Entropy of C^*-dynamical systems defined by bitstreams

by

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1 Introduction

From his study of binary shifts on the hyperfinite II_1-factor Powers [P] initiated a systematic study of sequences of symmetries, i.e. self-adjoint unitary operators, which pairwise either commute or anticommute. The problem was followed up by Price [Pr1], and it was shown in [PP] that the C^*-algebra generated by the symmetries was the CAR-algebra when the sequence was nonperiodic in a natural sense and of the form $M_n(\mathbb{C}) \otimes Z$ with Z abelian in the periodic case, see [E].

The sequence $(s_i)_{i\in\mathbb{N}}$ of symmetries is a natural representation of a bitstream, i.e. a sequence of 0's and 1's, by the requirement that s_i and s_j commute if the $|i-j|$'th element in the sequence is 0 and they anticommute if it is 1. Thus the problem has potential applications to the study of bitstreams.

In the present paper we shall change Power's problem slightly by considering sequences (s_i) with i ranging over all integers $\mathbb{Z}$. If σ is the shift $s_i \to s_{i+1}$ we obtain an automorphism of the C^*-algebra A generated by the s_i's and thus a C^*-dynamical system (A, σ, τ) where τ is a canonical invariant trace. The system (A, σ, τ) turns out to have a very rich ergodic theoretic structure. For example, in [NST] it was used to give a counter example to the additivity of the CNT-entropy [CNT] by exhibiting a sequence (s_i) such that the entropy $h_{\tau\otimes\tau}(\sigma\otimes\sigma) = \log 2$ while $h_\tau(\sigma) = 0$. We shall in the present paper initiate a systematic study of the C^*-dynamical system (A, σ, τ). Especially we shall consider the analogues of several concepts from classical (abelian) ergodic theory like K-systems, completely positive entropy, and computation of entropies. In addition we shall study the nonabelian concept of asymptotic abelianness. It turns out that for different choices of bitstreams (hence of the commutation relations) there is an abundance of examples with different properties; for example, we shall get an analogue of the celebrated theorem of Ornstein and Shields [OS] that there is an uncountable family of nonconjugate dynamical systems with completely positive entropy having the same value for the entropy $h_\tau(\sigma)$. Our main emphasis will be on entropy calculations. It turns out that in many cases with the sequence nonperiodic $h_\tau(\sigma) = \frac{1}{2}\log 2$, while in the periodic case it is always $\log 2$, and there is no value strictly between $\frac{1}{2}\log 2$ and $\log 2$. We have been unable to determine whether there are values strictly between 0 and $\frac{1}{2}\log 2$.

2 Basic results

If $X \subseteq \mathbb{N}$ we denote by $A(X)$ the C^*-algebra generated by a sequence $(s_n)_{n\in\mathbb{Z}}$ of symmetries satisfying the commutation relations

$$s_i s_j = (-1)^{g(|i-j|)} s_j s_i \,, \qquad i, j \in \mathbb{Z},$$

where g is the characteristic function of X. Note that g identifies X with the bitstream $(g(n))_{n\in\mathbb{Z}}$. The canonical trace τ on $A(X)$ is the one which takes the value 0 on all products $s_{i_1} s_{i_2} \cdots s_{i_k}$

with $i_1 < i_2 < \cdots < i_k$ and $\tau(1) = 1$. We denote by σ the automorphism on $A(X)$ determined by $\sigma(s_i) = s_{i+1}$. The entropy $h(\sigma) = h_\tau(\sigma)$ of σ with respect to τ as defined in [CNT] is the same as the entropy in the GNS-representation π due to τ of the finite von Neumann algebra $\pi(A(X))''$ as defined in [CS], see [CNT, Thm. VII.2]. Since $A(X)$ is an AF-algebra generated by the finite dimensional subalgebras $A[-n,n] = C^*(s_i : i = -n, -n+1, \dots, n)$ we can, due to the Kolmogoroff-Sinai Theorem [CS, Thm. 2], use the notation from [CS] freely on $A(X)$ to compute the entropy. By abuse of notation we say X is *periodic* (resp. *nonperiodic*) if $-X \cup \{0\} \cup X$ is periodic (resp. nonperiodic).

Proposition 2.1 *If X is nonperiodic then the entropy $h(\sigma) \leq \frac{1}{2}\log 2$.*

Proof We modify an argument of Price [Pr2]. For each $k \in \mathbb{Z}$ with $k < m$ put

$$A[k,m] = C^*(s_k, s_{k+1}, \dots, s_m).$$

Then $A[k,m]$ is a C^*-algebra of dimension 2^{m-k}. Fix $n \in \mathbb{N}$ and put $A_n = A[-n,n]$. Since $\bigcup_{n=1}^{\infty} A_n$ is norm dense in $A(X)$ it suffices by the Kolmogoroff-Sinai Theorem to show

$$H(A_n, \sigma) = \lim_{k \to m} \frac{1}{k+1} H(A_n, \sigma(A_n), \dots, \sigma^k(A_n)) \leq \frac{1}{2}\log 2.$$

Since $\sigma^j(A_n) \subset A[-n, n+k]$ for $0 \leq j \leq k$, it follows by Property C in [CS] that

$$H(A_n, \sigma(A_n), \dots, \sigma^k(A_n)) \leq H(A[-n, n+k]).$$

By [PP] $A[-n, n+k]$ is a factor for an infinite number of k's. Then $\dim A[-n, n+k] = 2^{2n+k+1}$ it is a factor of type $I_{2^{\frac{1}{2}(2n+k+1)}}$, hence

$$H(A[-n, n+k]) = \frac{1}{2}(2n+k+1)\log 2.$$

It follows that for this k,

$$\frac{1}{k+1} H(A_n, \sigma(A_n), \dots, \sigma^k(A_k)) \leq \frac{1}{2}\frac{2n+k+1}{k+1}\log 2,$$

and taking limits as $k \to \infty$

$$H(A_n, \sigma) \leq \frac{1}{2}\log 2.$$

QED

Corollary 2.2 *If X is nonperiodic then $h(\sigma \otimes \sigma) = \log 2$.*

Proof Since $\bigcup_{n=0}^{\infty} A_n \otimes A_n$ is uniformly dense in $A(X) \otimes A(X)$ and furthermore

$$C^*\Big(\bigcup_{j=0}^{k} \sigma^j(A_n) \otimes \sigma^j(A_n)\Big) = A[-n, n+k] \otimes A[-n, n+k]$$

the argument from the proof of Proposition 2.1 yields

$$H(A_n \otimes A_n, \sigma \otimes \sigma) \leq \log 2,$$

hence by the Kolmogoroff-Sinai Theorem $h(\sigma \otimes \sigma) \leq \log 2$. The converse inequality follows from the proof of [NST, Thm. 4.2]. QED

The periodic case is much simpler than the nonperiodic. We conclude this section with a discussion of this case. The results are independent of the following sections.

Suppose $X \subseteq \mathbb{N}$ is periodic with period p. If we write $g(-n) = g(n)$, $g(0) = 0$ for the characteristic function g of X then by periodicity

$$g(i+np) = g(i), \quad n \in \mathbb{Z}, \quad 0 \leq i \leq p-1.$$
$$g(i) + g(p-i) \equiv 0 \bmod 2, \quad 0 \leq i \leq p-1.$$

Let $t_0 = s_0 s_p$. Then for $i \in \mathbb{Z}$

$$s_i t_0 = (-1)^{g(i)+g(p-i)} t_0 s_i = t_0 s_i\,,$$

hence t_0 belongs to the center $Z(A(X))$ of $A(X)$, hence so does $t_j = \sigma^j(t_0)$ for all $j \in \mathbb{Z}$. Furthermore, if $j \in \mathbb{Z}$ then $j = i + kp$ for $0 \leq i \leq p-1$, $k \in \mathbb{Z}$, so we have

$$s_j = s_i t_i t_{i+p} \cdots t_{i+p(k-1)} = s_i t(j)$$

with $t(j) \in Z$ – the C^*-subalgebra of $Z(A(X))$ generated by the t_i's. If we let $A = C^*(s_0, \ldots, s_{p-1})$, then it follows that

$$A(X) = A \otimes Z.$$

We first compute the entropy of σ with respect to the canonical trace.

Proposition 2.3 *If X is periodic then $h(\sigma) = \log 2$. Moreover, if $A_n = A[-n,n]$ then $h(\sigma) = \lim_{n\to\infty} \frac{1}{2n+1} H(A_n)$.*

Proof Let $D_j = C^*(t_j)$. Then D_j is isomorphic to the diagonal 2×2 matrices, and $\tau|D_j$ is $\frac{1}{2}$ on each minimal projection. Furthermore $\tau(ab) = \tau(a)\tau(b)$ for $a \in D_i$, $b \in D_j$, $i \neq j$. Thus the restriction $\sigma|Z$ is the 2-shift, hence the entropy $h(\sigma|Z) = \log 2$, and therefore $h(\sigma) \geq h(\sigma|Z) = \log 2$.

For the converse inequality we recall from the proof of Proposition 2.1 that

$$H(A_n, \sigma(A_n), \ldots, \sigma^k(A_n)) \leq H(A[-n, n+k])$$

with $\dim H(A[-n,n+k]) = 2^{n+k+1}$. Thus we have

$$H(A_n, \sigma) \leq \lim_{k\to\infty} \frac{1}{k+1} \log 2^{n+k+1} = \log 2.$$

QED

Since $A(X) = A \otimes Z$, if τ_A is the restriction of τ to A, then $\tau_A \otimes \varphi$ is a trace on $A(X)$ for every state φ on Z.

Let $j_1 < j_2 < \cdots < j_k$, and write $j_\ell = i_\ell + k_\ell p$ with $0 \le i_\ell < p-1$, $k_\ell \in \mathbb{Z}$. With the notation introduced above $s_{j_\ell} = s_{i_\ell} t(i_\ell)$ with $t(i_\ell) \in Z$. Thus we have

$$s_{j_1} \ldots s_{j_k} = s_{i_1} \ldots s_{i_k} t(i_1) \ldots t(i_k),$$

so that

$$\tau_A \otimes \varphi(s_{j_1} \ldots s_{j_k}) = \tau_A(s_{i_1} \ldots s_{i_k}) \varphi(t(i_1) \ldots t(i_k)) = 0. \tag{1}$$

We have $\sigma(s_{p-1}) = s_p = s_0 t_0$. Using this we find as in (1) that

$$\tau_A \otimes \varphi(\sigma(s_{j_1} \ldots s_{j_n})) = 0.$$

If φ is σ-invariant then for $t \in Z$, $\tau_A \otimes \varphi(\sigma(t)) = \varphi(\sigma(t)) = \varphi(t) = \tau_A \otimes \varphi(t)$. Thus $\tau_A \otimes \varphi$ is σ-invariant.

Proposition 2.4 *Let notation be as above. Then the entropy $h_{\tau_A \otimes \varphi}(\sigma)$ of σ with respect to the σ-invariant trace $\tau_A \otimes \varphi$ is given by*

$$h_{\tau_A \otimes \varphi}(\sigma) = h_\varphi(\sigma | Z).$$

Proof Since $Z \subset A(X)$, and $h_{\tau_A \otimes \varphi}(\sigma | Z) = h_\varphi(\sigma | Z)$ we have

$$h_{\tau_A \otimes \varphi}(\sigma) \ge h_\varphi(\sigma | Z).$$

Note that for $n \in \mathbb{N}$ large enough

$$\widetilde{A}_n = A[-n, n+p] = A \otimes Z[-n, n],$$

where $Z[-n, m] = C^*(t_j : -n \le t \le m)$. Since

$$H_{\tau_A \otimes \varphi}(\widetilde{A}_n, \sigma(\widetilde{A}_n), \ldots, \sigma^{k-1}(\widetilde{A}_n)) \le H_{\tau_A \otimes \varphi}(C^*(\widetilde{A}_n, \ldots, \sigma^{k-1}(\widetilde{A}_n))),$$

and

$$C^*(\widetilde{A}_n, \ldots, \sigma^{k-1}(\widetilde{A}_n)) = A \otimes Z[-n, n+k-1],$$

we have, using that $\dim A = 2^p$,

$$\frac{1}{k} H_{\tau_A \otimes \varphi}(\widetilde{A}_n, \ldots, \sigma^{k-1}(\widetilde{A}_n)) \le \frac{1}{k}(p \log 2 + H_\varphi(Z[-n, n+k-1]))$$

Since in the limit

$$\lim_{k \to \infty} \frac{1}{k} H_\varphi(Z[-n, n+k-1]) \le h_\varphi(\sigma | Z),$$

we have

$$H_{\tau_A \otimes \varphi}(\widetilde{A}_n, \sigma) \le h_\varphi(\sigma | Z).$$

Since this holds for all n, $h_{\tau_A \otimes \varphi}(\sigma) \le h_\varphi(\sigma | Z)$. QED

3 Asymptotic abelianness

We shall in this section consider the case when the C^*-dynamical system $(A(X),\sigma)$ is asymptotically abelian. The following two versions will be of interest, see [DKS].

Definition 3.1 *Let (A,α) be a C^*-dynamical system (so α is a $*$-automorphism of the C^*-algebra A).*

(i) *We say (A,α) is* asymptotically abelian *if*

$$\lim_{n\to\infty} \|[\alpha^n(a),b]\| = 0 \qquad \text{for all } a,b\in A.$$

(ii) *We say (A,α) is* proximally asymptotically abelian *if there is a sequence $(n_i)_{i\in\mathbb{N}}$ with $n_i\to\infty$ as $i\to\infty$ such that*

$$\lim_{i\to\infty} \|[\alpha^{n_i}(a),b]\| = 0 \qquad \text{for all } a,b\in A.$$

In our case the concept of proximal asymptotic abelianness will be most important. This is due to the following result.

Proposition 3.2 *$(A(X),\sigma)$ is asymptotically abelian if and only if X is finite.*

Proof Suppose $(A(X),\sigma)$ is asymptotically abelian. Since

$$\|[s_0,\sigma^n(s_0)]\| = \|[s_0,s_n]\| = \begin{cases} 2 & \text{for } n\in X \\ 0 & \text{for } n\in\mathbb{N}\setminus X\end{cases}$$

it is immediate from asymptotic abelianness that there is $n_0\in\mathbb{N}$ such that $n\in\mathbb{N}\setminus X$ for $n\geq n_0$. Thus $X\subset\{1,2,\dots,n_0\}$.

Conversely, suppose X is finite, say $X\subset\{1,2,\dots,n_0\}$. Since $\bigcup_{n=1}^{\infty} A[-n,n]$ is uniformly dense in $A(X)$, in order to show $(A(X),\sigma)$ is asymptotically abelian, it suffices to show

$$\lim_{n\to\infty} \|[\sigma^n(a),b]\| = 0 \qquad \text{for } a,b\in A[-m,m].$$

Since $A[-m,m]$ is generated by s_i, $-m\leq i\leq m$, it suffices to consider the case when $a=s_i$, $b=s_j$, $-m\leq i,\ j\leq m$. Let $n>2(m+n_0)$. Then $\sigma^n(s_i)=s_{i+n}$, so it suffices to show $i+n-j\geq n_0$. But this is obvious since $i-j\geq -2m$. QED

If $k<m$ we denote by $[k,m]$ the interval $\{k,k+1,\dots,m\}\subset\mathbb{N}$.

Theorem 3.3 *$(A(X),\sigma)$ is proximally asymptotically abelian if and only if $\mathbb{N}\setminus X$ contains intervals of arbitrary large lengths, i.e. there exist intervals $[k_n,m_n]\subset\mathbb{N}\setminus X$ with $\lim_{n\to\infty}(m_n-k_n)=\infty$.*

Proof Assume $(A(X), \sigma)$ is proximally asymptotically abelian, and let (n_k) be a sequence such that $\|[\sigma^{n_k}(a), b]\| \to 0$ for all $a, b \in A(X)$. Given $m \in \mathbb{N}$ choose k_0 such that $[\sigma^{n_k}(s_i), s_j] = 0$ for $k \geq k_0$ and $i, j \in [-m, m]$. Then $n_k + i - j \in \mathbb{N} \setminus X$, so the interval $[-2m + n_k, 2m + n_k] \subset \mathbb{N} \setminus X$.

Conversely, if $[k_j, m_j] \subset \mathbb{N} \setminus X$ with $m_j - k_j$ arbitrarily large put $n_j = \frac{1}{2}(m_j + k_j)$. Given $m \in \mathbb{N}$ choose j so large that $4m \leq m_j - k_j$. Then for all $r, s \in [-m, m]$ we have $n_j + r - s = \frac{1}{2}m_j + \frac{1}{2}k_j + r - s \in \left[\frac{m_j}{2} + \frac{k_j}{2} - 2m, \frac{m_j}{2} + \frac{k_j}{2} + 2m\right]$ which by choice of j is contained in $[k_j, m_j]$. Thus as in the proof of Proposition 3.2, if $a, b \in A[-m, m]$ then $[\sigma^{n_j}(a), b] = 0$. It follows that $(A(X), \sigma)$ is proximally asymptotically abelian with sequence (n_i). QED

Remark 3.4 In [NST] it was presented an example of a shift σ with zero entropy. It is possible to prove that this shift is proximally asymptotically abelian.

Remark 3.5 There is a third definition of asymptotic abelianness which is more general than the two we have considered. Namely, one asumes that for a C^*-dynamical system (A, α, G), G a group, $\alpha: G \to \operatorname{Aut} A$ a representation, then for each $a \in A$ there is a sequence $(g_n(a))$ in G such that

$$\lim_{n \to \infty} \|[\alpha_{g_n(a)}(a), b]\| = 0 \qquad \text{for all } b \in A,$$

see [DKS]. In our case this generalization gives nothing new, since it is equivalent to proximal asymptotic abelianness. Indeed, applying the definition to $a = s_0$ there is a sequence (n_i) such that $\|s_0, s_{n_i}]\| \to 0$ as $i \to \infty$; in other words there is $i_0 \in \mathbb{N}$ such that $n_i \in \mathbb{N} \setminus X$ for $i \geq i_0$. Since we also have that

$$\|[s_0, s_{n_i+1}]\| = \|[s_0, \sigma^{n_i}(s_1)]\| \to 0,$$

it follows that $n_i + 1 \in \mathbb{N} \setminus X$, hence $[n_i, n_i + 1] \subset \mathbb{N} \setminus X$ for $i \geq i_1 \geq i_0$ for some i_1. Repeating the argument we can therefore for any $t \in \mathbb{N}$ find an interval $[k, m] \subset \mathbb{N} \setminus X$ such that $m - k \geq t$. Hence by Theorem 3.3 $(A(X), \sigma)$ is proximally asymptotically abelian.

4 K-systems

In classical ergodic theory the concept of a K-system is very important. Narnhofer and Thirring [NT] extended the notion to the noncommutative case, the definition being as follows.

Definition 4.1 *Let (A, α) be a C^*-dynamical system with an invariant tracial state τ. We say (A, α) is an* entropic K-system *if for each finite dimensional C^*-subalgebra $B \subset A$ (or rather $B \subset \pi_\tau(A)''$) we have*

$$\lim_{n \to \infty} H(B, \alpha^n) = H(B).$$

As was the case for the asymptotically abelian situation entropic K-systems are rare in our case.

Theorem 4.2 *Suppose X is nonperiodic. Then $(A(X), \alpha)$ is an entropic K-system if and only if X is finite.*

Proof Assume X is finite, say $X \subset [1,r]$, $r \in \mathbb{N}$. Suppose $B \subset A[-n,n]$ is a $*$-subalgebra, and let $t = 2n+1+r$. Then for $a \in B$ and $b \in \sigma^{jt}(B)$, $j \in \mathbb{N}$, we have $ab = ba$, and from the construction of τ, $\tau(ab) = \tau(a)\tau(b)$. Thus

$$\frac{1}{k}H(B, \sigma^t(B), \ldots, \sigma^{(k-1)t}(B)) = H(B) \qquad \text{for } k \in \mathbb{N},$$

so that in the limit

$$H(B, \sigma^t) = H(B). \tag{2}$$

If B is a general finite dimensional C^*-subalgebra of $A(X)$ we can approximate B by a finite dim subalgebra B_0 belonging to some $A[-n,n]$, and then apply [CNT; Prop. IV.3] to conclude (2).

Conversely assume $(A(X), \sigma)$ is an entropic K-system. Considering $A(X)$ in its imbedding in the hyperfinite II_1-factor R via the GNS-representation due to τ, we assume $A(X)'' = R$. Using strong approximation and [CNT, IV.3] one can prove that for any finite dimensional von Neumann subalgebra B of R we have (2), where σ is the extension of σ to R. Then [BN, Thm. 3.13] states that R is strongly asymptotically abelian with respect to σ, hence in particular

$$\lim_{n\to\infty} \|[\sigma^n(s_0), s_0]\|_2 = 0,$$

where $\|x\|_2 = \tau(x^*x)^{1/2}$. But

$$\|[s_n, s_0]\|_2 = \begin{cases} 2 & \text{if } n \in X \\ 0 & \text{f } n \in \mathbb{N} \setminus X. \end{cases}$$

Thus as before we conclude that X is finite. QED

It follows from the work of Choda [Ch] and Price [Pr2] that if $X \neq \emptyset$ is finite then $h(\sigma) = \frac{1}{2}\log 2$. Our next result shows that the entropy of σ is obtained as the mean of the entropies of the algebras $A[-n,n]$.

Proposition 4.3 *Let $X \neq \emptyset$ be finite. Then*

$$h(\sigma) = \lim_{n\to\infty} \frac{1}{2n+1} H(A[-n,n]).$$

Proof We follow the arguments of Choda [Ch]. By the Kolmogoroff-Sinai Theorem, which can be used by the arguments of our previous proof,

$$h(\sigma) = \lim_{n\to\infty} H(A[-n,n], \sigma).$$

Since $H(A[-n,n], \sigma) \le h(\sigma)$, given $\varepsilon > 0$ we have for large enough k and n,

$$\frac{1}{kp+1} H(A[-n,n], \sigma(A[-n,n]), \ldots, \sigma^{kp}(A[-n,n])) < h(\sigma) + \varepsilon,$$

where $p = 2n+1+t$ and $X \subset [0,t]$. Consequently

$$\frac{1}{kp+1} H(A[-n,n], \sigma^p(A[-n,n]), \ldots, \sigma^{kp}(A[-n,n])) < h(\sigma) + \varepsilon.$$

By the same arguments as in the proof of Theorem 4.2 the left side of the last inequality equals

$$\frac{k+1}{kp+1}H(A[-n,n]).$$

Thus in the limit as $k \to \infty$

$$\frac{1}{p}H(A[-n,n] \leq h(\sigma)+\varepsilon,$$

or since $p = 2n+1+t$, and ε is arbitrary.

$$\lim_{n\to\infty}\frac{1}{2n+1}H(A[-n,n]) \leq h(\sigma).$$

Conversely let $q = 2(n+t)$. Since $\sigma^j(A[-n,n]) \subset A[-n,n+kq]$ whenever $0 \leq j \leq kq$, we have

$$H(A[-n,n],\sigma(A[-n,n]),\ldots,\sigma^{kq}(A[-n,n]) \leq H(A[-n,n+kq]).$$

By choice of q and translation invariance of H

$$H(A[-n,n+kq]) = H(A[-(n+t)k-n,(n+t)k+n]),$$

so with $m = (n+t)k+n$

$$h(\sigma) = \lim_{m\to\infty} H(A[-m,m],\sigma) \leq \lim_{m\to\infty}\frac{1}{2m+1}H(A[-m,m]).$$

QED

By [NST] the tensor product formula does not hold in general for $(A(X),\sigma)$. However, when we have results like the above theorem, it can be proved.

Corollary 4.4 *Suppose $X_i \subset \mathbb{N}$ are finite, $i=1,2$. Denote by σ_i the shift on $A(X_i)$. Then $(A(X_1)\otimes A(X_2),\sigma_1\otimes\sigma_2)$ satisfies*

$$h(\sigma_1\otimes\sigma_2) = h(\sigma_1)+h(\sigma_2).$$

Proof Applying the above arguments we find

$$\lim_{n\to\infty}\frac{1}{2n+1}H(A_n^1\otimes A_n^2) = h(\sigma_1\otimes\sigma_2),$$

where $A_n^i = A_i[-n,n] \subset A(X_i)$ in obvious notation, $i=1,2$. By the theorem

$$\lim_{n\to\infty}\frac{1}{2n+1}H(A_n^1\otimes A_n^2) = \lim_{n}\frac{1}{2n+1}(H(A_n^1)+H(A_n^2)) = h(\sigma_1)+h(\sigma_2).$$

QED

Let α be the automorphism of $A(X)$ defined by $\alpha(s_i) = -s_i$. Let $A(X)_e$ denote the fixed point algebra of α and $A(X)_{\rm odd}$ the spectral subspace of -1. Then $A(X)_e$ is the C^*-algebra generated by the unitaries s_is_j, $i,j \in \mathbb{Z}$, and $A(X)_{\rm odd}$ is the closure of the linear span of products $s_{i_1}s_{i_2}\cdots s_{i_k}$ with k odd and $i_1 < i_2 < \cdots < i_k$. We call $A(X)_e$ the *even subalgebra* and elements in $A(X)_{\rm odd}$ for *odd elements.* It is evident that $A(X)_e$ is globally invariant under σ. We denote by σ_e the restriction of σ to $A(X)_e$.

Proposition 4.5 *Suppose $\mathbb{N} \setminus X$ is finite. Then we have:*

(i) *$(A(X)_e, \sigma_e)$ is an entropic K-system.*

(ii) *If $A_n^0 = C^*(s_i s_j\colon i, j \in [-n, n]) \subset A(X)_e$ then*

$$\lim_{n\to\infty} \frac{1}{2n+1} H(A_n^0) = h(\sigma_e) = \frac{1}{2}\log 2.$$

(iii) *If $X_i \subset \mathbb{N}$, $i = 1, 2$, and if $\mathbb{N} \setminus X_i$ is finite, then in obvious notation*

$$h(\sigma_e^1 \otimes \sigma_e^2) = h(\sigma_e^1) + h(\sigma_e^2).$$

Proof Choose $t \in \mathbb{N}$ such that $\mathbb{N} \setminus X \subset [0, t]$. Thus $s_i s_j$ and $s_k s_\ell$ commute whenever

$$\min\{|k - i|, |k - j|, |\ell - i|, |\ell - j|\} > t.$$

Thus by the arguments of the proof of Theorem 4.2 $(A(X)_e, \sigma_e)$ is an entropic K-system. Also by repreating the arguments from Theorem 4.3 we find

$$\lim_{n\to\infty} \frac{1}{2n+1} H(A_n^0) = h(\sigma_e).$$

Now it follows just as in Corollary 4.4 that the product formula (iii) holds. In particular when $X_i = X$, $i = 1, 2$, then

$$h(\sigma_e \otimes \sigma_e) = 2h(\sigma_e).$$

But, since the elements $s_i s_j \otimes s_i s_j$ and $s_k s_\ell \otimes s_k s_\ell$ pairwise commute for all i, j, k, ℓ, we have

$$\frac{1}{k+1} H(A_0, \sigma_e \otimes \sigma_e(A_0), \ldots, \sigma_e^k \otimes \sigma_e^k(A_0)) = \log 2, \qquad k \in \mathbb{N},$$

where A_0 is the abelian C^*-subalgebra of $A(X)_e \otimes A(X)_e$ generated by the $s_i s_j \otimes s_i s_j$. Thus

$$h(\sigma_e \otimes \sigma_e) \geq \log 2,$$

hence $h(\sigma_e) \geq \frac{1}{2}\log 2$. By Proposition 2.1, $h(\sigma) \leq \frac{1}{2}\log 2$ since X is nonperiodic, so that $h(\sigma_e) \leq h(\sigma) \leq \frac{1}{2}\log 2$. This completes the proof. QED

In particular we have shown (as also follows from [Pr2], since the assumption implies X is eventually periodic).

Corollary 4.6 *If $\mathbb{N} \setminus X$ is finite then $h(\sigma) = \frac{1}{2}\log 2$.*

Corollary 4.7 *Let $\mathbb{N} \setminus X$ be finite. Then*

$$h(\sigma) = \lim_{n\to\infty} \frac{1}{2n+1} H(A[-n, n]). \qquad (*)$$

Moreover, if $\mathbb{N} \setminus X_i$ is finite $i = 1, 2$, then

$$h(\sigma_1 \otimes \sigma_2) = h(\sigma_1) + h(\sigma_2).$$

Proof By our previous arguments it suffices to show (*). Since $s_i s_j = \pm s_0 s_i s_0 s_j$, it follows that

$$A_n^0 = C^*(s_0 s_j : j \in [-n, n]).$$

Since $s_0(s_0 s_j) = s_j$ it follows that

$$A[-n, n] = C^*(s_0, A_n^0),$$

which by the commutation relations is contained in $M_2(\mathbb{C}) \otimes A_n^0$. Thus we have

$$H(A_n^0) \leq H(A[-n, n]) \leq \log 2 + H(A_n^0),$$

hence the corollary follows from Proposition 4.5 (ii) and Corollary 4.6.

QED

5 Completely positive entropy

Definition 5.1 *We say* $(A(X), \sigma)$ *has* completely positive entropy *if* $H(B, \sigma) > 0$ *for all finite dimensional* C^**-subalgebras* B *of* $A(X)$.

In the classical case this is equivalent to $(A(X), \sigma)$ being an entropic K-system. It is immediate from our next lemma that an entropic K-system has completely positive entropy.

Lemma 5.2 *Let* $B \subset A(X)$ *be a finite dimensional* C^**-algebra. Then*

$$\frac{1}{n} H(B, \sigma^n) \leq H(B, \sigma), \qquad n \in \mathbb{N}.$$

Proof The lemma follows from the inequalities

$$\begin{aligned} \frac{1}{n} H(B, \sigma^n) &= \frac{1}{n} \lim_{k \to \infty} \frac{1}{k+1} H(B, \sigma^n(B), \ldots, \sigma^{nk}(B)) \\ &\leq \frac{1}{n} \lim_{k \to \infty} \frac{n}{kn+n} H(B, \sigma(B), \ldots, \sigma^{nk}(B)) \\ &= H(B, \sigma) \end{aligned}$$

QED

We shall in the present section show that not all systems $(A(X), \sigma)$ with completely positive entropy are entropic K-systems. Then we shall exhibit an uncountable family of nonconjugate systems $(A(X), \sigma)$ with completely positive entropy for which $h(\sigma) = \frac{1}{2} \log 2$, and thus obtain the analogue of the similar result in the classical case by Ornstein and Shields [OS].

Lemma 5.3 *Suppose* X *is nonperiodic and either* $X \subset \{1, 3, 5, \ldots\}$ *or* $X \supset \{1, 3, 5, \ldots\}$*. Then* $h(\sigma) = \frac{1}{2} \log 2$.

Proof Assume first $X \subset \{1, 3, 5, \ldots\}$. Let $A_0 = C^*(s_0)$, so A_0 is isomorphic to the diagonal 2×2 matrices. Since $s_{2i}s_{2j} = s_{2j}s_{2i}$ for all $i, j \in \mathbb{Z}$, the algebras $\sigma^{2i}(A_0)$ and $\sigma^{2j}(A_0)$ mutually commute, and σ^2 acts as the 2-shift on the abelian C^*-algebra D they generate. Thus $h(\sigma^2|D) = \log 2$. Hence

$$h(\sigma) = \frac{1}{2}h(\sigma^2) \geq \frac{1}{2}h(\sigma^2|D) = \frac{1}{2}\log 2.$$

The converse inequality follows from Proposition 2.1.

Suppose $X \supset \{1, 3, 5, \ldots\}$ and put $t_j = s_{2j-1}s_{2j}$, $j \in \mathbb{Z}$. Then a straightforward computation shows that the t_j's commute and that σ^2 acts as the 2-shift on the abelian C^*-algebra they generate. The proof is then completed as above. QED

Lemma 5.4 *Let $q > 1$ be an odd integer. Suppose $X \subset \{q^n : n \in \mathbb{N}\}$. Suppose $a \in A[0, q^n - 1]$ and $b \in A[2mq^n, (2m+1)q^n - 1]$, $m \in \mathbb{N}$. Then $ab = ba$ and $\tau(ab) = \tau(a)\tau(b)$.*

Proof We have

$$A[2mq^n, (2m+1)q^n - 1] = \sigma^{2mq^n}(A[0, q^n - 1]).$$

Thus there exists $c \in A[0, q^n - 1]$ such that $b = \sigma^{2mq^n}(c)$, hence a and b commute if

$$2mq^n + x - y \notin X \quad \text{whenever } x, y \in [0, q^n - 1],$$

or rather

$$2mq^n + x \neq q^{n+t} + y \qquad \text{for all } t \in \mathbb{Z},\ x, y \in [0, q^n - 1].$$

Assume on the contrary that we have equality for some $t \in \mathbb{Z}$. Then $x \neq y$ since q is odd. If $x > y$ then

$$0 < x - y = q^n(q^t - 2m).$$

If $t \geq 0$ then $q^t - 2m \in \mathbb{N}$, which means $x - y \geq q^n$, which is impossible since $x, y \in [0, q^n - 1]$. If $t < 0$ then $q^t - 2m < 0$, which is contrary to $x - y$ being positive.

Suppose next $x < y$. Then similarly

$$0 < y - x = q^n(2m - q^t),$$

and again we obtain $y - x \geq q^n$, contrary to assumption.

Finally, since $\tau(s_{i_1} \ldots s_{i_k}) = 0$ whenever $i_1 < \cdots < i_k$, and the linear span of such products of s_i's and the identity 1 is dense in $A(X)$, it is immediate from the first part of the proof that $\tau(ab) = \tau(a)\tau(b)$ whenever $a \in A[0, q^n - 1]$ and $b \in A[2mq^n, (2m+1)q^n - 1]$, $m \in \mathbb{N}$.

QED

Lemma 5.5 *Suppose $q > 1$ is an odd integer and $X \subset \{q^n : n \in \mathbb{N}\}$. Then for each finite dimensional C^*-subalgebra B of $A(X)$ we have*

$$\lim_{n \to \infty} H(B, \sigma^{2q^n}) = H(B)$$

Proof If $B \subset A\Big[-\frac{1}{2}(q^n-1), \frac{1}{2}(q^n-1)\Big]$ then in view of Lemma 5.4

$$\frac{1}{m+1}H(B, \sigma^{2q^n}(B), \ldots, \sigma^{2mq^n}(B)) = H(B), \qquad m \in \mathbb{N},$$

hence

$$H(B, \sigma^{2q^n}) = H(B). \tag{$*$}$$

Now suppose $B \not\subset \bigcup_{m=1}^{\infty} A[-m, m]$ with $m = \frac{1}{2}(q^n-1)$. Then we can as before consider $A(X)$ as a strongly dense subalgebra of the hyperfinite II_1-factor R. For every m as above we can construct a completely positive map $\gamma_m: B \to A[-m, m]$ such that $\gamma_m(a) \to a$ strongly as $m \to \infty$ (see proof of [CNT, VII.4]). Now it follows from $(*)$ and [CNT, VI.3] that

$$\lim_{n \to \infty} H(B, \sigma^{2q^n}) = H(B).$$

QED

Summing up we have:

Theorem 5.6 *Suppose $q > 1$ is an odd integer and $X \subset \{q^n : n \in \mathbb{N}\}$. Then we have:*

(i) $h(\sigma) = \frac{1}{2}\log 2$

(ii) *$(A(X), \sigma)$ is proximally asymptotically abelian.*

(iii) *$(A(X), \sigma)$ has completely positive entropy.*

(iv) *If X is infinite then $(A(X), \sigma)$ is not an entropic K-system.*

Proof (i) follows from Lemma 5.3. (ii) from Theorem 3.3, and (iii) from Lemmas 5.2 and 5.5. Finally, (iv) is a consequence of Theorem 4.2. QED

We conclude this section by showing the analogue of the result of Ornstein and Shields alluded to in the introduction to this section. We recall that if (A, α) and (B, β) are C^*-dynamical systems then they are *conjugate* if there is a $*$-isomorpism γ of A onto B such that $\beta = \gamma \circ \alpha \circ \gamma^{-1}$.

Theorem 5.7 *Let $q > 1$ be an odd integer and $X_i \subset \{q^n : n \in \mathbb{N}\}$, $i = 1, 2$. Let σ_i be the corresponding shifts on $A(X_i)$. If the set $\{n \in \mathbb{N} : q^n \in X_1 \cap (\mathbb{N} \setminus X_2)\}$ is infinite, then the systems $(A(X_1), \sigma_1)$ and $(A(X_2), \sigma_2)$ are nonconjugate.*

Proof. Denote by $\{n_1, n_2, \ldots\}$ the set of $n \in \mathbb{N}$ with $q^n \in X_1 \cap (\mathbb{N} \setminus X_2)$ and assume $n_1 < n_2 < n_3 < \cdots$ Then $q^{n_i} \notin X_2$ for $i \in \mathbb{N}$, and hence

$$[q^{n_i - 1} + 1, q^{n_i + 1} - 1] \subset \mathbb{N} \setminus X_2 .$$

Since

$$q^{n_i} - q^{n_i - 1} + 1 = q^{n_i - 1}(q-1) + 1 \to \infty \qquad \text{as } i \to \infty,$$

and $q^{n_i+1} - q^{n_i} - 1 \to \infty$ as $i \to \infty$, it follows that

$$\|[\sigma_2^{q^{n_i}}(x), y]\| \to 0 \qquad \text{as } i \to \infty$$

for all $x, y \in A[-n+1, n] \subset A(X_2)$ with n fixed. Thus for all $x, y \in A(X_2)$,

$$\lim_{i\to\infty} \|[\sigma_2^{q^{n_i}}(x), y]\| = 0.$$

On the other hand, since $q^{n_i} \in X_1$ we have

$$\|[\sigma_1^{q^{n_i}}(s_0), s_0]\| = \|[s_{q^{n_i}}, s_0]\| = 2.$$

It follows that σ_1 and σ_2 are nonconjugate. QED

Combining Theorems 5.6 and 5.7 we have

Corollary 5.8 *There is an uncountable family of pairwise nonconjugate systems $(A(X), \sigma)$ with completely positive entropy and entropy $h(\sigma) = \frac{1}{2}\log 2$.*

6 Shifts without completely positive entropy

It was shown in [NST] that there are shifts with zero entropy, hence in particular without completely positive entropy. We shall now exhibit examples when the entropy is positive, but the system does not have completely positive entropy. For this the following result is useful.

Theorem 6.1 *Let $k \in \mathbb{N}$ and suppose $X \subset k\mathbb{N}$ is nonperiodic.*

(i) *If k is even then $h(\sigma) = \frac{1}{2}\log 2$.*

(ii) *If k is odd then $h(\sigma) \geq \frac{k-1}{2k}\log 2$.*

(iii) *If $k\mathbb{N} \setminus X$ is finite then $h(\sigma) = \frac{1}{2}\log 2$.*

Proof If $k = 1$ then (ii) is obvious, and if $X = \mathbb{N}$ then (iii) follows for example from Corollary 4.7. We shall therefore assume $k \geq 2$. Put

$$A_j = C^*(s_{kn+j} : n \in \mathbb{Z}), \qquad j = 0, 1, \ldots, k-1.$$

If $i \neq j$ then

$$kn + j - km - i = k(n-m) + j - i \notin k\mathbb{Z}.$$

Hence $g(|k_n + j - k_m - i|) = 0$, and therefore

$$s_{kn+j}s_{km+i} = s_{km+i}s_{kn+j}\,.$$

Thus the C^*-algebras A_j, $j \in [0, k-1]$, commute pairwise. Now $A_0 = C^*(t_n, n \in \mathbb{Z})$ with $t_n = s_{nk}$. Since $[t_n, t_m] = 0$ if and only if $|m - n| \notin \frac{1}{k}X$, A_0 is of the form $A(Y)$ with

$Y = \frac{1}{k}X \subset \mathbb{N}$. Since X is nonperiodic, so is Y. Hence A_0, and therefore $A_j = \sigma^j(A_0)$ is isomorphic to the CAR-algebra [PP] and so

$$A(X) = \bigotimes_{j=0}^{k-1} A_j,$$

and σ^k leaves each A_j globally invariant. Thus

$$\sigma^k = \bigotimes_{j=0}^{k-1} \sigma^k|A_j\,.$$

Suppose first $k = 2m$ is even. Then

$$A(X) = (A_0 \otimes A_1) \otimes (A_2 \otimes A_3) \otimes \cdots \otimes (A_{2(m-1)} \otimes A_{2m-1})$$

Denote by

$$C_i = C^*(s_{kn+2i} \otimes s_{kn+2i+1} \colon n \in \mathbb{Z}) \subset A_{2i} \otimes A_{2i+1}\,,$$

$i \in [0, m-1]$. Then C_i is abelian, and σ^k restricted to C_i is the 2-shift, hence has entropy $\log 2$. Thus

$$\begin{aligned} h(\sigma) &= \frac{1}{k}h(\sigma^k) \\ &\geq \frac{1}{k}h(\sigma^k|C_0 \otimes \cdots \otimes C_{m-1}) \\ &= \frac{1}{k}\sum_{i=0}^{m-1} h(\sigma^k|C_i) \\ &= \frac{1}{k}m\log 2 \\ &= \frac{1}{2}\log 2\,. \end{aligned}$$

By Proposition 2.1, $h(\sigma) = \frac{1}{2}\log 2$.

If $k = 2m+1$, then similarly

$$A(X) = A_0 \otimes (A_1 \otimes A_2) \otimes \cdots \otimes (A_{2m-1} \otimes A_{2m})$$

hence by [SV, Lem. 3.4]

$$\begin{aligned} h(\sigma) &\geq \tfrac{1}{k}h(\sigma^k|A_0) + \tfrac{1}{k}\textstyle\sum_{i=1}^{m} h(\sigma^k|A_{2j-1} \otimes A_{2j}) \\ &\geq \tfrac{1}{k}m\log 2 \\ &= \tfrac{k-1}{2k}\log 2\,. \end{aligned} \qquad (*)$$

Finally, if $k\mathbb{N} \setminus X$ is finite with k odd, say $k = 2m+1$, then as above

$$A(X) = A_0 \otimes (A_1 \otimes A_2) \otimes \cdots \otimes (A_{2m-1} \otimes A_{2m})$$

In this case A_0 is the CAR-algebra and $h(\sigma^k|A_0) = \frac{1}{2}\log 2$ by Corollary 4.7. Thus by $(*)$

$$h(\sigma) \geq \frac{1}{2k}\log 2 + \frac{1}{k}m\log 2 = \frac{1}{2}\log 2,$$

completing the proof of the theorem. QED

Theorem 6.2 *There exists a nonperiodic set X such that σ has positive but not completely positive entropy.*

Proof. Let $X_0 \subset \mathbb{N}$ be a set such that the corresponding shift σ_0 has entropy zero, see [NST]. Put $X = 2X_0 = \{2n : n \in X_0\}$. By Theorem 6.1 $h(\sigma) = \frac{1}{2}\log 2$. With A_0 and A_1 as in the proof of the theorem, $A_0 = A(X_0)$ and $\sigma^2|A_0 = \sigma_0$, and similarly for A_1. Thus $h(\sigma^2|A_0) = 0 = h(\sigma^2|A_1)$. In particular, for all finite dimensional C^*-subalgebras $B \subset A_i$, $i = 0, 1$, $H(B, \sigma^2) = 0$. Let now $B = C^*(s_0)$ and $B_1 = \sigma(B) = C^*(s_1)$. Then we have by subadditivity of the entropy function H [CS, Property B]

$$H(B, \sigma(B), \ldots, \sigma^{2m}(B)) \leq H(B, \sigma^2(B), \ldots, \sigma^{2m}(B)) + H(B_1, \sigma^2(B_1), \ldots, \sigma^{2m-2}(B_1)).$$

Hence

$$H(B, \sigma) \leq \frac{1}{2} H(B, \sigma^2) + \frac{1}{2} H(B_1, \sigma^2) = 0.$$

Thus σ does not have completely positive entropy. QED

Remark 6.3 In analogy with the classical situation if $h(\sigma) > 0$ we say a σ-invariant C^*-subalgebra A_π of $A(X)$ is a *Pinsker subalgebra* if it satisfies the following conditions:

(i) $h(\sigma|A_\pi) = 0$

(ii) If B is a finite dimensional C^*-subalgebra of $A(X)$ such that $H(B, \sigma) = 0$, then $B \subset A_\pi$.

It is clear from (ii) that a Pinsker subalgebra, if it exists, is unique. The classical result of Pinsker [ME] states that there exists a nontrivial Pinsker subalgebra in the classical case when the transformation has positive entropy. In our case, however, this is false. Indeed, a counter example is provided by the situation in Theorem 6.2, since by its proof each C^*-subalgebra $C^*(s_i)$ would belong to the Pinsker algebra A_π if it exists, hence $A_\pi \supset C^*(s_i : i \in \mathbb{Z}) = A(X)$, a contradiction.

Remark 6.4 We used in the proof of Theorem 6.2 a system $(A(X_0), \sigma_0)$ with $h(\sigma_0) = 0$. A refinement of the argument in [NST] yields an uncountable family of such systems which are nonconjugate. We omit the rather technical proof.

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