EQUISINGULAR DEFORMATIONS OF SANDWICHED SINGULARITIES

TROND STØLEN GUSTAVSEN

1. Introduction

In the study of normal surface singularities one attaches to a normal surface singularity X, the dualgraph of resolution Γ. Given such Γ one may ask if this determines X. H. Laufer gives in [L2] an answer to this question by listing all dualgraphs with this property. However, generally there are families of normal surface singularities with the same dualgraph. If we want to use deformation theory to study such families, we are lead to a notion of equisingular deformations. A foundation for such theory was given by Wahl in [W1]. In this work, the functor of equisingular deformations, ES_X, of a normal surface singularity X, is defined. Moreover, it is proved that it is a subfunctor of the functor of all deformations.

The aim of this paper is to describe these equisingular deformations for sandwiched singularities.

A sandwiched singularity is a normal surface singularity which birationally dominates a smooth surface. Sandwiched singularities are rational and include all cyclic quotient singularities. More generally any rational singularity with reduced fundamental cycle is sandwiched. The sandwiched singularities were studied by several authors, among others, de Jong and van Straten [JS] and Spivakovsky [SP2]. In [JS] the deformation theory of sandwiched singularities is studied. Their results give a good understanding of the versal basespace, but only up to "up to equisingular deformations."

Of importance is a particular construction of sandwiched singularities. It was observed by Spivakovsky, [SP2], that a sandwiched singularity may be obtained from an embedded resolution of a plane curve singularity by contracting rational curves. This was used by de Jong and van Straten who considered one parameter δ-constant families of plane curves and showed that such corresponds to one parameter families of sandwiched singularities. We restrict our attention to equisingular families of plane curves. Doing this, we get an even better correspondence: We show that that to an equisingular flat deformation of a plane curve singularity there corresponds an equisingular (in the sense of Wahl, see [W1]) flat deformation of sandwiched singularities. Moreover all equisingular deformations of sandwiched singularities may be obtained this way. To be precise we show that there is a smooth map

$$ES_C \to ES_X$$

between the two functors of equisingular deformations. This map is the composite of a map

$$ES_C \to Def_Y,$$

where Y is a certain surface on which the sandwiched singularity X lies, with the restriction map. We are also able to describe the kernels of these two maps.

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Thus we are able to find many surface singularities which are not determined by their dualgraph. In fact we show (when the curve is irreducible) that if the embedded resolution of the plane curve is sufficiently non minimal, all of the equisingular moduli of the curve is transfered to the sandwiched singularity. This may be understood in terms of series of surfaces singularities: All surface singularities in a series come from the same curve singularity, the difference being only how non minimal the embedded resolution is. Our results then imply that the generic member of such a series has "the same amount of moduli" as the curve defining the series. They also show that the moduli problem of irreducible plane curve singularities is contained in the corresponding problem for rational surface singularities.

Central in our approach is what we would like to call *equinumerical* deformation of complete ideals. The sandwiched singularity lies on the blowup of the plane in a complete ideal, and the equinumerical deformation of complete ideals corresponds exactly to deformations of the surface which induce equisingular deformation of the singularities on the surface. It seems also that the connection between equisingular plane curves and deformation of complete ideals has not been noted earlier. This description of equisingularity of a deformation is interesting and particular useful for our purpose.

The sections are summarized as follows. Some preliminaries are given in section 2. We start section 3 on equisingular deformations by reviewing the construction of sandwiched singularities, and in subsection 3.3 we describe the different deformation functors which are involved. The observations in 3.3 are central in this paper. In 3.4 we study the tangent spaces of the deformation functors introduced in 3.3, and the relationship between them.

In section 4 we consider the generic case in the meaning described above. The main result is that for the case when the defining plane curve singularity is irreducible, there is a one to one correspondence between isomorphism classes of plane curve singularities with a given semigroup and normal surface singularities with a given dualgraph. In order to prove this we have to consider the possible dualgraphs for these singularities and consider the groups of automorphisms of these graphs. This is done in 4.1. In 4.3 we include the considerations on the tangent spaces in the generic case.

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2. Preliminaries

The field $\mathbb{C}$ of complex numbers is fixed as the ground field in the whole of this article.

2.1. **Deformations.** We are going to use the theory of deformation functors, [SCH], so let $\mathcal{C}$ be the category of local Artin $\mathbb{C}$-algebras, and let $\mathbb{C}[\epsilon] = \mathbb{C}[t]/(t^2)$ be the ring of dual numbers. Recall that a deformation of a $\mathbb{C}$-scheme Y over $A \in \mathcal{C}$ is a cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & \overline{Y} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{C} & \longrightarrow & \operatorname{Spec} A \end{array}$$

where $\overline{Y} \to \operatorname{Spec} A$ is flat. We will also speak of deformation classes where the the following equivalence is understood. Two deformations are equivalent if the corresponding schemes are isomorphic over A by an isomorphism which induces the identity when pulled back to $\mathbb{C}$. We denote by $Def_Y : \mathcal{C} \to$ Sets the functor of all deformation classes.

2.2. **Rational surface singularities.** Recall that a normal surface singularity $X = \operatorname{Spec} B$ is said to be rational if for any resolution $\pi : \widetilde{X} \to X$, $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$. The exceptional set $E \subset \widetilde{X}$ is a union of irreducible components $E_i \simeq \mathbb{P}^1$. There is a *fundamental cycle* supported on E which may be constructed as the unique smallest positive divisor $Z = \sum m_i E_i$ satisfying $Z \cdot E_i \leq 0$ for all irreducible component E_i. The embedding dimension of X equals $-Z^2 + 1$ and the multiplicity $e - 1 = -Z^2$.

Given a normal surface singularity we will often consider the minimal good resolution. A good resolution has the property that all exceptional curves intersects transversally and when they intersect, there are only two intersecting in one point. For rational surface singularities the minimal resolution is always the minimal good resolution.

2.3. **Dualgraphs for normal surface singularities.** It is common to describe the configuration of the exceptional curves in the minimal good resolution of a normal surface singularity, by a graph. This is called the dualgraph of resolution. This is an invariant of the singularity and accordingly one sometimes speaks of the dualgraph or only the graph of the singularity. Before we recall the definition, we fix some notations on graphs.

A weighted graph Γ is an ordered triple (v, e, w) such that e is a subset of unordered pair of elements in v and w is a map $w : v \to \mathbb{Z}$. We assume that v and e are finite sets and we refer to v as the vertices and to e as the edges of Γ. The map w gives the weights.

If Γ is a graph we write $v(\Gamma)$ for the vertices, $e(\Gamma)$ for the edges and $w(\Gamma) : v(\Gamma) \to \mathbb{Z}$ for the weights. We further write $xy \in e(\Gamma)$ for an edge with $x, y \in v(\Gamma)$. Two graphs Γ and Γ' are isomorphic if there is a bijection $\phi : v(\Gamma) \to v(\Gamma')$ such that $\phi(x)\phi(y) \in e(\Gamma')$ if and only if $xy \in e(\Gamma)$.

A subgraph $\Gamma' \subset \Gamma$ is an inclusion $v(\Gamma) \subset v(\Gamma)$ such that $xy \in e(\Gamma')$ if and only if $xy \in e(\Gamma)$ and such that $w(\Gamma') = w(\Gamma)_{|v(\Gamma')}$. A subgraph $\gamma \subset \Gamma$ is called a path in Γ if $v(\gamma) = \{x_0, \ldots, x_l\}$ and $e(\gamma) = \{x_0x_1, x_1x_2, \ldots, x_{l-1}x_l\}$.

Now let $X = \operatorname{Spec} B$ be a surface singularity and let $\pi : \widetilde{X} \to X$ be a resolution. Let $E = \cup_{i=1}^m E_i$ be the decomposition of the exceptional set into is irreducible components. We define a graph $\Gamma = \Gamma(\pi)$ as follows: Set

$$v(\Gamma) = \{E_1, \ldots, E_m\}$$

and define $e(\Gamma)$ by the condition that

$$E_iE_j \in e(\Gamma)$$

if and only if $i \neq j$ and $E_i \cap E_j \neq \emptyset$, Finally define the weights by

$$w(\Gamma)(E_i) = E_i^2.$$

Definition 2.1. The dualgraph of X is $\Gamma = \Gamma(X) = \Gamma(\pi)$, where π is the minimal good resolution of X.

2.4. **Plane curves.** By a plane curve singularity we will mean a scheme

$$C = \operatorname{Spec} \mathbb{C}[[x, y]]/(f).$$

To an irreducible plane curve singularity there corresponds a semigroup $S = S(C)$: Let

$$A = \mathbb{C}[[x, y]]/(f) \subset \overline{A} = \mathbb{C}[[t]]$$

where $\overline{A}$ is the normalization of A. On the quotientfield of $\overline{A}$, K, there is a canonical valuation

$$v : K \setminus \{0\} \to \mathbb{Z}.$$

The semigroup S is now the subsemigroup $v(\overline{A} \setminus \{0\})$ of $\mathbb{Z}$.

By a good embedded resolution of C we mean an embedded resolution such that the reduced total transform of C is a normal crossing divisor. In particular there are at maximum two irreducible components intersecting in a point.

3. Equisingular deformations of sandwiched singularities

A subclass of the rational surface singularities are the sandwiched singularities. A normal surface singularity $X = \operatorname{Spec}(B)$ is said to be a sandwiched singularity if there exist a birational map $X \to S$ to a smooth surface S. Cyclic quotient singularities and more generally rational surface singularities with reduced fundamental cycle are sandwiched singularities, see [SP2], [JS]. As remarked by Spivakovsky, see [SP2], there is however a general method of constructing sandwiched singularities from plane curve singularities as follows.

3.1. Construction of sandwiched singularities. Let

$$C = \operatorname{Spec} \mathbb{C}[[x,y]]/(f)$$

be a plane curve singularity, and let

$$\begin{array}{ccccccccc} S_n & \longrightarrow & S_{n-1} & \longrightarrow & \cdots & \longrightarrow & S_0 = \operatorname{Spec} \mathbb{C}[[x,y]] & & \\ \uparrow & & \uparrow & & & & \uparrow & & \\ C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_0 & = C & \end{array} \tag{3.3.1}$$

be an embedded (possibly non-minimal) good resolution of C, obtained by blowing up closed points, and where C_i is the strict transform of C in S_i. Given 3.3.1, there is a canonical sequence of blowups

$$Z = Z_n \xrightarrow{\pi_n} Z_{n-1} \xrightarrow{\pi_{n-1}} \cdots \longrightarrow Z_0 = \operatorname{Spec} \mathbb{C}[x,y] \tag{3.3.2}$$

which induces the first row of (3.3.1) by the base change

$$\operatorname{Spec} \mathbb{C}[[x,y]] \to \operatorname{Spec} \mathbb{C}[x,y].$$

Let π be the composition of the maps in (3.3.2), and let

$$E = \pi^{-1}(0) = \cup_{i=1}^{m} E_i$$

be the decomposition of the exceptional set into its irreducible components. Assume $E_r, \ldots, E_m$ are those with $E_i^2 = -1$. Now the intersection matrix of $E_1, \ldots, E_{r-1}$ is negative definite, so we may blow these curves down to obtain a surface Y. This surface is in fact algebraic, see [SP2]. Moreover, if (3.3.1) does not give the minimal good embedded resolution for any of the branches of C, Y will have a unique singular point, and one may show that up to analytic equivalence, this singularity depends only on the analytic equivalence class of C. In particular we may assume that that C is defined by a polynomial f. Thus we have $C' = V(f) \subset Z_0$ and we define $\widetilde{C} \subset Z$ to be the strict transform of C'. Put $\widetilde{X} = Z \setminus \widetilde{C}$. Then $\widetilde{X}$ is the minimal good resolution of an affine $X \subset Y$ which contains the singular point.

Let k be the number of analytic branches of C. We take $a \in \mathbb{N}^k$ and let a_i give the number of extra blowups needed in each branch to get $\pi : Z \to \operatorname{Spec} \mathbb{C}[x,y]$ from the minimal good resolution. The fact that $a_i \geq 1$ for $i = 1, \ldots, k$ corresponds to the assumption that that the embedded resolution is not minimal for any of the branches of C. We are in the following situation:

$$\begin{tikzcd}
\widetilde{C} \arrow[r, hook] \arrow[dd] & Z \arrow[dd] \arrow[dr] & & \widetilde{X} \arrow[ll, hook'] \arrow[d] \arrow[r, equal] & Z \setminus \widetilde{C} \\
& & Y \arrow[dl] & X \arrow[l, hook'] & \\
C' \arrow[r, hook] & \operatorname{Spec} \mathbb{C}[x,y] & & &
\end{tikzcd}$$

Finally, it is also possible to realize Y as the blowup of $\operatorname{Spec} \mathbb{C}[x,y]$ in a complete (see [ZA] and [ZS]) ideal $\mathfrak{q}$: Let C^* denote the total transform of C' in Z. Then $C^* = D + \widetilde{C}$, where

$$D = \sum m_i E_i.$$

Now let $\mathfrak{q} = H^0(Z, \mathcal{O}_Z(-D)) = H^0(\pi_* \mathcal{O}_Z(-D)) \subset \mathbb{C}[x,y]$. Then Y is the blowup of $\operatorname{Spec} \mathbb{C}[x,y]$ in $\mathfrak{q}$. For a proof of this, see [G].

Definition 3.1. We define $Z_{(C,a)} := Z$, $X_{(C,a)} := Y$ and $X_{(C,a)} := X$, where X, Y and Z are constructed as above. Also we set $\mathfrak{q}_{(C,a)} := \mathfrak{q}$.

Remark 3.2. All sandwiched singularities may be obtained as $X_{(C,a)}$ for a plane curve singularity C and a choice of the integers a, see [SP1],[JS]. Note that the a_i used here are the same as $l(i) - M(i)$ in the notation of [JS].

Remark 3.3. It is perhaps abuse of notation to speak of the sandwiched singularity $X_{(C,a)}$ as we rather should call Spec of the complete local ring in the singular point of $X_{(C,a)}$, for the singularity, as we did for curves. However in order to be compatible with the notation of [W2] on curves and with [W1] on surface singularities we will use this inconsistent notation. Also, an other reason to consider formal and not algebraic plane curve singularities, is that we need to speak about irreducible plane curve singularities, and this will always mean analytically irreducible.

We give an example of a sandwiched singularity.

Example 3.4. Let

$$f = x^{13} + y^5 \in R = \mathbb{C}[x,y],$$

$C = V(f) \subset \operatorname{Spec} R$, $a = 2$. Then

$$\pi^{-1}(0) = \cup_{i=1}^{8} E_i.$$

The dualgraph $\Gamma(\pi)$ is given in figure 1. The configuration of exceptional curves

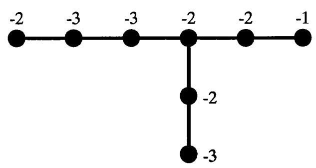


FIGURE 1. The graph $\Gamma(\pi)$.

has intersection matrix

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

and the negative of the inverse intersection matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 5 & 5 & 5 \\ 1 & 2 & 2 & 4 & 6 & 10 & 10 & 10 \\ 1 & 2 & 3 & 5 & 8 & 13 & 13 & 13 \\ 2 & 4 & 5 & 10 & 15 & 25 & 25 & 25 \\ 3 & 6 & 8 & 15 & 24 & 39 & 39 & 39 \\ 5 & 10 & 13 & 25 & 39 & 65 & 65 & 65 \\ 5 & 10 & 13 & 25 & 39 & 65 & 66 & 66 \\ 5 & 10 & 13 & 25 & 39 & 65 & 66 & 67 \end{bmatrix}.$$

From this matrix one may read off the multiplicity sequence for curves in $\operatorname{Spec} R$ such that $\pi : Z \to \operatorname{Spec} R$ gives an embedded resolution, see [LIP], and in particular we get that

$$D = 5E_1 + 10E_2 + 13E_3 + 25E_4 + 39E_5 + 65E_6 + 66E_7 + 67E_8.$$

The simple complete ideal $\mathfrak{q} = H^0(\operatorname{Spec} R, \pi_* \mathcal{O}_Z(-D))$ is

$$\left(y^6, xy^5, x^3y^4, x^6y^3, x^9y^2, x^{11}y, x^{14}, x^{13} + y^5\right),$$

see section 4.2. Embedding Y into $\mathbb{P}^n_{\operatorname{Spec} R} = \operatorname{Proj}(R[T_1, \ldots, T_n])$ by $\mathfrak{q}\mathcal{O}_Z$ gives $n = 7$ and one finds that Y is given by 49 elements in $R[T_1, \ldots, T_n]$. Calculating the fundamental cycle one finds that the embedding dimension of X is 6, so we do not get a minimal embedding this way.

3.2. **Equisingular deformations.** Oscar Zariski started the study of *equisingularity.* For plane curve singularities, equisingularity is well understood. In general, however, it is not easy to give the right definition. In this article we will only study plane curve singularities and rational surface singularities, and here Wahls definitions of equisingular deformations seems to be correct, see [W1] and [W2]. For the convenience of the reader we now recall these definitions.

3.2.1. *Plane curves.* Let $C = \operatorname{Spec} \mathbb{C}[[x, y]]/(f)$ be a plane curve singularity, and let

$$\begin{array}{ccccccc} S = S_n & \longrightarrow & S_{n-1} & \longrightarrow & \cdots & \longrightarrow & S_0 = \operatorname{Spec} \mathbb{C}[[x,y]] \\ \uparrow & & \uparrow & & & & \uparrow \\ C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_0 = C \end{array} \tag{3.3.3}$$

be a good embedded resolution of C, obtained by blowing up points. Let $s_{ij} : \operatorname{Spec} \mathbb{C} \to S_i$ define the points which are blown up. Let also $t_j : \operatorname{Spec} \mathbb{C} \to S$ define the ordinary double points of the reduced total transform of C. Then we define a deformation of C over $A \in \mathcal{C}$ with simultaneous embedded resolution, to be a deformation $\overline{C}$ of C, a commutative diagram

$$\begin{array}{ccccccc} \overline{S} = \overline{S}_n & \longrightarrow & \overline{S}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \overline{S}_0 \\ \uparrow & & \uparrow & & & & \uparrow \\ S = S_n & \longrightarrow & S_{n-1} & \longrightarrow & \cdots & \longrightarrow & S_0 \end{array} \tag{3.3.4}$$

and A-sections $\overline{s}_{ij} : \operatorname{Spec} A \to \overline{S}_i$ inducing s_{ij}, A-sections $\overline{t}_j : \operatorname{Spec} A \to \overline{S}$ inducing t_j such that

1. (3.3.4) gives an embedded resolution of the deformation $\overline{C}$ of C.
2. all the obvious diagrams commute and all sections are compatible
3. all A-sections are normally flat (i.e. they are defined by an ideal $\overline{\mathfrak{m}}$ such that $\overline{\mathfrak{m}}^n$ is A-flat for all $n > 0$.)
4. $\overline{S}_{i+1}$ is the blow up of $\overline{S}_i$ in the sections $\overline{s}_{ij}$.

Two such deformations,

$$\{\overline{C}, \overline{S}_i, \overline{s}_{ij}, \overline{t}_j\} \quad \text{and} \quad \{\overline{\overline{C}}, \overline{\overline{S}}_i, \overline{\overline{s}}_{ij}, \overline{\overline{t}}_j\}$$

are said to be isomorphic if there is an isomorphism (of deformations) of $\overline{S}_0$ and $\overline{\overline{S}}_0$ such that it

1. sends $\overline{C}$ to $\overline{\overline{C}}$
2. induces an isomorphism of (3.3.4) and the corresponding diagram for $\overline{\overline{S}}_0$
3. the sections are compatible with the isomorphisms.

The functor of equisingular deformations of C is defined as follows. Let for $A \in \mathcal{C}$,

$$ES_C(A) = \begin{array}{l} \text{Set of isomorphism clases of deformations } \overline{C} \text{ of } C \\ \text{with simulatanous embedded resolution} \end{array}$$

This definition coincides (up to isomorphism of functors) with the definition given in [W2], see [W2, 2.7, 2,12, 3.2, 7.3]. In particular is ES_C independent of the choice of embedded resolution, 3.3.3.

The functor of equisingular deformation of a plane curve possesses the following property:

Theorem 3.5 (Wahl, [W2, 7.4]). *$ES_C \subset Def_C$ is smooth subfunctor and has a hull.*

It is also possible to give another description of the functor of equisingular deformation, see 3.11, using complete ideals. This will be useful.

3.2.2. *Normal surface singularities.* For normal surface singularities the concept of equisingular deformations is more difficult than for plane curves. Wahl, [W1], tries to define equisingular deformation of normal surface singularities trough special deformations of the minimal good resolution. In the case of rational surface singularities the definition reduces to the following simple one.

Let $X = \operatorname{Spec}(B)$ be a rational surface singularity and let $\widetilde{X}$ be its minimal good resolution. Let $E = \cup E_i$ be the exceptional set. The functor of equisingular deformations of X, is defined by for $A \in \mathcal{C}$ letting

$$ES_X(A) = \begin{array}{l} \text{Set of isomorphism clases of deformations } \overline{\widetilde{X}} \text{ of} \\ \widetilde{X} \text{ to which the } E_i \text{ lifts} \end{array}$$

This is also (for rational surface singularities) the functor of simultaneous resolutions along normally flat sections, see [W1, Th. 5.16]. It has the following properties.

Theorem 3.6 (Wahl, [W1, 4.6]). *Let X be a rational surface singularity and let $\widetilde{X}$ be the minimal good resolution with exceptional set E. Then $ES_X \subset Def_X$ is a smooth subfunctor and has a hull. Moreover $ES_X(\mathbb{C}[\epsilon]) = H^1(\widetilde{X}, \theta(\log E)) \subset H^1(\widetilde{X}, \theta)$ where $\theta(\log E)$ is the dual of the sheaf of one forms with logarithmic poles along the exceptional set.*

3.3. **The deformation functors.** Let

$$C = \operatorname{Spec} \mathbb{C}[[x,y]]/(f)$$

be a plane curve singularity. We assume that f is a polynomial and consider f also as an element of $R = \mathbb{C}[x,y]$. Choose $a > 0$ and let $\mathfrak{q} = \mathfrak{q}_{(C,a)}$. Also let $C' = V(f) \subset \operatorname{Spec} R$. We will define several deformation functors, so again let $\mathcal{C}$ be the category of local Artin rings. The first functor to be defined is closely related to ES_C described above. It corresponds to deformations of the target of embedding disregarding the curve. When comparing with ES_C note also that the schemes involved in the following are of finite type.

Let

$$S_n \longrightarrow S_{n-1} \longrightarrow \cdots \longrightarrow S_0 = \operatorname{Spec} \mathbb{C}[[x,y]] \tag{3.3.5}$$

be obtained from an embedded resolution of C. We fix this once and for all. Let

$$Z = Z_n \xrightarrow{\pi_n} Z_{n-1} \xrightarrow{\pi_{n-1}} \cdots \longrightarrow Z_0 = \operatorname{Spec} R \tag{3.3.6}$$

be the canonical sequence of blowups which induces (3.3.5) via the base change

$$\operatorname{Spec} \mathbb{C}[[x,y]] \to \operatorname{Spec} \mathbb{C}[x,y].$$

Assume that

$$s_0 : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} R$$

define the origin in $\operatorname{Spec} R$ and let

$$s_{ij} : \operatorname{Spec} \mathbb{C} \to Z_i$$

define the other points which are blown up in (3.3.6). Choose $A \in \mathcal{C}$. Let

$$\overline{s}_0 : \operatorname{Spec} A \to \operatorname{Spec} R \otimes_{\mathbb{C}} A$$

be a normally flat A-section lifting s_0. Then this section is defined by

$$\overline{\mathfrak{m}} \subset R \otimes_{\mathbb{C}} A$$

such that $\overline{\mathfrak{m}}^i$ is A-flat for all i. Blow up $\operatorname{Spec} R \otimes_{\mathbb{C}} A$ in $\overline{\mathfrak{m}}$;

$$\overline{Z}_1 = \operatorname{Proj} \bigoplus_{i \geq 0} \overline{\mathfrak{m}}^i.$$

Then $\overline{Z}_1$ is a flat deformation of Z. We way may now speak about normally flat A-sections

$$\overline{s}_{1j} : \operatorname{Spec} A \to \overline{Z}_1,$$

lifting the s_{1j} and which are compatible with $\overline{s}_0$, that is, the composition with the blow down map to $\overline{Z}_0$ is $\overline{s}_0$. Now blow up these sections to obtain $\overline{Z}_2$. Continuing to choose liftings of s_{ij} and blow up we get a deformation of 3.3.6:

$$\begin{array}{ccccccccc} \overline{Z}_n & \longrightarrow & \overline{Z}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \overline{Z}_1 & \longrightarrow & \overline{Z}_0 = \operatorname{Spec} R \otimes_k A \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ Z_n & \longrightarrow & Z_{n-1} & \longrightarrow & \cdots & \longrightarrow & Z_1 & \longrightarrow & Z_0 = \operatorname{Spec} R. \end{array} \tag{3.3.7}$$

We will say that two such deformations,

$$\left\{\overline{Z}_i, \overline{s}_{ij}, \overline{s}_0\right\} \quad \text{and} \quad \left\{\overline{\overline{Z}}_i, \overline{\overline{s}}_{ij}, \overline{\overline{s}}_0\right\}$$

are isomorphic if there are isomorphisms (of deformations) $\phi_i : \overline{Z}_i \to \overline{\overline{Z}}_i$ such that we have commutative diagrams

$$\begin{array}{ccc} \overline{\overline{Z}}_i & \longrightarrow & \overline{\overline{Z}}_{i-1} \\ {\scriptstyle\phi_i}\big\downarrow & & {\scriptstyle\phi_{i-1}}\big\downarrow \\ \overline{Z}_i & \longrightarrow & \overline{Z}_{i-1} \end{array}$$

and such that

$$\phi_i \circ \overline{s}_{ij} = \overline{\overline{s}}_{ij}.$$

We now define the deformation functor G by letting

$$G(A) = \begin{array}{l}\text{Set of isomorphism clases of deformations}\\ \text{(with sections) as above.}\end{array}$$

There is an alternative description of this functor. Namely we define ES_Y by for $A \in \mathcal{C}$, letting

$$ES_Y(A) = \begin{array}{l}\text{Set of isomorphism classes of deformations } \overline{Z} \text{ of } Z\\ \text{to which the } E_i \text{ lift}\end{array}$$

and claim the following:

Proposition 3.7.

$$G \xrightarrow{\sim} ES_Y$$

Proof. See also [W2, Lemma 4.7]. The sections guarantees that the E_i lift so there is clearly a map $G \to ES_Y$. Now let $A \in \mathcal{C}$ and $\overline{Z}$ be a lifting of Z to Spec A to which the E_i lifts to $\overline{E_i} \subset \overline{Z}$. Let $\mathfrak{m} = (x, y) \subset R$. There is a cycle $D = \sum r_i E_i$, on Z such that $\mathfrak{m}\mathcal{O}_Z = \mathcal{O}_Z(-D)$ and $\mathfrak{m}^n = H^0(\mathcal{O}_Z(-nD))$. Further $H^1(\mathcal{O}_Z(-nD)) = 0$ for all n by [LIP, 12.1]. Let $\overline{D} = \sum r_i \overline{E_i}$ and put $\overline{\mathfrak{m}} = H^0(\mathcal{O}_{\overline{Z}}(-\overline{D}))$. Then $\overline{\mathfrak{m}}^n = H^0(\mathcal{O}_{\overline{Z}}(-n\overline{D}))$. By [W1, 0.4.4 and the proof of 5.13] $\overline{\mathfrak{m}}^n$ is A-flat for all $n \geq 0$. So, $\overline{\mathfrak{m}}$ defines a normally flat A-section, $\overline{s}_0 : \operatorname{Spec} A \to \operatorname{Spec} R \otimes_{\mathbb{C}} A$. Blow up $\overline{s}_0$ to get $\overline{Z}_1$. One proves easily that $\overline{\mathfrak{m}}\mathcal{O}_{\overline{Z}} = \mathcal{O}(-\overline{D})$ and in particular is $\overline{\mathfrak{m}}\mathcal{O}_{\overline{Z}}$ invertible, so $\overline{Z}$ dominates $\overline{Z}_1$. Now we may continue in obvious manner until we reach the following situation

$$\begin{array}{ccc} \overline{Z} & \xrightarrow{\ \alpha\ } & \overline{Z}_n \\ \big\uparrow & & \big\uparrow \\ Z & = & Z. \end{array}$$

But since α lifts the identity, it must be an isomorphism. Thus we have produced an element of $G(A)$ which maps to the class of $\overline{Z}$, so the map $G(A) \to ES_Y(A)$ is surjective. To prove injectivity, assume that

$$\phi : \overline{Z}^{(1)} \to \overline{Z}^{(2)}$$

is an isomorphism of deformations. One proves that if the E_i lifts, they lifts uniquely, so that ϕ takes the lifting of the -1-curves in $\overline{Z}^{(1)}$ to the liftings of the -1-curves in $\overline{Z}^{(2)}$. Thus the isomorphism ϕ "blows down" to isomorphisms

$$\phi_i : \overline{Z}_i^{(1)} \to \overline{Z}_i^{(2)}$$

and gives an isomorphism of the whole diagrams in which $\overline{Z}^{(1)}$ and $\overline{Z}^{(1)}$ sits. Now it is also clear that

$$\phi_i \circ \overline{s}_{ij}^{(1)} = \overline{s}_{ij}^{(2)},$$

so we have proved that the map is injective. □

To clarify the relationship to complete ideals, we define a third deformation functor F. Let D be such that $\mathfrak{q}\mathcal{O}_Z = \mathcal{O}_Z(-D)$. If the E_i lift, then so does D and its lifting is $\overline{D} = \Sigma r_i \overline{E}_i$, where $D = \Sigma r_i E_i$. So for $A \in \mathcal{C}$ we may define

$$F(A) = \begin{array}{l}\text{Set of isomorphism clases of flat liftings } \overline{\mathfrak{q}} \subset R \otimes_{\mathbb{C}} A \text{ of } \mathfrak{q} \\ \text{such that } \overline{\mathfrak{q}} = H^0(\mathcal{O}_{\overline{Z}}(-\overline{D})) \text{ where } \overline{Z} \in S(A).\end{array}$$

Remark 3.8. This functor could be called the equinumerical deformation functor of complete ideals. As in [SP1] one may define numerical invariants of complete ideals. Then deformations as above are exactly the deformations which keep these invariants fixed.

We will need the following lemma a couple of times.

Lemma 3.9. *Assume $\overline{Z}_1$ and $\overline{Z}_2$ are liftings of Z to* Spec A, $A \in \mathcal{C}$, *such that the corresponding deformation classes are in $ES_Y(A)$ and assume we have*

$$\begin{tikzcd} \overline{Z}_1 \arrow[d] & \overline{Z}_2 \arrow[d] \\ \overline{Y}_1 \arrow[r] & \overline{Y}_2 \end{tikzcd}$$

over Spec A *where $\overline{Y}_1$ and $\overline{Y}_2$ are deformations of Y. Then there exist a morphism*

$$\overline{Z}_1 \to \overline{Z}_2$$

making the diagram commutative.

Proof. The proof uses the vanishing result, [W3, Th. C], and is analogous to [W1, Prop. 1.12]. □

We claim:

Proposition 3.10.

$$ES_Y \xrightarrow{\sim} F$$

Proof. There is a morphism $ES_Y \to F$ sending $\overline{Z}$ to $\overline{\mathfrak{q}}$ which is surjective by definition if we know that $\overline{\mathfrak{q}}$ is a flat lifting. But $H^1(Z, \mathcal{O}_Z(-D)) = 0$. By [W1, 0.4.2] then also $H^1(\overline{Z}, \mathcal{O}_{\overline{Z}}(-\overline{D})) = 0$. From [W1, 0.4.3] it follows that $H^0(\mathcal{O}_{\overline{Z}}(-\overline{D}))$ is A-flat. In fact $\overline{\mathfrak{q}}^n = H^0(\mathcal{O}_{\overline{Z}}(-nD))$ is A-flat for all $n > 0$.

Next we must show that the map is injective. Let $\overline{Z}_1$ and $\overline{Z}_2$ be liftings giving the same $\overline{\mathfrak{q}}$. Let $\overline{Y}$ be the blow up of Spec $R \otimes_{\mathbb{C}} A$ in $\overline{\mathfrak{q}}$. Then $\overline{Y}$ is a deformation of Y and we have

$$\begin{tikzcd} \overline{Z}_i \arrow[d] & Z \arrow[l] \arrow[d] \\ \overline{Y} & Y \arrow[l] \end{tikzcd}$$

for $i = 1, 2$, since $\overline{Z}_i$ dominates $\overline{Y}$. Thus both $\overline{Z}_1$ and $\overline{Z}_2$ blow down to $\overline{Y}$. From 3.9 it follows that $\overline{Z}_1$ and $\overline{Z}_2$ are in the same deformation class. □

Assume we have a deformation $\overline{C}$ of C and that this has a simultaneous embedded resolution (with sections):

$$(3.3.8) \qquad \begin{tikzcd} S_n \arrow[r] \arrow[d] & S_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & S_0 \arrow[d] \\ \overline{S}_n \arrow[r] & \overline{S}_{n-1} \arrow[r] & \cdots \arrow[r] & \overline{S}_0 \end{tikzcd}$$

Let $\overline{s}_0$ be the first section. This is defined by an ideal

$$\overline{\mathfrak{m}} = (x - m_1, y - m_2) \subset \mathbb{C}[[x, y]] \otimes_{\mathbb{C}} A$$

where m_i are in the maximal ideal of A. Clearly we may lift this to give a section

$$\operatorname{Spec} A \to \overline{Z}_0 = \operatorname{Spec} R \otimes_{\mathbb{C}} A.$$

Blowing up this section and continuing, we get a corresponding (algebraic) simultaneous resolution of $\overline{C'}$. By forgetting we get an element of $ES_Y(A)$. In fact, in this way we define a transformation of functors

$$\rho : ES_C \to ES_Y.$$

This is a smooth map of deformation functors, but before we prove this, we connect equinumerical deformation of complete ideals to equisingular deformations of plane curve singularities. Recall that we are assuming that the defining curve $C = \operatorname{Spec} C[[x,y]]/(f)$ is defined by a polynomial f so that we may consider $C' = \operatorname{Spec} R/(f)$. Now consider a lifting $\overline{C'}$ of C'. Then, by pulling back, we get a deformation $\overline{C}$ of C.

Proposition 3.11. *In the notation above, $\overline{C} \in ES_C(A)$ if and only if the to $\overline{C}'$ corresponding $\overline{f} \in \overline{\mathfrak{q}}$ for a $\overline{\mathfrak{q}} \in F(A)$. Moreover the to $\overline{\mathfrak{q}}$ corresponding element in $G(A)$ gives the simultaneous embedded resolution for $\overline{C'}$.*

Proof. Assume $\overline{C} \in ES_C(A)$. As above, there is a corresponding embedded simultaneous resolution of $\overline{C}'$:

$$\begin{array}{ccccccc}
\overline{Z} = \overline{Z}_n & \longrightarrow & \overline{Z}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \overline{Z}_0 \\
\uparrow & & \uparrow & & & & \uparrow \\
\overline{C}_n & \longrightarrow & \overline{C}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \overline{C}_0 = \overline{C'}
\end{array}$$

Let $\overline{C}^*$ be the total transform of $\overline{C'}$. From [W2, Prop. 1.6] one easily shows that $\overline{C}^* = \overline{C}_n + \overline{D}$ where $\overline{D}$ is such that $\overline{\mathfrak{q}} = H^0(\mathcal{O}_{\overline{Z}}(-\overline{D}))$ is in $F(A)$. Thus $\mathcal{O}_{\overline{Z}}(-\overline{C}^*) \subset \mathcal{O}_{\overline{Z}}(-\overline{D})$. Since, $\pi_* \mathcal{O}_{\overline{Z}}(-\overline{C}^*) = (\overline{f})$, $\overline{f} \in \overline{\mathfrak{q}}$. Conversely if $\overline{f} \in \overline{\mathfrak{q}}$ then by proposition 3.10 and 3.7 there corresponds to $\overline{\mathfrak{q}}$ deformation as in (3.3.7). We only need to show that this is a simultaneous embedded resolution of $\overline{C'}$. The total transform of $\overline{C'}$ will be of the form $\overline{C}^* = \overline{C}_n + \overline{D}$ where again $\overline{D}$ is such that $\overline{\mathfrak{q}} = H^0(\mathcal{O}_{\overline{Z}}(-\overline{D}))$ and $\overline{C}_n$ is a lifting of C_n in $\overline{Z} = \overline{Z}_n$. Blowing down we get a lifting $\overline{C}_i$ of C_i at each step and $\overline{C}_0$ have to be $\overline{C'}$. Lastly from [W2, 1.9] the last sections (the $\overline{t}_j$) are uniquely determined by the lifting. The last part of the proposition is now clear. □

Remark 3.12. The proposition shows how equisingularity of a plane curve may be described in terms of equinumerical deformations of complete ideals.

Theorem 3.13. *The map ρ is smooth.*

Proof. See also [W2, Prop. 4.9, Th. 4.2]. Let $A_2 \to A_1$ be a small morphism in $\mathcal{C}$. Assume $\rho(A_1)(\overline{C}_1) = \overline{Z}_1$. Assume $\overline{Z}_2 \in ES_Y(A_2)$ and maps to $\overline{Z}_1$. We must prove that there is $\overline{C}_2 \in ES_C(A_2)$, lifting $\overline{C}_1$, such that $\rho(A_2)(\overline{C}_2) = \overline{Z}_2$. But the total transform $\overline{C}_1^* \subset \overline{Z}_1$ may be lifted locally trivially to a divisor $\overline{D} \subset \overline{Z}_2$. By the identification of ES_Y with the functor G, there is a map $\overline{\pi} : \overline{Z}_2 \to \operatorname{Spec} R \otimes_{\mathbb{C}} A_2$. Then $\overline{\pi}_* \mathcal{O}_{\overline{Z}_2}(-\overline{D})$ gives an ideal $\mathfrak{a}$ which one may show is a principal ideal which gives a lifting $\overline{C}_2$ of $\overline{C}_1$. Now proposition 3.11 (and its proof) shows that $\mathfrak{a} \subset \overline{\mathfrak{q}}$ and that $\overline{C}_2$ is in $ES_C(A_2)$, and that $\overline{C}_2$ maps to $\overline{Z}_2$. □

3.4. **The tangent spaces.** We first turn to the understanding of the kernel of the map ρ of the previous section. Let K denote the kernel of $\rho(\mathbb{C}[\epsilon])$, and recall that

$$ES_C(\mathbb{C}[\epsilon]) \subset T_C^1 = \mathbb{C}[[x,y]]/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}),$$

see theorem 3.5. We claim the following:

Proposition 3.14. *$K \subset T^1_C$ is given by the image of $\mathfrak{q}_{(C,a)}$ in T^1_C.*

Proof. Let $\overline{C} \in ES_C(\mathbb{C}[\epsilon])$. Clearly, up to infinitesimal automorphism, we may assume that $\overline{C}$ comes from a deformation $\overline{C'}$ of C'. Assume that $C' = V(f) \subset \operatorname{Spec} R$ and that $\overline{C'}$ corresponds to a lifting $\overline{f} \in R \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$ of f. By proposition 3.11 we get $\overline{f} \in \overline{\mathfrak{q}} \subset R \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$ and from the proof of 3.11 we see that $\overline{\mathfrak{q}}$ is the lifting of $\mathfrak{q}$ induced by $\rho(\mathbb{C}[\epsilon])(\overline{C})$. Now assume that this is the trivial deformation. By the identification of ES_Y with F, $\overline{\mathfrak{q}}$ is the trivial lifting of $\mathfrak{q}$. So if $\overline{f} = f + \epsilon g$ we must have $g \in \mathfrak{q}$. This shows that $K(\mathbb{C}[\epsilon])$ is contained in the image of $\mathfrak{q}$ in T^1_C.

On the other hand assume $g \in \mathfrak{q}$ and consider the lifting $\overline{f} = f + \epsilon g$ of f. First from proposition 3.11 this corresponds, by pulling back, to a $\overline{C} \in ES_C(\mathbb{C}[\epsilon])$. Also by proposition 3.11 we see that the corresponding simultaneous embedded resolution is the trivial one, so $\overline{C}$ is mapped to the trivial deformation $Z \times \operatorname{Spec} \mathbb{C}[\epsilon]$ by ρ. □

The tangentspace for the functor ES_Y itself is described by a cohomology group. Let again

$$\pi : Z \to \operatorname{Spec} R$$

and let $E = \pi^{-1}(0) = \cup_{i=1}^m E_i$ be the decomposition of the exceptional set in irreducible components. Let $\theta = \theta_Z$ be the tangent sheaf on Z. And let

$$\theta(\log E) \subset \theta$$

be the subsheaf of derivations taking the idealsheaf of E into itself. This is also the dual of the sheaf of one forms with logarithmic poles along E. Then from general theory we know that there is an identification

$$ES_Y(\mathbb{C}[\epsilon]) = H^1(Z, \theta(\log E)).$$

We now want to connect ES_Y to ES_X of the singularity $X \subset Y$. We do this first on tangent level. First let $\mathbb{T}^1_Y$ be the tangentspace of Def_Y, and let T^1_X be the tangent space of Def_X. Now, Y blow down to an affine so all H^2 are zero, so in our situation we have from the local global spectral sequence, see for instance [LAU], the short exact sequence

$$0 \to H^1(Y, \theta_Y) \to \mathbb{T}^1_Y \to T^1_X \to 0. \tag{3.3.9}$$

Here $H^1(Y, \theta_Y)$ is the subspace of $\mathbb{T}_Y$ corresponding to the locally trivial deformations of Y. We claim that there is a similar sequence involving the tangent spaces for ES_Y and ES_X. In fact, let $E' = \cup_{i=1}^{r-1} E_i$ (again $E_r, \ldots, E_n$ are those with $E_i^2 = -1$) then from theorem 3.6

$$ES_X(\mathbb{C}[\epsilon]) = H^1(\widetilde{X}, \theta(\log E')).$$

Here $\widetilde{X} \subset Z$ is the minimal resolution of X. We have the following:

Proposition 3.15. *There is an exact sequence*

$$0 \to H^1(Y, \theta_Y) \to H^1(Z, \theta(\log E)) \to H^1(\widetilde{X}, \theta(\log E')) \to 0.$$

Proof. Recall that there are exact sequences

$$0 \to \theta(\log E) \to \theta_Z \to \bigoplus_{i=1}^{m} N_{E_i} \to 0$$

and

$$0 \to \theta(\log E') \to \theta_Z \to \bigoplus_{i=1}^{r-1} N_{E_i} \to 0$$

of sheaves on Z. See [W1, Prop. 2.2]. We have $H^0(N_{E_i}) = 0$ for all i, so

$$H^1(Z, \theta(\log E')) = \ker\left(H^1(Z, \theta_Z) \to \bigoplus_{i=1}^{r-1} H^1(Z, N_{E_i})\right)$$

and

$$H^1(Z,\theta(\log E)) = \ker\left(H^1(Z,\theta_Z)\to\bigoplus_{i=1}^{m}H^1(Z,N_{E_i})\right).$$

By Riemann-Roch we have

$$\chi(N_{E_i}) = \chi(\mathcal{O}_{E_i}(E_i)) = E_i\cdot E_i + 1.$$

Thus if $r\le i\le m$ we have $H^1(Z,N_{E_i})=0$. This shows that $H^1(Z,\theta(\log E)) = H^1(Z,\theta(\log E'))$. From the long exact sequence of (local) cohomology, we have:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & \\
H^1(Z,\theta(\log E')) \arrow[r, "\alpha"] \arrow[d] & H^1(\widetilde{X},\theta(\log E')) \arrow[r] \arrow[d] & H^2_{\widetilde{C}}(Z,\theta(\log E')) \arrow[r] \arrow[d] & 0 \\
H^1(Z,\theta) \arrow[r, "\beta"] \arrow[d] & H^1(\widetilde{X},\theta) \arrow[r] \arrow[d] & H^2_{\widetilde{C}}(Z,\theta) \arrow[r] \arrow[d] & 0 \\
\oplus H^1(Z,N_{E_i}) \arrow[r, "\gamma"] \arrow[d] & \oplus H^1(\widetilde{X},N_{E_i}) \arrow[r] \arrow[d] & \oplus H^2_{\widetilde{C}}(Z,N_{E_i}) \arrow[r] & 0 \\
0 & 0 & &
\end{tikzcd}$$

We claim that all the modules on the right are zero. For example to show that $H^2_{\widetilde{C}}(Z,\theta)=0$ we use that

$$H^2_{\widetilde{C}}(Z,\theta) = \lim \operatorname{Ext}^2(\mathcal{O}_{n\widetilde{C}},\theta).$$

Taking $\operatorname{Hom}(-,\theta)$ of

$$0\to\mathcal{O}_Z(-n\widetilde{C})\to\mathcal{O}_Z\to\mathcal{O}_{n\widetilde{C}}\to 0$$

we get

$$\cdots\to\operatorname{Ext}^1(\mathcal{O}_Z,\theta)\xrightarrow{\rho}\operatorname{Ext}^1(\mathcal{O}_Z(-n\widetilde{C}),\theta)\to\operatorname{Ext}^2(\mathcal{O}_{n\widetilde{C}},\theta)\to 0$$

since

$$\operatorname{Ext}^2(\mathcal{O}_Z,\theta) = H^2(Z,\theta) = 0.$$

Considering the exact sequence

$$0\to\mathcal{O}_Z\to\mathcal{O}_Z(n\widetilde{C})\to\mathcal{O}_{n\widetilde{C}}(n\widetilde{C})\to 0$$

we identify the first map in

$$H^1(Z,\theta)\longrightarrow H^1(Z,\theta\otimes\mathcal{O}(n\widetilde{C}))\longrightarrow H^1(Z,\theta\otimes\mathcal{O}_{n\widetilde{C}}(n\widetilde{C}))\longrightarrow 0$$

with ρ. But the support of $\theta\otimes\mathcal{O}_{n\widetilde{C}}(n\widetilde{C})$ is $\widetilde{C}$, the normalization of C and is affine, so $H^1(Z,\theta\otimes\mathcal{O}_{n\widetilde{C}}(n\widetilde{C}))=0$. It follows that $\operatorname{Ext}^2(\mathcal{O}_{n\widetilde{C}},\theta)=0$.

Now consider the map γ. Since N_{E_i} has support on E_i and the sum is taken over $E_i\subset\widetilde{X}$, it follows that γ is an isomorphism. It follows that $\ker\alpha\simeq\ker\beta$, we show that $\ker\beta\simeq H^1(Y,\theta)$.

Let π_1 be the restriction of π to $\widetilde{X}$. From the Leray spectral sequence we get

$$0\to H^1(Y,(\pi_1)_*\theta)\to H^1(Z,\theta)\to H^0(Y,R^1(\pi_1)_*\theta)\to 0.$$

[BW, prop 1.2] states that $(\pi_1)_*\theta_Z=\theta_Y$, so we only need to show that

$$H^0(Y,R^1(\pi_1)_*\theta_Z) = H^1(\widetilde{X},\theta).$$

But $R^1(\pi_1)_*$ has support on the singular point so

$$H^0(Y, R^1(\pi_1)_*\theta_Z) = H^0(X, R^1(\pi_1)_*\theta_Z).$$

From [HAR, Prop. III.8.2] this equals to $H^1(\widetilde{X}, \theta)$. □

Now we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(Y,\theta_Y) & \longrightarrow & \mathbb{T}^1_Y & \longrightarrow & T^1_X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^1(Y,\theta_Y) & \longrightarrow & H^1(Z,\theta(\log E)) & \longrightarrow & H^1(\widetilde{X},\theta(\log E')) & \longrightarrow & 0.
\end{array}$$

From theorem 3.6 it follows that all vertical maps are injective.

Remark 3.16. From lemma 3.9 it follows that ES_Y is a subfunctor of Def_Y . Given a deformation $\overline{Y}$ of Y, we may always restrict to get a deformation $\overline{X}$ of X. It is clear from the definition of ES_Y and ES_X that if $\overline{Y}$ corresponds to a deformation for ES_Y, then $\overline{X}$ corresponds to a deformation for ES_X. Thus ES_Y is is the functor of deformations of Y inducing equisingular deformations on the singularity $X \subset Y$.

3.5. **The map $ES_C \to ES_X$.** It is now easy to see that there is a smooth map

$$ES_C \to ES_X.$$

The "restriction" map $ES_Y \to ES_X$ is smooth since from proposition 3.15 the map

$$ES_Y(\mathbb{C}[\epsilon]) \to ES_X(\mathbb{C}[\epsilon])$$

is surjective, see [BU]. So, we have:

Theorem 3.17. *There is a smooth map*

$$ES_C \to ES_X$$

of deformation functors.

We should now give some examples.

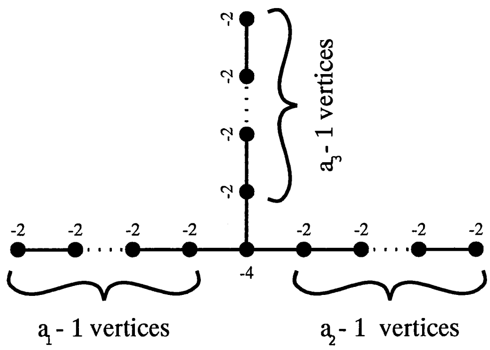
Example 3.18. Let

$$C = \operatorname{Spec} \mathbb{C}[[x,y]]/(x^3+y^3)$$

be three lines intersecting in a point. This has

$$ES_C(\mathbb{C}[\epsilon]) = 0.$$

Thus for any $a = (a_1, a_2, a_3) \in \mathbb{N}^3$ we have from 3.17 that $X = X_{(C,a)}$ is equisingular rigid. Indeed $X_{(C,a)}$ has the dualgraph



and is known to be taut, see [L2].

Example 3.19. Let

$$C = \operatorname{Spec} \mathbb{C}[[x,y]]/(x^4+y^4)$$

be four lines intersecting in a point. Letting $a = (1,1,1,1)$ we get a singularity $X \subset Y$ with the following dualgraph:

This is a cyclic quotient singularity and is know to be equisingular rigid, that is

$$\dim_{\mathbb{C}} H^1(\widetilde{X}, \theta(\log E')) = 0.$$

However, one finds (see section 4.2) that the corresponding complete ideal $\mathfrak{q}_{(C,a)}$ is

$$\left(x^4+y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\right).$$

Also x^2y^2 is a basis for

$$ES_C(\mathbb{C}[\epsilon]) \subset T^1_C = \mathbb{C}[[x,y]]/(x^3,y^3).$$

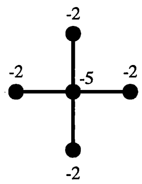
This is not in the image of $\mathfrak{q}$ so from 3.14,

$$ES_Y(\mathbb{C}[\epsilon]) = H^1(Z, \theta(\log E)) = 1.$$

This shows that (proposition 3.15)

$$\dim_{\mathbb{C}} H^1(Y, \theta) = 1.$$

This could be understood as a cross ratio between four exceptional curves in Y. Now, if we let $a = (2,2,2,2)$, we get a singularity with the following dualgraph:



This is known to have

$$\dim_{\mathbb{C}} H^1(\widetilde{X}, \theta(\log E')) = 1,$$

see [L2]. The corresponding complete ideal $\mathfrak{q}_{(C,a)}$ is

$$\left(x^4+y^4, x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6\right).$$

Thus we see that still, $ES_Y(\mathbb{C}[\epsilon])$ has dimension 1. So the moduli is now "swallowed up" by the singularity.

Example 3.20. Let

$$f = x^5 + y^{13} \in R$$

and choose $a = 3$. The graph of $X_{(C,a)}$ is given in figure 1. Using proposition we see that

$$ES_Y(\mathbb{C}[\epsilon])$$

is one dimensional. From [L2] we know that X is not equisingular rigid, thus it follows that $H^1(Y, \theta) = 0$. We have a one dimensional miniversal equisingular family

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \operatorname{Spec} \mathbb{C}[[t]], \end{array}$$

and there is a $\mathbb{C}^*$-action on this family and there are two orbits for this action. It is known, see [LP], that all curves with semigroup $\langle 5, 13\rangle$ is obtained as a equisingular deformation of f. It follows that all normal surface singularities with given dual graph are in this family and hence that there are to isomorphism classes of singularities with the given dual graph.

Remark 3.21. We have seen how the kernel of $ES_C \to ES_Y$ is nicely related to the curve and the complete ideal. It would be nice to have a similar description for the kernel of $ES_C \to ES_X$. This however seems to be difficult. Assume $\overline{Y}$ is a deformation of Y over $\operatorname{Spec}\mathbb{C}[\epsilon]$. This induces a deformation of $\overline{X}$ of X. Now there may be an infinitesimal automorphism of X which allow us to trivialize $\overline{X}$. Since $\overline{Y}$ blow down to $\operatorname{Spec}\mathbb{C}[x,y] \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$, also $\overline{X}$ does. However, the infinitesimal automorphism that trivializes $\overline{X}$ might not blow down. That is, it might not be possible to trivialize $\overline{X}$ *over* $\operatorname{Spec}\mathbb{C}[x,y] \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$. Thus to understand the kernel of $ES_C \to ES_X$ one must understand the infinitesimal automorphisms of X very well. We have done some calculations in this direction, but we are not able to discover a pattern. However the results of the next section shows that if the defining curve singularity is irreducible and a is large the smooth map of 3.17 is an isomorphism on tangent spaces.

4. Results in the case of large a.

In this section we will assume that the defining curve C is irreducible. Under this assumption we show that there is an a^*, which depends only on the topological type of C, such that if the a in the definition of $X = X_{(C,a)}$ is taken to be larger than this a^*, the moduli of the plane curve singularity C coincide with the moduli of the normal surface singularity X. To do this we need some notation and results concerning graphs of sandwiched singularities and complete ideals.

4.1. **Graphs of sandwiched singularities.** The possible dualgraphs for sandwiched singularities were classified by Spivakovsky, [SP2]. We will only consider the case when C is irreducible. Then the possible graphs may be well known, but to fix notation we give a review, following [SP1]. Let again C be an irreducible plane curve singularity and let $a \in \mathbb{N}$ be given. As in section 3.1 define

$$\pi : Z = Z_{(C,a)} \to \operatorname{Spec}\mathbb{C}[x,y].$$

We denote by $\Gamma^* = \Gamma_{(C,a)}$ the dual graph of this (non-minimal) resolution of "the smooth singularity." It is clear that the dual graph of $\Gamma = \Gamma(X)$ of $X = X_{(C,a)}$ is contained in Γ^* and that $\Gamma^* \setminus \Gamma$ consist of one vertex corresponding to the unique -1-curve in Z.

Remark 4.1. Given Γ, then Γ^* is not always uniquely determined, see [SP1, 2.4.8]. We will see, however, that if a is large enough, Γ^* is uniquely determined by Γ.

It is not difficult to see that Γ^* must have a form as described in figure 2. Each of the Γ_i for $1 \le i \le g$ are described in figure 3 and 4 depending on the parity of m_i. In both cases all vertices where the weight is not given, is taken to have weight -2. Further the integers g, m_i for $1 \le i \le g$ and $a_1^{(i)}, \dots, a_{m_i}^{(i)}$ are defined from the graph. The Γ_{g+1} is a line graph, where all vertices has weight -2 except the last (to the right) which has has weight -1. We define $m_{g+1} = 1$ and $a_1^{(g+1)}$ to be the length of this minus one.

So given an irreducible plane curve singularity, we have a collection of integers

$$\left\{a_j^{(i)}\right\}_{1 \le j \le m_i, 1 \le i \le g+1}.$$

These integers (together with g and $m_1, \dots, m_{g+1}$) are equivalent to $\Gamma^* = \Gamma_{(C,a)}$. They are also equivalent with the semigroup of C together with the number a. In fact we have the following. Put for every $1 \le i \le g+1$,

$$\beta_i' := a_1^{(i)} + \cfrac{1}{a_2^{(i)} + \cdots + \cfrac{1}{a_{m_i}^{(i)} + 1}}$$

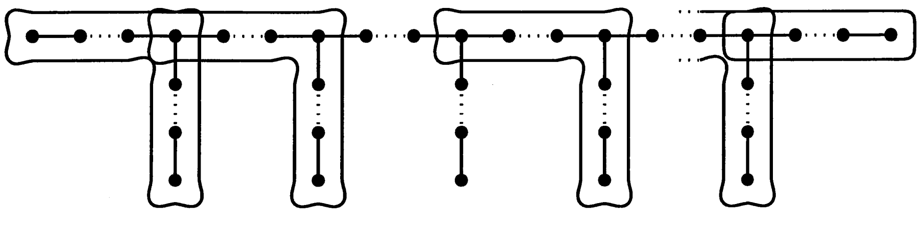


FIGURE 2. The shape of Γ^*.

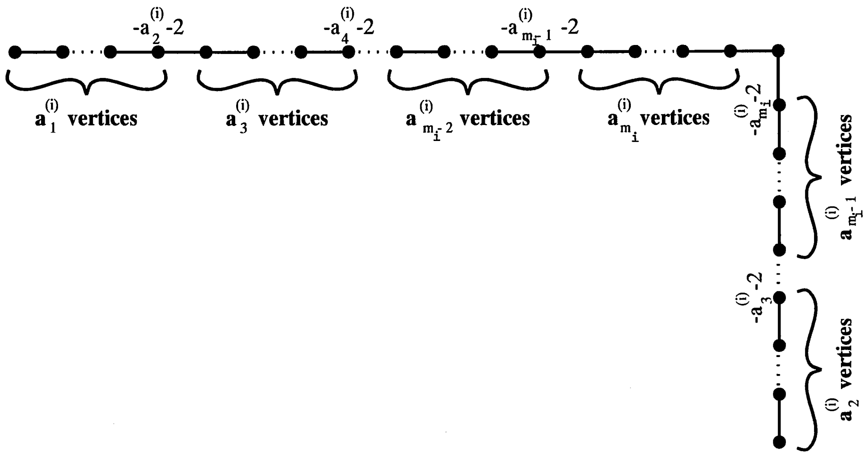


FIGURE 3. Γ_i for m_i odd.

Write $\beta_i' = \frac{p_i}{n_i}$, $p_i, n_i \in \mathbb{N}$, and $(p_i, n_i) = 1$. For $0 \leq i \leq g$, set $e_i = \Pi_{j=i+1}^{g} n_j$, $e_g = e_{g+1} = n_0 = \beta_0' = 1$. Now define

(4.4.1) $$\overline{\beta}_0 = e_0$$

(4.4.2) $$\overline{\beta}_i = (\beta_i' - 1)\, e_{i-1} + \overline{\beta}_{i-1} n_{i-1} \qquad \text{for } 1 \leq i \leq g+1.$$

The semigroup $S(C)$ of S is then the subsemigroup of $\mathbb{Z}$ generated by $\overline{\beta}_0, \ldots, \overline{\beta}_g$.

Example 4.2. Assume that C is a curve with semigroup $S(C)$ generated by two numbers p and q in $\mathbb{N}$, $(p, q) = 1$, $p < q$. Also let $a \in \mathbb{N}$ be given. Then $g = 1$, $m_2 = 1$, $\overline{\beta}_0 = p, \overline{\beta}_1 = q$ and $\overline{\beta}_2 = a + pq$. We also have

$$\frac{q}{p} = a_1^{(1)} + \cfrac{1}{a_2^{(1)} + \cdots + \cfrac{1}{a_{m_1}^{(1)} + 1}}$$

and $a_1^{(2)} = a$.

The following is an important observation.

Proposition 4.3. *In the notation above, consider $\Gamma \subset \Gamma^*$. Let*

$$\left\{a_j^{(i)}\right\}_{1 \leq j \leq m_i, 1 \leq i \leq g+1}$$

be the integers as defined from Γ^*. If

$$\left(a \neq a_2^{(g)} \text{ or } m_g \neq 2\right) \text{ and } \left(a \neq a_1^{(1)} \text{ or } m_1 \neq 2\right)$$

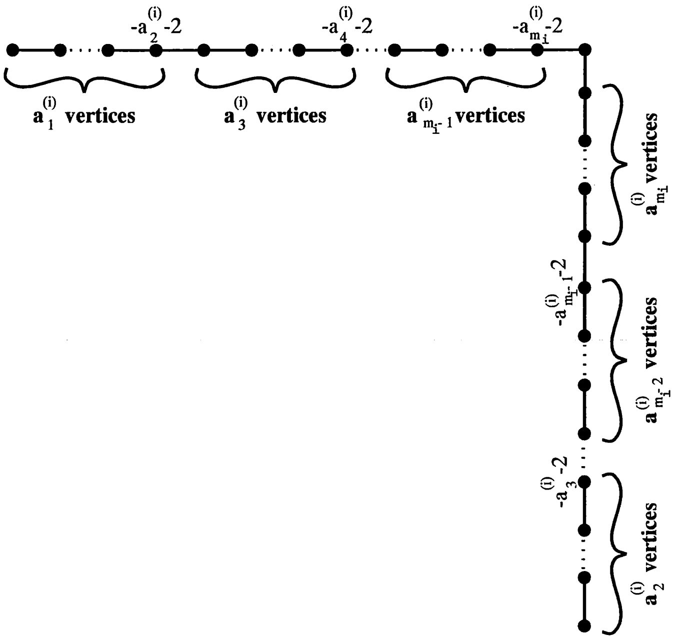


FIGURE 4. Γ_i for m_i even.

then there are no nontrivial automorphisms of Γ.

Proof. There are g stars in Γ. Only the leftmost and the rightmost stars have two arms containing an endvertex of Γ. So, an automorphism of Γ must preserve or interchange these stars. If they should be interchanged, it is clear from the description of Γ given above, that $m_1 = 2$ and $a_2^{(1)} = a_1^{(g+1)} = a$.

If the leftmost and the rightmost stars are not interchanged, it is easy to see from the description of Γ given above, that the only possibility for a non-trivial automorphism, is to interchange the two arms of the rightmost star which contains end vertices, and that this only may be done if $m_g = 2$ and $a = a_1^{(g+1)} = a_2^{(g)}$. $\square$

4.2. **Complete ideals.** We review some results on complete ideals to be found in [SP1]. First some notation. If B is any Noetherian domain, I is any ideal in B and $S \subset B$ is a subset, we set

$$\operatorname{mult}_I S = \max\{n \,|S \subset I^n\}.$$

Set R=$\mathbb{C}[[x,y]]$. Assume that

$$C = \operatorname{Spec} \mathbb{C}[[x,y]]/(f)$$

is an irreducible plane curve singularity, where $f \in R$.

Theorem 4.4. *Keep the notation above, and let $a \in \mathbb{N}$ be given. Define now the integers $g, \overline{\beta}_0, \ldots, \overline{\beta}_{g+1}$, and $n_0, \ldots, n_{g+1}$ as in section 4.1. Then there exist*

$$Q_0, \ldots, Q_{g+1} \in R$$

with the following properties

1. Q_0 *and* Q_1 *are parameters for* R.
2. Q_{g+1} *defines* C

3. $\mathfrak{q}_{(C,a)}$ *in* R *is generated by the following set*

$$\left\{ \prod_{j=0}^{g+1} Q_j^{\gamma_j} \,\middle|\, \sum_{j=0}^{g+1} \gamma_j \overline{\beta}_j \geq \overline{\beta}_{g+1} \right\}.$$

□

Now let $\mathfrak{a}$ be the ideal generated by the set

$$\left\{ \prod_{j=0}^{g} Q_j^{\gamma_j} \,\middle|\, \sum_{j=0}^{g} \gamma_j \overline{\beta}_j \geq \overline{\beta}_{g+1} \right\}$$

and set $f = Q_{g+1}$ to be the defining element of C. Then we have $\mathfrak{q}_{(C,a)} = (f) + \mathfrak{a}$. From [SP1] we also have that

$$\text{mult}_{\mathfrak{m}_R} \mathfrak{a} = \min \left\{ \sum_{j=0}^{g} \gamma_j \prod_{i=0}^{j-1} n_j \,\middle|\, \sum_{j=0}^{g} \gamma_j \overline{\beta}_j \geq \overline{\beta}_{g+1} \right\} \tag{4.4.3}$$

and since $\overline{\beta}_{g+1} = (a-1)e_g + \overline{\beta}_g n_g$ and $e_g, \overline{\beta}_0, \ldots, \overline{\beta}_g$ and n_g do not depend on a, we see that the multiplicity of $\mathfrak{a}$ only depends on the semigroup of C and on a. Further more the multiplicity of $\mathfrak{a}$ increases with a, that is, we may increase the multiplicity of $\mathfrak{a}$ beyond any limit, be increasing a.

4.3. **The tangent space of ES_Y when a is large.** We have the following:

Theorem 4.5. *Assume C is irreducible. Then there exist an a^* which depends only on the topological type of C, such that if $a \geq a^*$ and $Y = Y_{(C,a)}$, then*

$$ES_C(\mathbb{C}[\epsilon]) = ES_Y(\mathbb{C}[\epsilon]).$$

Proof. From, 3.14, we know that $ES_Y(\mathbb{C}[\epsilon])$ is the image of $ES_C(\mathbb{C}[\epsilon]) \subset T_C^1$ in $\mathbb{C}[x,y]/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) + \mathfrak{q}$. Here f defines C and $\mathfrak{q} = \mathfrak{q}_{(C,a)}$. We will prove that there exist an a^* such that $a \geq a^*$ implies

$$\mathfrak{q} \subset (f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}).$$

Defining $\mathfrak{a}$ as section 4.2 we must prove $\mathfrak{a} \subset (f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. But it is easy to see that $(x,y)^\mu \subset (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, and from (4.4.3) we may choose a^* such that $a \geq a^*$ gives $\mathfrak{a} \subset (x,y)^\mu$. It is known that μ is an topological invariant of C. □

This gives us, at first sight, information on the moduli only of the surface Y. But in fact, in the next section we will prove the following:

Theorem 4.6. *Assume C is irreducible with semigroup S, and let Γ be the dualgraph of $X_{(C,a)}$. Then there exist an a^*, depending only on S, such that if $a \geq a^*$ the isomorphismclasses of plane curve singularities with semigroup S are in one to one correspondence with the isomorphismclasses of (the complete local ring of) normal surface singularities with dualgraph Γ, by the construction in section 3.1.*

Now assume C is given and that $a \geq a^*$. Let $Y = Y_{(C,a)}$. We claim that it follows from the theorem that $H^1(Y, \theta) = 0$. In fact, assume $H^1(Y, \theta) \neq 0$. Since $H^2(Y, \theta) = 0$ there is a nontrivial deformation of Y. So in the analytic category there is a nontrivial flat family Y_t where $Y_0 = Y$. There must be a Y_s in this family which is not isomorphic to Y_0, since if all fibers are isomorphic, the family is trivial by a result of Fisher and Grauert, see [FG]. But since the singularity on these are the same, it follows from the preceding theorem that the defining curves are isomorphic. But this is not possible unless Y_0 and Y_s are isomorphic. Thus from proposition 3.15, we also have

Theorem 4.7. *Assume C is irreducible. Then there exist an a^* which depends only on the topological type of C, such that if $a \geq a^*$ and $X = X_{(C,a)}$, then*

$$ES_C(\mathbb{C}[\epsilon]) = ES_X(\mathbb{C}[\epsilon]).$$

4.4. **The proof of theorem 4.6.** We introduce some notation. Let C be an irreducible plane curve singularity, and let $x \in X = X_{(C,a)}$ be the singular point. Then we define

$$P_{(C,a)} = \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}.$$

By pulling back via

$$\operatorname{Spec} \mathbb{C}[[x,y]] \to \operatorname{Spec} \mathbb{C}[x,y],$$

leaving $P_{(C,a)}$ unchanged, we may assume that $Z_{(C,a)}, Y_{(C,a)}$ and $X_{(C,a)}$ blow down to $\operatorname{Spec} \mathbb{C}[[x,y]]$. In the following we will keep to this view, so let $R = \mathbb{C}[[x,y]]$. Thus defining $\mathfrak{a} = \mathfrak{a}_{(C,a)}$ as in section 4.2 and letting $a_1, \dots, a_n$ generate $\mathfrak{a}$, we find that $X_{(C,a)} = \operatorname{Spec} A$, where

$$A = R\left[\frac{a_1}{f}, \dots, \frac{a_n}{f}\right]$$

and where f defines C. Thus $P_{(C,a)} = \operatorname{Spec} \hat{A}$. A main part is to prove the following statement:

Proposition 4.8. *Let S be the semigroup of an irreducible plane curve singularity. Then there exist an a^* such that if C and D are irreducible curve singularities with semigroup S, a and b are greater than a^* and $P_{(C,a)}$ and $P_{(D,b)}$ are isomorphic then $a = b$ and C and D are isomorphic.*

Proof. Assume $P_{(C,a)}$ and $P_{(D,b)}$ are isomorphic. Thus:

$$\begin{array}{ccc}
\widetilde{P_{(C,a)}} & \xrightarrow[\sim]{\psi} & \widetilde{P_{(D,b)}} \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \rho_1} \\
P_{(C,a)} & \xrightarrow[\sim]{\phi} & P_{(D,b)} \\
{\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \rho_0} \\
\operatorname{Spec} R & & \operatorname{Spec} R
\end{array}$$

Here, π_1 and ρ_1 are minimal resolutions. These are products of quadratic transformations. Let Γ be the dualgraph of $P_{(C,a)}$. Then Γ is also the dualgraph of $P_{(D,a)}$. Since C and D have the same semigroup, $\Gamma_{(C,1)} = \Gamma_{(D,1)}$. But $\Gamma_{(C,a)}$ and $\Gamma_{(D,b)}$ must have the same number of vertices. It follows from section 4.1, that $a = b$ and that $\Gamma_{(C,a)} = \Gamma_{(D,b)}$. Let $\Gamma^* = \Gamma_{(C,a)}$. We may view ψ as giving an automorphism of $\Gamma \subset \Gamma^*$, but from proposition 4.3, by choosing $a = b$ large, this must be the identity. This means (in the notation of section 4.1) that the curves corresponding (via the construction of section 3.1) to the endvertices of $\Gamma_1 \subset \Gamma_{(C,a)}$ in $\widetilde{P_{(C,a)}}$ are mapped by ψ to the curves corresponding to the endvertices of $\Gamma_1 \subset \Gamma_{(D,b)}$ in $\widetilde{P_{(D,b)}}$.

Let $P_{(C,a)} = \operatorname{Spec} \hat{A}$ as above, and let $P_{(D,a)} = \operatorname{Spec} \hat{B}$, where

$$B = R\left[\frac{b_1}{g}, \dots, \frac{b_m}{g}\right].$$

Here $g \in R$ defines D and $\mathfrak{q}_{(D,b)} = (g) + \mathfrak{b}$ where $\mathfrak{b} = (b_1, \dots, b_m)$ and $\operatorname{mult}_{\mathfrak{m}_R} \mathfrak{b}$ may, as in section 4.2, be made arbitrary large by choosing $b = a$ large. Let Φ denote the isomorphism $\hat{A} \to \hat{B}$ induced by ϕ. Consider now, in the notation of section 4.1, the subgraph $\Gamma_1 \subset \Gamma^*$. We may assume that the parameters x and y of R are such chosen, that the strict transforms in $Z_{(C,a)}$ respectively intersect the exceptional set in the irreducible components corresponding to the end vertices of $\Gamma_1 \subset \Gamma^*$

and no other components of the exceptional set. Thus the the strict transforms of the curves in $P_{(D,b)}$ defined by the images $\Phi(x)$ and $\Phi(y)$ must respectively intersect each of the curves corresponding to the endvertices of $\Gamma_1 \subset \Gamma_{(D,b)}$ in $\widetilde{P_{(D,b)}}$. From this, it is easy to see that $\Phi(x) = u_1p_1$ and $\Phi(y) = u_2p_2$ where u_1 and u_2 are units in $\hat{B}$ and where p_1 and p_2 are regular parameters for R. Let $\mathfrak{c} = \left(\frac{b_1}{g}, \ldots, \frac{b_m}{g}\right)$. Then we may write $u_i = u'_i + v_i$ (for $i = 1, 2$) where u'_i are units in R and $v_i \in \mathfrak{c}$. Choose $f_1 \in \mathfrak{q}_{(C,a-1)}$ generic, that is, such that the strict transform of $V(f_1)$ in $Z_{(C,a)}$ intersects the exceptional set only in the irreducible component which in turn intersects the component with selfintersection -1. It follows that the strict transform in $\widetilde{P_{(D,a)}}$ of the curve in $P_{(D,b)}$ defined by the image $h = \Phi(f_1)$ intersect the irreducible component of the exceptional set with the same property. It follows easily that $h = u_3g_1$ where $g_1 \in \mathfrak{q}_{(D,a-1)} \subset R$ and u_3 is an unit in $\hat{B}$. Define an automorphism $\theta : R \to R$ by $x \mapsto u'_1p_1, y \mapsto u'_2p_2$, and consider $\theta(f_1) - g_1 = \theta(f_1) - h + h - g_1$. By definition of θ it is clear that $\theta(f_1) - h \in \mathfrak{c} \subset \hat{B}$. Write $u_3 = u'_3 + v_3$, where $u'_3 \in R$ is a unit and $v_3 \in \mathfrak{c}$. Thus $h = u'_3g_1 + v_3g_1$. Redefining g_1 we may assume $h = g_1 + c$, where $c \in \mathfrak{c}$. This gives $h - g_1 = c \in \mathfrak{c}$. Hence $\theta(f_1) - g_1 \in \mathfrak{c} \cap R$. Since it is clear that $\operatorname{mult}_{\mathfrak{m}_R} \mathfrak{c} \cap R \geq \operatorname{mult}_{\mathfrak{m}_R} \mathfrak{b} - \operatorname{mult}_{\mathfrak{m}_R} g$, it follows by choosing $a = b$ large, that $\theta(f_1) - g_1 \in \mathfrak{m}_R^\mu$, where μ is the Milnor number of f, f_1, g and g_1. (These have all the same Milnor number since they have the same semigroup.) From Mathers theorem it follows that $R/(\theta(f_1))$ and $R/(g_1)$ are isomorphic. We also get $f - f_1 \in \mathfrak{m}_R^\mu$ and $g - g_1 \in \mathfrak{m}_R^\mu$, so that it follows that $R/(f)$ and $R/(g)$ are isomorphic. $\square$

Remark 4.9. In the notation of the proof, we may take a^* such that $a \geq a^*$ gives

$$\operatorname{mult}_{\mathfrak{m}_R} \mathfrak{b} - \operatorname{mult}_{\mathfrak{m}_R} g = \operatorname{mult}_{\mathfrak{m}_R} \mathfrak{a} - \operatorname{mult}_{\mathfrak{m}_R} f \geq \mu,$$

and from section 4.2 it follows that this depends only on the semigroup S. It also follows that we may choose a^* such that if $a \geq a^*$, the hypothesis of proposition 4.3. Clearly this only depends on the numbers $\{a_j^{(i)}\}$ and hence only on the semigroup S.

In order to conclude the proof of theorem 4.6 let S be a semigroup for a plane irreducible curve. Let Γ be the graph obtained form $\Gamma^* = \Gamma_{(C,a)}$ by deleting the vertex with weight -1. From the proposition and the remark above, it remains now only to show that any sandwiched singularity with dualgraph Γ may be obtained from a irreducible plane curve singularity with semigroup S. This is however clear from, [SP2, Prop. 1.11].

Example 4.10. In general the number a^* which make the proof work, will be large. If we take the semigroup S to be generated by 5 and 13 as in example 3.4, the proof gives $a^* = 612$. The actual bound is probably much smaller.

References

[BU] R.O. Buchweitz, *Déformations de diagrammes, deploiments et singularités très rigides, liaison algébrique* Thèses, Université Paris VII, 1981.

[BW] D. Burns, J. Wahl, *Local contributions to global deformations of surfaces*, Invent. math. **26** (1974), 67-88.

[FG] W. Fisher, H. Grauert, *Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten*, Nachr. Akad. Wiss. Gö ttingen II. Math.-Phys. Kl., 1965, 88-94.

[G] T. S. Gustavsen, *Moduli for plane curve singularities, complete ideals and sandwiched surface singularities*, In preparation.

[HAR] R. Hartshorne, *Algebraic Geometry*, Springer Verlag, 1977.

[HER] M. Herrmann, S. Ikeda, U. Orbanz, *Equimultiplicity and blowing up*, Springer Verlag, 1988.

[HUS] C. Huneke, J. D. Sally, *Birational Extensions in Dimension Two and Integrally closed Ideals,* J. of Algebra, **115** (1988), 481-500.

[JS] T. de Jong, D. van Straten, *Deformation Theory of Sandwiched Singularities*, Preprint, (1995).

[L1] H. B. Laufer, *Normal Two-Dimensional Singularities,* Princeton University Press, (1971).

[L2] H. B. Laufer, *Taut Two-Dimensional Singularitites,* Math. Ann. **205** (1973), 131-164.

[LAU] O. A. Laudal, *Formal moduli of algebraic structures*, Lecture Notes in Math., no. 754, Springer Verlag, 1979.

[LP] O. A. Laudal, G. Pfister, *Local Moduli and Singularities,* Lecture Notes in Math., no. 1310, Springer Verlag, 1988.

[LIP] J. Lipman, *Rational Singularities with application to algebraic surfaces and unique factorisation,* Publ. Math. IHES **36** (1969), 195-279.

[SCH] M. Schlessinger, *Functors on Artin rings*, Trans. A.M.S., **130** (1968), 208-222.

[SP1] M. Spivakovsky, *Valuations in function fields of surfaces,* Am. J. of Math. **112** (1990), 107-156

[SP2] M. Spivakovsky, *Sandwiched Singularities and Desingularization of Surfaces by normalized Nash Transformations,* Ann. of Math. **131** (1990), 411-491.

[W1] J. Wahl, *Equisingular deformations of normal surface singularities, I,* Ann. of Math. **104** (1976), 325-356.

[W2] J. Wahl, *Equisingular deformations of plane algebroid curves,* Trans. A.M.S. **193** (1974), 143-170.

[W3] J. Wahl, *Vanishing Theorems for Resolutions of Surface Singularities* Invent. Math. **31** (1974), 17-41.

[ZA] O. Zariski, *Polynomial ideals defined by infinitely near base points,* Am. J. of Math. **60** (1938), 151-204.

[ZS] O. Zariski and P. Samuel, *Commutative Algebra*, Springer Verlag, 1960.

Matematisk institutt, PB 1053 Blindern, N-0316 OSLO, Norway
E-mail address: stolen@math.uio.no