Homotopy of a pair of approximately commuting unitaries in a simple C*-algebra

Ola Bratteli
Department of Mathematics, University of Oslo,
P. O. Box 1053 Blindern,
N–0316 Oslo, Norway

George A. Elliott
Mathematics Institute, Universitetsparken 5, DK–2100 Copenhagen Ø, Denmark
and
Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 3G3

David E. Evans
Department of Mathematics,
University of Wales,
Swansea SA2 8PP,
Wales, U.K.

Akitaka Kishimoto
Department of Mathematics,
Hokkaido University,
Sapporo, 060 Japan

Abstract

It is shown that, for a class of unital C*-algebras including purely infinite simple C*-algebras, real rank zero simple $A\mathbb{T}$ algebras, and AF algebras, if u and v are almost commuting unitaries where u has trivial K_1-class, and a certain K_0-valued obstruction associated to the pair u, v is trivial, then u can be deformed to 1 through a path of unitaries in the algebra almost commuting with v, and the length of the path can be estimated by a universal constant. This result is used to identify the obstruction with Loring's Bott element associated to the pair u, v and also to prove the more universal statement that if (u_i, v_i), $i = 1, 2$, are two pairs of almost commuting unitaries with $[u_i]_1$, $[v_i]_1$, and $\mathrm{Bott}(u_i, v_i)$ each independent of i, then one pair can be deformed into the other along a path of pairs of almost commuting unitaries in the algebra, the length of the path being bounded by a universal constant.

1 Introduction

This paper originated in the course of the classification of purely infinite simple C*-algebras obtained as inductive limits of direct sums of algebras of the form $M_k \otimes \mathcal{O}_n \otimes C(\mathbb{T})$, where M_k is the C*-algebra of $k \times k$ matrices, $\mathcal{O}_n$ is the Cuntz algebra of order n [Cun1], and $\mathbb{T}$ is the circle; see [BEEK]. A similar (slightly less general) result was obtained at about the same time by Lin and Phillips [LP]. A more general result was announced subsequently by Kirchberg and Phillips ([Kir], [Phi]). The present paper contains certain results of independent interest which were used in [BEEK] (and also in [ElR]).

As usual, if p is a projection and u is a partial unitary in a C*-algebra A, then $[p]_0, [u]_1$ will denote their canonical images in $K_0(A), K_1(A)$, respectively. If A is unital, $\mathcal{U}(A)$ will denote the unitary group in A, and $\mathcal{U}_0(A)$ the connected component of 1 in $\mathcal{U}(A)$.

If p is a projection in A, let $d(p)$ denote its Murray–von Neumann equivalence class, and let $D(A)$ denote the local semigroup of such equivalence classes with the preorder defined by $d(p) \leq d(q')$ if there exist projections $p', q' \in A$ with $d(p') = d(p)$, $d(q') = d(q)$ and $p' \leq q'$, [Zha2]. When A has real rank zero, it follows from [Zha2], Theorem 1.1, that $D(A)$ has the Riesz decomposition property with respect to this preorder, i.e. if $x, y, z \in D(A)$ and $x \leq y + z$, then there are elements $x_1, x_2 \in D(A)$ with $x_1 + x_2 = x$, $x_1 \leq y$ and $x_2 \leq z$. If A furthermore is simple, any nonzero element in $D(A \otimes K)$ is dominated by a multiple of any other, i.e. $D(A \otimes K)$ is simple. Here K denotes the C*-algebra of compact operators on a separable infinite dimensional Hilbert space.

See [BP] for the definition and elementary properties of real rank zero C*-algebras.

The class of C*- algebras we will be mostly concerned with in this paper is the following.

Definition. A C*-algebra A will be said to be a K_1-simple real rank zero C*-algebra if A is unital and separable and

1. A has real rank zero.

2. Any two non-zero projections in A, or in a matrix algebra over A, have the same K_0-class if and only if they are Murray–von Neumann equivalent.

3. For any $k_1 \in K_1(A)$ and any non-zero projection $p \in A$, there exists a unitary $u \in A$ such that $(1-p)u = u(1-p) = 1-p$ and $[u]_1 = k_1$.

4. $D(A \otimes K)$ is simple, or $K_1(A) = 0$.

The requirement 3 could more concisely be formulated as the condition that the canonical map

$$\mathcal{U}(pAp)/\mathcal{U}_0(pAp) \to K_1(A)$$

be surjective for each non-zero projection $p \in A$.

Note that, since any separable simple C*-algebra is stably isomorphic to its cut-downs by projections ([Bro]), we have $K_*(A) = K_*(pAp)$, and for a simple algebra it would therefore suffice to assume that the map

$$\mathcal{U}(pAp)/\mathcal{U}_0(pAp) \to K_1(pAp)$$

is surjective for each non-zero projection $p \in A$.

A useful fact that we will often use is that this last map also is injective for any real rank zero C*-algebra A. To see this one first notes that pAp is of real rank zero [BP], from which it follows that the map is injective by [Lin1]. Also, by [Lin1], the elements in $\mathcal{U}_0(pAp)$ with finite spectrum are dense, and hence (by injectivity) any element $u \in \mathcal{U}(pAp)$ with $K_1(u) = 0$ can be connected to 1 in $\mathcal{U}(pAp)$ by a rectifiable path of length at most $\pi + \varepsilon$.

Special classes of K_1-simple real rank zero C*-algebras are

1. Unital purely infinite simple C^*-algebras.

2. Unital real rank zero simple $A\mathbb{T}$-algebras.

3. Unital AF-algebras (not necessarily simple).

Recall from [Cun2] that a simple C*-algebra is called purely infinite if every non-zero hereditary sub-C*-algebra contains an infinite projection. Such a C*-algebra has real rank zero by [Zha1], and the properties 2 and 3 hold by [Cun2]. Recall from [Ell3] and [LR] that an $A\mathbb{T}$-algebra is an inductive limit of finite direct sums of algebras of the form $M_n \otimes C(\mathbb{T})$. The cut-down of a simple real rank zero $A\mathbb{T}$-algebra by a projection is again such an algebra, and since $K_1(A) \cong \mathcal{U}(A)/\mathcal{U}_0(A)$ for such an algebra A, the property 3 follows, while 2 is trivial. For an AF-algebra A, $K_1(A) = 0$, and the properties 1 to 3 are straightforward. In the last two cases (but not in the first) the property 2 is even true without the qualification "non-zero".

Our main result states, briefly, that if u and v are two unitaries in a K_1-simple real rank zero C*-algebra A that almost commute, and $[u] = 0$ in

$K_1(A)$, and a certain K_0-valued obstruction $\mathrm{Bott}(u, v)$ is zero, then u can be deformed to 1 along a path u_t in the unitary group of A of length less than a universal constant in such a way that u_t almost commutes with v along the path. The precise statement is given in Theorem 8.1. The obstruction $\mathrm{Bott}(u, v)$ is the Bott element associated to the two unitaries as defined by Loring in [Lor]. It is defined whenever $\|uv - vu\| \le \delta_0$, where δ_0 is a universal constant. It is defined as the K_0-class

$$\mathrm{Bott}(u, v) = \left[\chi_{[\frac{1}{2},\infty)}(e(u, v))\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right],$$

where $e(u, v)$ is a self-adjoint element of $M_2(A)$ of the form

$$e(u, v) = \begin{pmatrix} f(v) & h(v)\,u + g(v) \\ u^*h(v) + g(v) & 1 - f(v) \end{pmatrix},$$

where f, g, h are certain universal real-valued continuous functions on $\mathbb{T}$. The condition $\|uv - vu\| \le \delta_0$ ensures that $e(u, v)$ has a spectral gap near $\frac{1}{2}$. $\mathrm{Bott}(u, v)$ is a homotopy invariant within the class of pairs (u, v) where it is defined, and it has the properties

$$\mathrm{Bott}\left(\bigoplus_{j=1}^{n} u_j, \bigoplus_{j=1}^{n} v_j\right) = \sum_{i=1}^{n} \mathrm{Bott}(u_j, v_j)$$

whenever $\|u_j v_j - v_j u_j\| \le \delta_0$,

$$\mathrm{Bott}(u, v) = -\mathrm{Bott}(v, u),$$

and

$$\mathrm{Bott}(u_1 u_2 \ldots u_n, v) = \sum_{j=1}^{n} \mathrm{Bott}(u_j, v)$$

when $\|u_j v - v u_j\| \le \delta_0/n$; see [ElR].

The road to Theorem 8.1 is long and somewhat tortuous, and leads past a series of more special homotopy lemmas. In all of these the Bott element occurs in the guise of a certain more concrete obstruction to the deformation. Let us first give a rough description of this more concrete obstruction and then proceed to specifics: In defining the obstruction, one may instead of considering v consider the unital endomorphism λ defined by $\lambda(x) = vxv^*$, $x \in A$. One assumes that u has finite spectrum and that $\|\lambda(u) - u\|$ is small. Let t_1, t_2, t_3, t_4 be a sequence of points of $\mathbb{T}$ numbered in counter-clockwise order such that the distances $|t_i - t_{i-1}|$ are much greater than $\|\lambda(u) - u\|$. Let

Q and E denote the spectral projections of u corresponding to the half-open intervals $(t_1, t_3]$ and $(t_2, t_4]$, respectively. Since $\|\lambda(u) - u\|$ is much smaller than the distance between the t_i's, $\lambda(Q)$ approximately commutes with E, and hence $\lambda(Q)\,E$ is approximately a projection. Let $[\lambda(Q)\,E]_0$ denote the K_0-class of that projection. For a general unital endomorphism λ one now defines the isospectral obstruction $F(\lambda, u)$ as the image of $[\lambda(Q)\,E]_0 - [QE]_0$ in $K_0(A)/\mathrm{Im}(\lambda_* - 1)$, but if $\lambda = \mathrm{Ad}(v)$, then $\lambda_* = 1$ and the isospectral obstruction is just

$$\mathrm{Isospec}(u, v) = F(\lambda, u) = [\lambda(Q)\,E]_0 - [QE]_0\,.$$

We shall show that $\mathrm{Isospec}(u, v)$ is independent of the choice of the points t_1, t_2, t_3, t_4, and is also a homotopy invariant for pairs of almost commuting unitaries, in Theorem 4.1. We shall use the isospectral obstruction in proving Theorem 8.1, and only afterwards identify the isospectral obstruction with the Bott element, in Section 9.

Independently of all this we will, in Section 11, prove the so-called tail lemma, stating that if $\{u_t\}, \{v_t\}$, $t \in [0,1]$, are two continuous paths of unitaries in a unital purely infinite simple C*-algebra A such that $[u_t]_1 = [v_t]_1$ and all u_t, v_t have full spectrum, and $\varepsilon > 0$, then there is a continuous path $\{w_t\}$ of unitaries in A such that $\|w_t u_t w_t^* - v_t\| \le \varepsilon$ for all $t \in [0,1]$, and if $u_0 = v_0 = 1$, then $\{w_t\}$ can be chosen with $w_0 = 1$. This is used in [BEEK] and [ElR]. We emphasize that this is not true for a finite C*-algebra, not even for a one-point path.

Huaxin Lin has pointed out an alternative proof of the basic homotopy lemma, Theorem 8.1, in the purely infinite case. One first uses a theorem of Lin to approximate the unitaries u and v by exactly commuting unitaries. This is always possible in a purely infinite simple unital C*-algebra, by [Lin2]. Then one uses Lin's [Lin3] and Dadarlat's [Dad] classification of injective unital morphisms from $C(X)$ to A, where X is a closed subset of $\mathbb{T}^2$ and A is a purely infinite simple unital C*-algebra. In this case, $KL(C(X), A) = KK(C(X), A) = \mathrm{Hom}(K_*(C(X), K_*(A)))$. Hence this classification states, in the present case, that two such morphisms φ and ψ are approximately unitarily equivalent if, and only if, $K_*(\varphi) = K_*(\psi)$, where approximate unitary equivalence means that there is a sequence $\{u_n\}$ of unitaries in A such that

$$\|u_n \varphi(a) u_n^* - \psi(a)\| \to 0$$

for all $a \in C(X)$. Hence, in order to prove the basic homotopy lemma, it is enough to prove it for some exact morphism model with the correct K-theory data and the correct joint spectrum of u, v in $\mathbb{T}^2$. (By the K-theory data is meant the classes $K_1(u), K_1(v)$, and $\mathrm{Bott}(u,v) \in K_0(A)$.) A model for which the desired homotopy property holds within the class of exact morphisms is constructed in [LS, Lemma 4.4].

Our proof of Theorem 8.1 has the advantage of being valid beyond the purely infinite case. Our techniques (in particular, the concrete identification of the Bott element) may be of interest in themselves—and in any case are used in [BEEK] (and also in [ElR]).

We mention finally that the Basic Homotopy Lemma can be extended to have the following more symmetric form:

Theorem 1.1 (The Super Homotopy Lemma) *For any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: Let A be a K_1-simple real rank zero C^*-algebra, and let u_0, v_0, u_1, and v_1 be unitaries in A with the properties*

$$\begin{aligned}&[u_0]_1 = [u_1]_1, \quad [v_0]_1 = [v_1]_1,\\ &\|u_0v_0 - v_0u_0\| < \delta, \quad \|u_1v_1 - v_1u_1\| < \delta,\\ &\mathrm{Bott}(u_0, v_0) = \mathrm{Bott}(u_1, v_1).\end{aligned}$$

It follows that there exist continuous rectifiable paths $u(t)$, $v(t)$ of unitaries in A such that

$$\begin{aligned}&u(0) = u_0, \quad u(1) = u_1,\\ &v(0) = v_0, \quad v(1) = v_1,\\ &\|u(t)v(t) - v(t)u(t)\| < \varepsilon,\\ &\mathrm{Length}(u(t)) < 18\pi + \varepsilon,\\ &\mathrm{Length}(v(t)) < 18\pi + \varepsilon.\end{aligned}$$

This theorem follows from the Basic Homotopy Lemma by an argument which will be given in Section 12. It should be noted that, strictly speaking, the Basic Homotopy Lemma is not a special case of the Super Homotopy Lemma, not only because of the estimate on the length of the paths, but more importantly, because of the presence in the general case of two paths. (In [BEEK] and [ElR], it is essential to keep one unitary fixed.)

2 Some spectral theoretic lemmas

Lemma 2.1 *Let $\varepsilon > 0$, and let I_1, I_2 be two closed intervals in $\mathbb{T}$ (resp. in $[-1, 1]$) such that I_2 is contained in the interior of I_1. Then there exists $\delta > 0$*

with the following property : If u_1, u_2 are unitary operators (resp. self-adjoint operators of norm at most 1) on a Hilbert space such that

$$\|u_1 - u_2\| < \delta$$

then

$$\|P_1(I_1)\, P_2(I_2) - P_2(I_2)\| < \varepsilon$$

where $P_i(\cdot)$ is the projection valued measure on $\mathbb{T}$ (resp. $[-1,1]$) defined by u_i by spectral theory.

Proof. We prove the statements for unitaries. The proofs of the statements for self-adjoints are very similar.

Define a continuous function $g : \mathbb{T} \to [0,1]$ by

$$g(z) = \begin{cases} 1 \quad \text{when} \quad z \in I_2, \\ 0 \quad \text{when} \quad z \notin I_1, \\ \text{linear interpolation in } I_1 \backslash I_2. \end{cases}$$

By the Stone–Weierstrass Theorem, there exists a trigonometric polynomial

$$h(z) = \sum_{|n| \leq N} a_n z^n$$

such that

$$\|g - h\|_\infty < \frac{\varepsilon}{6}\,.$$

Set

$$\delta = \varepsilon \Big/ \Big(6 \sum_{|n| \leq N} |n|\, |a_n| \Big)$$

and choose u_1, u_2 as in the statement of the lemma. Then, by spectral theory,

$$\|g(u_i) - h(u_i)\| < \frac{\varepsilon}{6}$$

for $i = 1, 2$. Furthermore,

$$\begin{aligned} \|h(u_1) - h(u_2)\| &= \Big\| \sum_{|n| \leq N} a_n (u_1^n - u_2^n) \Big\| \\ &\leq \sum_{|n| \leq N} |a_n| \Big\| \sum_{k=0}^{n-1} u_1^k (u_1 - u_2)\, u_2^{n-k-1} \Big\| \\ &\leq \sum_{|n| \leq N} |a_n|\, n\delta < \varepsilon/6\,. \end{aligned}$$

Thus,

$$\begin{aligned}
&\|g(u_1) - g(u_2)\| \\
&\leq \|g(u_1) - h(u_1)\| + \|h(u_1) - h(u_2)\| + \|h(u_2) - g(u_2)\| \\
&< 3\varepsilon/6 = \varepsilon/2\,.
\end{aligned}$$

Since

$$\begin{aligned}
P_1(I_1)\, g(u_1) &= g(u_1), \\
g(u_2)\, P_2(I_2) &= P_2(I_2),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\|P_1(I_1)\, P_2(I_2) - P_2(I_2)\| \\
&= \|P_1(I_1)\, g(u_2)\, P_2(I_2) - g(u_2)\, P_2(I_2)\| \\
&\leq \|P_1(I_1)(g(u_2) - g(u_1))\, P_2(I_2)\| \\
&\quad + \|P_1(I_1)\, g(u_1)\, P_2(I_2) - g(u_2)\, P_2(I_2)\| \\
&< \varepsilon/2 + \|g(u_1)\, P_2(I_2) - g(u_2)\, P_2(I_2)\| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon\,.
\end{aligned}$$

□

Lemma 2.2 *There exists a positive function $\eta(\delta_1, \delta_2)$ defined for $\delta_1, \delta_2 > 0$ such that η is increasing in δ_1, decreasing in δ_2, and $\lim_{\delta_1 \to 0} \eta(\delta_1, \delta_2) = 0$, with the following property : If I_1, I_2 are any two closed intervals in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (resp. in $[-1, 1]$) such that the δ_2-neighbourhood of I_2 is contained in I_1, and if u_1, u_2 are unitary operators (resp. self-adjoint operators of norm at most 1) such that*

$$\|u_1 - u_2\| < \delta_1$$

then

$$\|P_1(I_1)\, P_2(I_2) - P_2(I_2)\| < \eta(\delta_1, \delta_2)$$

where $P_i(\cdot)$ is the projection valued measure on $\mathbb{T}$ (resp. $[-1, 1]$) defined by u_i by spectral theory.

Proof. By compactness (or elementary considerations), there exists a finite number of closed intervals $I_{1,n}, I_{2,n}$ with the property that the $\delta_2/2$-neighbourhood of $I_{2,n}$ is contained in $I_{1,n}$ for any n, and if I_1, I_2 is any pair of intervals satisfying the hypotheses of the lemma, then there exists an n such that

$$I_2 \subseteq I_{2,n} \subseteq I_{1,n} \subseteq I_1\,.$$

For given $\varepsilon > 0$, let $\delta(\varepsilon)$ denote the minimum of the finite number of δ's obtained in Lemma 2.1 by letting I_1, I_2 there range over $I_{1,n}, I_{2,n}$ for the finite number of n's. We may assume that $\delta(\varepsilon)$ is a decreasing function of δ, and inverting this we obtain the function $\varepsilon = \eta(\delta_1, \delta_2)$ referred to in the lemma. From

$$P_1(I_{1,n}) \leq P_1(I_1),$$

$$P_2(I_2) \leq P_2(I_{2,n}),$$

follows

$$\|P_1(I_1)\, P_2(I_2) - P_2(I_2)\| \leq \|P_1(I_{1,n})\, P_2(I_{2,n}) - P_2(I_{2,n})\| < \varepsilon\ . \qquad \square$$

Lemma 2.3 *Let e, p be projections in a unital* C*- *algebra A, and assume that*

$$\|ep - p\| \leq \varepsilon < \frac{1}{2}\,.$$

Then there exists a unitary element $u \in A$ such that

$$\|u - 1\| \leq 6\varepsilon\,,$$

$$upu^* \leq e\ ,$$

and

$$\|upu^* - p\| \leq 2\varepsilon\,.$$

Proof. This is a restatement of Lemma 2.1 of [Ell2]. □

Lemma 2.4 *Let e, p, q be projections in a unital* C*- *algebra A. Assume that*

$$q \leq e\,,$$

$$\|ep - p\| \leq \varepsilon < \tfrac{1}{6}\,,$$

and

$$\|pq - q\| \leq \varepsilon\,.$$

Then there exists a unitary element $u \in A$ such that

$$\|u - 1\| \leq 24\varepsilon\,,$$

$$q \leq upu^* \leq e\ ,$$

and

$$\|upu^* - p\| \leq 8\varepsilon\,.$$

Proof. By Lemma 2.3, there is a unitary $u_1 \in A$ such that

$$\|u_1 - 1\| \leq 6\varepsilon\,,$$

$$p_1 = u_1 p u_1^* \leq e\ ,$$

and

$$\|p_1 - p\| \leq 2\varepsilon\,.$$

Now p_1 and q are projections in the unital C*- algebra eAe, and

$$\|p_1 q - q\| \quad \leq \|(p_1 - p)\, q\| + \|pq - q\|$$

$$\leq 2\varepsilon + \varepsilon = 3\varepsilon\,.$$

Thus,

$$\|(e-q)(e-p_1) - (e-p_1)\| = \|qp_1 - q\| \leq 3\varepsilon\,.$$

Applying Lemma 2.3 again, we obtain a unitary $u_2 \in eAe$ such that

$$u_2(e-p_1)\, u_2^* \leq e - q\,,$$

$$\|u_2 - e\| \leq 6 \cdot 3\varepsilon = 18\varepsilon\,,$$

and

$$\|u_2(e-p_1)\, u_2^* - (e-p_1)\| \leq 2 \cdot 3\varepsilon = 6\varepsilon\,.$$

Hence,

$$q \leq u_2 p_1 u_2^*$$

and

$$\|u_2 p_1 u_2^* - p_1\| \leq 6\varepsilon\,.$$

Thus, with

$$u = (u_2 + (1-e))\, u_1\,,$$

u is a unitary element in A with

$$\|u - 1\| \quad \leq \|u_2 - e\| + \|u_1 - 1\|$$

$$\leq 18\varepsilon + 6\varepsilon = 24\varepsilon\,,$$

and

$$q \leq u_2 p_1 u_2^* = upu^* \leq e$$

and

$$\|upu^* - p\| = \|u_2 p_1 u_2^* - p\|$$

$$\leq \|u_2 p_1 u_2^* - p_1\| + \|p_1 - p\|$$

$$\leq 6\varepsilon + 2\varepsilon = 8\varepsilon\,.$$

□

Lemma 2.5 *Let A be a unital* C*- *algebra and let $p_0, \ldots, p_n$ and $e_0, \ldots, e_n$ be two families of mutually orthogonal projections in A with*

$$\sum_{k=0}^{n} p_k = 1 = \sum_{k=0}^{n} e_k .$$

Assume that

$$\|p_k - e_k\| < \varepsilon < 1 .$$

It follows that there exists a unitary element $u \in A$ such that

$$u p_k u^* = e_k$$

for $k = 0, \ldots, n$, and

$$\|u - 1\| \leq (n+1)2\varepsilon .$$

Thus, if $\varepsilon < 1/(n+1)$, there is a continuous path u_t in the unitary group of A of length at most $(n+1)7\varepsilon$ with $u_0 = 1, u_1 = u$.

Proof. By [Eff, Corollary A8.3] there are partial isometries u_k in A with

$$u_k u_k^* = e_k , \quad u_k^* u_k = p_k$$

and

$$\|u_k - p_k\| \leq 2\|e_k - p_k\| < 2\varepsilon .$$

Put

$$u = \sum_{k=0}^{n} u_k .$$

Then u has the desired properties. The last statement follows from spectral theory. We have $u = e^{ih}$ where $h = h^* \in A$ and

$$\left| e^{i\|h\|} - 1 \right| = \|u - 1\| < 2 ,$$

$$\|h\| \leq \pi \|u - 1\| < (n+1)2\pi\varepsilon < (n+1)7\varepsilon . \qquad \square$$

Lemma 2.6 *Let A be a K_1-simple real rank zero* C*-*algebra. If $p, q,\ e_1, e_2$ are projections in A with $p \lneqq e_i \lneqq q$ and $K_0(e_1) = K_0(e_2)$, then there exists a path $u_t \in \mathcal{U}((q-p)A(q-p))$ of length at most 3.15 such that $\mathrm{Ad}(u_t)$ connects $e_1 - p$ with $e_2 - p$.*

Proof. We may assume $p = 0$, $q = 1$ by replacing A by $(q-p)A(q-p)$, and then e_1, e_2 are unitarily equivalent by the property 2 among the properties of A listed in the Definition in Section 1. So there is a $w \in \mathcal{U}(A)$ with $e_2 = we_1w^*$.

By the property 3 there is a unitary $v \in A$ such that $v(1-e_1) = (1-e_1)v = 1 - e_1$ and $K_1(v) = -K_1(w)$. Replace w by wv. Then $we_1w^* = e_2$ and $K_1(w) = 0$. By [Lin1], w may be approximated by a unitary with finite spectrum, and so there exists a path u_t in the unitary group of A of length slightly larger than π such that $u_0 = 1$, $u_1 = w$. (Cf. Section 1.) □

Lemma 2.7 *Let q, p be projections in a C*-algebra A that approximately commute:*

$$\|qp - pq\| \leq \varepsilon < 1/6 .$$

Then there exist projections $q_1, p_1 \in A$ and a partial isometry $u \in A$ such that

$$\begin{aligned}
&q_1 \leq q , \\
&p_1 \leq p , \\
&u^*u = q_1 , \\
&uu^* = p_1 , \\
&\|q_1 - qpq\| \leq \frac{1 - \sqrt{1-4\varepsilon}}{2} \leq 2\varepsilon , \\
&\|p_1 - pqp\| \leq \frac{1 - \sqrt{1-4\varepsilon}}{2} \leq 2\varepsilon , \\
&\|p_1 - q_1\| \leq 6\varepsilon , \\
&\|u - q_1\| \leq 12\varepsilon .
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&(qpq)^2 - qpq \\
&= qpqpq - qpq \\
&= q(pq - qp)pq
\end{aligned}$$

and so

$$\|(qpq)^2 - qpq\| \leq \varepsilon < 1/4 .$$

Applying a continuous function which is 0 on $\left[0, \frac{1-\sqrt{1-4\varepsilon}}{2}\right]$ and 1 on $\left[\frac{1+\sqrt{1-4\varepsilon}}{2}, 1\right]$ to qpq we obtain a projection q_1 near qpq such that $q_1 \leq q$ and

$$\|q_1 - qpq\| \leq \frac{1 - \sqrt{1-4\varepsilon}}{2} \leq 2\varepsilon .$$

The projection p_1 is obtained by applying the same function to pqp and then

$$\|p_1 - pqp\| \leq 2\varepsilon \ .$$

Thus,

$$\begin{aligned} \|p_1 - q_1\| &\leq \|p_1 - pqp\| + \|pqp - qpq\| + \|qpq - q_1\| \\ &\leq 2\varepsilon + \|(pq - qp)p\| + \|q(qp - pq)\| + 2\varepsilon \\ &\leq 6\varepsilon \ . \end{aligned}$$

Since $6\varepsilon < 1$, the existence of the partial isometry u now follows from [Eff, Lemma A8.2]. □

Lemma 2.8 *Let $N \in \{2, 3, 4, \ldots\}$ and let $0 \leq \varepsilon < 1/12$. Let A be a unital C*-algebra, and let q_i, p_i be two families of projections in A indexed by $i \in \mathbb{Z}/N\mathbb{Z}$. Assume that*

$$\sum_i q_i = \sum_i p_i = 1 \, ,$$

and that q_i is approximately contained in $p_i + p_{i+1}$, i.e.,

$$\|(p_i + p_{i+1})q_i - q_i\| \leq \varepsilon \ ,$$

and that p_i is approximately contained in $q_{i-1} + q_i$, i.e.,

$$\|(q_{i-1} + q_i)p_i - p_i\| \leq \varepsilon \ .$$

Also assume that each pair of projections p_i, q_i approximately commutes, i.e.,

$$\|q_i p_i - p_i q_i\| \leq \varepsilon \ .$$

It follows that each q_i and each p_i has a decomposition as a sum of two projections,

$$\begin{aligned} q_i &= q_{i1} + q_{i2} \ , \\ p_i &= p_{i1} + p_{i2} \ , \end{aligned}$$

in such a way that

$$\begin{aligned} \|q_{i1} - p_{i1}\| &\leq 6\varepsilon \ , \\ \|q_{i2} - p_{i+1,2}\| &\leq 12\varepsilon \ . \end{aligned}$$

Furthermore, there exists a unitary operator $u \in A$ such that

$$\begin{aligned} u\, q_{i1} u^* &= p_{i1} \ , \\ u\, q_{i2} u^* &= p_{i+1,2} \ , \end{aligned}$$

for $i \in \mathbb{Z}/N\mathbb{Z}$, and

$$\|u - 1\| \leq 36N\varepsilon \, .$$

Proof. Using the approximate permutability of p_i and q_i, by Lemma 2.7 there exist for each $i \in \mathbb{Z}/N\mathbb{Z}$ projections q_{i1}, p_{i1} in A such that

$$\begin{aligned} q_{i1} &\leq q_i \ , \\ p_{i1} &\leq p_i \ , \\ \|q_{i1} - q_i p_i q_i\| &\leq 2\varepsilon \ , \\ \|p_{i1} - p_i q_i p_i\| &\leq 2\varepsilon \ , \\ \|p_{i1} - q_{i1}\| &\leq 6\varepsilon \ . \end{aligned}$$

Now put

$$\begin{aligned} q_{i2} &= q_i - q_{i1} \ , \\ p_{i2} &= p_i - p_{i1} \ . \end{aligned}$$

The elements q_{i2} and p_{i2} are projections, and

$$\begin{aligned} q_i &= q_{i1} + q_{i2} \ , \\ p_i &= p_{i1} + p_{i2} \ . \end{aligned}$$

Let us check the estimate on $p_{i+1,2} - q_{i2}$. We have

$$\begin{aligned} p_{i+1,2} - q_{i2} &= p_{i+1} - p_{i+1,1} - q_i + q_{i1} \\ &= p_{i+1} - p_{i+1}(q_i + q_{i+1})p_{i+1} + p_{i+1}(q_i + q_{i+1})p_{i+1} \\ &\quad -p_{i+1,1} + p_{i+1}q_{i+1}p_{i+1} - p_{i+1}q_{i+1}p_{i+1} \\ &\quad -q_i + q_i(p_i + p_{i+1})q_i - q_i(p_i + p_{i+1})q_i \\ &\quad +q_{i1} - q_i p_i q_i + q_i p_i q_i \ , \end{aligned}$$

and so

$$\begin{aligned} \|p_{1+1,2} - q_{i2}\| &\leq \|p_{i+1} - p_{i+1}(q_i + q_{i+1})p_{i+1}\| \\ &\quad +\|p_{i+1,1} - p_{i+1}q_{i+1}p_{i+1}\| \\ &\quad +\|q_i - q_i(p_i + p_{i+1})q_i\| \\ &\quad +\|q_{i1} - q_i p_i q_i\| \\ &\quad +\|p_{i+1}q_i p_{i+1} - q_i p_{i+1} q_i\| \\ &\leq \|p_{i+1}\| \, \|(q_i + q_{i+1})p_{i+1} - p_{i+1}\| \\ &\quad +\|p_{i+1,1} - p_{i+1}q_{i+1}p_{i+1}\| \\ &\quad +\|q_i\| \, \|(p_i + p_{i+1})q_i - q_i\| \\ &\quad +\|q_{i1} - q_i p_i q_i\| \\ &\quad +\|p_{i+1}q_i p_{i+1} - q_i p_{i+1} q_i\| \\ &\leq \varepsilon + 2\varepsilon + \varepsilon + 2\varepsilon + \|p_{i+1}q_i p_{i+1} - q_i p_{i+1} q_i\| \\ &= 6\varepsilon + \|p_{i+1}q_i p_{i+1} - q_i p_{i+1} q_i\| \ . \end{aligned}$$

To estimate the last term, note that as

$$\|(p_i + p_{i+1})q_i - q_i\| \leq \varepsilon \ ,$$

and thus

$$\|q_i(p_i + p_{i+1}) - q_i\| \leq \varepsilon \ ,$$

we have

$$\|[q_i, p_i + p_{i+1}]\| \leq 2\varepsilon \, .$$

But as $\|[q_i, p_i]\| \leq \varepsilon$, we obtain

$$\|q_i p_{i+1} - p_{i+1} q_i\| \leq 3\varepsilon.$$

Thus,

$$\begin{aligned}
&\|p_{i+1} q_i p_{i+1} - q_i p_{i+1} q_i\| \\
&\quad \leq \|p_{i+1} q_i p_{i+1} - q_i p_{i+1}\| \\
&\quad + \|q_i p_{i+1} - q_i p_{i+1} q_i\| \\
&\quad = \|[p_{i+1}, q_i] p_{i+1}\| + \|q_i [q_i, p_{i+1}]\| \\
&\quad \leq 3\varepsilon + 3\varepsilon = 6\varepsilon.
\end{aligned}$$

Thus we obtain the desired estimate

$$\|p_{i+1,2} - q_{i2}\| \leq 6\varepsilon + 6\varepsilon = 12\varepsilon.$$

Now, as $12\varepsilon < 1$, it follows from [Eff, Lemma A8.2] that there exist partial isometries u_{i1}, u_{i2} for $i \in \mathbb{Z}/N\mathbb{Z}$ with

$$\begin{aligned}
u_{i1}^* u_{i1} &= q_{i1}, \\
u_{i1} u_{i1}^* &= p_{i1}, \\
u_{i2}^* u_{i2} &= q_{i2}, \\
u_{i2} u_{i2}^* &= p_{i+1,2}, \\
\|u_{i1} - q_{i1}\| &\leq 12\varepsilon, \\
\|u_{i2} - q_{i2}\| &< 24\varepsilon.
\end{aligned}$$

Put

$$u = \sum_{i \in \mathbb{Z}/N\mathbb{Z}} (u_{i1} + u_{i2}).$$

Then u is unitary,

$$\begin{aligned}
u q_{i1} u^* &= p_{i1}, \\
u q_{i2} u^* &= p_{i+1,2},
\end{aligned}$$

and

$$\begin{aligned}\|u-1\| &\leq \sum_{i\in\mathbb{Z}/N\mathbb{Z}} \|u_{i1}-q_{i1}\| + \|u_{i2}-q_{i2}\| \\ &\leq (12+24)N\varepsilon = 36N\varepsilon\,.\end{aligned}$$

□

We will also need the following variant of Lemma 2.2.

Lemma 2.9 *There exists a positive function $\eta'(\delta_1,\delta_2)$ defined for $\delta_1,\delta_2>0$ such that η' is increasing in δ_1, decreasing in δ_2, and $\lim_{\delta_1\to 0}\eta'(\delta_1,\delta_2)=0$, with the following property: If I_1, I_2 are any two closed intervals in $\mathbb{T}$ such that the distance between any endpoint of I_1 and any endpoint of I_2 is at least δ_2, and u_1, u_2 are unitary operators such that*

$$\|u_1-u_2\|<\delta_1,$$

then

$$\|P_1(I_1)P_2(I_2)-P_2(I_2)P_1(I_1)\|<\eta'(\delta_1,\delta_2)$$

where $P_i(\cdot)$ is the projection valued measure on $\mathbb{T}$ defined by u_i by spectral theory.

Proof. The proof is similar to those of Lemmata 2.1 and 2.2. The details are left to the reader. □

3 Local connectedness of the unitaries with finite spectrum in a C*-algebra

If A is a unital C*-algebra, let $\mathcal{U}_F(A)$ denote the set of unitaries in A with finite spectrum. Clearly $\mathcal{U}_F(A)\subseteq\mathcal{U}_0(A)$. In this section we will prove that $\mathcal{U}_F(A)$ is locally connected in the following sense:

Proposition 3.1 *For any $\varepsilon>0$ there exists a $\delta>0$ with the following property: If A is a unital C*-algebra and $v_0, v_1\in\mathcal{U}_F(A)$ are elements with $\|v_0-v_1\|<\delta$, then there exists a continuous rectifiable path $t\in[0,1]\mapsto v_t\in\mathcal{U}_F(A)$ connecting v_0 and v_1, of length $<\varepsilon$.*

Proof. Given $\varepsilon > 0$ we first choose an $N \in \mathbb{N}$ such that

$$\frac{3\pi}{N} < \frac{\varepsilon}{2}$$

and then set

$$\varepsilon' = \frac{\varepsilon}{72N\pi},$$

so that

$$\frac{3\pi}{N} + \pi 36 N\varepsilon' < \varepsilon.$$

We may assume that the functions η and η' given by Lemmata 2.2 and 2.9 are the same (by taking the maximum of the two functions), and let us then choose $\delta > 0$ such that

$$\eta\left(\delta, \frac{\pi}{N}\right) \leq \varepsilon'.$$

Let us show that δ has the desired property.

Denote q_k by the spectral projection of v_0 corresponding to the half-open interval $\left[\frac{2k}{2N}, \frac{2(k+1)}{2N}\right)$ in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and by p_k the spectral projection of v_1 corresponding to the interval $\left[\frac{2k-1}{2N}, \frac{2k+1}{2N}\right)$.

Lemmas 2.2 and 2.9 then imply that

$$\begin{aligned} &\|(p_i + p_{i+1})q_i - q_i\| \leq \varepsilon', \\ &\|(q_{i-1} + q_i)p_i - p_i\| \leq \varepsilon', \\ &\|q_i p_i - p_i q_i\| \leq \varepsilon'. \end{aligned}$$

Hence by Lemma 2.8, we have decompositions into projections

$$\begin{aligned} q_i &= q_{i1} + q_{i2}, \\ p_i &= p_{i1} + p_{i2} \end{aligned}$$

and a unitary $u \in A$ such that

$$\begin{aligned} u q_{i1} u^* &= p_{i1}, \\ u q_{i2} u^* &= p_{i+1,2}, \end{aligned}$$

and

$$\|u - 1\| \leq 36 N\varepsilon' = \frac{\varepsilon}{2\pi}.$$

Consequently, u has the form

$$u = e^{ih}$$

where $h = h^* \in A$ and $\|h\| < \frac{\varepsilon}{4}$. We shall now construct the path v_t from v_0 to v_1 in five steps.

Step 1. Deform v_0 to the unitary $v_{\frac{1}{5}} = \sum_{k=1}^{N} e^{2\pi i \frac{2k+1}{2N}} q_k$ by keeping the spectral projections fixed and moving the eigenvalues in $\left[\frac{2k}{2N}, \frac{2(k+1)}{2N}\right)$ linearly to the one point $\frac{2k+1}{2N}$. Thus the path v_t between $t=0$ and $t=\frac{1}{5}$ has length at most $\frac{\pi}{N}$.

Step 2. Deform $v_{\frac{1}{5}}$ to the unitary

$$v_{\frac{2}{5}} = \sum_{k=1}^{N} \left(e^{2\pi i \frac{4k+1}{4N}} q_{k,1} + e^{2\pi i \frac{4k+3}{4N}} q_{k,2} \right)$$

by moving the one eigenvalue inside each projection $q_{k,j}$. This path has length $\frac{\pi}{2N}$.

Step 3. Deform $v_{\frac{2}{5}}$ to the unitary

$$v_{\frac{3}{5}} = \sum_{k=1}^{N} \left(e^{2\pi i \frac{4k+1}{4N}} p_{k,1} + e^{2\pi i \frac{4k+3}{4N}} p_{k+1,2} \right)$$

by applying $\mathrm{Ad}(e^{i5(t-\frac{2}{5})h})$ to $v_{\frac{2}{5}}$ and letting t run from $\frac{2}{5}$ to $\frac{3}{5}$. This path has length at most $2\|h\| \leq \frac{\varepsilon}{2}$.

Step 4. Deform $v_{\frac{3}{9}}$ to

$$v_{\frac{4}{5}} = \sum_{k=1}^{n} e^{2\pi i \frac{k}{N}} p_k$$

by moving the eigenvalue within each $p_{k,j}$. This path has length $\frac{\pi}{2N}$.

Step 5. Deform $v_{\frac{4}{5}}$ to v_1 by moving $e^{2\pi i \frac{k}{N}}$ to the appropriate eigenvalue on each eigenprojection of v_1. This path has length at most $\frac{\pi}{N}$.

Of course, Steps 4 and 5 are more or less the reversal of Steps 2 and 1, respectively.

In this way we have deformed v_0 to v_1 in $\mathcal{U}_F(A)$ along a path of total length at most $\frac{\pi}{N} + \frac{\pi}{2N} + \frac{\varepsilon}{2} + \frac{\pi}{2N} + \frac{\pi}{N} < \varepsilon$. $\square$

4 The isospectral obstruction of a unitary operator with respect to an endomorphism

This is defined by the following theorem. Note that if the C*-algebra A is not of real rank zero, the obstruction $F(\lambda, u)$ can nevertheless be defined on unitaries u of finite spectrum with $\|\lambda(u) - u\| < \varepsilon_0$, and still have properties 1 and 3. This follows from the proof.

Theorem 4.1 *There is a universal constant* $\varepsilon_0 > 0$ *and a function* $D : (0, \varepsilon_0] \to \mathbb{R}_+$ *such that* $\lim_{\varepsilon \to 0} D(\varepsilon) = 0$ *with the following property:*

If A *is a unital real rank zero* C**-algebra,* λ *is a unital endomorphism of* A *and* u *is a unitary in* $\mathcal{U}_0(A)$ *with*

$$\|\lambda(u) - u\| < \varepsilon_0$$

then there exists an element

$$F(\lambda, u) \in K_0(A)/\mathrm{Im}(\lambda_* - 1)$$

with the following properties:

1. $u \mapsto F(\lambda, u)$ *is invariant under homotopy of* u *among the allowed* u*'s in* $\mathcal{U}_0(A)$.

2. If $u_1, \ldots, u_n$ *are unitaries with* $\sum_{i=1}^{n} \|\lambda(u_i) - u_i\| < \varepsilon_0$ *then*

$$F(\lambda, u_1 \ldots u_n) = \sum_{i=1}^{n} F(\lambda, u_i).$$

3. If u *has finite spectrum,* $\varepsilon = \|\lambda(u) - u\| < \varepsilon_0$*, and* $\{t_1, \ldots, t_4\}$ *is a sequence of points of* $\mathbb{T}$ *in counter-clockwise order such that*

$$|t_i - t_{i-1}| > D(\varepsilon)$$

for $i = 1, \ldots, 4$ *(with* $t_0 = t_4$*), then, with* Q *(resp.* E*) the spectral projection of* u *corresponding to* $(t_1, t_3]$ *(resp.* $(t_2, t_4]$*),* $\lambda(Q)E$ *is close to a projection, and, with* $[\lambda(Q)E]_0$ *denoting the class of that projection in* $K_0(A)$*,*

$$F(\lambda, u) = q([\lambda(Q)E]_0 - [QE]_0) \ ,$$

where $q : K_0(A) \to K_0(A)/\mathrm{Im}(\lambda_* - 1)$ *is the quotient map.*

Proof. We will use the property 3 to define $F(\lambda, u)$ when u has finite spectrum. Using Lemma 2.9, it is clear that we can find ε_0 and $D(\varepsilon)$, with $D(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that $\lambda(Q)E$ in 3 is close to a projection. The distance to a projection will be small if $\eta'(\varepsilon, D(\varepsilon))$ is sufficiently small, where η' is the function given in Lemma 2.9. The K_0-class of this projection is independent of the choice of projection—for given $t_1, \ldots, t_4$—since two close projections are equivalent. (There is no need to ensure that $\eta'(\varepsilon, D(\varepsilon)) \to 0$ as $\varepsilon \to 0$, only that it be less than a certain universal constant.)

In order to verify that $F(\lambda, u)$ defined in this way has the desired properties, we must choose $D(\varepsilon)$ is such a way that $\eta(\varepsilon, D(\varepsilon)) \to 0$ as $\varepsilon \to 0$ where η is the function given by Lemma 2.2. This requirement is compatible with the preceding one.

Provided that $|t_i - t_{i-1}| > D(\varepsilon)$, a variation in t_1 does not affect QE at all and, by Lemma 2.2, affects $\lambda(Q)E$ almost not at all; hence the definition 3 of $F(\lambda, u)$ is invariant under variations in t_1. Similarly, the definition is invariant under variations in t_4, which affect E but not QE. If t_3 is increased from t_3 to $t_3 + \delta$, one obtains

$$F(t_1, t_2, t_3 + \delta, t_2) = F(t_1, t_2, t_3, t_4) + [\lambda(\Delta)]_0 - [\Delta]_0$$

where

$$F(t_1, t_2, t_3, t_4) = [\lambda(Q)E]_0 - [QE]_0$$

and Δ is the spectral projection of u corresponding to $(t_3, t_3 + \delta]$. Hence the class of F within $K_0(A)/\mathrm{Im}(\lambda_* - 1)$ is not affected by variations in t_3. A similar calculation shows that F (itself) does not depend on t_2—because the change in E coming from a change in t_2 is contained in Q, and approximately contained in $\lambda(Q)$.

We have proved that for a given u with finite spectrum, the definition 3 is independent of $t_1, \ldots, t_4$, provided that $|t_i - t_{i-1}| > D(\varepsilon)$.

Now, let u_t be a continuous path of unitaries with finite spectra and $\|\lambda(u_t) - u_t\| < \varepsilon_0$. Then $F(\lambda, u_t)$ is constant around any t such that the spectrum of u_t is disjoint from $t_1, \ldots, t_4$, which we fix. If one of the eigenvalues of u_t crosses one of the points t_1, t_2, t_3, t_4, then one of the projections E or Q will make a jump, but the two terms in

$$[\lambda(Q)E]_0 - [QE]_0$$

will then each jump by an equally large amount in $K_0(A)$ at the points t_1, t_2, and t_4, and an equally large amount up to $\mathrm{Im}(\lambda_* - 1)$ at t_3. Thus, $F(\lambda, u_t)$ is independent of t in any case with $\|\lambda(u) - u\| < \varepsilon_0$.

Now, a general $u \in \mathcal{U}_0(A)$ can be approximated arbitrarily well by a $v \in \mathcal{U}_0(A)$ with finite spectrum by [Lin1]. Do this, and define

$$F(\lambda, u) = F(\lambda, v).$$

To verify that this definition is consistent, we have to show that

$$F(\lambda, v_0) = F(\lambda, v_1)$$

whenever v_0, v_1 are two such finite spectrum approximations to u. But if v_0, v_1 are two such approximations, there is a path v_t of unitaries with finite spectrum connecting them such that v_t is a good approximation to u for all t, by the local connectedness of $\mathcal{U}_F(A)$ given by Proposition 3.1. Hence, by the previous argument, the definition of $F(\lambda, u)$ is independent of the approximant $v \in \mathcal{U}_F(A)$ if the approximation is good enough. Also, the property 1 in the statement of Theorem 4.1, that $u \mapsto F(\lambda, u)$ is homotopy invariant, is now clear. It remains to establish the property 2, and by induction it is enough to do this for $n = 2$. For this, note that

$$F(\lambda, u_1 u_2) = F\left(\lambda \otimes 1_2, \begin{pmatrix} u_1 u_2 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

where 1_2 is the identity automorphism of M_2. Now define

$$u(\theta) = \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for $0 \le \theta \le \pi/2$ and note that

$$u(0) = \begin{pmatrix} u_1 u_2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad u\left(\frac{\pi}{2}\right) = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

and that

$$\|\lambda \otimes 1_2(u(\theta)) - u(\theta))\| \le \|\lambda(u_1) - u_1\| + \|\lambda(u_2) - u_2\| < \varepsilon_0$$

for all θ, since $\lambda \otimes 1_2$ acts trivially on the rotation matrix. By the already established homotopy invariance it follows that

$$\begin{aligned} F(\lambda, u_1 u_2) &= F\left(\lambda \otimes 1_2, \begin{pmatrix} u_1 u_2 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= F\left(\lambda \otimes 1_2, \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}\right) \\ &= F(\lambda, u_1) + F(\lambda, u_2). \end{aligned}$$

□

5 The isospectral homotopy lemma in the case of a large spectral gap

In this section we will consider the case that u has finite spectrum with a large gap somewhere, or, equivalently, the case that u is replaced by a self-adjoint operator h with finite spectrum. In this case the obstruction discussed in Sections 1 and 4 is not present.

In order to state the lemma we need some notation. Let $h = h^* \in A$ be an element such that $0 \leq h \leq 1$, and assume that h has finite spectrum. Let $N \in \mathbb{N}$, and let p_k denote the spectral projection of h corresponding to the interval

$$\left[\frac{N-k+1}{N}, 1\right] \quad \text{for} \quad k = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, N+\frac{1}{2}, N+1 .$$

(We will need the half-integer values of k in the course of the proof of Lemma 5.1.) Then

$$0 = p_{\frac{1}{2}} \leq p_1 \leq p_{\frac{3}{2}} \leq \ldots \leq p_N \leq p_{N+\frac{1}{2}} \leq p_{N+1} = 1 ,$$

where the first inequality is strict if $1 \in \mathrm{Sp}\, h$, and all the others are strict if h has no spectral gap larger than or equal to $\frac{1}{2N}$ in [0,1]. Put

$$h(N) = \frac{1}{N} \sum_{k=1}^{N} p_k .$$

It follows from spectral theory that $h(N) \in \mathrm{C}^*(h) \subseteq A$,

$$\|h - h(N)\| \leq \frac{1}{N} ,$$

and the spectrum of $h(N)$ is contained in $\left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}\right\}$.

Lemma 5.1 *Given $\varepsilon > 0$ and $N \in \mathbb{N}$ with $\varepsilon < \frac{1}{16(N+1)}$ choose $\delta > 0$ such that*

$$\eta\left(\delta, \frac{1}{2N}\right) \leq \varepsilon$$

where $\eta(\cdot, \cdot)$ is the (version for self-adjoint elements of the) function defined in Lemma 2.2. Let A be a K_1-simple real rank zero C^*-algebra. Let $h = h^* \in A$ be an element with finite spectrum such that $0 \leq h \leq 1$. Let w be a unitary element in A such that

$$\|whw^* - h\| < \delta .$$

It follows that there exists a continuous path $t \mapsto u_t$ in $\mathcal{U}(A)$ of length at most $6.3 + 101(N+1)\,\varepsilon$ such that

$$u_0 = 1\,,$$

$$u_1 h(N) u_1^* = wh(N)\,w^*\,,$$

and

$$\|u_t h(N)\,u_t^* - h(N)\| < \frac{6}{N} + 64(N+1)\,\varepsilon + 2\delta$$

for all t.

Proof. For notational convenience, assume that N is even and that $1 \in \mathrm{Spec}\,(h)$. (The remedies to remove these conditions are left to the reader.)

Consider first the case that h has no spectral gap longer than or equal to $\frac{1}{2N}$ in $[0,1]$. We have the strict inequalities

$$0 = p_{\frac{1}{2}} \lneqq p_1 \lneqq p_{\frac{3}{2}} \lneqq p_2 \lneqq \cdots \lneqq p_{N+1} = 1$$

from the spectral gap assumption and $1 \in \mathrm{Sp}\,(h)$. Put

$$k(N) = wh(N)w^* = \frac{1}{N}\sum_{i=1}^{N} e_i$$

where

$$e_i = wp_i w^*\,,$$

and, consistently, put

$$k = whw^*\,.$$

As $\|k - h\| < \delta$, it follows from Lemma 2.2 that

$$\|p_k e_{k-\frac{1}{2}} - e_{k-\frac{1}{2}}\| \le \eta\left(\delta, \tfrac{1}{2N}\right) \le \varepsilon\,,$$

$$\|e_k p_{k-\frac{1}{2}} - p_{k-\frac{1}{2}}\| \le \eta\left(\delta, \tfrac{1}{2N}\right) \le \varepsilon$$

for $k = 1, \frac{3}{2}, 2, \ldots, N+1$. It follows from Lemma 2.4 that there exist unitaries u_k, v_k in A such that

$$e_{k-\frac{1}{2}} \le u_k p_k u_k^* \le e_{k+\frac{1}{2}}$$

and

$$p_{k-\frac{1}{2}} \le v_k e_k v_k^* \le p_{k+\frac{1}{2}}$$

for $k = 1, 2, \ldots, N$, and such that, with

$$p'_k = u_k p_k u_k^*$$

and

$$e'_k = v_k e_k v_k^*$$

we have

$$\|p_k - p'_k\| \leq 8\varepsilon$$

and

$$\|e_k - e'_k\| \leq 8\varepsilon\,.$$

Now, introduce

$$h'(N) = \frac{1}{N}(p_1 + e'_2 + p_3 + e'_4 + \ldots + e'_N)\,,$$

$$k'(N) = \frac{1}{N}(p'_1 + e_2 + p'_3 + e_4 + \ldots + e_N)\,.$$

We shall find a continuous path u_t of unitaries such that $\mathrm{Ad}(u_t)$ deforms

$$h(N) = \frac{1}{N}(p_1 + p_2 + p_3 + p_4 + \ldots + p_N)$$

into

$$h'(N) = \frac{1}{N}(p_1 + e'_2 + p_3 + e'_4 + \ldots + e'_N)$$

when $0 \leq t \leq \frac{1}{3}$, and further into

$$k'(N) = \frac{1}{N}(p'_1 + e_2 + p'_3 + e_4 + \ldots + e_N)$$

when $\frac{1}{3} \leq t \leq \frac{2}{3}$, and further into

$$k(N) = \frac{1}{N}(e_1 + e_2 + e_3 + e_4 + \ldots + e_N)$$

when $\frac{2}{3} \leq t \leq 1$. Let us start with $0 \leq t \leq \frac{1}{3}$. Since

$$e'_{2k} = v_{2k} e_{2k} v_{2k}^* = v_{2k} w p_{2k} (v_{2k} w)^*$$

we have $K_0(e'_{2k}) = K_0(p_{2k})$, and it follows from Lemma 2.6 that there exists a path in $\mathcal{U}((p_{2k+1} - p_{2k-1})A(p_{2k+1} - p_{2k-1}))$ of length at most 3.15 connecting

$p_{2k} - p_{2k-1}$ with $e'_{2k} - e_{2k-1}$ provided the differences $p_{2k+1} - p_{2k}$, $p_{2k} - p_{2k-1}$, $p_{2k+1} - e'_{2k}$, $e'_{2k} - p_{2k-1}$ are all non-zero. The first two are so, and

$$p_{2k-1} < p_{2k-\frac{1}{2}} \leq e'_{2k} \leq p_{2k+\frac{1}{2}} < p_{2k+1} \ ,$$

and so $p_{2k+1} - e'_{2k}$ and $e'_{2k} - p_{2k-1}$ are also non-zero. Hence Lemma 2.6 applies to give a path of unitaries in $(p_{2k+1} - p_{2k-1})A(p_{2k+1} - p_{2k-1})$ of length at most 3.15 connecting p_{2k} to e'_{2k}. Since the projections $p_{2k+1} - p_{2k-1}$ are mutually orthogonal for the permitted k's, we obtain by addition a unitary path of length at most 3.15 connecting $h(N)$ to $h'(N)$. (Note that it is at this point that the full weight of the assumption that A is a K_1-simple real rank zero C*-algebra is used.)

$k'(N)$ is connected to $k(N)$ in a similar manner, so it remains to connect $h'(N)$ to $k'(N)$. To this end we use the estimates

$$\|p'_{2k-1} - p_{2k-1}\| \leq 8\varepsilon \, ,$$

$$\|e_{2k} - e'_{2k}\| \leq 8\varepsilon$$

to deduce

$$\|(e_{2k} - p'_{2k-1}) - (e'_{2k} - p_{2k-1})\| \leq 16\varepsilon$$

and

$$\|(p'_{2k+1} - e_{2k}) - (p_{2k+1} - e'_{2k})\| \leq 16\varepsilon \, .$$

Thus, by Lemma 2.5, there exists a unitary $u \in A$ such that

$$e_{2k} - p'_{2k-1} = u(e'_{2k} - p_{2k-1})\, u^*$$

and

$$p'_{2k+1} - e_{2k} = u(p_{2k+1} - e'_{2k})\, u^* \, ,$$

and for $k = 1, \ldots, N/2$

$$\|u - 1\| \leq (N+1)2 \cdot 16\varepsilon \, .$$

Thus u can be deformed from 1 along a path of length at most

$$\pi(N+1) \cdot 32\varepsilon < 101(N+1)\,\varepsilon \, .$$

Thus, altogether, we obtain a path u_t of length

$$3.15 + 101(N+1)\,\varepsilon + 3.15$$

connecting $h(N)$ with $wh(N)w^*$. This proves the existence of a path u_t with the properties required in the statement of Lemma 5.1 except for the estimate of $\|u_t h(N)\, u_t^* - h(N)\|$. To verify this, note that by construction,

$$u_t h(N) u_t^* = \frac{1}{N}(p_1 + e_{2,t} + p_3 + e_{4,t} + \ldots)$$

when $0 \le t \le \frac{1}{3}$, where the elements $e_{2k,t}$ are projections such that

$$p_{2k-1} < e_{2k,t} < p_{2k+1}$$

for $k = 1, \ldots, N/2$, $0 \le t \le \frac{1}{3}$. Thus, if

$$h''(N) = \frac{2}{N}(p_1 + p_3 + \ldots + p_{N-1})$$

then $h''(N)$ commutes with $u_t h(N) u_t^*$ and by spectral theory

$$\|u_t h(N) u_t^* - h''(N)\| \le \frac{1}{N}\,.$$

Thus (first putting $t = 0$ above)

$$\|u_t h(N) u_t^* - h(N)\| \le \frac{2}{N}$$

for $0 \le t \le \frac{1}{3}$. If $\frac{1}{3} \le t \le \frac{2}{3}$, then

$$\|u_t - u_{\frac{1}{3}}\| \le 32(N+1)\,\varepsilon$$

and hence

$$\begin{aligned}\|u_t h(N) u_t^* - h(N)\| &\le \|u_t h(N) u_t^* - u_{\frac{1}{3}} h(N) u_{\frac{1}{3}}^*\| \\ &\quad + \|u_{\frac{1}{3}} h(N) u_{\frac{1}{3}}^* - h(N)\| \\ &\le 64(N+1)\,\varepsilon + \frac{2}{N}\end{aligned}$$

when $\frac{1}{3} \le t \le \frac{2}{3}$. Finally, if $\frac{2}{3} \le t \le 1$, then

$$\|u_t k(N) u_t^* - k(N)\| \le \frac{2}{N}$$

by the same reasoning as for $h(N)$ when $0 \le t \le \frac{1}{3}$, but as

$$\begin{aligned} \|h(N) - k(N)\| &\le \|h(N) - h\| + \|h - k\| + \|k - k(N)\| \\ &\le \frac{1}{N} + \delta + \frac{1}{N} = \frac{2}{N} + \delta \end{aligned}$$

we obtain

$$\|u_t h(N) u_t^* - h(N)\| \le \frac{2}{N} + 2\left(\frac{2}{N} + \delta\right) = \frac{6}{N} + 2\delta$$

for $\frac{2}{3} \le t \le 1$. Assembling the three estimates, we have

$$\|u_t h(N) u_t^* - h(N)\| \le \frac{6}{N} + 64(N+1)\,\varepsilon + 2\delta$$

for $0 \le t \le 1$.

Now let us remove the assumption that every spectral gap is of length longer than or equal to $\frac{1}{2N}$ in $[0,1]$. In this case, certain of the projections p_1, $p_{\frac{3}{2}}, \ldots$ may be equal. Suppose, for instance, that $p_k = p_{k+\frac{1}{2}}$, for some $k = 1, \frac{3}{2}, \ldots, N + \frac{1}{2}$. Then also $e_k = e_{k+\frac{1}{2}}$, and as

$$\|p_{k+\frac{1}{2}} e_k - e_k\| \le \varepsilon\ , \quad \|e_{k+\frac{1}{2}} p_k - p_k\| \le \varepsilon\ ,$$

we have

$$\|p_k e_k - e_k\| \le \varepsilon\ , \quad \|e_k p_k - p_k\| \le \varepsilon\ .$$

Hence by Lemma 2.4 with $q = e$,

$$\|p_k - e_k\| \le 8\varepsilon\ .$$

Of course, also, for such k,

$$\|p_{k+1} - e_{k+1}\| \le 8\varepsilon\ .$$

It follows that the proof may be completed as before, with $p'_k = e_k$ when k as above is an odd integer, and $e'_k = p_k$ when k as above is an even integer—and the applications of Lemma 2.6 which are no longer possible are in fact unnecessary, as the projections to be connected are equal. □

6 The isospectral homotopy lemma

In this section we will consider the case that u has spectrum without large gaps. (For completeness, we shall consider the other case, too—to verify that it reduces to Lemma 5.1 above.) If w is another unitary element such that $\|wu - uw\|$ is small, we define the isospectral obstruction of u with respect to w as

$$\operatorname{Isospec}(w, u) = F(\operatorname{Ad}(w), u) \in K_0(A)$$

where F was defined in Theorem 4.1. Since u has no large spectral gaps, $\operatorname{Isospec}(w, u)$ may be non-zero.

The following lemma could be proved by using triviality of the isospectral obstruction to split u into two parts, with spectra contained in two large disjoint arcs of $\mathbb{T}$ with union $\mathbb{T}$, thus reducing the problem to the large spectral gap situation covered by Lemma 5.1. However, we prefer a more local approach, dividing the circle into a large number of arcs, which nevertheless have great length compared to the distance between u and wuw^*.

Lemma 6.1 *For any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property:*

Let A be a K_1-simple real rank zero C-algebra, let u, w be unitaries in A, with $[u]_1 = 0$, and assume that*

$$\|wu - uw\| < \delta \; .$$

Assume that

$$\operatorname{Isospec}(w, u) = 0 \; .$$

It follows that there exists a unitary $u' \in A$, and a continuous path $t \mapsto z_t$ in the unitary group of A of length at most 7, such that

$$\begin{aligned} z_0 &= 1\,, \\ z_1 u' z_1^* &= w u' w^*, \\ \|u - u'\| &\leq \varepsilon, \end{aligned}$$

and

$$\|z_t u' z_t^* - u'\| \leq \varepsilon$$

for all $t \in [0, 1]$. If u has finite spectrum, we may take $u' \in C^*(u)$.

Proof. Since the set of unitaries with finite spectrum is dense in $\mathcal{U}_0(A)$ [Lin1], replacing ε by $\frac{\varepsilon}{2}$ we may assume at the outset that u has finite spectrum.

Now, choose $N \in \mathbb{N}$ so large that

$$\frac{4\pi}{2N} < \frac{\varepsilon}{2} ,$$

and set

$$\varepsilon' = \frac{\varepsilon}{2 \cdot 36 N\pi} .$$

Then choose $\delta > 0$ so small that

$$\eta \vee \eta' \left(\delta, \frac{\pi}{4N}\right) \leq \varepsilon' ,$$

where $\eta \vee \eta'$ denotes the maximum of the function η of Lemma 2.2 and the function η' of Lemma 2.9. If u has a spectral gap greater than or equal to $\frac{1}{4N}$ in length, say from $1 - \frac{1}{2N}$ to 1 in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, choose δ so that in addition $\|whw^* - h\|$ is small in the sense of Lemma 5.1, where $0 \leq h \leq 1 - \frac{1}{4N}$ and $e^{2\pi i h} = u$. The conclusion of Lemma 6.1 then follows from Lemma 5.1.

We may suppose, therefore, that u has no such spectral gap. Let q_i, $i \in \mathbb{Z}_{2N} := \mathbb{Z}/2N\mathbb{Z}$ denote the spectral projection of u corresponding to the spectral interval $\left[\frac{i}{2N}, \frac{i+1}{2N}\right)$ in $\mathbb{T}$, so that $q_i \neq 0$, and put $p_i = wq_iw^*$. Next, define

$$\begin{aligned} Q_i &= q_{2i} + q_{2i+1} , \qquad & i \in \mathbb{Z}_N , \\ P_i &= p_{2i-1} + p_{2i} , \qquad & i \in \mathbb{Z}_N . \end{aligned}$$

Thus, Q_i is the spectral projection of u corresponding to the spectral interval $\left[\frac{i}{N}, \frac{i+1}{N}\right)$ and P_i is the spectral projection of wuw^* corresponding to $\left[\frac{2i-1}{2N}, \frac{2i+1}{2N}\right)$. As $\|wuw^* - u\| < \delta$ it follows from the choice of δ and Lemmas 2.2 and 2.9 that

$$\begin{aligned} &\|(P_i + P_{i+1})Q_i - Q_i\| \leq \varepsilon' , \\ &\|(Q_{i-1} + Q_i)P_i - P_i\| \leq \varepsilon' , \\ &\|Q_iP_i - P_iQ_i\| \leq \varepsilon' . \end{aligned}$$

¿From these estimates and Lemma 2.8, it follows that each Q_i, P_i has a decomposition as a sum of two projections,

$$\begin{aligned} Q_i &= q'_{2i} + q'_{2i+1} , \\ P_i &= p'_{2i-1} + p'_{2i} , \end{aligned}$$

and there exists a unitary operator $v \in A$ such that

$$v q_i' v^* = p_i'$$

for all $i \in \mathbb{Z}_{2N}$, with

$$\|v - 1\| \leq 36N\varepsilon' = \frac{\varepsilon}{2\pi} \ .$$

We may suppose that $\frac{\varepsilon}{2\pi} < 2$. Furthermore, q_0' for example is constructed as an approximant of $P_0 Q_0 \approx Q_0 P_0$. But since $F(\mathrm{Ad}w, u) = 0$ by assumption, it follows (cf. Theorem 4.1) that

$$K_0(q_0') = K_0(q_0) \ .$$

Similarly, the vanishing of the isospectral obstruction implies that

$$K_0(q_i') = K_0(q_i)$$

and

$$K_0(p_i') = K_0(p_i)$$

for all $i \in \mathbb{Z}_{2N}$.

Note that, as well as assuming that each q_i is non-zero, we may also assume that each q_i' is non-zero. For example, q_0' which is approximately $P_0 Q_0$ is non-zero because not only is q_0 non-zero, but the subprojection p of q_0 corresponding to the right-hand half of the interval $\left[\frac{0}{2N}, \frac{1}{2N}\right)$ to which q_0 corresponds is non-zero, and because, furthermore, p is approximately contained in P_0, to within ε', by the choice of δ.

By Lemma 2.6, there exists a path of unitaries in $Q_i A Q_i$ of length at most 3.15 connecting q_{2i} to q_{2i}', and thereby q_{2i+1} to q_{2i+1}'. Adding these unitaries, we get a path of unitaries in A of length at most 3.15 connecting q_i to q_i' for each $i \in \mathbb{Z}_{2N}$. Similarly there is a path connecting p_i' to p_i. Finally, as $\|v - 1\| \leq \frac{\varepsilon}{2\pi} < 2$, v has the form $v = e^{ih}$ where $h = h^* \in A$ and $\|h\| \leq \frac{\varepsilon}{4}$, and the path $t \mapsto e^{ith}$, $t \in [0,1]$, of unitaries connects q_i' to p_i' for each i, and has length $\|h\| \leq \frac{\varepsilon}{4}$. Let z_t denote the composition of the three paths, so that

$$z_0 = 1, \qquad z_{\frac{1}{3}} q_i z_{\frac{1}{3}}^* = q_i' \ ,$$
$$z_{\frac{2}{3}} q_i z_{\frac{2}{3}}^* = p_i', \qquad z_1 q_i z_1^* = p_i \ .$$

The path z has total length less than $3.15 + \varepsilon/2 + 3.15 \leq 7$.

Now, put

$$u' = \sum_{j \in \mathbb{Z}_{2N}} e^{2\pi i \frac{2j+1}{4N}} q_j \ .$$

Then

$$\|u - u'\| \leq \frac{2\pi}{4N} \leq \frac{\varepsilon}{8} \leq \varepsilon$$

and

$$\begin{aligned} wu'w^* &= \sum_{j\in\mathbb{Z}_{2N}} e^{2\pi i \frac{2j+1}{4N}} p_j \\ &= z_1 u' z_1^* . \end{aligned}$$

For $0 \leq t \leq \frac{1}{3}$ we have

$$z_t Q_i z_t^* = Q_i \ .$$

Thus, putting

$$u'' = \sum_{j\in\mathbb{Z}_{2N}} e^{2\pi i \frac{2j+1}{2N}} Q_j$$

we have $\|u'' - u'\| \leq \frac{2\pi}{4N} \leq \frac{\varepsilon}{8}$, and

$$z_t u'' z_t^* = u''$$

for $0 \leq t \leq \frac{1}{3}$, whence

$$\|z_t u' z_t^* - u'\| \leq 2\frac{\varepsilon}{8} = \frac{\varepsilon}{4}$$

for those t. Now, since the length of the path $t \mapsto z_t$ between $t = \frac{1}{3}$ and $t = \frac{2}{3}$ is at most $\frac{\varepsilon}{4}$, we obtain

$$\|z_t u' z_t^* - u'\| \leq \frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \frac{3}{4}\varepsilon$$

Finally, putting

$$u''' = \sum_{j\in\mathbb{Z}_{2N}} e^{2\pi i \frac{j}{N}} P_j$$

and

$$u'''' = \sum_{j\in\mathbb{Z}_{2N}} e^{2\pi i \frac{j}{N}} (q_{2j-1} + q_{2j}) \ ,$$

we have

$$z_t u'''' z_t^* = u'''$$

for $\frac{2}{3} \leq t \leq 1$, and

$$\|u''' - u'\| \leq \frac{\varepsilon}{8} , \qquad \|u'''' - u'\| < \frac{\varepsilon}{8} \ .$$

Hence,

$$\|z_t u' z_t^* - u'\| \leq \frac{3}{4}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon$$

for $\frac{2}{3} \leq t \leq 1$. □

7 The basic homotopy lemma in the case of trivial K_1

This lemma is

Lemma 7.1 *For any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property: If A is a K_1-simple real rank zero* C*-*algebra and u, v are unitaries in A with*

$$[u]_1 = [v]_1 = 0 \ ,$$
$$\|uv - vu\| < \delta \ ,$$
$$\mathrm{Isospec}(v, u) = 0 \ ,$$

then there exists a continuous rectifiable path u_t of unitaries in A with

$$u_0 = 1 , \qquad u_1 = u \ ,$$
$$\|[v, u_t]\| \leq \varepsilon \ ,$$

and

$$\mathrm{Length}(u_t) \leq 4\pi + 1 \ .$$

An important property of K_1-simple real rank zero C*-algebras needed in the proof is the approximate decomposition of the following lemma.

Lemma 7.2 *If A is a K_1-simple real rank zero* C*-*algebra, and u is a partial unitary in A with full spectrum (or approximately full spectrum), and $k \in K_1(A)$, then u approximately has a decomposition as an orthogonal sum $u_1 + u_2$ of partial unitaries such that $[u_1]_1 = k$.*

Proof of Lemma 7.1 from Lemma 7.2. By the isospectral homotopy lemma (Lemma 6.1) there exists, for sufficiently small, but universal, δ, a finite spectrum approximation v' to v and a path $t \mapsto z_t$ in the unitary group of A of length at most 7 such that

$$z_0 = 1 \ ,$$
$$z_1 v' z_1^* = u v' u^* \ ,$$
$$\|v - v'\| \leq \frac{\varepsilon}{3} \ ,$$
$$\|z_t v' z_t^* - v'\| \leq \frac{\varepsilon}{3}$$

for all $t \in [0, 1]$. In particular, if

$$w = z_1^* u$$

then w commutes with v', and since v' has finite spectrum, w has a decomposition

$$w = \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{pmatrix}$$

over the spectral projections of v'. We know that

$$[w_1]_1 + \ldots + [w_n]_1 = [w]_1 = [z_1^*]_1 + [u]_1 = 0 + 0 = 0 \; .$$

If now each w_i had K_1-class 0, then u could be deformed to 1 in the approximate commutant of v as follows:

$$u \longrightarrow z_t^* u \longrightarrow z_1^* u = w \longrightarrow 1$$

where w is deformed along a path of length $\pi + \varepsilon$ in the exact commutant of v'. This always works if $K_1(A) = 0$, for example if A is an AF-algebra, so we may replace $4\pi + 1$ by $3\pi + 1$ in that case.

If, however, not every w_i has trivial K_1-class, this argument has to be modified as follows: First, if some of the w_i have trivial K_1-class, their spectra may not be the whole circle, but by deforming each w_i along a path of length $\pi + \varepsilon$ inside the appropriate spectral projection of v', we may assume that each w_i has full spectrum. Such a deformation is possible by Lemma 7.3, below, and the condition 3 in the definition of K_1-simple real rank zero C*-algebras. The total deformation of w then takes place along a path of unitaries commuting with v' of length at most $\pi + \varepsilon$.

After this we modify w along a short path as follows: Let e_i denote the spectral projection of v' where w_i lives, and define a decomposition $e_i = e_{i1} + e_{i2}$ of e_i and partial unitaries w_{i1}, w_{i2} with $w_{ij}^* w_{ij} = w_{ij} w_{ij}^* = e_{ij}$ as follows: Put $e_{12} = e_1$, $e_{11} = 0$, $w_{12} = w_1$. When the pair e_{i2}, w_{i2} has been constructed, proceed as follows: Let $w_{i+1,1} + w_{i+1,2}$ be an approximate decomposition of w_{i+1} as in Lemma 7.2, with $[w_{i+1,1}]_1 = -[w_{i,2}]_1$. Proceed in this way until $i+1 = n$; then, as $[w]_1 = 0$, we necessarily have $[w_{n,2}]_1 = 0$. After this small deformation, each $w_{i,2} + w_{i+1,1}$ is a partial unitary of trivial K_1-class, and can therefore be deformed to $e_{i,2} + e_{i+1,1}$ along a path of length at most $\pi + \varepsilon$. But since $e_{i,2} + e_{i+1,1} \leq e_i + e_{i+1}$, all the unitaries along this path will approximately commute with v', and thus with v. Summing up, we can deform u to 1 along a path of length $2\pi + \pi + \pi + \varepsilon$ in the approximate commutant of v. This proves Lemma 7.1 apart from

Proof of Lemma 7.2. If $K_1(A) = 0$, Lemma 7.2 is trivial, so we may assume $K_1(A) \neq 0$. Then, by hypothesis, $D(A \otimes K)$ is simple. Thus the set of nonzero elements in $D(A)$ is downward directed. Let now u be the (partial) unitary given in the statement of the lemma. Since u has (at least approximately) full spectrum, we may approximate u by another partial unitary v with spectrum the nth roots of unity,

$$v = \sum_{k=0}^{n-1} e^{2\pi i \frac{k}{n}} e_k$$

where the projections $e_k \neq 0$. By downward directedness of the non-zero part of $D(A)$, choose a non-zero projection p which is equivalent to part of each of the n eigenprojections e_k of v, i.e., such that there exists $v_k \in A$ with

$$v_k v_k^* \leq e_k \,, \qquad v_k^* v_k = p.$$

By simplicity (of $D(A \otimes K)$) and real rank zero, there is a projection $q \in A$ such that $q \neq 0$, $q \leq p$ and $p - q \neq 0$. Let $k \in K_1(A)$ be given as in the statement of the lemma, and choose, using the property 3 in the definition, a partial unitary w_1 inside q with $[w_1]_1 = k$ and a partial unitary w_2 inside $p - q$ with $[w_2]_1 = -k$. Then $w = w_1 + w_2$ is a partial unitary inside p with $[w_1] = 0$. Thus, by [Lin1] again, we may approximate w by a partial unitary w_3 with the same support p as w and spectrum contained in the nth roots of 1,

$$w_3 = \sum_{k=0}^{n-1} e^{2\pi i \frac{k}{n}} f_k$$

Define a partial isometry V between p and part of $\sum_k e_k$ by

$$V = \sum_{k=0}^{n-1} v_k f_k$$

and put

$$\widetilde{w}_i = V w_i V^*$$

for $i = 1, 2, 3$. Then $K_1(\widetilde{w}_1) = k$, $K_1(\widetilde{w}_2) = -k$ and $\widetilde{w}_3 \approx \widetilde{w}_1 + \widetilde{w}_2$. Furthermore, if $E = VV^*$, then

$$Ev = vE = \widetilde{w}_3$$

since $v_k f_k v_k^* \leq v_k p v_k^* \leq e_k$. Hence $\widetilde{w}_1$, which is an approximate direct summand of $\widetilde{w}_3$, is an approximate direct summand of v and thus of u. □

Lemma 7.3 *Let A be a unital* C*-*algebra containing unitary v with $[v]_1 \neq 0$. Then A contains a unitary u with full spectrum and $[u]_1 = 0$.*

Proof. As $[v]_1 \neq 0$, v has full spectrum. Let $f: \mathbb{T} \to \mathbb{T}$ be a continuous function of winding number 0 such that f is surjective, for example $f(z) = \exp(\pi(z - \overline{z})/2)$. Then $u = f(v)$ does the job. □

8 The basic homotopy lemma

We are now ready to prove the main result of this paper.

Theorem 8.1 *For any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property: If A is a K_1-simple real rank zero C^*-algebra, and u, v are unitaries in A with*

$$\begin{aligned} &[v]_1 = 0 \ , \\ &\|vu - uv\| < \delta \ , \\ &\mathrm{Isospec}(u, v) = 0 \ , \end{aligned}$$

then there exists a rectifiable path v_t of unitaries in A with

$$\begin{aligned} &v_0 = v \, , \quad v_1 = 1 \ , \\ &\|[u, v_t]\| < \varepsilon \ , \\ &\mathrm{Length}(v_t) \leq 5\pi + 1 \ . \end{aligned}$$

Remark. If $K_1(A) = 0$, then this is also true with the estimate $\mathrm{Length}(v_t) \leq 3\pi + 1$ because of Lemma 7.1 (and its proof).

Proof. By Lemma 7.1, we may assume $[u]_1 \neq 0$. In particular, $K_1(A) \neq 0$, and so $D(A \otimes K)$ is simple by definition. By Lemma 8.2, to follow, for any $\varepsilon' > 0$ there exists $\delta' > 0$ such that with u and v as above—with $\delta > 0$ to be specified—, there is a projection $E \in A$ and unitaries $u_1, v_1 \in A$ with

$$\begin{aligned} &\|u - u_1\| < \varepsilon' \, , \qquad \|v - v_1\| < \varepsilon' \, , \\ &[u_1, E] = 0, \qquad [v_1, E] = 0 \, , \\ &u_1 v_1 E = v_1 u_1 E \, , \\ &K_1(u_1 E) = 0 \qquad \text{and} \quad \mathrm{Spec}(u_1 E) + (0, \varepsilon') = \mathbb{T} \, , \\ &\mathrm{Spec}(v_1) \quad \text{is finite.} \end{aligned}$$

Next, applying Lemma 7.2 to $u_1 E$, with ε' small relative to a fixed $\delta > 0$ (to be specified), we find a projection $E_2 \leq E$ and an approximant u_2 to u_1, commuting with E_2, such that $u_2(1-E) = u_1(1-E)$ and $K_1(u_2E_2) = -[u]_1$. Putting $E_1 = 1 - E$, $E_3 = E - E_2$, we have

$$\begin{aligned}
&\|u - u_1\| < \delta, \quad \|u - u_2\| < \delta, \quad \|v - v_1\| < \delta \ , \\
&E_1 + E_2 + E_3 = 1 \ , \\
&[u_2, E_i] = 0 \ , \\
&[v_1, E_1] = 0, \quad [u_1, E_1] = 0 \ , \\
&[u_2E_1]_1 = [u]_1 \ , \\
&[u_2E_2]_1 = -[u]_1 \ , \\
&v_1u_1(E_2 + E_3) = u_1v_1(E_2 + E_3) \ , \\
&\mathrm{Spec}(u_1(E_2 + E_3)) \ \text{ is finite} \ , \\
&\mathrm{Spec}(v_1(E_2 + E_3)) \ \text{ is finite.}
\end{aligned}$$

It follows from the last three properties that $\mathrm{Isospec}(u_1(E_2 + E_3), v_1(E_2 + E_3)) = 0$. Hence $\mathrm{Isospec}(u_1E_1,\ v_1E_1) = 0$, and since u_1, u_2 are norm close one has that $\mathrm{Isospec}(u_2E_1 + u_2E_2, v_1E_1 + E_2) = 0$. Thus by Lemma 7.1 applied to the pair $u_2E_1 + u_2E_2$, $v_1E_1 + E_2$, for a given $\varepsilon > 0$, choosing $\delta > 0$ sufficiently small one obtains a path $w_t^{(1)}$ of unitaries such that

$$\begin{aligned}
&w_t^{(1)}E_3 = E_3 \ , \\
&w_0^{(1)} = v_1E_1 + E_2 + E_3 \ , \\
&w_1^{(1)} = 1 \ , \\
&\left\| \left[u_2E_1 + u_2E_2 + E_3, w_t^{(1)}\right] \right\| < \frac{\varepsilon}{3} \ , \\
&\mathrm{Length}(w_t^{(1)}) < 4\pi + \frac{1}{2} \ .
\end{aligned}$$

It follows from the finiteness of $\mathrm{Spec}(v_1(E_2+E_3))$ and the equation $v_1u_1(E_2+E_3) = u_1v_1(E_2 + E_3)$ that there is a path $w_t^{(2)}$ of unitaries such that

$$\begin{aligned}
&w_t^{(2)}E_1 = E_1 \ , \\
&w_0^{(2)} = v_1(E_2 + E_3) + E_1 \ , \\
&w_1^{(2)} = 1 \ , \\
&[u_1(E_2 + E_3) + E_1, w_t^{(2)}] = 0 \ , \\
&\mathrm{Length}(w_t^{(2)}) \le \pi \ .
\end{aligned}$$

Hence $w_t = w_t^{(1)}w_t^{(2)}$ satisfies

$$\|[u, w_t]\| = \|[u, w_t^{(1)}]w_t^{(2)} + w_t^{(1)}[u, w_t^{(2)}]\| \le \frac{\varepsilon}{3} + 2\delta \ ,$$

$$
\begin{aligned}
&w_0 = v_1\ , \\
&w_1 = 1\ , \\
&\text{Length}(w) < 5\pi + \frac{1}{2}\ .
\end{aligned}
$$

Since $\|v - v_1\| < \delta$ we can find a path connecting v and v_1 of length at most $\pi\delta/2$, and approximately commuting with u to within 2δ. Choose δ in such a way that also $\delta < \frac{\varepsilon}{6}$ and $\pi\delta < 1$. $\square$

Lemma 8.2 *For any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property: Let A be a real rank zero* C*-*algebra. For any two unitaries $u, v \in A$ with* $\mathrm{Spec}(u) = \mathbb{T}$*, $[v]_1 = 0$ and $\|[u,v]\| < \delta$, there exist unitaries $u_1, v_1 \in A$ and a projection $E \in A$ such that*

$$
\begin{aligned}
&\|u - u_1\| < \varepsilon\,, \quad \|v - v_1\| < \varepsilon\,, \\
&[u_1, E] = 0\,, \quad [v_1, E] = 0\,, \\
&u_1 v_1 E = v_1 u_1 E\,, \\
&\mathrm{Spec}(u_1 E) \textit{ is finite and } \mathrm{Spec}(u_1 E) + (0, \varepsilon) = \mathbb{T}\,, \\
&\mathrm{Spec}(v_1) \textit{ is finite.}
\end{aligned}
$$

Proof. By [Lin1], we may suppose that $\mathrm{Spec}(v)$ is finite.

Choose a sufficiently large $n \in \mathbb{N}$. Let a and b be continuous functions on $\mathbb{T}$ such that $0 \le a \le 1$, $0 \le b \le 1$, and

$$
\begin{aligned}
&\sum_{k=0}^{n^2-1} a(e^{2\pi i k/n^2}\cdot) = 1\,, \\
&\mathrm{supp}(b) \subset (e^{-2\pi i/n^2}, e^{2\pi i/n^2}), \\
&ab = a\,.
\end{aligned}
$$

Note that $a(1) = 1$. Since the self-adjoint element

$$
\sum_{k=0}^{n^2-1} a(e^{2\pi i l/n}u)a(e^{2\pi i k/n^2}v)a(e^{2\pi i l/n}u) = a(e^{2\pi i l/n}u)^2
$$

has norm 1 for $l = 0, 1, \ldots, n-1$, there is a k_l such that the element

$$
x_l = a(e^{2\pi i l/n}u)a(e^{2\pi i k_l/n^2}v)a(e^{2\pi i l/n}u)
$$

has norm at least $1/n^2$. Set

$$
e_l = b(e^{2\pi i l/n}u)b(e^{2\pi i k_l/n^2}v)b(e^{2\pi i l/n}u)\,.
$$

Then

$$\|e_l x_l - x_l\| < \varepsilon_0$$

where

$$\varepsilon_0 = \sup_{\lambda,\mu\in\mathbb{T}} \|[b(\lambda v), a(\mu u)]\|\,.$$

Note that ε_0 is dominated by the value of a function at $\delta = \|[v,u]\|$ which converges to zero as $\delta \downarrow 0$.

Since

$$\varepsilon_0 \geq \|x_l(1-e_l)x_l\| \geq (1-\|e_l\|)\|x_l\|^2 \geq \frac{1}{n^2}(1-\|e_l\|)$$

it follows that

$$\|e_l\| \geq 1 - n^2\varepsilon_0\,.$$

We shall assume that $n^2\varepsilon_0$ is sufficiently close to zero.

Denote by E_l the spectral projection of v corresponding to the interval $(e^{-2\pi i(k_l/n+1/n^2)},\ e^{-2\pi i(k_l/n-1/n^2)})$. Then $\|b(e^{2\pi il/n}u)E_l b(e^{2\pi il/n}u)\| \geq 1 - n^2\varepsilon_0$. Denote by B_l the hereditary C*-subalgebra of A generated by all $h(b(e^{2\pi il/n}u)E_l b(e^{2\pi il/n}u))$ with h satisfying

$$\operatorname{supp} h \subset [1-(n^2+1)\varepsilon_0, \infty)\,.$$

Since B_l is a non-zero C*-subalgebra of a C*-algebra of real rank zero, by [BP] there is a non-zero projection $p_k \in B_l$. Then

$$p_k b(e^{2\pi il/n}u)E_l b(e^{2\pi il/n}u)p_l \geq (1-(n^2+1)\varepsilon_0)p_l\,,$$

and so

$$\begin{aligned}
\|p_l - b(e^{2\pi il/n}u)p_l\|^2 &\leq \|p_l(1-2b(e^{2\pi il/n}u) + b(e^{2\pi il/n}u)^2)p_l\| \\
&\leq \|p_l(1-b(e^{2\pi il/n}u)^2 p_l\| \leq (n^2+1)\varepsilon_0\,, \\
\|p_l - E_l p_l\|^2 &= \|p_l - p_l E_l p_l\| \\
&\leq \|p_l - p_l b(e^{2\pi il/n}u)E_l b(e^{2\pi il/n}u)p_l\| \\
&\quad +2\|p_l - b(e^{2\pi il/n}u)p_l\| \\
&\leq (n^2+1)\varepsilon_0 + 2\sqrt{(n^2+1)\varepsilon_0} = \varepsilon_1\,.
\end{aligned}$$

Let f be a continuous function $\mathbb{T}$ onto $\mathbb{T}$ such that

$$f|[e^{2\pi i(l/n-1/n^2)}\ ,\ e^{2\pi i(l/n+1/n^2)}] = e^{2\pi il/n}$$

for $l = 0, 1, \ldots, n-1$. With a suitable choice of f, $u_1 = f(u)$ satisfies $\|u - u_1\| < 2\pi/n^2$. Note that $u_1 p_l = e^{-2\pi i l/n} p_l$ and $\{p_l; l = 0, \ldots, n-1\}$ are mutually orthogonal. Set $E = \sum_{l=0}^{n-1} p_l$. Then

$$u_1 E = E u_1 = \sum_{l=0}^{n-1} e^{-2\pi i l/n} p_l \,.$$

Set $I = \bigcup_{l=0}^{n-1} (e^{-2\pi i(k_l+1)/n^2}, e^{-2\pi i(k_l-1)/n^2})$. Define a function g on $\mathbb{T}$ as follows: $g(\lambda) = \lambda$ for $\lambda \notin I$ and $g(\lambda) = e^{\pi i(\alpha+\beta)}$ on a connected component $(e^{2\pi i\alpha}, e^{2\pi i\beta})$ of I. Then $v' = g(u)$ satisfies $\|v - v'\| \le 2\pi(n+1)/n^2$, since the length of a connected component of I is less than or equal to $2\pi(n+1)/n^2$.

Let $\{I_i\}$ denote the set of connected components of I. Denote by F_i the spectral projection of v corresponding to I_i. Set

$$G_i = \sum_{e^{-2\pi i l/n} \in I_i} p_l$$

Then it follows that

$$\|G_i - F_i G_i\| \le n\sqrt{\varepsilon_1} \,.$$

Denote by U_i the partial isometry obtained from the polar decomposition of $F_i G_i$. Then

$$\begin{aligned} \|U_i - G_i\| &= \|F_i G_i (G_i F_i G_i)^{-1/2} - G_i\| \\ &\le n\sqrt{\varepsilon_1} + \frac{1}{\sqrt{1 - n\sqrt{\varepsilon_1}}} - 1 \\ &< 2n\sqrt{\varepsilon_1} \end{aligned}$$

(if $n\sqrt{\varepsilon_1}$ is sufficiently small). Set $V_0 = \sum U_i$. Since V_0 is a partial isometry close to the initial projection $\sum G_i = \sum_{l=0}^{n-1} p_l$, we have a partial isometry V_1 with initial projection $1 - V_0^* V_0$ and final projection $1 - VV^*$ such that V_1 is close to $1 - V_0^* V_0$ up to a constant multiple of $2n^2\sqrt{\varepsilon_1}$. Set $V = V_0 + V_1$ and set

$$v_1 = V^* v' V$$

Then $\|v - v_1\| \le 2\pi(n+1)/n^2 + Cn^2\sqrt{\varepsilon_1}$ for some constant C, and $\mathrm{Spec}(v_1)$ is finite.

If $e^{-2\pi i l/n} \in I_i = (e^{2\pi i\alpha}, e^{2\pi i\beta})$, then

$$v_1 p_l = e^{\pi i(\alpha+\beta)} p_l \,.$$

Thus it follows that $v_1E = Ev_1$ and

$$u_1v_1E = v_1u_1E\,.$$

On taking $n \in \mathbb{N}$ sufficiently large and then taking $\delta > 0$ sufficiently small, the conclusion follows. □

9 Identification of the isospectral obstruction with the Bott class

If A is a unital C*-algebra, and u, v are unitaries in A with $\|vu - uv\| < \varepsilon_0$, where ε_0 is a certain universal constant, we described the Bott class $\mathrm{Bott}(v, u) \in K_0(A)$ in Section 1. If in addition A has real rank zero and $u \in \mathcal{U}_0(A)$, or if u simply has finite spectrum, we have defined the isospectral obstruction of u with respect to v as

$$\mathrm{Isospec}(v, u) = F(\mathrm{Ad}(v), u) \in K_0(A)\ ,$$

where F is as defined in Theorem 4.1. We will now prove that these two K_0 valued invariants coincide whenever the basic homotopy lemma, Theorem 8.1 (or even just Lemma 7.1), is stably valid, i.e., valid for any matrix algebra over A. (Note that this holds if A is K_1-simple, as then any matrix algebra over A is also K_1-simple.)

Theorem 9.1 *There exists a universal constant $\varepsilon_0 > 0$ with the following property:*

If A is a K_1-simple real rank zero C*-*algebra, and u, v are unitaries in A with $[u]_1 = 0$ and*

$$\|vu - uv\| < \varepsilon_0\ ,$$

then

$$\mathrm{Isospec}(v, u) = \mathrm{Bott}(v, u)\ .$$

To prove Theorem 9.1 we need a lemma which we first state in a form that will be used in Section 10.

Lemma 9.2 *Assume that A is a purely infinite simple unital* C*-*algebra. Let $k_0 \in K_0(A)$ and let $u \in A$ be a unitary with $[u]_1 = 0$ and* $\mathrm{Spec}(u) = \mathbb{T}$. *Then for any sufficiently small $\varepsilon > 0$ there is a unitary $v \in A$ such that*

$$\|vuv^* - u\| < \varepsilon$$

and

$$\mathrm{Isospec}(u, v) = k_0\ .$$

Proof. Let $n \in \mathbb{N}$ be such that $\frac{2\pi}{n} < \frac{\varepsilon}{2}$. Then there is a unitary $u_1 \in A$ with $\mathrm{Spec}(u_1) = \{e^{2\pi ik/n}; k = 0, 1, \ldots, n-1\}$ such that $\|u - u_1\| < \varepsilon/2$. If

$$u_1 = \sum_{k=0}^{n-1} e^{2\pi ik/n} p_k$$

is the spectral decomposition of u_1, choose a non-zero subprojection e_k of p_k such that $[e_k]_0 = k_0$, and a partial isometry w_k such that

$$w_k^* w_k = e_k\,, \qquad w_k w_k^* = e_{k+1}$$

for $k = 0, 1, \ldots, n-1$ with $e_n = e_0$. (Cf. [Cun2].) Set

$$v = \sum_{k=0}^{n-1} w_k + 1 - \sum_{k=0}^{n-1} e_k\,.$$

Then v satisfies the required properties. □

Note that Lemma 9.2 only holds in this form when A is purely infinite. When A is an AF-algebra, the degree ε of commutation of u and v will impose restrictions on the range of $\mathrm{Isospec}(v, u)$, and in general this range will be a small subset of $K_0(A)$ of the form $D_\varepsilon - D_\varepsilon$, where D_ε is a hereditary subset of the dimension range. However, to prove Theorem 9.1, we just need a version of Lemma 9.2 where the unitaries are allowed to lie in a matrix algebra $M_n(A) = M_n \otimes A$ over A.

Lemma 9.3 *Let A be a unital C^*-algebra, let $\varepsilon > 0$, and let $k_0 \in K_0(A)$ and $k_1 \in K_1(A)$. Then there exist $n \in \mathbb{N}$ and unitaries $u, v \in M_n(A)$ such that u has finite spectrum, $[v]_1 = k_1$,*

$$\|vuv^* - u\| < \varepsilon$$

and

$$\mathrm{Isospec}(v, u) = \mathrm{Bott}(v, u) = k_0.$$

Proof. Find projections e_1, e_2 in a matrix algebra $M_m(A)$ over A such that

$$k_0 = [e_1]_0 - [e_2]_0.$$

Let $l \in \mathbb{N}$ be large and denote $e_{i,k}$ by the projection in $M_{lm}(A)$ which is represented by a diagonal $l \times l$ matrix over $M_m(A)$ with e_i in the kth entry on the diagonal and 0 elsewhere, $k = 0, \cdots, l-1$, $i = 1, 2$. Set

$$u_i = \sum_{k=0}^{l-1} e^{2\pi ik/l} e_{i,k} + (1 - \sum_{k=0}^{l-1} e_{i,k})$$

for $i = 1, 2$, and let $v_1 = v_2$ denote the unitary shift matrix in $M_l \otimes 1 \subseteq M_{lm}(A)$ such that

$$v_i e_{i,k} v_i^* = e_{i,k+1}$$

for $i = 1, 2$ [Voi]. Then

$$\mathrm{Isospec}(v_i, u_i) = [e_i]_0$$

for $i = 1, 2$. But by [ExL, Theorem 4.1] we also have

$$\mathrm{Bott}(v_i, u_i) = [e_i]_0.$$

Set $n = 2lk$. Then $M_n(A)$ contains $M_{lm}(A) \oplus M_{lm}(A)$ as a unital subalgebra, and we may define unitary operators u, v in $M_n(A)$ by

$$\begin{aligned} u &= u_1 \oplus v_2, \\ v &= v_1 \oplus u_2. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Isospec}(v, u) &= \mathrm{Isospec}(v_1, u_1) + \mathrm{Isospec}(u_2, v_2) \\ &= [e_1]_0 - [e_2]_0 \\ &= k_0, \end{aligned}$$

and, correspondingly,

$$\mathrm{Bott}(v, u) = k_0.$$

Furthermore,

$$\begin{aligned} v^* u v - u &= (v_1 u_1 v_1 - u_1) \;\oplus\; (u_2 v_2 u_2 - v_2) \\ &= ((e^{-2\pi i/l} - 1) u_1) \oplus ((e^{2\pi i/l} - 1) v_2) \end{aligned}$$

and so

$$\|v^* u v - u\| \le |e^{-2\pi i/l} - 1| < \frac{2\pi}{l}.$$

Thus, on choosing l large enough that $\frac{2\pi}{l} < \varepsilon$, and that $\mathrm{Isospec}(v_1, u_1)$ and $\mathrm{Bott}(v_1, u_1)$ are defined, the conclusion of Lemma 9.3 follows. □

Proof of Theorem 9.1. Let u, v be the given pair in the theorem (with ε_0 sufficiently small), and $\mathrm{Isospec}(v, u) = k_0 \in K_0$. By Lemma 9.3 there exists an $n \in \mathbb{N}$ and unitaries $v_1, u_1 \in M_n(A)$ with $\mathrm{Bott}(v_1, u_1) = \mathrm{Isospec}(v_1, u_1) = -k_0$, $[u_1]_1 = 0$, $[v_1]_1 = -[v]_1 = -k_1$, and $\|v_1 u_1 v_1^* - u_1\| < \varepsilon$.

Now put u, v into the upper left hand corner of $M_{n+1}(A)$, and u_1, v_1 simultaneously into the remaining block, to obtain unitaries

$$u_2 = u \oplus u_1 \quad , \quad v_2 = v \oplus v_1$$

such that

$$\begin{aligned} &\|v_2u_2 - u_2v_2\| < \varepsilon_0, \\ &[u_2]_1 = [u]_1 + [u_1]_1 = 0 + 0 = 0, \\ &[v_2]_1 = [v]_1 + [v_1]_1 = [v]_1 - [v]_1 = 0, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Isospec}(v_2, u_2) &= \mathrm{Isospec}(v, u) + \mathrm{Isospec}(v_1, u_1) \\ &= k_0 - k_0 = 0. \end{aligned}$$

By Lemma 7.1, u_2 can be connected to 1 by a continuous path of unitaries which all almost commute with v_2. But since both Bott and Isospec are invariant under homotopy of almost commuting unitaries, it follows that

$$\mathrm{Bott}(v_2, u_2) = \mathrm{Bott}(v_2, 1) = 0$$

and

$$\mathrm{Isospec}(v_2, u_2) = \mathrm{Isospec}(v_2, 1) = 0 \ ,$$

and as, furthermore,

$$\begin{aligned} \mathrm{Bott}(v_2, u_2) &= \mathrm{Bott}(v, u) + \mathrm{Bott}(v_1, u_1), \\ \mathrm{Isospec}(v_2, u_2) &= \mathrm{Isospec}(v, u) + \mathrm{Isospec}(v_1, u_1), \\ \mathrm{Bott}(v_1, u_1) &= \mathrm{Isospec}(v_1, u_1), \end{aligned}$$

it follows that

$$\mathrm{Bott}(v, u) = \mathrm{Isospec}(v, u),$$

which is the conclusion of Theorem 9.1. □

10 The tail lemma for trivial K_1

In this section and the following one we will prove a result which is independent of the basic homotopy lemma, and which is used in [EllR]. Note that this result definitely does not hold for an AF-algebra. The reason is that two unitaries with full spectrum in such an algebra are not necessarily approximately unitary equivalent. In order that two full spectrum unitaries u, v should be approximately unitary equivalent we would have to assume that the "distribution of K_0 over the spectrum" of the two unitaries is approximately the same, as formulated more precisely in [Ell1].

Lemma 10.1 (The tail lemma when K_1 is trivial.) *For any $\delta > 0$ there exists $\varepsilon > 0$ with the following property: Let $\{u_t\}, \{v_t\}$, $t \in [0,1]$, be two continuous paths of unitaries in a purely infinite simple unital* C*-*algebra A such that $[u_t]_1 = [v_t]_1 = 0$ and such that* $\mathrm{Spec}(u_t) = \mathbb{T}$ *and* $\mathrm{Spec}(v_t)$ *is δ-dense, for all $t \in [0,1]$. Then there is a continuous path $\{w_t\}$ of unitaries in A such that $\|w_t u_t w_t^* - v_t\| < \varepsilon$ for all $t \in [0,1]$. If $u_0 = v_0$, $\{w_t\}$ can be chosen with* $w_0 = 1$.

Proof. First choose $\delta_0 > 0$ such that if two unitaries $u, v \in A$ are such that $[u]_1 = [v]_1 = 0$, $\|vuv^* - u\| < \delta_0$, and $\mathrm{Isospec}(u,v) = 0$, then there is a continuous path $\{v_t\}$ of unitaries in A such that $v_0 = v$, $v_1 = 1$, and $\|v_t u v_t^* - u\| < \varepsilon/2$. We may suppose that $\delta_0 < \varepsilon/4$. Let $\delta \in (0, \delta_0/12\pi)$ and let $\{u_t\}, \{v_t\}$ be as in the statement of the lemma. There exists an $n \in \mathbb{N}$ such that if $|s-t| \le 1/n$, then $\|u_s - u_t\| < \delta_0/3$ and $\|v_s - v_t\| < \delta_0/3$. As in [Ell1] (using [Lin1]) one obtains a unitary $x_0 \in A$ such that

$$\|x_0 u_0 x_0^* - v_0\| < 4\pi\delta < \delta/3 .$$

(If $u_0 = v_0$ we may choose $x_0 = 1$.) Set $w_t = x_0$ for $t \in \left[0, \frac{1}{n}\right]$. Then

$$\|w_t u_t w_t^* - v_t\| < \delta_0 < \varepsilon$$

for $t \in \left[0, \frac{1}{n}\right]$. Now suppose that we have defined w_t for $t \in \left[0, \frac{k}{n}\right]$ such that $\|w_t u_t w_t^* - v_t\| < \varepsilon$ and $\left\|w_{\frac{k}{n}} u_{\frac{k}{n}} w_{\frac{k}{n}}^* - v_{\frac{k}{n}}\right\| < \delta_0$. There is a unitary $x_k \in A$ such that

$$\left\|x_k u_{\frac{k}{n}} x_k^* - v_{\frac{k}{n}}\right\| < \min\left\{\frac{\delta_0}{3}, \delta_0 - \left\|w_{\frac{k}{n}} u_{\frac{k}{n}} w_{\frac{k}{n}}^* - v_{\frac{k}{n}}\right\|\right\}.$$

Note that

$$\left\|w_{\frac{k}{n}}^* x_k u_{\frac{k}{n}} x_k^* w_{\frac{k}{n}}^* - u_{\frac{k}{n}}\right\| < \delta_0 .$$

By Lemma 9.2 we can choose x_k so that $\mathrm{Isospec}\left(u_{\frac{k}{n}}, w_{\frac{k}{n}}^* x_k\right) = 0$. Then there is a continuous path $\{y_t; t \in [0,1]\}$ of unitaries such that $y_0 = 1$, $y_1 = w_{\frac{k}{n}}^* x_k$ and $\left\|y_t u_{\frac{k}{n}} y_t^* - u_{\frac{k}{n}}\right\| < \varepsilon/2$. For $t \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$ set

$$w_t = w_{\frac{k}{n}} y_{nt-k} .$$

Then w_t is continuous at $t = \frac{k}{n}$ and for $t \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$,

$$\begin{aligned}\|w_t u_t w_t^* - v_t\| &< \frac{2\delta_0}{3} + \left\|w_t u_{\frac{k}{n}} w_t^* - v_{\frac{k}{n}}\right\| \\ &< \frac{2\delta_0}{3} + \frac{\varepsilon}{2} + \left\|w_{\frac{k}{n}} u_{\frac{k}{n}} w_{\frac{k}{n}}^* - v_{\frac{k}{n}}\right\| \\ &< \frac{2\delta_0}{3} + \frac{\varepsilon}{2} + \delta_0 < \varepsilon\end{aligned}$$

and

$$\begin{aligned}&\left\|w_{\frac{k+1}{n}} u_{\frac{k+1}{n}} w_{\frac{k+1}{n}}^* - v_{\frac{k+1}{n}}\right\| = \left\|x_k u_{\frac{k+1}{n}} x_k^* - v_{\frac{k+1}{n}}\right\| \\ &\quad < \frac{2\delta_0}{3} + \left\|x_k u_{\frac{k}{n}} x_k^* - v_{\frac{k}{n}}\right\| \\ &\quad < \delta_0 .\end{aligned}$$

□

11 The tail lemma for non-trivial K_1

Assume throughout this section that A is a purely infinite simple unital C*-algebra. We first need

Lemma 11.1 *Let B be a hereditary* C**-subalgebra of $C[0,1] \otimes A$ such that $B(t) \neq 0$ for $t \in [0,1]$. Then for any $k_0 \in K_0(A)$ there is a non-zero projection $e \in B$ such that $[e(t)] = k_0$.*

Proof. For each $t_0 \in [0,1]$ there is a non-zero projection $p \in B(t_0)$ such that $[p]_0 = k_0$ and $p \neq 1$. There is a positive $x_{t_0} \in B$ such that $x_{t_0}(t_0) = p$. Since $x_{t_0}(t)$ is close to a projection for t around t_0, we may assume that there is an open interval I_{t_0} containing t_0 such that $x_{t_0}(t)$ is a projection for $t \in I_{t_0}$. Hence there exist $t_0 = 0 < t_1 < t_2 < \ldots < t_n = 1$ and $x_i \in B$, $i = 0, 1, \ldots, n-1$ such that $x_i(t)$ is a non-zero projection of class k_0 with $x_i(t) \neq 1$ for $t \in [t_i, t_{i+1}]$.

Fix i. Since $[x_i(t_{i+1})] = [x_{i+1}(t_{i+1})]$, there is a unitary $w \in B(t_{i+1}) + \mathbb{C}1$ such that $w x_{i+1}(t_{i+1}) w^* = x_i(t_{i+1})$. We may suppose that $[w_i]_1 = 0$. Then there is a unitary $\widetilde{w} \in B + \mathbb{C}1$ such that $\widetilde{w}(t_{i+1}) = w$ and $\widetilde{w}(t) = 1$ for $t \geq t_{i+2}$. Set $\tilde{x}_{i+1} = \widetilde{w} x_{i+1} \widetilde{w}^*$, and let $\tilde{x}_0 = x_0$. Note that $\tilde{x}_i \in B$.

Define $e \in A$ by

$$e(t) = \tilde{x}_i(t), \quad t \in [t_i, t_{i+1}] .$$

Since $e(t) \in B(t)$, it follows that $e \in B$. Thus e is as required. □

Lemma 11.2 *For $k = 0, 1, \dots, N-1$ let $e_k, p_k \in C[0,1] \otimes A$ be non-zero projections such that $[e_k(t)] = [p_k(t)]$ and $\sum_{k=0}^{N-1} e_k(t) \lneqq 1$ and $\sum_{k=0}^{N-1} p_k(t) \lneqq 1$. Then there exists a unitary $w \in C[0,1] \otimes A$ such that $we_kw^* = p_k$, for $k = 0, 1, \dots, N-1$. If $e_k(0) = p_k(0)$, w can be chosen with $w(0) = 1$.*

Proof. We may suppose that the e_ks are constant and that $e_k(t) = p_k(0)$. Then we have to find a unitary $w \in C[0,1] \otimes A$ such that $w(t)p_k(0)w(t)^* = p_k(t)$. For t close to zero set

$$x(t) = \sum_{k=0}^{N} p_k(t)p_k(0) \ ,$$

where $p_N(t) = 1 - \sum_{k=0}^{N-1} p_k(t)$. Since $x(t)$ is close to 1 the unitary $u(t)$ obtained from the polar decomposition of $x(t)$ has the properties that $u(t)p_k(0)u(t)^* = p_k(t)$ and $t \mapsto u(t)$ is continuous where $u(t)$ is defined. We just repeat this procedure. □

Theorem 11.3 (The tail lemma when K_1 is non-trivial.) *Let A be a purely infinite simple unital C*-algebra. Let $\{u_t\}, \{v_t\}$, $t \in [0,1]$ be two continuous paths of unitaries in A such that $[u_t]_1 = [v_t]_1 \neq 0$. For any $\varepsilon > 0$ there is a continuous path $\{w_t\}$ of unitaries in A such that $\|w_tu_tw_t^* - v_t\| < \varepsilon$ for all $t \in [0,1]$. If $u_0 = v_0$, $\{w_t\}$ can be chosen with $w_0 = 1$.*

Proof. We regard $u = \{u_t\}$ and $v = \{v_t\}$ as elements of $C[0,1] \otimes A$. We may suppose that u is constant and $u_t = v_0$ for $t \in [0,1]$, since any two unitaries $u_0, v_0 \in A$ with $[u_0]_1 = [v_0]_1 \neq 0$ are approximately unitarily equivalent [Ell1].

For $\varepsilon > 0$ choose $\delta > 0$ as in Lemma 10.1 and let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \delta$. Let $w_k = e^{2\pi ik/N}$ for $k = 0, \dots, N-1$. Let f_k be a non-zero continuous function on $\mathbb{T}$ whose support lies in a small neighbourhood of w_k, and let B_k denote the hereditary C*-subalgebra of $C[0,1] \otimes A$ generated by $f_k(v)$. By Lemma 11.1 there is a non-zero projection e_k of B_k with $[e_k(t)]_0 = 0$. By making $\mathrm{supp}(f_k)$ sufficiently small one obtains a unitary $\tilde{v}_k \in C[0,1] \otimes A$ such that $\|v - \tilde{v}_k\| < \varepsilon$ and

$$\tilde{v}_k e_k(t) = w_k e_k(t) \,.$$

Set $e_N = 1 - \sum_{k=0}^{N-1} e_k$. Choose non-zero subprojections F_1, F_2 of $1 - e_N(0)$ such that $F_1 + F_2 \leq 1 - e_N(0)$ and $[F_1]_0 = [F_2]_0 = [1]_0$, and set $G =$

$1 - e_N(0) - F_1 - F_2$. Choose partial isometries V_1, V_2 such that $V_iV_i^* = F_i$, $V_i^*V_i = 1 - e_N(0)$. There is a unitary $w_0 \in C[0,1] \otimes A$ such that

$$w_0^* e_k(0) w_0 = e_k \, ,$$

where we regard an element of A as a constant function in $C[0,1] \otimes A$. Hence

$$w_0 \tilde{v} w_0^* = x + D \, ,$$

where $D = \sum_{k=0}^{N-1} w_k e_k(0)$ is a unitary in $(1 - e_N(0))A(1 - e_N(0))$ and $x(t)$ is a unitary in $e_N(0) A \, e_N(0)$. Consider the unitary element

$$y = V_1 x(0) V_1^* + V_2 x^* V_2^* + G$$

of $C[0,1] \otimes (1 - e_N(0))A(1 - e_N(0))$. Since $[y]_1 = 0$, there is a unitary $w_1 \in C[0,1] \otimes A$ such that $w_1 e_N(0) = e_N(0)$ and

$$\|w_1(x + D)w_1^* - (x \oplus y)\| < \varepsilon \, .$$

Set $w_2 = V_1 + V_1^* + 1 - e_N(0) - F_1$. Then

$$\begin{aligned} w_2(x+y)w_2^* &= w_2(x + V_1 x(0) V_1^* + V_2 x^* V_2^* + G) w_2^* \\ &= x(0) + V_1 x V_1^* + V_2 x^* V_2^* + G = x(0) + z \, . \end{aligned}$$

There is a unitary w_3 such that $w_3 e_N(0) = e_N(0)$ and

$$\|w_3(x(0) + z)w_3^* - (x(0) + D)\| < \varepsilon \, .$$

Set $w = w_3 w_2 w_1 w_0$. Then

$$\|w \tilde{v} w^* - \tilde{v}(0)\| < 2\varepsilon \, ,$$

and hence

$$\|w v w^* - v(0)\| < 4\varepsilon \, .$$

Now we want to make $w(0) = 1$. First, we may assume that $w_0(0) = 1$. Second, since $w_1 e_N(0) = e_N(0) = w_3 e_N(0)$ and $\mathrm{Ad}\, w_1(0)(D) = y(0)$, $\mathrm{Ad}\, w_3(0)(y(0)) = D$, we may assume that $w_1(0) = w_3(0)^*$. Third, since $\mathrm{Ad}\, w_2$ interchanges x and $V_1 x(0) V_1^*$, and $x(t) + V_1 x(0) V_1^*$ is almost diagonal for t close to 0, we may assume that $w_2(0) = 1$, allowing at most an error

$$\|w_2(x + y)w_2^* - (x(0) + z)\| < \varepsilon$$

Hence we may choose w such that $w(0) = 1$ and $\|w v w^* - v(0)\| < 5\varepsilon$. $\square$

12 Proof of the Super Homotopy Lemma

In this section we will prove Theorem 1.1.

Case 1. $[u_0]_1 = 0 = [v_0]_1$ and $\mathrm{Bott}(v_0, u_0) = 0$

Let us first assume that $[u_0]_1 = 0 = [v_0]_1$ and $\mathrm{Bott}(v_0, u_0) = 0$. If we choose $\delta > 0$ as in Lemma 7.1, there exist rectifiable paths $u'(t), v'(t)$ of unitaries in A with

$$v'(0) = v_0,\ v'(1) = 1,\ \|[u_0, v'(t)]\| < \varepsilon,$$
$$\mathrm{Length}(u'(t)) < 4\pi + \varepsilon'$$

(for a given $\varepsilon' > 0$), and

$$u'(0) = 1,\ u'(1) = u_1,\ \|[u'(t), v_1]\| < \varepsilon,$$
$$\mathrm{Length}(u'(t)) < 4\pi + \varepsilon'.$$

By [Lin1] there exist paths $u''(t), v''(t)$ of unitaries in A such that

$$u''(0) = u_0,\ u''(1) = 1$$
$$\mathrm{Length}(u''(t)) < \pi + \varepsilon',$$

and

$$v''(0) = 1,\ v''(1) = v_1,$$
$$\mathrm{Length}(v''(t)) < \pi + \varepsilon'.$$

Now compose these paths, as follows:

$$u(t) = \begin{cases} u_0 & \text{for } 0 \le t \le \frac{1}{4}, \\ u''(4t-1) & \text{for } \frac{1}{4} \le t \le \frac{2}{4}, \\ 1 & \text{for } \frac{2}{4} \le t \le \frac{3}{4}, \\ u'(4t-3) & \text{for } \frac{3}{4} \le t \le 1, \end{cases}$$

$$v(t) = \begin{cases} v'(4t) & \text{for } 0 \le t \le \frac{1}{4}, \\ 1 & \text{for } \frac{1}{4} \le t \le \frac{2}{4}, \\ v''(4t-2) & \text{for } \frac{2}{4} \le t \le \frac{3}{4}, \\ v_1 & \text{for } \frac{3}{4} \le t \le 1. \end{cases}$$

One checks that the paths $u(t), v(t)$ satisfy the conditions of the conclusion of Theorem 1.1 with

$$\begin{aligned} \mathrm{Length}(u(t)) &< 5\pi + 2\varepsilon', \\ \mathrm{Length}(v(t)) &< 5\pi + 2\varepsilon' . \end{aligned}$$

Case 2. $[u_0]_1 = 0 = [v_0]_1$ and $\mathrm{Bott}(v_0, u_0)$ arbitrary

Next, let us assume that $[u_0]_1 = 0 = [v_0]_1$ but make no assumption on the K_0-obstruction $\mathrm{Bott}(v_0, u_0)$. Let us show that the pair u_0, v_0 is homotopic to the direct sum of three pairs, the first and second of very special form—Voiculescu pairs as considered in the proofs of Lemmas 9.2 and 9.3, with obstructions x and y where x and y are positive in $K_0(A)$ and $x - y = \mathrm{Bott}(v_0, u_0)$ (so the third pair has zero obstruction)—and all three pairs with trivial K_1. (The proof will be then completed by putting the pair u_1, v_1 also in the standard form, and making a simple comparison of the two standard forms.)

First of all, using the Riesz decomposition property for Murray–von Neumann equivalence classes, somewhat in the manner of the proof of Lemma 7.2, let us show that if a K_0-class can be expressed inside finitely many projections—as the difference in each case of the K_0-classes of two subprojections—to be assumed, for technical reasons, to be non-zero—then it can be expressed simultaneously inside all, using the same Murray–von Neumann equivalence classes (now no longer required to be non-zero). By induction, it is enough to consider the case of two given projections, provided that only inside one of them is the expression of the given K_0-class assumed to be in terms of non-zero projections. Let, then, e and f be projections, let e_1 and e_2 be subprojections of e, and let f_1 and f_2 be non-zero subprojections of f, and suppose that

$$[e_1]_0 - [e_2]_0 = [f_1]_0 - [f_2]_0 \, .$$

By the property 2 in the definition of K_1-simplicity, as $e_1 \oplus f_2$ has the same K_0-class in $M_2(A)$ as $e_2 \oplus f_1$, and since neither of these projections is zero,

$$d(e_1) + d(f_2) = d(e_2) + d(f_1) \, ,$$

where $d(p)$ denotes the Murray–von Neumann equivalence class of projection $p \in A$. Hence, by the Riesz decomposition property, there exist Murray–von Neumann equivalence classes of projections in A, x_{ij}, $i, j \in \{1, 2\}$, such that

$$d(e_1) \;=\; x_{11} + x_{12} \, ,$$

$$
\begin{aligned}
d(f_2) &= x_{21}+x_{22} \ , \\
d(e_2) &= x_{11}+x_{21} \ , \\
d(f_1) &= x_{12}+x_{22} \ .
\end{aligned}
$$

Then the equivalence classes x_{12} and x_{21} provide the desired representation of the given K_0-class inside both e and f. (x_{12} and x_{21} are both less than $d(e)$ and $d(f)$, and the difference $x_{12}-x_{21}$ in $K_0(A)$ is the given K_0-class.)

Let us now construct a pair u_0', v_0' of the special form described above, to which the pair u_0, v_0 is homotopic. As in the proof of Lemma 6.1, we may suppose that both u_0 and u_1 have finite spectrum.

Note first that, if δ is sufficiently small, the K_0-obstruction $\mathrm{Bott}(v_0, u_0)$ fits inside the spectral projection of u_0 corresponding to an arbitrarily small interval, in the sense of being the difference of two subprojections of this projection. This follows directly from the definition; see Theorem 4.1. If δ is sufficiently small, both of the subprojections may be chosen to be non-zero, as shown in the proof of Lemma 6.1. By the preceding paragraph, it follows that the obstruction may be represented simultaneously inside the spectral projections corresponding to finitely many intervals—by pairs of projections belonging to the same pair of Murray–von Neumann equivalence classes (now possibly zero)—if δ is sufficiently small in a way depending only on the size of the smallest interval.

Now, as in the proof of Lemma 6.1, with u_0 and v_0 in place of u and w, with N and ε' as in that argument, and with δ both as small as there and as small as required above, with respect to intervals of length $\frac{1}{2N}$, let Q_i, P_i, q_i, p_i, q_i' , p_i', and v be as constructed in that argument. We may suppose that $\frac{\varepsilon}{2\pi} < 2$, so that $v = e^{ih}$ with h self-adjoint and of norm at most $\frac{\varepsilon}{4}$. Replacing v_0 by v^*v_0, then, which involves an initial homotopy of v_0 along a path of length at most $\frac{\varepsilon}{4}$, we may suppose that $v = 1$, so that $p_i' = q_i'$, provided that we include this amount in the total length of the path. Note then that

$$
\begin{aligned}
Q_i &= q_{2i}+q_{2i+1} &&= q_{2i}'+q_{2i+1}' \ , \\
P_i &= p_{2i-1}+p_{2i} &&= q_{2i-1}'+q_{2i}' \ .
\end{aligned}
$$

(Recall that $p_i = v_0 q_i v_0^*$.) With x and y the equivalence classes of projections such that the difference, $x - y$, of the images of x and y in K_0 is $\mathrm{Bott}(v_0, u_0)$, and members of x and y may be found inside the spectral projections of u_0 corresponding to all the standard half-open intervals of length $\frac{1}{2N}$—namely, the projections q_i—choose a projection $e_{2i-1} \leq q_{2i-1}$ belonging to the Murray–von Neumann class x and a projection $e_{2i} \leq q_{2i}$ belonging to the class y. For any choice of partial isometries w_{2i-1} from e_{2i-1} to e_{2i+1}

and w_{2i} from e_{2i} to e_{2i+2}, the pair $u_0(\sum e_{2i-1})$, $\sum w_{2i-1}$ has obstruction x, and the pair $u_0(\sum e_{2i})$, $\sum w_{2i}$, which is orthogonal to it, has obstruction $-y$. Let us now show how to choose the partial isometries w_k in such a way that the pair u_0, v_0 is (manifestly) homotopic to the sum of these two pairs and a third pair—necessarily with obstruction zero. The sum of all three pairs then satisfies the requirements for u'_0, v'_0.

First, deform the projection q'_{2i}, within Q_i, to $q_{2i}-e_{2i}+e_{2i+1}$, and thereby also the complementary projection q'_{2i+1}, also within Q_i, to $q_{2i+1}-e_{2i+1}+e_{2i}$ (using the method of Lemma 2.6). This may be done by a path of unitaries of length at most 3.15, which should then also be used to deform v_0, by multiplying on the left. (Note that then v_0 takes the projection $q_{2i-1}+q_{2i}$ onto the projection

$$(q_{2i-1}-e_{2i-1})+e_{2i-2}+(q_{2i}-e_{2i})+e_{2i+1}\ .$$

After multiplying v_0 on the right by a unitary connected to 1 by a path of length at most 3.15 (again as in Lemma 2.6), we may assume that v_0 takes e_{2i} to e_{2i-2} and e_{2i-1} to e_{2i+1}. The restriction of v_0 to e_k now has the properties required of w_k. Note that u_0 has remained fixed so far; let us deform it by a short path to a linear combination of the projections q_k: $v'_0 = \sum_{k=1}^{2N} e^{2\pi i k/2N} q_k$. The total length of the deformation of v_0 to v'_0 is at most two times 3.15 plus $\frac{\varepsilon}{4}$.

Similarly, the pair u_1, v_1 may be deformed to a pair u'_1, v'_1 in what might be called standard form—the direct sum of two Voiculescu pairs, with obstructions x and $-y$ respectively, and a third one.

We do not know that all three summands are non-zero in each case, but we do know (as a consequence of our assumption that the classes x and y in the two cases are the same) that the projections determining the first summands in the two cases are Murray–von Neumann equivalent, and similarly for the projections determining the second summands. We may also easily ensure in the construction that the third summand is non-zero in each case, by choosing the projection e_k to be a proper subprojection of q_k for each k. (Recall that e_k could be chosen inside the spectral projection corresponding to an arbitrarily small interval, and in particular a subinterval of the interval determining q_k.) Hence as in the proof of Lemma 2.6, there is a path of unitaries of length at most 3.15 transforming the projections determining the direct summands of u_1, v_1 onto the corresponding projections for u_0, v_0.

Thus, if we allow the total lengths of the deformations of u_0, v_0 and u_1, v_1 to u'_0, v'_0 and u'_1, v'_1 to be $4(3.15)+\frac{\varepsilon}{4}$, we may suppose that the direct summands of the standard pairs u'_0, v'_0 and u'_1, v'_1 are determined by the same projections.

By the case of zero K_0-obstruction, considered above, the two third summands are homotopic, by a path of length at most $5\pi + 1$. Let us consider the question whether the two first summands are homotopic. In fact, they are unitarily equivalent; they are both described in terms of the standard matrix units of two full matrix algebras of the same order with equivalent minimal projections. It would be desirable to choose a unitary with trivial K_1-class tranforming one pair onto the other; however, such a unitary exists, inside the unit of this summand, if, and only if, the K_1-class of a unitary which does make the transformation is divisible by the order of the matrix algebra.

Accordingly, we must first homotope the third summands, in each case, to the trivial pair, $1, 1$; this involves two paths of length $5\pi + 1$. Remembering that these amounts must be added later, we may assume that the third summands of u'_0, v'_0 and u'_1, v'_1 are the trivial pair $1, 1$, inside a non-zero projection. This gives us enough room to carry out the needed homotopies between the two first summands, and the two second summands. Breaking the third projection up into two non-zero components, and adding the first projection to the first summand and the second to the second, we may add a unitary to each of the unitaries carrying out the transformation between the two pairs—using the property 3 of Section 1—in the first and second summands to make it have trivial K_1-class. As A has real rank zero, and so also any cut-down of A, the unitaries making the transformation in the two enlarged first summands may now be connected by a path of length at most 3.15 to 1 (cf. proof of Lemma 2.6).

Thus, the standard pairs u'_0, v'_0 and u'_1, v'_1 may be connected by a path of length $11\pi + \varepsilon$.

In fact, the preceding case may be realized—by continuation of the induction, the equivalence classes x and y may be chosen to lie inside the appropriate spectral projections of u_1 as well as those of u_0—and so to enter simultaneously into the construction of the standard pairs u'_0, v'_0 and u'_1, v'_1.

Case 3. $[u_0]_1, [v_0]_1$, $\mathrm{Bott}(v_0, u_0)$ all arbitrary

It is now enough to reduce the general case to the case that the unitaries of both pairs have trivial K_1-class.

As in the proof of Lemma 7.1, we may assume that $K_1(A) \neq 0$, so that $D(A \otimes K)$ is simple, and the set $D(A)$ of Murray–von Neumann equivalence classes of non-zero projections is downward directed in the natural order. Hence, passing to subprojections, we may suppose that the projections E_0

and E_1 given by Lemma 12.1, below, applied to the pairs u_0, v_0 and u_1, v_1, are equivalent. Since we may also assume that they are proper (i.e., $\neq 1$), by Lemma 2.6 they are connected by a path of unitaries of length at most 3.15. Transforming the pair u_1, v_1 by this path reduces us to the case $E_0 = E_1$, at the cost of a homotopy in u_1 and v_1 of length at most two times 3.15.

Perturbing both pairs u_0, v_0 and u_1, v_1 by a small amount, we may suppose that they commute with the projection $E_0 = E_1 =: E$, and that inside E all four unitaries are scalars. Since A is K_1-simple with $K_1(A) \neq 0$, A has no minimal non-zero projections. Since, as above, $D(A)$ is downward directed, it follows that E contains four orthogonal equivalent projections, F_1, F_2, F_3, F_4, and replacing E by $\sum F_i$ we may suppose that $E = \sum F_i$. By the property 3 of the definition of K_1-simplicity, there exist unitaries w and z in F_3AF_3 and F_4AF_4 respectively with

$$[w]_1 = [u_0]_1 \ (= [u_1]_1) \quad , \quad [z]_1 = [v_0]_1 \ (= [v_1]_1) \ .$$

Put copies of w^{-1} and z^{-1} inside F_1 and F_2 respectively, and note that the unitaries

$$\begin{bmatrix} w^{-1} & & & \\ & 1 & & \\ & & w & \\ & & & 1 \end{bmatrix} , \quad \begin{bmatrix} 1 & & & \\ & z^{-1} & & \\ & & 1 & \\ & & & z \end{bmatrix}$$

commute, have trivial K_0-obstruction, and trivial K_1-classes, and hence by Lemma 7.1 are homotopic via a path of approximately commuting pairs to the pair $1, 1$, of length at most $4\pi + \varepsilon$. Note that the restrictions of the pairs u_0, v_0 and u_1, v_1 to $E = \sum F_i$ are also homotopic to $1, 1$ by such a path, of length at most π. Therefore, keeping track of these path lengths, we may assume that the pairs u_0, v_0 and u_1, v_1 restricted to E are both the pair above.

Next, consider the restrictions of the pairs u_0, v_0 and u_1, v_1 to the projection $(1 - E) + F_1 + F_2$. They fulfil the hypotheses of the theorem (with A replaced by its cut-down to $(1 - E) + F_1 + F_2$), and belong to the case of trivial K_1 considered above, and so are homotopic each other through a path of such pairs inside this projection, of length at most $18\pi + \varepsilon$. Since, inside the complement of this projection, namely, $F_3 + F_4$, the pairs are equal, the desired homotopy has been obtained. □

Lemma 12.1 *For any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property: Let A be a K_1-simple real rank zero* C*-*algebra. For any two unitaries u, $v \in A$ with*

$$\|uv - vu\| < \delta$$

there exists a non-zero projection $E \in A$ and scalars λ, $\mu \in \mathbb{T}$ such that

$$\|(u-\lambda)E\| < \varepsilon\ , \quad \|(v-\mu)E\| < \varepsilon\ .$$

(Necessarily, u and v then approximately commute with E, to within 6ε.)

Proof. The proof is similar to that of Lemma 8.2. (And the present result may be used in place of Lemma 8.2 in the proof of Theorem 8.1.)

If (f_i) is a partition of unity consisting of continuous functions of u, and (g_j) a second partition of unity consisting of functions of v, then, for sufficiently small δ, $(f_i g_j)$ is close to being a partition of unity also: the relation $0 \leq f_i g_j \leq 1$ holds approximately and the relation $\sum f_i g_j = 1$ holds exactly. With f_1, $f_2, \ldots$ chosen to be supported on adjacent, slightly overlapping subintervals of $\mathbb{T}$, we may suppose both that $\|f_i\| = 1$ and that uf_i is close to a scalar multiple of f_i for each i. Similarly, we may suppose that $\|g_j\| = 1$ and vg_j is close to a scalar multiple of g_j for each j.

Fix i, and note that, for sufficiently small δ, the sum $\sum_j g_j^{1/2} f_i g_j^{1/2}$ is close to f_i. Since only adjacent terms in this sum (labelled cyclically) are not orthogonal, and the product of two adjacent elements $g_j^{1/2}$ has norm at most $\frac{1}{2}$, and since $\|f_i\| = 1$, it follows that at least one of the elements $g_j^{1/2} f_i g_j^{1/2}$, $j = 1, 2, \ldots$, must have norm at least $\frac{1}{3}$—at least if the number of j's is even, as we may suppose. (We may divide the elements $g_j^{1/2} f_i g_j^{1/2}$ into two groups consisting of even and odd terms, the members of each group being then mutually orthogonal. The sum of the elements of both groups, being close to f_i, having norm close to 1, and in particular at least $\frac{2}{3}$, the sum of the elements of at least one group must have norm at least $\frac{1}{3}$, and this norm is attained by at least one element in this group of orthogonal elements.) Set $g_j^{1/2} f_i g_j^{1/2} = h$ with i and j as above, so that $\|h\| \geq \frac{1}{3}$. If δ is sufficiently small, then u and v are both close to scalars on h. Since A has real rank zero, there exists a non-zero projection E in A such that h is arbitrarily close to $\|h\|$ on E. Hence, as $\|h\| \geq \frac{1}{3}$, u and v are arbitrarily close to scalars on E, as desired. □

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