OPTIMAL STOCHASTIC INTERVENTION CONTROL
WITH APPLICATION TO THE EXCHANGE RATE

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Abstract

We formulate a mathematical model for the optimal control of the exchange rate under uncertainty. The control consists of a combination of

(i) a (continuous) stochastic control

and

(ii) an impulse control.

We give general sufficient conditions for its solution. The results are applied to the following situation:

Suppose that a government has two means of influencing the foreign exchange rate of its own currency:

(i) At all times t the government can choose the domestic interest rate r_t.

(ii) At selected times the government can intervene in the foreign exchange market by buying or selling large amounts of foreign currency.

We assume that the exchange rate is stochastic and that there are given costs involved in these two actions. It is also costly to have an exchange rate which deviates too much from a given central parity m. How does the government apply its two means of influence in order to keep the exchange rate as stable as possible with minimal expected costs?

We formulate this problem mathematically as a combined stochastic control (i) and impulse control (ii) problem, and we discuss the solution in a specific example.

1 Introduction

The research on target zone exchange rate regimes has progressed rapidly over the last few years. With a target zone regime the exchange rate is allowed to move within a specified band, and it is then customary to assume that the central bank intervenes to prevent the exchange rate from moving outside the band. Krugman (1987) introduced the standard target zone model. Here the exchange rate depends on both "fundamentals" and on expectations about its future values. See Svensson (1992a), Bertola (1994) and Garber and Svensson (1996) for a review of the relevant later literature.

This paper contributes to this literature by deriving the optimal intervention policy by the central bank in order to stabilize the exchange rate within its band. This policy comprises both interventions in the foreign exchange market, at selected times only, and a continuous control of the domestic interest rate level. We should however point out that our analysis below only depend on one of these instruments being used discretely, and the other continuously. Thus instead assuming interventions as occurring continuously, while changes in the interest rate occurring more periodically, is also possible to consider in our mathematical model.

We set up a model of optimal intervention policy for the central bank in a small economy to find the optimal deviation of the exchange rate from the central parity of its band. We then derive an optimal interest rate differential between domestic and foreign interest rates (the foreign interest rate level is although taken as given), and the optimal time and amount of central-bank intervention in the foreign exchange market. The analysis ignores the issue of whether interventions are sterilized or unsterilized. Note also that we assume that speculators do not have information which allows them to infer the size or the timing of jumps in the exchange rate. This implies that the central bank never announces the amount of intervention, and speculators cannot anticipate risk-free profits at an infinite rate nor can they compete away possible arbitrage profits by exactly counteracting the intervention amount.

Our mathematical model consists of a combination of stochastic control and impulse control and we give sufficient conditions that a given function is the minimal expected cost function for the central bank and that the corresponding strategy is optimal. The model is illustrated by a specific example. The mathematical model presented has however greater generality, and can be applied to other economics problems that involve the combination of stochastic control and impulse control.

2 A Brief Review of the Target-Zone Literature

The initial Krugman ((1987), (1991)) model was based on a number of crucial assumptions. One is that the exchange rate target zone is perfectly credible, in the sense that market agents believe that the lower and upper edges of the band will remain fixed forever. Another is that the target zone is defended with interventions at its edges, and no interventions take place when the exchange rate is strictly inside the band (i.e. no intramarginal

interventions). Following Krugman's methodology, Svensson (1991) finds a deterministic, non-linear inverse relationship between the exchange rate deviations from parity and interest rate differentials, which is flatter and less-nonlinear for longer maturities, and which follows from the exchange rate band being credible. Most empirical evidence has however put in question the assumptions and conclusions of these models.

A number of questions remains unanswered, particularly following the last European exchange rate crisis in the fall of 1992. Among these are the issue of the optimal policy to maintain the currency band, and the optimal width of the currency band once adopted. The current paper deals with these two questions. The literature contains a few analysis related to ours. Svensson (1992b) studies a model of optimal intervention policy in a small open economy, where the central bank minimizes a weighted sum of interest rate and exchange rate variability using as its only instrument foreign exchange interventions. He assumes a credible band, with a negative trade-off between interest rate smoothing and exchange rate variability. He then finds that there exists an optimal negative trade-off between exchange rate and interest rate variability. Miller and Zhang (1994) attempt to derive an optimal target zone, and assume that the costs of intervening in the foreign exchange rate market are proportional to the size of the intervention (as in Avesani (1991)). They then find that the optimal policy is to stay within a given target zone, using marginal and infinitesimal interventions at the boundaries of the target zone in order to obtain a reflecting barrier.

By contrast to these papers, in our model the cost of each intervention is greater than a fixed, positive minimum, no matter how small the intervention. This leads to an optimal policy with *discrete* intervention episodes.

The model in the literature that most closely resembles ours is probably Jeanblanc-Picqué (1993). She uses the mathematical theory of impulse control to show that there exists an optimal intervention policy (impulse control) which – under given intervention costs – forces the given (constant-drift Brownian motion) diffusion for the exchange rate to stay within a band $[a, b]$. The optimal policy is shown to be the following: When the process is in (a, b), no interventions should be made. When the process reaches the value a or b, respectively, an intervention should be made which makes the exchange rate to jump inside its band to points α or β, respectively, where $a < \alpha \leq \beta < b$.

Our approach differs from that of Jeanblanc-Picqué mainly in two ways. First, we do not necessarily require that the process stays within a given band $[a, b]$, but there is a given function which determines the cost of leaving a certain implicit exchange rate band. Secondly and more importantly, we allow for *two* types of control, namely both discrete-time foreign exchange interventions, and continuous-time interest rate control. Mathematically, this leads to a combination of impulse control and (continuous) stochastic control. Moreover, we assume that the currency band is not necessarily credible and may be exposed to speculative attacks. In particular, when the domestic currency moves above (below) its central parity, there may be expectations of further depreciations (appreciations), leading to higher (lower) domestic interest rates. The central bank may permit the domestic interest rate to rise to a certain level above the foreign interest rate. When an optimal level is attained, the central bank applies its second instrument for affecting the exchange

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rate, namely foreign exchange interventions. We determine the optimal timing and amount of such interventions. We then derive an optimal distance of the exchange rate from its parity at which interventions are to take place, and the overall costs of applying the two instruments are minimized.

3   A mathematical model for the optimal control of the exchange rate

Here we will present a model based on the theory of combined stochastic control, of how the government can minimize the total costs of large exchange rate deviation from its central parity, the costs of interventions and of the interest rate differentials. For more background and details we refer to Brekke and Øksendal (1996).

Let Y_t = the exchange rate at time t (the number of domestic currency units required to buy a unit of foreign currency). If Y_t is high then the domestic currency is *weak*; if Y_t is low then the currency is *strong*.

We denote the *central parity* by m. The government tries to keep Y_t within an optimal interval containing m. For this purpose the government has two control possibilities:

(i) By choosing the domestic *interest rate* r_t. Authorities will set a higher interest rate to compensate the investors for a weak national currency, so that these investors get higher return by investing in the domestic country. If this policy is successful, that is, investors obtain enough interest income to compensate them for a depreciating currency, investors will begin to buy the domestic currency, and as a consequence this currency becomes more valuable and the exchange rate goes down and moves back toward the central parity. This type of control $r = (r_t)_{t \in \mathbf{R}}$ is called *the continuous control*. The set of all continuous controls is denoted by $\mathcal{U}$.

(ii) At selected times the domestic country can use its international reserves to intervene in the foreign exchange market. The effect of intervening by buying (selling) foreign currency is to make the national currency weaker (stronger). So the national currency becomes less (more) valuable and the exchange rate goes up (down). This kind of control is applied only at discrete, selected (stochastic) times θ_j and with selected amounts ξ_j at these times. The double sequence

$$v = (\theta_1, \theta_2, \cdots, \theta_N; \xi_1, \xi_2, \cdots, \xi_N)$$

is called the *impulse* control/intervention. Here $N \leq \infty, \theta_k \leq \theta_{k+1}$ and $\theta_k \to \infty$ as $k \to N$ (so if N is finite then $\theta_N = \infty$). The set of all impulse controls is denoted by $\mathcal{V}$.

The pair $w = (r, v) \in \mathcal{U} \times \mathcal{V}$ is called a *combined stochastic* control. We set $\mathcal{W} = \mathcal{U} \times \mathcal{V}$.

A rapidly fluctuating, unpredictable exchange rate is bad for the country because of the uncertainty that it creates to the corresponding market participants. On the other hand, the application of the controls $w = (r, v)$ to stabilize the exchange rate is also costly and therefore one tries to apply the controls in an optimal way. High domestic interest rates can cause high social costs and it can be very undesirable at times when the country is facing some recession for example. Certainly, the amount of reserves that a country possesses is not unlimited either, even though the ERM countries have large facilities of credit lines for borrowing reserves. All of this is however costly.

Let $\bar{r}_t$ denote the foreign interest rate at time t. We assume that if $r_t = \bar{r}_t$ and there are no central bank interventions in the foreign exchange market, then the exchange rate Y_t will behave like a Brownian motion, B_t and one can expect to be in a pure float exchange rate regime.

Let $-F(r_t - \bar{r}_t)$ denote the effect on the exchange rate produced by the interest rate differential $r_t - \bar{r}_t$. It is natural to assume that F has graphically the following form:

F (r-r̄)

(r-r̄)
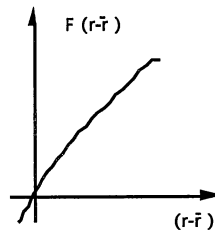
More precisely, F is concave, increasing and we have

$$F(r - \bar{r}) > 0 \Leftrightarrow r - \bar{r} > 0.$$

The form that we assume for F tells us that the effect of the interest rate differentials on the exchange rate decreases as the interest rate differentials increases. Therefore high levels of the domestic interest rate in relation to the foreign one are ineffective. When this becomes so, the central bank may consider it optimal to intervene in the foreign exchange market.

Let $\gamma(\xi)$ denote the effect on the exchange rate obtained by intervening by buying (if $\xi > 0$) or selling (if $\xi < 0$) the amount ξ of foreign currency. Again it is natural to assume that γ is concave, perhaps linear. Moreover, $\gamma(\xi) > 0 \Leftrightarrow \xi > 0$.

γ

ξ
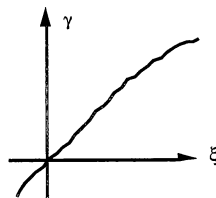
Similarly here, the marginal effectiveness of a central bank intervention in the foreign exchange market may decrease with the absolute amount of intervention.

Hence, if the combined stochastic control $w = (r_t, (\theta_1, \theta_2, \cdots; \xi_1, \xi_2, \cdots))$ is applied to the exchange rate Y_t, it gets the form

$$(3.1) \qquad Y_t := Y_t^{(w)}(\omega) := y - \int\limits_0^t F(r_s - \bar{r}_s)ds + \sigma B_t(\omega) + \sum_{j:\theta_j \leq t} \gamma(\xi_j)$$

where $\sigma > 0$ is a constant, $\bar{r}_s$ the foreign interest rate and $B_t(\omega); \omega \in \Omega$ denotes Brownian motion.

Suppose that the discount rate is $\rho > 0$ and that the cost rate for the society of having the exchange rate Y_t is $K(Y_t - m)$, where $K(x) \geq 0$ for all x. Let $R(r_t - \bar{r}_t) \geq 0$ be the cost rate of having an interest rate differential $r_t - \bar{r}_t$ and suppose that the cost of applying the impulse control ξ_j at time θ_j is $L(\xi_j) > 0$.

Put $x = (s, y)$. Then the total, discounted expected cost of applying the combined intervention control $w = (r_t, (\theta_1, \cdots; \xi_1, \cdots))$ is

$$(3.2) \qquad J^w(s,y) = E^y \left[\int\limits_s^T e^{-\rho t}(K(Y_t - m) + R(r_t - \bar{r}_t))dt + \sum_{j:\theta_j \leq T} L(\xi_j)e^{-\rho\theta_j} \right]$$

where $T \leq \infty$ is a given (fixed) future time and E^y denotes expectation with respect to the probability law of Y_t starting at y. Notice that we do not a priori assume symmetric costs for intervening (buying and selling foreign currency). In reality it may be the case that the latter is more costly than the former because the loss of reserves is more negative to the country than the accumulation of reserves.

PROBLEM 1.

Find
$$(3.3) \qquad \Lambda(x) := \inf_{w \in \mathcal{W}} J^w(x)$$

where $\mathcal{W}$ denotes the set of all combined stochastic controls. Moreover, find $w^* \in \mathcal{W}$ such that
$$\Lambda(x) = J^{w^*}(x)$$

i.e. w^* is a corresponding optimal combined stochastic control.

The function $J^w(x)$ is the total expected cost that one incurs by starting from the state $x = (s, y)$ and applying the control w. Therefore, $\Lambda(x)$ represents the minimal total expected cost when the state of the system starts at x.

$\Lambda(x)$ is called the *value* of the system at state x.

SOLUTION METHOD TO PROBLEM 1

From now on we assume that $\bar{r}_t = \bar{r}(t)$ is deterministic and we only consider *Markov* interest rate controls, i.e. interest rate controls of the form

$$r_t(\omega) = r(t, Y_t(\omega)),$$

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for some function $r : \mathbf{R}^2 \to \mathbf{R}$.

Then if there are no (impulse controls) interventions in the foreign exchange market by the central bank, the process

$$(3.4) \qquad X_t = X_t^{(r)} = \begin{bmatrix} s + t \\ Y_t^{(r)} \end{bmatrix} ; \ t \geq 0 ; \ X_0 = \begin{bmatrix} s \\ y \end{bmatrix} = x$$

will be an Ito diffusion whose generator $A^{(r)}$ coincides (on the space $C_0^2(\mathbf{R}^2)$ of twice continuously differentiable functions on $\mathbf{R}^2$ with compact support) with the partial differential operator

$$(3.5) \qquad L^r(s,y) = L^{r(s,y)} f(s,y) = \frac{\partial f}{\partial s} - F(r(s,y) - \bar{r}(s)) \frac{\partial f}{\partial y} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial y^2},$$

which is defined for all functions $f : \mathbf{R}^2 \to \mathbf{R}$ for which the derivatives involved exist at $x = (s,y)$.

If $w = (r, v)$ is a combined stochastic control we put (with a slight abuse of notation)

$$X_t^{(w)} = \begin{bmatrix} s + t \\ Y_t^{(w)} \end{bmatrix} ; \ t \geq 0 ; \ X_0 = \begin{bmatrix} s \\ y \end{bmatrix} = x.$$

The probability law of $X_t^{(w)}$ is denoted by $Q^{x,w}$ and the expectation w.r.t. $Q^{x,w}$ is denoted by $E^{x,w}$ or just E^x.

A continuous function $\phi : \mathbf{R} \to \mathbf{R}$ is called *stochastically* C^2 w.r.t. the Ito diffusion $X_t = X_t^{(r)}$ if the following *generalized Dynkin formula* holds:

$$(3.6) \qquad E^x[\phi(X_{\tau'})] = E^x[\phi(X_\tau)] + E^x[\int_\tau^{\tau'} L^{r(X_t)} \phi(X_t) dt],$$

for all *stopping times* $\tau(\omega), \tau'(\omega)$ satisfying

$$\tau \leq \tau' \leq \min\{R, \inf\{t > 0; \ |X_t| > R\}\},$$

for some $R < \infty$.

REMARK. This concept was introduced in Brekke and Øksendal (1991). There it was proved that, under certain conditions, a function ϕ which is C^1 (continuously differentiable) everywhere and C^2 (twice continuously differentiable) outside a "thin" set (w.r.t. the Green measure $G^X(\cdot)$ of X_t) is stochastically C^2.

Define the switching operator $\mathcal{M}$ by

$$(3.7) \qquad \mathcal{M}h(s,y) = \inf_\xi \{h(s, y + \gamma(\xi)) + L(\xi)e^{-\rho s}\}$$

for all Borel functions $h : \mathbf{R}^2 \to \mathbf{R}$. $\mathcal{M}$ is a nonlinear operator, $h \to \mathcal{M}h$, mapping bounded measurable functions into bounded measurable functions. Suppose that for each (s, y) the

infimum in (3.7) is achieved by at least one $\hat{\xi} = \xi(s, y) \in \mathbf{R}$ and let $\hat{\xi} = \mathcal{S}_h(s, y)$ be a measurable selection of such $\hat{\xi}'s$.

If h is a cost function then we may regard $\mathcal{M}h(x)$ as the minimal cost we can achieve by an intervention in the foreign exchange market at x, assuming that the central bank must intervene. Such an intervention of size ξ transforms the state from y to $y + \gamma(\xi)$.

Using a result of Brekke and Øksendal (1996) we now get:

THEOREM 1.

(I) Suppose there exists a continuous function $\phi : \mathbf{R}^2 \to [0, \infty)$ with the following properties:

(3.8) ϕ is stochastically C^2 w.r.t. $X_t^{(r)}$ for all $r : \mathbf{R}^2 \to \mathbf{R}$

(3.9) $\phi \leq \mathcal{M}\phi$ on $\mathbf{R}^2$

(3.10) $L^r\phi(s, y) + e^{-\rho s}(K(y - m) + R(r - \bar{r})) \geq 0$

       for a.a. (s, y) w.r.t. the Green measure for $X_t^{(r)}$, for all $r = r(s, y) : \mathbf{R}^2 \to \mathbf{R}$.

   Then

$$\phi(x) \leq J^w(x) \quad \text{for all} \quad w \in \mathcal{W}$$

(II) Suppose that – in addition to (3.8)–(3.10) – there exists a function $\hat{r} : \mathbf{R}^2 \to \mathbf{R}$ such that the *minimal value zero* 0 of the left hand side of (3.10) is attained, i.e.

(3.11) $L^{\hat{r}(s,y)}\phi(s, y) + e^{-\rho s}(K(y - m) + R(\hat{r}(s, y) - \bar{r}_s)) = 0$

for all $(s, y) \in D$, where

(3.12) $D := \{x; \phi(x) < \mathcal{M}\phi(x)\}.$

Define the following impulse control $\hat{v} = (\hat{\theta}_1, \hat{\theta}_2, \cdots; \hat{\xi}_1, \hat{\xi}_2, \cdots)$ inductively as follows: Put $\hat{\theta}_0 = 0$ and

(3.13) $\hat{\theta}_{k+1} = \inf\{t > \hat{\theta}_k; X_t^{(k)} \notin D\}, \qquad k = 0, 1, 2, \cdots,$

where $X_t^{(k)}$ is the result of applying the control

$$(\hat{r}_t, (\hat{\theta}_1, \cdots, \hat{\theta}_k, \infty; \hat{\xi}_1, \hat{\xi}_2, \cdots, \hat{\xi}_k, \cdot))$$

to X_t, and

(3.14) $\hat{\xi}_{k+1} = \mathcal{S}_\phi(X_{\theta_{k+1}^-}^{(k)}); \quad k = 0, 1, 2, \cdots,$

(where $X_{\theta_{k+1}^-} = \lim_{t \uparrow \theta_k} X_t$).

   Put $\hat{w} = (\hat{u}, \hat{v})$ and suppose that

(3.15) $\lim_{k \to \infty} \hat{\theta}_k = \infty \qquad \text{a.s. } Q^{x, \hat{w}} \text{ for all } x \in \mathbf{R}$

and that

(3.16)
$$\lim_{k\to\infty} E^x[\phi(X^{(\hat{w})}_{\hat{\theta}^-_{k+1}})] = 0, \quad \text{for all } x \in \mathbf{R}.$$

Then

(3.17)
$$\Phi(x) = \Lambda(x)$$

and the *optimal* combined stochastic control is

(3.18)
$$w^* = \hat{w} \in \mathcal{W}.$$

REMARK. D defined by (3.13) is called *the continuation region*. No intervention in the foreign exchange market should be made while X_t is in D, only the optimal continuous control (interest rate) $\hat{r}_t$. When X_t reaches ∂D we apply an (impulse control) intervention control (according to (3.15) above), which will bring X_t back into D (inside the currency band). Thus the optimal amount $\hat{\xi}_{k+1}$ to sell (buy) at time $\hat{\theta}_{k+1}$ is the (measurably selected) value of ξ which gives the minimum of

(3.19)
$$g(\xi) := \phi(\theta_{k+1}, X^{(k)}_{\theta^-_{k+1}} + \gamma(\xi)) + L(\xi)e^{-\rho\theta_{k+1}}$$
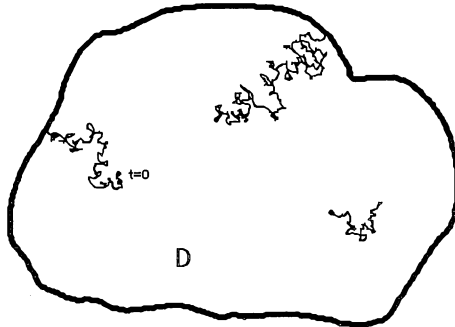
For a proof see Brekke and Øksendal (1996), Theorem 3.1. For a related result (which, however, is insufficient for our application) see Theorem 2 in Perthame (1984).

4   A special case

As an illustration we consider the special case when the functions in the model have the following forms, which seem reasonable as a first approximation for the model:

(4.1)   The function $F(x)$ which gives the effect of the interest rate differential x on the exchange rate, is given by

$$F(x) = bx \quad (b > 0 \text{ constant})$$

(4.2)   The function $\gamma(\xi)$ which gives the effect on the exchange rate by the intervention of size ξ is given by

$$\gamma(x) = ax \quad (a > 0 \text{ constant})$$

(4.3)   The function $R(x)$ which gives the cost rate of the interest rate differential x is given by

$$R(x) = \beta x^2 \quad (\beta > 0 \text{ constant})$$

(4.4)   The foreign interest rate $\bar{r}_t$ is constant. By translation of the axis of units we may then assume for simplicity that

$$\bar{r}_t = 0.$$

(4.5)   The cost of an intervention of size ξ is given by the function

$$L(\xi) = \lambda|\xi| + c \quad (\lambda \geq 0, \ c > 0 \text{ constants}).$$

(4.6)   The central parity m is constant. For simplicity we may assume that the units are chosen such that

$$m = 0.$$

(4.7)   The discount rate ρ is constant.

(4.8)   The cost rate $K(y)$ of having the exchange rate y is an even function (i.e. $K(y) = K(-y)$).

REMARK. The constant c represents a fixed, basic cost for each (impulse control) intervention (independent of ξ_j). Each such intervention is a large operation involving several people (who can otherwise use their time on other activities) and several hours of discussions and considerations. It therefore represents a cost of at least $c > 0$, no matter the size of the operation. Note that $c > 0$ implies the impossibility of applying such interventions continuously.

In this case the problem becomes (with $T = \infty$):

PROBLEM 2. Find $\Lambda(x)$ and w^* such that

(4.9) $$\Lambda(x) = \inf_{w \in \mathcal{W}} J^w(x) = J^{w^*}(x)$$

where

(4.10) $$J^w(x) = J^w(s, y) = E^x[\int_s^\infty e^{-\rho t}(K(Y_t) + \beta r_t^2)dt + \sum_j e^{-\rho\theta_j}(c + \lambda|\xi_j|)]$$

and

(4.11) $$Y_t = Y_t^{(w)}(\omega) = y - \int_0^t br_s ds + \sigma B_t(\omega) + a \sum_{j:\theta_j \leq t} \xi_j.$$

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SOLUTION TO PROBLEM 2

Then equation (3.11) becomes

$$(4.12) \qquad \frac{\partial \phi}{\partial s} - ba\frac{\partial \phi}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 \phi}{\partial y^2} + e^{-\rho s}(K(y) + \beta\alpha^2) \geq 0$$

for a.a. $y \in \mathbf{R}$ and all $\alpha \in U = \mathbf{R}$.

The minimum of the function

$$(4.13) \qquad g(\alpha) := -ba\frac{\partial \phi}{\partial y} + \beta\alpha^2 e^{-\rho s}$$

is obtained when

$$(4.14) \qquad \alpha = \frac{b}{2\beta} \cdot \frac{\partial \phi}{\partial y} e^{\rho s}.$$

Note that this value of α is our candidate for the optimal interest rate control $\hat{r}(s, y)$.

For this value of α the left hand side of (4.12) is required to be equal to 0 in D:

$$(4.15) \qquad \frac{\partial \phi}{\partial s} - \frac{b^2}{2\beta}\left(\frac{\partial \phi}{\partial y}\right)^2 e^{\rho s} + \frac{1}{2}\sigma^2\frac{\partial^2 \phi}{\partial y^2} + e^{-\rho s}K(y) + \frac{b^2}{4\beta}\left(\frac{\partial \phi}{\partial y}\right)^2 e^{\rho s} = 0 \quad \text{in } D$$

or

$$(4.16) \qquad \frac{\partial \phi}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 \phi}{\partial y^2} - \frac{b^2}{4\beta}\left(\frac{\partial \phi}{\partial y}\right)^2 e^{\rho s} + e^{-\rho s}K(y) = 0 \quad \text{in } D.$$

As a candidate for the minimal cost function $\Lambda(s, y)$ we try a function ϕ of the form

$$(4.17) \qquad \phi(s, y) = e^{-\rho s}\psi(y)$$

for a suitable function ψ (to be determined) and we assume by symmetry of K that D has the form

$$(4.18) \qquad D = \mathbf{R} \times (-\eta, \eta) \quad \text{for suitable } \eta \in (0, \infty).$$

The interval $(-\eta, \eta)$ can be interpreted as the band of "admissible" exchange rate values, or the "no intervention band" (where there will not be any central bank interventions in the foreign exchange market). Hence the exchange rate values $-\eta$ and η are the "trigger" levels for intervention in the foreign exchange market by the central bank.

Then from (4.16) we get

$$(4.19) \qquad \frac{1}{2}\sigma^2\psi''(y) - \frac{b^2}{4\beta}(\psi'(y))^2 - \rho\psi(y) + K(y) = 0; \qquad y \in (-\eta, \eta)$$

The equation (4.19) has the form

$$(4.20) \qquad \psi''(y) - A(\psi'(y))^2 - B\psi(y) = -C(y)$$

where

(4.21)
$$A = \frac{b^2}{2\beta\sigma^2}, \quad B = \frac{2\rho}{\sigma^2} \quad \text{and} \quad C(y) = \frac{2K(y)}{\sigma^2}.$$

By symmetry it is enough to consider the case when $y \geq 0$. From the general theory of ordinary differential equations (see e.g. Birkhoff and Rota (1989), Theorem 8, p. 190) we know that there exists $T > 0$ (the explosion time) such that for any choice of initial values $f(0), f'(0)$ the equation (4.20) has a unique solution $\psi = f(y)$ for $y < T$. In particular, for each given $z > 0$ we let $f(y) = f_z(y)$; $0 \leq y < T$ be the unique solution of

(4.22)
$$\begin{cases} f''(y) - A(f'(y))^2 - Bf(y) = -C(y); & 0 < y < T \\ f(0) = z, \quad f'(0) = 0 \end{cases}$$

We now assume that there exists $z > 0$ such that for at least two different values of $y \in (0, T)$, say $y_1 < y_2 < T$, we have
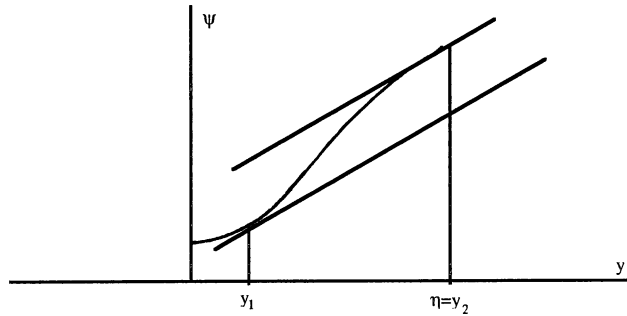
(4.23)
$$f_z'(y_1) = f_z'(y_2) = \frac{\lambda}{a}.$$

From now on we choose $y_1 < y_2$ to be the two smallest positive numbers with this property.
We also assume that

(4.24)
$$f_z''(y_1) > 0.$$

We now choose η (the exchange rate trigger level for intervention) as follows:

(4.25)
$$\eta = y_2$$

and as our candidate for the value function ψ in the interval $[0, \eta]$ we choose

(4.26)
$$\psi(y) = f_z(y); \quad 0 \leq y \leq \eta.$$

REMARK. To illustrate that the situation (4.23), (4.24) is not unusual, consider the special case with $b = A = 0$, $B = 1$ and $C(y) = y^2$ in (4.20). Then the solution of (4.22) is

$$f_z(y) = (z - 2)\cosh(y) + y^2 + 2$$

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and we see that the equation (in y)

$$f'_z(y) = (z - 2)\sinh(y) + 2y = \frac{\lambda}{a}$$

has two solutions $y_1, y_2 > 0$ if $0 < z < 2$ and

(4.27) $$2\operatorname{Arccosh}(\frac{2}{2-z}) < \frac{\lambda}{a} + \sqrt{z(4-z)}.$$

Moreover, if (4.27) holds then

$$f''_z(y_1) > 0, \quad f''_z(y_2) < 0.$$

(To see this, consider the function

$$g(y) := (z - 2)\sinh(y) + 2y - \frac{\lambda}{a})$$

Next, we consider the switching operator, which in this case gets the form

$$\mathcal{M}\phi(s, y) = \inf_\xi\{\phi(s, y + a\xi) + (\lambda|\xi| + c)e^{-\rho s}\}$$

In terms of $\psi(y) = e^{\rho s}\phi(s, y)$ we get the operator

(4.28) $$\mathcal{M}\psi(y) = \inf_\xi\{\psi(y + a\xi) + \lambda|\xi| + c\}$$

Again by symmetry it is enough to consider the case $y > 0$. The first order condition for the minimum of the function

$$h(\xi) := \psi(y + a\xi) + \lambda|\xi| + c; \quad -\frac{y}{a} < \xi < 0$$

gives that at a point $\hat{\xi} = \hat{\xi}(y) < 0$ where the minimum in (4.28) is attained we have

$$\psi'(y + a\hat{\xi}) = \frac{\lambda}{a}$$

By (4.23), (4.24) we get

$$y + a\hat{\xi}(y) = y_1, \quad \text{i.e.} \quad \hat{\xi}(y) = \frac{1}{a}(y_1 - y) < 0.$$

In particular, if $y = \eta$ (the trigger level) we get by (3.15) that

(4.29) $$\hat{\xi}(\eta) = \frac{1}{a}(y_1 - \eta) < 0$$

is the optimal intervention size and this pushes the exchange rate from η back to the level y_1:

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(4.30)
$$y_1 = \eta + a\hat{\xi}(\eta).$$

Hence, since
$$\psi(y) = \mathcal{M}\psi(y) \qquad \text{for } y \geq \eta,$$

we have

(4.31)
$$\begin{aligned}
\psi(y) &= \psi(y + a\hat{\xi}(\eta)) + \lambda|\hat{\xi}(\eta)| + c \\
&= \psi(y_1) - \lambda\hat{\xi}(\eta) + c \\
&= \left(\psi(y_1) + c - \frac{\lambda y_1}{a}\right) + \frac{\lambda}{a}y \qquad \text{for } y \geq \eta.
\end{aligned}$$

Thus we have a definition of our solution candidate $\psi(y)$ as follows (see (4.26))

(4.32)
$$\psi(y) = \begin{cases} f_z(y) & \text{for } 0 \leq y \leq \eta = y_2 \\ f_z(y_1) + c + \frac{\lambda}{a}(y - y_1) & \text{for } \eta < y \end{cases}$$

and $\psi(-y) = \psi(y)$, $y \in \mathbf{R}$.

Continuity and smoothness (C^1) at $y = \eta$ give the requirements

(4.33)
$$f_z(\eta) = f_z(y_1) + c + \frac{\lambda}{a}(\eta - y_1)$$

(4.34)
$$\psi'(\eta) = \frac{\lambda}{a} \qquad \text{(follows from (4.23))}$$

All the conditions of Theorem 1 are now satisfied, including (3.16) (since $y_1 < \eta$). Therefore we can summarize the above as follows:

THEOREM 2 For $z > 0$ let $f_z(y)$; $0 \leq y < T$ be the unique solution of the differential equation (4.22). Suppose (4.23) and (4.24) hold. Put $\eta = y_2$ and define

(4.35)
$$\psi(y) = \psi_z(y) = \begin{cases} f_z(y) \; ; & 0 \leq y \leq \eta \\ f_z(y_1) + c + \frac{\lambda}{a}(y - y_1) \; ; & \eta < y \end{cases}$$

and
$$\psi(-y) = \psi(y); \quad y \in \mathbf{R}.$$

Suppose there exists a value of $z > 0$ such that

(4.36)
$$f_z(\eta) = f_z(y_1) + c + \frac{\lambda}{a}(\eta - y_1).$$

Then with this value of z we get that the function

(4.37)
$$\Lambda(x) := \phi(s, y) := e^{-\rho s}\psi_z(y)$$

is the value function of Problem 2.

Moreover, the corresponding optimal combined stochastic control $w^* = (r_t^*, v^*) = (r_t^*, (\theta_1^*, \theta_2^*, \ldots; \xi_1^*, \xi_2^*, \ldots))$ is the following:

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a) When $|Y_t| < |\eta|$ use no intervention control (no buying or selling of foreign currency), only the optimal continuous control (interest rate) given by (4.14):

$$(4.38) \qquad r_t^* = r^*(y) = \frac{b}{2\beta}\psi'(y) = \frac{b}{2\beta}f_z'(y)$$

(Note that such a value of r^* will introduce a drift in the exchange rate Y_t given by (4.11) towards the 'best' value $Y_t = 0$).

b) When Y_t reaches the value $-\eta$ or η, we make an intervention (by buying or selling foreign currency, respectively) of the size $\hat{\xi}(\eta) > 0$ or $-\hat{\xi}(\eta)$ required to bring Y_t to the value $-y_1$ or y_1, respectively. Thus the optimal impulse control is defined inductively by

$$(4.39) \qquad \theta_1^* = \inf\{t > 0; \ |Y_t| = \eta\}$$

and

$$(4.40) \qquad \theta_k^* = \inf\{t > \theta_{k-1}^*; \ |Y_t| = \eta\}; \quad k = 2, 3, \ldots$$

with optimal impulses ξ_k^* given by

$$(4.41) \qquad \begin{cases} \xi_k^* = \hat{\xi}(\eta) > 0 & \text{(required to bring } Y_t \text{ to } -y_1 \text{ if } Y_{\theta_k^{*-}} = -\eta) \\ \xi_k^* = -\hat{\xi}(\eta) < 0 & \text{(required to bring } Y_t \text{ to } y_1 \text{ if } Y_{\theta_k^{*-}} = \eta) \end{cases}$$

where $\hat{\xi}(\eta)$ is defined in (4.29).

REMARK. The above example was discussed primarily to illustrate the content of the general model. Unfortunately, even in this simple case we are unable to find an exact, analytic solution. Nevertheless we can describe a few characteristics of the optimal strategy. We mention two examples:

(I) Note that by (4.38) the optimal interest rate $r^*(y)$ is proportional to $\psi'(y)$. Therefore the maximal value of $r^*(y)$ occurs at the point $y = \hat{y}$ where $\psi'(y)$ is maximal. By (4.23) and (4.24) this occurs at a point $\hat{y} \in (y_1, y_2)$ such that

$$(4.42) \qquad f_z''(\hat{y}) = 0.$$

Note in particular that $y_1 < \hat{y} < y_2 = \eta$. Thus we get the perhaps surprising conclusion that $r^*(y)$ is *not* maximal at the trigger level η for intervention, but before this level is reached. This means that (in our model) the interest rate control r_t^* becomes relatively costly/ineffective (and hence should be reduced) as the exchange rate increases beyond $\hat{y}$ and approaches the trigger level for intervening.

(II) It is an interesting question how the optimal strategy depends on the minimum cost of intervention, c. The value of ψ at a given point y will depend on c, so we have $\psi(y) = \psi(c, y)$. Similarly the numbers y_1 and η depends on c: $y_1 = y_1(c), \eta = \eta(c)$. Therefore (4.36) can be written

$$\psi(c, \eta(c)) = \psi(c, y_1(c)) + c + \frac{\lambda}{a}(\eta(c) - y_1(c)).$$

Assume that the functions $\psi(c, y), \eta(c), y_1(c)$ are continuously differentiable for $(c, y) \in (0, \infty) \times \mathbf{R}$. Then we get

(4.43)
$$\begin{aligned}
D_1\psi(c, \eta(c)) + D_2\psi(c, \eta(c)) \cdot \eta'(c) = \\
= D_1\psi(c, y_1(c)) + D_2\psi(c, y_1(c)) \cdot y_1'(c) + 1 + \tfrac{\lambda}{a}(\eta'(c) - y_1'(c))
\end{aligned}$$

where D_1, D_2 denote differentiation w.r.t. first and second variables, respectively.

Now by (4.23) we have

$$D_2\psi(c, \eta(c)) = \frac{\lambda}{a} = D_2\psi(c, y_1(c))$$

and hence (4.43) leads to

(4.44)
$$D_1\psi(c, \eta(c)) = D_1\psi(c, y_1(c)) + 1 \qquad \text{for all } c > 0.$$

In particular, this implies that the value function is more sensitive to an increase in the minimum cost c of intervening (by impulse control) in the foreign exchange market near the trigger value η than at y_1. This is natural since an approach of the exchange rate towards η makes intervention indispensible and a fixed cost c therefore plays an important role.

(III) Since $\psi(c, y)$ decreases when c decreases for fixed y, we can define $\psi(0, y) = \lim_{c \to 0} \psi(c, y)$. Suppose that

(4.45)
$$\eta(0) := \lim_{c \to 0} \eta(c) \quad \text{and} \quad y_1(0) := \lim_{c \to 0} y_1(c)$$

exist and that $\psi(c, y)$ is a continuously differentiable function for $(c, y) \in [0, \infty) \times \mathbf{R}$. Then by (4.44) we get

(4.46)
$$D_1\psi(0, \eta(0)) = D_1\psi(0, y_1(0)) + 1.$$

In particular, we get the surprising conclusion that $\eta(0) \neq y_1(0)$.

We conclude that, if the functions $\psi(c, y)$, $\eta(c)$, $y_1(c)$ are continuously differentiable for $(c, y) \in (0, \infty) \times \mathbf{R}$ and (4.45) holds, then one of the following two situations occurs (possibly both):

Either

(4.47)
$$\lim_{c \to 0} D_1\psi(c, \eta(c)) = \lim_{c \to 0} D_1\psi(c, y_1(c)) = \infty$$

or

(4.48) $y_1(0) \neq \eta(0).$

If (4.48) holds, this means that no matter how small the minimum intervention cost $c > 0$ is, the optimal strategy remains of impulse control type with the size of the jump (from $\eta(c)$ to $y_1(c)$) bounded away from zero.

The conclusion is surprising, since if we start out with the assumption that $c = 0$, then it is reasonable to expect that an optimal strategy will be to intervene infinitesimally (if $\lambda > 0$) every time the exchange rate Y_t reaches certain trigger values $\pm\tilde{y}$. (See e.g. Krugman (1991), Froot and Obstfeld (1991a) and see also the related problem discussed in Davis and Normann (1990).) So if this is correct, then we would expect to have $y_1(0) = \eta(0) = \tilde{y}$ and the resulting optimal exchange rate process to be a Brownian motion in $[-\tilde{y}, \tilde{y}]$ reflected at the boundary $\pm\tilde{y}$. (We emphasize that if $c > 0$ then by (4.10) infinitesimal interventions are not optimal.)

Statement (4.47) is not quite so dramatic, but still expresses a non-smooth relation between the intervention cost c and the value function ψ near $c = 0$.

FINAL REMARKS

The main purpose of this paper has been to find a mathematical formulation of the problem of controlling optimally (under uncertainty) the exchange rate by means of

a) the domestic interest rate

and

b) interventions in the form of buying or selling large amounts of foreign currency.

We have proposed a mathematical model consisting of a combined stochastic control/impulse control problem and we have given a sufficient condition for its solution in terms of quasivariational Hamilton-Jacobi-Bellman inequalities (3.8)–(3.16). In general there seems to be little hope of obtaining explicit solutions. However, the model may still give some new insight. It is a demanding task to find efficient numerical solution methods. We leave this for future research.

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References

Avesani, R. (1990), "Endogeneous determined target zone and central bank optimal policy with limited reserves", Mimeo, Presented at CEPR/NBER Conference on Exchange Rates and Currency Bands, University of Warwick.

Bellman, R. (1957), "Dynamic Programming", Princeton University Press.

Bensoussan, A. and Lions, J.L. (1984), "Impulse Control and Quasi-Variational Inequalities", Gauthier-Villars.

Bertola, G. (1994), "Continuous-time models of exchange rate and intervention", In F. van der Ploeg, ed., Handbook of International Macroeconomics, Basil Blackwell, London.

Birkhoff, G. and Rota, G.-C. (1989), "Ordinary Differential Equations", Fourth Edition, J. Wiley & Sons.

Brekke, K.A. and Øksendal, B. (1991), "The high contact principle as a sufficiency condition for optimal stopping". In D. Lund and B. Øksendal (eds.): Stochastic Models and Option Values", North-Holland 1991, pp. 187–208.

Brekke, K.A. and Øksendal, B. (1994), "Optimal switching in an economic activity under uncertainty", SIAM Journal of Control and Optimization, 32, 1021–1036.

Brekke, K.A. and Øksendal, B. (1996), "A verification theorem for combined stochastic control and impulse control", Preprint, Dept. of Mathematics, University of Oslo 1996.

Davis, M.H.A. and Normann, A.R. (1990), "Portfolio selection with transaction costs", Math. Op. Res. 15 (4), 676–713.

Froot, K.A. and Obstfeld, M. (1991a), "Exchange-rate dynamics under stochastic regime shifts: A unified approach", Journal of International Economics, 31, 203–229.

Garber, P.M. and Svensson, L.E.O. (1996), "The operation and collapse of fixed exchange rate regimes", in Kenneth Rogoff and Gene Grossman, eds., Handbook of International Economics (North-Holland, Amsterdam).

Jeanblanc-Picqué, M. (1993), "Impulse control method and exchange rate", Mathematical Finance, 8, 161–177.

Krugman, P. (1987), "Trigger strategies and price dynamics in equity and foreign exchange markets", NBER Working paper # 2459.

Krugman, P (1991), "Target zones and exchange rate dynamics", Quarterly Journal of Economics, 106, 311–325.

Miller, M. and Lei Zhang (1994), "Optimal target zones: How an exchange rate mechanism can improve upon discretion", CEPR Discussion Paper #1031.

Øksendal, B. (1995), "Stochastic Differential Equations", Fourth edition, Springer-Verlag.

Perthame, B. (1984), "Continuous and impulse control of diffusion processes in $\mathbf{R}^N$", Nonlinear Analysis, Theory, Methods and Applications, 8, 1227–1239.

Svensson, L.E.O. (1991), "The term structure of interest rate differentials in a target zone: Theory and Swedish data", Journal of Monetary Economics 28, 87–116.

Svensson, L.E.O. (1992a), "An interpretation of recent research on exchange rate target zones", The Journal of Economic Perspectives, 6, 119–144.

Svensson, L.E.O. (1992b), "Why exchange rate bands? Monetary independence in spite of fixed exchange rates", Working Paper, Institute for International Economic Studies, Stockholm University.