NON-COMMUTATIVE ALGEBRAIC GEOMETRY

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Contents

Introduction

The theory of schemes, founded by Chevalley, [3], Serre, [12, FAC] and Grothendieck, [5, SGA-EGA], is based upon the idea of a ringed space. The geometric picture consists of a space of points, a topology, and a structure sheaf of rings of "functions" on this topology. The space being, locally, the set of prime ideals of a commutative ring, extending the Gelfand-Šilov correspondence between maximal ideals in the C*-algebra of continous complex valued functions on a compact space, and the points of this space. The notion of localization in commutative rings, which gives rise to the Zariski topology, also furnishes the structure sheaf of commutative rings on this topology. This is the notion of affine scheme, the category of which is dual to the category of commutative rings. Moreover, representations, or modules of the ring we start out with, form a category equivalent to the category of sheaves of modules (or representations) of the structure sheaf. The crucial point in this equivalence is the *Serre theorem*, namely the vanishing of the higher cohomology groups of a sheaf of quasicoherent modules on an affine scheme, and the identification of the ring A with the ring of global sections of the structure sheaf. From this affine starting point we may globalize, by glueing together affine pieces to obtain the notion of scheme, a space with a topology and a structure sheaf of commutative rings, locally isomorphic to an affine scheme. This made commutative algebra part of a vast generalization of the classical, reduced, algebraic geometry dating back to Descartes and Fermat. This also permitted the use of methods of differential geometry in algebra and algebraic geometry. Taylor series, infinitesimals, differential operators, etc. got a natural algebraic foundation, and turned out to be of great importance for the development of modern algebraic geometry.

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In his introduction to the work of the laureat, when Grothendieck got the Fields medal, Dieudonné [4] stressed the importence of the notion of representable functor in scheme theory. Moduli theory is based on this notion, and with it some of the most spectacular discoveries of this century, together with a new and very active collaboration between mathematics and physics. However, it seems to me that this moduli theory now forces upon us a revision of the scheme theory, and the introduction of a non-commutative theory, capable of including non-commutative as well as commutative algebra in one *Geometry*, thereby completing Descartes programme. In fact, it should now be obvious to anybody working in moduli theory, that the notion of representability in scheme theory, is grossly inadequat for the purpose of studying families of algebraic objects, their invariants and their moduli spaces. There is little help in the introduction of algebraic spaces, stacks or other notions of that sort. The reason, I claim, is that the theory of moduli is really non-commutative.

Now, a non-commutative version of differential geometry has been around for decades, spured by the needs of quantum mechanics and by the needs of invariant theory. In fact, there is a flora of proposed non commutative geometries, see, e.g. [2], [11], and [1]. The first ones were based on the notion of operator algebras. Von Neumanns work on quantum mechanics created a geometry where points, in some sense, were replaced by states or pure states in C^* -algebras. Working on foliations, Connes, [2] has, in a most convincing way, developed a theory of quotientspaces, or orbitspaces, related to the theory of moduli, which trancends the classical geometry. However, the basic notions of point, as prime ideal in a ring, the topology, and the structure sheaf disappeare in this model. Even the Leibnitz-Newtonian idea of infinitesimal neighbourhoods of the points vanishes. There is also the super theory, brought forward by the russian school, where the essential new ingrediant is the extension of the notion of symmetry group. Most of these developements are within differential geometry, and are dependent upon differential geometric techniques. However, there are also purely algebraic attempts at the construction of non-commutative schemes, with the conservation of the notion of a set of points, corresponding to some ideals, with topology and structure sheaf, see e.g.[11]. The common aspect of these models have been that they do not include non-reduced algebras, and therefore cannot treat 0-dimensional *schemes*, and subsequently contain no infinitesimal theory.

Using the notion of non-commutative deformation of modules, worked out in [9], we shall here embark upon a purely algebraic construction of a non-commutative algebraic geometry, and show that it is, at least in some cases, usefull for the understanding of invariant- and moduli problems. The fundamental ideas are very general and only dependent upon a reasonable abelian category of objects. Since the process of generalizing to this case will be clear, I shall assume that we are given a k-algebra A as above, and that we are considering the category C of right A-modules. As a model we shall take the scheme construction Spec(A) when A is a commutative finite type k-algebra. A closed point of Spec(A) is a finite dimensional simple module of A, i.e. the residue field $k(t)$ of a closed point t considered as an A-module. We know that the completion of A at t is the hull of the deformation functor $Def(k(t))$. Thus the infinitesimal algebraic structure of the regular functions of Spec(A) at t is completely determined by the deformation functor of $k(t)$.

Since moreover A and therefore Spec(A) are, morally, determined by the family of completions at the different closed points, we shall try to copy this procedure for the case of a general k-algebra A.

Consider first the 0-dimensional case. This is the subject of the paper *A Generalized Burnside Theorem*, see [10]. Let A be a finite dimensional k-algebra, k algebraically closed, and V=$\{V_i\}$ the (finite) family of (simple) modules . We shall consider each module of this family as a point, and we shall consider the non commutative formal moduli H(V) as the structure sheaf of the 0-dimensional scheme V. The infinitesimal neighbourhood of a point of V, the analogue of the completion of a commutative k-algebra A at the point $\mathfrak{t}$, see above, is the subalgebra $H_{i,i}$ of H(V), the hull of the deformation functor Def V_i. However, here is a change, the structure sheaf H(V) on V, defining the structure of our geometry, is not the sum of the algebras $H_{i,i}$ corresponding to the points, as it is in the commutative case. The *infinitesimal interactions* of the points translates into the components $H_{i,j}$ of H(V). By definition of the ring of *observables O(V)*, see Chapter 2, there is a morphism of k-algebras

$$\eta : A \longrightarrow O(V) := (H_{i,j} \otimes Hom_k(V_i, V_j))$$

and we have seen that if V is the family of all simple modules then this morphism is an isomorphism. This is our *Serre theorem* in the 0-dimensional non-commutative algebraic geometry. Notice that in the construction of H(V) we have only used the structure of the abelian category (of A-modules) in which we consider our family of objects V. To recover A, i.e. in the construction of the ring of observables, we must also know the dimensions of the different "points" V_i of the non-commutative scheme V, i.e. we must know the forgetfull functor π of A-mod into k-vector spaces.

This is now going to be the basis for the construction of a non commutative scheme theory, generalizing the classical one. For each *diagram* c of C, and for each forgetful functor π, we associate a k-algebra of *observables*, $O(c, \pi)$, a k-algebra of *geometric observables* $O_\triangle(c, \pi)$, and a canonical homomorphism $A \longrightarrow O_\triangle(c, \pi)$. In particular, if A is a commutative k-algebra, the points of Spec(A) may be identified with members of the family of irreducible modules V= Spec(A)=$\{A/\mathfrak{p} | \mathfrak{p}$ a prime of A$\}$. Moreover, we shall consider this family of A-modules with their obvious canonical morphisms, obtaining a diagram (really an ordered set) c=*Spec*(A), of A-mod. Then the imbedding of the classical algebraic geometry (defined on a field k), into the proposed non-commutative algebraic geometry, is taken care of by the,

Conjecture. *(See paragraph 3) Let A be any commutative k-algebra, essentially of finite type. Then the canonical morphism of k-algebras*

$$\eta : A \longrightarrow O_\triangle(Spec(A), \pi)$$

is an isomorphism.

The Descartes program. The problem of defining the notion *GEOMETRY* is an old one. Since we celebrate, this year, the $400^{th.}$ anniversary of Rene Descartes, it is maybe proper to link the above ideas to the ideas of the founder of algebraic geometry. His main idea was to associate to the space, coordinate functions whose numerical values characterise the *points* of the space and their free movements in *space*. Now, this is taken care of in the *scheme theory* of today for the closed points.

For the other points, the custom of today is to consider the different components of the Hilbert space of the scheme in question.

The non-commutative algebraic geometry that I shall propose, is grown out of this carthesian idea, considered in relation to the problem of moduli in algebra. Given an algebraic object, we know via deformation theory what we should mean by its (infinitesimal) movents, or change of states, and we would like to find an algebra of *coordinate operators*, with the property that the *eigenvalues* of the different representations of these operators characterise the possible *states* of the object, and such that the *local structure* of the algebra, contains all the information about the possible movements, or changes of states, of the algebraic object, including the abrupt changes observed in families for which some discrete invariant jumps.

This is analogous to the set-up of quantum theory, where our algebraic object is replaced by the (platonic?) idea of some reality *out there*, a fundamental particle, or a hydrogene atom, say. The state space is a module, or representation, of a ring of generalized coordinate functions, the *observables* operating on the state space. If the simple modules of this ring had been all isomorphic to some field of *numbers* we would have been able to extend the Descartes programme to our new *space* of *states of the reality.* When, however, the algebra of observables is non-commutative and the simple representations may be of infinite dimensions on the base-field, the numerical trick of Descartes does not function. We are left with a new description of the space of realities, in which measurement must be redefined, and time and dynamics rethought.

I claim that the non-commutative algebraic geometry I am proposing, may contribute to a better understanding of this situation, and that the basics of Descartes program will prevail.

1.Homological preparations.

1.1.Exts and Hochschild cohomology. Let k be an algebraically closed field, and let A be a k-algebra. Denote by A-mod the category of right A-modules and consider the exact forgetful functor

$$\pi : A - mod \longrightarrow k - mod$$

Given two A-modules M and N, we shall always use the identification

$$\sigma^i : Ext^i_A(M, N) \simeq HH^i(A, Hom_k(M, N) \text{ for } i \geq 0$$

If L_* and F_* are A-free resolutions of M and N respectively, and if an element

$$\xi \in Ext^1_A(M, N)$$

is given in Yoneda form, as

$$\xi = \{\xi_n\} \in \prod_n Hom_A(L_n, F_{n-1})$$

then $\sigma^1(\xi)$ is gotten as follows. Let σ be a k-linear section of the augmentation morphism

$$\rho : L_0 \longrightarrow M$$

and let for every $a \in A$ and $m \in M$, $\sigma(ma) - \sigma(m)a = d_0(x)$. Then,

$$\sigma^1(\xi)(a,m) = -\mu(\xi_1(x))$$

where

$$\mu : F_0 \longrightarrow N$$

is the augmentation morphism of F_*. Then,

$$\sigma^1(\xi) \in Der_k(A, Hom_k(M,N))$$

and its class in $HH^1(A, Hom_k(M,N))$ represents ξ.

Recall the spectral sequence associated to a change of rings. If $\pi : A \longrightarrow B$ is a surjectiv homomorphism of commutative k-algebras, M a B-module and N an A-module, then $Ext^*_A(M,N)$ is the abuttment of the spectral sequence given by,

$$E_2^{p,q} = Ext^p_B(M, Ext^q_A(B,N)).$$

There is an exact sequence,

$$0 \longrightarrow E_2^{1,0} \longrightarrow Ext^1_A(M,N) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0}$$

Which, for a B-module N, considered as an A-module, implies the exactness of

$$0 \longrightarrow Ext^1_B(M,N) \longrightarrow Ext^1_A(M,N)$$
$$\longrightarrow Hom_B(M, Hom_B(I/I^2, N)) \longrightarrow Ext^2_B(M,N)$$

where I=kerπ. The corresponding exact sequence,

$$0 \to HH^1(B, Hom_k(M,N)) \to HH^1(A, Hom_k(M,N))$$
$$\to Hom_{A\otimes A^{op}}(B, Hom_k(M,N))$$

in the non commutative case is induced by the sequence ,

$$0 \to Der_k(B, Hom_k(M,N)) \to Der_k(A, Hom_k(M,N))$$
$$\to Hom_{A\otimes A^{op}}(B, Hom_k(M,N))$$

Notice that in general we do not know that the last morphism is surjective. This, however, is true if B=A/rad(A), where rad(A) is the radical of A, and A is a finite dimensional, i.e. an artinian, k-algebra. In this case, B is semisimple and the surjectivity above follows from the Wedderburn-Malcev theorem, see e.g. [10]. Notice also that in the commutative case,

$$Hom_{A\otimes A^{op}}(B, Hom_k(M,N)) \simeq Hom_B(I/I_2, Hom_B(M,N))$$

as it must, since for $\phi \in Hom_B(M,N)$, $a \in A$, and $b \in I$, ab=ba, and therefore

$$(a\phi)b = \phi(ab) = \phi(ba) = (\phi a)(b)$$

This implies that for $B = A/\mathfrak{p}$, $M = A/\mathfrak{p}$, $N = A/\mathfrak{q}$, where $\mathfrak{p} \subseteq \mathfrak{q}$ are (prime) ideals of A,

$$Ext^1_A(A/\mathfrak{p}, A/\mathfrak{q}) \simeq Hom_A(\mathfrak{p}/\mathfrak{p}^2, A/\mathfrak{q})$$

and, in particular

$$Ext^1_A(A/\mathfrak{q}, A/\mathfrak{q}) \simeq Hom_A(\mathfrak{q}/\mathfrak{q}^2, A/\mathfrak{q}) = N_{\mathfrak{q}},$$

the normal bundle of $V(\mathfrak{q})$ in Spec(A). If $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \neq \mathfrak{p}$ we find,

$$Ext^1_A(A/\mathfrak{p}, A/\mathfrak{q}) \simeq Ext^1_{A/\mathfrak{q}}(A/\mathfrak{p}, A/\mathfrak{q}).$$

In [6], chapter 1., we considered the cohomology of a category c with values in a bifunctor, i.e. in a functor defined on the category *mor c*. It is easy to see that this is an immediate generalization of the projective limit functor and its derivatives, or , if one likes it better, the obvious generalization of the Hochschild cohomology of a ring. In fact, for every small category c and for every bifunctor,

$$G : c \times c \longrightarrow Ab$$

contravariant in the first variable, and covariant in the second, we may consider the complex,

$$D^*(c, G)$$

where,

$$D^p(c, G) = \prod_{c_0 \to c_1 \cdots \to c_p} G(c_0, c_p)$$

where the indices are strings of morphisms $\psi_i : c_i \to c_{i+1}$ in c, and the differential,

$$d^p : D^p(c, G) \longrightarrow D^{p+1}(c, G)$$

is defined as usual,

$$(d^p\xi)(\psi_1, \dots, \psi_i, \psi_{i+1}, \dots, \psi_{p+1}) = \psi_1\xi(\psi_2, \dots, \psi_{p+1})$$
$$+ \sum_{i=1}^{p}(-1)^i\xi(\psi_1, \dots, \psi_i \circ \psi_{i+1}, \dots, \psi_{p+1}) + (-1)^{p+1}\xi(\psi_1, \dots, \psi_p)\psi_{p+1}.$$

As shown in [6], the cohomology of this complex is the higher derivatives of the projective limit functor $\varprojlim^{(*)}_{mor\ c}$ applied to the covariant functor

$$G : mor\ c \longrightarrow Ab.$$

This is the "Hochschild" cohomology of the category c, denoted

$$H^*(c, G) := H^*(D^*(c, G)).$$

Example 1. Let c be a multiplicative subset of a ring R, considered as a category with one object, then

$$H^0(c, Hom(-,-)) = \{\phi \in R | \phi\psi = \psi\phi \text{ for all } \psi \in c\},$$

i.e. the commutant in R of c.

Example 2. Let A be a commutative k-algebra of finite type, k algebraically closed, and let $\underline{Spec}(A)$ be the subcategory of A-mod consisting of the modules $A/\mathfrak{p}$, where p runs through Spec(A), the morphisms being only the obvious ones. There is an obvious functor $Hom_\pi(-,-)$ defined on $mor\ \underline{Spec}(A)$, and it is easy to see that the obvious homomorphism

$$\eta(\underline{Spec}(A), \pi) : A \longrightarrow H^0(\underline{Spec}(A), Hom_\pi(-,-))$$

identifies $A/rad(A)$ with $H^0(\underline{Spec}(A), Hom_\pi(-,-))$. If, however, A is a local k-algebra, essentially of finite type, then this is no longer true in general. To remedy this situation we shall in the next paragraph introduce, and study a generalization $O(\underline{Spec}(A), \pi)$ of

$$O_0(\underline{Spec}(A), \pi) := H^0(\underline{Spec}(A), Hom_\pi(-,-))$$

defined in terms of the non-commutative deformation theory introduced in [9].

1.2. The category of A-G-modules.. Let A be any k-algebra and let $g : A \to A$ be an automorphism. Given an A-module M_i, i=1,2 consider an automorphism of k-modules $\nabla_g^i : M_i \to M_i$, such that for $m_i \in M_i$ and $a \in A$ we have,

$$\nabla_g^i(m_i a) = \nabla_g^i(m_i) g(a) \text{ for i=1,2}$$

i.e. such that ∇_g^i is g-linear. Then there is an automorphism,

$$\theta_g^p := \theta_g^p(\nabla^1, \nabla^2) : Ext_A^p(M_1, M_2) \longrightarrow Ext_A^p(M_1, M_2)$$

induced via the isomorphism,

$$Ext_A^p(M_1, M_2) \simeq HH^p(A, Hom_k(M_1, M_2))$$

by the g^{-1}-linear automorphism of bi-modules,

$$\zeta_g : Hom_k(M_1, M_2) \longrightarrow Hom_k(M_1, M_2)$$

defined by,

$$\psi \longmapsto \nabla_g^1 \circ \psi \circ \nabla_{g^{-1}}^2.$$

Notice that we compose morphisms in the natural order. For $a \in A$ we compute,

$$\zeta_g(g(a)\psi) = \nabla_g^1 \circ g(a)\psi \circ \nabla_{g^{-1}}^2 = a(\nabla_g^1 \circ \psi \circ \nabla_{g^{-1}}^2) = a\zeta_g(\psi)$$

$$\zeta_g(\psi g(a)) = \nabla_g^1 \circ \psi g(a) \circ \nabla_{g^{-1}}^2 = (\nabla_g^1 \circ \psi \circ \nabla_{g^{-1}}^2)a = \zeta_g(\psi)a.$$

This implies that there is an automorphism of Hochschild cohomology,

$$\zeta_g^p : HH^p(A, Hom_k(M_1, M_2)) \longrightarrow HH^p(A, Hom_k(M_1, M_2))$$

defined on cochain form by,

$$\xi^p \longmapsto \{(a_1, a_2, \ldots, a_p) \mapsto \nabla_g^1 \circ \xi^p(g(a_1), \ldots, g(a_p)) \circ \nabla_{g^{-1}}^2\}.$$

In particular the automorphism,

$$\zeta_g^1 : Ext_A^1(M_1, M_2) \longrightarrow Ext_A^1(M_1, M_2)$$

is induced by the map

$$\zeta_g^1 : Der_k(A, Hom_k(M_1, M_2)) \longrightarrow Der_k(A, Hom_k(M_1, M_2))$$

defined by

$$\zeta_g^1(\delta)(a) = \nabla_g^1 \circ \delta(g(a)) \circ \nabla_{g^{-1}}^2.$$

When $\mathfrak{p} \subseteq A$ is a g-invariant ideal of A contained in the annihilator of M_2, we know that the restriction of the derivations of $Der_k(A, Hom_k(M_1, M_2))$ to $\mathfrak{p}$ induces an isomorphism,

$$Hom_A(\mathfrak{p}/\mathfrak{p}^2, Hom_A(A/\mathfrak{p}, M_2)) \simeq Ext_A^1(A/\mathfrak{p}, M_2)$$

such that the automorphism ζ_g^1 takes the form,

$$\zeta_g^1(\psi)(x) = \nabla_{g^{-1}}^2(\psi(gx)) \text{ for } x \in \mathfrak{p}/\mathfrak{p}^2.$$

Suppose $\xi \in Ext_A^1(M_1, M_2)$ is represented by the exact sequence of A-modules,

$$(*) \qquad 0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

Since the g-linear automorphisms $\nabla_g^i : M_i \to M_i$ correspond to an A-linear isomorphism,

$$\nabla_g^i : M_i \to M_i \otimes_{g^{-1}} A$$

we deduce from $(*)$ the exact sequence of A-modules,

$$(**) \qquad 0 \longrightarrow M_2 \otimes_{g^{-1}} A \longrightarrow E \otimes_{g^{-1}} A \longrightarrow M_1 \otimes_{g^{-1}} A \longrightarrow 0$$

which represents the element $\zeta_g^1(\xi) \in Ext_A^1(M_1, M_2)$. The ζ_g^1-invariant elements ξ of $Ext_A^1(M_1, M_2)$ therefore corresponds to the extensions $(*)$ for which there exists an isomorphism

$$(***) \qquad \nabla_g : E \longrightarrow E \otimes_{g^{-1}} A$$

compatible with the ∇_g^i, for i=1,2. Another way of viewing this is to look at $\xi_g^1(\xi)-\xi$ as an obstruction for the existence of such an isomorphism (**).

Given one $\nabla_g : E \longrightarrow E \otimes_{g-1} A$ compatible with the ∇_g^i, another ∇_g' will differ from the first one by the composition Γ_g of the homomorphism $E \longrightarrow M_1$ and some A-linear map $\alpha : M_1 \to M_2 \otimes_{g^{-1}} A$, and any such Γ_g added to $(***)$, will again be compatible with the ∇_g^i, for i=1,2. In the category of (A-g)-modules, we therefore find,

$$Ext^1_{A-g}(M_1, M_2) \simeq Ext^1_A(M_1, M_2)^{\zeta_g} \oplus Hom_A(M_1, M_2 \otimes_{g-1} A)/\sim$$

The equivalence $\sim$ identifies (E', ∇_g') and (E'', ∇_g'') if there exists an isomorphism of extensions $\zeta : E \simeq E''$ compatible with the ∇'s. Since

$$\nabla_g^2 : Hom_A(M_1, M_2) \simeq Hom_A(M_1, M_2 \otimes_{g^{-1}} A)$$

the equivalence relation $\sim$ is trivial.

Now, suppose G is a group acting on the k-algebra A, i.e. suppose there exists a homomorphism of groups,

$$\rho : G \longrightarrow Aut_k(A).$$

Consider A-modules M_i, i=1,2, with G-actions compatible with ρ, i.e. homomorphisms

$$\nabla^i : G \longrightarrow Aut_k(M_i)$$

such that for $g \in G$, $m_i \in M_i$, and $a \in A$,

$$\nabla_g^i(m_i a) = \nabla_g^i(m_i) g(a) \text{ for i=1,2}$$

where we denote by g(a) the action of ρ(g) on $a \in A$.

Given an invariant $\xi \in Ext^1_A(M_1, M_2)$ under the action of the group G, as explained above, there exists for every $g \in G$ an isomorphism

$$\nabla_g : E \longrightarrow E \otimes_{g^{-1}} A$$

Since

$$(E \otimes_{g_1^{-1}} A) \otimes_{g_2^{-1}} A = E \otimes_{(g_1 g_2)^{-1}} A$$

we find an obstruction for the existence of a homomorphism of groups,

$$\nabla : G \longrightarrow Aut_k(E)$$

compatible with the ∇^i's which is a 2-cocycle of G with values in the G-bimodule $Hom_A(M_1, M_2)$,

$$(g_1, g_2) \longmapsto (\nabla_{g_1} \circ \nabla_{g_2} - \nabla_{g_1 g_2}).$$

When the corresponding 2-class,

$$\sigma_\xi \in H^2(G, Hom_A(M_1, M_2))$$

vanish, there exists a ∇ and the set of such will be a torsor under

$$H^1(G, Hom_A(M_1, M_2))$$

.

Proposition 1.2. *Suppose $H^i(G, Hom_A(M_1, M_2)) = 0$ for i=1,2, then,*

$$Ext^1_{A-G}(M_1, M_2) \simeq Ext^1_A(M_1, M_2)^G.$$

Notice that a 1-coboundary of the form

$$g \longmapsto (g\alpha - \alpha)$$

corresponds to an automorphism $\theta_\alpha : E \longrightarrow E$ inducing an automorphism of (E, ∇_g).

1.3. The category of A- $\mathfrak{g}$-modules. Suppose

$$\rho : \mathfrak{g} \longrightarrow Der_k(A)$$

is a Lie-Cartan pair, i.e. an A-linear map and a k-Lie homomorphism. We shall treat this as the tangent map of a Lie-group action ρ studied in the previous section. Let M_i, i=1,2 be A-modules with $\mathfrak{g}$-integrabel connections

$$\nabla^i : \mathfrak{g} \longrightarrow End_k(M_i),$$

and consider for every $\delta \in \mathfrak{g}$ and every $\psi \in Hom_k(M_1, M_2)$ the map

$$\delta \longmapsto \nabla^1_\delta \psi - \psi \nabla^2_\delta$$

This defines a Lie algebra homomorphism,

$$\rho : \mathfrak{g} \longrightarrow End_k(Hom_k(M_1, M_2))$$

such that $\rho(\delta a) = a\rho(\delta) - \rho(\delta)a$. Let $D \in Der_k(A, Hom_k(M_1, M_2))$. The map

$$a \longmapsto \nabla_\delta(D)(a) := D(\delta(a)) + \nabla^1_\delta D(a) - D(a)\nabla^2_\delta$$

is a derivation, and we obtain a connection

$$\nabla : \mathfrak{g} \longrightarrow End_k(Ext^1_A(M_1, M_2))$$

As above, every $\xi \in Ext^1_A(M_1, M_2)^{\mathfrak{g}}$ is associated to an obstruction,

$$\sigma(\xi) \in H^2(\mathfrak{g}, Hom_k(M_1, M_2))$$

which vanish if and only if there exists an integrabel connection on the middle term E of the exact sequence representing ξ,

$$0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

compatible with the connections ∇^i on M_i. The set of isomorphism classes of such (ξ, ∇) is then a torsor under

$$H^1(\mathfrak{g}, Hom_A(M_1, M_2))$$

Proposition 1.3. *Suppose*

$$H^1(\mathfrak{g}, Hom_A(M_1, M_2)) = 0 \text{ for i=1.2}$$

then,

$$Ext^1_{A-\mathfrak{g}}(M_1, M_2) = Ext^1_A(M_1, M_2)^{\mathfrak{g}}$$

2. Non-commutative schemes.

2.1. Trivializations and observables.. Let C be any abelian category with Massey products. The last proviso is satisfied if C has enough projectives, but there are other cases where Massey products exist even though projectives are scarce. See [13] for an exposition of the Massey product structure in the category of all O_X-modules for X a scheme defined on some field **k**. Let $c \subseteq C$ be a diagram. Assume there exists a functor

$$\pi : C \longrightarrow B$$

such that

$$Btextisabelian \tag{1}$$

and

$$Ext^1_B(-,-) = 0 \tag{2}$$

Definition 2.1.1. Any such functor π will be called a *trivialization* of c.

.

Example 1. The obvious example of this set up is the following: Let A be any k-algebra, k a field, and let C= A-mod and

$$\pi : A - mod \longrightarrow k - mod.$$

the forgetful functor. Then π will be a trivialization for any diagram

$$c \subseteq C = A - mod.$$

Given any trivialization π of $c \subseteq C$, consider the k-algebra

$$O_0(c, \pi) := H^0(c, Hom_\pi)$$

defined in Chap.1, where

$$Hom_\pi : mor\ c \longrightarrow Ab.$$

is the functor defined by

$$Hom_\pi(\psi) = Hom_B(\pi(c_1), \pi(c_2))$$

for $\psi : c_1 \longrightarrow c_2$ in c.

Definition 2.1.2. $O_0 := O_0(c, \pi)$ is the k-algebra of immediate *observables*

It is clear that O_0 acts on each object $\pi(c) \in B$, $c \in ob\ c$, in the sense that there is a canonical k-algebra homomorphism

$$O_0 \longrightarrow \mathrm{End}_B(\pi(c))$$

such that the image diagram

$$im\ \pi|_c \subseteq B$$

becomes a diagram of O_0 -representations.

In the example above, we obtain for every diagram $c \subseteq A - mod$, a k-algebra $O_0(c,\pi)$ acting on every A-module in c such that c becomes a diagram of $O_0(c,\pi)$-modules. Moreover there is a canonical homomorphism of k-algebras

$$\eta_0 : A \longrightarrow O_0(c,\pi)$$

which is, in an obvious sense, a universal "extension" of the algebra A, with respect to the diagram c. Since we have,

$$c \subseteq O_0 - mod$$

and since the trivialization π induces a trivialization,

$$\pi_0 : O_0 - mod \longrightarrow k - mod$$

we may performe the construction of trivial observables, once more. We obtain,

$$O_0(c,\pi_0) = O_0(c,\pi) = O_0$$

This implies that the operation of constructing trivial observables, is a closure operation.

Example 2. Consider any commutative k-algebra A of finite type. Recall from paragraph 1., that if $c = Spec(A)$, then

$$\eta_0 : A \longrightarrow O_0(Spec(A),\pi)$$

induces an isomorphism,

$$A/radA \simeq O_0(Spec(A),\pi)$$

provided k is algebraically closed. Now, let Ind(A) be the full subcategory of A-mod fefined by the indecomposible modules and let Prim(A) denote the diagram of the form A/$\mathfrak{q}$, where $\mathfrak{q}$ is a primary ideal, and where the included morphisms are the obvious ones. It is easy to see that the canonical homomorphism

$$\eta_0 : A \longrightarrow O_0(Prim(A),\pi)$$

is an isomorphism. Notice that there is a generalized Zariski topology on Prim(A), defined as follows. Let a $\in$ A and consider the full subdiagram D(a) of Prim(A) defined by the objects V for which a is not a zerodivisor. Obviously D(a)$\cap$D(b)=D(ab) and D(a) is simply the localization of Prim(A) at a. There are canonical isomorphisms

$$O_0(D(a),\pi) \simeq A_{(a)} = O_S(D(a)|_{Spec(A)})$$

where S is the affine scheme Spec(A), and where O_S is the structure sheaf. This shows that there exists a ringed space (Prim(A), O_P), and a continous map

$$S = Spec(A) \longrightarrow Prim(A) = P$$

inducing isomorphisms of the structure sheaves

$$O_S \simeq O_P.$$

The problem with Prim(A) is that it is too big, that the topology is too coarse, and that it has unsatisfactory functorial properties. On the other hand, Spec(A) seems to be too small, since the trivial observables for Spec(A) omit the nilpotents of A. These problems stem from the trivial nature of the trivial observables. In the construction of O_0, we use only the trivial categorical structure of A-mod, restricted to c. To get to the goal, we have to take into account the infinitesimal structure of the category A-mod, i.e. the abelian structure of A-mod, and, in particular, the family of multiple extensions of the objects of c.

The goal is to construct, for every diagram c as above, two extensions of $O_0(c, \pi)$, which we shall denote $O(c, \pi)$, respectively $O_\Delta(c, \pi)$, a canonical homomorphism

$$O(c, \pi) \longrightarrow O_\Delta(c, \pi)$$

and an extension of η_0,

$$\eta_\Delta : A \longrightarrow O_\Delta(c, \pi)$$

with good functorial properties, extending the notion of *structure sheaf* into non-commutative algebra, providing us with a generalized, non-commutative, algebraic geometry, in which the presently unsolvable problems in invariant theory, and moduli theory, find solutions. We shall be guided by two "principles":

(2.1) Given any diagram c of some category of A-modules, we shall consider the objects of c as points, and the morphisms as incidences between these points. c may be provided with an action of a Lie group or Lie algebra, and there may also exist internal operations, like tensor products, defined in c, or some other reasonable extra structure. This is a *geometry*, and we want to construct an algebraic representation of the *geometry*. Such a representation should include an algebra $O(c, \pi)$ extending A, and to which the diagram c extends, with all its geometric properties. Moreover, $O(c, \pi)$ should contain all the parameters needed to describe the *dynamics* of the *geometry*, as f.ex., all deformations of the objects and the morphisms within c.

(2.2) If classical quantum theory is concerned with realities, for which there exist a mathematical model, one should consider the algebra of observables (in quantum theory) as the moduli space of these realities.

So consider a general diagram c in C=A-mod, trivialized by the functor π. Assume first that c is finite. Let $V =: |c| = \{V_{i i=1,2,..r}$, be the family of objects, and construct the non-commutative formal moduli $H(V) = (H_{i,j})$ as in [9]. Let $\tilde{V}$ be the versal family and consider the k-algebra

$$O(|c|, \pi) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_\pi(V_i, V_j))$$

and the k-algebra homomorphism,

$$\eta : A \longrightarrow O(|c|, \pi)$$

defined by the action of A on $\tilde{V}$, which, by construction, commutes with the action of H. Recall that for an artinian algebra A, and for the family V of all the simple A-modules η is an isomorphism, (see [10]). In this paper, $O(|c|, \pi)$ was denoted A(V)). Notice that, by definition of the terms, there is a unique morphism of k-algebras,

$$\rho : O(|c|, \pi) \longrightarrow O_0(|c|, \pi)$$

which, together with η and η_0 form a commutative diagram. Therefore **c** is, in an obvious sense, a diagram of $O(|c|, \pi)$-modules. Notice that if $c_1 \subseteq c_2$, there exist a canonical homomorphism

$$H(|c_2|) \longrightarrow H(|c_1|)$$

which is easily seen to have a section. If c is infinite we put

$$O(|c|, \pi) = \varprojlim_{c' \subseteq c} O(|c'|, \pi)$$

where c' runs through all finite subdiagrams of c. The k-algebra we are heading for is now a subalgebra of $O(|c|, \pi)$, singled out by a set of conditions, induced by the $\pi - incidences$ of our geometry, i.e. by the morphisms ϕ of our diagram for which $\pi(\phi)$ is not an isomorphism.Let

$$\phi_{i,j} : V_i \longrightarrow V_j \tag{2.3}$$

be a morphism of **c**. Since $Def(V_l) = H(\{V_l\})$, it follows that there are canonical surjective homomorphisms

$$H_{l,l} \longrightarrow Def(V_l), l = i, j.$$

Here Def(V) denote the formal moduli of the A-module V, in the non-commutative sense, see [9]. Now, the morphism $\phi_{i,j}$ induces maps

$$H_{i,j} \longrightarrow H_{l,l}, l = i, j$$

These are, respectively, left and right linear on $H_{l,l}$, for l=i,j. Both morhisms are defined in terms of Massey products with $\phi_{i,j}$, see [8, 10, 11]. Moreover, it follows from the construction of [6], properly generalized to the non-commutative case, that the formal moduli for the diagram (2.3) is

$$H(\phi_{i,j}) := Def(\phi_{i,j}) = Def(V_i) \otimes_{H_{i,j}} Def(V_j) \tag{2.4}$$

In particular, there exists a universal lifting of $\phi_{i,j}$,

$$\Phi_{i,j} : H(\phi_{i,j}) \otimes V_i \longrightarrow H(\phi_{i,j}) \otimes V_j$$

and morphisms,

$$\iota_l : H_{l,l} \otimes End_\pi(V_l) \longrightarrow H(\phi_{i,j}) \otimes End_\pi(V_l),\ l = i, j.$$

which, in its turn, induce morphisms,

$$\nu_l : H_{l,l} \otimes End_\pi(V_l) \longrightarrow H(\phi_{i,j}) \otimes Hom_\pi(V_i, V_j),\ l = i, j. \tag{2.5}$$

The first condition to be imposed on $O(c, \pi)$ is that the element $\{\alpha_{p,q}\}$ of $O(c, \pi)$ be such that $\alpha_{l,l}$, for l=i, j maps to the same element in $H(\phi_{i,j}) \otimes Hom_\pi(V_i, V_j)$. This amounts to the the condition,

$$\Phi_{i,j} \circ \tilde{\alpha_{j,j}} = \tilde{\alpha_{i,i}} \circ \Phi_{i,j} \tag{$D_{i,j}$}$$

where $\tilde{\alpha_{l,l}} = \iota_l(\alpha_{l,l})$. Moreover, the conditions $(D_{i,j})$ should be satisfied for all choices of liftings $\Phi_{i,j}$ of of the morphisms $\phi_{i,j}$ of c.

Obviously, the set of elements of $(H_{i,j}(|c|) \otimes Hom_\pi(V_i, V_j))$ satisfying these conditions form a sub k-algebra. Now, more generally, the morphism $\phi_{i,j}$ induces morphisms

$$H_{p,j} \longrightarrow H_{p,i} \text{ and } Hom_\pi(V_p, V_i) \longrightarrow Hom_\pi(V_p, V_j)$$

The first morphism is left linear with respect to $H_{p,p}$. These morphisms induce morphisms,

$$H_{p,i} \otimes Hom_\pi(V_p, V_i) \longrightarrow H_{p,i} \otimes Hom_\pi(V_p, V_j) \longleftarrow H_{p,j} \otimes Hom_\pi(V_p, V_j) \tag{2.6}$$

The second sets of conditions to be imposed on$O(c, \pi)$, imply that for $\alpha \in O(c, \pi)$, the "coordinates" $\alpha_{p,i}$ and $\alpha_{p,j}$ maps to the same element for the morphisms in (2.6), i.e.

$$(id_{H p,i} \otimes \phi_{i,j})(\alpha_{p,i}) = (\phi^*_{i,j} \otimes id_{Hom_\pi(V_p,V_j)})(\alpha_{p,j}) \tag{$R_{p,i,j}$}$$

It is easy to see that these conditions also define a sub k-algebra of $(H_{i,j}(|c|) \otimes Hom_\pi(V_i, V_j))$.

Definition 2.1.3. The k-algebra of *geometric observables* $O_\Delta(c, \pi)$ of the diagram c, is the sub algebra of $(H_{i,i}(|c|, \pi) \otimes End_\pi(V_i))$ defined by the conditions (D) above.

Clearly, the morphisms,

$$A \longrightarrow (H_{i,j} \otimes Hom_\pi(V_i, V_j)) \longrightarrow (H_{i,i} \otimes End_\pi(V_i)) \tag{2.7}$$

induce a canonical homomorphism of k-algebras,

$$\eta_\Delta : A \longrightarrow O_\Delta(c, \pi) \tag{2.8}$$

We now define the notion of observables in general,

Definition 2.1.4. The k-algebra of *observables* $O(c,\pi)$ of the diagram c, is the subalgebra of $(H_{i,j}(|c|)\otimes Hom_\pi(V_i,V_j))$ for which the conditions (D), and (R) are satisfied.

We have already shown that

$$O(c,\pi)\subseteq (H_{i,j}(|c|)\otimes Hom_\pi(V_i,V_j)).$$

is a sub k-algebra. Therefore there is a canonical homomorphism of k-algebras,

$$\rho_\Delta : O(c,\pi)\longrightarrow O_\Delta(c,\pi)$$

which composed with the morphism

$$\kappa : O_\Delta(c,\pi)\longrightarrow O_0(c,\pi)$$

is the obvious morphism

$$\rho_0 : O(c,\pi)\longrightarrow O_0(c,\pi).$$

However, the morphism

$$\eta_\Delta : A\longrightarrow O_\Delta(c,\pi)$$

cannot necessarily be lifted to a morphism,

$$\eta=\eta(c,\pi) : A\longrightarrow O(c,\pi)\subseteq (H_{i,j}(|c|)\otimes Hom_\pi(V_i,V_j)).$$

The reason is that in the construction of the versal family $\tilde{V}$, in particular in the definition of the right action of A on $\tilde{V}$, we made some choices of derivations representing the classes of $Ext^1_A(V_i,V_j)$, see the Remark 1. below. And there are no reasons why these choices should behave properly with respect to the actions of the morphisms (2.6). In fact there are obstructions for this to be true.

Definition 2.1.5. The diagram c is called *conservative* or *Mittag Leffler* if

(i) π considered as a presheaf on c is flabby

i.e. if for any saturated sub diagram c_0 of c, the canonical morphism,

$$\varprojlim_{c}(\pi)\longrightarrow \varprojlim_{c_0}(\pi)$$

is surjectiv.

Theorem 2.1. *Suppose c is a conservative subcategory of A-mod. For every object V_i of c there is a sequence of obstructions,*

$$o\in Ext^2_c(Ext^1_A(V_i,-),Hom_A(V_i,-))$$
$$o^n_0(A,c)\in \varprojlim_{c}{}^{(1)}(Ext^1_A(V_i,-))$$

such that if these are zero, for $n \geq 1$, there is a morphism,

$$\eta(c,\pi) : A \longrightarrow O(c,\pi) \subseteq (H_{i,j} \otimes Hom_\pi(V_i, V_j))$$

lifting η_Δ. Conversely, if there exists such a morphism $\eta(c,\pi)$, then all obstructions vanish.

Proof. To understand the relations D and R, in the definition of $O(c,\pi)$ above, one has to go back to the construction of the non-commutative deformations of a family of A-modules. The formal versal family $\tilde{V} = (H_{i,j} \otimes V_j)$, is a left H-module, and a right A-module. The right multiplication by an element $a \in A$, commutes with the action of H. This induces the homomorphism

$$\eta(c,\pi) : A \longrightarrow End_H((H_{i,j} \otimes V_j)) = (H_{i,j} \otimes Hom_\pi(V_i, V_j))$$

Recall that the action of a at the tangent level of $(H_{i,j} \otimes V_j)$, is given by,

$$v_i.a = v_i a + \Sigma_{l,j} t_{i,j}(l) \otimes \psi_{i,j}(l)(a, v_i)$$

where $\{\psi_{i,j}(l)\}_l$, is a family of derivations, $\psi_{i,j}(l) \in Der_k(A, Hom_k(V_i, V_j)$ representing a basis of $Ext^1_A(V_i, V_j)$, and where $\{t_{i,j}(l)\}_l$ is the dual basis of $E_{i,j}$.

If $\phi_{i,p} : V_j \longrightarrow V_p$ is a morphism of c, then there are maps

$$\phi_{j,p} : Ext^1_A(V_i, V_j) \longrightarrow Ext^1_A(V_i, V_p)$$

mapping $\psi_{i,j}(l)$ to $\psi_{i,j}(l)\phi_{j,p}$. This map and its dual

$$E_{i,j} \longleftarrow E_{i,p}$$

induce maps,

$$E_{i,p} \otimes Hom_\pi(V_i, V_p) \longrightarrow E_{i,j} \otimes Hom_\pi(V_i, V_p)$$

and

$$E_{i,j} \otimes Hom_\pi(V_i, V_j) \longrightarrow E_{i,j} \otimes Hom_\pi(V_i, V_p)$$

such that the elements

$$\Sigma_l t_{i,p}(l) \otimes \psi_{i,p}(l)(a, v_i) \in E_{i,p} \otimes Hom_\pi(V_i, V_p)$$

and

$$\Sigma_l t_{i,j}(l) \otimes \psi_{i,j}(l)(a, v_i) \in E_{i,j} \otimes Hom_\pi(V_i, V_j)$$

map to

(2.9) $\quad \Sigma_l t_{i,p}(l)\phi_{j,p} \otimes \psi_{i,p}(l)(a, v_i)$ *respectively* $\Sigma_l t_{i,j}(l) \otimes \psi_{i,j}(l)(a, v_i)\phi_{j,p}$

in $E_{i,j} \otimes Hom_\pi(V_i, V_p)$. The elements of (2.9) are not necessary equal, the difference being sums of tensor products, in which the right hand side factors, are trivial derivations. In fact, consider for any $V_i \in c$, the functor $Hom_A(V_i, -)$, on c. In the construction of $\tilde{V}$, we picked for every base element $\tilde{\psi_{i,j}}(l)$ of $Ext^1_A(V_i, V_j)$ a representative denoted $\psi_{i,j}(l) \in Der_k(A, Hom_k(V_i, V_j))$. If we pick another representative, the difference will be a trivial divisor, i.e. given by a linear map $\kappa_{i,j} \in Hom_k(V_i, V_j)$, such that $a \in A$ maps to $a \circ \kappa_{i,j} - \kappa_{i,j} \circ a$. So consider the exact sequences,

$$(1) \qquad 0 \longrightarrow Hom_A(V_i, -) \longrightarrow Hom_\pi(V_i, -) \longrightarrow K(V_i, -) \longrightarrow 0$$

and the exact sequence,

$$(2) \qquad 0 \longrightarrow K(V_i, -) \longrightarrow Der_k(A, Hom_\pi(V_i, -)) \longrightarrow Ext^1_A(V_i, -) \longrightarrow 0$$

as exact sequences of presheaves on c.

What we really want is a section of the last epimorphism. I.e. we want the exact sequence (2) to split. This however is the same as saying that the corresponding element

$$\xi \in Ext_c(Ext^1_A(V_i, -), K(V_i, -))$$

vanish. Since we have the general lemma,

Lemma 2.1.6. *Let F be a flabby presheaf of k-vectorspaces defined on c. Then F is injectiv.*

Proof. Let $G \subseteq H$ be presheaves on c, and assume given a morphism

$$\tau : G \longrightarrow F$$

Consider the set of extensions $\tau_\alpha : G_\alpha \longrightarrow F$ of τ, $G_\alpha \subseteq H$. There exists a maximal extension $\tau_0 : G_0 \longrightarrow F$. In the usual way we prove that if $G_0 \not\subseteq H$., then we may extend it further, say by picking an object V_i such that $G_0(V_i) \not\subseteq H(V_i)$ and an element $h_i \in H(V_i)$ but such that $h_i \notin G_0(V_i)$. Extend $\tau_0(V_i)$ arbitrarily, by picking a value $\tau_1(V_i)(h_i)$, and consider the subset c_0 of those objects V_j of c for which there exist a morphism $\phi_{i,j} : V_i \longrightarrow V_j$ such that $g_j := H(\phi_{i,j})(h_i) \in G_0(V_j)$. The family $\tau_0(V_j)(h_j) - F(\phi_{i,j})(\tau_1(h_i))$ obviously defines an element of $\varprojlim_{c_0} F$. By assumption there is an element $f_i \in F(V_i)$ mapping onto this family. Now correct the definition of $\tau_1(h_i)$ by adding f_i, and we have found an extension τ_1 of τ_0, proving that $G_0 = H$, thus proving that there exists an extension of τ to H, i.e. that F is injectiv. □

Now, by assumption, $Hom_\pi(V_i, -)$ is flabby, therefore injective, which implies that

$$Ext^1_c(Ext^1_A(V_i, -), Hom_\pi(V_i, -)) = 0.$$

Using the exact sequence (1), and denoting o(A,c), the image of ξ in

$$Ext^2_c(Ext^1_A(V_i, -), Hom_A(V_i, -))$$

we see that ξ vanish if $o(A, c) = 0$. Thus there exists a section of (2). This proves that the action of A on $\tilde{V}$ is compatible with the action of the morphisms of c, at the tangent level. Since for every vanishing Massey product, the choice of the corresponding lifting, $\eta(i_1, i_2 \ldots, i_r)$, see [7-9], is only determined up to a cocycle, the corresponding problems on the higher order levels, will lead to similar problems. For each level the choice of coherent η's is met by an obstruction

$$o^n(A,c) \in \varprojlim_{c}{}^{(1)}(Ext^1_A(V_i,-))$$

the vanishing of which makes it possible to define the action of A on $\tilde{V}$ compatible with the action of the morphisms of c. □

Remark 2.1.7. If c is conservative then

$$H^p(c, Hom_\pi(-,-)) = Ext^p_c(\pi,\pi) = 0, for\ p \geq 1$$

Example 3. (i): Let A be a commutative k-algebra, and put $c = Spec(A)$. The projective system $(Hom_\pi(V_i,-)$ is flabby on Spec(A), i.e. for any finitely generated saturated subset Λ of Spec(A), smaller than an element V_i, the natural map of V_i on the projective limit of $Hom_\pi(V_i,-)$ on Λ, is surjective. It follows from Lemma (2.1.6) that although $H^0(c, Hom_\pi(-,-)) = A/rad$ we find $H^n(c, Hom_\pi(-,-)) = 0$ for $n \geq 1$.

(ii) By [9] we know that the exact sequences of A-modules (1) and (2) defines an element of,

$$\begin{aligned} & Ext^2_c(Ext^1_A(V_i,-), Hom_A(V_i,-)) \\ & = H^2(c, Hom_A(Ext^1_A(V_i,-), Hom_A(V_i,-)) \end{aligned} \tag{2.10}$$

Picking a free resolution of V_i, and considering the two spectral sequences of the double complex

$$\begin{aligned} & D^*(c, Hom_A(Hom_A(L_*,-), Hom_A(V_i,-))) \\ & = D^*(c, L_* \otimes_A Hom_A(-, Hom_A(V_i,-))) \end{aligned}$$

we see that if

$$\varprojlim_{c}{}^{(p+1)} Tor^A_{(p)}(V_i,-) \tag{2.11}$$

vanish, then so does (2.10).

In particular we find the following,

Corollary. *If A is an irreducible commutative k-algebra, then there exists a morphism,*

$$\eta(Spec(A),\pi) : A \longrightarrow O(Spec(A),\pi) \subseteq (H_{i,j} \otimes Hom_\pi(V_i,V_j))$$

lifting η_Δ.

Notice that the construction of $O(c,\pi)$ is functorial, in the following sense: Let

$$\phi : C_1 \longrightarrow C_2$$

be any exact functor relating two abelian, admissible categories. Let c_i be a diagram of C_i, for i=1.2, such that ϕ restricts to a functor between c_1 and c_2. Moreover, assume given trivializations π_i of $(C_i,\ c_i)$, commuting with ϕ. Then there is a natural commutative diagram of k-algebras,

$$((*))\qquad \begin{array}{ccc} O(c_2,\pi_2) & \xrightarrow{\phi} & O(c_1,\pi_1) \\ {\scriptstyle\rho_2}\downarrow & & {\scriptstyle\rho_1}\downarrow \\ O_0(c_2,\pi_2) & \xrightarrow{\phi_0} & O_0(c_1,\pi_1) \end{array}$$

In particular, when c_0 is a subdiagram of c, there is a canonical *projection* morphism,

$$\phi : O(c,\pi) \longrightarrow O(c_0,\pi_0)$$

The construction of $O(c,\pi)$ and the morphism η, is unique, up to isomorphism.

Notice that the discrete diagram $|c|$ is a subdiagram of c, inducing the canonical injection,

$$O(c,\pi) \longrightarrow O(|c|,\pi) = (H_{i,j} \otimes Hom_\pi(V_i,V_j))$$

2.2. Non-commutative schemes.

Let c be diagram of A-mod. Above we considered the objects of c as *points*, and the morphisms as incidences in our *geometry.* This is in tune with the general setup, see the Introduction. The first invariant of such a geometry is the dimension. There are several possible definitions, in particular the obvious,

Definition 2.2.1. The Krull dimension of c, denoted dim c, is the maximal length of a chain of composable morphisms of c.

Then we consider the most general notion of scheme,

Definition 2.2.2. Let A be any k-algebra. A conservative diagram of A-modules c is called a scheme for A, if the the obstructions of Theorem (2.1) vanish, and the morphism

$$\eta(c,\pi) : A \longrightarrow O(c,\pi)$$

is an isomorphism.

Example 1. According to the Gerneralized Burnside Theorem, the family of simple modules V form a (0-dimensional) scheme for any finite dimensional k-algebra, when k is algebraically closed.

It turns out that in many applications, it is convenient to work with the morphism category $M(c) = mor\ c$.It is then natural to extend the notion of points of our geometry to include the morphisms of c, the incidences. Let $M(c)$ be all the objects of $mor\ c$ and let $\phi_{i,j} \in M(c)$, and consider the

Definition 2.2.3. The local k-algebra of our geometry at the point $\phi_{i,j}$ is the algebra $O(\phi_{i,j},\pi) := Def(\phi_{i,j})$.

It is clear that there are localization morphisms,

$$O(c,\pi) \longrightarrow O(\phi_{i,j},\pi)$$

and

$$O_\Delta(c,\pi) \longrightarrow O(\phi_{i,j},\pi).$$

Definition 2.2.4. Let A be any k-algebra. A diagram of A-modules c is called a geometric scheme for A, if the morphism

$$\eta_\Delta(c,\pi) : A \longrightarrow O_\Delta(c,\pi)$$

is an isomorphism.

Example 2. We shall see, in paragraph 3, that $Spec(A)$, $Prim(A)$, and more generally, any quiver in the sense of Auslander-Reiten, is a geometric scheme for A.

In general we may prove for $O_\Delta(c,\pi)$ as well as for $O(c,\pi)$ the following property,

Theorem 2.2. *Let A be any k-algebra, and let c be any diagram of A-modules such that the conclusion of Theorem (2.1) hold. Then c may be considered a diagram of $O(c,\pi)$-modules. Let V_1 and V_2 be two objects of* **c**. *Then,*

(i) $$Hom_{O(c,\pi)}(V_1,V_2) \subseteq Hom_A(V_1,V_2)$$

and

(ii) $$Ext^1_{O(c,\pi)}(V_1,V_2) \longrightarrow Ext^1_A(V_1,V_2)$$

is a surjection. If these morphisms are isomorphisms, then c is a scheme for $O(c,\pi)$.

Proof. Since by assumption, there is a natural morphism,

$$\eta(c) : A \longrightarrow O(c,\pi) \subseteq (H_{i,j} \otimes Hom_\pi(V_i,V_j)).$$

There is therefore a natural homomorphism of r-pointed k-algebras,

$$\iota : H \longrightarrow H(O)$$

where H(O) is the formal moduli of $|c|$ as a family of $O = O(c,\pi)$-modules. Moreover, since $(H_{i,j} \otimes V_j)$ is a family of O-modules (from the right), the action of O commuting with the action of H (from the left), it is clear that there exists a natural morphism

$$\nu : H(O) \longrightarrow H$$

inducing this family of O-modules. Naturality implies that the composition of ι and ν is an isomorphism, and in particular, the identity on the tangent level. But this is (ii). Since (i) is trivial, we just have to notice that if (i) and (ii) are equalities, then ι will be a surjection, as well as an injection, therefore an isomorphism, and the formation of $O(c,\pi)$ for **c** as a diagram of O-modules will be the same as the formation of $O(c,\pi)$ for **c** as a diagram of A-modules. The proviso that (i) be an equality, is there to make sure that the conditions (D) in the definition of $O(c,\pi)$ for **c** as a diagram of O-modules, are the same as (D) for A. In particular we need that the lifted morphisms Φ in the first case are the same as those used for the construction of O. □

It follows from this theorem that, for any finite family V of finite dimensional A-modules, V is a scheme for $O(V,\pi)$. In particular V becomes the family of simple $O(V,\pi)$-modules.

By construction, the ring of observables $O(-,\pi)$, is a presheaf on the ordered set of diagrams of A-mod. It follows that there are no problems in globalizing the notions of schemes for an algebra A, introducing toplogies in the classical way. In particular, the construction of the observables $O_\Delta(-,\pi)$, applied to the obvious diagrams of the category $O_X - Mod$, where X is a k-scheme, gives us a globalization procedure.

2.3 Infinitesimal structures on schemes. Let c be a diagram of A-mod. Put $X = mor\ c$, and consider a point x=$\phi_{i,j} : V_i \longrightarrow V_j$ of X.

Definition 2.3.1. Given a point $x \in X$, we put

$$T_{X,x} = ker\{Ext^1_A(V_i, V_i) \times Ext^1_A(V_j, V_j) \longrightarrow Ext^1_A(V_i, V_j)\}$$

and we shall call it the *big tangent space* of X at x.

There is a canonical map

$$Der_k(A, A) \longrightarrow T_{X,x} \tag{2.11}$$

the composition of the natural map,

$$Der_k(A, A) \longrightarrow Der_k(A, End_k(V_i)) \times Der_k(A, End_k(V_j))$$

and the surjection

$$Der_k(A, End_k(V_i)) \times Der_k(A, End_k(V_j)) \longrightarrow Ext^1_A(V_i, V_i) \times Ext^1_A(V_j, V_j)$$

It is clear that this composition ends up in $T_{X,x}$.

For every $\delta \in Der_k(A, A)$, let $\delta(x) \in T_{X,x}$, be the image of δ in $T_{X,x}$. Thus $Der_k(A, A)$ is a right A-module of *vector fields* defined on X. Suppose now that c is a scheme for A, then;

Definition 2.3.2. Given a point x=$\phi_{i,j} : V_i \longrightarrow V_j$ of $X = mor\ c$, we shall say that x is non-singular if the map (2.10) is surjective

3. The commutative case.

3.1. The main Theorem. To show that the non-commutative algebraic geometry, introduced above is a bona fide extension of classical algebraic geometry, we have to prove that, for commutative k-algebras A,

$$\eta_\Delta(Spec(A), \pi) : A \longrightarrow O_\Delta(Spec(A), \pi) \tag{S}$$

is an isomorphism. This is still a conjecture in general, having been checked in many cases, see below.

Moreover, we have to show that the construction of the observables $O(c, \pi)$, applied to the obvious subcategories c of the category $O_X - Modules$, where X is a k-scheme, gives us a globalization procedure.

According to paragraph 2 above, the globalization procedure is a consequence of (S), so we are left with (S).

Notice that in this paragraph we shall assume that A is a commutative k-algebra essentially of finite type on an algebraically closed field. The extension of the theory to include schemes on general base rings, seems difficult.

Theorem 3.1. *Let A be any commutative k-algebra, essentially of finite type. Then the canonical morphism of k-algebras*

$$\eta_\Delta : A \longrightarrow O_\Delta(Spec(A), \pi)$$

is an injection.

Proof. We have already seen that the natural morphism,

$$\rho : O(mor\ Spec(A), \pi) \longrightarrow O_0(Spec(A), \pi) \cong A/radA$$

is surjectiv. Moreover, for any closed point $x \in Spec(A)$, let, as above, $k(x)$ be the corresponding simple A-module, and consider the k-algebra homomorphisms,

$$\eta : A \longrightarrow O_\Delta(Spec(A), \pi)$$

and

$$\rho_x : O_\Delta(Spec(A), \pi) \longrightarrow O(k(x), \pi)$$

The composition is the map of A into the completion of A at x.

$$A \longrightarrow O(k(x), \pi) \cong A_x$$

Since this is true for all closed points of Spec(A), η_Δ must be injective. □

Theorem 3.2. *Let A be any irreducible reduced commutative k-algebra, essentially of finite type. Then the canonical morphism of k-algebras*

$$\eta_\Delta : A \longrightarrow O_\Delta(Spec(A))$$

is an isomorphism.

Proof. The composition of the natural morphisms,

$$A \longrightarrow O_\Delta(Spec(A)) \longrightarrow O_0(Spec(A), \pi) \cong A/radA$$

is an isomorphism. Moreover, for every $V_i = A/\mathfrak{p}_i$ it is clear that the morphism

$$\phi_{0,i} : V_0 = A \longrightarrow A/\mathfrak{p}_i$$

induces maps

$$End_\pi(A) \longrightarrow H_{i,i} \otimes Hom_\pi(A, A/\mathfrak{p}_i)$$

and

$$H_{i,i} \otimes End_\pi(A/\mathfrak{p}_i) \longrightarrow H_{i,i} \otimes Hom_\pi(A, A/\mathfrak{p}_i).$$

Since the last one is obviously injective, and since the coordinate ring at V_0 is A, represented as right actions on A, it is clear that η_Δ is an isomorphism. □

4. Invariant theory and moduli

Consider a k-algebra A, and for the purpose I have in mind now, we may assume A to be commutative, k to be algebraically closed, and the ring of observables associated to a diagram c of A-modules to be the $O_\Delta(c, \pi)$ defined above.

Suppose that there is a Lie-algebra $\mathfrak{g}$ of vectorfields (i.e. derivations), acting on A. Consider the category, C of A-$\mathfrak{g}$ -modules, i.e. A-modules with integrable $\mathfrak{g}$-covariant derivations. In this category we define the trivialization functor,

$$\pi : C \longrightarrow \mathrm{k} - mod$$

by

$$\pi_{\mathfrak{g}}(V) = H^0(\mathfrak{g}, V)$$

Recall that when $\mathfrak{g}$ is semisimple then $\pi_{\mathfrak{g}}$ is exact. Moreover, Ext, in this category, is then simply the $\mathfrak{g}$-invariants of the Ext in A-mod. The exactness of $\pi_{\mathfrak{g}}$, is, however, not neccessary for the construction of the ring of observables. Therefore, given any diagram c of C, say $Spec(A - \mathfrak{g})$, corrsponding to the ordered set of $\mathfrak{g}$-invariant prime ideals of A, we may consider the ring of observables $O_\Delta(c, \pi_{\mathfrak{g}})$ In particular, we pose,

Definition 4.1. The quotient of Spec(A) with $\mathfrak{g}$, denoted by Spec(A)/$\mathfrak{g}$, is the presheaf of k-algebras of observables,

$$O_\Delta(Spec(A - \mathfrak{g}), \pi_{\mathfrak{g}})$$

There is now a notion of moduli for singularities, in this *non-commutative*, sense. In some easy cases, one gets exactly what one would expect, see paragraph 6. The computations rely heavily on the computations of Massey products for modules, and on the computations of cohomology of Lie algebras.

Notice that in the above formalism, I might have considered, instead of the Lie-algebra $\mathfrak{g}$, any group G, and for that matter, any other reasonable superstructure on the category of A-modules.

That this invariant theory fits with the classical invariant theory, is shown by the following result,

Theorem 4.1. *Let A be any irreducible and reduced commutative k-algebra of finite type, and $\mathfrak{g}$ a semi-simple Lie-algebra of vectorfields (i.e. derivations), acting on A. Assume that the geometric quotient of Spec(A) with $\mathfrak{g}$ exists, and is affine. Then it coincides with the Spec(A)/$\mathfrak{g}$, defined above.*

Proof. By assumption, the diagram $Spec(A^{\mathfrak{g}})$ induces the diagram $Spec(A - \mathfrak{g})$. Moreover the trivialization $\pi_{\mathfrak{g}}$ maps the diagram $Spec(A - \mathfrak{g})$ onto the diagram $Spec(A^{\mathfrak{g}})$, or rather, to the image of this diagram under the canonical trivialization π. But then the exactnes of $\pi_{\mathfrak{g}}$ and the smoothness of the morphism of the geometric quotient, proves that the formal moduli of the family $|Spec(A-\mathfrak{g})|$ in the category of $A-\mathfrak{g}$-mod is isomorphic to the corresponding formal moduli of the family $|Spec(A-\mathfrak{g})|$ in the category of $A^{\mathfrak{g}}$-mod. Since the trivializations coincide, the Theorem 3.3 shows that

$$O_\Delta(Spec(A - \mathfrak{g}), \pi_{\mathfrak{g}}) \cong A^{\mathfrak{g}}$$

which is exactely what we wanted. □

5. Tensor products and quantum groups

Let c be a subcategory of A-mod, with a trivializing functor π. Suppose given a tensor product on the category c, i.e. bi-functor

$$\otimes : c \times c \longrightarrow c$$

consistent with π, which is a faithful imbedding, with some extra structure. In particular there should exist natural isomorphisms,

$$\alpha_{-,-,-} : ((- \otimes -) \otimes -) \simeq (- \otimes (- \otimes -)$$

satisfying the Mac Lane pentagon,

$$id_X \otimes \alpha_{Y,Z,W} \circ \alpha_{X,Y \otimes Z,W} \circ \alpha_{X,Y,Z} \otimes idW$$
$$= \alpha_{X,Y,Z \otimes W} \circ \alpha_{X \otimes Y,Z,W}$$

Consider the exact functor,

$$\Delta : c \to c \times c,$$

defined by $\Delta(V) = V \times V$. Then there exist homomorphisms of k-algebras,

$$O(c,\pi) \to O(c \otimes c,\pi) \to O(c \times c,\pi) \to O(c,\pi) \tag{1}$$

such that,

$$O(c \times c,\pi) \simeq O(c,\pi) \otimes O(c,\pi)$$

and the last morphism of (1) is the multiplication morphism of the k-algebra $O(c,\pi)$.

The Mac Lane pentagon garanties that the first morphism of (1) becomes an associative co-algebra structure on $O(c,\pi)$. Clearly any extra functorial *symmetry* one may want to consider on c, will show up in the corresponding k-algebra $O(c,\pi)$.

6. Examples

6.1. The non-commutative projective line. Let $A = k[x_0, x_1]$, and consider the usual k^*-action. We shall compute the space Spec(A)$/k^*$. The subcategory Spec(A-k^*) of A-k^*-modules consists of the origin V_3, the lines through the origin $V_2(l)$, and the generic point V_1. The trivializing functor (see paragraph 4),

$$\pi : A - k^* - mod \longrightarrow k - mod$$

has the values,

$$\pi(V_1) = k, \ \pi(V_2(l)) = k, \ \pi(V_3) = k$$

Therefore there are no π-incidences, and the non-commutative orbit space is given by the hull of the deformation functor, i.e. by $(H_{i,j})$. Since $H^p(k^*, -) = 0$ for p$\geq$1, we may use the Proposition (1.2), and we obtain,

$$Ext_{A-k^*}(V_i, V_j) = Ext_A(V_i, V_j)^{k^*}.$$

It is easy to compute the different ext-groups, we find:

$$Ext^1_A(V_i, V_j) = 0, \text{for i=1, j=1,2,3.}$$
$$Ext^1_A(V_2(l), V_1) = V_2(l) = A/(\alpha x_0 + \beta x_1)$$
$$Ext^1_A(V_2(l), V_2(l)) = V_2(l)$$
$$Ext^1_A(V_2(l), V_2(l')) = 0 \quad \text{if } l \neq l'$$
$$Ext^1_A(V_2(l), V_3) = V_3 = k$$
$$Ext^1_A(V_3, V_1) = 0$$
$$Ext^1_A(V_3, V_2(l)) = V_3 = k$$
$$Ext^1_A(V_3, V_3) = k^2$$

Using the results of paragraph 1.2. we obtain for the invariants

$$Ext^1_A(V_i, V_j)^{k^*} = 0, \text{ for i=1, j=1,2,3.}$$
$$Ext^1_A(V_2(l), V_1)^{k^*} = k \text{ represented by } \xi = l$$
$$Ext^1_A(V_2(l), V_2(l))^{k^*} = k \text{ represented by } \xi = l$$
$$Ext^1_A(V_2(l), V_2(l'))^{k^*} = 0 \text{ if } l \neq l'$$
$$Ext^1_A(V_2(l), V_3)^{k^*} = 0$$
$$Ext^1_A(V_3, V_1)^{k^*} = 0$$
$$Ext^1_A(V_3, V_2(l))^{k^*} = 0$$
$$Ext^1_A(V_3, V_3)^{k^*} = 0$$

The corresponding quotient becomes the infinite matrix algebra of the form,

$$Spec(A)/k^* := O(Spec(A - k^*), \pi) = \begin{pmatrix} k & 0 & 0 \\ k[[t_2(l)]]t_{2,1} & k[[t_2(l)]] & 0 \\ 0 & 0 & k \end{pmatrix}$$

where l runs through all the points in the ordinary projective line. We observe that the special point, corresponding to the isolated orbit, i.e. the origo, stays isolated, even infinitesimally. There are, however, adjacencies between the formal points corresponding to the lines through the origo, and the *generic* point corresponding to the generic point of the ordinary projective line.

Suppose that we localize, say in x_0, i.e. that we restrict to the

$$Spec(A_{\{x_0\}} - k^*) = \{V_1 = A_{\{x_0\}}, V_2(l) = A_{\{x_0\}}/(l)\}$$

then we find,

$$\pi(V_1) = k[x_1/x_0], \; \pi(V_2(l)) = k$$

and therefore the π-incidences,

$$\pi(V_1) \longrightarrow \pi(V_2(l))$$

for all l. The exts in the new category looks like,

$$
\begin{aligned}
&ExtA^1_{\{x_0\}}(V_i,V_j)^{k^*}=0, \text{ for i=1, j=1,2.}\\
&ExtA^1_{\{x_0\}}(V_2(l),V_1)^{k^*}=k \text{ represented by } \xi=l\\
&ExtA^1_{\{x_0\}}(V_2(l),V_2(l))^{k^*}=k \text{ represented by } \xi=l\\
&ExtA^1_{\{x_0\}}(V_2(l),V_2(l'))^{k^*}=0 \text{ if } l\neq l'
\end{aligned}
$$

With this we find,

$$
Spec(A_{\{x_0\}})/k^* = O(Spec(A_{\{x_0\}}-k^*,\pi) = \begin{pmatrix} f(x_1/x_0) & 0 \\ \psi(f(x_1/x_0))t_{2,1} & f(x_1/x_0)] \end{pmatrix}
$$

where ψ is some derivation of $Der_k(k[x_1/x_0])$ and f runs through $k[x_1/x_0]$ in,

$$
\begin{pmatrix} End_k(k[x_1/x_0]) & 0 \\ k[[x_1/x_0]]t_{2,1} & k[[t_2(l)]] \end{pmatrix}
$$

as expected, see the Theorem (4.1).

It is therefore clear that the non-commutative version of the projective line *contains* the geometric projective line.

If we consider, instead of the action by the group k^*, the action of the Lie algebra $\mathfrak{g}$ generated by the Euler vectorfield $\delta_0 = x_0\frac{\partial}{\partial x_0}+x_1\frac{\partial}{\partial x_1}$, we get a different picture steming from the fact that $\mathfrak{g}$ has cohomology. The subcategory Spec(A-$\mathfrak{g}$) of A-$\mathfrak{g}$-modules consists of the origin V_3, the lines through the origin $V_2(l)$, and the generic point V_1. The trivializing functor

$$
\pi : A-\mathfrak{g}-mod \longrightarrow k-mod
$$

has the values,

$$
\pi(V_1)=k,\ \pi(V_2(l))=k,\ \pi(V_3)=k
$$

Since there are no π-incidences, the non-commutative orbit space $Spec(A)/\mathfrak{g}$ is given by the hull of the deformation functor, i.e. by $(H_{i,j})$, as above. However, here we cannot use the result (4.1), since for most $\mathfrak{g}$-modules V, $H^1(\mathfrak{g},V)=V/\delta_0V\neq 0$. In fact we get,

$$
\begin{aligned}
&Ext^1_{A-\mathfrak{g}}(V_i,V_j)=Ext^1_A(V_i,V_j)^{\mathfrak{g}}\oplus H^1(\mathfrak{g},Hom_A(V_i,V_j))\\
&Ext^2_{A-\mathfrak{g}}(V_i,V_j)=H^1(\mathfrak{g},Ext^1_A(V_i,V_j))
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&Ext^1_{A-\mathfrak{g}}(V_1,V_j) = H^1(\mathfrak{g}, Hom_A(V_1,V_j)) = k \text{ for j=1,2,3.}\\
&Ext^1_{A-\mathfrak{g}}(V_2(l),V_j) = Ext^1_A(V_2,V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_2,V_j))\\
&= k\oplus 0 \text{ for j=1}\\
&= k\oplus k \text{ for } V_j = V_2(l)\\
&= 0\oplus 0 \text{ for } V_j = V_2(l') \ l\neq l'\\
&= 0\oplus k \text{ for j=3}\\
&Ext^1_{A-\mathfrak{g}}(V_3,V_j) = Ext^1_A(V_3,V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_3,V_j))\\
&= 0\oplus 0 \text{ for j=1}\\
&= 0\oplus 0 \text{ for j=2}\\
&= 0\oplus k \text{ for j=3}\\
&Ext^2_{A-\mathfrak{g}}(V_i,V_j) = H^1(\mathfrak{g}, Ext^1_A(V_i,V_j))\\
&= 0 \text{ for i=1, j=1,2,3.}\\
&= k \text{ for i=2, j=1}\\
&= k \text{ for} V_i = V_2(l), V_j = V_2(l)\\
&= 0 \text{ for } V_i = V_2(l), V_j = V_2(l'),\ l\neq l'\\
&= k \text{ for i=3, j=3}
\end{aligned}
$$

It follows that $Spec(A)/\mathfrak{g}$ is given by the rather complicated looking k-algebra, generated by,

$$
\begin{pmatrix} k[[t_1]] & t_{1,2}(l) & t_{1,3}(l)\\ u_{2,1}(l) & k[[t_2(l),u_2(l)]] & t_{2,3}(l)\\ 0 & 0 & k[[t_3]] \end{pmatrix}
$$

with some relations.

6.2. The moduli space of simple singularities, the A_2 case.. We shall consider the Weierstrass family $F := F(t_0,t_1,x,y) = x^3-y^2+t_1x+t_0$, parametrized by the k-algebra $A := k[t_0,t_1]$, and the corresponding Kodaira-Spencer kernel $\mathfrak{g}\subseteq Der_k(A)$, generated by,

$$
\begin{aligned}
\delta_0 &= 3t_0\frac{\partial}{\partial t_0} + 2t_1\frac{\partial}{\partial t_1}\\
\delta_1 &= 2t_1^2\frac{\partial}{\partial t_0} - 9t_0\frac{\partial}{\partial t_1}
\end{aligned}
$$

We claim that the *moduli space* consisting of the three singularities in the family F, is given as the quotient space $Spec(A)/\mathfrak{g}$. We must therfore consider the diagram $Spec(A-\mathfrak{g})$, consisting of the 3 $A-\mathfrak{g}$-modules,$V_1 = k[t_0,t_1]$, $V_2 = k[t_0,t_1]/(\Delta)$, where $\Delta = 27t_0^2+4t_3^3$ is the discriminant of F, and finally $V_3 = k$ coresponding to origo.

As above we find that

$$\pi = H^0(\mathfrak{g},-) : A-\mathfrak{g}-mod \longrightarrow k-mod$$

defines three points,

$$\pi(V_1) = k,\ \pi(V_2) = k,\ \pi(V_3) = k$$

with no incidences. Since it is easy to see that

$$H^2(\mathfrak{g}, Hom_A(V_i, V_j)) = 0$$

we find,

$$Ext^1_{A-\mathfrak{g}}(V_i, V_j) = H^0(\mathfrak{g}, Ext^1_A(V_i, V_j)) \oplus H^1(\mathfrak{g}, Hom_A(V_i, V_j))$$

which implies,

$$\begin{aligned}
&Ext^1_{A-\mathfrak{g}}(V_1, V_j) = H^1(\mathfrak{g}, Hom_A(V_1, V_j)) = k \text{ for j=1,2,3.}\\
&Ext^1_{A-\mathfrak{g}}(V_2, V_j) = H^0(\mathfrak{g}, Ext^1_A(V_2, V_j)) \oplus H^1(\mathfrak{g}, Hom_A(V_2, V_j))\\
&= k \oplus 0 \text{ for j=1}\\
&= 0 \oplus k \text{ for } j = 2\\
&= 0 \oplus k \text{ for j=3}\\
&Ext^1_{A-\mathfrak{g}}(V_3, V_j) = Ext^1_A(V_3, V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_3, V_j))\\
&= 0 \oplus 0 \text{ for j=1}\\
&= 0 \oplus 0 \text{ for j=2}\\
&= 0 \oplus k \text{ for j=3}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&Ext^2_{A-\mathfrak{g}}(V_i, V_j) \subseteq H^0(\mathfrak{g}, Ext^2_A(V_i, V_j)) \oplus H^1(\mathfrak{g}, Ext^1_A(V_i, V_j))\\
&= 0 \text{ for all i,j=1,2,3.}
\end{aligned}$$

The moduli space is therefore given by the k-algebra freely generated by,

$$\begin{pmatrix} k[[t_{1,1}]] & t_{1,2} & t_{1,3} \\ 0 & k[[t_{2,2}]] & t_{2,3} \\ 0 & 0 & k[[t_{3,3}]] \end{pmatrix}$$

which has a *reduced* quotient, given by the matrices of the form,

$$\begin{pmatrix} k & kt_{1,2} & kt_{1,3} \oplus kt_{1,2}t_{2,3} \\ 0 & k & kt_{2,3} \\ 0 & 0 & k \end{pmatrix}$$

which is the k-algebra of the non-commuting adjacency diagram corresponding to the Weierstrass family, see [Laudal [10]],

$$\begin{aligned}
&t_{2,3} : \text{cusp} \rightarrow \text{node}\\
&t_{1,2} : \text{node} \rightarrow \text{ellipt}\\
&t_{1,2}t_{2,3} : \text{cusp} \longrightarrow \text{ellipt}\\
&t_{1,3} : \text{cusp} \rightarrow \text{ellipt}
\end{aligned}$$

Notice that $\mathfrak{g}$ is a rank 2 A-module, such that we may expect to find exact sequences of $A-\mathfrak{g}$-modules,

$$0 \longrightarrow A \longrightarrow \mathfrak{g} \longrightarrow A \longrightarrow 0$$
$$0 \longrightarrow A/(\Delta) \longrightarrow \mathfrak{g} \otimes_A A/(\Delta) \longrightarrow A/(\Delta) \longrightarrow 0$$
$$0 \longrightarrow A/(t_0,t_1) \longrightarrow \mathfrak{g} \otimes_A k(o) \longrightarrow A/(t_0,t_1) \longrightarrow 0$$

explaining the diagonal tangent structure of the quotient space,

$$\begin{pmatrix} k[[t_{1,1}]] & t_{1,2} & t_{1,3} \\ 0 & k[[t_{2,2}]] & t_{2,3} \\ 0 & 0 & k[[t_{3,3}]] \end{pmatrix}.$$

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