ON ORIENTATION AND DYNAMICS IN OPERATOR ALGEBRAS. PART I

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Introduction

An old problem in operator algebras, motivated by physics, is to determine which (partially) ordered normed linear spaces can be the self-adjoint part of a C*-algebra or a von Neumann algebra. This is implicit in several papers of Segal [28, 29] and Kadison [20,21,24], and it was explicitly raised for von Neumann algebras by Sakai [27] and for C*-algebras by Sherman [30].

The self-adjoint elements of such algebras are used to represent bounded observables in algebraic models of quantum mechanics. However, the self-adjoint part A of a C*-algebra is not closed under the given associative product, but only under the symmetrized product ("Jordan product")

$$a \circ b = \frac{1}{2}(ab + ba) = \frac{1}{2}\big((a+b)^2 - a^2 - b^2\big). \tag{1}$$

This product makes A into a (real) Jordan algebra, and it has been proposed to model quantum mechanics on Jordan algebras rather than associative algebras. This approach is corroborated by the fact that many physically relevant properties of observables are adequately described by Jordan constructs. Knowing an element of A, we can express not only the expectation value of the correponding observable, but its entire probability law which is given by spectral functional calculus, and in turn by the squaring operation $a \mapsto a^2$.

The Jordan algebra approach to quantum mechanics was initiated by Jordan, von Neumann and Wigner in [17] where they introduced and studied finite dimensional "formally real" Jordan algebras. The restriction to finite dimensions was removed by von Neumann [18]. Jordan operator algebras (linear spaces of self-adjoint operators on a Hilbert space closed under the Jordan product) were first studied by Topping [34] and Størmer [32]. The general definitions of JB-algebras and JBW-algebras (together with a Gelfand-Naimark type representation theorem) were given by Alfsen, Shultz and Størmer [4] and by Shultz [31] respectively. These algebras are defined axiomatically as (real) Jordan algebras which are also Banach spaces, subject to suitable conditions connecting Jordan product and norm. The self-adjoint part of a C*-algebra or a von Neumann algebra is a special case of a JB-algebra or a von Neumann algebra respectively. Not all JB-algebras or JBW-algebras arise in this fashion (cf. [13, Ch.3-4]), but nevertheless they have enough structure to effectively model quantum mechanical observables.

However, it is an important feature of quantum mechanics that the physical variables play a dual role, as observables and as generators of transformation groups. The observables are random variables with a specified probability law in each state of the quantum system, while the generators determine one-parameter groups of transformations of observables (Heisenberg picture) or states (Schrödinger picture).

Both aspects can be adequately dealt with in the C*-algebra or von Neumann algebra formulation of quantum mechanics. An element a in the self-adjoint part A of such an algebra represents an observable whose probability law is determined by spectral theory as indicated above, while an element h in A determines the one-parameter group $\alpha_t : b \mapsto e^{iht} b e^{-iht}$ (equivalently $d\alpha_t(b)/dt = i[h,b]$), which represents the time evolution of the observable b. The spectral functional calculus is a Jordan construct, but the generation of one-parameter groups cannot be expressed in terms of the symmetrized product. Instead it is determined by the anti-symmetrized product in A, which we will write as follows

$$a \star b = \frac{i}{2}[a,b] = \frac{i}{2}(ab - ba). \tag{2}$$

Thus the decomposition of the associative product into its *Jordan part* and its *Lie part*

$$ab = a \circ b - i(a \star b) \tag{3}$$

separates the two aspects of a physical variable.

To solve the characterization problem, we must find appropriate conditions for an ordered normed linear space A, under which it is possible to define an associative product on $A + iA$ making this space a C*-algebra or a von Neumann algebra. By the discussion above, this problem can be divided in two parts: first to construct the Jordan part of the associative product, then the Lie part when the Jordan part is known.

By a theorem of Kadison [21] the ordering and the norm of a C*-algebra determine the Jordan part of the product. However, they do not determine the product itself, since the opposite algebra has the same ordering and norm but differs in the sign of the Lie part of the product. Thus, to go from the Jordan structure to the C*-structure, we must make a choice for the Lie part of the product.

It was Connes who first realized that a concept of orientation was relevant in this context [8]. In this paper he studies ordered linear spaces associated with sigma-finite von Neumann algebras. (These are the von Neumann algebras which have a faithful normal state (cf. e.g. [33, Prop. II.3.19]), or equivalently the ones which admit a faithful representation with a separating and cyclic vector ξ.) Connes concludes with a characterization of such spaces (Theorem 5.8). Here he shows that a "complex ordered linear space with order unit" (E, E^+) is isomorphic to the pair (M, M^+) for a sigma-finite von Neumann algebra M iff there exists a self-adjoint form s on E such that the completion of the cone E^+ with respect to s has the following properties: (i) it is "self-polar", (ii) it is "facially homogeneous", (iii) it is "orientable". (All three properties are defined in Connes' paper.) In the development leading up to this result, it is shown that the completion of M^+ with respect to a self-polar form s is independent of s (Theorem 2.1), and that it can be identified with the natural cone $P^\natural_\xi$ of Tomita-Takesaki theory, which can be abstractly characterized by the three properties (i),(ii),(iii) above (Theorem 5.2).

In [5] Bellissard and Iochum showed that the properties (i) and (ii) above characterize the natural cone associated in an analogous fashion with a (sigma-finite) JBW-algebra. (See also [14].) Thus in the context of such cones, Connes' notion of orientation is exactly what is needed to move from (sigma-finite) JBW-algebras to (sigma-finite) von Neumann algebras.

It follows from results of Kadison [20] that the self-adjoint part of a C*-algebra is isometrically isomorphic, as an ordered normed linear space, to the space $A(K)$ of all w*-continuous affine functions on the state space K. Similarly, the self-adjoint part of a von Neumann algebra is isometrically isomorphic to the space of all bounded affine functions on the normal state space. In view of this, characterizing the self-adjoint part of a C*-algebra (von Neumann algebra) is equivalent to characterizing the state space of a C*-algebra (the normal state space of a von Neumann algebra). This was accomplished for C*-algebras by Alfsen, Hanche-Olsen and Shultz in [3] and for von Neumann algebras by Iochum and Shultz in [15]. Here too the program proceeds by way of Jordan algebras (JB-algebras and JBW-algebras respectively), but it does not involve Tomita-Takesaki theory.

The result in [3] is based on two earlier papers of Alfsen and Shultz [1] and [2]. In [1] they gave conditions on the facial structure of a compact convex set guaranteeing that $A(K)$ admits a satisfactory spectral theory and functional calculus. This gives a candidate for a Jordan product in $A(K)$ defined in terms of squares as in equation (1). Then in [2, Th. 7.2] they gave necessary and sufficient conditions that this product be bilinear, in which case it makes $A(K)$ a JB-algebra. The key condition is the "Hilbert ball axiom" which says that each face of K which is generated by two extreme points, must be affinely isomorphic to the unit ball of a Hilbert space. (These Hilbert spaces can be of arbitrary finite or infinite dimension for a general JB-algebra. But for a C*-algebra they are of dimension 3 or 1, so for the characterization of C*-algebras the relevant condition is a "3-ball axiom" rather than a general "Hilbert ball axiom".)

As in Connes' paper, the final step is to move from Jordan structure to associative structure (here from JB-algebras to C*-algebras). Again the key role is played by a concept of "orientability". In [3] this concept is geometric; one requires that all the "facial 3-balls" (alluded to above) can be oriented in a continuous fashion with respect to the w*-topology. This can always be done locally, so that the requirement is that these local choices can be pieced together in a continuous way to give a global orientation. (See [3, §7] for the precise definition of orientation and [3, Th. 8.4] for the main result characterizing state spaces of C*-algebras.) Note also that if K is the state space of a C*-algebra, then there is a 1-1 correspondence between all "global orientations" on K and all those associative products on $A + iA$ which organize this space to a C*-algebra inducing the same Jordan product on A as the original algebra [3, Cor.8.5].

In [15], Iochum and Shultz first characterize the normal state space of a JBW-algebra by conditions closely related to Connes' facial homogeneity axiom. Then they characterize the self-adjoint part of a von Neumann algebra among all JBW-algebras. This is more difficult than the similar problem for C*-algebras in one respect, and easier in another. Since the (generally non-compact) normal state space of a von Neumann algebra may be devoid of extreme points, one must use a modified and more complicated version of the 3-ball axiom in this case. On the other hand, no orientability condition is needed in [15] to solve the characterization problem for the normal state space of a von Neumann algebra. (Nevertheless, here too one can define a notion of "global orientation" in 1-1 correspondence with associative products in the same way as for C*-algebras, but now orientability is automatic, as we will see in Part II.)

The concept of orientation introduced by Connes in [8] and that introduced by ourselves in [3] are completely different in character and relate to different contexts, but the purpose is quite similar: to pass from Jordan structure to associative structure. More specifically, Connes' "orientability axiom" provides a passage from JBW-algebras to von Neumann algebras, while our "orientability axiom" (together with the "3-ball axiom") provides a passage from JB-algebras to C*-algebras. One of the main purposes of the current paper is to relate these two notions.

In the present Part I we will give a solution of the characterization problem which relates to dynamics and applies equally well in the C* and the von Neumann case. In the forthcoming Part II we will develop a general theory of orientation for C*-algebras and von Neumann algebras, which is of geometric nature and bridges the two original approaches to orientation for such algebras.

We will now give a brief survey of the content of the paper.

We begin by a short summary of definitions and results from the theory of JB-algebras and JBW-algebras (with reference to [13] for proofs). All results are direct generalization of known results for C*-algebras and von Neumann algebras, so readers mainly interested in the theory of orientation for C*-algebras and von Neumann algebras can avoid the intricacies of Jordan algebras.

Then we will transfer Connes' notion of orientation from the natural cone of a JBW-algebra A to the algebra itself, and we will call the resulting notion a "Connes orientation on A". Like the original concept, such an orientation is a complex structure on the Lie algebra of "order derivations" modulo its center (Definition 16).

Our next step will be to introduce yet another notion which is closely related to that of a Connes orientation. This notion, which makes sense both in the JB and the JBW context (and in particular in the C* and the von Neumann context) will be called a "dynamical correspondence". It is defined to be a (suitably axiomatized) correspondence which assigns a "skew order derivation" ψ_a to each element a of the given algebra A (Definition 17). The skew order derivations are generators of one-parameter groups of unital order automorphisms of A, and by duality also of one-parameter groups of motions of the state space K of A. Thus a dynamical correspondance gives the elements of A a double identity, which reflects the dual role of physical variables as observables and as generators of a one-parameter group of motions of the state space. Hence the name "dynamical correspondence". (For related notions, see [6, 7, 9, 25].)

To motivate the definition of a dynamical correspondence, we explain the geometrical meaning of this notion in the case of the 2×2 matrix algebra (which models a 2-level quantum system, cf. e.g. [26, Ch.15]). Here the state space is a Euclidean 3-ball, and a self-adjoint element of the algebra acts as an affine function on the ball. This function attains its maximum and its minimum at two antipodal points, and the corresponding one-parameter group consists of rotations about the diameter between these two points (in either one of the two possible directions depending on orientation). The geometric meaning of a dynamical correspondence in the general case will be explained in Part II.

Note that a Connes orientation is defined in the context of JBW-algebras, while a dynamical orientation is defined in the general context of unital JB-algebras (which include JBW-algebras as a special case). However, for JBW-algebras, it is shown that each Connes orientation determines a unique dynamical correspondence (Proposition 21), and conversely that each dynamical orientation on a JBW-algebra arises in this way from a unique Connes orientation on the algebra (Corollary 24).

Our main result is Theorem 23 by which a unital JB-algebra A is the self-adjoint part of a C*-algebra iff there exists a dynamical correspondence on A, in which case there is a natural 1-1 map from the dynamical correspondences on A to those C*-products on $A + iA$ which induce the given

Jordan structure on A. The same conclusions hold with "JBW" in place of "JB" and "von Neumann" in place of "C*".

In Part II we will concentrate on C*-algebras and von Neumann algebras, for which we will define our general notion of orientation. Like the orientation in [3], it is defined geometrically, but here without use of extreme points. Nevertheless, the definition is "local" in that one prescribes an orientation on "small subsystems" and then requires that the choice varies continuously. This local geometric notion of orientation provides a unified framework for studying the passage from Jordan structure to associative structure in C*-algebras and von Neumann algebras. In this framework we will describe the geometry of dynamical correspondences and complete the process of relating the various notions studied in Part I.

Order Derivations

We begin by giving the definition of JB-algebras and JBW-algebras and some of their basic properties. (A comprehensive treatment of such algebras can be found in [13]).

1. Definition. A *JB-algebra* is a real Jordan algebra A which is also a Banach space such that the Jordan product and the norm are connected by the following conditions for a, b in A

(i) $\|a \circ b\| = \|a\| \circ \|b\|$
(ii) $\|a^2\| = \|a\|^2$
(iii) $\|a^2\| \leq \|a^2 + b^2\|$.

A JB-algebra with identity element 1 is said to be *unital.* In this paper, we will always assume our JB-algebras are unital.

2. Definition. A *JBW-algebra* is a JB-algebra A which is the dual of a Banach space A_*. The space A_* is unique [13, Th.4.4.16], and it is called the *predual* of A.

The self-adjoint part of a C*-algebra is a JB-algebra (for the product $a \circ b = \frac{1}{2}(ab + ba)$), and the self-adjoint part of a von Neumann algebra is a JBW-algebra. In fact, many of the basic constructs for these associative algebras can be carried over to their Jordan counterparts. Note in this connection that since a JBW-algebra is a special case of a JB-algebra, all definitions given for JB-algebras also apply to JBW-algebras.

A unital JB-algebra A is a complete order unit space with positive cone $A^+ = \{a^2 \mid a \in A\}$ such that for $a \in A$

$$-1 \leq a \leq 1 \;\Rightarrow\; 0 \leq a^2 \leq 1, \tag{4}$$

and this actually characterizes unital JB-algebras among all real Jordan algebras with identity [13, Prop. 3.1.6].

A linear functional ρ on a JB-algebra A is said to be *positive*, denoted $\rho \geq 0$, if $\rho(a) \geq 0$ for all $a \in A$. The set of all positive functionals in A^* is a w*-closed (convex) cone, denoted by $(A^*)^+$. If A is a unital JB-algebra, then the set of all $\rho \in (A^*)^+$ such that $\rho(1) = 1$ is a w*-compact convex set called the *state space* of A. Elements of the state space are called *states*, and the extreme points of the state space are called *pure states*.

A JB-algebra is *monotone complete* if every upper bounded increasing net $\{a_\alpha\}$ has a least upper bound a in A. A bounded linear functional ρ on A is called *normal* if $\rho(a_\alpha) \to \rho(a)$ for each net $\{a_\alpha\}$ as above. The positive normal linear functionals form a *separating set* for A if for every non-zero $a \in A$ there exists a positive normal linear functional ρ such that $\rho(a) \neq 0$. (This fact together with monotone completeness characterize the JBW-algebras among all JB-algebras, and this characterization is taken as the definition in [13].)

The predual A_* of a JBW-algebra A can be identified with the subspace of A^* which consists of all normal linear functionals [13, Th.4.4.16]. We will use the term *σ-weak topology* to denote the topology on A determined by the duality with A_* (the $\sigma(A, A_*)$-topology in Bourbaki's terminology). Thus, the σ-weakly continuous linear functionals are precisely the normal ones. Note that every JBW-algebra A is unital [13, Lem.4.1.7]. The convex set of normal states on A is called the *normal state space* of A. If A is a JB-algebra, then A^{**} can be made into a JBW-algebra in such a way that the state space of A is identified with the normal state space of A^{**} [13, §4.4]. Furthermore, multiplication is separately σ-weakly continuous on a JBW-algebra [13, Cor. 4.1.6], so we can often make use of the σ-weak density of A in A^{**}.

We will now introduce an order theoretic concept of derivation which plays an important role in Connes' paper [8]. It can be defined in the general context of ordered linear spaces, but we will only give the definition for JB-algebras. But first some motivating remarks.

Derivations occur in many different contexts. What is common for various derivations δ, is the fact that they are linear operators generating a one-parameter group of maps $e^{t\delta}$ which preserve the algebraic structure under study. In our present context, we are focusing on the order structure, ignoring the multiplicative aspect. Therefore the Leibniz rule is not relevant here.

3. Definition. A bounded linear operator δ on a JB-algebra A is called an *order derivation* if $e^{t\delta}(A^+) \subset A^+$ for all $t \in \mathbf{R}$, or what is equivalent, if $\{e^{t\delta}\}_{t\in\mathbf{R}}$ is a one-parameter group of order automorphisms.

4. Lemma. *An order derivation δ on a JBW-algebra A is σ-weakly* *continuous.*

Proof. We will first show that the map $\phi = e^{t\delta}$ is σ-weakly continuous for given $t \in \mathbf{R}$, or what is the same, that $\rho \circ \phi \in A_*$ for every $\rho \in A_*$. Since ϕ is an order automorphism and ρ is a normal linear functional, then $\rho \circ \phi$ is also a normal linear functional. Thus $\rho \circ \phi \in A_*$ as desired.

The order derivation δ is the norm limit of $t^{-1}(e^{t\delta} - 1)$ when $t \to 0$, so we can find a sequence $\{\psi_n\}$ of σ-weakly continuous linear maps such that $\|\psi_n - \delta\| \to 0$. Let $\rho \in A_*$. Then $\rho \circ \psi_n \in A$ for every n and $\|\rho \circ \psi_n - \rho \circ \delta\| \to 0$. Since A_* is complete, $\rho \circ \delta \in A_*$. Since $\rho \in A$ was arbitrary, δ is σ-weakly continuous. □

We will give a necessary and sufficient condition that a linear operator be an order derivation. The idea behind this criterion is the following: If δ is an order derivation, then the orbit of $e^{t\delta}$ through a boundary point of the cone A^+ cannot proceed to the exterior of A^+, nor to the interior, so the velocity vector must lie in the tangent space. Connes turned this heuristic argument into a precise criterion for self-polar cones [8, Lemma 5.3]. Later on Hanche-Olsen and Evans generalized it to arbitrary cones with the *nearest point property*, i.e. to cones for which the minimum distance from an arbitrary point to a point in the cone is effectively attained [12]. In this form it can be applied also in our context, as the positive cone of a unital JB-algebra, and in fact of every order unit space, has the nearest point property. This is an elementary result, which is certainly well known. But since we have not been able to find a good reference, we include the proof.

5. Lemma. *The positive cone A^+ of an order unit space A has the nearest point property.*

Proof. Let $a \notin A^+$. Observe that there exists $\lambda \in \mathbf{R}^+$ such that $a + \lambda 1 \in A^+$. In fact, we can take $\lambda = \|a\|$, since $a \geq -\|a\|1$ (by the definition of order unit [13, 1.2.1]). Set

$$\lambda_0 = \inf\{\lambda \in R \mid a + \lambda 1 \in A^+\}. \tag{5}$$

Since the positive cone A^+ is closed, there is an element $b = a + \lambda_0 1 \in A^+$. We claim that b is a nearest point for a, i.e. that $\|c - a\| \geq \|b - a\| = \lambda_0$ for every $c \in A^+$.

Let $c \in A^+$ and set $\lambda = \|c - a\|$. Then $c \leq a + \lambda 1$ (again by the definition of the order unit). Since $c \in A^+$, also $a + \lambda 1 \in A^+$. Hence $\lambda \geq \lambda_0$ as claimed. □

6. Lemma. *A bounded linear operator δ on a unital JB-algebra is* *an order derivation iff the following implication holds for all $a \in A^+$ and $\rho \in (A^*)^+$*

$$(6) \qquad \rho(a) = 0 \;\Rightarrow\; \rho(\delta a) = 0.$$

Proof. By Theorem 1 of [12] the quoted statement holds in the context of ordered Banach spaces with the nearest point property. By Lemma 5 it can be applied in our case. □

We will denote the Jordan multiplier determined by an element b of a JB-algebra A by δ_b. Thus for all $a \in A$

$$(7) \qquad \delta_b(a) = b \circ a.$$

7. Lemma. *Let A be a unital JB-algebra. Then δ_b is an order derivation for every $b \in A$.*

Proof. Suppose $\rho = 0$ for $a \in A^+$ and $\rho \in (A^*)^+$. By the Cauchy-Schwartz inequality for JB-algebras [13, 3.6.2],

$$(8) \qquad \|\rho(\delta_b a)\|^2 = \|\rho(b \circ a)\|^2 \leq \rho(b^2)\rho(a^2).$$

Generally $a^2 \leq \|a\|a$ for every $a \in A^+$. In fact, the Jordan triple product (defined in [13, 2.3.2]) determines an order preserving map $a \mapsto \{cac\}$ for every $c \in A$ [13, 3.3.6], so if we evaluate $a^{\frac{1}{2}}$ by spectral theory [13, 3.2.4], we can write

$$a^2 = \{a^{\frac{1}{2}} a a^{\frac{1}{2}}\} \leq \{a^{\frac{1}{2}}(\|a\|1)a^{\frac{1}{2}}\} = \|a\|a.$$

Now it follows from (8) that

$$\|\rho(\delta_b a)\|^2 \leq \rho(b^2)\|a\|\rho(a) = 0.$$

By Lemma 6, δ_b is an order derivation. □

8. Definition. An order derivation δ on a unital JB-algebra A is *self-adjoint* if $\delta = \delta_a$ for some $a \in A$, and it is *skew-adjoint* (or just *skew*) if $\delta(1) = 0$.

Our next lemma shows, among other things, that the skew order derivations are the *Jordan derivations*, i.e. the bounded linear operators δ which satisfy the Leibniz rule

(9) $$\delta(a \circ b) = (\delta a) \circ b + a \circ (\delta b).$$

9. Lemma. *Let A be a unital JB-algebra with state space K and let δ be an order derivation on A. For every $t \in \mathbf{R}$ let $\alpha_t = e^{t\delta}$ and let α_t^* be the dual map defined on A^* by $(\alpha_t^* \rho) = \rho(\alpha_t(a))$ for $\rho \in A^*$ and $a \in A$. Now the following are equivalent:*

(i) δ *is skew*
(ii) $\alpha_t(1) = 1$ *for all* t
(iii) α_t *is a Jordan automorphism for all* t
(iv) δ *is a Jordan derivation*
(v) $\alpha_t^*(K) \subset K$ *for all* t.

Proof. *(i)* $\Rightarrow$ *(ii)* Use the exponential series for $e^{t\delta}$.

(ii) $\Rightarrow$ *(iii)* By a known theorem of Kadison, every unital order automorphism of a C*-algebra is a Jordan automorphism [21]. The same result is in fact valid for a JB-algebra. It can most easily be obtained from Theorem 12.13 of [1], by which the Jordan square $a^2 = a \circ a$ of an element a of A (and hence every Jordan product $a \circ b$) is completely determined by the ordering and the order unit (via the spectral integral $\int \lambda^2 de_\lambda$ of [1, equation (8.28)]). Thus, if α_t is a unital order automorphism, then it is also a Jordan automorphism.

(iii) $\Rightarrow$ *(iv)* Since α_t is a Jordan automorphism, then $\alpha(a \circ b) = \alpha(a) \circ \alpha(b)$ for $a, b \in A$. By the standard argument

$$\delta(a \circ b) = \lim_{t \to 0} t^{-1}(\alpha_t(a) \circ \alpha_t(b) - a \circ b) = (\delta a) \circ b + a \circ (\delta b).$$

(iv) $\Rightarrow$ *(i)* By Leibniz' rule $\delta(1) = \delta(1 \circ 1) = 2(\delta(1)$. Hence $\delta(1) = 0$.

(ii) $\Leftrightarrow$ *(v)* Trivial. □

For our next proof we shall need two elementary results valid for elements x, y, z in a unital Banach algebra. The first is the equation

(10) $$\lim_{n \to \infty} \|(1 + n^{-1}x)^n - e^x\| = 0,$$

which follows from the continuity of the holomorphic functional calculus. The second is the inequality

(11) $$\|y^n - z^n\| \leq n \cdot \mathrm{Max}\{\|y\|, \|z\|\}^{n-1} \|y - z\|,$$

which follows from the decomposition

$$y^n - z^n = y^{n-1}(y - z) + y^{n-2}(y - z)z + \cdots + (y - z)z^{n-1}.$$

We will denote the set of order derivations of a JB-algebra A by $D(A)$.

10. Proposition. *The set $D(A)$ of order derivations of a unital JB-algebra A is a real linear space closed under Lie brackets $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$.*

Proof. The fact that $D(A)$ is closed under linear operations, follows directly from Lemma 6.

To show that $D(A)$ is closed under Lie brackets, it suffices to show that $[\delta_1, \delta_2]$ is an order derivation for a given pair $\delta_1, \delta_2 \in D(A)$.

By looking at the first few terms of the exponential series involved, we see that

$$e^{t\delta_1}e^{t\delta_2}e^{-t\delta_1}e^{-t\delta_2} = 1 + t^2[\delta_1, \delta_2] + t^4\phi_t, \tag{12}$$

where $\|\phi_t\|$ is bounded for t in a neighbourhood of 0.

Set $t_n = n^{-\frac{1}{2}}$ and define $\alpha_n = e^{t_n\delta_1}e^{t_n\delta_2}e^{-t_n\delta_1}e^{-t_n\delta_2}$, $\beta_n = 1+t_n^2[\delta_1, \delta_2]$ and $\gamma_n = \phi_{t_n}$ for $n = 1, 2, \cdots$. Now

$$\alpha_n - \beta_n = n^{-2}\gamma_n \tag{13}$$

for $n = 1, 2, \cdots$, and $\{\|\gamma_n\|\}$ is a bounded sequence.

Clearly $(\alpha_n)^n$ is an order automorphism for every n. It follows from (10) that $\|(\beta_n)^n - \exp[\delta_1, \delta_2]\| \to 0$ when $n \to \infty$. Thus we only have to show that $\|(\alpha_n)^n - (\beta_n)^n\| \to 0$ when $n \to \infty$.

We will prove this by applying (11) with α_n and β_n in place of y and z, and we begin by showing that $\{\|\alpha_n\|^n\}$ and $\{\|\beta_n\|^n\}$ are bounded sequences. By (12)

$$\|\alpha_n\| \le 1 + n^{-1}\|[\delta_1, \delta_2]\| + n^{-2}\|\gamma_n\|$$

for $n = 1, 2, \cdots$. We will assume $[\delta_1, \delta_2] \neq 0$ (otherwise there is nothing to prove). Let $\lambda > 1$ be arbitrary. Then for sufficiently large n

$$\|\alpha_n\| \le 1 + \lambda n^{-1}\|[\delta_1, \delta_2]\| \le \exp\big(\lambda n^{-1}\|\delta_1, \delta_2\|\big),$$

which gives

$$\|\alpha_n\|^n \le \exp\big(\lambda\|[\delta_1, \delta_2]\|\big).$$

Since $\lambda > 1$ was arbitrary,

$$\overline{\lim_{n\to\infty}}\|\alpha_n\|^n \le \exp\big(\|[\delta_1, \delta_2\|\big).$$

For every n

$$\|\beta_n\| \leq 1 + n^{-1}\|[\delta_1, \delta_2]\|,$$

and then also

$$\|\beta_n\|^n \leq \exp\bigl(\|[\delta_1, \delta_2]\|\bigr).$$

Let $M > \exp\|[\delta_1, \delta_2]\| \geq 1$. By the inequalities above, $\|\alpha_n\|^n \leq M$ and $\|\beta_n\|^n \leq M$, and then also $\|\alpha_n\|^{n-1} \leq M$ and $\|\beta_n\|^{n-1} \leq M$, for sufficiently large n. Now it follows from (11) and (13) that

$$\|(\alpha_n)^n - (\beta_n)^n\| \leq nM\|\alpha_n - \beta_n\| \leq n^{-1}M\|\gamma_n\|$$

for large n. Thus $\|(\alpha_n)^n - (\beta_n)^n\| \to 0$ when $n \to \infty$, and we are done. □

11. Lemma. *Every order derivation δ on a unital JB-algebra A can be decomposed uniquely as the sum of a self-adjoint and a skew derivation, namely $\delta = \delta_a + \delta'$ where $a = \delta(1)$.*

Proof. Set $a = \delta(1)$ and $\delta' = \delta - \delta_a$. Then $\delta'(1) = a - a \circ 1 = 0$, so $\delta = \delta_a + \delta'$ is a decomposition of the desired type. If $\delta = \delta_b + \delta''$ is another such decomposition, then evaluation at 1 gives $\delta(1) = b \circ 1 = b$, so $b = a$. Therefore the decomposition is unique. □

To each $\delta \in D(A)$ we will associate the *adjoint* order derivation δ^* defined by $\delta^* = \delta_a - \delta'$ where $\delta = \delta_a + \delta'$ as above. Thus $\delta \in D(A)$ is self-adjoint iff $\delta^* = \delta$.

Two elements a, b of a JB-algebra A are said to *operator commute* if the operators δ_a, δ_b commute, i.e. if $(a \circ x) \circ b = a \circ (x \circ b)$ for all $x \in A$. If A is the self-adjoint part of a C*-algebra, then a, b operator commute iff $ab = ba$ [11, Lem. 5.1]. The set of all elements in a JB-algebra A which operator commute with every other element of A is called *the center* of A, and it will be denoted by $Z(A)$. Note that $Z(A)$ is an associative subalgebra of A. We will also denote by $Z(D(A))$ the center of the Lie algebra $D(A)$, i.e. the set of all $\delta \in D(A)$ such that $[\delta, \delta'] = 0$ for every other element δ' of $D(A)$.

12. Lemma. *If δ is a skew order derivation on a unital JB-algebra A and $z \in Z(A)$, then*

(i) $e^{t\delta} z = z$ *for all* $t \in \mathbf{R}$
(ii) $\delta z = 0$.

Proof. Recall that the bidual A^{**} is a JBW-algebra. By σ-weak continuity of multiplication in each variable separately, the bidual map $(e^{t\delta})^{**}$ is also a Jordan automorphism. Furthermore, again by σ-weak continuity of multiplication, the center of A will be contained in the center of A^{**}, so it suffices to prove the lemma for the special case where A is a JBW-algebra. Then it is enough to prove the lemma for the case where z is a central idempotent, i.e. $z^2 = z$. Since δ is skew, then $e^{t\delta}$ is a Jordan automorphism, so $e^{t\delta}z$ is also a central idempotent for $t \in R$. Thus $e^{t\delta}z - z$ is the difference of two central projections, so $\|e^{t\delta}z - z\|$ is either zero or one. Since $\|e^{t\delta}z - z\|$ is a continuous function of t which is zero when $t = 0$, it must be zero for all t. This proves (i).

Now also $\delta z = \lim_{t\to 0} t^{-1}(e^{t\delta}z - z) = 0$, which proves (ii). □

13. Lemma. *If A is a unital JB-algebra, then*

$$Z(D(A)) = \{\delta_z \mid z \in Z(A)\}. \tag{14}$$

Proof. Assume first that $\delta \in Z(D(A))$. In particular $[\delta, \delta_a] = 0$ for every $a \in A$. Let $z = \delta(1)$. Then for every $a \in A$

$$\delta(a) = \delta\delta_a(1) = \delta_a\delta(1) = \delta_a(z) = a \circ z = z \circ a.$$

Hence $\delta = \delta_z$. Also $\delta_z\delta_a = \delta_a\delta_z$, so $z \in Z(A)$.

Assume next that $z \in Z(A)$. By the definition of $Z(A)$), δ_z commutes with every self-adjoint order derivation δ_a. Therefore we only have to show that δ_z commutes with every skew derivation δ. But such a derivation is a Jordan derivation, so it follows from the Leibniz rule (9) and Lemma 12 (ii) that

$$\delta\delta_z(x) = \delta(z \circ x) = (\delta z) \circ x + z \circ (\delta x) = \delta_z\delta(x)$$

for every $x \in A$. Thus $\delta\delta_z = \delta_z\delta$ as desired. □

If A is the self-adjoint part of a C*-algebra $\mathcal{A}$, then we will assign to each $d \in \mathcal{A}$ a linear operator δ_d on A defined by

$$\delta_d(x) = \frac{1}{2}(dx + xd^*) \tag{15}$$

for $x \in A$.

14. Lemma. *If A is the self-adjoint part of a C^*-algebra $\mathcal{A}$ and $\delta = \delta_d$ for $d \in \mathcal{A}$, then for $x \in A$ and $t \in \mathcal{R}$*

$$exp(2t\delta)(x) = e^{td}xe^{td^*}. \tag{16}$$

In particular if $\delta = \delta_a$ *for* $a \in A$ *(self-adjoint case), then*

$$exp(2t\delta)(x) = e^{ta}xe^{ta}, \tag{17}$$

and if $\delta = \delta_{ib}$ *for* $b \in A$ *(skew case), then*

$$exp(2t\delta)(x) = e^{itb}xe^{-itb}. \tag{18}$$

Proof. Consider the left and right multiplication operators $L_d : x \mapsto dx$ and $R_d : x \mapsto xd^*$ defined on $\mathcal{A}$ for $d \in \mathcal{A}$. Since L_d and R_d^* commute, then for $x \in A$

$$\exp(2t\delta_d)(x) = \exp(tL_d + tR_{d^*})(x) = \exp(tL_d)\exp(tR_{d^*})(x) = e^{td}xe^{td^*}.$$

This proves (16), from which (17) and (18) both follow. □

15. Proposition. *If* A *is the self-adjoint part of a von Neumann algebra* $\mathcal{A}$*, then the order derivations of* A *are the operators* δ_d *on* A *defined above, and an order derivation* δ *is self-adjoint (skew) iff it is of the form* δ_d *for* d *self-adjoint (skew).*

Proof. An order derivation is self-adjoint iff it is of the form δ_a for $a \in A$ (by definition). Clearly also, an order derivation is skew if it is of the form δ_{ib} (i.e. of the form δ_d with d skew). Conversely we will show that an arbitrary skew order derivation δ is of this form

For every $t \in R$ the map $e^{t\delta}$ is a Jordan automorphism of A. We extend it by (complex) linearity to all of $\mathcal{A}$. By a theorem of Kadison [22] there is a central projection c such that $e^{t\delta}$ acts as a *-isomorphism from $c\mathcal{A}$ into $\mathcal{A}$ and a *-anti-isomorphism from $(1-c)\mathcal{A}$ into $\mathcal{A}$. By Lemma 9, $e^{t\delta}$ fixes c. Hence $e^{t\delta}$ is a *-automorphism of $c\mathcal{A}$ and a *-anti-automorphism of $(1-c)\mathcal{A}$. Applying $e^{t\delta}$ twice, we observe that $e^{2t\delta}$ acts as a *-automorphism also on $(1-c)\mathcal{A}$. Since $t \in R$ was arbitrary, this means that $e^{t\delta}$ is in fact a *-automorphism of $\mathcal{A}$ for every $t \in R$. Thus by the Kadison-Sakai theorem the generator δ of the one-parameter group $\{e^{t\delta}\}_{t\in R}$ is an inner derivation on $\mathcal{A}$, i.e. $\delta(x) = \frac{1}{2}(hx - xh)$ for some $h \in \mathcal{A}$ and all $x \in A$ [19, Ex.8.7.55].

Let $h = a + ib$ where $a, b \in A$. Then for each $x \in A$

$$\delta(x) = -i\delta_{ia}(x) + \delta_{ib}(x),$$

and since $\delta(x) \in \mathcal{A}$ then $\delta_{ia}(x) = 0$ and $\delta(x) = \delta_{ib}(x)$. Thus $\delta = \delta_{ib}$.

If $d \in \mathcal{A}$, say $d = a + ib$, then δ_d is the sum of the order derivations δ_a and δ_{ib}, so δ_d is also an order derivation. Conversely we will show that every order derivation is of this form.

Assume that δ is an arbitrary order derivation on A and consider the decomposition $\delta = \delta_a + \delta'$ established in Lemma 11. By the argument above, $\delta' = \delta_{ib}$ for some $b \in A$. Thus for every $x \in A$

$$\delta(x) = \frac{1}{2}(ax + xa) + \frac{i}{2}(bx - xb) = \frac{1}{2}\big((a + ib)x + x(a + ib)^*\big) = \delta_{a+ib}(x),$$

so δ is of the desired form. □

The results above can easily be dualized to the predual $\mathcal{A}_*$ of the von Neumann algebra $\mathcal{A}$. If δ is an order derivation of A and $\alpha_t = e^{t\delta}$, then we consider the dual operator $(\alpha_t)_*$ on the self-adjoint part A_* of $\mathcal{A}_*$ for $t \in \mathcal{R}$. Generally $\{(\alpha_t)_*\}_{t\in R}$ is a one-parameter group of order automorphisms of A_*, and if δ is skew then each $(\alpha_t)_*$ leaves the normal state space invariant (Lemma 9 (v)), so $\{(\alpha_t)_*\}_{t\in R}$ is a one-parameter group of affine automorphisms of the normal state space.

The orbits of $\{(\alpha_t)_*\}_{t\in R}$ can be easily visualized in the case where $\mathcal{A}$ is the 2×2 matrix algebra. Here the (normal) state space is a Euclidean 3-ball, and the pure state space is the surface of the ball, i.e. a Euclidean 2-sphere. If $a \in \mathcal{A}$ is self-adjoint and has two distinct eigenvalues $\lambda_1 < \lambda_2$ corresponding to (unit) eigenvectors ξ_1, ξ_2, then the vector states $\omega_{\xi_1}, \omega_{\xi_2}$ are antipodal points on the sphere (South Pole and North Pole on Fig.1). If $\delta = \delta_{ia}$ (the skew case), then $(\alpha_t)_*$ is a rotation of the ball by an angle $t(\lambda_1 - \lambda_2)/2$ about the diameter $[\omega_{\xi_1}, \omega_{\xi_2}]$. Thus the one-parameter group $\{(\alpha_t)_*\}_{t\in R}$ represents a rotational motion with rotational velocity $(\lambda_1 - \lambda_2)/2$ about this diameter, and the orbits on the sphere are the "parallel circles" (in planes orthogonal to $[\omega_{\xi_1}, \omega_{\xi_2}]$). If $\delta = \delta_a$ (the self-adjoint case), then the orbits will take us out of the state space. But this can be remedied by a normalization, i.e. by considering the parametric curves $t \mapsto \|(\alpha_t)_*\sigma\|^{-1}(\alpha_t)_*\sigma$ instead of $t \mapsto (\alpha_t)_*\sigma$. These are the "longitudinal semi-circles" on the sphere (in planes through $[\omega_{\xi_1}, \omega_{\xi_2}]$). The proof of these facts is elementary matrix calculation and will be omitted.

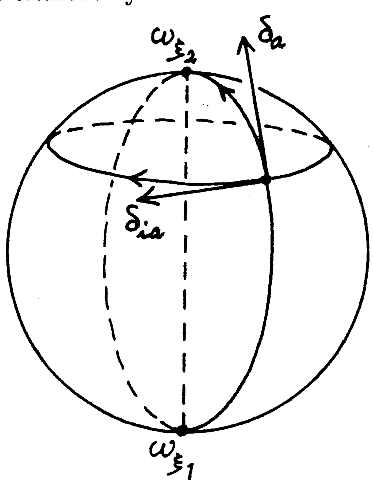

Fig.1

In the example above we can easily see how the one-parameter group is determined by the geometry in the self-adjoint case, and we can also see what indeterminacy there is in the skew case. The self-adjoint element $a \in \mathcal{A}$ determines a real valued affine function $\hat{a} : \omega \mapsto \omega(a)$ on the state space. This function attains its minimum λ_1 at ω_{ξ_1} and its maximum λ_2 at ω_{ξ_2}. In the self-adjoint case the orbits are the longitudinal semi-circles traced out in the direction from ω_{ξ_1} to ω_{ξ_2}. In the skew case the orbits are the parallel circles, but they can be traced out in two possible directions "eastbound" and "westbound". The mere knowledge of the affine function $\hat{a}$ does not tell us which direction to choose. This would require a specific orientation of the ball (right handed or left handed around $\overrightarrow{\omega_{\xi_1}\omega_{\xi_2}}$).

Connes orientations and dynamical correspondences

We will now transfer Connes' concept of orientation [8, Def. 4.11] to the context of JBW-algebras. The idea is to axiomatize the map $\delta_d \mapsto \delta_{id}$ of $D(A)$ into itself where A is the self-adjoint part of a von Neumann algebra $\mathcal{A}$, or rather the corresponding map which is obtained when elements of $D(A)$ are identified modulo $Z(D(A))$.

To simplify the notation, we will write $\tilde{D}(A)$ in place of $D(A)/Z(D(A))$. We will also denote the equivalence class of an element δ of $D(A))$ modulo $Z(D(A))$ by $\tilde{\delta}$. Note that the involution $\tilde{\delta}^* = \widetilde{(\delta^*)}$ is well defined on $\tilde{D}(A))$, for if $\tilde{\delta}_1 = \tilde{\delta}_2$ then $\delta_1 - \delta_2 = \delta_z$ for some $z \in Z(A)$ (Lemma 13), Hence $\delta_1^* - \delta_2^* = \delta_z^* = \delta_z$, so $\widetilde{(\delta_1^*)} = \widetilde{(\delta_2^*)}$.

16. Definition. A *Connes orientation* on a JBW-algebra A is a complex structure on $\tilde{D}(A)$, which is compatible with Lie brackets and involution, i.e. a linear operator I on $\tilde{D}(A)$ which satisfies the requirements

(i) $I^2 = -1$ (the identity map).
(ii) $[I\tilde{\delta}_1, \tilde{\delta}_2] = [\tilde{\delta}_1, I\tilde{\delta}_2] = I[\tilde{\delta}_1, \tilde{\delta}_2]$
(iii) $I(\tilde{\delta}^*) = -(I\tilde{\delta})^*$.

If A is the self-adjoint part of a Von Neumann algebra $\mathcal{A}$, then it is easily verified that $I : \tilde{\delta}_d \mapsto \tilde{\delta}_{id}$ (with $d \in \mathcal{A}$) is a Connes orientation of A. We call it *the Connes orientation induced on A from $\mathcal{A}$.* (Note that the map $I_0 : \delta_d \mapsto \delta_{id}$ from $D(A)$ into itself is not well-defined, since d is not determined by δ_d if d is not known to be self-adjoint.)

An alternative approach is to take the basic properties of the map $a \mapsto \delta_{ia}$ where a is a self-adjoint element of a von Neumann algebra $\mathcal{A}$ as axioms for a map ψ which assigns to each element a in a general unital JB-algebra A a skew order derivation ψ_a on A. Geometrically ψ assigns to each real valued affine function on the state space K of A a one-parameter group of affine automorphisms of K. Such a one-parameter-group describes a motion of states, and we will call ψ a "dynamical correspondence". The precise definition is the following:

17. Definition. A *dynamical correspondence* on a unital JB-algebra A is a map $\psi : a \mapsto \psi_a$ from A into the set of skew order derivations on A which satisfies the requirements

(i) $[\psi_a, \psi_b] = -[\delta_a, \delta_b]$ for $a, b \in A$
(ii) $\psi_a a = 0$ for all $a \in A$.

A dynamical correspondence on a JB-algebra A will be called *complete* if it maps A onto the set of all skew order derivations on A.

It is easily verified that condition (ii) above is equivalent to the statement that $\exp(t\psi_a)$ fixes a for all $a \in A$ and all $t \in R$. This property is easily visualized in the 2×2 matrix example, and it will play an important role in the geometrical investigations in Part II.

We will also state the definition of a dynamical correspondence in another form. In this connection we shall need the following lemma, which will also be needed later.

18. Lemma. *Let A be a unital JB-algebra and let ψ be a map from A into the set of all skew order derivations on A. Then for all pairs $a, b \in A$*

$$[\psi_a, \delta_b] = \delta_{\psi_a b}. \tag{19}$$

Proof. Since ψ_a is skew, it is a Jordan derivation. Hence for all $c \in A$

$$\psi_a(b \circ c) = (\psi_a b) \circ c + b \circ (\psi_a c),$$

which can be rewritten

$$\psi_a \delta_b c - \delta_b \psi_a c = \delta_{\psi_a b} c.$$

This gives (19). □

Note that linearity of ψ is not listed among the requirements in the proposition below, since it follows from the other requirements.

19. Proposition. *Let A be a unital JB-algebra and let $\psi : A \mapsto \psi_a$ be a map from A into the set of skew order derivations of A. Then ψ is a dynamical correspondence iff the following requirements are satisfied for $a, b \in A$*

(i) $[\psi_a, \psi_b] = -[\delta_a, \delta_b]$
(ii) $[\psi_a, \delta_b] = [\delta_a, \psi_b]$.

Proof. Assume first that ψ is a dynamical correspondence. Condition (i) above is trivially satisfied as it is identical with condition (i) of Definition 17.

By condition (ii) of Definition 17 and Lemma 18, then for all $a \in A$

$$[\psi_a, \delta_a] = \delta_{\psi_a a} = \delta_0 = 0.$$

By linearity of ψ, then for all $a, b \in A$

$$0 = [\psi_{a+b}, \delta_{a+b}] = [\psi_a, \delta_b] + [\psi_b, \delta_a],$$

which gives

$$[\psi_a, \delta_b] = -[\psi_b, \delta_a] = [\delta_a, \psi_b],$$

and proves condition (ii) above.

Assume next that ψ satisfies conditions (i) and (ii) above. Condition (i) of Definition 17 is trivially satisfied, and it follows from condition (ii) above and Lemma 18 that for all $a \in A$

$$\delta_{\psi_a a} = [\psi_a, \delta_a] = [\delta_a, \psi_a] = -[\psi_a, \delta_a].$$

Hence $\psi_a a = \delta_{\psi_a a} 1 = 0$, so condition (ii) of Definition 17 is also satisfied.

It remains to show that ψ is linear. By Lemma 18 and condition (ii) above, then for all a, b

$$\delta_{\psi_a b}(1) = [\psi_a, \delta_b](1) = [\delta_a, \psi_b](1) = -[\psi_b, \delta_a](1) = -\delta_{\psi_b a}(1).$$

Hence

$$\psi_a b = -\psi_b a, \tag{20}$$

from which it follows that $a \mapsto \psi_a b$ is a linear map from A into $D(A)$ for each fixed $b \in A$. $\square$

In the above proof we have actually also shown that if ψ is a dynamical correspondence on A, then equation (20) above holds for all pairs $a, b \in A$. We shall make more use of this equation later.

We will now explain the relationship between Connes orientations and dynamical correspondences, and we begin with the following:

20. Lemma. *If ψ is a dynamical correspondence on a unital JB-algebra A, then the kernel of $\psi : A \to D(A)$ consists of all self-adjoint order derivations δ_z where $z \in Z(A)$.*

Proof. If $a \in \ker \psi$, then it follows from Definition 17 (i) that $[\delta_a, \delta_b] = -[\psi_a, \psi_b] = 0$ for all $b \in A$. Hence $a \in Z(A)$.

Conversely, if $z \in Z(A)$ then $\delta_z \in D(Z(A))$ (Lemma 13), so it follows from Lemma 18 and Proposition 19 (ii) that for all $b \in A$

$$\delta_{\psi_z b} = [\psi_z, \delta_b] = [\delta_z, \psi_b] = 0.$$

Thus $\psi_z b = 0$ for all $b \in A$, so $z \in \ker \psi$. □

21. Proposition. *A Connes orientation I on a JBW-algebra A determines a complete dynamical correspondence ψ such that $\psi_a \in I(\tilde{\delta}_a)$ for all $a \in A$, and ψ is the only dynamical correspondence with this property.*

Proof. Let $a \in A$. We will first show that there exists a unique $\delta \in I(\tilde{\delta}_a)$ such that δ is skew.

Choose $\delta_1 \in I(\tilde{\delta}_a)$. By Definition 16 (iii)

$$\tilde{\delta}_1^* = I(\tilde{\delta}_a)^* = -I(\tilde{\delta}_a^*) = -I(\tilde{\delta}_a) = -\tilde{\delta}_1.$$

Hence $\tilde{\delta}_1^* + \tilde{\delta}_1 = \tilde{0}$, so $\delta_1^* + \delta_1 = \delta_z$ for some $z \in Z(A)$ (Lemma 13). Now define $\delta = \delta_1 - \frac{1}{2}\delta_z = \frac{1}{2}(\delta_1 - \delta_1^*)$. Thus δ is skew. Also

$$\delta = \delta_1 - \frac{1}{2}\delta_z \in I(\tilde{\delta}_1) = I(\tilde{\delta}_a).$$

If δ_1 is an arbitrary skew order derivation such that $\delta' \in I(\tilde{\delta}_a)$, then $\delta - \delta'$ is both central and skew. By Lemma 13 $\delta - \delta' = 0$, so δ is the unique skew order derivation in $I(\tilde{\delta}_a)$.

Denote the unique skew order derivation in $I(\tilde{\delta}_a)$ by ψ_a for each $a \in A$. Clearly $\psi : a \mapsto \psi_a$ is a linear map from A into $D(A)$.

Let $a, b \in A$. By Definition 16 (i),(ii)

$$[I\tilde{\delta}_a, I\tilde{\delta}_b] = I[\tilde{\delta}_a, I\tilde{\delta}_b] = I^2[\tilde{\delta}_a, \tilde{\delta}_b] = -[\tilde{\delta}_a, \tilde{\delta}_b].$$

Thus for some $z \in Z(A)$

$$[\psi_a, \psi_b] = -[\delta_a, \delta_b] + \delta_z.$$

Since ψ_a and ψ_b are skew, then $[\psi_a, \psi_b](1) = 0$. Also $[\delta_a, \delta_b](1) = a \circ b - b \circ a = 0$. Hence $z = \delta_z(1) = 0$. Thus $[\psi_a, \psi_b] = -[\delta_a, \delta_b]$, so ψ satisfies condition (i) of Definition 17.

To show that ψ also satisfies condition (ii) of Definition 17, we first observe that for all $a \in A$ then by Definition 16 (ii)

$$[\tilde{\psi}_a, \tilde{\delta}_a] = [I\tilde{\delta}_a, \tilde{\delta}_a] = I[\tilde{\delta}_a, \tilde{\delta}_a] = 0.$$

By Lemma 13 there exists $z \in Z(A)$ such that $[\psi_a, \delta_a] = \delta_z$. Now δ_z is a commutator of two bounded linear operators on A and it commutes with each of them, so it follows from the Kleinecke-Shirokov Theorem [10, p.128] that δ_z is quasi-nilpotent, i.e.

$$\lim_{n\to\infty} \|{\delta_z}^n\|^{\frac{1}{n}} = 0.$$

But ${\delta_z}^n(1) = z^n$ for all n: Hence

$$\|{\delta_z}^{2^n}\|^{2^{-n}} \geq \|z^{2^n}\|^{2^{-n}} = \|z\|,$$

so $z = 0$. (The norm-closed subalgebra generated by a single element and 1 is isometrically isomorphic to $C(X)$ for some compact Hausdorff space X [13, 3.2.4], so $\|z^{2^n}\| = \|z\|^{2^n}$). Thus $[\psi_a, \delta_a] = 0$. Now it follows from Lemma 18 that $\delta_{\psi_a a} = 0$, and hence also $\psi_a a = 0$, so we have verified condition (ii) of Definition 17. Thus ψ is a dynamical correspondence.

To show that ψ is complete, we consider an arbitrary skew order derivation δ and we will show that $\delta = \psi_a$ for some $a \in A$. More specifically, we choose an arbitrary $\delta_1 \in I(-\tilde{\delta})$ and we will show that δ_1 is self-adjoint, i.e. of the form $\delta_1 = \delta_a$ for $a \in A$, and that $\psi_a = \delta$.

By Definition 16 (iii) and the fact that δ is skew,

$$\tilde{\delta}_1^* = -(I\tilde{\delta})^* = I(\tilde{\delta}^*) = I(-\tilde{\delta}) = \tilde{\delta}_1.$$

Thus $\delta_1^* - \delta_1 = \delta_z$ for some $z \in Z(A)$. Applying both sides to 1, we get $(\delta_1^* - \delta_1)(1) = z$. Since $\delta_1^* - \delta_1$ is skew, then $z = 0$. Thus $\delta_1^* = \delta_1$ so δ_1 is self-adjoint, i.e. $\delta_1 = \delta_a$ for some $a \in A$.

By definition $\delta_a \in I(-\tilde{\delta})$, so $\tilde{\delta}_a = -I\tilde{\delta}$. Then by Definition 16 (i), $I\tilde{\delta}_a = -I^2\tilde{\delta} = \tilde{\delta}$. Thus $\delta \in I\tilde{\delta}_a$, so δ is the unique skew order derivation in $I\tilde{\delta}_a$; in other words $\delta = \psi_a$. With this we have shown that ψ is a complete dynamical correspondence.

The uniqueness is clear, since ψ_a is the only skew order derivation in $I\tilde{\delta}_a$. □

The concept of a Connes orientation is defined for JBW-algebras, while the concept of a dynamical correspondence is defined for unital JB-algebras, so the two concepts cannot be equivalent. Note however, that it will follow from our main theorem that a dynamical correspondence on a JBW-algebra is necessarily complete and is derived from a Connes orientation as in Proposition 21 (Corollary 25). Thus the two concepts are in fact equivalent in the context of JBW-algebras.

The main theorem

We are now ready to prove our main theorem which relates dynamical correspondences to associative products.

22. Definition. Let A be a unital JB-algebra. A *C*-product (W*-product) compatible with* A is an associative product $(x, y) \mapsto xy$ on the complex linear space $A + iA$ which induces the given Jordan product on A and organizes $A + iA$ to a C*-algebra (von Neumann algebra) with the involution $(a + ib)^* = a - ib$ and the norm $\|x\| = \|x^*x\|^{1/2}$.

Note that if a JB-algebra A is the self-adjoint part of a C*-algebra $\mathcal{A}$, then we can transfer the product and the norm from $\mathcal{A}$ to $A + iA$ by the representation $x = a+ib$ (where $x \in \mathcal{A}$ and $a, b \in A$). This organizes $A+iA$ to a C*-algebra with the properties in the definition above. Thus, a JB-algebra is the self-adjoint part of a C*-algebra iff there exists a C*-product compatible with A on $A + iA$. Similarly in the JBW-context.

23. Theorem. *A unital JB-algebra is (Jordan isomorphic to) the self-adjoint part of a C*-algebra iff there exists a dynamical correspondence on A. In this case each dynamical correspondence ψ on A determines a unique C*-product compatible with A such that for $a, b \in A$*

$$\psi_a b = \frac{i}{2}(ab - ba), \tag{23}$$

and each C-product compatible with A arises in this way from a unique dynamical correspondence ψ on A. The same conclusions hold with "JBW" in place of "JB" and "W*" or "von Neumann" in place of "C*".*

Proof. Assume first that A admits a dynamical correspondence ψ. By equation (20) of Proposition 19 we can define an anti-symmetric bilinear product $(a, b) \mapsto a \star b$ on A by writing

$$a \star b = \psi_a b. \tag{24}$$

Next define a bilinear map $(a, b) \mapsto ab$ from $A \times A$ into $A + iA$ (considered as a real linear space) by writing

$$ab = a \circ b - i(a \star b). \tag{25}$$

This map can be uniquely extended to a bilinear product on $A + iA$ (considered as a complex linear space). We will show that this product is associative. By linearity, it suffices to prove the associative law

$$a(cb) = (ac)b \tag{26}$$

for $a, b, c \in A$.

Writing out (26) by means of (25), we get

$$\begin{aligned} &a \circ (c \circ b) - i\big(a \star (c \circ b)\big) - i\big(a \circ (c \star b)\big) - a \star (c \star b) \\ &= (a \circ c) \circ b - i\big((a \circ c) \star b\big) - i\big((a \star c) \circ b\big) - (a \star c) \star b. \end{aligned}$$

Separating real and imaginary terms (and using the anti-symmetry of the $\star$-product), we get two equations. The first one can be written as follows

$$a \star (b \star c) - b \star (a \star c) = -a \circ (b \circ c) + b \circ (a \circ c), \tag{27}$$

and the second one as follows

$$a \star (b \circ c) - b \circ (a \star c) = a \circ (b \star c) - b \star (a \circ c). \tag{28}$$

The left hand side of (27) is nothing but $[\psi_a, \psi_b](c)$ and the right hand side of (27) is nothing but $-[\delta_a, \delta_b](c)$. Similarly the left hand side of (28) is $[\psi_a, \delta_b](c)$ and the right hand side of (28) is $[\delta_a, \psi_b](c)$. Thus these two equations follow directly from the characterization of a dynamical correspondence in Proposition 19.

We must also show that the bilinear product on $A + iA$ is compatible with the involution, i.e. that $(xy)^* = y^*x^*$ for $x, y \in A + iA$. By linearity it suffices to show that $(ab)^* = ba$ for $a, b \in A$. But this follows directly from the antisymmetry of the $\star$-product, as

$$(ab)^* = \big(a \circ b - i(a \star b)\big)^* = a \circ b + i(a \star b) = b \circ a - i(b \star a) = ba.$$

We have now shown that $A + iA$ is an associative *-algebra.

By the definition of the involution, the self-adjoint part of $A+iA$ is A. Thus by (25) and the anti-symmetry of the $\star$-product,

$$\frac{1}{2}(ab + ba) = a \circ b$$

for all pairs $a, b \in A$. Thus the associative product in $A + iA$ induces the given Jordan product on A.

We will now show that $x^*x \in A^+$ for every $x \in A + iA$. The closed Jordan subalgebra $C(x^*x)$ of A generated by the self-adjoint element x^*x and 1 is associative, hence isometrically isomorphic to the real commutative Banach algebra $C(X)$ for a compact Hausdorff space X [13, Th.3.2.2]. Thus $x^*x = a - b$ where a, b are two positive elements of $C(x^*x)$ such that $a \circ b = 0$. By the definition of the Jordan triple product [13, 2.3.2] and the associativity of $C(x * x)$,

$$\{bab\} = 2b \circ (b \circ a) - (b \circ b) \circ a = b \circ (b \circ a) = 0.$$

But since the Jordan product in A is equal to that induced from the associative algebra $A + iA$, the same is true for the Jordan triple product. Therefore also $bab = 0$.

Calculating in the associative *-algebra $A + iA$, we now find that

$$(xb)^*(xb) = bx^*xb = b(a-b)b = -b^3. \tag{29}$$

Since $0 \leq b \in C(x^*x)$, then $b^3 \geq 0$ and $b^3 = 0$ iff $b = 0$. Thus in order to prove $b = 0$, it suffices to show that $(xb)^*(xb) = 0$. For brevity we set $y = xb$, and we will show that $y^*y = 0$.

By (29) $y^*y \in -A^+$. Write $y = c + id$ where $c, d \in A$ and calculate

$$yy^* + y^*y = 2(c^2 + d^2),$$

which gives

$$yy^* = 2(c^2 + d^2) - y^*y \in A^+.$$

Thus we have shown that

$$y^*y \in -A^+ \quad \text{and} \quad yy^* \in A^+. \tag{30}$$

Following [13, 3.2.9] we define the *inverse* of an element $a \in A$ to be its inverse in $C(a)$ (if it exists), and we denote it by a^{-1}. Note that this definition is equivalent to the usual definition of inverse in Jordan algebras; namely that $a' \in A$ is the inverse of a if a' satisfies $a \circ a' = 1$ and $a^2 \circ a' = a$ [4, Prop.2.4].

Furthermore it is easily shown that $a \in A$ is invertible with inverse a' in the Jordan algebra A iff $a \in A$ is invertible with inverse a' in the Jordan algebra $A + iA$ iff a is invertible with inverse a' in the associative *-algebra $A + iA$. (For the last equivalence, see [16, p.51].)

By definition, a real number λ is in the spectrum of an element a of the JB-algebra A, in symbols $\lambda \in \mathrm{sp}(a)$, if $\lambda 1 - a$ is non-invertible in A. Thus, by the above, $\lambda \in \mathrm{sp}(a)$ iff $\lambda 1 - a$ is non-invertible in the associative algebra $A + iA$.

Calculating in the associative algebra $A + iA$ (in the same way as in the proof of [19, Prop.3.2.8]), we find that if $\lambda 1 - y^*y$ is invertible and $\lambda \neq 0$, then

$$(\lambda 1 - yy^*)\big(y(\lambda 1 - y^*y)^{-1}y^* + 1\big) = \lambda 1,$$

so $\lambda 1 - yy^*$ is also invertible. With this we have shown that

$$\mathrm{sp}(y^*y)\backslash\{0\} = \mathrm{sp}(yy^*)\backslash\{0\}.$$

By (30) $\mathrm{sp}(y^*y) = \mathrm{sp}(yy^*) = \{0\}$, and by the spectral theorem for JB-algebras [13, Th.3.2.4] $y^*y = yy^* = 0$. Hence $b = 0$, and then $x^*x \in A^+$ as claimed.

Define now for $x \in A + iA$

$$\|x\| = \|x^*x\|^{\frac{1}{2}}. \tag{31}$$

Extend each state ρ on A to a complex linear functional on $A + iA$. Note that since the states separate the points of A, their extensions will separate the points of $A + iA$. Define for $x \in A + iA$

$$(x \mid y)_\rho = \rho(y^*x). \tag{32}$$

Since $x^*x \in A^+$ for all $x \in A + iA$, this is a semi-definite inner product. Construct now the GNS-representations (π_ρ, H_ρ) in the usual way, and let (π, H) be the direct sum of all such representations. If x is a non-zero element of $A + iA$, then there exists a state ρ such that $\rho(x^*x) \neq 0$, and then $\pi_\rho(x) \neq 0$. Thus π is a *-isomorphism of $A + iA$ into $\mathcal{B}(H)$. In particular, π restricts to a Jordan isomorphism of A into the self-adjoint part of $\mathcal{B}(H)$, so it follows from [13, Prop.3.4.3] that $\|\pi(a)\| = \|a\|$ for $a \in A$. By (31) and the corresponding equation for the norm of $\mathcal{B}(H)$, we now have for $x \in A + iA$

$$\|\pi(x)\| = \|\pi(x^*x)\|^{\frac{1}{2}} = \|x^*x\| = \|x\|. \tag{33}$$

Thus (30) defines a norm on $A + iA$ which makes $\pi : A + iA \to \mathcal{B}(H)$ an isometric *-isomorphism. Clearly this norm satisfies the C*-condition $\|x\|^2 = \|x^*x\|$, and we will now show that $A+iA$ is complete for this norm.

Pulling back the corresponding inequalities from $\mathcal{B}(H)$, we have for every $x = a + ib \in A + iA$

$$\max(\|a\|, \|b\|) \leq \|x\| \leq \|a\| + \|b\|.$$

Since A is complete in the order unit norm, the space $A + iA$ must be complete in the norm (31), and then be a C*-algebra.

It follows from (24),(25) and the anti-symmetry of the $\star$-product that

$$\frac{i}{2}(ab - ba) = \frac{1}{2}(a \star b - b \star a) = \psi_a(b).$$

Thus we have constructed a C*-product compatible with A which satisfies the requirement (23). This product is unique since the compatibility with the Jordan algebra A determines the self-adjoint part $a \circ b$ in (25), and the requirement (23) determines the skew part $a \star b$ in (25).

Assume next that $A + iA$ is equipped with a C*-product $(x, y) \mapsto xy$ compatible with the JB-algebra A. Now we define a map ψ from A into

$D(A)$ by equation (23), and we prove by straightforward calculation that the two requirements (i),(ii) of Definition 17 are satisfied. Thus ψ is a dynamical correspondence on A. Also we can recover the given associative product by substituting $\psi_a b$ for $a \star b$ in (25), so this product arises from ψ by the construction in the first part of the proof.

It only remains to specialize to JBW-algebras. In this connection we must prove that if A is a JBW-algebra and $(x, y) \mapsto xy$ is a C*-product on $A + iA$ which is compatible with A, then $A + iA$ is a von Neumann algebra. The quickest proof of this fact is based on Kadison's theorem that a monotone complete C*-algebra with a separating family of normal states is a von Neumann algebra [23] (or [19, Ex.7.6.38]). In fact, the conditions on monotone completeness and separation by normal states are both imposed on the self-adjoint part of the C*-algebra, and since they are satisfied for the JBW-algebra A, there is nothing more to prove. □

Remark. The proof above shows that if A is a JB-algebra and $A+iA$ is equipped with an associative product such that $a + ib \mapsto a - ib$ is an involution, then $A + iA$ can be normed (in a necessarily unique way) to become a C*-algebra.

24. Corollary. *A JBW-algebra A is the self-adjoint part of a von Neumann algebra iff there exists a Connes orientation on A. In this case each Connes orientation I on A determines a unique W*-product compatible with A such that for* $d \in A + iA$

$$I(\tilde{\delta}_d) = \tilde{\delta}_{id}, \tag{34}$$

and each W-product compatible with A arises in this way from a unique Connes orientation on A.*

Proof. Assume first that A admits a Connes orientation I. By Proposition 21 there exists a dynamical correspondence ψ on A such that $\psi_a \in I(\tilde{\delta}_a)$ for $a \in A$. Construct the corresponding W*-product in $A + iA$ as in Theorem 23.

Let $d \in A$, say $d = a + ib$ where $a, b \in A$. Then to verify (34) we observe that for all $c \in A$

$$\delta_{ib} c = \frac{i}{2}(bc - cb) = \psi_b c,$$

so $\delta_{ib} = \psi_b$. Since $\tilde{\psi}_b = I(\tilde{\delta}_b)$, then

$$I(\tilde{\delta}_{ib}) = I(\tilde{\psi}_b) = I^2(\tilde{\delta}_b) = -\tilde{\delta}_b,$$

which gives

$$I(\tilde{\delta}_d) = I(\tilde{\delta}_a) + I(\tilde{\delta}_{ib}) = \tilde{\psi}_a - \tilde{\delta}_b = \tilde{\delta}_{ia} - \tilde{\delta}_b = \tilde{\delta}_{id}, \tag{35}$$

and establishes (34). Recall that (as shown in the proof of Proposition 21) there is a unique skew-adjoint order derivation in $I(\tilde{\delta}_{id})$. Thus any two W*-products compatible with A and satisfying (34) induce the same Lie multiplication map δ_{id} for each $d \in A$, as well as the same Jordan multiplication (inherited from A), and thus coincide.

Assume next that $A + iA$ is equipped with a C*-product compatible with the JBW-algebra A. Thus A is the self-adjoint part of a von Neumann algebra $\mathcal{A}$, and by Proposition 15 each order derivation on A is of the form δ_d for some $d \in \mathcal{A}$. Now we can define a map I from $\tilde{D}(A)$ into itself by (34), and we prove by straightforward calculation that the requirements (i),(ii),(iii) of Definition 16 are satisfied. Thus I is a Connes orientation on A. Clearly I is the unique Connes orientation on A for which equation (34) is satisfied. □

25. Corollary. *A dynamical correspondence ψ on a JBW-algebra A is necessarily complete , and there is a unique Connes orientation I on A such that $\psi_a \in I(\tilde{\delta}_a)$ for all $a \in A$.*

Proof. Let ψ be a dynamical correspondence on a JBW-algebra A. By Theorem 23 there is a unique W*-product in A such that (23) holds. In other words, A is the self-adjoint part of a von Neumann algebra $\mathcal{A}$ which is unique (up to a *-isomorphism) under the requirement $\psi_a = \delta_{ia}$ for all $a \in A$. Now it follows from Corollary 24 that there is a Connes orientation I on A such that $I(\tilde{\delta}_a) = \tilde{\delta}_{ia} = \tilde{\psi}_a$. Thus $\psi_a \in I(\tilde{\delta}_a)$ for all $a \in A$.

By the same argument as the one leading up to (35), $I(\tilde{\delta}_d) = \tilde{\delta}_{id}$ for all $d \in \mathcal{A}$. Since each order derivations on A is of the form δ_d for some $d \in \mathcal{A}$ (Proposition 15), this proves that I is uniquely determined by the requirement

$$\tilde{\psi}_a = I(\tilde{\delta}_a) = \tilde{\delta}_{ia} \quad \text{for } a \in A. \tag{36}$$

By Proposition 21 there is a unique dynamical correspondence on A for which (36) holds, and this correspondence is complete. Thus ψ is complete, and we are done. □

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