A Mahlo-universe of effective domains with totality

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Abstract

We construct a typed hierarchy of effective algebraic domains with totality of hight the first recursively Mahlo ordinal. The hierarchy is based on the empty type and the domains for singleton, boolean values and natural numbers, and it is closed under dependent sums and products of effectivly parameterised families of types, and under universes closed under any continuous operator that generates a universe.

1 Introduction

Given a type theory, there are several ways to produce a semantics for the theory. An important distinction will be between intuitionistic and classical semantics. It is an understandable view that a classical semantics for an intuitionistic type theory is of little value. On the other hand, if one construct some natural, classical semantics for a type-theory, by interpreting the type operators described in the theory, one gets an impression of the classical strength of these operators, even inside a constructive environment. In [9] we constructed a hierarchy of domains with totality and density having the complexity of Kleene-recursion in the functional ${}^3E$. Taking the hereditarily effective version of this hierarchy, the complexity will be that of the first non-recursive ordinal ω_1^{CK}. In a sense this is the minimal complexity of effective dependent sums and products, and corresponds to the minimal model of KP-set theory.

If we introduce the W-type constructor, the corresponding minimal classical model will be of complexity the first recursively inaccessible ordinal. This is essentially established in Normann [10]. There are corresponding results of equivalent proof-theoretical strength between certain extensions of KP-theory and certain intuitionistic type theories, see Griffor and Rathjen [3] and Setzer [15]. Type theory with one universe, but without the W-operator corresponds to KP-theory with one admissible, while type theory with one universe *and* the W-operator corresponds to KP-theory with one recursively inaccessible ordinal. The correspondance between results on proof-theoretical strength and on the complexity of the minimal models can not be a coincidence. With this paper we will extend this pattern one step.

When we use the term *minimal model* this is not accurate. It is of course possible to construct semantics of these type theories of semicomputable complexity; we introduce formal interpretations of every type proved to exist and every object proved to be of that type. Since the set of proofs is computable, the model will be semicomputable. What we mean by minimal here, is that we view the constructors used in the theory, we then make classical interpretations of these constructors inside the category of effective domains with totality, and finally consider the minimal hierarchies closed under these constructors.

In this paper we will go the other way around. In the previous estimates of complexity we have somehow viewed transfinite computations as transfinite propositions, and we have used standard (and not so standard) *propositions as types* translations to express termination and value of transfinite computations in the typed hierarchy. Here we will consider computations relative to the functional known as *the superjump* defined by Gandy [2], and see what sort of type constructors will be required in order to express these computations as properties of the typed hierarchy. A. Setzer on one hand, and E.R. Griffor and M. Rathjen on the other, has given a proof-theoretical analysis (unpublished) of Mahloness, and their results and the results of this paper supplement each other.

Domains

The domains in this paper will be *algebraic domains* or *Scott-Ershov-domains* as defined in Stoltenberg-Hansen, Lindström and Griffor [16]. We view a domain element as an ideal in a partially ordered set of finitary compacts. Thus we use the symbol "$\sqsubseteq$" for the ordering relation on any domain, including the domain of continuous functions. We will assume that the reader is familiar with the theory of domains as presented in e.g. [16]

Acknowledgements

My work on this started when I visited the University of Uppsala in September '95. At several occations I had discussions with Ulrich Berger on a related problem, representing computations in the superjump in a typed hierarchy of domains with dense totality. Many of the ideas used in this paper grew from the discussions with him. The workshop in Uppsala in April '96 was most stimulating, as was discussions with Griffor and Setzer earlier on. My final inspiration for working out this approach came through Rathjen's visit to Oslo in October '96. It is still my belief that the original approach via uniformly dense universes can be carried out, and a proof along these lines will give a much more fine structured analysis of the computations relative to the superjump.

2 The Mahlo-universe

In this section we will introduce a typed hierarchy of effective domains with totality. We first introduce the underlying domains via a standard fix-point construction. Following the format of the papers Kristiansen and Normann [6, 7] and Normann [9, 10, 11] we will construct one domain T, being the domain of *type descriptions*, and an interpretation map $I(t)$ interpreting each $t \in T$ as a domain. See also Berger [1] or Waagbø [17] for a discussion.
I will be a parameterisation in the sense of Palmgren and Stoltenberg-Hansen [14]. As a part of our construction, we will give interpretations of universes. Here we will follow ideas from Berger [1], though we will not be bothered with problems of density.

Definition 1 We let O, B and N be atomic, maximal elements of T.
The interpretations will be

$I(O) = \{\bot\}$

$I(B) = \{\bot, tt, ff\}$, i.e. the flat domain of boolean values.

$I(N) = \mathbb{N}_\perp$, i.e. the flat domain of natural numbers.

If $t \in T$ and $F : I(t) \to T$ is continuous, we call (t, F) a *T-parameterisation*, or just a parameterisation.
We then let $\Pi(t, F)$ and $\Sigma(t, F)$ be in T.
The interpretations will be the corresponding dependent product and dependent sum as defined in e.g. [14], in [6] or in [1].

If $\Phi : T \to T$ is continuous, we let $(U, \Phi) \in T$.
We define $I(U, \Phi)$ and the map $\rho : I(U, \Phi) \to T$ below. We let J be the composition of ρ and I. J will be the interpretation map on $I(U, \Phi)$.
ρ and J will be independent of Φ.

> O, B and N are in S with ρ being the identity
>
> (S, J) are closed under Π and Σ in the same way as (T, I) is.
> $\rho(\Pi(s, F)) = \Pi(\rho(s), \rho \circ F)$ and analogue for Σ.
>
> If $\Psi \subseteq \Phi$ and $s \in I(U, \Psi)$, we let $o(\Psi, s) \in I(U, \Phi)$ with
> $\rho(o(\Psi, s)) = \Psi(\rho(s))$.
> Here o is just a formal symbol, we could use (o, Ψ, s) instead.

Remark 1 T is technically defined via the least solution of the set of equations above. It is clear that the domain T and all parameterisations involved will be effective in the sense that there is an enumeration of the compacts such that the ordering, the consistency relation and the least upper bound operator on compacts all are computable.

From now on we will restrict ourselves to effective objects in the domains, i.e. objects where the set of compact approximations are r.e.

We are now ready to define the hierarchy of well-formed types and the set of total objects in each well-formed type:

Definition 2 By induction on the countable ordinal α we define $T_\alpha \subseteq T$ and the total elements $\bar{I}(t)$ for $t \in T_\alpha$ as follows:

O, B and N are in T_α for all α with the obvious set of total elements.

If α is a limit ordinal, then $T_\alpha = \bigcup_{\beta < \alpha} T_\beta$.

If $\alpha = \beta + 1$ we let

$\Sigma(t, F) \in T_\alpha$ if $t \in T_\beta$ and $F(x) \in T_\beta$ for all $x \in \bar{I}(t)$.

$\Pi(t, F) \in T_\alpha$ if $t \in T_\beta$ and $F(x) \in T_\beta$ for all $x \in \bar{I}(t)$.

In both cases the total objects are defined in the obvious way.
If $\Phi : T \to T$ we define $I_\alpha(U, \Phi)$ as the union of the inductively defined sets $I_{\gamma,\alpha}$ for $\gamma < \alpha$ as follows:

We perform the same inductive definition as for T_α with one restriction and one suplement.
The restriction is that we do not ad any element s to $I_{\gamma+1,\alpha}(U, \Phi)$ unless $\rho(s) \in T_\beta$.
The suplement is that if $\Psi \subseteq \Phi$, $s \in I_{\gamma,\alpha}(U, \Psi)$ and $\Psi(t) \in T_\beta$ for all $t \in T_\beta$ with $t \subseteq \rho(s)$, then $o(\Psi, s) \in I_{\gamma',\alpha}(U, \Phi)$ for all γ' with $\gamma < \gamma' < \alpha$.

We then let $(U, \Phi) \in T_\alpha$ if $I_\alpha(U, \Phi)$ is closed under Π and Σ and if $\Phi(t) \in T_\beta$ whenever $t \subseteq \rho(s)$, $t \in T_\beta$ and $s \in I_\alpha(U, \Phi)$. In this case $I_\alpha(U, \Phi) = \bar{I}(U, \Phi)$.

Finally we let

$$\bar{T} = \bigcup_{\alpha < \omega_1} T_\alpha$$

Remark 2 The intuition is that we ad a well-formed universe to our hierarchy when we have an operator that maps well-formed types to well-formed types; at least when the operator is restricted to its own closure. The definition of totality of $o(\Psi, s)$ is as it is in order to ensure certain regularity properties discussed in the next section.

Our main result will be that this hierarchy will have the first recursively Mahlo ordinal ρ_0 as its closure ordinal. In our hierarchy there is no way to 'tell in advance' when we may form a universe from an operator, we can only do so when it is established that we have a closure of the operator. This aspect is in our view the essence of the Mahlo property, and any axiomatised version of Mahlo type theory must take this into account.

Lemma 1 *If $s \in T_{\alpha+1} \setminus T_\alpha$, then $\alpha < \rho_0$.*

Proof
Closure under Π and Σ requires admissibility.
If $\Phi : T \to T$ is such that $(U, \Phi) \in T_{\alpha+1} \setminus T_\alpha$, we consider the ordinal function $\hat{\Phi} : \alpha \to \alpha$ defined by

$$\hat{\Phi}(\gamma) = \mu\beta[\forall s \in I_{\gamma,\alpha}(s \in T_\beta)]$$

By construction, α will be the least admissible ordinal closed under $\hat{\Phi}$, so $\alpha < \rho_0$.

Remark 3 We might ad the W-operator as one of our basic operators inside each universe. The closure ordinals of each universe would then be recursively inaccessible, without this changing the argument.

3 Digression

The hierarchy investigated in this paper is an extension of the single-valued version of the hierarchy from Normann [11]. Waagbø [17] uses a similar hierarchy to give an interpretation of basic intuitionistic type theory.

In both these papers natural equivalence relations on the set of well-formed type expressions and on the total elements of equivalent types are constructed. These equivalence relations will correspond to objects having the same extentional interpretation. In this digression we will indicate how these equivalence relations and their characterisations can be extended to the universes. We will not need this result for the characterisation of the complexity. Thus we do not give all the details. A more general treatment of the kind of results established here will be given in the forthcomming Normann [13].

In this digression it is of no importance if we consider the effective hierarchy or the full hierarchy. All arguments will hold in both cases.

We let D be the domain where each interpretation $I(t)$ will be a subdomain of D. We will use the notation D_0, $I_0(t)$, T_0 etc. for the set of compacts in each domain.

Lemma 2 *There is a partial map $\nu : D_0 \to T_0$ such that for all $t \in T$ and $\delta \in D_0$:*

$$\textit{If } \delta \in I_0(t) \textit{ then } \nu(\delta) \subseteq t \textit{ and } \delta \in I_0(\nu(\delta)).$$

Proof
The proof of Normann [11] can be used in all cases except for the new universe operator, and the extension to that case is simple by recursion on the possible elements of $I(U, \Phi)$, we simply collect the amount of Φ used in the formation of the compact.

Corollary 1 *For each t_1 and t_2 in T we have that*

$$I_0(t_1 \cap t_2) = I_0(t_1) \cap I_0(t_2).$$

Definition 3 Let X and Y be domains. We say that X is a *full subdomain* of Y if X is a subdomain of Y and for all $x \in X$ and $y \in Y$, if $y \subseteq x$ in the sense of Y, then $y \in X$.

Lemma 3 *Let $\Phi \subseteq \Phi_1$. Then $I(U, \Phi)$ is a full subdomain of $I(U, \Phi_1)$.*

The proof is trivial

Definition 4 a) If $s \subseteq t \in T$ and $x \in I(s)$ we let x^t be the minimal extension of x to an element of $I(t)$

b) If $s \subseteq t \in T$ and $y \in I(t)$, we let y_s be the restriction of y to an element of $I(s)$.

Theorem 1 a) *If $t \in \bar{T}$ and $t \subseteq t_1$, then*

- **i)** $t_1 \in \bar{T}$.
- **ii)** *If $x \in \bar{I}(t)$, then $x^{t_1} \in \bar{I}(t_1)$.*
- **iii)** *If $y \in \bar{I}(t_1)$, then $y_t \in \bar{I}(t)$.*

b) *If t_1 and t_2 are in $\bar{T}$ and $\{t_1, t_2\}$ is bounded, then $t_1 \cap t_2 \in \bar{T}$.*

c) *If t_1 and t_2 are as above, x_1 and x_2 are total in $I(t_1)$ and $I(t_2)$ resp. and $\{x_1, x_2\}$ is bounded, then $x_1 \cap x_2$ is total in $I(t_1 \cap t_2)$.*

Proof
We prove the theorem by simultanous induction on the rank of t and the rank of the bound on t_1 and t_2. Sufficient methods are given in Normann [11] or in Waagbø [17] in all cases except the formation of universes, so let us consider this case.

We first sketch the proof of a). Let $\Phi \subseteq \Phi_1$.
We prove ii) and iii) by recursion on the formation of the universes. As a consequence of iii) we obtain that the construction of $\bar{I}(U, \Phi_1)$ will be closed, so $(U, \Phi_1) \in \bar{T}$. Thus i) will hold.

In the proof of ii) and iii), the only new case is the use of the operator. For ii) this case is trivial, since $I(U,\Phi)$ is a full subdomain of $I(U,\Phi_1)$ and we actually prove that every total object in $I(U,\Phi)$ also is total as an element of $I(U,\Phi_1)$.

In order to prove iii) we need the following

Claim

Let Ψ_1 and Ψ_2 be bounded and let $s_1 \in I_\alpha(U,\Psi_1)$ and $s_2 \in I_\alpha(U,\Psi_2)$. Assume that $\{s_1, s_2\}$ is bounded. Then

$$s_1 \cap s_2 \in I_\alpha(U, \Psi_1 \cap \Psi_2)$$

Proof of claim:

We prove this for $I_{\gamma,\alpha}(U,\Psi)$ by induction on γ uniformly for all Ψ, with $\alpha = \beta + 1$.

For all cases but the use of the operator, we can use the methods from Normann [11].

So, let

$$s_1 = o(\Psi_1', t_1) \in I_{\gamma+1,\alpha}(U,\Psi_1)$$

$$s_2 = o(\Psi_2', t_2) \in I_{\gamma+1,\alpha}(U,\Psi_2)$$

where $\{\Psi_1, \Psi_2\}$ is bounded and $\{s_1, s_2\}$ is bounded.

Then the t's and the Ψ''s are bounded and we may use the induction hypothesis to obtain

$$t_1 \cap t_2 \in I_{\gamma,\alpha}(U, \Psi_1' \cap \Psi_2')$$

and thus that $\rho(t_1 \cap t_2) \in T_\beta$.

By assumption $\Psi_1'(\rho(t_1 \cap t_2)) \in T_\beta$ and $\Psi_2'(\rho(t_1 \cap t_2)) \in T_\beta$, so $(\Psi_1' \cap \Psi_2')(\rho(t_1 \cap t_2)) \in T_\beta$. It follows that

$$s_1 \cap s_2 = o(\Psi_1' \cap \Psi_2', t_1 \cap t_2) \in I_{\gamma+1,\alpha}(U, \Psi_1 \cap \Psi_2)$$

This ends the proof of the claim.

We now prove iii) in this case.

Let $t_1 = (U,\Psi_1)$, $t = (U,\Psi)$ and $y = o(\Psi_1, x)$, where $\Psi_1 \subseteq \Phi_1$.

Let $\Psi = \Phi \cap \Psi_1$. By the *claim* Ψ is total and $o(\Psi, x_{(U,\Psi)}) \subseteq y_t$. By the induction hypothesis, $o(\Psi, x_{(U,\Psi)})$ is total, so $y_t \in \bar{I}(U,\Phi_1)$. This ends the proof of a).

The proofs of b) and c) follow the same pattern and are omitted. All cases except the universe formation are as in Normann [11] and the remaining case is mainly taken care of by the *claim*. This ends the proof of the theorem.

Definition 5 We define two binary relations $\sim$ on $\bar{T}$ and $\approx$ on $\Sigma(t \in \bar{T})\bar{I}(t)$ as follows:

$O \sim O$ etc. for atomic elements of $\bar{T}$.

$(N, 17) \approx (N, 17)$ etc. for atomic total elements of atomic types.

$\Sigma(s, F) \sim \Sigma(t, G)$ if $s \sim t$ and for all $x \in \bar{I}(s)$ and all $y \in \bar{I}(t)$, if $(s, x) \approx (t, y)$, then $F(x) \sim G(y)$.
In this case $(\Sigma(s, F), (x, u)) \approx (\Sigma(t, G), (y, v))$ if $(s, x) \approx (t, y)$ and $(F(x), u) \approx (G(y), v)$.

$\Pi(s, F) \sim \Pi(t, G)$ if $s \sim t$ and for all $x \in \bar{I}(s)$ and all $y \in \bar{I}(t)$, if $(s, x) \approx (t, y)$, then $F(x) \sim G(y)$.
In this case $(\Pi(s, F), f) \approx (\Pi(t, G), g)$ if $(F(x), f(x)) \approx (G(y), g(y))$ whenever $(s, x) \approx (t, y)$.

In the case (U, Φ) we first define a relation $\hat{\approx}$ between elements of $\bar{I}(U, \Phi_1)$ and elements of $\bar{I}(U, \Phi_2)$ for arbitrary Φ_1 and Φ_2 (omitting some indices on $\hat{\approx}$) as follows:
For base types we define $\hat{\approx}$ as the identity.
We let $\Sigma(s_1, F_1) \hat{\approx} \Sigma(s_2, F_2)$ if $s_1 \hat{\approx} s_2$ and whenever x_1 and x_2 are total in $I(\rho(s_1))$ and $I(\rho(s_2))$ resp., and $(\rho(s_1), x_1) \approx (\rho(s_2), x_2)$, then $F_1(x_1) \hat{\approx} F_2(x_2)$.
We define $\hat{\approx}$ for Π-types in the analogue way.
$o(\Psi_1, s_1) \hat{\approx} o(\Psi_2, s_2)$ if $s_1 \hat{\approx} s_2$ and for all $t_1 \subseteq \rho(s_1)$ and $t_2 \subseteq \rho(s_2)$, if $t_1 \sim t_2$, then $\Psi_1(t_1) \sim \Psi_2(t_2)$.

We then let $(U, \Phi_1) \sim (U, \Phi_2)$ if for all s_1, s_2, t_1 and t_2, if $s_1 \hat{\approx} s_1$, $t_1 \subseteq \rho(s_1)$, $t_2 \subseteq \rho(s_2)$ and $t_1 \sim t_2$, then $\Phi_1(t_1) \sim \Phi_2(t_2)$.

Remark 4 The idea is to define the obvious notion of extentional equality by recursion on the inductive definition of $\bar{T}$. We will prove that these relations indeed are equivalence relations. A consequence will be that every object (function, parameterisation) will be extentional.

Theorem 2 a) *Given* $s_1, s_2 \in \bar{T}$ *we have*

$$s_1 \sim s_2 \Leftrightarrow s_1 \cap s_2 \in \bar{T}$$

b) *Given* $s_1, s_2 \in \bar{T}$*,* $x_1 \in \bar{I}(s_1)$ *and* $x_2 \in \bar{I}(s_2)$ *we have*

$$(s_1, x_1) \approx (s_2, x_2) \Leftrightarrow s_1 \sim s_2 \wedge x_1 \cap x_2 \in \bar{I}(s_1 \cap s_2)$$

c) $\sim$ *and* $\approx$ *are equivalence relations.*

d) *If* $t_1 \sim t_2$ *then* $\text{rank}(t_1) = \text{rank}(t_2)$.

Proof
We prove a) and b) by induction. c) follows as in Normann [11] and d) is proved by a simple induction.
The cases *Base type*, Σ-*type* and Π-*type* is handled as in Normann [11]. In order to handle the last case we need the following
Claim
Let $s_1 = (U, \Phi_1)$ and $s_2 = (U, \Phi_2)$ be total, and let $\hat{\approx}$ be the relation between elements of $\bar{I}(s_1)$ and $\bar{I}(s_2)$ given in the definition.
Then for each $t_1 \in I_{\gamma,\alpha}(s_1)$ and $t_2 \in I_{\gamma,\alpha}(s_2)$ we have

$$t_1 \hat{\approx} t_2 \Leftrightarrow t_1 \cap t_2 \in I_{\gamma,\alpha}(s_1 \cap s_2)$$

Proof of claim
We use induction on γ. The basic types, Σ-types and Π-types are handled as in [11].
So let $t_1 = o(\Psi_i, t_1') \in I_{\gamma+1,\alpha}(U, \Phi_1)$ and $t_2 = o(\Psi_2, t_2') \in I_{\gamma+1,\alpha}(U, \Phi_2)$.
First assume that $t_1 \hat{\approx} t_2$. Then $t_1' \hat{\approx} t_2'$ and by the induction hypothesis, $t_1' \cap t_2' \in I_{\gamma,\alpha}(\Phi_1 \cap \Phi_2)$. It follows that $t_1' \cap t_2' \in I_{\gamma,\alpha}(\Psi_1 \cap \Psi_2)$.
Let $t \subseteq \rho(t_1' \cap t_2')$ with $t \in T_\beta$ $(\alpha = \beta + 1)$. Then $\Psi_1(t) \sim \Psi_2(t)$. It follows that $(\Psi_1 \cap \Psi_2)(t) \in T_\beta$, and as a consequence we obtain

$$o(\Psi_1 \cap \Psi_2, t_1' \cap t_2') \in I_{\gamma+1,\alpha}(U, \Phi_1 \cap \Phi_2).$$

Conversly, assume that

$$o(\Psi_1 \cap \Psi_2, t_1' \cap t_2') \in I_{\gamma+1,\alpha}(U, \Phi_1 \cap \Phi_2).$$

Then $t'_1 \cap t'_2 \in I_{\gamma,\alpha}(U, \Phi_1 \cap \Phi_2)$, so by the induction hypothesis $t'_1 \hat{\approx} t'_2$.
Let $s'_1 \subseteq \rho(t'_1)$ and $s'_2 \subseteq \rho(t'_2)$ be such that $s'_1 \sim s'_2$.
Then $s'_1 \cap s'_2 \in T_\beta$ and $\rho(t'_1 \cap t'_2) \in T_\beta$. These two objects are consistent, so by Theorem 1 we have that

$$s' = s'_1 \cap s'_2 \cap \rho(t'_1 \cap t'_2) \in T_\beta.$$

By assumption, $(\Psi_1 \cap \Psi_2)(s') \in T_\beta$, since $o(\Psi_1 \cap \Psi_2, t'_1 \cap t'_2)$ is total.
Clearly

$$(\Psi_1 \cap \Psi_2)(s') \subseteq \Psi_1(s'_1) \cap \Psi_2(s'_2) \in T_\beta$$

so $\Psi_1(s'_1) \sim \Psi_2(t'_2)$.
This shows that $o(\Psi_1, t'_1) \hat{\approx} o(\Psi_2, t'_2)$, and the claim is proved.

We now prove a). b) will be a direct consequence of a) and the claim.

1. Let $(U, \Phi_1) \sim (U, \Phi_2)$.
Let $t_1 \in \bar{I}(U, \Phi_1)$ and $t_2 \in \bar{I}(U, \Phi_2)$ with $t_1 \hat{\approx} t_2$.
By the claim $t_1 \cap t_2 \in \bar{I}(U, \Phi_1 \cap \Phi_2)$.
We then argue as in the case $o(\Psi, t)$ in the proof of the claim and see that if $t \subseteq \rho(t_1 \cap t_2)$ with $t \in T_\beta$ we have that $(\Phi_1 \cap \Phi_2)(t) \in T_\beta$. This establishes $\Rightarrow$.

2. Let $(U, \Phi_1 \cap \Phi_2)$ be total.
Let $t_1 \hat{\approx} t_2$, and let $t'_1 \subseteq \rho(t_1)$ and $t'_2 \subseteq \rho(t_2)$ with $t'_1 \sim t'_2$.
Then, as for the corresponding case in the proof of the claim,

$$(\Phi_1 \cap \Phi_2)(t'_1 \cap t'_2) \in T_\beta$$

This establishes $\Leftarrow$.

End of proof.

4 Simulation

In section 6 we will be simulating computations in the functional S, *The Superjump*, which will be defined in section 5. This will involve the simulation of natural numbers obtained as the result of transfinite computations.

There are of course various formats that we could choose for the simulation of such computations. In Normann [9, 12] we have used the so called representations. The representation technique will require that the total objects in each type is dense, a property that we certainly do not have.

Here we will use the same method of simulation that was introduced in the unpublished [8] and reused in [10].

Definition 6 **a)** Let $t \in \bar{T}$ and let $\nu : \bar{I}(t) \to \mathbb{N}_\perp$ be continuous.
We say that (t, ν) *simulates* the number n if $\bar{I}(t) \neq \emptyset$, ν is total and constant n on $\bar{I}(t)$.

b) If $t = (U, \Phi) \in \bar{T}$ and $s \in \bar{I}(t)$, we say that (s, ν) *t-simulates* n if $(\rho(s), \nu)$ simulates n.

We will in reality be working with operations on simulations, but technically we can only deal with operations on T. In this section we will see that any operator on the set of simulations can be translated to an operator on T. First we translate a simulation to an element of T.

Definition 7 **a)** If $t_1 = t_{tt}$ and $t_2 = t_{ff}$ are two elements in T, we let $t_1 \oplus t_2 = \Sigma(B, t_x)$

b) We let $N_0 = B$, and recursively we let $N_{k+1} = N_k \oplus B$ be elements of T.

c) Let $t \in T$ and let $\nu : I(t) \to \mathbb{N}_\perp$ be continuous.
We let $[t, \nu]$ be $\Pi(t, F)$, where $F(x) = N_{\nu(x)}$.

Lemma 4 *There is a continuous map C (for collapse) such that for any $t \in T$*

i) *$C(t)$ is of the form $[t_1, \nu]$.*

ii) *If $t = [t_1, \nu]$, then $C(t) = t$.*

Proof
If t is not a product, we just select some code for a simulation, e.g. the product of the constant B over N.
If t is a product $\Pi(t_1, F)$, we construct the parameterisation G that for each $x \in I(t_1)$ is $F(x)$ if $F(x) = N_k$ for some k, and is B if $F(x)$ is inconsistent with all N_k, undefined otherwise.

Lemma 5 *Let C be as above.*
If $t \in \bar{T}$, then $C(t) \in \bar{T}$.

The proof is left for the reader.

Definition 8 A *pre-simulation* will be a pair (t, ν), where $t \in \bar{T}$ and $\nu : I(t) \to \mathbb{N}_\perp$ is total.

It is clear that there is a bicontinuous 1-1 correspondance between pre-simulations and products of total parameterisations with values of the form N_k. From now on we will work with pre-simulations and simulations, but consider all operators constructed as operators on T, composing with C for objects t that do not directly correspond to pre-simulations.

Our next step will be to transfer all pre-simulations to simulations, not altering the simulated value in case the input already is a simulation.

Lemma 6 *There is a continuous total map ϕ from the set of pre-simulations to the set of simulations such that if (t, ν) simulates n, then $\phi(t, \nu)$ simulates n.*

Proof
Let (t, ν) be given. For x and y in $I(t)$, let
$B(x, y) = B$ if $\nu(x) = \nu(y) \in \mathbb{N}$
$B(x, y) = O$ if $\nu(x) \in \mathbb{N}$, $\nu(y) \in \mathbb{N}$, but $\nu(x) \neq \nu(y)$.
$B(x, y) = \perp_T$ otherwise.
Let $C = \Pi(x \in I(t))\Pi(y \in I(t))B(x, y)$, and let t_1 be a code for

$$(I(t) \times C) \oplus ((I(t) \times C) \to \mathbb{N}_\perp))$$

Let $\nu_1(left(z_1, z_2)) = \nu(z_1)$ and let $\nu_1(right(z)) = 0$.
It is easy to se that the left hand side contains total elements if and only if the right hand side does not. The left hand side contains total elements if and only if both C and $I(t)$ does so, and this is the case exactly when $\bar{I}(t) \neq \emptyset$ and ν is total and constant on $\bar{I}(t)$.
This ends the proof of the lemma.

As a consequence of this lemma we will operate on simulations and produce new simulations, blowing the operators up to operators from the set of total pre-simulations to the set of simulations by composing with ϕ. In the sequel we will do so without further explanation on what is really going on.

We may extend the notion of simulating a number to simulation of a function:

Definition 9 Let $f : \mathbb{N} \to \mathbb{N}$ be a function. A *simulation* of f will be a family $\{(t_n, \nu_n)\}_{n\in\mathbb{N}}$ where (t_n, ν_n) is a simulation of $f(n)$ for each n.

Remark 5 We will show that the functions f that can be simulated by objects in $\bar{T}$ will be exactly the functions that appears in Gödel's L before the first recursively Mahlo ordinal ρ_0. The easy inclusion is given by Lemma 1.

5 The Superjump

Our method for proving the main theorem will be by simulating computations in the type 3 functional S known as the superjump. Kleene [5] gave a general definition of computations $\{e\}(\vec{\phi}) \approx n$ where e is a natural number, and $\vec{\phi}$ is a sequence of total functionals of pure finite type. The definition is given as an inductive definition with 9 clauses, S1 - S9, and for each clause we give an index coding the clause, the signature of the arguments and the indices of the immediate subcomputations. Below we will give the definition without specifying the construction of each index. For a more complete definition, we refer to the original paper [5] or to any textbook on the subject. We omit clause S5, covering primitive recursion, since this clause can be reduced to the 8 other clauses. Here x will denote a natural number

Definition 10 The relation $\{e\}(\vec{\phi}) = n$ is inductively defined as follows:

S1 $\{e\}(x, \vec{\phi}) = x + 1$

S2 $\{e\}(\vec{\phi}) = q$

S3 $\{e\}(x, \vec{\phi}) = x$

S4 $\{e\}(\vec{\phi}) = \{e_1\}(\{e_2\}(\vec{\phi}), \vec{\phi})$

S6 $\{e\}(\vec{\phi}) = \{e_1\}(\tau(\vec{\phi}))$ where τ is a permutation.

S7 $\{e\}(x, f, \vec{\phi}) = f(x)$

S8 $\{e\}(\psi, \vec{\phi}) = \psi(\lambda\xi\{e_1\}(\xi, \psi, \vec{\phi}))$

S9 $\{e\}(x, \vec{\phi}, \vec{\psi}) = \{x\}(\vec{\phi})$

Remark 6 Non-terminating computations will be introduced via S9. In S4 we will assume that $\{e_2\}(\vec{\phi})$ terminates and gives a value y, and then that $\{e_1\}(y, \vec{\phi})$ terminates. In S8, ψ must be of type $k+2$ and ξ will range over the total objects of type k. We will assume that $\{e_1\}(\xi, \psi, \vec{\phi})$ will terminate for all ξ.

Gandy [2] introduced the type three functional S defined as follows

Definition 11 Let F be a total functional of type 2, and let e be a natural number.
$S(F, e) = 0$ if $\{e\}(F) = 0$, otherwise $S(e, F) = 1$

We call S *The Superjump* since it is a jump operator for arbitrary functionals of type 2.

Harrington [4] showed that a function f is computable in S if and only if $f \in L_{\rho_0}$ where ρ_0 is the first recursively Mahlo ordinal. We will show that any function f that is computable in S can be simulated by an element of $\bar{T}$, and thereby show that every $f \in L_{\rho_0} \cap \mathbb{N}^{\mathbb{N}}$ can be simulated. The other way around follows from Lemma 1.

6 Simulating computations

The aim

In this section we will use the notation $\{e\}^S(f_1, \ldots, f_n)$, or simply the notation $\{e\}^S(\vec{f})$ for a computation relative to S. Here each f_i will either be a function or a number, which sort will be clear from the index.
Our notation for computations relative to a type two functional F will be similar, $\{d\}^F(\vec{n})$.

Uniformly in each index e for a computation $\{e\}^S(\vec{f})$ we will construct a continuous operator $\Phi_e : T^n \to T$ such that if $\vec{t} \in \bar{T}^n$ are simulations of $\vec{f}$, and $\{e\}^S(\vec{f}) = m$, then $\Phi_e(\vec{t}) \in \bar{T}$ and $\Phi_e(\vec{t})$ is a simulation of m.

We will define the operators using the fix-point theorem for domains, but we will simultanously give the induction steps needed in order to prove that our construction works.

We will use the following notational conventions: When t is an element of T representing a simulation of a number, we let the simulation be the pair

(t', ν) where this notation will commute with the use of indices. Similarily, the operators Φ_e will be split into two operators ϕ_e and μ_e giving the object and the function of the simulation $\Phi_e(\vec{t})$.
We will only use this notation when it simplifies our construction.

Basic computations

There are four clauses giving the basic computations.
$\{e\}^S(x, \vec{f}) = x + 1$. If (s, ν) is a simulation for x, we use $(s, \nu + 1)$ (with the obvious meaning of $\nu + 1$) as a simulation of $x + 1$.
The cases S2 and S3 are even more trivial.
In the case S7, we let (s, ν) be a simulation of x, and $\{(s_i, \nu_i)\}_{i \in \mathbb{N}}$ be a simulation of f. Then $\phi = \Sigma(s, s_{\nu(x)})$ and $\mu(x, y) = \nu_{\nu(x)}(y)$ will be a simulation of $f(x)$.
It is trivial to show that our constructions of simulation for basic computations work.

Composition, permutation, enumeration

The three cases S4, S6 and S9 are fairly simple.

S4 $\{e\}(\vec{f}) = \{e_1\}(\{e_2\}(\vec{f}), \vec{f})$.
We assume that we have constructed Φ_{e_1} and Φ_{e_2}
We might then simply use the composition $\Phi_{e_1}(\Phi_{e_2}(\vec{t}), \vec{t})$, but it will be an advantage to code in the subsimulations more explicitly.
Let $c(n)$ be some canonical simulation of the number n. We then let

$$\phi_e(\vec{t}) = \Sigma(x \in I(\phi_{e_2}(\vec{t})))\phi_{e_1}(c(\mu_{e_2}(\vec{t})(x)), \vec{t})$$

and we let

$$\mu_e(\vec{t})(x, y) = \mu_{e_1}(c(\mu_{e_2}(\vec{t})(x)), \vec{t})(y)$$

S6 $\{e\}(\vec{f}) = \{e_1\}(\tau(\vec{f}))$. Let

$$\Phi_e(\vec{t}) = \Phi_{e_1}(\tau(\vec{t}))$$

.

S9 $\{e\}(e', \vec{f}, \vec{g}) = \{e'\}(\vec{f})$. Let

$$\phi_e(t, \vec{t}, \vec{r}) = \Sigma(x \in I(t'))\phi_{\nu(x)}(\vec{t})$$

$$\mu_e(x, y) = \mu_{\nu(x)}(\vec{t})(y).$$

Application

$\{e\}^S(d, \vec{f}) = S(d, \lambda g\{e_1\}^S(g, \vec{f}))$
We let $F(g) = \{e_1\}^S(g, \vec{f})$ in this section.
Recall that $S(d, F) = 0$ if $\{d\}(F) = 0$, while $S(d, F) = 1$ otherwise.
We will do the final proof by induction on the length of the computations in S, so we may assume that F is total, and consequently deduce from the induction hypothesis that if $t \in \bar{T}$ and $\vec{t}$ are simulations of $\vec{f}$, then

$$\Phi(t) = \Phi_{e_1}(t, \vec{t})$$

will also be an element of $\bar{T}$. Here Φ will depend continuously on the choice of $\vec{t}$. We then have $(U, \Phi) \in \bar{T}$.

Our first step will be to construct simulations s of $\{d\}^F(\vec{n})$ inside $\bar{I}(U, \Phi)$ uniformly in simulations for $\vec{n}$ in the same universe. By this we will mean that $\rho(s)$ is a simulation in $\bar{T}$.
For each d being an index accepting a type two functional and k numbers as inputs, we construct a continuous function $\Psi_d : (I(U, \Phi))^k \to I(U, \Phi)$ transforming a simulation $\vec{s}$ of $\vec{n}$ to a simulation of $\{d\}^F(\vec{n})$ whenever the latter terminates.
S7 does not apply here, and all cases exept S8 is handled exactly as in the major construction.
S8: $\{d\}^F(\vec{n}) = F(\lambda m\{d_1\}^F(m, \vec{n}))$
By the induction hypothesis it is trivial to construct a simulation s for
$g = \lambda m\{d_1\}(m, \vec{n})$, i.e. $\rho(s)$ is a simulation of g in $\bar{T}$.
By the grand induction hypothesis and the construction of Φ, $\Phi(\rho(s))$ will be a simulation of $F(g)$. We then use $o(\Phi, s)$ as a simulation of
$F(g) = \{d\}(F, \vec{n})$ in $\bar{I}(U, \Phi)$.
This ends our construction.

The second step will be to use this to construct a type that contains total elements if and only if $\{d\}(F) = 0$. The idea is to take any element x in

$\bar{I}(U,\Phi)$ and ask if x is a simulation of $\{d\}(F)$ constructed as in the first step. The reason why this will work is that $\bar{I}(U,\Phi)$ is inductively defined, and by recursion on this induction we can compare the object with the ones used to simulate the results of computations.

Lemma 7 *There is a continuous function σ defined on sequences of the form $(d,\vec{n},s)$ where $s \in \bar{I}(U,\Phi)$ such that $\sigma(d,\vec{n},s)$ is a simulation of 1 if s can be extended to a simulation of $\{d\}^F(\vec{n})$ obtained from a simulation in $\bar{I}(U,\Phi)$ for $\vec{n}$, and $\sigma(d,\vec{n},s)$ is a simulation of 0 otherwise.*

Proof
We define σ by the 7 cases corresponding to S1-4, S6, S8 and S9, but the proof that σ fullfills the lemma will be by induction on the rank of s in $\bar{I}(U,\Phi)$.
We will not give all the details. In section 4 we established how the use of quantifiers or boolean combinations can be transfered to continuous operations on simulations. Thus when we in the construction below 'check' something, we mean that we construct a simulation of the truth value of the statement.

We now give a scetch of the argument, noticing that the constructions will be by recursion on the inductive definition of $\bar{I}(U,\Phi)$.
S1: $\{d\}^F(x,\vec{n}) = x+1$
We simply have to check if s is a simulation of $x+1$.
S2 and S3 are equally simple.
S4: $\{d\}^F(\vec{n}) = \{d_1\}^F(\{d_2\}^F(\vec{n}),\vec{n})$
In this case, s' has to be a sum $\Sigma(s_1',F)$ where s_1 is a simulation of $\{d_2\}^F(\vec{n})$ and for each total x, $F(x)$ is a simulation of the appropriate $\{d_1\}^F(m,\vec{n})$. This can be checked.
In the case S8, $\{d\}^F(\vec{n}) = F(\lambda m\{d_1\}^F(m,\vec{n}))$, we must have that $s = o(\Phi',s_1)$ for some $\Phi' \subseteq \Phi$, where s_1 can be extended to a simulation of $\lambda m\{d_1\}^F(m,\vec{n})$. By the induction hypothesis, this can be checked.
The other cases are also easy, we can first check if s is locally of the correct form, and then use the induction hypothesis to check if the subtypes are the simulations we want them to be.
This ends the proof.

The third stage will be to use this to construct a simulation of

$$\{e\}^S(d,\vec{f}) = S(d,\lambda g\{e_1\}^S(g,\vec{f}))$$

in the case S8. But we simply have to ask if there is any element s of $\bar{I}(U, \Phi)$ such that $\sigma(d, \cdot, s)$ is a simulation of 1 and s is a simulation of 0. If this is the case, the simulation we construct should be a simulation of 0, otherwise it should be a simulation of 1. We have already developed standard techniques to do this inside $\bar{T}$.

The Main Theorem

We are now ready to state our main result:

Theorem 3 *We use the notation from the paper.*
The first recursively Mahlo ordinal ρ_0 is the least ordinal α such that $T_{\alpha+1} = T_\alpha$.

Proof
By lemma 1, ρ_0 is an upper bound for this least α.
On the other hand it is easy to see that $T_\beta \in L_{\omega+\beta+1}$ for all β, and if f has a simulation in T_β, then $f \in L_{\omega+\beta+\omega}$.
If f is computable in S, then by our main construction, f can be simulated in $\bar{T}$. By Harrington [4] ρ_0 is the least α such that $f \in L_\alpha$ whenever f is computable in S. The theorem follows.

Remark 7 In this proof we have used Harrington's result because it was available. The advantage was that we would never have to worry if the operators Φ considered in the construction actually defined universes. If we were to use some other system for the first recursively Mahlo, say viewing S as a monotone partial operator defined on any d and partial F containing at least enough information to determine $S(d, F)$, we would have to be more careful.

The use of Harrington's result indicates that we might form a Mahlo-hierarchy in an impredicative way; it is sufficient for our purpose to construct universes from operators that will remain total at the end of the construction.

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