A Moving Boundary Value Problem in a Stochastic Medium

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August 1997

Abstract

Suppose a fluid is pumped into a heterogeneous medium in $\mathbb{R}^d$ at a known rate and that there is a known initial wet region $D(0)$ (region filled with fluid) at time $t = 0$. What is the wet region, $D(t)$, at a later time t and what is the pressure, $p(t, x)$, at this time and at the point $x \in D(t)$? We propose a mathematical model for this problem when the region is *stochastic*, i.e., has a stochastic permeability represented by some (positive) white noise functional. The model has the form of a family of stochastic variational inequalities. We show that there is a unique stochastic, weak (in Baiocchi sense) solution of this problem and we discuss its properties.

1 Introduction

The flow of an incompressible fluid in a heterogeneous, isotropic porous medium is described by the following two equations (a) and (b).

(a) Darcy's law

$$\vec{q} = -\frac{1}{\mu} K \nabla p, \tag{1.1}$$

where $\vec{q} = \vec{q}(t, x)$ is the (seepage) velocity at time t at the point $x \in \mathbb{R}^d$, $p = p(t, x)$ is the pressure of the fluid, μ is the viscosity, and $K = K(x) \geq 0$ is the permeability of the medium.

(b) The continuity equation

$$\frac{\partial \theta}{\partial t} = -\operatorname{div}(\rho \vec{q}) + \xi, \tag{1.2}$$

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where $\theta = \theta(t,x)$ is saturation of the fluid, $\xi = \xi(t,x)$ is the source rate of the fluid, and ρ is the density of the fluid.

Combining (1.1) and (1.2) we obtain

$$\frac{\partial \theta}{\partial t} = \operatorname{div}(K\nabla p) + \xi, \tag{1.3}$$

where we have let $\rho = \mu = 1$.

This equation and its various interpretations have been studied by many authors as a moving boundary problem in some form. See e.g. [2], [3], [11], [13], [14], and [15] and the references therein.

The purpose of this paper is to discuss a model for fluid flow in the case when the explicit values of the permeability is not known, only its probabilistic distribution. It is then natural to represent K by some random quantity and try to solve the corresponding *stochastic* moving boundary value problem (see Section 3).

The problem of this paper may be regarded as a generalization of the problem of solving the stochastic pressure equation in a fixed region. Consult [5], [7], and [16] for discussions of this equation.

We begin in Section 2 by defining the spaces involved and providing some results. In the following section we give a stochastic interpretation of (1.3) and show that this leads to a family of stochastic variational inequalities. We also prove these variational inequalities have a unique solution. Finally, in Section 4 we discuss a few properties of the solution.

2 Basic Definitions and Theorems

Let $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$ denote the Schwartz functions on $\mathbb{R}^d$, endowed with the usual Fréchet topology, and let $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$ be its dual endowed with the weak-$*$ topology. Then the Bochner-Minlos theorem ensures the existence of a probability measure μ on the Borel sets $\mathscr{B} := \mathscr{B}(\mathscr{S}')$ of $\mathscr{S}'$ satisfying

$$\int_{\mathscr{S}'} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = e^{-\frac{1}{2}||\phi||^2} \text{ for every } \phi \in \mathscr{S}, \tag{2.1}$$

where $||\phi||^2 = \int_{\mathbb{R}^d} \phi(x)^2\, dx$. The triple $(\mathscr{S}', \mathscr{B}, \mu)$ is called the white noise probability space.

By applying (2.1) to the inverse Fourier transform of the Fourier transform of a function $f \in C_0^\infty(\mathbb{R})$ we obtain the identity

$$E[f(\langle\cdot,\phi\rangle)] = \frac{1}{\sqrt{2\pi}\,||\phi||} \int_{\mathbb{R}} f(x) \exp(-\frac{x^2}{2||\phi||^2})\, dx. \tag{2.2}$$

If $f(x) = x^2$ is approximated from below, on an increasing sequence of compact sets whose union is $\mathbb{R}$, then (2.2) yields the isometry

$$E[\langle\cdot,\phi\rangle^2] = ||\phi||^2 \tag{2.3}$$

in the limit. Using (2.3) we may extend the action of ω to $\phi \in L^2(\mathbb{R}^d)$ by defining

$$\langle \omega, \phi \rangle := \lim_{i \to \infty} \langle \omega, \phi_i \rangle \text{ (limit in } L^2(\mu)),$$

where $\{\phi_i\}_{i=1}^\infty \subset \mathscr{S}$ is any sequence of functions converging to ϕ in $L^2(\mathbb{R}^d)$.

Let

$$h_i(x) = (-1)^i e^{x^2/2} \frac{d^i}{dx^i} e^{-x^2/2}, \; i \in \mathbb{N}_0 := \{0, 1, \ldots\}$$

denote the Hermite polynomials and define the Hermite functions by

$$\xi_i(x) := \pi^{-1/4}((i-1)!)^{-1/2} e^{-x^2/2} h_{i-1}(\sqrt{2}\, x), \; i \in \mathbb{N} := \{1, 2, \ldots\}.$$

Suppose $\delta^{(i)} = (\delta_1^{(i)}, \delta_2^{(i)}, \ldots, \delta_d^{(i)})$ is multi-index number i in some fixed ordering of all $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}^d$ satisfying $\delta_1^{(i)} + \cdots + \delta_d^{(i)} \leq \delta_1^{(j)} + \cdots + \delta_d^{(j)}$ if $i \leq j$. Throughout this paper $\{e_i\}_{i=1}^\infty \subset \mathscr{S}$ denotes the orthonormal basis for $L^2(\mathbb{R}^d)$ given by

$$e_i := \xi_{\delta_1^{(i)}} \otimes \xi_{\delta_2^{(i)}} \otimes \cdots \otimes \xi_{\delta_d^{(i)}}.$$

Define

$$H_\alpha(\omega) := \prod_{i=1}^\infty h_{\alpha_i}(\theta_i(\omega)) \tag{2.4}$$

for all $\alpha \in I$, where

$$I := \{\alpha = (\alpha_1, \alpha_2, \ldots) \in (\mathbb{N}_0)^{\mathbb{N}} \; : \; \alpha_i = 0 \text{ for all but a finite number of } i\}$$

and $\theta_i(\omega) := \langle \omega, e_i \rangle$ for $i \in \mathbb{N}$. It is shown in [4] that

$$\{H_\alpha(\omega) : \alpha \in I\}$$

forms an orthogonal basis for $L^2(\mu) := L^2(\mathscr{S}', \mathscr{B}, \mu)$, with $E[H_\alpha H_\beta] = \delta_{\alpha,\beta}\alpha!$, where $\alpha! := \prod_{j=1}^\infty \alpha_j!$. Observe that the notation in [4] differs somewhat from ours.

We now turn to the definition of the Kondratiev Hilbert spaces. For $-1 \leq \rho \leq 1$ and $k \in \mathbb{R}$ we consider the inner product spaces $(\mathscr{S})_{\rho,k}$ defined by

$$(\mathscr{S})_{\rho,k} := \{f = \sum_\alpha f_\alpha H_\alpha \; : \; f_\alpha \in \mathbb{R} \text{ and } \|f\|_{\rho,k} < \infty\},$$

where $\|\cdot\|_{\rho,k}$ is the norm associated with the inner product

$$(f, g)_{\rho,k} := \sum_\alpha f_\alpha g_\alpha (\alpha!)^{1+\rho} (2\mathbb{N})^{\alpha k}. \tag{2.5}$$

In (2.5) we have used the notation

$$(2\mathbb{N})^\alpha := \prod_{i=1}^\infty (2i)^{\alpha_i}$$

for $\alpha \in I$. It is not difficult to verify the following proposition:

Proposition 2.1 *For every pair (ρ, k) with $-1 \leq \rho \leq 1$ and $k \in \mathbb{R}$, $(\mathcal{S})_{\rho,k}$ equipped with the inner product (2.5) is a separable Hilbert space.*

Remark 2.2 *When $\rho \in [-1, 0)$ and $k < 0$ an element in $(\mathcal{S})_{\rho,k}$ is a formal sum, i.e., the sum $\sum_\alpha f_\alpha H_\alpha$ does not necessarily converge in $L^1(\mu)$.*

The reason these spaces are not considered for $|\rho| > 1$ is that in this case it is not possible to define the $\mathcal{S}$-transform (see [9]).

From the definition of $(\mathcal{S})_{\rho,k}$ we see that $(\mathcal{S})_{0,0} \cong L^2(\mu)$. Moreover, if $0 \leq \rho \leq 1$,

$$(\mathcal{S})_{\rho,k} \subset (\mathcal{S})_{\rho,\ell} \subset (\mathcal{S})_{\rho,0} \subseteq (\mathcal{S})_{-\rho,0} \subset (\mathcal{S})_{-\rho,-\ell} \subset (\mathcal{S})_{-\rho,-k} \tag{2.6}$$

for every $k > \ell > 0$. The following proposition shows $(\mathcal{S})_{-\rho,-k}$ is the dual of $(\mathcal{S})_{\rho,k}$.

Proposition 2.3 *If $0 \leq \rho \leq 1$ and $k \geq 0$, then $(\mathcal{S})_{-\rho,-k}$ is the dual space of $(\mathcal{S})_{\rho,k}$, where the action is defined by*

$$\langle F, f\rangle := \sum_\alpha F_\alpha f_\alpha \alpha!$$

for $F = \sum_\alpha F_\alpha H_\alpha \in (\mathcal{S})_{-\rho,-k}$ and $f = \sum_\alpha f_\alpha H_\alpha \in (\mathcal{S})_{\rho,k}$.

Proof: If $F = \sum_\alpha F_\alpha H_\alpha \in (\mathcal{S})_{-\rho,-k}$ and $f = \sum_\alpha f_\alpha H_\alpha \in (\mathcal{S})_{\rho,k}$, then

$$|\langle F, f\rangle| \leq \sum_\alpha \left[F_\alpha^2(\alpha!)^{1-\rho}(2\mathbb{N})^{-k\alpha}\right]^{1/2} \cdot \left[f_\alpha^2(\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha}\right]^{1/2} \leq \|F\|_{-\rho,-k}\|f\|_{\rho,k}.$$

Hence any element in $(\mathcal{S})_{-\rho,-k}$ defines a continuous linear functional on $(\mathcal{S})_{\rho,k}$.

Conversely, if F is a continuous linear functional on $(\mathcal{S})_{\rho,k}$, Riesz' theorem implies there is a $g = \sum_\alpha g_\alpha H_\alpha \in (\mathcal{S})_{\rho,k}$ such that $F(f) = (g, f)_{\rho,k}$. It is easily verified that

$$F(f) = (g, f)_{\rho,k} = \langle G, f\rangle$$

where $G = \sum_\alpha g_\alpha(\alpha!)^\rho(2\mathbb{N})^{k\alpha}H_\alpha \in (\mathcal{S})_{-\rho,-k}$. Therefore $(\mathcal{S})_{-\rho,-k}$ is the dual of $(\mathcal{S})_{\rho,k}$, as claimed. □

As usual we let $(\mathcal{S})_1 := \cap_{k=0}^\infty(\mathcal{S})_{1,k}$ endowed with the projective limit topology and $(\mathcal{S})_{-1} := \cup_{k=0}^\infty(\mathcal{S})_{-1,-k}$ endowed with the inductive limit topology, denote the Kondratiev spaces. We refer to [17] for a comparison between the spaces $(\mathcal{S})_{\pm 1}$ and $(\mathcal{N})_{\pm\infty}$ used in [9] and [10].

We now introduce a family of Hilbert spaces which has turned out to be useful when stochastic partial differential equations are investigated. Fix an open set $D \subseteq \mathbb{R}^d$ and let $(\cdot,\cdot)_{m,D}$, or simply $(\cdot,\cdot)_m$ if D is clear from the context, denote the usual inner product on the real Sobolev space $H^m(D)$ for $m \in \mathbb{N}_0$. We define the inner product

$$(f, g)_{\rho,k,H^m(D)} := \sum_\alpha (f_\alpha, g_\alpha)_{m,D}(\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \tag{2.7}$$

on the set of functions of the form

$$f(x) = \sum_\alpha f_\alpha(x)H_\alpha,$$

where $f_\alpha \in H^m(D)$ for every $\alpha \in I$.

Definition 2.4 *Let* $(\mathscr{S})_{\rho,k,H^m(D)}$ *(resp.* $(\mathscr{S})_{\rho,k,H_0^m(D)}$*) denote the set of functions*

$$\{f(x) = \sum_\alpha f_\alpha(x) H_\alpha \quad : \quad f_\alpha \in H^m(D)\ \forall\alpha\ \textit{(resp. } H_0^m(D)\ \forall\alpha\textit{), and}$$
$$\|f\|_{\rho,k,H^m(D)} := (f,f)^{1/2}_{\rho,k,H^m(D)} < \infty\}$$

equipped with the inner product (2.7).

Recall that $H_0^m(D)$ is defined as the completion of $C_0^\infty(D)$ with respect to the $\|\cdot\|_{m,D}$-norm and that $L^2(D) = H_0^0(D)$. Hence $(\mathscr{S})_{\rho,k,H^m(D)}$ and $(\mathscr{S})_{\rho,k,H_0^m(D)}$ are equipped with the same norm and $(\mathscr{S})_{\rho,k,H_0^0(D)} = (\mathscr{S})_{\rho,k,L^2(D)}$. The latter simplifies the statements in some of the propositions that ensue.

Proposition 2.5 *If* $-1 \le \rho \le 1$ *and* $k \in \mathbb{R}$*, then* $(\mathscr{S})_{\rho,k,H^m(D)} \cong (\mathscr{S})_{\rho,k} \otimes H^m(D)$ *and* $(\mathscr{S})_{\rho,k,H_0^m(D)} \cong (\mathscr{S})_{\rho,k} \otimes H_0^m(D)$ *for* $m \in \mathbb{N}_0$*. Moreover,* $(\mathscr{S})_{\rho,k,H^m(D)}$ *and* $(\mathscr{S})_{\rho,k,H_0^m(D)}$ *are separable Hilbert spaces.*

Proof: Fix $-1 \le \rho \le 1$, $k \in \mathbb{R}$, and $m \in \mathbb{N}_0$. Since $H^m(D)$ is a separable Hilbert space, it has an orthonormal basis $\{b_i\}_{i=1}^\infty$. We claim $\{J_\alpha b_i\}$ is an orthonormal basis for $(\mathscr{S})_{\rho,k,H^m(D)}$, where $J_\alpha = (\alpha!)^{-(1+\rho)/2}(2\mathbb{N})^{-k\alpha/2} H_\alpha$ for every multi-index α. $\{J_\alpha b_i\}$ is clearly an orthonormal set, to show completeness let $f = \sum_\alpha f_\alpha J_\alpha \in (\mathscr{S})_{\rho,k,H^m(D)}$ and suppose $(f, J_\alpha b_i)_{\rho,k,H^m(D)} = 0$ for all α and i. For each fixed α, we obtain $(f_\alpha, b_i)_{H^m(D)} = 0$ for every i which implies $\|f_\alpha\|_{H^m(D)} = 0$. Hence $\|f\|_{\rho,k,H^m(D)} = 0$ and $\{J_\alpha b_i\}$ is an orthonormal basis.

Define
$$U : J_\alpha b_i \mapsto J_\alpha \otimes b_i,$$
then U maps a countable orthonormal basis for $(\mathscr{S})_{\rho,k,H^m(D)}$ onto a countable orthonormal basis for $(\mathscr{S})_{\rho,k} \otimes H^m(D)$. Since U extends uniquely to an isomorphism from $(\mathscr{S})_{\rho,k,H^m(D)}$ onto the Hilbert space $(\mathscr{S})_{\rho,k} \otimes H^m(D)$, we have proved the proposition for $(\mathscr{S})_{\rho,k,H^m(D)}$. The proof for $(\mathscr{S})_{\rho,k,H_0^m(D)}$ is similar. □

If $m \in \mathbb{N}_0$ we recover the inclusions (2.6)

$$(\mathscr{S})_{\rho,k,H^m(D)} \subset (\mathscr{S})_{\rho,\ell,H^m(D)} \subset (\mathscr{S})_{\rho,0,H^m(D)}$$
$$\subseteq (\mathscr{S})_{-\rho,0,H^m(D)} \subset (\mathscr{S})_{-\rho,-\ell,H^m(D)} \subset (\mathscr{S})_{-\rho,-k,H^m(D)}$$

when $\rho \in [0,1]$ and $k > \ell > 0$. For fixed ρ and k

$$(\mathscr{S})_{\rho,k,H^{m_1}(D)} \subseteq (\mathscr{S})_{\rho,k,H^{m_2}(D)}$$

if m_1, $m_2 \in \mathbb{N}_0$ and $m_1 \ge m_2$. Combining these results we see that

$$(\mathscr{S})_{\rho_1,k_1,H^{m_1}(D)} \subseteq (\mathscr{S})_{\rho_2,k_2,H^{m_2}(D)}$$

when $\rho_1 \ge \rho_2$, $k_1 \ge k_2$, and $m_1 \ge m_2$. Moreover, $(\mathscr{S})_{\rho_1,k_1,H^{m_1}(D)}$ is a proper subset of $(\mathscr{S})_{\rho_2,k_2,H^{m_2}(D)}$ if and only if one of the inequalities are strict. Similar results hold for $(\mathscr{S})_{\rho,k,H_0^m(D)}$.

So far we have constructed the Hilbert spaces $(\mathcal{S})_{\rho,k,H^m(D)}$ and $(\mathcal{S})_{\rho,k,H_0^m(D)}$ and considered basic relations between the spaces. We now turn to study properties and define operations on elements from them.

One of the reasons for introducing $(\mathcal{S})_{\rho,k,H^m(D)}$ and $(\mathcal{S})_{\rho,k,H_0^m(D)}$ for $m \geq 1$, is that elements from these spaces can be differentiated.

Definition 2.6 *Let $D \subseteq \mathbb{R}^d$ be an open set and $m \in \mathbb{N}$. For $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq m$ we define*

$$\frac{\partial^\beta}{\partial x^\beta} f(x) := \sum_\alpha \frac{\partial^\beta f_\alpha}{\partial x^\beta}(x) H_\alpha,$$

for any $f = \sum_\alpha f_\alpha H_\alpha \in (\mathcal{S})_{\rho,k,H^m(D)}$. $\partial^\beta f_\alpha / \partial x^\beta$ is interpreted in the usual $L^2(D)$ sense. We often write ∂_x^β or ∂^β for $\partial^\beta/\partial x^\beta$.

With this definition

$$\partial_x^\beta : (\mathcal{S})_{\rho,k,H^m(D)} \to (\mathcal{S})_{\rho,k,H^{m-|\beta|}(D)}$$

is a continuous linear operator, if $|\beta| \leq m$.

Example: The smoothed white noise process is usually defined as

$$W(\phi, x, \omega) := \langle \omega, \phi_x \rangle,$$

where $\phi_x(\cdot) := \phi(\cdot - x)$ and $\phi \in L^2(\mathbb{R}^d)$. For each $x \in \mathbb{R}^d$ we obtain from (2.3) that

$$\langle \omega, \phi_x \rangle = \langle \omega, \sum_{i=1}^\infty (\phi_x, e_i)_{0,\mathbb{R}^d} e_i \rangle = \sum_{i=1}^\infty (\phi_x, e_i)_{0,\mathbb{R}^d} H_{\varepsilon_i}(\omega) \text{ (equality in } L^2(\mu) = (\mathcal{S})_{0,0}),$$

where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots)$ denotes the multi-index whose only nonzero entry is a 1 in the ith position. Lebesgue's monotone convergence theorem implies

$$\|\langle \omega, \phi_x \rangle\|_{\rho,k,0}^2 = \sum_{i=1}^\infty \int_D (\phi_x, e_i)_{0,\mathbb{R}^d}^2 \, dx \, (2\mathbb{N})^{k\varepsilon_i} = \int_D \left[\sum_{i=1}^\infty (\phi_x, e_i)_{0,\mathbb{R}^d}^2 (2\mathbb{N})^{k\varepsilon_i} \right] dx.$$

Since $\sum_{i=1}^\infty (\phi_x, e_i)_{0,\mathbb{R}^d}^2 = \|\phi\|_{0,\mathbb{R}^d}^2 < \infty$ for every $x \in \mathbb{R}^d$, the sum on the right hand side cannot converge for all $\phi \in L^2(\mathbb{R}^d)$ unless $(2\mathbb{N})^{k\varepsilon_i}$ is uniformly bounded in i. Hence, if we define

$$W_{\phi_x} := \sum_{i=1}^\infty (\phi_x, e_i)_{0,\mathbb{R}^d} H_{\varepsilon_i},$$

then $W_{\phi_x} \in (\mathcal{S})_{\rho,-k,L^2(D)}$ for all $\phi \in L^2(\mathbb{R}^d)$ if $-1 \leq \rho \leq 1$, $k \geq 0$, and D is a bounded open set in $\mathbb{R}^d$. Moreover, $W_{\phi_x}(\omega) = W(\phi, x, \omega)$ in $(\mathcal{S})_{0,0,L^2(D)}$.

More generally, viewing $(\phi_x, e_i)_{0,\mathbb{R}^d}$ as the convolution of a tempered distribution and a Schwartz function gives

$$\partial_x^\beta (\phi_x, e_i)_{0,\mathbb{R}^d} = (\partial_x^\beta \phi_x, e_i)_{0,\mathbb{R}^d}$$

for every $\beta \in \mathbb{N}_0^d$, and Definition 2.6 implies

$$\partial_x^\beta W_{\phi_x} = \sum_{i=1}^{\infty} (\partial_x^\beta \phi_x, e_i)_{0,\mathbb{R}^d} H_{\varepsilon_i}.$$

Since $\partial^\beta \phi \in \mathscr{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, summation over all $|\beta| \le m$ shows $W_{\phi_x} \in (\mathscr{S})_{\rho,-k,H^m(D)}$ for all $-1 \le \rho \le 1$, $k \ge 0$, $m \in \mathbb{N}_0$, and bounded open sets $D \subset \mathbb{R}^d$.

Suppose $\phi \in \mathscr{S}$ and let $\tilde{\phi}(z) := \phi(-z)$, then $\langle \omega, \phi_x \rangle = \omega * \tilde{\phi}(x)$. Since $\omega * \tilde{\phi}(x)$ is a C^∞-function with polynomial growth, W_{ϕ_x} cannot in general belong to $(\mathscr{S})_{\rho,-k,L^2(D)}$ for unbounded sets D. For the same reason W_{ϕ_x} does not generally belong to $(\mathscr{S})_{\rho,-k,H_0^m(D)}$. We recall the following result from [18].

Lemma 2.7 $\sum_\alpha (2\mathbb{N})^{-k\alpha} < \infty$ *if and only if* $k > 1$.

Thus, if $D \subseteq \mathbb{R}^d$ is unbounded, $W_{\phi_x} \in (\mathscr{S})_{\rho,-k,H^m(D)}$ when $-1 \le \rho \le 1$, $k > 1$, $m \in \mathbb{N}_0$ and $\phi \in \mathscr{S}$. Hence the above results can be extended to unbounded open sets $D \subseteq \mathbb{R}^d$ by requiring $k > 1$. Note also that

$$\frac{\partial^\beta}{\partial x^\beta} W_{\phi_x}(\omega) = \frac{\partial^\beta}{\partial x^\beta} \langle \omega, \phi_x \rangle = \frac{\partial^\beta}{\partial x^\beta} (\omega * \tilde{\phi})(x) = \omega * (\partial^\beta \tilde{\phi})(x) = (-1)^{|\beta|} \langle \omega, \partial_x^\beta \phi_x \rangle$$

for every ω and $\phi \in \mathscr{S}$. This means that the derivative of W_{ϕ_x} in the sense of Definition 2.6 coincides with the usual one.

A closely related process is the singular white noise

$$W_x := \sum_{i=1}^{\infty} e_i(x) H_{\varepsilon_i}.$$

Formally, W_x is W_{δ_x} where δ_x is Dirac's point mass at x. It is easily seen from Lemma 2.7 that $W_x \in (\mathscr{S})_{\rho,-k,L^2(D)}$ for any open set $D \subseteq \mathbb{R}^d$, $-1 \le \rho \le 1$, and $k > 1$.

We would also like to compute the expectation of elements in $(\mathscr{S})_{\rho,k,H^m(D)}$ for general ρ, k, and $m \in \mathbb{N}_0$. Inspired by [5] we observe that if $f = \sum_\alpha f_\alpha H_\alpha \in (\mathscr{S})_{0,0,L^2(D)}$, then

$$E[f] = E[f \cdot 1] = \sum_\alpha f_\alpha E[H_\alpha \cdot 1] = \sum_\alpha f_\alpha E[H_\alpha \cdot H_{(0,0,\ldots)}] = f_{(0,0,\ldots)} \in L^2(D),$$

using $E[H_\alpha H_\beta] = \delta_{\alpha,\beta} \alpha!$. It is therefore reasonable to define:

Definition 2.8 *For* $f = \sum_\alpha f_\alpha H_\alpha \in (\mathscr{S})_{\rho,k,H^m(D)}$ *(resp.* $(\mathscr{S})_{\rho,k,H_0^m(D)}$*), we define the* (generalized) expectation *to be the deterministic function*

$$E[f] := f_{(0,0,\ldots)}(x) : D \to \mathbb{C},$$

which belongs to $H^m(D)$ *(resp.* $H_0^m(D)$*).*

We use the term generalized expectation since an $f \in (\mathcal{S})_{\rho,k,H^m(D)}$ not necessarily is integrable with respect to the measure μ.

Before we can proceed further we need the Wick product.

Definition 2.9 *If $f = \sum_\alpha f_\alpha H_\alpha$ and $g = \sum_\alpha g_\alpha H_\alpha$ are two formal series, we define their* Wick product, *$f \diamond g$, to be the formal series*

$$f \diamond g := \sum_{\alpha,\beta} f_\alpha g_\beta H_{\alpha+\beta} = \sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \Big) H_\gamma.$$

Some useful properties of the Wick product are:

(i) $f,\ g \in (\mathcal{S})_1 \Rightarrow f \diamond f \in (\mathcal{S})_1$

(ii) $f,\ g \in (\mathcal{S})_{-1} \Rightarrow f \diamond g \in (\mathcal{S})_{-1}$

Recall that $L^2(\mu)$ is not closed under Wick multiplication. To see this let, for example, $f = H_{\varepsilon_1}$ then $g \mapsto f \diamond g$ is a densely defined unbounded linear operator on $L^2(\mu)$. If $f,\ g \in (\mathcal{S})_{0,0,L^2(D)}$ we have the additional problem that $f_\alpha g_\beta$ need not belong to $L^2(D)$. To provide conditions on f such that $g \mapsto f \diamond g$ gives a continuous linear operator on $(\mathcal{S})_{-1,k,L^2(D)}$ we introduce the Banach space $\mathcal{F}_\ell$. For open $D \subseteq \mathbb{R}^d$ and $\ell \in \mathbb{R}$ we let

$$\mathcal{F}_\ell(D) := \{ f(x) = \sum_\alpha f_\alpha(x) H_\alpha \ : \ f_\alpha(x) \text{ is measurable on } D \text{ for every } \alpha \text{ and }$$
$$\|f\|_{\ell,*} := \operatorname*{ess\,sup}_{x\in D} \Big(\sum_\alpha |f_\alpha(x)| (2\mathbb{N})^{\ell\alpha} \Big) < \infty \}.$$

We suppress the set, D, in the notation whenever it is clear from the context.

Proposition 2.10 (Proposition 5 in [16]) *Let D be an open subset of $\mathbb{R}^d$ and $\ell \in \mathbb{R}$. Then $f \in \mathcal{F}_\ell$ defines a continuous linear operator on $(\mathcal{S})_{-1,k,L^2(D)}$ by $g \mapsto f \diamond g$ when $k \le 2\ell$. Moreover*

$$\|f \diamond g\|_{-1,k,L^2(D)} \le \|f\|_{k/2,*} \|g\|_{-1,k,L^2(D)} \le \|f\|_{\ell,*} \|g\|_{-1,k,L^2(D)} \ \textit{for } g \in (\mathcal{S})_{-1,k,L^2(D)}.$$

Corresponding results for the ordinary Kondratiev Hilbert spaces with no x dependence can be found in [17].

W_{ϕ_x} belongs to $\mathcal{F}_\ell(D)$ for arbitrary $\phi \in L^2(\mathbb{R}^d)$, open set $D \subseteq \mathbb{R}^d$, and $\ell < -1/2$. To see this note that by Schwarz' inequality and Lemma 2.7

$$\|W_{\phi_x}\|_{\ell,*} = \operatorname*{ess\,sup}_{x\in D} \sum_{i=1}^{\infty} |(\phi_x, e_i)_{0,\mathbb{R}^d}| (2\mathbb{N})^{\ell\varepsilon_i} \le \|\phi\|_{0,\mathbb{R}^d} \left(\sum_\alpha (2\mathbb{N})^{2\ell\alpha} \right)^{1/2} < \infty,$$

when $2\ell < -1$. Using 22.14.17 in [1] to obtain $|e_i(x)| \le (C\pi^{-1/4})^d$ for $x \in \mathbb{R}^d$ and $i \in \mathbb{N}$, where $C \approx 1.086435$, it can be shown that the singular white noise $W_x \in \mathcal{F}_\ell$ if $\ell < -1$.

When we turn to consider the stochastic moving boundary value problem it will be important to be able to determine when the bilinear form $b(g_1, g_2) = (f \diamond g_1, g_2)_{-1,k,L^2(D)}$ is strictly positive, that is, if there exists $C > 0$ such that $b(g, g) \geq C\|g\|^2_{-1,k,L^2(D)}$ for every $g \in (\mathscr{S})_{-1,k,L^2(D)}$. Let

$$\mathscr{P}_\ell(D) := \{f \in \mathscr{F}_\ell(D) \,:\, \exists A > 0 \text{ such that } (f_{(0,0,\ldots)}g, g)_{0,D} \geq A\|g\|^2_{0,D}\, \forall g \in L^2(D)\}.$$

Note that $f \in \mathscr{F}_\ell$ ensures $b(\cdot,\cdot)$ is continuous on $(\mathscr{S})_{-1,k,L^2(D)}$ for $k \leq 2\ell$. The second condition is necessary, otherwise $b(\cdot,\cdot)$ would fail to be strictly positive on the subspace $\{g(x)H_{(0,0,\ldots)} : g \in L^2(D)\}$ of $(\mathscr{S})_{-1,k,L^2(D)}$. The following proposition shows the conditions are also sufficient, provided k is small enough.

Proposition 2.11 (Proposition 6 in [16]) *Let $D \subseteq \mathbb{R}^d$ be open and $f \in \mathscr{P}_\ell(D)$ for some real ℓ. Then there exist constants $L = L(f) \leq 2\ell$ and $C = C(f) > 0$ such that*

$$(f \diamond g, g)_{-1,-k,L^2(D)} \geq C\|g\|^2_{-1,-k,L^2(D)} \text{ for every } g \in (\mathscr{S})_{-1,-k,L^2(D)},$$

when $-k < L$.

A useful result for what follows is:

Proposition 2.12 (Proposition 7 in [17]) *Let $D \subseteq \mathbb{R}^d$ be open and $\ell \in \mathbb{R}$.*

(i) If $f,\, g \in \mathscr{F}_\ell(D)$, then $\|f \diamond g\|_{\ell,*} \leq \|f\|_{\ell,*}\|g\|_{\ell,*}$.

(ii) Suppose $G(y) = \sum_{n=0}^\infty c_n y^n$ is analytic for $-R < y < R$ where $R > 0$. If $f \in \mathscr{F}_\ell(D)$ with $\|f\|_{\ell,*} < R$ then

$$G^\diamond(f) := \sum_{n=0}^\infty c_n f^{\diamond n} \in \mathscr{F}_\ell(D)$$

and

$$\|G^\diamond(f)\|_{\ell,*} \leq \sum_{n=0}^\infty |c_n|\|f\|^n_{\ell,*} < \infty.$$

(iii) If $f \in \mathscr{F}_\ell(D)$, then $\exp^\diamond f := \sum_{n=0}^\infty f^{\diamond n}/n! \in \mathscr{P}_\ell(D)$.

Remark 2.13 *Similar results are readily obtained for the ordinary Kondratiev Hilbert spaces $(\mathscr{S})_{-1,k}$.*

3 A Stochastic Interpretation of the Moving Boundary Value Problem

Before we begin our treatment of the stochastic moving boundary value problem we discuss the concept of *positivity* for a generalized stochastic process.

Definition 3.1 *We say* $h \in (\mathscr{S})_{-1,-k,H_0^1(D)}$ *is* positive on D *and write* $h \geq 0$ *if*

$$\langle h(x), \psi \rangle \geq 0, \quad a.e. \ x \in D,$$

for any $\psi \in (\mathscr{S})_1$ *such that* $\psi \geq 0$.

From [9] any $\psi \in (\mathscr{S})_1$ has a version which is defined pointwise. In the definition we assume such a version is chosen and $\psi \geq 0$ means that $\psi(\omega) \geq 0$ for every $\omega \in \mathscr{S}'$.

If $X \in (\mathscr{S})_{-1}$ is positive in the sense of Definition 3.1, then by the main result in [10], X can be represented by a positive measure ν_X on $(\mathscr{S}'(\mathbb{R}^d), \mathscr{B})$, in the sense that

$$\langle X, \psi \rangle = \int_{\mathscr{S}'(\mathbb{R}^d)} \psi(\omega) \, d\nu_X(\omega) \text{ for } \psi \in (\mathscr{S})_1.$$

Moreover, in this case we have

$$E[X] = \int_{\mathscr{S}'(\mathbb{R}^d)} 1 \, d\nu_X(\omega) = \nu_X(\mathscr{S}'(\mathbb{R}^d)) < \infty.$$

Hereafter we assume that $U \subset \mathbb{R}^d$ is open and bounded. Whenever we write spaces or inner products without mentioning the underlying set, we have the set U in mind.

Suppose $K \in \mathscr{P}_\ell$ for some real ℓ. In applications as, for example, oil flow in porous rock one usually assumes $K(x)$ is independent, lognormally distributed, and stationary. In [5] it is shown that $K(x) = \exp^\diamond W_x$ has these properties and that $K(x) = \exp^\diamond W_{\phi_x}$ satisfies a weak independence in addition to being lognormally distributed and stationary. Recall from Proposition 2.12 that $\mathscr{P}_\ell$ contains $\exp^\diamond W_x$ and $\exp^\diamond W_{\phi_x}$ for $\ell < -1$ and $\ell < -1/2$, respectively. Hence $\mathscr{P}_\ell$ contains the most interesting examples of stochastic permeability from a physical point of view.

In the stochastic version of (1.3) we regard $\theta(t,x)$, and $p(t,x)$ as $(\mathscr{S})_{-1,-k,H_0^1((0,T)\times U)}$-valued stochastic processes and get

$$\frac{\partial \theta}{\partial t} = \operatorname{div}(K \diamond \nabla p) + \xi, \tag{3.1}$$

where $\diamond$ denotes the Wick product and $\xi \in (\mathscr{S})_{-1,-k,L^2((0,T)\times U)}$.

We refer the reader to [5] and [6] for discussions on the use of the Wick product in stochastic differential equation models like this. Here we just remark that the spaces $(\mathscr{S})_{-1}$ and $(\mathscr{S})_{-1,-k}$ seem well suited as general solution spaces for stochastic partial/ordinary differential equations, and that on these spaces the Wick product is a well-defined and well-behaved operation in the sense of Proposition 2.10.

It turns out that the positivity concept defined in Definition 3.1 is too strong for equations of the type (3.1). Even in the static case when ξ, and hence θ and p, do not depend on t, Hu [7] has shown that the (unique) solution found in [5] not necessarily is positive, but only *weakly positive.* In view of this we shall consider a weaker positivity concept than the one in Definition 3.1, also for the more general equation (3.1). Our positivity concept, *pseudopositivity* (defined below), is different from Hu's weak positivity.

The following operator will play a crucial role: If

$$F = \sum_{\gamma} c_\gamma H_\gamma \in (\mathscr{S})_{-1},$$

then we define

$$\hat{F} = \hat{F}^{(\ell)} = \sum_{\gamma} c_\gamma (2\mathbb{N})^{-\ell\gamma} H_\gamma \tag{3.2}$$

for all $\ell \in \mathbb{R}$. Note that $\hat{F}^{(\ell)} \in (\mathscr{S})_{-1}$ for any real number ℓ.

Example 3.2 *Suppose* $F(\omega) = \exp^{\diamond}\langle \omega, \phi \rangle$ *for some* $\phi \in \mathscr{S}(\mathbb{R}^d)$, *then*

$$F(\omega) = \exp^{\diamond}\langle \omega, \sum_{k=1}^{\infty} a_k e_k \rangle,$$

where $a_k = (\phi, e_k)_{L^2(\mathbb{R}^d)}$ *for* $k \in \mathbb{N}$. *Hence*

$$\begin{aligned} F(\omega) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} a_k H_{\varepsilon_k}(\omega) \right)^{\diamond n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{\gamma_1+\gamma_2+\cdots=n} \frac{n!}{\gamma_1!\gamma_2!\cdots} a_1^{\gamma_1} a_2^{\gamma_2} \cdots H_{\gamma_1\varepsilon_1+\gamma_2\varepsilon_2+\cdots}(\omega) \right) \\ &= \sum_{n=0}^{\infty} \sum_{|\gamma|=n} \frac{1}{\gamma!} a^\gamma H_\gamma(\omega) \\ &= \sum_{\gamma} \frac{1}{\gamma!} a^\gamma H_\gamma(\omega), \end{aligned}$$

where $a = (a_1, a_2, \ldots)$. *Therefore*

$$\hat{F}^{(\ell)}(\omega) = \sum_{\gamma} \frac{1}{\gamma!} a^\gamma (2\mathbb{N})^{-\ell\gamma} H_\gamma(\omega) = \exp^{\diamond}\langle \omega, \check{\phi} \rangle, \tag{3.3}$$

where

$$\check{\phi}(x) = \check{\phi}^{(\ell)}(x) = \sum_{k=1}^{\infty} a_k (2k)^{-\ell} e_k(x). \tag{3.4}$$

Similarly, if

$$F(\omega) = \exp^{\diamond} W_y(\omega) = \exp^{\diamond}\left(\sum_{k=1}^{\infty} e_k(y) H_{\varepsilon_k}(\omega)\right),$$

then

$$\hat{F}^{(\ell)}(\omega) = \exp^{\diamond}\left(\sum_{k=1}^{\infty}(2k)^{-\ell} e_k(y) H_{\varepsilon_k}(\omega)\right), \quad y \in \mathbb{R}. \tag{3.5}$$

A similar notation is applied to elements of the spaces $(\mathscr{S})_{-1,-k,H^m(D)}$.

We now define the positivity concept we will apply with in this paper:

Definition 3.3 *(a) We say that* $\eta \in (\mathscr{S})_{-1}$ *is* strongly positive *and write* $\eta \trianglerighteq 0$ *if*

$$\hat{\eta}^{(\ell)}(\omega) \geq 0 \textit{ for all } \ell \in \mathbb{R}.$$

The same notation is applied to elements of $(\mathscr{S})_{-1,L^2(D)}$.

(b) We say that $X \in (\mathscr{S})_{-1,-k,H_0^1(D)}$ *is* pseudopositive *and write* $X \succeq 0$ *if*

$$(X, \eta)_{-1,-k,L^2} \geq 0$$

for all $\eta \in (\mathscr{S})_{-1,-k,C_0^\infty(D)}$ *such that* $\eta \trianglerighteq 0$.

Example 3.4 *From Example 3.2 we see that the Wick exponentials* $\eta(\omega) = \exp^{\diamond}\langle\omega, \phi\rangle$ *are strongly positive.*

Remark 3.5 *In the setting of* generalized Wiener functionals *the concept of strongly positive elements was first introduced by Nualart and Zakai in [12].*

3.1 The Deterministic Moving Boundary Value Problem

A useful property of the Wick product is that

$$E[X \diamond Y] = E[X]\, E[Y]$$

for all $X, Y \in (\mathscr{S})_{-1}$, where $E[\cdot]$ denotes the generalized expectation (see e.g. Definition 2.8, [5], and [6]). If X, Y, and $X \diamond Y$ are μ-integrable functions on $\mathscr{S}'(\mathbb{R}^d)$, the generalized expectation coincides with the usual expectation.

Suppose we have found a solution $p(t,x) \in (\mathscr{S})_{-1,-k,H_0^1((0,T)\times U)}$ of (3.1) with $p(t,x) \succeq 0$. Taking the generalized expectation we get the equation

$$\frac{\partial \theta_0}{\partial t} = \operatorname{div}(K_0 \nabla p_0) + \xi_0, \tag{3.6}$$

where $\theta_0 = E[\theta(t,x)]$, $p_0(t,x) = E[p(t,x)] \geq 0$, etc. This is a classical, deterministic moving boundary value problem, which we now briefly discuss.

If we assume that for all t and each point x expected saturation $\theta_0(t,x)$ is either 0 at x (the point x is "dry") or maximal $\lambda_0(x) > 0$ (the point x is "wet"), we can define the wet region at time t by

$$D(t) := \{x;\ \theta_0(t,x) = \lambda_0(x)\}.$$

Then one can show that a natural interpretation of (3.6) is the following set of three equations in the unknowns $p_0(t,x) \geq 0$, $D(t) \subset \mathbb{R}^d$:

$$\operatorname{div}(K_0(x)\nabla p_0(t,x)) = -\xi_0(t,x),\ x \in D(t) \tag{3.7}$$

$$p_0(t,x) = 0,\ x \in \partial D(t) \tag{3.8}$$

$$\lambda_0(x)\frac{d}{dt}(\partial D(t)) = -N^T K_0(x)\nabla p_0(t,x),\ x \in \partial D(t) \tag{3.9}$$

where N is the exterior unit normal to $\partial D(t)$ at $x \in \partial D(t)$ and N^T is its transposed. The deterministic moving boundary value problem is: Given the functions $K_0(x)$, $\lambda_0(x)$, and $\xi_0(t,x)$ and the initial domain $D(0) \subset \mathbb{R}^d$, find $p_0(t,x) \geq 0$ and $\{D(t);\ t \geq 0\}$ such that (3.7), (3.8), and (3.9) hold, in some (weak or strong) sense. We refer the reader to the discussions in [2], [3], [11], [13], [14], and [15] regarding various weak concepts and their solutions. The weak stochastic solution concept we will develop below, is a natural stochastic analogue of the deterministic weak solution of type (D1) defined in [14]. We mention the following result.

Theorem 3.6 (Theorem 2.6 and Theorem 2.13 in [14]) *Assume that $D(0)$ is a non-empty bounded domain in $\mathbb{R}^d$ and that $\lambda_0(x)$ and $\xi_0(t,x)$ are bounded positive functions on $\mathbb{R}^d$. Moreover, assume*

$$\operatorname{supp}\xi_0(t,\cdot) \subset D(0) \text{ for all } t \geq 0$$

and that there exists $q \in (1,\infty)$ such that

$$K_0(x) \text{ is a } q\text{-admissible weight.}$$

Then there exists $T > 0$ and a bounded domain $U \subset \mathbb{R}^d$ such that a unique positive (D1)-weak solution $u_0(t,x) \in W_0^{1,2}(U;K_0)$ exists for $t \in [0,T]$. This solution $u_0(t,x)$ is related to the solution $\{p_0(t,x), D(t)\}$ of (3.7), (3.8), and (3.9) by

$$u_0(t,x) = \int_0^t p_0(s,x)\,ds \text{ (the Baiocchi transformation)}$$

and

$$D(t) = D(0) \cup \{x : u_0(t,x) > 0\} \subset U \text{ for } 0 \leq t \leq T.$$

Moreover, $s \leq t$ implies $D(s) \subset D(t)$ for $0 \leq s,t \leq T$.

Remark 3.7 *The set of p-admissible weights includes the set of Muckenhoupt A_p weights (with the same p). In particular, all bounded measurable functions which are bounded away from 0 are p-admissible for all $p \in (1,\infty)$.*

3.2 The Stochastic Moving Boundary Value Problem

We now discuss an interpretation of the stochastic moving boundary value problem (3.1), i.e.,

$$\frac{\partial \theta}{\partial t} = \operatorname{div}(K \diamond \nabla p) + \xi. \tag{3.10}$$

To avoid working on unbounded sets we fix $T < \infty$, a bounded open set $U \subset \mathbb{R}^d$, and regard (3.10) as an equation in $(\mathscr{S})_{-1,H_0^1} = (\mathscr{S})_{-1,H_0^1((0,T)\times U)}$. We assume that there exists $k_0 \in \mathbb{N}$ such that

$$\text{the } \textit{stochastic permeability } K(x) \in (\mathscr{S})_{-1,-k_0,H_0^1(U)}, \tag{3.11}$$

$$\text{the } \textit{stochastic source rate } \xi(t,x) \in (\mathscr{S})_{-1,-k_0,H_0^1((0,T)\times U)}, \text{ and} \tag{3.12}$$

$$\text{the } \textit{stochastic maximal saturation } \lambda(x) \in (\mathscr{S})_{-1,-k_0,H_0^1(U)} \tag{3.13}$$

are given, in addition to an *initial wet region*

$$D(0) \subset\subset U. \tag{3.14}$$

We seek the following two quantities

$$\text{the } \textit{stochastic pressure } p(t,x) \in (\mathscr{S})_{-1,-k_0,H_0^1((0,T)\times U)}, \tag{3.15}$$

and

$$\text{the } \textit{stochastic saturation } \theta(t,x) \in (\mathscr{S})_{-1,-k_0,H_0^1((0,T)\times U)}. \tag{3.16}$$

We make the following assumptions:

(Assumption 1) $K \in \mathscr{P}_\ell(U)$ for some $\ell \in \mathbb{R}$

(Assumption 2) $u(t,x) \succeq 0$ for all $(t,x) \in [0,T) \times U$

where $u(t,x)$ is the *stochastic Baiocchi transform*

$$u(t,x) = \int_0^t p(s,x)\, ds. \tag{3.17}$$

In addition it is necessary to postulate a relation between the saturation $\theta(t,x)$ and $u(t,x)$. In the deterministic case it is assumed that

$$\text{either } \theta(t,x) = \lambda(x) \text{ or } \theta(t,x) = 0$$

and that

$$\operatorname{supp} u(t,\cdot) \subseteq D(t) := \{x;\ \theta(t,x) = \lambda(x)\}.$$

As a stochastic analogue of this, we propose the following two assumptions:

(Assumption 3) $(\lambda(\cdot)-\theta(t,\cdot),\psi(\cdot))_{-1,-k_0,L^2}\geq 0$ for all $\psi\succeq 0$, all $t\geq 0$, and

(Assumption 4) $(\lambda_\gamma(\cdot)-\theta_\gamma(t,\cdot),u_\gamma(t,\cdot))_{L^2}=0$ for all γ, all $t\geq 0$.

The *stochastic wet region* will then be parameterized by the strongly positive test functions $\eta\trianglerighteq 0$, $\eta\in(\mathscr{S})_{-1,-k}$ and defined by

$$V(t,\eta)=\{x;\ (\lambda(x)-\theta(t,x),\eta)_{-1,-k}=0\}. \tag{3.18}$$

(Intuitively, $V(t,\eta)$ is the wet region at time t if the system is "observed" by applying the stochastic test function η.)

In general, if $F\in(\mathscr{S})_{-1}$ we write

$$F=\sum_\gamma F_\gamma H_\gamma$$

for the chaos expansion of F (with H_γ as in (2.4)).

We will now deduce a natural weak/variational interpretation of (3.10) in terms of a family of *stochastic variational inequalities* for $u(t,x)$: The time space variational form of (3.10) is that for all $k\geq k_0$ we have

$$(\frac{\partial\theta}{\partial t},v)_{-1,-k,L^2((0,T)\times U)}=(\mathrm{div}\,(K\diamond\nabla p),v)_{-1,-k,L^2((0,T)\times U)}+(\xi,v)_{-1,-k,L^2((0,T)\times U)},$$

for every $v\in(\mathscr{S})_{-1,-k,C_0^\infty(\mathbb{R}\times\mathbb{R}^d)}$. Integration by parts gives

$$\begin{aligned}&\sum_\gamma(\frac{\partial\theta_\gamma}{\partial t},\phi_\gamma(t)\psi_\gamma(x))(2\mathbb{N})^{-k\gamma}\\ =&-\sum_\gamma(\sum_{\alpha+\beta=\gamma}K_\alpha\nabla p_\beta,\phi_\gamma(t)\nabla\psi_\gamma(x))(2\mathbb{N})^{-k\gamma}+\sum_\gamma(\xi_\gamma,\phi_\gamma(t)\psi_\gamma(x))(2\mathbb{N})^{-k\gamma},\end{aligned} \tag{3.19}$$

where we have used the notation (with $H_\gamma=H_\gamma(\omega)$ as in (2.4))

$$\theta(t,x)=\sum_\gamma\theta_\gamma(t,x)H_\gamma,\quad K(x)=\textstyle\sum_\gamma K_\gamma(x)H_\gamma,$$
$$p(t,x)=\sum_\gamma p_\gamma(t,x)H_\gamma,\quad \xi(t,x)=\textstyle\sum_\gamma\xi_\gamma(t,x)H_\gamma,$$

and the test functions $v\in(\mathscr{S})_{-1,-k,C_0^\infty(\mathbb{R}\times\mathbb{R}^d)}$ are represented by

$$v(t,x)=\sum_\gamma\phi_\gamma(t)\psi_\gamma(x)H_\gamma.$$

Now

$$\begin{aligned}A_\gamma &:= (\frac{\partial\theta_\gamma}{\partial t},\phi_\gamma(t)\psi_\gamma(x))_{L^2((0,T)\times U)} = -(\theta_\gamma,\phi_\gamma'(t)\psi_\gamma(x))_{L^2((0,T)\times U)}\\ &= -\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\theta_\gamma(t,x)\psi_\gamma(x)\,dx\right)\phi_\gamma'(t)\,dt = \int_{\mathbb{R}}\frac{d}{dt}\left(\int_{\mathbb{R}^d}\theta_\gamma(t,x)\psi_\gamma(x)\,dx\right)\phi_\gamma(t)\,dt.\end{aligned}$$

By a standard approximation procedure we may put $\phi_\gamma(s) = \chi_{[0,t]}(s)$, which gives

$$A_\gamma = \int_0^t \frac{d}{ds}\left(\int_{\mathbb{R}^d} \theta_\gamma(s,x)\psi_\gamma(x)\,dx\right) ds = \int_{\mathbb{R}^d} (\theta_\gamma(t,x) - \theta_\gamma(0,x))\psi_\gamma(x)\,dx.$$

Substituting into (3.19) we obtain

$$\sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} K_\alpha \nabla u_\beta, \nabla\psi_\gamma)(2\mathbb{N})^{-k\gamma} = \sum_\gamma (\bar{\xi}_\gamma(t,x) - \theta_\gamma(t,x) + \theta_\gamma(0,x), \psi_\gamma(x))(2\mathbb{N})^{-k\gamma}, \tag{3.20}$$

where $u = u(t,x)$ is given by the (stochastic) Baiocchi transform (3.17) and

$$\bar{\xi}(t,x) = \int_0^t \xi(s,x)\,ds.$$

Now define

$$g(t,x) = \sum_\gamma (\bar{\xi}_\gamma(t,x) - \theta_\gamma(t,x) + \theta_\gamma(0,x))H_\gamma = \bar{\xi}(t,x) - \theta(t,x) + \theta(0,x)$$

and

$$f(t,x) = \sum_\gamma (\bar{\xi}_\gamma(t,x) - \lambda_\gamma(x) + \theta_\gamma(0,x))H_\gamma = \bar{\xi}(t,x) - \lambda(x) + \theta(0,x).$$

Note that $f(t,x)$ does not depend on $\theta(t,x)$ and that

$$g(t,x) - f(t,x) = \lambda(x) - \theta(t,x).$$

Therefore by Assumption 3, we have

$$(g(t,x) - f(t,x), \psi(x))_{-1,-k,L^2} \geq 0 \tag{3.21}$$

for all $\psi \in (\mathscr{S})_{-1,-k,H_0^1}$ such that $\psi \succeq 0$. Moreover, by Assumption 4 we get

$$(g(t,x) - f(t,x), u(t,x))_{-1,-k,L^2} = 0. \tag{3.22}$$

Define the *stochastic bilinear form* $\mathscr{E}_k(\cdot,\cdot)$ on $(\mathscr{S})_{-1,-k,H_0^1}$ by

$$\mathscr{E}_k(v,w) = (K \diamond \nabla v, \nabla w)_{-1,-k,L^2} = \sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} K_\alpha \nabla v_\beta, \nabla w_\gamma)(2\mathbb{N})^{-k\gamma}, \tag{3.23}$$

for v, $w \in (\mathscr{S})_{-1,-k,H_0^1}$. Then (3.20) can be written, with $u(t,x)$ as in (3.17),

$$\mathscr{E}_k(u(t,\cdot),\psi) = (g(t,\cdot),\psi)_{-1,-k,L^2}; \ \psi \in (\mathscr{S})_{-1,-k,H_0^1} \tag{3.24}$$

and combining this with (3.21) and (3.22) we get

$$\mathscr{E}_k(u(t,\cdot),\psi) \geq (f(t,\cdot),\psi)_{-1,-k,L^2} \text{ for all } \psi \in (\mathscr{S})_{-1,-k,H_0^1} \text{ with } \psi \succeq 0$$

and

$$\mathscr{E}_k(u(t,\cdot),u(t,\cdot)) = (f(t,\cdot),u(t,\cdot))_{-1,-k,L^2}.$$

Summing up we have deduced that the assumptions (3.11)-(3.16) and Assumption 1–4 lead to the following *stochastic variational inequality* for $u(t,\cdot) = \int_0^t p(s,\cdot)\,ds$, for all $k \geq k_0$:
Define

$$\mathbb{M}_k = \{\psi \in (\mathscr{S})_{-1,-k,H_0^1};\ \psi \succeq 0\}. \tag{3.25}$$

Note that $\mathbb{M}_k$ is a closed convex subset of $(\mathscr{S})_{-1,-k,H_0^1}$. Then for all $t \in [0,T)$ we have

$$u(t,\cdot) \in \mathbb{M}_k \tag{3.26}$$

and

$$\mathscr{E}_k(u(t,\cdot),\psi) \geq (f(t,\cdot),\psi)_{-1,-k,L^2} \text{ for all } \psi \in \mathbb{M}_k \tag{3.27}$$

and

$$\mathscr{E}_k(u(t,\cdot),u(t,\cdot)) = (f(t,\cdot),u(t,\cdot))_{-1,-k,L^2}. \tag{3.28}$$

This makes the following definition natural:

Definition 3.8 *We say that $u(t,x)$ is a* weak, stochastic solution *of the stochastic moving boundary value problem (3.10) if there exists $k_0 \in \mathbb{N}$ such that $u(t,\cdot)$ satisfies the variational inequality (3.26)-(3.28) for all $k \geq k_0$ and all $t \in [0,T)$.*

Remark 3.9 *(1) Note that the connection between the solution $u(t,x)$ of (3.26)-(3.28) and the pressure process, $p(t,x)$, in (3.10) is given by the Baiocchi transform (3.17), i.e.,*

$$u(t,x) = \int_0^t p(s,x)\,ds.$$

(2) Also note that in the variational inequality (3.26)-(3.28), time, t, is reduced to a parameter whose value can be assumed fixed in $[0,T)$.

There is an equivalent, but sometimes more convenient formulation of (3.26)-(3.28).

Definition 3.10 *We say $u(t,x)$ is a weak, stochastic solution of the moving boundary problem (3.10) if there exists $k_0 \in \mathbb{N}$ such that $u(t,\cdot)$ satisfies the following variational inequality for all $k \geq k_0$ and all $t \in [0,T)$:*

$$u(t,\cdot) \in \mathbb{M}_k, \tag{3.29}$$

where $\mathbb{M}_k$ is defined in (3.25) and

$$\mathscr{E}_k(u(t,\cdot),v-u(t,\cdot)) \geq (f(t,\cdot),v-u(t,\cdot))_{-1,-k,L^2} \text{ for all } v \in \mathbb{M}_k. \tag{3.30}$$

Proof of the equivalence of Definitions 3.8 and 3.10: Assume that (3.27) and (3.28) hold, then by subtracting them we get (3.30)

Conversely, if (3.30) holds then by choosing $v(\cdot) = 2u(t, \cdot)$ and $v(\cdot) = u(t, \cdot)/2$ we get (3.28). Finally, choosing $v(\cdot) = u(t, \cdot) + \psi(\cdot)$ gives (3.27). □

We proceed to show that there is a unique solution $u(t, x)$ of the family of stochastic variational inequalities (3.29)-(3.30) for $t \in [0, T)$ and for all $k \geq k_0$, if k_0 is sufficiently large. First we establish the following general result. (For simplicity we suppress t in the following.)

Theorem 3.11 *Suppose $K \in \mathscr{P}_\ell(U)$ for some $\ell \in \mathbb{R}$ and that $f \in (\mathscr{S})_{-1,L^2}$. Then there is $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ there exists a unique $u^{(k)} \in \mathbb{M}_k$ such that*

$$\mathscr{E}_k(u^{(k)}, v - u^{(k)}) \geq (f, v - u^{(k)})_{-1,-k,L^2} \text{ for all } v \in \mathbb{M}_k, \tag{3.31}$$

where

$$\mathscr{E}_k(w, \psi) = (K \diamond \nabla w, \nabla \psi)_{-1,-k,L^2} = \sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} K_\alpha \nabla w_\beta, \nabla \psi_\gamma \Big)(2\mathbb{N})^{-k\gamma}. \tag{3.32}$$

Moreover, there exists a $C < \infty$ such that if u and $\bar{u}$ are the solutions corresponding to f and $\bar{f}$, respectively, then

$$\|u - \bar{u}\|_{-1,-k,H_0^1} \leq C\|f - \bar{f}\|_{-1,-k,H^{-1}} \leq C\|f - \bar{f}\|_{-1,-k,L^2}. \tag{3.33}$$

C does not depend on f, $\bar{f}$, k, or $\mathbb{M}_k$.

Proof: Since $K \in \mathscr{P}_\ell(U)$, Proposition 2.11 implies that there is an $L \leq 2\ell$ such that if $k \geq k_0 = -L + 1 > -L$, then (using Poincaré's inequality) we have that

$$\mathscr{E}_k(v, v) \geq A \cdot (v, v)_{-1,-k,H_0^1} \text{ for all } v \in (\mathscr{S})_{-1,-k,H_0^1},$$

where $A > 0$ does not depend on v or $k \geq k_0$. Moreover, we know that $f \in (\mathscr{S})_{-1,-k,L^2}$ for k large enough. Therefore, Theorem 2.1 in Chapter 2 in [8] applied to the Hilbert space $(\mathscr{S})_{-1,-k,H_0^1}$ ensures that there is a unique $u^{(k)} \in \mathbb{M}_k$ such that (3.31) holds. Moreover, (3.33) holds with $C = 1/A$. □

Remark 3.12 *This result gives a unique solution $u^{(k)} \in \mathbb{M}_k$ of (3.31) for each $k \geq k_0$. Note, however, that Definition 3.10 requires a $u(t, \cdot)$ which solves* all *the variational inequalities (3.31) simultaneously for $k \geq k_0$. This is achieved by the following argument:*

Suppose $u := u^{(k)}$ solves the variational inequality (3.29)-(3.30) for a given k. We claim that u also solves the variational inequality corresponding to $k + \ell$, when $\ell \geq 0$. To show this it suffices to verify that

$$u \in \mathbb{M}_{k+\ell}, \tag{3.34}$$

$$\mathscr{E}_{k+\ell}(u,v) \geq (f,v)_{-1,-(k+\ell),L^2} \text{ for all } v \in \mathbb{M}_{k+\ell}, \tag{3.35}$$

and

$$\mathscr{E}_{k+\ell}(u,u) = (f,u)_{-1,-(k+\ell),L^2}. \tag{3.36}$$

Property (3.34) is clear, since $\mathbb{M}_k \subseteq \mathbb{M}_{k+\ell}$. To check (3.35), we note that by (3.27) we have

$$\mathscr{E}_{k+\ell}(u,v) = \mathscr{E}_k(u,\hat{v}^{(\ell)}) \geq (f,\hat{v}^{(\ell)})_{-1,-k,L^2} = (f,v)_{-1,-(k+\ell),L^2},$$

for all $v \in \mathbb{M}_{k+\ell}$, since $v \in \mathbb{M}_{k+\ell} \Rightarrow \hat{v}^{(\ell)} \in \mathbb{M}_k$.

Moreover, using (3.24) and Assumption 4 we get

$$\mathscr{E}_{k+\ell}(u,u) = \mathscr{E}_k(u,\hat{u}^{(\ell)}) = (g,\hat{u}^{(\ell)})_{-1,-k,L^2} = (f,\hat{u}^{(\ell)})_{-1,-k,L^2} = (f,u)_{-1,-(k+\ell),L^2},$$

which proves (3.36). We have proved

Theorem 3.13 *Suppose $K \in \mathscr{P}_\ell(U)$ for some $\ell \in \mathbb{R}$ and that $f \in (\mathscr{S})_{-1,L^2}$. Then there exists $k_0 \in \mathbb{N}$ and $u(t,\cdot) \in (\mathscr{S})_{-1,-k_0,H_0^1}$ such that $u(t,\cdot)$ solves* all *the variational inequalities (3.31) simultaneously for all $k \geq k_0$.*

Hence the stochastic distribution process, $u(t,x)$, is the (unique) weak stochastic solution of the stochastic moving boundary problem (3.10), according to Definition 3.8.

4 Some Properties of the Solution

In the previous section we started from the stochastic partial differential equation for the moving boundary value problem and deduced that a (weak) solution must solve a family of stochastic variational inequalities. We then established existence and uniqueness of solution for these inequalities. In this last section we investigate to what extent we can deduce physical properties of the moving boundary from this solution.

First note that the *generalized expectation of* $u(t,x)$,

$$u_0(t,x) = E[u(t,x)] = \langle u(t,x), 1\rangle$$

solves the *deterministic* moving boundary value problem obtained by replacing $K(x)$, $\lambda(x)$, and $\xi(t,x)$ by their generalized expectations $K_0(x)$, $\lambda_0(x)$, and $\xi_0(t,x)$, respectively. In this case the (deterministic) wet region at time t is given by

$$D_0(t) = D(0) \cup \operatorname{supp} u_0(t,\cdot) \tag{4.1}$$

(see e.g. [14]).

To construct the saturation $\theta(t,x)$ from $u(t,x)$ we proceed as follows: Fix $t \in [0,T)$, $k \geq k_0$ and consider the mapping

$$v \mapsto \mathscr{E}_k(u,v) \text{ for } v \in (\mathscr{S})_{-1,-k,H_0^1},$$

where $\mathscr{E}_k(\cdot,\cdot)$ is defined by (3.32). By Proposition 2.10 this is a bounded linear functional on $(\mathscr{S})_{-1,-k,H_0^1}$. Hence there exists a $g = g(t,\cdot) \in (\mathscr{S})_{-1,-k,H^{-1}}$ such that

$$\mathscr{E}_k(u,v) = (g,v)_{-1,-k,L^2} \text{ for } v \in (\mathscr{S})_{-1,-k,H_0^1}. \tag{4.2}$$

Note that g does not depend on $k \geq k_0$.

From (3.27) we conclude that

$$(g(t,\cdot),\psi)_{-1,-k,L^2} \geq (f(t,\cdot),\psi)_{-1,-k,L^2} \tag{4.3}$$

for all $\psi \succeq 0$.

Now we define the *stochastic saturation* at time t, $\theta(t,\cdot)$, by

$$\theta(t,x) = \lambda(x) - g(t,x) + f(t,x), \tag{4.4}$$

then

$$(\theta(t,\cdot),\psi)_{-1,-k,L^2} \leq (\lambda,\psi)_{-1,-k,L^2} \text{ for all } \psi \succeq 0. \tag{4.5}$$

Thus we have recovered the physical property that we set up as Assumption 3 in the original moving boundary value problem.

To recover Assumption 4, note that for all $k \geq k_0$ and $\ell \geq 0$ we have

$$(\lambda(\cdot) - \theta(t,\cdot), u(t,\cdot))_{-1,-(k+\ell),L^2} = (g(t,\cdot) - f(t,\cdot), u(t,\cdot))_{-1,-(k+\ell),L^2} = 0,$$

by (4.2) and (3.36). Hence

$$\sum_\gamma (\lambda_\gamma(\cdot) - \theta_\gamma(t,\cdot), u_\gamma(t,\cdot))_{L^2} (2\mathbb{N})^{-(k+\ell)\gamma} = 0$$

for all $\ell \geq 0$, which is only possible if

$$(\lambda_\gamma(\cdot) - \theta_\gamma(t,\cdot), u_\gamma(t,\cdot))_{L^2} = 0 \text{ for all } \gamma.$$

Thus we have recovered the basic physical properties we set up for the stochastic moving boundary value problem in Assumption 3 and Assumption 4, based on our weak solution, $u(t,x)$, of the stochastic variational inequalities. However, many important questions remain. Some of them are:

(Q1) Is $u(t,x) \geq 0$ in the sense of Definition 3.1? As pointed out earlier (Section 3) Hu [7] has shown that in general the answer to this is no. However, it might be yes in some important cases. If $u(t,x) \geq 0$ we can regard $u(t,x)$ as a *measure* $u(t,x,\cdot)$ on $\mathscr{S}'(\mathbb{R}^d)$ and it is then (in view of Theorem 3.6) natural to define the *stochastic wet region at time* t by

$$D(t,\omega) = D(0) \cup \{x \in \mathbb{R}^d;\ \omega \in \operatorname{supp} u(t,x,\cdot)\}.$$

In general we can only give ω-averages of the wet region, in the following sense: If $\eta \in (\mathscr{S})_{-1,-k}$ and $\eta \trianglerighteq 0$, define *the* η*-averaged wet region* by

$$D(t,\eta) = D(0) \cup \{x;\ (u(t,x),\eta)_{-1,-k} > 0\}. \tag{4.6}$$

What are the properties of $D(t,\eta)$ for $t \geq 0$?

(Q2) What is the relation between the two 'wet regions' $D(t,\eta)$ defined by (4.6) and $V(t,\eta)$ defined by (3.18)? In the deterministic case we have $D(t) = V(t)$ (see Theorem 3.6).

(Q3) Is there a tractable constructive/numerical method for computing the coefficients $u_\gamma(t,x)$ of $u(t,x)$?

(Q4) If $\gamma = 0$ we have the following *conservation of mass formula*

$$\int_{D_0(t)-D(0)} \lambda_0(x)\,dx = \int_{D_0(t)} \bar{\xi}_0(t,x)\,dx. \tag{4.7}$$

The right hand side represents the amount of fluid being pumped into the medium up to time t, while the left hand side represents the (added) volume of the fluid in the wet region at this time. To prove (4.7) choose $\psi = \psi_0 = \chi_{V_0(t)}(x)$ (deterministic) in (3.24), where $V_0(t)$ is a neighborhood of $\overline{D_0(t)}$. Then we get

$$\begin{aligned} 0 &= \int_{V_0(t)} g_0(t,x)\,dx = \int_{V_0(t)} (f_0(t,x) + \lambda_0(x) - \theta_0(t,x))\,dx \\ &= \int_{V_0(t)} (\bar{\xi}_0(t,x) - \theta_0(t,x) + \theta_0(0,x))\,dx \\ &\to \int_{D_0(t)} \bar{\xi}_0(t,x)\,dx - \int_{D_0(t)} \lambda_0(x)\,dx + \int_{D(0)} \lambda_0(x)\,dx \end{aligned}$$

as $V_0(t) \downarrow D_0(t)$. □

Can one find a similar conservation of mass formula in the stochastic case?

Acknowledgements: One of the authors, B. Øksendal, has been supported by VISTA, a research cooperation between the Norwegian Academy of Science and Letters and Den Norske Stats Oljeselskap A.S. (Statoil).

The other author, G. Våge, has been financed by the Norwegian Academy of Science and Letters and the DFG.

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