THE COATES–SINNOTT CONJECTURE AND EIGENSPACES OF K–GROUPS

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Abstract. Let E signify a totally real Abelian number field with a prime power conductor and ring of p–integers R_E for a prime p. Let G denote the Galois group of E over the rationals, and let χ be a p–adic character of G of order prime to p. The odd–primary results in this paper depend on the Bloch–Kato conjecture, while the two–primary results are non–conjectural. Theorem A calculates, under a minor restriction on χ, the Fitting ideals of $\mathrm{K}_n(R_E;\mathbb{Z}_p)(\chi)$ over $\mathbb{Z}_p[G](\chi)$. Here we require that $n \equiv 2 \bmod 4$. These Fitting ideals are principal, and generated by a Stickelberger element. This gives a partial verification and also a strong indication of the Coates–Sinnott conjecture. We also discuss (co)–descent for higher K–groups, and prove in Theorem B a Hilbert Theorem 90 type of result for the transfer map in higher K–theory of number fields.

Contents

1. Introduction and summary of results
2. Eigenspace techniques
3. (Co)–descent for K–theory of number rings
4. Fitting ideals for K–groups of totally real number rings
5. More on the structure of the K–groups

1. Introduction and summary of results

Let E/F be a Galois extension of number fields. The Galois group $Gal(E/F)$ acts on various Abelian groups associated to the extension E/F. Galois module structure theory deals with these groups by exploring the extra structure coming from the $Gal(E/F)$–action. Classical examples of such groups are Picard groups, differents and units. This paper will focus on the Galois module structure for the algebraic K–groups of number rings. These are a priori just Abelian groups, but our ability to understand their deeper structure depends on the imposed Galois module structure.

The genesis of this paper was to investigate the Coates–Sinnott conjecture posed in [CS]. Given an integer $f > 1$ and an integer a relatively prime to f, one defines

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the partial zeta function by

$$\zeta_f(s,a) = \sum_{k\equiv a \bmod f} k^{-s}$$

where $Re(s) > 1$. The function $\zeta_f(s,a)$ can be extended to a meromorphic function over the whole complex plane. Let E be an Abelian number field of conductor f, and choose a primitive fth root of unit ζ_f. For every positive a with $(a,f)=1$ we denote by $\sigma_a \in Gal(E/\mathbb{Q})$ the restriction to E of the automorphism of $\mathbb{Q}(\zeta_f)$ that maps ζ_f to its ath power. For $n \geq 0$, the nth Stickelberger element relative to E is defined as the sum:

$$\theta_n = \sum_{(a,f)=1, 1\leq a<f} \zeta_f(-n,a)\sigma_a^{-1}.$$

Many arithmetic properties of a number field are concealed in its zeta function. For K–groups of rings of integers in number fields one expects a similar phenomenon. Coates and Sinnott announced in their Inventiones Mathematicae paper [CS] from 1974 the following conjecture.

Conjecture 1.1. *(Coates–Sinnott) Let b be a positive integer with $(b,f)=1$. The element $\theta_n(b) = w_{n+1}(\mathbb{Q})(b^{n+1}-\sigma_b)\theta_n$ annihilates $\mathrm{K}_{2n}(\mathcal{O}_E)$ where $w_n(\mathbb{Q})$ denotes the largest integer m such that $(\mathbb{Z}/m)^*$ has exponent dividing n, and $\mathcal{O}_E$ denotes the ring of integers of the field E.*

Background and motivation. Conjecture 1.1 was inspired by the Lichtenbaum conjectures, and earlier work by Coates and Lichtenbaum [CL]. Moreover, Coates and Sinnott proved in [CS] – with an additional assumption on b – that the element $\theta_1(b)$ annihilates the p–primary subgroup of $\mathrm{K}_2(\mathcal{O}_E)$ for all odd primes p. If E is totally real and n is even, then the conjecture is totally insipid since the Coates–Sinnott element $\theta_n(b)$ is trivial. The classical Stickelberger theorem in algebraic number theory fits beautifully into this picture, having Conjecture 1.1 as a natural generalization. One conspicuous attempt to establish the odd primary part of the Coates–Sinnott conjecture is the Annals of Mathematics paper [Ba1] of Banaszak from 1992. Theorems A and C in loc sit give evidence for Conjecture 1.1, and the same goes for Theorem 5 in [Ba2]. Théorème 2–2 of Nguyen in [Ng] tells us that the Coates–Sinnott element $\theta_n(b)$ annihilates the étale cohomology group $\mathrm{H}^2_{\text{ét}}(\mathcal{O}_E[\frac{1}{p}];\mathbb{Z}_p(n+1))$. This gives a very strong evidence for Conjecture 1.1. In fact, if the Bloch–Kato conjecture for number fields is true for an odd prime p, then Théorème 2–2 in [Ng] would imply the p–primary part of the Coates–Sinnott conjecture. This follows from the well known fact that the Bloch–Kato conjecture for number fields at an odd prime p implies the famous longstanding Quillen–Lichtenbaum conjecture for number fields at p. See Theorem 1.2 for more precise statements.

Our approach to Conjecture 1.1 is based on recent results of Kolster, Rognes and Weibel which give a cohomological interpretation of two–primary algebraic K–groups, techniques introduced by Greither [Gri] in order to study modules of projective dimension one, and the main conjecture of Iwasawa theory due to Mazur and Wiles [MW]. We point out that [RW] is based on the recent flourishing activity in motivic cohomology, cf. the papers [BL], [FV], [SV] and [Vo1] written by Bloch, Friedlander, Lichtenbaum, Suslin and Voevodsky – alone and in collaboration – and of course Voevodsky's proof of the Milnor conjecture at the prime 2 in [Vo2].

This paper is based on and could not have been written without the benefit of all these extensive works. Instead of studying the entire algebraic K–group we will try to pin down each of its eigenspaces. For this we try to determine the size and structure of the eigenspaces for the action of $Gal(E/\mathbb{Q})$ on the even algebraic K–groups for E a totally real Abelian number field of prime power conductor.

Main results. Given a prime p we let $\omega : Gal(\mathbb{Q}(\mu_{2p})/\mathbb{Q}) \longrightarrow \mathbb{Z}_p^*$ denote the Teichmüller character. For an Abelian number field E and a p–adic character χ of $Gal(E/\mathbb{Q})$ we think of χ as a character of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ which is trivial on $Gal(\overline{\mathbb{Q}}/E)$. Here we write $\overline{\mathbb{Q}}$ for the algebraic closure of the rationals. Further we let $\mathbb{Q}_\chi$ denote the part of $\overline{\mathbb{Q}}$ fixed by the kernel of χ. Let M be a finitely presented module over a commutative ring R with identity. We denote the first Fitting ideal of M over R by $\mathrm{Fit}_R(M)$. Recall that the latter is contained in the annihilator ideal of the R–module M. Given a Galois module M and a character χ as above, we denote by $M(\chi)$ the χ–part of M. For more details on χ–parts, see Section 2. The two–primary conclusion in the following result is non–conjectural, while the odd-primary result depends on the Bloch–Kato conjecture.

Theorem A. *Let E be a totally real Abelian number field with ring of p–integers R_E, and with a prime power conductor. Let G denote the Galois group of E over the rationals, and let P be its p–Sylow subgroup. Write $G = \Delta \times P$, and let $\chi : \Delta \longrightarrow \overline{\mathbb{Q}}_p^*$ be a character. Suppose that $\chi^{-1}\omega^{\frac{n}{2}+1}$ is a nontrivial character of $Gal(\mathbb{Q}_{\chi^{-1}\omega^{\frac{n}{2}+1}}/\mathbb{Q})$ where $n \equiv 2 \bmod 4$. Assume the Bloch–Kato conjecture is true for E at the odd rational prime p. Then we have*

$$\mathrm{Fit}_{\mathbb{Z}_p[G](\chi)} \mathrm{K}_n(R_E;\mathbb{Z}_p)(\chi) = (\theta_{\frac{n}{2}}(\chi)) \text{ for } n \equiv 2 \bmod 4$$

$$\mathrm{Fit}_{\mathbb{Z}_2[G](\chi)} \mathrm{K}_n(R_E;\mathbb{Z}_2)(\chi) = (\theta_{\frac{n}{2}}(\chi)) \text{ for } n \equiv 2 \bmod 8$$

$$\mathrm{Fit}_{\mathbb{Z}_2[G](\chi)} \mathrm{K}_n(R_E;\mathbb{Z}_2)(\chi) = (\frac{\theta_{\frac{n}{2}}(\chi)}{2}) \text{ for } n \equiv 6 \bmod 8$$

where $\theta_{\frac{n}{2}}$ is the $\frac{n}{2}$th Stickelberger element relative to E.

Theorem A gives a partial verification of the two–primary part of Conjecture 1.1. The elements $\theta_n(b)$ constructed by Coates and Sinnott are quite different from the generators of the Fitting ideals in Theorem A. This refinement should not be too surprisingly since we consider eigenspaces of the K–groups. Theorem A also gives, modulo the Bloch–Kato conjecture, a refinement of the odd–primary part of Conjecture 1.1.

In Section 3 we discuss (co)–descent for algebraic K–groups of number fields and their corresponding number rings. Most noteworthy we establish a Hilbert Theorem 90 for higher algebraic K–groups. See [MS] for the case of K_2, and [Vo2] for a similar result in higher Milnor K–theory of fields.

Theorem B. *(Hilbert Theorem 90) Let E/F be a Galois extension of number fields with Galois group Γ. Let r_∞ denote the number of real ramified primes in E/F and let $n \geq 1$. If the Bloch–Kato conjecture holds for both E and F at the odd prime p, then the transfer map $\mathrm{tr}_\Gamma : \mathrm{K}_{2n}(E)_\Gamma \longrightarrow \mathrm{K}_{2n}(F)$ is bijective on the p–primary part. For $n \equiv 3 \bmod 4$ the transfer map tr_Γ is bijective on the 2-primary part. The same map is injective with cokernel isomorphic to $(\mathbb{Z}/2)^{r_\infty}$ for $n \equiv 1 \bmod 4$ on the two–primary part. For $n \equiv 0, 2 \bmod 4$, the two–primary part of the kernel of* tr_Γ

has order less or equal to 2^{r_∞} *while the cokernel is trivial. Here we assume that* $\mathrm{K}_{2n}(R_A;\mathbb{Z}_2)$ *is isomorphic to* $\mathrm{H}^2_{\text{ét}}(R_A;\mathbb{Z}_2(n+1))$ *for* $A = E, F$ *if* $n \equiv 2 \bmod 4$.

Our odd–primary results are to our knowledge still conjectural, and they depend on the Bloch–Kato conjecture which we will now briefly explain. A written account of the following can be found in the not so easily accessible source [Ko]. The Bloch–Kato conjecture for F at an odd prime p predicts that the Galois symbol $\mathrm{K}^M_n(F)/p^\nu \longrightarrow \mathrm{H}^n(F;\mathbb{Z}/p^\nu(n))$ from Milnor K–theory of F to Galois cohomology of F is an isomorphism for all $n \geq 2$. Let $\mathcal{M}$ indicate the motivic cohomology introduced in [FV] and [Vo1]. From Suslin's identification in [Su] of Bloch's higher Chow groups with motivic cohomology $\mathcal{M}$ one finds that the Bloch–Lichtenbaum spectral sequence for F amounts to a third quadrant spectral sequence:

$$\mathrm{E}_2^{m,n} = \mathrm{H}^{m-n}_{\mathcal{M}}(F;\mathbb{Q}_p/\mathbb{Z}_p(-n))) \Longrightarrow \mathrm{K}_{-m-n}(F;\mathbb{Q}_p/\mathbb{Z}_p).$$

Suslin and Voevodsky proved in [SV] that the Bloch–Kato conjecture for F at p implies that motivic cohomology $\mathcal{M}$ coincide with étale cohomology in a certain range. Briefly, this means that the Bloch–Lichtenbaum spectral sequence for F can be rewritten as:

$$\mathrm{E}_2^{m,n} = \begin{cases} \mathrm{H}^{m-n}_{\text{ét}}(F;\mathbb{Q}_p/\mathbb{Z}_p(-n)) & \text{for } n \leq m \leq 0 \\ 0 & \text{otherwise} \end{cases} \Longrightarrow \mathrm{K}_{-m-n}(F;\mathbb{Q}_p/\mathbb{Z}_p).$$

This conjectural spectral sequence collapses directly at the E_2–page, and standard techniques with localization sequences deliver

Theorem 1.2. *Let* F *be a number field with ring of* S*–integers* R_F *where* S *is a finite set of primes in* F *that contains the infinite ones and the* p*–adic ones for an odd rational prime* p*. If the Bloch–Kato conjecture is true for* F *at* p*, then the Quillen–Lichtenbaum conjecture holds for* F *at* p*. In other words, the Chern classes induce isomorphisms* $\mathrm{K}_{2n-m}(R_F)\otimes\mathbb{Z}_p \xrightarrow{\cong} \mathrm{H}^m_{\text{ét}}(R_F;\mathbb{Z}_p(n))$ *for* $m = 1, 2$ *and* $n \geq 2$.

Method of proof. To prove Theorem B we use the the Tate spectral sequence for étale cohomology and the identification of K-groups with étale cohomology groups. The two–primary parts of these results lend on results from [RW], which we now quote parts of. Parts (a) and (b) of Theorem 1.3 are a reformulation of Theorem 6.13 in [RW], part (c) follows from Theorems 6.3 and 6.7 in [RW] and the Bockstein exact sequence.

Theorem 1.3. *(Rognes–Weibel) We have the following results for* $n \geq 0$.

(a) Let F *be a totally imaginary number field. Then there exist isomorphisms* $\mathrm{K}_{2n}(R_F)\{2\} \cong \mathrm{H}^2_{\text{ét}}(R_F;\mathbb{Z}_2(n+1))$ *and* $\mathrm{K}_{2n+1}(R_F)\{2\} \cong \mathrm{H}^1_{\text{ét}}(R_F;\mathbb{Z}_2(n+1))$.

(b) Let F *be a number field with at least one real embedding. Then*

1) $\mathrm{K}_{8n}(R_F)\{2\} \cong \mathrm{H}^2_{\text{ét}}(R_F;\mathbb{Z}_2(4n+1))$.

2) $\mathrm{K}_{8n+2}(R_F)\{2\} \cong \mathrm{H}^2_{\text{ét}}(R_F;\mathbb{Z}_2(4n+2))$.

3) $\mathrm{K}_{8n+4}(R_F)\{2\}$ *surjects onto* $\mathrm{H}^2_{\text{ét}}(R_F;\mathbb{Z}_2(4n+3))$.

4) $\mathrm{K}_{8n+6}(R_F)\{2\} \cong \mathrm{H}^2_{+}(R_F;\mathbb{Z}_2(4n+4))$ *where the* $+$ *indicates positive étale cohomology.*

(c) Both (a) and (b) hold for F *as well.*

The latter result is also important for Theorem A. The proof of Theorem A is the most technical part of the paper. We lend on recent results of Greither in [Gri], and the main conjecture of Iwasawa theory [MW]. The philosophy is to restrict our attention to eigenspaces of the K–groups. In fact, Theorem A is not valid integrally as illustrated in Remark 5.23. We remark that readers familiar with the important work of Dwyer and Friedlander [DF] on étale K–theory recognize from Theorem 1.2 that the conjectural odd–primary results in Theorem A are genuine theorems for the étale K–groups of R_E.

In Section 5 we generalize previous results on K_2 due to Tate [Ta]. If enough roots of unity are in the field we find, due to an easy trick with étale cohomology groups, a periodicity behavior of the K–groups $K_{2n}(F)$ which reduces everything down to $K_2(F)$ (Theorem 5.8(b)). Theorem A combined with Nakayama's lemma gives the minimal number of generators of the χ–eigenspace of the even K–groups in terms of Iwasawa theory (Theorem 5.1).

Organization of the paper. Section 2 is preparatory where we draw on the basic ideas of considering eigenspaces. We also hope to motivate a bit why one should focus on eigenspaces by including Proposition 2.4 where we discuss the two–primary algebraic K–theory of a cyclic cubic number field. In Section 3 we look at Galois extensions of number fields, and try to compare the (co)–invariants of the algebraic K–theory of the extension field with that of the base field. These considerations culminate with a proof of Theorem B, which we for obvious reasons call Hilbert Theorem 90 for higher algebraic K-groups of number fields. The entire Section 4 is devoted to prove Theorem A. From the knowledge of the Fitting ideals of the algebraic K–groups we find structural results for these groups in Section 5. Further we give direct generalizations of classical theorems on K_2 due to Tate.

2. Eigenspace techniques

In this section we introduce some notation, and discuss as an example the two–primary algebraic K–theory of a cyclic cubic number field.

Let G be a finite Abelian group and let p be a prime number not dividing the order of G. In the following we will discuss the structure of finitely generated $\mathbb{Z}_p[G]$–modules. Let $\chi : G \longrightarrow \overline{\mathbb{Q}}_p^*$ denote a p–adic character. Two such characters are considered equivalent if they are $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$–conjugate. We choose the notation $\mathcal{O}_\chi = \mathbb{Z}_p[\chi]$ for the p–adic integers extended with the values of χ. This is the ring of integers in a finite unramified extension of $\mathbb{Q}_p$. In result we have that $\mathcal{O}_\chi$ is a discrete valuation ring, and hence a principal ideal domain. The ring $\mathcal{O}_\chi$ is also a $\mathbb{Z}_p[G]$–algebra via the rule $g \cdot x := \chi(g)x$ for $g \in G$ and $x \in \mathcal{O}_\chi$. We have a decomposition

$$\mathbb{Z}_p[G] \cong \bigoplus \mathcal{O}_\chi \tag{2.1}$$

where the sum is over all p–adic characters of G, modulo equivalence. From the decomposition (2.1) we see that the ring $\mathbb{Z}_p[G]$ is semisimple. Let M be a finitely generated $\mathbb{Z}_p[G]$–module. The χ–part $M(\chi)$ of M is defined as the χ–eigenspace of M, namely $M(\chi) = M \otimes_{\mathbb{Z}_p[G]} \mathcal{O}_\chi = \{x \in M \mid g \cdot x = \chi(g)x \text{ for all } g \in G\}$. The crux with taking eigenspaces is the decomposition

$$M \cong \bigoplus M(\chi) \tag{2.2}$$

where the sum in (2.2) is over the finite set of representatives of equivalence classes of p–adic characters of G. Each $M(\chi)$ is an $\mathcal{O}_\chi$–module. So in order to understand the $\mathbb{Z}_p[G]$–module structure of M it suffices to study the structure of its eigenspaces. The following proposition is an immediate consequence of the structure theorem for finitely generated modules over a principal ideal domain.

Proposition 2.3. *The χ–part of a finite $\mathcal{O}_\chi$–module M has p–rank a multiple of the extension degree $f_\chi = [\mathcal{O}_\chi : \mathbb{Z}_p]$. This equals in turn the multiplicative order of p in $(\mathbb{Z}/f_\chi)^*$. Moreover, we have an isomorphism $\mathcal{O}_\chi/p^n \cong (\mathbb{Z}/p^n)^{f_\chi}$ of Abelian groups.*

Let M be a finite $\mathcal{O}_\chi$–module. Proposition 2.3 tells us that M is isomorphic to a finite direct sum of copies of $(\mathbb{Z}/p^n)^{f_\chi}$ as an Abelian group.

We illustrate the above with the example of a cyclic cubic number field F with Galois group Γ over the rationals. The number of 2–adic characters of Γ modulo equivalence is just two. Let ζ_3 be a primitive 2–adic third root of unity. In (2.1) we have a direct summand $\mathbb{Z}_2$ that corresponds to the trivial character χ_0 and a direct summand $\mathbb{Z}_2[\zeta_3]$ that corresponds to the single nontrivial 2–adic character χ. Note that $f_\chi = 2$.

Proposition 2.4. *Let F be a cyclic cubic number field with ring of p–integers R_F. The 2–ranks of the class group $\mathrm{Pic}(R_F)$ and of the narrow class group $\mathrm{Pic}_+(R_F)$ are both even. In the following we assume that $n \geq 2$. For $p = 2$, or p and n both odd, we have that the 2–rank of $\mathrm{K}_n(R_F)$ is even for $n \equiv 0,4,5,6 \bmod 8$ and odd for $n \equiv 1,2,3,7 \bmod 8$. For p odd the 2–rank of $\mathrm{K}_n(R_F)$ is even for $n \equiv 2 \bmod 8$ and odd for $n \equiv 0,4,6 \bmod 8$.*

Proof. The χ_0–parts of $\mathrm{Pic}_+(R_F)$ and $\mathrm{Pic}(R_F)$ are both trivial since $\mathrm{Pic}_+(\mathbb{Z}[\frac{1}{p}]) = \mathrm{Pic}(\mathbb{Z}[\frac{1}{p}]) = 0$. Further $\mathrm{Pic}_+(R_F)(\chi)$ and $\mathrm{Pic}(R_F)(\chi)$ are finite $\mathbb{Z}_2[\zeta_3]$–modules. Their two–ranks are even being multiples of $f_\chi = 2$, see Proposition 2.3. By the same argument we find that the two–rank of the torsion subgroup of $\mathrm{K}_n(R_F)(\chi)$ is even. The two–primary algebraic K–theory of $\mathbb{Z}[\frac{1}{2}]$ is known from [RW] and [W2]. Combining these we find the list for $p = 2$ above. Let p be an odd prime and $n \geq 1$. Then

$$\mathrm{rk}_2\, \mathrm{K}_{2n}(\mathbb{Z}[\frac{1}{p}]) = \begin{cases} 1 & \text{for } n \equiv 0,2,3 \bmod 4, \\ 2 & \text{for } n \equiv 1 \bmod 4. \end{cases}$$

The proof of this is easy but tedious, and implies our claim. □

3. (Co)–descent for K–theory of number rings

In this section we will study Galois descent and co–descent properties for the higher algebraic K–groups of number rings. We will also spend some pages on the transfer map in algebraic K–theory. The main result is Theorem B.

We start by introducing notation. Let F be a number field. Let S be a finite set of primes of F including the primes lying above a fixed rational prime p, and the infinite ones. For a prime $\wp$ in F we denote its residue field by $F[\wp]$, and we let R_F be the ring of S–integers in F.

For any $i \in \mathbb{Z}$, let $\mathrm{H}^n_{\text{ét}}(R;\mathbb{Z}_p(i))$ denote the continuous p–adic étale cohomology of R with coefficients in the sheaf $\mathbb{Z}_p(i)$ as constructed by Jannsen in [Ja]. Here R is a unital ring containing p as an invertible element. An equivalent definition can be found in Definition 2.8 of [DF]. This group sits in a short exact sequence $0 \to \lim^1 \mathrm{H}^{n-1}_{\text{ét}}(R;\mathbb{Z}/p^\nu(i)) \to \mathrm{H}^n_{\text{ét}}(R;\mathbb{Z}_p(i)) \to \lim_\nu \mathrm{H}^n_{\text{ét}}(R;\mathbb{Z}/p^\nu(i)) \to 0$, where lim denotes the inverse limit functor. The rightmost étale cohomology group is called the p–adic étale cohomology of R with coefficients in the sheaf $\mathbb{Z}_p(i)$. The $\lim^1$–term is often trivial, e.g., for rings of S–integers and finite fields.

We recall for future reference the following important results.

Theorem 3.1. *Let $\mathbb{F}_q$ be the finite field with q elements, and $n \geq 1$. Then $\mathrm{K}_{2n}(\mathbb{F}_q)$ is the trivial group. Let p be a prime number such that $(p,q)=1$. Then we have natural isomorphisms $\mathrm{K}_{2n-1}(\mathbb{F}_q)\otimes\mathbb{Z}_p \cong \mathrm{H}^1_{\text{ét}}(\mathbb{F}_q,\mathbb{Z}_p(n)) \cong \mathbb{Z}_p/(q^n-1)$.*

Proof. See [Q1] and [So]. □

Theorem 3.2. *For $n \geq 1$ we have $\mathrm{K}_{2n+1}(R_F) \cong \mathrm{K}_{2n+1}(F)$ and a short exact sequence $0 \to \mathrm{K}_{2n}(R_F) \to \mathrm{K}_{2n}(F) \to \bigoplus_{\wp\notin S}\mathrm{K}_{2n-1}(F[\wp]) \to 0$.*

Proof. See [So] and [W1]. □

We have an analogous result in étale cohomology.

Proposition 3.3. *For $i \geq 2$ we have $\mathrm{H}^1_{\text{ét}}(R_F,\mathbb{Z}_p(i)) \cong \mathrm{H}^1_{\text{ét}}(F,\mathbb{Z}_p(i))$ and a short exact sequence:*

$$0 \to \mathrm{H}^2_{\text{ét}}(R_F,\mathbb{Z}_p(i)) \to \mathrm{H}^2_{\text{ét}}(F,\mathbb{Z}_p(i)) \to \bigoplus_{\wp\notin S}\mathrm{H}^1_{\text{ét}}(F[\wp],\mathbb{Z}_p(i-1)) \to 0.$$

Proof. Note that $\lim^1 \mathrm{H}^0_{\text{ét}}(F,\mathbb{Z}/p^\nu(i))$ is trivial. The isomorphism in the proposition follows by passing to the inverse limit with respect to ν in the localization sequence in étale cohomology, cf. Proposition 1 in [So]. This works since the limit functor is left exact, and $\mathrm{H}^0_{\text{ét}}(F[\wp],\mathbb{Z}_p(i-1))$ is trivial.

To prove the second assertion, we pass to the colimit in the localization sequence for étale cohomology. This preserves exactness and we get a short exact sequence:

$$0 \to \mathrm{H}^1_{\text{ét}}(R_F,\mathbb{Q}_p/\mathbb{Z}_p(i)) \to \mathrm{H}^1_{\text{ét}}(F,\mathbb{Q}_p/\mathbb{Z}_p(i)) \to \bigoplus_{\wp\notin S}\mathrm{H}^0_{\text{ét}}(F[\wp],\mathbb{Q}_p/\mathbb{Z}_p(i-1)) \to 0.$$

All three terms in the above exact sequence are p–primary Abelian torsion groups. For any such group we let Div denote its maximal divisible subgroup. The right term is a direct sum of finite groups, so its divisible part is trivial and we get the exact sequence:

$$0 \longrightarrow \mathrm{H}^1_{\text{ét}}(R_F,\mathbb{Q}_p/\mathbb{Z}_p(i))/\operatorname{Div} \longrightarrow \mathrm{H}^1_{\text{ét}}(F,\mathbb{Q}_p/\mathbb{Z}_p(i))/\operatorname{Div} \longrightarrow$$
$$\longrightarrow \bigoplus_{\wp\notin S}\mathrm{H}^0_{\text{ét}}(F[\wp],\mathbb{Q}_p/\mathbb{Z}_p(i-1)) \longrightarrow 0.$$

Further $\mathrm{H}^1_{\text{ét}}(F,\mathbb{Q}_p/\mathbb{Z}_p(i))/\operatorname{Div} \cong \mathrm{H}^2_{\text{ét}}(F,\mathbb{Z}_p(i))$ from Proposition 2.3 in [Ta], and likewise for R_F. Finally we have $\mathrm{H}^0_{\text{ét}}(F[\wp],\mathbb{Q}_p/\mathbb{Z}_p(i-1)) \cong \mathrm{H}^1_{\text{ét}}(F[\wp],\mathbb{Z}_p(i-1))$. □

Let E/F be a Galois extension of number fields with Galois group Γ. Let T be the set of primes lying above S in E, and let R_E be the ring of T–integers in E. Let $n \geq 1$ and let γ denote the natural map from $\mathrm{K}_{2n}(F)$ to $\mathrm{K}_{2n}(E)^\Gamma$. In the following key proposition we compare γ with the induced map γ_S from $\mathrm{K}_{2n}(R_F)$ to $\mathrm{K}_{2n}(R_E)^\Gamma$. Let $\mathcal{M}(R)$ denote the exact category of finitely generated R–modules for a ring R. The nth K′–group of R is defined as $\mathrm{K}_n(\mathcal{M}(R))$. It is fundamental that ordinary algebraic K–theory coincide with K′–theory for regular rings.

Proposition 3.4. *We have an exact sequence*

$$0 \longrightarrow \ker \gamma_S \longrightarrow \ker \gamma \longrightarrow \bigoplus_{\wp \notin S} \ker \left(\mathrm{K}_{2n-1}(F[\wp]) \xrightarrow{e_\wp} \mathrm{K}_{2n-1}(F[\wp])\right) \longrightarrow$$

$$\longrightarrow \operatorname{coker} \gamma_S \longrightarrow \operatorname{coker} \gamma \longrightarrow \bigoplus_{\wp \notin S} \mathrm{K}_{2n-1}(F[\wp])/e_\wp$$

where $e_\wp$ denotes the ramification index of $\wp$ in E/F.

Proof. We can – from Theorem 3.2 – make the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{K}_{2n}(R_F) & \longrightarrow & \mathrm{K}_{2n}(F) & \longrightarrow & \bigoplus_{\wp \notin S} \mathrm{K}_{2n-1}(F[\wp]) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma_S} & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{K}_{2n}(R_E)^\Gamma & \longrightarrow & \mathrm{K}_{2n}(E)^\Gamma & \longrightarrow & (\bigoplus_{\overline{\wp} \notin T} \mathrm{K}_{2n-1}(E[\overline{\wp}]))^\Gamma. & & \end{array}$$

The localization sequence is natural with respect to pairs like (F, S) and (E, T). Moreover, the action of Γ on the K–groups of the finite fields can be described in the following way. Consider the decomposition group $\Gamma_{\overline{\wp}}$ of $\overline{\wp}$ where $\overline{\wp}$ is one of the primes $\overline{\wp}_i$ lying above $\wp$ in E. This group surjects onto the Galois group G of the extension $E[\overline{\wp}]/F[\wp]$. The Γ–action is given by:

$$(\bigoplus_{\overline{\wp}_i} \mathrm{K}_{2n-1}(E[\overline{\wp}_i]))^\Gamma = (\bigoplus_{\overline{\wp}_i} (\mathrm{K}_{2n-1}(E[\overline{\wp}_i]))^{\Gamma_{\overline{\wp}}})^{\Gamma/\Gamma_{\overline{\wp}}}.$$

Here $\Gamma_{\overline{\wp}}$ acts on $\mathrm{K}_{2n-1}(E[\overline{\wp}_i])$ via the natural action of G and the surjection above. The group $\Gamma/\Gamma_{\overline{\wp}}$ acts by permuting the $\#\Gamma/\Gamma_{\overline{\wp}}$ factors. By a result of Quillen, see p.585 [Q1] or Theorem 3.1, we have that $\mathrm{K}_{2n-1}(E[\overline{\wp}_i])^G \cong \mathrm{K}_{2n-1}(F[\wp])$. Thus we can identify $\mathrm{K}_{2n-1}(F[\wp])$ with $(\bigoplus_{\overline{\wp_i}} \mathrm{K}_{2n-1}(E[\overline{\wp}]))^\Gamma$.

The map on the right hand side in the diagram is nothing but multiplication by the ramification index, as C. A. Weibel kindly pointed out to us. See also [Ge], Corollary 1.11 [Gi], Proposition 1.2 [Sh] and p.276 [So]. Let R'_F and R'_E be such that $Spec(R'_F) = Spec(R_F) \setminus \{\wp\}$ and $Spec(R'_E) = Spec(R_E) \setminus (\{\overline{\wp}\})_{\overline{\wp}|\wp}$. The commutative diagram of exact categories

$$\begin{array}{ccccc} \mathcal{M}(F[\wp]) & \longrightarrow & \mathcal{M}(R_F) & \longrightarrow & \mathcal{M}(R'_F) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{\overline{\wp}|\wp} \mathcal{M}(R_E/\overline{\wp}^{e_\wp}) & \longrightarrow & \mathcal{M}(R_E) & \longrightarrow & \mathcal{M}(R'_E) \end{array} \quad (3.5)$$

induces the maps between the localization sequences. Here the left vertical map is tensoring over R_F with R_E, and $F[\wp] \otimes_{R_F} R_E \cong \bigoplus_{\overline{\wp}|\wp} R_E/\overline{\wp}^{e_\wp}$. On K′–theory we obtain diagrams of fibrations, and we need to describe the map:

$$\mathrm{K}_{2n-1}(F[\wp]) \longrightarrow (\bigoplus_{\overline{\wp}|\wp} \mathrm{K}'_{2n-1}(R_E/\overline{\wp}^{e_\wp}))^\Gamma \cong (\bigoplus_{\overline{\wp}|\wp} \mathrm{K}_{2n-1}(E[\overline{\wp}]))^\Gamma \cong \mathrm{K}_{2n-1}(F[\wp]).$$

Here the left most isomorphism follows since K′–theory is immune to quotients of nilpotent ideals, cf. Corollary 2 in § 5 of [Q2]. For an object $\mathcal{P}$ in $\mathcal{M}(F[\wp])$ we now form the characteristic filtration:

$$0 = \overline{\wp}^{e_\wp}(\mathcal{P} \otimes_{R_F} R_E) \subset \cdots \subset \overline{\wp}(\mathcal{P} \otimes_{R_F} R_E) \subset \mathcal{P} \otimes_{R_F} R_E.$$

The quotients in this filtration are all isomorphic to $\mathcal{P} \otimes_{F[\wp]} E[\overline{\wp}]$. Additivity for characteristic filtrations – see Corollary 2 in § 3 of [Q2] – gives the claim. An application of the snake lemma then concludes the proof. □

Corollary 3.6. *Assume that* $((\#F[\wp])^n - 1, e_\wp) = 1$ *for all ramified primes not in* S*. Then* $\ker \gamma_S \cong \ker \gamma$ *and* $\operatorname{coker} \gamma_S \cong \operatorname{coker} \gamma$*. When* E *is Abelian over* $\mathbb{Q}$*, the same results hold for the* χ*–components if* χ *is a nontrivial character of order prime to* p*, and* $\mathrm{K}_{2n-1}(F[\wp])$ *is fixed by* $Gal(F/\mathbb{Q})$ *for all the ramified primes* $\wp$ *in* E/F *not in* S*.*

Proof. The equality $\#\,\mathrm{K}_{2n-1}(F[\wp]) = (\#F[\wp])^n - 1$ and Proposition 3.4 give the first claim. For the second part, we assume that $\mathrm{K}_{2n-1}(F[\wp])$ is fixed by $Gal(F/\mathbb{Q})$ for all the ramified primes $\wp$ in E/F not in S. Thus on χ–components we have the equality:

$$\bigoplus_{\wp \notin S} \ker\left(\mathrm{K}_{2n-1}(F[\wp]) \xrightarrow{e_\wp} \mathrm{K}_{2n-1}(F[\wp])\right)(\chi) =$$

$$\{x \in \bigoplus_{\wp \notin S} \ker\left(\mathrm{K}_{2n-1}(F[\wp]) \xrightarrow{e_\wp} \mathrm{K}_{2n-1}(F[\wp])\right) \mid x = \chi(g)x \;\; \forall \;\, g \in Gal(F/\mathbb{Q})\}.$$

This sum amounts to the trivial group since χ is a nontrivial character. Proposition 3.4 gives us the isomorphisms $\ker \gamma_S(\chi) \cong \ker \gamma(\chi)$ and $\operatorname{coker} \gamma_S(\chi) \cong \operatorname{coker} \gamma(\chi)$. Here we write $\gamma(\chi)$ and $\gamma_S(\chi)$ for the maps induced respectively by γ and γ_S on the χ–part. □

We refer to [AGV] for the following result.

Theorem 3.7. *(Lyndon-Hochschild-Serre spectral sequence) Assume that* R_E/R_F *is a finite Galois extension of number rings with Galois group* Γ*. The group* Γ *acts on the groups* $\mathrm{H}^n_{\text{ét}}(R_E; \mathbb{Z}/p^\nu(i))$*, and there exists a first quadrant spectral sequence of cohomological type*:

$$\mathrm{E}_2^{r,s} = \mathrm{H}^r(\Gamma; \mathrm{H}^s_{\text{ét}}(R_E; \mathbb{Z}/p^\nu(i))) \Longrightarrow \mathrm{H}^{r+s}_{\text{ét}}(R_F; \mathbb{Z}/p^\nu(i)).$$

In [Ka1] Bruno Kahn introduced positive Galois cohomology following ideas of Kato and Milne. Positive étale cohomology is defined likewise in [C–S]. We indicate positive étale cohomology with the lower index $+$. The positive groups come with an exact sequence $\cdots \to \mathrm{H}^n_+(F; \mathbb{Z}_2(i)) \to \mathrm{H}^n_{\text{et}}(F; \mathbb{Z}_2(i)) \to \bigoplus \mathrm{H}^n_{\text{et}}(F_\wp; \mathbb{Z}_2(i)) \to \mathrm{H}^{n+1}_+(F; \mathbb{Z}_2(i)) \to \cdots$ from [Ka1], where we sum over the infinite primes in F and $F_\wp$ denotes the completion of F at the place $\wp$. An identical exact sequence where the field F is replaced by its ring of S–integers is established in [C–S]. In particular, we have short exact sequences $0 \to \bigoplus_{\wp|\infty}(\mathbb{Z}_2)\wp \to \mathrm{H}^1_+(F; \mathbb{Z}_2(i)) \to \mathrm{H}^1_{\text{et}}(F; \mathbb{Z}_2(i)) \to 0$ and $0 \to \mathrm{H}^2_+(F; \mathbb{Z}_2(i)) \to \mathrm{H}^2_{\text{et}}(F; \mathbb{Z}_2(i)) \to \bigoplus_{\wp \text{ real}}(\mathbb{Z}/2)\wp \to 0$ for $i \geq 2$ even. Identical exact sequences are obtained by replacing F by a ring of S–integers. There is also a Lyndon–Hochschild–Serre spectral sequence for positive étale cohomology, cf. [Ka1]. Also the Tate spectral sequence considered in [Ka2] exists for the positive groups. We write the proof of the following result for étale cohomology, with the positive case being just a verbatim.

Theorem 3.8. *Let E/F be a Galois extension of number fields with Galois group Γ. Assume that S contains the primes lying above the rational prime (2), and the ramified primes in E/F. We have the following results for $n \geq 1$.*

(a) Let F be a totally imaginary number field. Then there exists an exact sequence

$$0 \to \mathrm{H}^1(\Gamma; M) \to \mathrm{K}_{2n}(R_F; \mathbb{Z}_2) \to \mathrm{K}_{2n}(R_E; \mathbb{Z}_2)^\Gamma \to \mathrm{H}^2(\Gamma; M) \to 0$$

where $M = \mathrm{K}_{2n+1}(R_F; \mathbb{Z}_2)$.

(b) Let F be a real number field, and assume that $\mathrm{K}_{2n}(R_A; \mathbb{Z}_2)$ is isomorphic to $\mathrm{H}^2_{\text{ét}}(R_A; \mathbb{Z}_2(n+1))$ for $A = E, F$ if $n \equiv 2 \bmod 4$. Then there exists an exact sequence

$$0 \to \mathrm{H}^1(\Gamma; M) \to \mathrm{K}_{2n}(R_F; \mathbb{Z}_2) \to \mathrm{K}_{2n}(R_E; \mathbb{Z}_2)^\Gamma \to \mathrm{H}^2(\Gamma; M) \to \mathrm{H}^3_{\text{ét}}(R_F; \mathbb{Z}_2(n+1))$$

where M denotes $\mathrm{H}^1_{\text{ét}}(R_E; \mathbb{Z}_2(n+1))$ for $n \equiv 0, 1, 2 \bmod 4$. For $n \equiv 3 \bmod 4$ we have the same exact sequence as in part (a), where M now denotes $\mathrm{H}^1_+(R_E; \mathbb{Z}_2(n+1))$.

Proof. The ramification assumption makes sure that R_E/R_F is a Galois extension with Galois group Γ. We pass to the limit over ν in the spectral sequences of Theorem 3.7, where we may assume throughout the proof that the twist $i \geq 2$. The $\lim^1$–term is trivial for number rings, so the emerging spectral sequence is:

$$\mathrm{E}_2^{r,s} = \mathrm{H}^r(\Gamma; \mathrm{H}^s_{\text{ét}}(R_E; \mathbb{Z}_2(i))) \Longrightarrow \mathrm{H}^{r+s}_{\text{ét}}(R_F; \mathbb{Z}_2(i)).$$

Recall that $\mathrm{H}^0_{\text{ét}}(R_E; \mathbb{Z}_2(i)) = 0$. Thus all terms on the line $\mathrm{E}_2^{r,0}$ vanish. We derive the following picture of the E_2–page:

$$\begin{array}{c|cccc}
 & \mathrm{E}_2^{0,2} & \mathrm{E}_2^{1,2} & \mathrm{E}_2^{2,2} & \cdots \\
s & \mathrm{E}_2^{0,1} & \mathrm{E}_2^{1,1} & \mathrm{E}_2^{2,1} & \cdots \\
 & 0 & 0 & 0 & \cdots \\
\hline
 & & r & &
\end{array}$$

We read off the isomorphism $\mathrm{H}^1_{\text{ét}}(R_E; \mathbb{Z}_2(i))^\Gamma \cong \mathrm{H}^1_{\text{ét}}(R_F; \mathbb{Z}_2(i))$, and the exact sequence:

$$0 \longrightarrow \mathrm{H}^1(\Gamma; \mathrm{H}^1_{\text{ét}}(R_E; \mathbb{Z}_2(i))) \longrightarrow \mathrm{H}^2_{\text{ét}}(R_F; \mathbb{Z}_2(i)) \longrightarrow$$
$$\longrightarrow \mathrm{H}^2_{\text{ét}}(R_E; \mathbb{Z}_2(i))^\Gamma \longrightarrow \mathrm{H}^2(\Gamma; \mathrm{H}^1_{\text{ét}}(R_E; \mathbb{Z}_2(i))) \longrightarrow \mathrm{H}^3_{\text{ét}}(R_F; \mathbb{Z}_2(i)).$$

Theorem 1.3 finishes the proof if R_F is the ring of two–integers in F, and R_E is the ring of two–integers in E. However, the cited result remains valid for arbitrary rings of S–integers where S contains the dyadic primes. For this, one can for example combine Theorems 3.1 and 3.2 with Proposition 3.3. □

By the same argument as in the proof of Theorem 3.8, we also obtain

Theorem 3.9. *Let E/F be a Galois extension of number fields with Galois group Γ. Assume that S contains the primes lying above the odd rational prime (p), and the ramified primes in E/F. If the Bloch–Kato conjecture is true for both E and F at p, then we have an exact sequence*

$$0 \to \mathrm{H}^1(\Gamma; M) \to \mathrm{K}_{2n}(R_F; \mathbb{Z}_p) \to \mathrm{K}_{2n}(R_E; \mathbb{Z}_p)^{\Gamma} \to \mathrm{H}^2(\Gamma; M) \to 0$$

where M denotes $\mathrm{K}_{2n+1}(R_E; \mathbb{Z}_p)$ and $n \geq 1$.

The layers in a $\mathbb{Z}_p$–extension of F give typical examples where we can apply Theorems 3.8 and 3.9. We hope to discuss the algebraic K–theory of such extensions more carefully in a forthcoming paper.

Corollary 3.10. *Let E be a totally real Abelian number field, let Γ be the p–Sylow subgroup of $Gal(E/\mathbb{Q})$, let F be the fixed field of Γ and let $j \geq 0$. Let χ be a p–adic character of $Gal(F/\mathbb{Q})$. Assume that the character $\chi^{-1}\omega^{4j+2}$ is nontrivial. With the same assumptions as in Theorems 3.8 and 3.9 we have an isomorphism $\mathrm{K}_{8j+2}(R_F; \mathbb{Z}_p)(\chi) \cong \mathrm{K}_{8j+2}(R_E; \mathbb{Z}_p)(\chi)^{\Gamma}$ induced by $\gamma_S(\chi)$.*

Proof. Theorems 3.8 and 3.9 give that both the kernel and the cokernel of the map γ_S is given by Γ–cohomology groups of $\mathrm{H}^1_{\text{ét}}(R_E; \mathbb{Z}_p(4j+2))$. Recall that E is totally real, so $\mathrm{H}^1_{\text{ét}}(R_E; \mathbb{Z}_p(4j+2)) \cong \mathrm{H}^0_{\text{ét}}(E; \mathbb{Q}_p/\mathbb{Z}_p(4j+2))$ from the Bockstein exact sequence. The imposed assumption on χ implies $\mathrm{H}^0_{\text{ét}}(E; \mathbb{Q}_p/\mathbb{Z}_p(4j+2))(\chi) = 0$ for all primes p (Tate cohomology commutes with χ–components, cf. Proposition 1 of [Co]). This implies our claim. □

Let E/F be a Galois extension of number fields with Galois group Γ. Let $\mathrm{tr} : \mathrm{K}(E) \longrightarrow \mathrm{K}(F)$ denote the algebraic K–theory transfer map. We denote the restriction of tr to $\mathrm{K}(R_E)$ by tr_T. Likewise we let tr_Γ signify the restriction of tr to $\mathrm{K}(E)_\Gamma$. We now prove Theorem B.

Proof of Theorem B. First we assume that p is an odd prime, or that $p = 2$ and F is a totally imaginary number field. The Bloch–Lichtenbaum spectral sequence from [BL] is by construction compatible with transfers, and therefore the isomorphism $\mathrm{K}_{2n+1}(E; \mathbb{Q}_p/\mathbb{Z}_p) \cong \mathrm{H}^1_{\text{ét}}(E; \mathbb{Q}_p/\mathbb{Z}_p(n+1))$ is also compatible with transfers. The Tate second quadrant spectral sequence

$$\mathrm{E}_2^{-r,s} = \mathrm{H}_r(\Gamma; \mathrm{H}^s_{\text{ét}}(E; \mathbb{Q}_p/\mathbb{Z}_p(n+1))) \Longrightarrow \mathrm{H}^{-r+s}_{\text{ét}}(F; \mathbb{Q}_p/\mathbb{Z}_p(n+1))$$

delivers an isomorphism:

$$\mathrm{H}^1_{\text{ét}}(E; \mathbb{Q}_p/\mathbb{Z}_p(n+1))_\Gamma \cong \mathrm{H}^1_{\text{ét}}(F; \mathbb{Q}_p/\mathbb{Z}_p(n+1)).$$

Next consider the Bockstein exact sequence:

$$\begin{array}{ccccccc}
\mathrm{K}_{2n+1}(E)_\Gamma \otimes \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & \mathrm{K}_{2n+1}(E; \mathbb{Q}_p/\mathbb{Z}_p)_\Gamma & \longrightarrow & \mathrm{K}_{2n}(E)\{p\}_\Gamma & \longrightarrow & 0 \\
\downarrow & & \downarrow \cong & & \downarrow & & \\
\mathrm{K}_{2n+1}(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & \mathrm{K}_{2n+1}(F; \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & \mathrm{K}_{2n}(F)\{p\} & \longrightarrow & 0.
\end{array}$$

The vertical map in the middle of this diagram is an isomorphism by the previous arguments. Thus the right vertical map is surjective, and its kernel is divisible.

Theorem 3.2 tells us that any divisible subgroup of $\mathrm{K}_{2n}(E)$ is trivial. This gives our claim.

Now let $p = 2$, and let F be a real number field. Since $\mathrm{H}^q_+(E, \mathbb{Q}_p/\mathbb{Z}_p(n+1))$ is the trivial group for $q \geq 2$, the Tate spectral sequence for totally positive cohomology delivers an isomorphism:

$$\mathrm{H}^1_+(E, \mathbb{Q}_p/\mathbb{Z}_p(n+1))_\Gamma \cong \mathrm{H}^1_+(F, \mathbb{Q}_p/\mathbb{Z}_p(n+1)).$$

As in the previous case, we derive the isomorphism:

$$\mathrm{H}^2_+(E; \mathbb{Z}_2(n+1))_\Gamma \cong \mathrm{H}^2_+(F; \mathbb{Z}_2(n+1)).$$

Theorem 1.3 gives our claim for $n \equiv 3 \bmod 4$. Next we let $n \equiv 1 \bmod 4$. Using the remarks before Theorem 3.8, and observing that $\mathrm{H}_1(\Gamma, \oplus_{\wp' \text{ real}} \frac{\mathbb{Z}}{2}\wp')$ is trivial because $(\oplus_{\wp' \text{ real}} \frac{\mathbb{Z}}{2}\wp')$ is an induced Γ-module, we let $M = \mathbb{Z}_2(n+1)$ and form the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^2_+(E, M)_\Gamma & \longrightarrow & \mathrm{H}^2_{\text{ét}}(E, M)_\Gamma & \longrightarrow & (\oplus_{\wp' \text{ real}} \frac{\mathbb{Z}}{2}\wp')_\Gamma & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^2_+(F, M) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(F, M) & \longrightarrow & \oplus_{\wp \text{ real}} \frac{\mathbb{Z}}{2}\wp & \longrightarrow & 0.
\end{array}$$

The left vertical map is an isomorphism. The right vertical map is clearly injective with cokernel equal to a direct sum of copies of $\mathbb{Z}/2$ indexed over the real ramified primes of F which ramify in E/F. The snake lemma and Theorem 1.3 therefore conclude the proof in the case $n \equiv 1 \bmod 4$.

Next we treat the cases $n \equiv 0, 2 \bmod 4$. Then $\mathrm{H}^1_{\text{ét}}(\mathbb{R}, \mathbb{Z}_2(n+1)) \cong \mathbb{Z}/2$ and $\mathrm{H}^2_{\text{ét}}(\mathbb{R}, \mathbb{Z}_2(n+1)) = 0$, for $\mathbb{R}$ the real numbers. Again, using the remarks preceding Theorem 3.8, we form the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
(\oplus_{\wp' \text{ real}} \frac{\mathbb{Z}}{2}\wp')_\Gamma & \longrightarrow & \mathrm{H}^2_+(E, \mathbb{Z}_2(n+1))_\Gamma & \longrightarrow & \mathrm{H}^2_{\text{ét}}(E, \mathbb{Z}_2(n+1))_\Gamma & \longrightarrow & 0 \\
\downarrow & & \downarrow \cong & & \downarrow & & \\
\oplus_{\wp \text{ real}} \frac{\mathbb{Z}}{2}\wp & \longrightarrow & \mathrm{H}^2_+(F, \mathbb{Z}_2(n+1)) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(F, \mathbb{Z}_2(n+1)) & \longrightarrow & 0.
\end{array}$$

The central vertical map is an isomorphism. This implies that the right vertical map is surjective. Moreover, the kernel of the right vertical map has order less or equal to the order of the cokernel of the left vertical map, which equals 2^{r_∞}. These observations combined with Theorem 1.3 conclude our proof. □

Proposition 3.11. *Let E/F be a Galois extension of number fields with Galois group Γ and let $n \geq 1$. Then we have an isomorphism* $\mathrm{coker}\,(\mathrm{tr}_T)_\Gamma \cong \mathrm{coker}\ \mathrm{tr}_\Gamma$ *on the $2n$th* K*-group, and a short exact sequence*:

$$0 \longrightarrow \ker(\mathrm{K}_{2n}(R_E)_\Gamma \longrightarrow \mathrm{K}_{2n}(E)_\Gamma) \longrightarrow \ker\,(\mathrm{tr}_T)_\Gamma \longrightarrow \ker\ \mathrm{tr}_\Gamma \longrightarrow 0.$$

Moreover, if E/F is a cyclic extension then we have the following exact sequence:

$$\bigoplus_{\wp \notin S} \mathrm{K}_{2n-1}(F[\wp])/e_\wp \longrightarrow \mathrm{K}_{2n}(R_E)_\Gamma \xrightarrow{(\mathrm{tr}_T)_\Gamma} \mathrm{K}_{2n}(R_F) \longrightarrow \mathrm{coker}\ \mathrm{tr}_\Gamma \longrightarrow 0.$$

Proof. Consider the following commutative diagram of Γ–modules with exact rows:

$$\begin{array}{ccccccccc} & & \mathrm{K}_{2n}(R_E)_\Gamma & \longrightarrow & \mathrm{K}_{2n}(E)_\Gamma & \longrightarrow & (\bigoplus_{\overline{\wp}\notin T}\mathrm{K}_{2n-1}(E[\overline{\wp}]))_\Gamma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\mathrm{tr}_T)_\Gamma} & & \downarrow{\scriptstyle \mathrm{tr}_\Gamma} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{K}_{2n}(R_F) & \longrightarrow & \mathrm{K}_{2n}(F) & \longrightarrow & \bigoplus_{\wp\notin S}\mathrm{K}_{2n-1}(F[\overline{\wp}]) & \longrightarrow & 0. \end{array}$$

The former diagram – with the same notation as in Proposition 3.4 – originates from the following commutative diagram of exact categories:

$$\begin{array}{ccccc} \bigoplus_{\overline{\wp}\notin T}\bigoplus_{\overline{\wp}|\wp}\mathcal{M}(E[\overline{\wp}]) & \longrightarrow & \mathcal{M}(R_E) & \longrightarrow & \mathcal{M}(E) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{\wp\notin S}\mathcal{M}(F[\wp]) & \longrightarrow & \mathcal{M}(R_F) & \longrightarrow & \mathcal{M}(F). \end{array}$$

The left vertical map is forgetful, and induces the transfer map from $\mathrm{K}(E[\overline{\wp}])$ to $\mathrm{K}(F[\wp])$ in algebraic K–theory. We can identify $(\bigoplus_{\overline{\wp}|\wp}\mathrm{K}_{2n-1}(E[\overline{\wp}]))_\Gamma$ with $\mathrm{K}_{2n-1}(F[\overline{\wp}])$, and the transfer map appearing on the right hand side of the first diagram is surjective. For this see Lemme 9 of [So]. Thus the transfer map is an isomorphism, in fact equal to the identity map. An application of the snake lemma then gives an isomorphism $\operatorname{coker}(\mathrm{tr}_T)_\Gamma \cong \operatorname{coker}\mathrm{tr}_\Gamma$ and the claimed short exact sequence.

Now for the case of a cyclic extension. Under the identification above we have that the norm map $(\bigoplus_{\overline{\wp}|\wp}\mathrm{K}_{2n-1}(E[\overline{\wp}]))_\Gamma \to \mathrm{K}_{2n-1}(F[\wp]) \to (\bigoplus_{\overline{\wp}|\wp}\mathrm{K}_{2n-1}(E[\overline{\wp}]))^\Gamma$ corresponds to the map $\mathrm{K}_{2n-1}(F[\wp]) \longrightarrow \mathrm{K}_{2n-1}(F[\wp])$ given by raising an element to the power of $e_\wp$. In fact the first map is an isomorphism and the second one is multiplication by $e_\wp$, see the last part of the proof of Proposition 3.4. This gives the isomorphism $\widehat{\mathrm{H}}^0(\Gamma,\bigoplus_{\overline{\wp}\notin T}\mathrm{K}_{2n-1}(E[\overline{\wp}])) \cong \bigoplus_{\wp\notin S}\mathrm{K}_{2n-1}(F[\wp])/e_\wp$. From the snake lemma we find the exact sequence:

$$\mathrm{H}_1(\Gamma,(\bigoplus_{\overline{\wp}\notin T}\mathrm{K}_{2n-1}(E[\overline{\wp}]))) \to \mathrm{K}_{2n}(R_E)_\Gamma \xrightarrow{\mathrm{tr}_T} \mathrm{K}_{2n}(R_F) \to \operatorname{coker}\mathrm{tr}_\Gamma \to 0.$$

But we have

$$\mathrm{H}_1(\Gamma,\bigoplus_{\overline{\wp}\notin T}\mathrm{K}_{2n-1}(E[\overline{\wp}]) \cong \widehat{\mathrm{H}}^{-2}(\Gamma,\bigoplus_{\overline{\wp}\notin T}\mathrm{K}_{2n-1}(E[\overline{\wp}])) \cong \widehat{\mathrm{H}}^{0}(\Gamma,\bigoplus_{\overline{\wp}\notin T}\mathrm{K}_{2n-1}(E[\overline{\wp}]))$$

where the first isomorphism comes from the definition of Tate cohomology groups, and the second holds because Γ is cyclic. □

Corollary 3.12. *Assume that E/F is a cyclic extension. Let $n \geq 1$, let tr_Γ : $\mathrm{K}_{2n}(E)_\Gamma \to \mathrm{K}_{2n}(F)$, and suppose that the two natural numbers $(\#F[\wp])^n - 1$ and $e_\wp$ are relatively prime for all ramified primes $\wp$ not in S. Then* $\ker(\mathrm{tr}_T)_\Gamma$ *is trivial. If E is Abelian over $\mathbb{Q}$, then the same result holds for the χ–components if χ is a nontrivial character of order prime to p, and $\mathrm{K}_{2n-1}(F[\wp])$ is fixed by Gal$(F/\mathbb{Q})$ for all the ramified primes $\wp$ in E/F not in S.*

Proof. The argument is the same as in Corollary 3.6. □

4. Fitting ideals for K–groups of totally real number rings

The main goal of this section is to prove Theorem A. In fact we will prove a somewhat stronger result in Theorem 4.1. We obtain Theorem A as a consequence of the identification of the algebraic K–groups appearing in Theorem A with étale cohomology groups. We adopt the same notation as in Section 1.

Theorem 4.1. *Let E be a totally real Abelian number field with ring of p–integers R_E, and with a prime power conductor. Let G denote the Galois group of E over the rationals, and let $n \equiv 2 \bmod 4$ be a positive integer. Let P be the p–Sylow subgroup of G and write $G = \Delta \times P$. Let $\chi : \Delta \longrightarrow \overline{\mathbb{Q}}_p^*$ be a p–adic character. Suppose that $\chi^{-1}\omega^{\frac{n}{2}+1}$ is a nontrivial character of $Gal(\mathbb{Q}_{\chi^{-1}\omega^{\frac{n}{2}+1}}/\mathbb{Q})$. Then*

$$\mathrm{Fit}_{\mathbb{Z}_p[G](\chi)} \mathrm{H}^2_{\text{ét}}(R_E; \mathbb{Z}_p(\frac{n}{2}+1))(\chi) = (\theta_{\frac{n}{2}}(\chi))$$

where $\theta_{\frac{n}{2}}$ is the $\frac{n}{2}$th Stickelberger element relative to E.

Remark 4.2. *Observe that if $p = 2$ or 3 then ω^2 is the trivial character and the condition $\chi^{-1}\omega^{\frac{n}{2}+1} \neq 1$ in Theorem 4.1 is equivalent to $\chi \neq 1$, because $\frac{n}{2}+1$ is an even integer.*

For an introduction to Fitting ideals and their main properties, see the appendix of [MW]. Fix a commutative unital ring R, and a finitely generated R–module M. Choose a presentation $0 \to A \xrightarrow{\phi} R^n \to M \to 0$. Then form the $n \times n$ matrices $\Phi = (\phi(a_1), ..., \phi(a_n))^t$ where $(a_1, ..., a_n)$ runs through all n–tuples of elements in A. The Fitting ideal $\mathrm{Fit}_R M$ of M over R is defined as the ideal in R generated by the elements $\det(\Phi)$.

The Fitting ideal is in general not multiplicative in short exact sequences, but we have the following useful lemma.

Lemma 4.3. *Let R be a commutative ring and let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely presented R–modules. Suppose that C has a free resolution of length one. Then $\mathrm{Fit}_R B = \mathrm{Fit}_R A \cdot \mathrm{Fit}_R C$.*

Proof. See Lemma 3 in [CG]. □

Let E_∞ be the cyclotomic $\mathbb{Z}_p$–extension of E. Write Γ for the Galois group $Gal(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$. Let M_p and $M_{p,\infty}$ be the maximal Abelian p–extension of E_∞ which are unramified outside primes over p and unramified outside p–adic and infinite primes, respectively. Let $X_p = Gal(M_p/E_\infty)$, $X_{p,\infty} = Gal(M_{p,\infty}/E_\infty)$ and $\Gamma' = Gal(E_\infty/E)$. If p is odd then $M_p = M_{p,\infty}$ and $X_p = X_{p,\infty}$. Let $G_{p,\infty}$ be the Galois group of the maximal extension of E unramified outside the primes above p and the infinite ones. Throughout Section 4, the ring R_E will denote the ring of p–integers of E.

Lemma 4.4. *Let E be a totally real Abelian number field. For all even i we have $\mathrm{H}^2_{\text{ét}}(R_E; \mathbb{Z}_p(i)) \cong \mathrm{Hom}((X_{p,\infty}(-i))_{\Gamma'}, \mathbb{Q}_p/\mathbb{Z}_p)$.*

Proof. To prove the isomorphism, we refer to p.53 in [RW]. Shortly one shows that

$$\begin{aligned}\mathrm{Hom}((X_{p,\infty}(-i))_{\Gamma'}, \mathbb{Q}_p/\mathbb{Z}_p) &\cong \mathrm{Hom}(X_{p,\infty}, \mathbb{Q}_p/\mathbb{Z}_p(i))^{\Gamma'} \\ &\cong \mathrm{H}^1(G_{p,\infty}; \mathbb{Q}_p/\mathbb{Z}_p(i)) \\ &\cong \mathrm{H}^2_{\text{ét}}(R_E; \mathbb{Z}_p(i)).\end{aligned}$$ □

We put $E_0 = E$ – corresponding to characters of first kind – if p^2 does not divide the conductor of E, and $E_0 = E \cap \mathbb{Q}(\zeta_p)$ if p^2 divides the conductor of E. We identify Γ with the group E_∞/E_0. We have $[E : E_0] = p^e$ for some e and $\Gamma' = \Gamma^{p^e}$. We set $E_0' = E_0(\zeta_p + \zeta_p^{-1})$. We denote by P' the p–Sylow subgroup of $G' = Gal(E_0'/\mathbb{Q})$ and write $G' \cong P' \times \Delta'$. For all p–adic characters ξ of Δ' we denote the discrete valuation ring $\mathbb{Z}_p[\Delta'](\xi) = \mathbb{Z}_p(\xi)$ by $\mathcal{O}_\xi$. Let X_p' and $X_{p,\infty}'$ be the central Iwasawa modules associated to E_0', and let Λ denote $\mathbb{Z}_p[\Gamma]$. For all nontrivial characters ξ as above Greither proved that $X_p'(\xi)$ is a $\Lambda_\xi[P'] := \mathcal{O}_\xi \otimes_{\mathbb{Z}_p} \Lambda[P']$–module of projective dimension one, see Proposition 5.1 in [Gri]. For the case when p divides the conductor of E see the comments in advance of Theorem 7.6 in [Gri]. Moreover, Greither also determined its $\Lambda_\xi[P']$ Fitting ideal, see the proof of Theorem 7.4 in [Gri] and also [Wi]. We summarize these results.

Theorem 4.5. *(Greither) The $\Lambda_\xi[P']$–module $X_p'(\xi)$ has projective dimension one for all nontrivial even ξ as above. The Fitting ideal $\mathrm{Fit}_{\Lambda_\xi[P']} X_p'(\xi)$ is principal and generated by $\frac{1}{2}F_\xi$. The element $F_\xi \in \Lambda_\xi[P']$ is characterized, modulo units, by the equations $\psi(F_\xi) = G_\psi(T)$ for all characters ψ of $Gal(E_0'/\mathbb{Q})$ extending ξ. Here $G_\psi(T) \in \mathbb{Z}_p[\psi][[T]]$ is the unique power series such that $L_p(1-s,\psi) = G_\psi(\kappa(\gamma)^s - 1)$ where γ is a topological generator of Γ and $\kappa : \Gamma \to \mathbb{Z}_p^*$ is the cyclotomic character defined by $x \cdot \zeta = \zeta^{\kappa(x)}$ for any $\zeta \in \mathbb{Z}_p(1)$.*

For an even character ξ the Iwasawa module $X_p'(\xi)$ is $\Lambda_\xi[P']$–torsion, see § 5 and § 6 of Chapter 5 in [La]. As a consequence of the famous Ferrero–Washington theorem in [FW] – saying that the Iwasawa invariant μ_p vanishes for Abelian number fields – and the structure theorem for Iwasawa modules – e.g., Theorem 13.12 of [Wa] – one obtains that $X_p'(\xi)/p$ has finite order. A characterization of the Fitting ideal of $X_p'(\xi)$ is given in Lemma 3.7 of [Gri].

Lemma 4.6. *(Greither) Let M be a $\Lambda_\xi[P']$–torsion module of projective dimension less or equal to 1. Suppose that M/p is finite and that $\phi \in \Lambda_\xi[P']$ is such that $\mathrm{Fit}_{\Lambda_\xi(\psi)} M = (\psi(\phi))$ for all characters ψ of G'. Then $\mathrm{Fit}_{\Lambda_\xi[P']} M = (\phi)$.*

Let F be a totally real Abelian number field. Let $G(F)_2$ be the Galois group of the maximal Abelian two–extension of F unramified outside dyadic primes. Let $G(F)_{2,\infty}$ be the Galois group of the maximal Abelian two–extension of F unramified outside dyadic and infinite primes. Global class field theory gives us a short exact sequence of Galois–modules:

$$0 \longrightarrow \mathbb{Z}/2[Gal(F/\mathbb{Q})] \longrightarrow G(F)_{2,\infty} \longrightarrow G(F)_2 \longrightarrow 0. \tag{4.7}$$

Let E_n' be the extension of E' of degree 2^n contained in the cyclotomic $\mathbb{Z}_2$–extension E_∞'. We insert $F = E_n'$, the nth layer in the cyclotomic $\mathbb{Z}_2$–extension of E', in (4.7) and then pass to the inverse limit. The inverse system $\mathbb{Z}/2[Gal(E_n'/\mathbb{Q})]$ is surjective, so we get a short exact sequence:

$$0 \longrightarrow \mathbb{Z}/2[Gal(E_\infty'/\mathbb{Q})] \longrightarrow X_{2,\infty}' \longrightarrow X_2' \longrightarrow 0. \tag{4.8}$$

Proposition 4.9. *With the same notation as in Theorem 4.5 we have the equality $\mathrm{Fit}_{\Lambda_\xi[P']} X_{p,\infty}'(\xi) = (F_\xi)$.*

Proof. This is immediate for p odd from Theorem 4.5, since then $X_{p,\infty}' = X_p'$ and 2 is clearly a unit in $\Lambda_\xi[P']$. Next let $p = 2$. Since ξ is nontrivial we also have

$E = E_0$ and $P = P'$. Let us consider the ξ–parts of the short exact sequence (4.8). Clearly we have $\mathrm{Fit}_{\Lambda_\xi[P]}\mathbb{Z}/2[Gal(E_\infty/\mathbb{Q})] = (2)$. By Proposition 5.1 of [Gri], the $\Lambda_\xi[P']$–module $X_2'(\xi)$ has projective dimension equal to one. The ring $\Lambda_\xi[P']$ is local, so every projective module is free. Hence we have a resolution $0 \to \Lambda_\xi[P']^{n_1} \to \Lambda_\xi[P']^{n_2} \to X_2'(\xi) \to 0$. Since $X_2'(\xi)$ is a torsion $\Lambda_\xi[P']$–module we must have $n_1 = n_2$. By Theorem 4.5 we have $\mathrm{Fit}_{\Lambda_\xi[P']} X_2'(\xi) = (F_\xi/2)$. An application of Lemma 4.3 to the ring $R = \Lambda_\xi[P']$ and the exact sequence (4.8) gives the assertion $\mathrm{Fit}_{\Lambda_\xi[P']} X_{2,\infty}'(\xi) = (F_\xi)$. $\square$

Next we determine the $\mathcal{O}_{\chi^{-1}}[P]$–Fitting ideal of the dual of $\mathrm{H}^2_{\text{ét}}(R_E;\mathbb{Z}_p(i))$ for all even i. Recall that $\Lambda_{\chi^{-1}}[P'] \cong \mathcal{O}_{\chi^{-1}}[P'][[T]]$, where we identify a generator γ of Γ with $1+T$. For any nontrivial character ξ of Δ', the element $F_{\xi^{-1}}$ can be thought of as a power series with coefficients in $\mathcal{O}_{\xi^{-1}}[P']$.

Proposition 4.10. *With the assumptions and notations as above we have for all positive integers $n \equiv 2 \bmod 4$:*

$$\mathrm{Fit}_{\mathcal{O}_{\chi^{-1}}[P]}\mathrm{Hom}(\mathrm{H}^2_{\text{ét}}(R_E;\mathbb{Z}_p(\frac{n}{2}+1)),\mathbb{Q}_p/\mathbb{Z}_p)(\chi^{-1}) =$$

$$(F_{\chi^{-1}\omega^{\frac{n}{2}+1}}(\kappa(\gamma)^{\frac{n}{2}+1}(1+T)-1) \bmod ((1+T)^{p^e}-1)).$$

Proof. The condition on n implies that $\frac{n}{2}+1$ is even. By Lemma 4.4 we have:

$$\mathrm{Hom}(\mathrm{H}^2_{\text{ét}}(R_E;\mathbb{Z}_p(\frac{n}{2}+1)),\mathbb{Q}_p/\mathbb{Z}_p)(\chi^{-1}) \cong X_{p,\infty}(-\frac{n}{2}-1)(\chi^{-1})_{\Gamma'}.$$

Furthermore we find:

$$\begin{aligned}(\mathrm{Fit}_{\mathcal{O}_{\chi^{-1}}[P]}(X_{p,\infty}(-\frac{n}{2}-1)(\chi^{-1}))_{\Gamma'}) &= (\mathrm{Fit}_{\Lambda_{\chi^{-1}}[P']} X_{p,\infty}(-\frac{n}{2}-1)(\chi^{-1}))_{\Gamma'}\\ &= (\mathrm{Fit}_{\Lambda_{\chi^{-1}}[P']} X'_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1})(-\frac{n}{2}-1))_{\Gamma'}\\ &= (F_{\chi^{-1}\omega^{\frac{n}{2}+1}}(\kappa(\gamma)^{\frac{n}{2}+1}(1+T)-1))_{\Gamma'}\\ &= (F_{\chi^{-1}\omega^{\frac{n}{2}+1}}(\kappa(\gamma)^{\frac{n}{2}+1}(1+T)-1)\\ &\quad \bmod ((1+T)^{p^e}-1)).\end{aligned}$$

The first equality follows from the isomorphism $(\Lambda_{\chi^{-1}}[P'])_{\Gamma'} \cong \mathcal{O}_{\chi^{-1}}[P]$ and well known properties of Fitting ideals. The second equality is clear since Δ' acts on $\mathbb{Z}_2(-\frac{n}{2}-1)$ via $\omega^{-\frac{n}{2}-1}$. Proposition 4.9 gives the equality:

$$\mathrm{Fit}_{\Lambda_{\chi^{-1}\omega^{\frac{n}{2}+1}}[P']} X'_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1}) = F_{\chi^{-1}\omega^{\frac{n}{2}+1}}(T).$$

The action we consider is twisted $-(\frac{n}{2}+1)$ times. In result we have to replace T by $\kappa(\gamma)^{\frac{n}{2}+1}(1+T)-1$. This justifies the third equality. The last equality follows from the identification of the generator of Γ' with $(1+T)^{p^e}-1$. $\square$

Lemma 4.11. *Write α_χ^* for $F_{\chi^{-1}\omega^{\frac{n}{2}+1}}(\kappa(\gamma)^{\frac{n}{2}+1}(1+T)-1) \bmod ((1+T)^{p^e}-1)$. For all characters ψ of G extending χ and all positive integers $n \equiv 2 \bmod 4$ we have the equality*

$$\psi^{-1}(\alpha_\chi^*) = \psi(u_n\theta_{\frac{n}{2}}) = -(1-p^{\frac{n}{2}}\psi^{-1}(p))\frac{B_{\frac{n}{2}+1,\psi^{-1}}}{\frac{n}{2}+1}$$

in $\mathbb{Q}_p(\psi)$, where u_n is a unit of $\mathcal{O}_\chi[P]$. Here we write $\psi^{-1}(p)$ for $\psi^{-1}(Frob_p)$, where $Frob_p$ is the Frobenius element of G associated to p.

Proof. Write $\psi = \psi'\rho$ where ψ' is a character of $Gal(E_0/\mathbb{Q})$ and ρ is a character of $Gal(E/E_0)$. By the characterization of $F_{\chi^{-1}\omega^{\frac{n}{2}+1}}$ and Theorems 4.2 and 5.11 in [Wa] we find:

$$\begin{aligned}\psi^{-1}(\alpha_\chi^*) &= G_{\psi'^{-1}\omega^{\frac{n}{2}+1}}(\kappa(\gamma)^{\frac{n}{2}+1}\rho(\gamma)-1)\\ &= L_p(-\frac{n}{2},\psi^{-1}\omega^{\frac{n}{2}+1})\\ &= (1-p^{\frac{n}{2}}\psi^{-1}(p))L(-\frac{n}{2},\psi^{-1})\\ &= -(1-p^{\frac{n}{2}}\psi^{-1}(p))\frac{B_{\frac{n}{2}+1,\psi^{-1}}}{\frac{n}{2}+1}.\end{aligned}$$

On the other hand, let f be the conductor of E and let $\zeta_f \in \overline{\mathbb{Q}}$ be a primitive fth root of unity. Let $\sigma_a \in Gal(\mathbb{Q}(\zeta_f)/\mathbb{Q})$ with $a \in (\mathbb{Z}/f)^*$ be the element given by $\sigma_a\zeta_f = \zeta_f^a$. For any integer $i \geq 1$ we have the formula

$$\zeta_f(1-i,\sigma_a) = -\frac{f^{i-1}}{i}B_i(a/f) \tag{4.12}$$

where B_i is the ith Bernoulli polynomial. This follows from Theorem 4.2 in [Wa] with $s = 1-i$. In fact, the function $\zeta(s, a/f)$ in loc cit is equal to $f^s\zeta_f(s,\sigma_a)$ for $Re(s) > 1$ and hence for all s by analytic continuation. Now let $u_n = 1 - p^{\frac{n}{2}}\sigma_p^{-1}$. Using formula (4.12) with $i = \frac{n}{2}+1$ we compute:

$$\begin{aligned}\psi(u_n\theta_{\frac{n}{2}}) &= (1-p^{\frac{n}{2}}\psi^{-1}(p))(-\frac{f^{\frac{n}{2}}}{\frac{n}{2}+1}\sum_{a=1}^{f}B_{\frac{n}{2}+1}(a/f)\psi^{-1}(a))\\ &= -(1-p^{\frac{n}{2}}\psi^{-1}(p))\frac{B_{\frac{n}{2}+1,\psi^{-1}}}{\frac{n}{2}+1}.\end{aligned}$$

This concludes the proof of Lemma 4.11. □

Let A be any $\mathbb{Z}_p[G]$–module. We denote by $\mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ the dual of A as an Abelian group with the action $\sigma\varphi(a) = \varphi(\sigma^{-1}a)$ for $\sigma \in G$, $a \in A$ and $\varphi : A \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$. We denote by $\mathrm{Hom}^*(A, \mathbb{Q}_p/\mathbb{Z}_p)$ the dual of A as an Abelian group with the action $\sigma\varphi(a) = \varphi(\sigma a)$ where σ, a and φ are as above. We now briefly recall some properties of the rings $\mathcal{O}_\chi[P]$. Let a be a nonzero divisor in $\mathcal{O}_\chi[P]$. The ring $\mathcal{O}_\chi[P]/a$ is a zero–dimensional Gorenstein ring. For definitions see Proposition 4 in the appendix of [MW]. In particular, let A be an $\mathcal{O}_\chi[P]$–module of finite order annihilated by a. Then A is an $\mathcal{O}_\chi[P]/a$–module and we have an isomorphism of $\mathcal{O}_\chi[P]$–modules:

$$\mathrm{Hom}_{\mathcal{O}_\chi[P]}(A, \mathcal{O}_\chi[P]/a) \cong \mathrm{Hom}^*(A, \mathbb{Q}_p/\mathbb{Z}_p). \tag{4.13}$$

The functor $\mathrm{Hom}^*(-, \mathbb{Q}_p/\mathbb{Z}_p)$ is a contravariant exact functor on the category of $\mathcal{O}_\chi[P]$–modules of finite order. By Proposition 1 in the appendix of [MW] we have:

$$\mathrm{Fit}_{\mathcal{O}_\chi[P]} A = \mathrm{Fit}_{\mathcal{O}_\chi[P]} \mathrm{Hom}^*(A, \mathbb{Q}_p/\mathbb{Z}_p). \tag{4.14}$$

For any $\mathbb{Z}_p[G]$–module A we denote by A^* the $\mathbb{Z}_p[G]$–module which is the same as A as an Abelian group, but where $g \in G$ acts via g^{-1}. If A is an $\mathcal{O}_{\chi^{-1}}[P]$–module and $\mathrm{Fit}_{\mathcal{O}_{\chi^{-1}}[P]} A = (\alpha)$ with $\alpha \in \mathcal{O}_{\chi^{-1}}[P]$, then A^* is an $\mathcal{O}_\chi[P]$–module and $\mathrm{Fit}_{\mathcal{O}_\chi[P]} A^* = (\alpha^*)$, where (α^*) is an element of $\mathcal{O}_\chi[P]$ such that $\psi(\alpha^*) = \psi^{-1}(\alpha)$ for any character ψ of G extending χ.

Proof of Theorem 4.1. Let M denote $\mathbb{Z}_p(\frac{n}{2}+1)$ and let W denote $\mathbb{Q}_p/\mathbb{Z}_p$. From (4.14) we have:

$$\mathrm{Fit}_{\mathcal{O}_\chi[P]} \mathrm{H}^2_{\text{ét}}(R_E; M)(\chi) = \mathrm{Fit}_{\mathcal{O}_\chi[P]} \mathrm{Hom}^*(\mathrm{H}^2_{\text{ét}}(R_E; M)(\chi), W).$$

Furthermore we find that:

$$\begin{aligned}\mathrm{Fit}_{\mathcal{O}_\chi[P]} \mathrm{Hom}^*(\mathrm{H}^2_{\text{ét}}(R_E; M)(\chi), W) &= \mathrm{Fit}_{\mathcal{O}_\chi[P]} \mathrm{Hom}(\mathrm{H}^2_{\text{ét}}(R_E; M)(\chi), W)^* \\ &= \mathrm{Fit}_{\mathcal{O}_\chi[P]} (\mathrm{Hom}(\mathrm{H}^2_{\text{ét}}(R_E; M), W)(\chi^{-1}))^* \\ &= (F_{\chi^{-1}\omega^{\frac{n}{2}+1}}(\kappa(\gamma)^{\frac{n}{2}+1}(1+T)-1) \\ &\quad \bmod ((1+T)^{p^e}-1))^*.\end{aligned}$$

The last equality follows by Proposition 4.10. In particular the Fitting ideal $\mathrm{Fit}_{\mathcal{O}_\chi[P]} \mathrm{H}^2_{\text{ét}}(R_E; M)(\chi)$ is principal and generated by:

$$\alpha_\chi = F_{\chi^{-1}\omega^{\frac{n}{2}+1}}(\kappa(\gamma)^{\frac{n}{2}+1}(1+T)-1) \bmod ((1+T)^{p^e}-1)^*.$$

Let ψ be a character of G which extends χ. From Lemma 4.11 we know that $\psi(\alpha_\chi) = \psi(u_n\theta_{\frac{n}{2}})$. Since $\mathcal{O}_\chi[P] \cong \mathcal{O}_\chi[T]/((1+T)^{\#P}-1)$, by mapping a generator of P to $1+T$, both α_χ and $u_n\theta_{\frac{n}{2}}$ can be viewed as polynomials in $\mathbb{Q}_p(\chi)[1+T]$ of degree $< \#P$ whose values agree on all the $\#P$th roots of unit. Since the Vandermonde determinant formed by the powers of the $\#P$th roots of unity does not vanish, these two polynomials must be equal. Thus we find:

$$\mathrm{Fit}_{\mathcal{O}_\chi[P]} \mathrm{H}^2_{\text{ét}}(R_E; \mathbb{Z}_p(\frac{n}{2}+1))(\chi) = (\alpha_\chi) = (\theta_{\frac{n}{2}}(\chi)). \quad \square$$

Proof of Theorem A. We apply Theorem 1.3. Let $n \equiv 2 \bmod 8$. The isomorphism $\mathrm{K}_n(R_E; \mathbb{Z}_2) \cong \mathrm{H}^2(R_E; \mathbb{Z}_2(\frac{n}{2}+1))$ implies that $\mathrm{Fit}_{\mathcal{O}_\chi[P]} \mathrm{K}_n(R_E; \mathbb{Z}_2)(\chi) = (\theta_{\frac{n}{2}}(\chi))$. On the other hand for $n \equiv 6 \bmod 8$ there is a short exact sequence:

$$0 \to \mathrm{K}_n(R_E; \mathbb{Z}_2)(\chi) \to \mathrm{H}^2(R_E; \mathbb{Z}_2(\frac{n}{2}+1))(\chi) \to \mathbb{Z}_2[G]/2(\chi) \to 0. \tag{4.15}$$

The module $\mathbb{Z}_2[G]/2(\chi) \cong \mathcal{O}_\chi[P]/2$ has the resolution $0 \to \mathcal{O}_\chi[P] \xrightarrow{2} \mathcal{O}_\chi[P] \to \mathcal{O}_\chi[P]/2 \to 0$. We now apply Lemma 4.3 to $R = \mathcal{O}_\chi[P]$ and the exact sequence (4.15), to obtain the equality of ideals $2\,\mathrm{Fit}_{\mathcal{O}_\chi[P]} \mathrm{K}_n(R_E; \mathbb{Z}_2)(\chi) = (\theta_{\frac{n}{2}}(\chi))$. This gives our claim since 2 is not a zero divisor in $\mathcal{O}_\chi[P]$.

The odd–primary claim is immediate from Theorems 1.2 and 4.1. $\square$

We conclude this section by giving a formula for the order of the eigenspaces for the higher K–groups of totally real Abelian number fields.

Corollary 4.16. *Assume the hypothesis of Theorem* A. *For all natural numbers* $n \equiv 2 \bmod 4$ *we have the order formulas*

$$\#\mathrm{K}_n(R_E;\mathbb{Z}_2)(\chi) \sim_2 \prod_{\chi'\sim\chi}\prod_{\psi'|\chi'} \frac{1}{2^{a_n}} L(-\frac{n}{2},\psi'^{-1}) \sim_2 \prod_{\chi'\sim\chi}\prod_{\psi'|\chi'} \frac{1}{2^{a_n}} \frac{B_{\frac{n}{2}+1,\psi'^{-1}}}{(\frac{n}{2}+1)}$$

with a_n *equal to* 0 *or* 1 *respectively if* $n \equiv 2$ *or* $n \equiv 6$ *modulo* 8, *and*

$$\#\mathrm{K}_n(R_E;\mathbb{Z}_p)(\chi) \sim_p \prod_{\chi'\sim\chi}\prod_{\psi'|\chi'} L(-\frac{n}{2},\psi'^{-1}) \sim_p \prod_{\chi'\sim\chi}\prod_{\psi'|\chi'} \frac{B_{\frac{n}{2}+1,\psi'^{-1}}}{(\frac{n}{2}+1)}$$

for p *odd. The symbol* $a \sim_p b$ *means that both sides have the same* p*–adic valuation. The first product runs over the characters* χ' *which are* $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*–conjugate to* χ*. The symbol* $\psi' \mid \chi'$ *means that* ψ' *is a character of* G *extending* χ'*.*

Proof. We prove the first assertion. The odd–primary result is similar, and will be left to the reader. We have $\#\mathrm{K}_n(R_E;\mathbb{Z}_2)(\chi) = \#(\mathbb{Z}_2/\mathrm{Fit}_{\mathbb{Z}_2}\mathrm{K}_n(R_E;\mathbb{Z}_2)(\chi))$. Let d_χ be the order of χ, and let $\zeta_{\#Pd_\chi}$ be a primitive $\#Pd_\chi$th root of unit. The ring $\mathcal{O}_\chi[P]$ is a free $\mathbb{Z}_2$–module, and the connection between Fitting ideals over the rings $\mathcal{O}_\chi[P]$ and $\mathbb{Z}_2$ is given in Proposition 6, § 9.4, Ch. III of [Bo]. Let N denote the norm map for fields. Applying Theorem A we get:

$$\begin{aligned}\mathrm{Fit}_{\mathbb{Z}_2}\mathrm{K}_n(R_E;\mathbb{Z}_2)(\chi) &= \mathrm{N}_{\mathbb{Q}_2}^{\mathbb{Q}_2(\zeta_{\#Pd_\chi})}\mathrm{Fit}_{\mathcal{O}_\chi[P]}\mathrm{K}_n(R_E;\mathbb{Z}_2)(\chi)\\ &= (\mathrm{N}_{\mathbb{Q}_2}^{\mathbb{Q}_2(\zeta_{\#Pd_\chi})}\frac{1}{2^{a_n}}\theta_{\frac{n}{2}}(\chi)).\end{aligned}$$

The proof of Lemma 4.11 shows that we have the equality of ideals:

$$(\mathrm{N}_{\mathbb{Q}_p}^{\mathbb{Q}_p(\zeta_{\#Pd_\chi})}\theta_{\frac{n}{2}}(\chi)) = (\prod_{\chi'\sim\chi}\prod_{\psi'|\chi'}\frac{B_{\frac{n}{2}+1,\psi'^{-1}}}{(\frac{n}{2}+1)}).$$

This implies our claim. □

5. More on the structure of the K–groups

In this section we will derive some consequences of Theorem A concerning the actual group structure of the higher K–groups $\mathrm{K}_n(R_E;\mathbb{Z}_p)(\chi)$.

Let χ be a p–adic character of Δ. The ring $\mathcal{O}_\chi[P]$ is local, with maximal ideal $\mathfrak{m}$ generated by p and the augmentation ideal of P, and we have $\mathcal{O}_\chi[P]/\mathfrak{m} \cong \mathcal{O}_\chi/p$. Given an $\mathcal{O}_\chi[P]$–module M, we know by Nakayama's lemma that its minimal number of generators is the dimension of the $\mathcal{O}_\chi/p$–vector space $M/\mathfrak{m}M$. Given an even natural number n we want to study the minimal number of generators for the higher K–groups in degree n viewed as $\mathcal{O}_\chi[P]$–modules. Here we stress that $\chi^{-1}\omega^{\frac{n}{2}+1}$ is a nontrivial character.

Let $G_p(\chi^{-1}\omega^{\frac{n}{2}+1})$ and $G_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1})$ be the $\chi^{-1}\omega^{\frac{n}{2}+1}$–component of the Galois group of the maximal Abelian p–extension of $\mathbb{Q}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$ which is unramified outside the p–adic or unramified outside the p–adic and infinite primes respectively. Observe that if $n \equiv 2 \bmod 4$, then $\mathbb{Q}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$ is a totally real Abelian number field. In this case Leopoldt's conjecture is true, and implies that if $\chi^{-1}\omega^{\frac{n}{2}+1}$ is nontrivial, then both $G_{p,}(\chi^{-1}\omega^{\frac{n}{2}+1})$ and $G_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1})$ have finite order (see Theorem 5.2, Ch. 5 in [La]). We denote by $\mathcal{O}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$ the ring of integers of $\mathbb{Q}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$.

Theorem 5.1. *Assume the hypothesis of Theorem* A. *Then the minimal number of generators of the $\mathcal{O}_\chi[P]$-module $\mathrm{K}_n(R_E;\mathbb{Z}_p)(\chi)$ equals the minimal number of generators of the $\mathcal{O}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$-module $G_p(\chi^{-1}\omega^{\frac{n}{2}+1})$ if $p=2$ and $n\equiv 6 \bmod 8$, and of the $\mathcal{O}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$-module $G_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1})$ otherwise.*

To prove Theorem 5.1 we will combine Theorem A with the following lemma.

Lemma 5.2. *Let M be a finitely generated $\mathcal{O}_\chi[P]$-module, and suppose that the Fitting ideal $\mathrm{Fit}_{\mathcal{O}_\chi[P]} M$ is principal and generated by a nonzero divisor. Then the minimal number of generators of M equals the minimal number of generators of $\mathrm{Hom}(M,\mathbb{Q}_p/\mathbb{Z}_p)$ viewed as $\mathcal{O}_{\chi^{-1}}[P]$-modules.*

Proof. Proposition 4 in [CG] implies that there exists an exact sequence $0\to\mathcal{O}_\chi[P]^k\to\mathcal{O}_\chi[P]^k\to M\to 0$. Let f be a generator for the Fitting ideal of the $\mathcal{O}_\chi[P]$-module M. Consider the following commutative diagram whose vertical arrows are induced by multiplication by f:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_\chi[P]^k & \xrightarrow{\varphi} & \mathcal{O}_\chi[P]^k & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_\chi[P]^k & \xrightarrow{\varphi} & \mathcal{O}_\chi[P]^k & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Observe that f is not a zero divisor in $\mathcal{O}_\chi[P]$ and that $fM=0$. An application of the snake lemma to the diagram above gives the exact sequence:

$$0\longrightarrow M\longrightarrow(\mathcal{O}_\chi[P]/f)^k\longrightarrow(\mathcal{O}_\chi[P]/f)^k. \tag{5.3}$$

Let $M^{\#}=\mathrm{Hom}^*(M,\mathbb{Q}_p/\mathbb{Z}_p)$ and let k' be the minimal number of generators of $M^{\#}$. Now apply $\mathrm{Hom}^*(-,\mathbb{Q}_p/\mathbb{Z}_p)$ to (5.3). Since $\mathrm{Hom}^*(\mathcal{O}_\chi[P]/f,\mathbb{Q}_p/\mathbb{Z}_p)\cong\mathcal{O}_\chi[P]/f$ by the Gorenstein property (4.13), we derive that the $\mathcal{O}_\chi[P]$-module $M^{\#}$ can be generated by k elements, therefore $k'\le k$. By (4.14) also $M^{\#}$ has Fitting ideal generated by f. By reversing the role of M and $M^{\#}$ we find that $k\le k'$, i.e., $k=k'$. Therefore the minimal number of generators of the $\mathcal{O}_{\chi^{-1}}[P]$-module $\mathrm{Hom}(M,\mathbb{Q}_p/\mathbb{Z}_p)\cong(M^{\#})^*$ is also k. □

Proof of Theorem 5.1. We let k denote the minimal number of generators of the group $\mathrm{H}^2_{\text{ét}}(R_E;\mathbb{Z}_p(\frac{n}{2}+1))(\chi)$. From Lemma 5.2 we find that the minimal number of generators of the $\mathcal{O}_{\chi^{-1}}[P]$-module $\mathrm{Hom}(\mathrm{H}^2_{\text{ét}}(R_E;\mathbb{Z}_p(\frac{n}{2}+1));\mathbb{Q}_p/\mathbb{Z}_p)(\chi^{-1})$ is also k. By Lemma 4.4 this last module is isomorphic to $X_{p,\infty}(-\frac{n}{2}-1)_{\Gamma'}(\chi^{-1})$. By applying Nakayama's lemma to the local ring $\Lambda_{\chi^{-1}}[P']$, we get that the $\Lambda_{\chi^{-1}}[P']$-module

$$X_{p,\infty}(-\frac{n}{2}-1)(\chi^{-1})\cong X'_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1})(-\frac{n}{2}-1)$$

has minimal number of generators equal to k. This implies that the minimal number of generators of the $\Lambda_{\chi^{-1}\omega^{\frac{n}{2}+1}}[P']$-module $X'_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1})$ is also k. Observe now that $G_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1})\cong(X'_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1}))_{\Gamma\times P'}$. Nakayama's lemma applied this time to the ring $\Lambda_{\chi^{-1}\omega^{\frac{n}{2}+1}}[P']$ implies that the minimal number of generators of the $\mathcal{O}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$-module $G_{p,\infty}(\chi^{-1}\omega^{\frac{n}{2}+1})$ is k. We can repeat all the above by changing cohomology groups with positive cohomology groups and removing all the symbols ∞, to get that the two $\mathcal{O}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$–modules $\mathrm{H}^2_+(R_E;\mathbb{Z}_p(\frac{n}{2}+1))(\chi)$ and $G_p(\chi^{-1}\omega^{\frac{n}{2}+1})$ have the same number of generators. As in the proof of Theorem A, the identification of cohomology groups with K groups gives our claim. $\square$

Let Pic_+ denote the narrow Picard group.

Corollary 5.4. *Assume the hypothesis of Theorem* A. *Suppose that:*

$$\mathrm{Pic}_+(\mathcal{O}_{\chi^{-1}\omega^{\frac{n}{2}+1}})(\chi^{-1}\omega^{\frac{n}{2}+1}) \text{ is a cyclic } \mathcal{O}_{\chi^{-1}\omega^{\frac{n}{2}+1}} - \text{module.}$$

Moreover, we assume that there is only one prime lying above p in $\mathbb{Q}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$. Then $\mathrm{K}_n(R_E;\mathbb{Z}_p)(\chi) \cong \mathcal{O}_\chi[P]/(\theta_{\frac{n}{2}}(\chi))$ for p odd. When $p=2$, we have isomorphisms $\mathrm{K}_n(R_E;\mathbb{Z}_2)(\chi) \cong \mathcal{O}_\chi[P]/(\theta_{\frac{n}{2}}(\chi))$ for $n \equiv 2 \bmod 8$ and $\mathrm{K}_n(R_E;\mathbb{Z}_2)(\chi) \cong \mathcal{O}_\chi[P]/(\frac{\theta_{\frac{n}{2}}(\chi)}{2})$ for $n \equiv 6 \bmod 8$.

Proof. Let $F = \mathbb{Q}_{\chi^{-1}\omega^{\frac{n}{2}+1}}$ and $\xi = \chi^{-1}\omega^{\frac{n}{2}+1}$. For all places $\wp$ of F, we denote by $U_\wp$ the ring of $\wp$–adic integers in the completion of F at $\wp$. By global class field theory we have an isomorphism

$$G_{p,\infty} \cong \frac{C_F}{\prod_{\wp\nmid p\infty} U^*_\wp \cdot \prod_{\wp|\infty} \mathbb{R}_+}$$

where C_F is the idele class group of F and $\mathbb{R}_+$ are the real positive numbers. Since $\mathrm{Pic}_+(\mathcal{O}_F)$ is the narrow ideal class group of F, we get a short exact sequence

$$0 \longrightarrow \frac{\prod_{\wp|p} U^*_\wp}{\mathcal{O}^*_{F,+}}(\xi) \longrightarrow G_{p,\infty}(\xi) \longrightarrow \mathrm{Pic}_+(\mathcal{O}_F)(\xi) \longrightarrow 0 \tag{5.5}$$

where the totally positive units $\mathcal{O}^*_{F,+}$ of F embed diagonally in $\prod_{\wp|p} U^*_\wp$. We note that $\prod_{\wp|p} U^*_\wp(\xi)$ is a cyclic $\mathcal{O}_\xi$–module. The first term in (5.5) is trivial, because ξ is not the trivial character and there is only one prime above p. Our assumption then implies that $G_{p,\infty}(\xi)$ is a cyclic $\mathcal{O}_\xi$–module. The $\mathcal{O}_\xi$–module $G_p(\xi)$ is also cyclic, because it is a surjective image of $G_{p,\infty}(\xi)$. Theorem 5.1 then gives that $\mathrm{K}_n(R_E;\mathbb{Z}_p)(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$–module, and therefore it is isomorphic to $\mathcal{O}_\chi[P]/\,\mathrm{Fit}_{\mathcal{O}_\chi[P]}\,\mathrm{K}_n(R_E;\mathbb{Z}_p)(\chi)$. An application of Theorem A concludes the proof. $\square$

We observe that with a slight modification of the above proof for $p=2$ and $n \equiv 6 \bmod 8$ we could have weaken the hypothesis of Corollary 5.4 in this case by requiring only the cyclicity of the ordinary Picard group instead of the narrow one.

Corollary 5.4 gives us a computation of the χ–components of higher K–groups in many cases.

An easier accessible object than the ordinary tame kernel is the modified tame kernel, or if we prefer the narrow K_2–group. The reason for introducing the narrow tame kernel is that we do not want the real embeddings to have any impact on our K–groups. We will beef up the definition of the narrow tame kernel to higher K–theory in Definition 5.6. Like in the classical K_2 setting, we want the quotient of the tame kernel by the narrow tame kernel to be an elementary two–group of rank the number of real embeddings of the field. Let F be a number field. The notion of a higher narrow tame kernel for F and R_F makes sense only in degrees $8j+2$, where $j \geq 0$. This fact is reflected in Theorems A and 5.8, cf. also the list of the 2–rank of $\mathrm{K}_n(R_F)$ displayed in Theorem 7.11 of [RW].

Definition 5.6. *Let $j \geq 0$. We define the two-primary part of the $(8j+2)$th narrow tame kernel $\mathrm{K}^{+}_{8j+2}(R_F)$ of F as $\mathrm{H}^2_+(R_F;\mathbb{Z}_2(4j+2))$. In the same way we define the two-primary part of the $(8j+2)$th narrow* K*-group $\mathrm{K}^{+}_{8j+2}(F)$ of F as $\mathrm{H}^2_+(F;\mathbb{Z}_2(4j+2))$.*

Remark 5.7. *The group $\mathrm{K}_2^+(R_F)$ has been around for a long time, cf. the papers* [Gr1], [Gr2] *and* [Ke]. *As for K_2 we have the exact sequence $0 \to \mathrm{K}^{+}_{8j+2}(F) \to \mathrm{K}_{8j+2}(F) \to \bigoplus_{\wp \text{ real}}(\mathbb{Z}/2)\wp \to 0$. This also holds with F replaced by R_F, cf. the exact sequences prior to Theorem* 3.8. *On odd-primary parts there is no difference between higher narrow and ordinary algebraic* K*-groups.*

We will next demonstrate that the Bloch–Kato conjecture for F tells us more about the higher algebraic K-groups of F. Let $(\bigoplus A_i)_0$ denote the subgroup of the direct sum $\bigoplus A_i$ consisting of the elements $a = (a_i)$ such that – in additive notation – $\sum a_i = 0$. We let μ_n signify the group of nth roots of unity, and like before we let S be a finite set of primes in F containing the infinite ones S_∞ and the p–adic ones for a fixed rational prime p. Finally we let S_c denote the complex primes of F. Part (d) of the following theorem lends on results from [RW] and the comments in the last part of the proof of Theorem 3.8. We point out that Theorem 5.8 is a direct generalization of classical K_2–results due to Tate, cf. the famous Theorem 6.2 in [Ta].

Theorem 5.8. *Assume that the Bloch–Kato conjecture is true for F at p if p is an odd rational prime. Then we have the following results, where $n \geq 1$, $\nu \geq 1$ and p is odd in (a) - (c).*

(a) We have an isomorphism $\mathrm{K}_{2n}(F)\{p\} \cong \mathrm{H}^2_{\text{ét}}(F;\mathbb{Z}_p(n+1))$ for $n \geq 1$, and there exists a map $\delta : \mathrm{H}^1_{\text{ét}}(F;\mathbb{Z}/p^\nu(n)) \longrightarrow \mathrm{K}_{2n}(F)\{p\}$ whose image is the exponent p^ν subgroup of $\mathrm{K}_{2n}(F)$. The same holds for R_F.

(b) Assume that F contains a primitive p^νth root of unity. Then we have periodic isomorphisms $\mathrm{K}_{2n}(F)/p^\nu \cong (\mathrm{K}_2(F)/p^\nu)(n-1)$ and also $\mathrm{K}_{2n}(R_F)/p^\nu \cong (\mathrm{K}_2(R_F)/p^\nu)(n-1)$ of Galois modules.

(c) With the same assumption as in (b), we have a natural short exact sequence of Galois modules:

$$0 \longrightarrow \mathrm{Pic}(R_F)/p^\nu(n) \longrightarrow \mathrm{K}_{2n}(R_F)/p^\nu \longrightarrow (\bigoplus_{S\setminus S_c} \frac{\mathbb{Z}}{p^\nu}\wp)_0(n) \longrightarrow 0.$$

(d) Let $p = 2$. Note that in (b) and (c) one has $\nu = 1$ if F is a real number field. Parts (a), (b) and (c) are non-conjecturally true if F is totally imaginary, or F is real and $n \equiv 0, 1 \bmod 4$. Let F be a real number field from now on. If $\mathrm{K}_{2n}(F)\{2\} \cong \mathrm{H}^2_{\text{ét}}(F;\mathbb{Z}_2(n+1))$ for $n \equiv 2 \bmod 4$, and likewise for R_F, then parts (a) - (c) are true for $n \equiv 2 \bmod 4$. For $n \equiv 3 \bmod 4$, (a) - (c) remain true where we replace étale cohomology by positive étale cohomology, the functor K_2 by the functor K_2^+, the Picard group by the narrow Picard group and S_c with S_∞.

Proof. (a) The Block–Lichtenbaum spectral sequence provides an isomorphism $\mathrm{K}_{2n+1}(F;\mathbb{Q}_p/\mathbb{Z}_p) \cong \mathrm{H}^1_{\text{ét}}(F;\mathbb{Q}_p/\mathbb{Z}_p(n+1))$, see the discussion before Theorem 1.2. If we quotient both modules in this isomorphism by their divisible part, we obtain our first claim. Next we make the diagram:

$$(5.9)\qquad \begin{array}{ccccc} & & \mathrm{K}_{2n}(F)\{p\} & \xrightarrow{p^\nu} & \mathrm{K}_{2n}(F)\{p\} \\ & \nearrow^{\delta} & \downarrow\cong & & \downarrow\cong \\ \mathrm{H}^1_{\text{ét}}(F;\mathbb{Z}/p^\nu(n+1)) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(F;\mathbb{Z}_p(n+1)) & \xrightarrow{p^\nu} & \mathrm{H}^2_{\text{ét}}(F;\mathbb{Z}_p(n+1)). \end{array}$$

The lower part of (5.9) forms part of the long exact sequence in étale cohomology associated with the extension $0 \to \mathbb{Z}_p(n+1) \xrightarrow{p^\nu} \mathbb{Z}_p(n+1) \to \mathbb{Z}/p^\nu(n+1) \to 0$. The existence of a map δ with the desired property is now clear.

(b) Recall that $\mathrm{H}^3_{\text{ét}}(F;\mathbb{Z}_p(n+1))$ is the trivial group for all n. We now find the isomorphisms:

$$\begin{aligned} \mathrm{K}_{2n}(F)/p^\nu &\cong \mathrm{H}^2_{\text{ét}}(F;\mathbb{Z}_p(n+1))/p^\nu \cong \mathrm{H}^2_{\text{ét}}(F;\mathbb{Z}/p^\nu(n+1)) \\ &\cong \mathrm{H}^2_{\text{ét}}(F;\mathbb{Z}/p^\nu(2))(n-1) \cong (\mathrm{H}^2_{\text{ét}}(F;\mathbb{Z}_p(2))/p^\nu)(n-1) \\ &\cong (\mathrm{K}_2(F)/p^\nu)(n-1). \end{aligned}$$

The second isomorphism follows directly from the Bockstein exact sequence, while the third isomorphism follows from the hypothesis on F. Precisely the same kind of arguments apply for R_F. See also the remark concluding the proof of Theorem 3.8.

(c) The claimed exact sequence of Galois modules for $n=\nu=1$ is Theorem 6.2 of [Ta]. The same proof extends to the case $n=1$ and $\nu\geq 1$. Part (b) lifts that result to the higher K–groups.

(d) Our assertions follow from Theorem 1.3. If F is totally imaginary, or F is real and $n\equiv 1,3 \bmod 4$ then we need no addition arguments than for p odd. In part (c) with $n\equiv 3 \bmod 4$ we employ the narrow version of Theorem 6.2 in [Ta], see Theorem 3.6 in [Ke].

Now for the case F real and $n\equiv 0,2 \bmod 4$. Part (a) is as in the odd case, and part (c) follows from (b). To prove (b), recall that $\mathrm{H}^3_{\text{ét}}(F;\mathbb{Z}_2(n+1)) \cong \bigoplus_{\wp \text{ real}}(\mathbb{Z}/2)_\wp$ for $n\geq 2$ even. The Bockstein exact sequence diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{H}^2_{\text{ét}}(F;\mathbb{Z}_2(n+1))/2 & \longrightarrow & \mathrm{H}^2_{\text{ét}}(F;\frac{\mathbb{Z}}{2}(n+1)) & \longrightarrow & \bigoplus_{\wp \text{ real}}\frac{\mathbb{Z}}{2}\wp & \longrightarrow & 0 \\ & & \downarrow\cong & & & & \\ & & \mathrm{H}^2_{\text{ét}}(F;\frac{\mathbb{Z}}{2}(2))(n-1) & \longrightarrow & \bigoplus_{\wp \text{ real}}\frac{\mathbb{Z}}{2}\wp & \longrightarrow & 0 \end{array}$$

arising from the identification of $\mathbb{Z}/2(n+1)$ with $\mathbb{Z}/2(2)(n-1)$ as sheaves on the étale site over $Spec(F)$ gives the claimed periodicity. Likewise for R_F. □

The $n=1$ case of the following result is the narrow version of Theorem 6.2 in [Ta], see Theorem 3.6 in [Ke].

Proposition 5.10. *Let F be a real number field. Then we have isomorphisms $\mathrm{K}^+_{8j+2}(F)/2 \cong \mathrm{K}^+_2(F)/2$ and $\mathrm{K}^+_{8j+2}(R_F)/2 \cong \mathrm{K}^+_2(R_F)/2$. Moreover, there exists a natural short exact sequence*:

$$(5.11)\qquad 0 \longrightarrow \mathrm{Pic}_+(R_F)/2 \longrightarrow \mathrm{K}^+_{8j+2}(R_F)/2 \longrightarrow (\bigoplus_{S\setminus S_\infty} \mathbb{Z}/2)_0 \longrightarrow 0.$$

In particular, we have the 2–rank formula:

$$\mathrm{rk}_2\ \mathrm{K}^+_{8j+2}(R_F) = \mathrm{rk}_2\ \mathrm{Pic}_+(R_F) + \#(S \setminus S_\infty) - 1.$$

Proof. The argument is the same as in Theorem 5.8(d) for $n \equiv 3 \bmod 4$. This time the twist does not matter, since we reduce modulo 2. □

The $n = 1$ case of the following result is Theorem 5.4 in [Ke].

Corollary 5.12. *Assume that the Bloch–Kato conjecture is true for F at the odd rational prime p. If $\zeta_p \notin F$, then we have a short exact sequence*

$$0 \longrightarrow (\mathrm{Pic}(R_{F(\zeta_p)})/p(n))^\Gamma \longrightarrow \mathrm{K}_{2n}(R_F)/p \longrightarrow (\bigoplus_{\wp \in S\setminus S_c} \frac{\mathbb{Z}}{p}\wp)_0(n)^\Gamma \longrightarrow 0$$

of Γ–modules where Γ signifies the Galois group $Gal(F(\zeta_p)/F)$ and $n \geq 1$.

Proof. From Theorem 5.8 we find the short exact sequence

$$0 \to (\mathrm{Pic}(R_{F(\zeta_p)})/p(n))^\Gamma \to (\mathrm{K}_{2n}(R_{F(\zeta_{p^\nu})})/p)^\Gamma \to (\bigoplus_{S\setminus S_c} \frac{\mathbb{Z}}{p}\wp)_0(n)^\Gamma \to 0$$

of Γ–modules since the orders of Γ and p are relatively prime. For the same reason we find that $\mathrm{K}_{2n}(R_F)/p \xrightarrow{\cong} (\mathrm{K}_{2n}(R_{F(\zeta_{p^\nu})})/p)^\Gamma$ from the projection formula in algebraic K–theory. □

Proposition 5.13. *Assume the hypothesis of Theorem* A *and let $p = 2$. Then for $n \equiv 2 \bmod 8$:*

$$\mathrm{Fit}_{\mathcal{O}_\chi[P]}\ \mathrm{K}^+_n(R_E)(\chi) = (\frac{\theta_{\frac{n}{2}}(\chi)}{2}).$$

In particular, the element $\frac{1}{2}\theta_{\frac{n}{2}}(\chi)$ is contained in $\mathcal{O}_\chi[P]$.

Proof. Consider the short exact sequence defining $\mathrm{K}^+_n(R_E)(\chi)$:

$$0 \longrightarrow \mathrm{K}^+_n(R_E)(\chi) \longrightarrow \mathrm{K}_n(R_E)(\chi) \longrightarrow \mathbb{Z}_2[G]/2(\chi) \longrightarrow 0. \tag{5.14}$$

The same argument as in the proof of Theorem A in the case $p = 2$ and $n \equiv 6 \bmod 8$ applied to (5.14) gives our claim. □

The next proposition generalizes Proposition 6.2 in [Ke].

Proposition 5.15. *Let E/F be a Galois extension of number fields, with Galois group Γ. Then the transfer map induces an isomorphism $\mathrm{K}^+_{8j+2}(E)_\Gamma \xrightarrow{\cong} \mathrm{K}^+_{8j+2}(F)$.*

Suppose that Γ is solvable and for $\wp$ a prime in F let $e_\wp$ denote its ramification index in E/F. If $(e_\wp, (\#F[\wp])^{4j+1} - 1) = 1$ for all ramified primes $\wp$ not in S, then the transfer map induces an isomorphism $\mathrm{K}^+_{8j+2}(R_E)_\Gamma \xrightarrow{\cong} \mathrm{K}^+_{8j+2}(R_F)$.

Let now E be a totally real Abelian number field of prime power conductor f, $F \subset E$ with $\Gamma = Gal(E/F)$ of 2-power order, and χ a nontrivial character of $G = Gal(E/\mathbb{Q})$ of order prime to 2. Then the transfer map $\mathrm{K}^+_{8j+2}(R_E) \longrightarrow \mathrm{K}^+_{8j+2}(R_F)$ induces an isomorphism $\mathrm{K}^+_{8j+2}(R_F)(\chi) \cong \mathrm{K}^+_{8j+2}(R_E)(\chi)_\Gamma$.

Proof. We have an isomorphism $\mathrm{H}^2_+(E;\mathbb{Z}_2(4j+2))_\Gamma \cong \mathrm{H}^2_+(F;\mathbb{Z}_2(4j+2))$ from the Tate spectral sequence for positive étale cohomology, cf. the proof of Theorem B. The odd–primary claim is contained in the same result.

The assumption that Γ is solvable means that we may assume that Γ is cyclic. Consider the diagram:

$$\begin{array}{ccccccccc} & & \mathrm{K}^+_{8j+2}(R_E)_\Gamma & \longrightarrow & \mathrm{K}^+_{8j+2}(E)_\Gamma & \longrightarrow & (\bigoplus_{\overline{\wp}\notin T}\mathrm{K}_{8j+1}(E[\overline{\wp}]))_\Gamma & \longrightarrow & 0 \\ & & \downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & \mathrm{K}^+_{8j+2}(R_F) & \longrightarrow & \mathrm{K}^+_{8j+2}(F) & \longrightarrow & (\bigoplus_{\wp\notin S}\mathrm{K}_{8j+1}(F[\wp])) & \longrightarrow & 0. \end{array}$$

The right hand side is an isomorphism, cf. the proof of Proposition 3.11. Thus we are reduced to show that $\mathrm{H}_1(\Gamma;\bigoplus_{\overline{\wp}|\wp}\mathrm{K}_{8j+1}(E[\overline{\wp}]))$ is the trivial group for all $\wp\notin S$. Since Γ is cyclic we have

$$\mathrm{H}_1(\Gamma,\bigoplus_{\overline{\wp}|\wp}\mathrm{K}_{8j+1}(E[\overline{\wp}]))\cong\widehat{\mathrm{H}}^{-2}(\Gamma,\bigoplus_{\overline{\wp}|\wp}\mathrm{K}_{8j+1}(E[\overline{\wp}]))\cong\widehat{\mathrm{H}}^{0}(\Gamma,\bigoplus_{\overline{\wp}|\wp}\mathrm{K}_{8j+1}(E[\overline{\wp}]))$$

where the first isomorphism comes from the definition of Tate cohomology groups. We have that $\widehat{\mathrm{H}}^{0}(\Gamma,\bigoplus_{\overline{\wp}|\wp}\mathrm{K}_{8j+1}(E[\overline{\wp}]))\cong\mathrm{K}_{8j+1}(F[\wp])/e_\wp$ where $e_\wp$ denotes the ramification index of $\wp$. Our hypothesis together with Theorem 3.1 give that this last group is trivial. This implies the second assertion.

Now for the last claim. Since $E\subset\mathbb{Q}(\zeta_f+\zeta_f^{-1})$, there is only one prime of F which ramifies in the extension E/F, and it is fixed by $Gal(F/\mathbb{Q})$. Proposition 3.12 then gives $\mathrm{K}_{8j+2}(R_E)(\chi)_\Gamma\cong\mathrm{K}_{8j+2}(R_F)(\chi)$. The proof now proceeds as in the second part, by employing the short exact sequence mentioned in Remark 5.7 for R_E and R_F. □

Corollary 5.16. *Assume the hypothesis of Theorem* A *with* $p=2$, *and let* $E_0=E^P$. *Then* $\mathrm{K}^+_{8j+2}(R_E)(\chi)$ *is a cyclic* $\mathcal{O}_\chi[P]$*–module if and only if* $\mathrm{K}^+_{8j+2}(R_{E_0})(\chi)$ *is a cyclic* $\mathcal{O}_\chi$*–module.*

Proof. The group $\mathrm{K}^+_{8j+2}(R_E)(\chi)$ is a module over the local ring $\mathcal{O}_\chi[P]$. Thus – from Nakayama's lemma – the number of generators of $\mathrm{K}^+_{8j+2}(R_E)(\chi)$ is nothing but the dimension of the $\mathcal{O}_\chi/2$–vector space $\mathrm{K}^+_{8j+2}(R_E)(\chi)_P/2$. Proposition 5.15 gives at once the isomorphism $\mathrm{K}^+_{8j+2}(R_E)(\chi)_P/2\cong\mathrm{K}^+_{8j+2}(R_{E_0})(\chi)/2$. □

Next we consider the cyclicity of $\mathrm{K}_{8j+2}(R_{R_{E_0}})(\chi)$ as an $\mathcal{O}_\chi$–module. For any finite $\mathcal{O}_\chi$–module M, we define its χ–rank $\mathrm{rk}_\chi M$ as the dimension of the $\mathcal{O}_\chi/2$–vector space $M/2$.

Lemma 5.17. *Let* F *be a real Abelian number field and let* $G=Gal(F/\mathbb{Q})=P\times\Delta$ *where* P *is the* 2*–Sylow subgroup of* G. *For any* 2*–adic character* χ *of* P *we have:*

$$\mathrm{rk}_\chi\,\mathrm{K}^+_{8j+2}(R_F)(\chi)=\mathrm{rk}_\chi\,\mathrm{Pic}_+(R_F)(\chi)+\mathrm{rk}_\chi(\bigoplus_{\wp|2}\mathbb{Z}/2)(\chi)-\mathrm{rk}_\chi\,\mathbb{Z}/2(\chi).$$

Observe that $\mathbb{Z}/2(\chi)$ *is trivial unless* χ *is the trivial character.*

Proof. For this we take χ–parts in (5.11). The claim follows since rk_χ is additive on exact sequences of finite $\mathcal{O}_\chi/2$–modules. □

Proposition 5.18. *Assume the hypotheses of Corollary* 5.16, *and moreover that the χ-rank of $\mathrm{K}^+_{8j+2}(R_{E_0})(\chi)$ is less than* 2. *Then:*

$$\mathrm{K}^+_{8j+2}(R_E)(\chi) \cong \frac{\mathcal{O}_\chi[P]}{(\theta_{4j+1}(\chi)/2)}.$$

Proof. By hypothesis $\mathrm{K}^+_{8j+2}(R_{E_0})(\chi)$ is a cyclic $\mathcal{O}_\chi$-module. Corollary 5.16 implies that the group $\mathrm{K}^+_{8j+2}(R_E)(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. Thus

(5.19) $\quad \mathrm{K}^+_{8j+2}(R_E)(\chi) \cong \mathcal{O}_\chi[P]/\operatorname{Fit}_{\mathcal{O}_\chi[P]}\mathrm{K}^+_{8j+2}(R_E)(\chi) \cong \mathcal{O}_\chi[P]/(\theta_{4j+1}(\chi)/2)$

where the last isomorphism follows from Proposition 5.13. □

Proposition 5.20. *Suppose that E is a totally real Abelian number field and let $G = Gal(E/\mathbb{Q}) = P \times \Delta$ where P is the 2–Sylow subgroup of G. Let χ be any 2–adic character of Δ and suppose that $\mathcal{O}^*_{E,+}(\chi) = (\mathcal{O}^*_E)^2(\chi)$, where $\mathcal{O}^*_{E,+}$ are the totally positive units of R_E. Then $\mathrm{K}_{8j+2}(R_E)(\chi) \cong \mathrm{K}^+_{8j+2}(R_E)(\chi) \bigoplus \mathcal{O}_\chi[P]/2$.*

Proof. The hypothesis means that the signature map $Sign : R^*_F \longrightarrow \mathbb{Z}_2[G]/2(\chi) \cong \mathcal{O}_\chi[P]/2$ is surjective. We want to show that (5.14) is split. We do this in two steps, first the classical K_2–case and then in general.

Let 1_χ be the unit of the ring $\mathcal{O}_\chi[P]/2$ and let $u \in R^*_E$ be such that $Sign(u) = 1_\chi$. The Steinberg symbol $\{-1, u\} \in \mathrm{K}_2(E)$ lies in the tame kernel $\mathrm{K}_2(R_E)$ and has order 2. Mapping 1_χ to $\{-1,u\}(\chi)$ induces a section of the map $\mathrm{K}_2(R_E)(\chi) \longrightarrow \mathcal{O}_\chi[P]/2$, and done with the K_2–case.

Since $\mathbb{Z}_2/4(2) \cong \mathbb{Z}_2/4(4j+2)$, we find an isomorphism $\mathrm{H}^2_{\text{ét}}(E;\mathbb{Z}_2(4j+2))/4 \cong \mathrm{H}^2_{\text{ét}}(E;\mathbb{Z}_2(2))/4$ by arguing as in the last part of the proof of Theorem 5.8(d). Therefore by the above we know that we can define a section of $\mathrm{K}_{8j+2}(R_E)(\chi)/4 \to \mathcal{O}_\chi[P]/2$ by sending 1_χ to the class of some element $\alpha \in \mathrm{K}_{8j+2}(R_E)(\chi)$. It is now easy to check that we can modify α in order to get an element of $\mathrm{K}_{8j+2}(R_E)(\chi)$ of order 2 which gives a section $\mathcal{O}_\chi[P]/2 \to \mathrm{K}_{8j+2}(R_E)(\chi)$. □

Lemma 5.17, Propositions 5.18 and 5.20 allow us to give explicit calculations of $\mathrm{K}_{8j+2}(R_E)(\chi)$ in many cases where χ is not the trivial character. Precisely the same argument as in the proof of Proposition 5.20 shows that the extension $0 \to \mathrm{K}^+_{8j+2}(F) \to \mathrm{K}_{8j+2}(F) \to \bigoplus_{\wp \text{ real}}(\mathbb{Z}/2)_\wp \to 0$ is split for any number field F, since this is true for K_2 by an argument with Steinberg symbols. Clearly, the same holds for R_F under the assumption $\mathcal{O}^*_{F,+} = (\mathcal{O}^*_F)^2$.

Let E be a prime power conductor field, and let χ_0 be the trivial 2–adic character of the odd part of $Gal(E/\mathbb{Q})$. Then $\mathrm{K}_{8j+2}(R_E)(\chi_0) \cong \mathrm{K}_{8j+2}(R_F) \otimes \mathbb{Z}_2$ where $F = E^\Delta$ has degree $\#P$ over $\mathbb{Q}$. Thus to study χ_0–components we may assume that E is a totally real number field of prime power conductor p^n such that $Gal(E/\mathbb{Q})$ is a cyclic group P of 2–power order. Then $\mathrm{Pic}_+(R_E) \otimes \mathbb{Z}_2 \cong 0$. In fact there exists a surjection from $\mathrm{Pic}(\mathcal{O}_K) \otimes \mathbb{Z}_2$ where K is the field of conductor p^n and degree $2^{ord_2(p^n - p^{n-1})}$ over the rationals. Recall that $\mathrm{Pic}(\mathcal{O}_K) \otimes \mathbb{Z}_2 = 0$ by Theorem 10.4 in [Wa], and thus the hypothesis of Proposition 5.20 are satisfied. Hence we find $\mathrm{K}_{8j+2}(R_E) \otimes \mathbb{Z}_2 \cong (\mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2) \bigoplus \mathbb{Z}_2[P]/2$. We are thus reduced to study the two–primary part of $\mathrm{K}^+_{8j+2}(R_E)$. Lemma 5.17 implies that $\mathrm{rk}_2\, \mathrm{K}^+_{8j+2}(R_E) = s - 1$ where s is the number of dyadic primes of E. Thus $\mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2$ is the trivial group if $s = 1$. There is also a formula for the number of elements in $\mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2$.

Proposition 5.21. *If the hypothesis of Theorem* A *are satisfied, then we have the equality* $\# \mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2 \sim_2 \prod_{\psi \neq 1} \frac{1}{2} L(-4j-1, \psi^{-1})$ *where* ψ *runs over all the* 2*–characters of* P*, modulo equivalence.*

Proof. Apply Corollary 4.16. □

We now determine the Galois module structure of $\mathrm{K}^+_{8j+2}(R_E)$ for the number field $E = \mathbb{Q}(\sqrt{p})$ where $p \equiv 1 \bmod 4$ is a rational prime.

Proposition 5.22. *Let* $p \equiv 1 \bmod 4$ *be a rational prime and let* $E = \mathbb{Q}(\sqrt{p})$*. Let* $P = Gal(E/\mathbb{Q})$ *be generated by* σ*, and let* ψ *be the nontrivial character of* P*. Then:*

$$\mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2 \cong \frac{\mathbb{Z}_2[\sigma]}{(1+\sigma, \frac{1}{2}L(-4j-1,\psi))}.$$

Proof. By Lemma 5.17 we know that $\mathrm{rk}_2 \, \mathrm{K}^+_{8j+2}(R_E) \leq 1$. Actually the two–rank equals 1 if $p \equiv 1 \bmod 8$ and 0 if $p \equiv 5 \bmod 8$. Hence $\mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2$ is a cyclic $\mathbb{Z}_2$–module. Since $\mathrm{K}^+_{8j+2}(\mathbb{Z}) \otimes \mathbb{Z}_2$ is the trivial group we derive that $(1+\sigma)\mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2 = 0$. Moreover, from Proposition 5.21 we derive the equality $\frac{1}{2}L(-4j-1,\psi)\mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2 = 0$. We have shown that the ideal $I = (1+\sigma, \frac{1}{2}L(-4j-1,\psi))$ annihilates the cyclic $\mathbb{Z}_2[\sigma]$–module $\mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2$, and hence we are done since $\#\mathbb{Z}_2[\sigma]/I = \# \mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2$. □

Remark 5.23. *The ideal* I *in the proof of Proposition* 5.22 *is the* $\mathbb{Z}_2[\sigma]$*–Fitting ideal of* $\mathrm{K}^+_{8j+2}(R_E) \otimes \mathbb{Z}_2$*. Since the ideal* I *is not principal for* $p \equiv 1 \bmod 8$*, we see that it is impossible to extend Corollary* 5.13*, and therefore Theorem* A*, to the trivial character.*

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