is a contradiction. Consequently, $C_B > C_A$ implies that $\bar{q}_B > \bar{q}_A$.

For the converse, suppose $C_B \leq C_A$ and let $\bar{q}_B > \bar{q}_A$. By symmetry, it is sufficient to consider the case when $C_B = C_A$, which in the same manner leads to a contradiction. $\square$

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