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Infinitesimal incidence in noncommutative algebraic geometry,
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Arne B. Sletsjøe

Abstract. We study the geometry of associative (noncommutative) algebras. The concept of interaction or infinitesimal incidence is explored by considering non-vanishing of certain Ext-groups. As an application non-commutative conic sections are investigated.

0. Introduction

The fundamental idea of this report is the following: Let k be some field and let Alg_k be the category of associative k -algebras. As a generalization of classical (commutative and reduced) algebraic geometry we define noncommutative algebraic geometry as the dual Alg_k° of the category of associative algebras. Thus the geometrical objects are the algebras $R \in \text{Obj}(Alg_k)$, and morphisms are reversed k -algebra homomorphisms. Our purpose is to try to give the algebraic objects a geometrical interpretation, just in the way we do in the commutative setting. Using a geometrical language, geometrical concepts and hopefully geometrical insight we will try to come closer to an understanding of the algebraic objects.

With our definition the geometrical object associated to an algebra always exists. The problem is whether the geometrical object has a natural geometrical interpretation in terms of classical geometric intuition. If it has not, it would be rather difficult to accept the term geometrical object.

A geometrical object can only be visualized through its geometrical properties. To enlighten the different properties we need some geometrical tools. Our basic tools are the concepts of points, symmetries, incidence and interaction (or infinitesimal incidence). Once we have studied and understood the geometry of an object we can look at the embeddings of this element into other objects. In this way we can build up the knowledge of geometrical objects in geometrical terms.

Let R be some k -algebra and let $X_R = \text{Hom}_{Alg_k}(R, k)$ be the set of k -algebra homomorphisms of R on k . It is well-known from Hilbert Nullstellensatz that if R is commutative and reduced X_R determines R . For an object R of the category alg_{red} we can identify X_R with the dual object R° . Thus $X_- : alg_{red} \rightarrow alg_{red}^\circ$ has two geometrical interpretations. Since $k \in \text{Obj}(alg_{red})$ and $X_k = \{e\}$, this is a natural definition for the concept of a point. The map $\phi : R \rightarrow k$ gives an inclusion $X_k \simeq \{pt\} \hookrightarrow X_R$ and X_R has a natural interpretation as the parametrization of the set points of X_R . On the other hand X_- defines an anti-equivalence between alg_{red} and the dual category alg_{red}° , saying that X_R is the geometrical space associated to the algebra R . Combining these two points of view we obtain the seemingly obvious fact that a geometrical object is made up by and completely determined by its points. For noncommutative or non-reduced algebras this will no more be true. One way of handling this problem is to endow the geometrical objects with some structure sheaf and say that this is a part of the geometry. Another way of doing this is to say that the geometry is more than the classical points, i.e. allowing new points with non-trivial symmetry group and accept that interactions between points, or more general between subsets, are part of the geometry. This will be our point of view.

In his paper [Ko] from 1992 Kontsevich suggests an approximation to noncommutative geometry by considering the set X_R^n of homomorphisms from the k -algebra R to the matrix algebra $M_n(k)$. This is a variety and if R is finitely generated, so is X_R^n . This approach is further studied by Kontsevich and Rosenberg [KR] and Le Bruyn [Lb]. In the search for a non-commutative algebraic geometry Kontsevich and Rosenberg propose that "A noncommutative structure of some kind on R , generalizing a commutative structure, should give an analogous commutative structure on all schemes X_R^n , $n \geq 1$ ". A crucial point in our approach to noncommutative algebraic geometry is that the geometry of an algebraic object should be understood through its building-up from known geometrical objects, their incidence relations and interactions. Thus, we have to restrict the source category for the functor \mathcal{X}_R , i.e. the choice of building blocks. Any choice must be justified either by classical algebraic geometry methods or by iterated methods in the setting of this paper. For every choice of building blocks we want to find a subcategory of algebras, such that the objects in this subcategory is determined by their relations with the building blocks.

The report is organized as follows. The first part is the general part where we study the geometrical objects corresponding to noncommutative algebras. We also give a kind of Nullstellensatz for extensions of closed points. In the second part we continue investigating extensions, and in the third and last part we study the geometry of the noncommutative plane conic sections.

1. Algebras as geometrical objects

1.1. Let k be a field and let R be any associative k -algebra. Consider the functor

$$\mathcal{X}_R = \text{Hom}_{\text{Alg}_k}(R, -) : \text{Alg}_k \longrightarrow \text{sets}$$

of k -algebra homomorphisms of R . This functor is called the **affine scheme** of R . For any k -algebra S the set $\mathcal{X}_R(S) = \text{Hom}_{\text{Alg}_k}(R, S)$ is the set of S -**points** of \mathcal{X}_R . A k -algebra map $\phi : R \rightarrow R'$ induces a natural transformation

$$\phi^* : \mathcal{X}_{R'} \longrightarrow \mathcal{X}_R$$

given by $\phi^*(\psi) = \psi \circ \phi$. The functor \mathcal{X}_R determines the algebra R up to isomorphism in the following sense:

PROPOSITION (1.2.). *Let $\phi : R \rightarrow R'$ be a k -algebra map of two associative k -algebras such that $\phi^* : \mathcal{X}_{R'} \rightarrow \mathcal{X}_R$ is an isomorphism. Then $R \simeq R'$.*

Proof. By assumption the map $\phi^*(R) : \mathcal{X}_{R'}(R) \rightarrow \mathcal{X}_R(R)$ is an isomorphism. Thus we can find $\psi \in \mathcal{X}_{R'}(R)$ such that $\phi^*(R)(\psi) = \text{id}_R$, i.e. we can find $\psi : R' \rightarrow R$ such that $\psi \circ \phi = \text{id}_R$. Now consider the opposite composition, $\phi \circ \psi : R' \rightarrow R'$. We have

$$\phi^*(R')(\phi \circ \psi) = (\phi \circ \psi) \circ \phi = \phi \circ (\psi \circ \phi) = \phi \circ \text{id}_R = \text{id}_{R'} \circ \phi = \phi^*(R')(\text{id}_{R'})$$

By assumption $\phi^*(R')$ is an isomorphism and hence injective. Thus $\phi \circ \psi = \text{id}_{R'}$, proving that ϕ is an automorphism. \square

If the algebra R is commutative the functor \mathcal{X}_R is the ordinary affine scheme. The set $\mathcal{X}_R(k)$ is the set of k -points of $\text{Spec}(R)$, i.e. the maximal ideals of R . Other choices for integral domains S produces prime ideals of R of various height. Let $R_{ab} = R/([R, R])$ be the quotient of R by the two-sided ideal $([R, R])$ generated by all commutators in R . For any commutative k -algebra S we obviously have $\mathcal{X}_R(S) = \mathcal{X}_{R_{ab}}(S)$. If $S = k$ we use the notation $X_R = \mathcal{X}_R(k)$ for the closed points of $\text{Spec}(R_{ab})$. The surjection $R \rightarrow R_{ab}$ induces a natural inclusion $\mathcal{X}_{R_{ab}} \hookrightarrow \mathcal{X}_R$. Thus the classical geometric object is part of the more general geometry. Notice that $X_{R_{ab}} = X_R$ so the closed points of the two functors $\mathcal{X}_{R_{ab}}$ and \mathcal{X}_R are the same.

1.3. The above construction defines a functor

$$\mathcal{X} : \text{Alg}_k \rightarrow \text{sets}^{\text{Alg}_k}$$

given by $\mathcal{X}(R) = \mathcal{X}_R(-)$. For any associative algebra S the functor \mathcal{X} induces in an obvious way a contravariant functor

$$\mathcal{G}_S : \text{Alg}_k \rightarrow \text{sets}$$

given by $\mathcal{G}_S(-) = \mathcal{X}_{(-)}(S)$. Thus as $\mathcal{X}_R(-)$ gives the affine geometry of R , $\mathcal{G}_S(-)$ is the S -point functor, returning the S -points $\mathcal{G}_S(R) = \mathcal{X}_R(S)$ of an argument algebra R . An other way of saying this is that R or rather \mathcal{X}_R is the algebraic object we want to study, and the righthand S represents the geometrical tool we are using in our investigation.

DEFINITION (1.4.). *Let $\underline{\mathcal{C}}$ be some subcategory of the category of associative algebras and let $\iota : \underline{\mathcal{C}} \hookrightarrow \text{Alg}_k$ be the inclusion functor. The subcategory $\underline{\mathcal{C}}$ is called a **geometrical quotient** of Alg_k if there exists a left adjoint functor $g : \text{Alg}_k \rightarrow \underline{\mathcal{C}}$ for the inclusion functor.*

Let $\underline{\mathcal{C}}$ be a geometrical quotient of Alg_k . The composition $\sigma_{\underline{\mathcal{C}}} = \iota g$ is a projection functor of Alg_k on $\underline{\mathcal{C}}$.

1.5. For any family of associative k -algebras \mathcal{S} let $\text{Alg}_k^{\mathcal{S}}$ be a maximal geometrical quotient of Alg_k such that $\mathcal{G}_S(-) \circ \sigma_{\mathcal{S}} = \mathcal{G}_S(-)$ for all $S \in \mathcal{S}$. Here we have used the notation $\sigma_{\mathcal{S}} = \sigma_{\text{Alg}_k^{\mathcal{S}}}$. In general such a geometrical quotient does not exist. If it exists it is not obvious whether it is unique.

DEFINITION (1.6.). Let \mathcal{S} be a family of associative k -algebras such that a maximal geometrical quotient $\sigma_{\mathcal{S}} : \text{Alg}_k \rightarrow \text{Alg}_k^{\mathcal{S}}$ exists. Then the pair $(\mathcal{S}, \sigma_{\mathcal{S}})$ is called a **geometry pair**.

On the other hand, given any projection functor $\sigma : \text{Alg}_k \rightarrow \text{Alg}_k$, let $\underline{\mathcal{C}} = \text{Alg}_k^{\sigma}$ be the fixed subcategory. We call a family \mathcal{S} of k -algebras a **representation model** for $\underline{\mathcal{C}}$ if $\sigma = \sigma_{\mathcal{S}}$, i.e. $(\mathcal{S}, \sigma_{\mathcal{S}})$ is a geometry pair.

1.7. **Example.** For $\mathcal{S} = \{k\}$ we have $\text{Alg}_k^k \simeq \text{alg}_k$ is the subcategory of commutative, reduced algebras and the projection is the algebra quotient where we divide out by the radical of the commutator ideal. On the other side, for $\mathcal{S} = \text{alg}_k$ the category of commutative algebras we have $\text{Alg}_k^{\text{alg}_k} = \text{alg}_k$ and the projection is the identity.

1.8. The geometrical interpretation of a k -algebra homomorphism $\phi : R \rightarrow S$ is as an S -point of R , i.e. an inclusion $\phi^* : \mathcal{X}_S \hookrightarrow \mathcal{X}_R$. In the special case $R = S$ an automorphism $\alpha : R \rightarrow R$ induces an automorphism $\alpha^* : \mathcal{X}_R \rightarrow \mathcal{X}_R$ of the functor \mathcal{X}_R . In the geometrical setting automorphisms are called **symmetries**. In general the set \mathcal{X}_S will have non-trivial symmetry group. For any automorphism $\alpha : S \rightarrow S$ the two S -points ϕ and $\alpha \circ \phi$ represents the same geometrical impact, the difference sits in the internal structure of S . The symmetry group $\Gamma_S = \text{Aut}_{\text{Alg}_k}(S)$ obviously acts on $\mathcal{G}_S(R)$. The set $\mathcal{G}_S(R)/\Gamma_S$ is the **S -geometry** of R .

1.9. A special case of associative k -algebras are the matrix-rings $M_n(S)$ and their subrings $T_n(S)$ of (upper) triangular matrices. We shall use the notation $\mathcal{G}_k^n(-) = \mathcal{G}_{M_n(k)}(-)$ for the $M_n(k)$ -points of any affine scheme, and we let

$$\mathcal{X}_R^n(-) = \text{Hom}_{\text{Alg}_k}(R, M_n(-))$$

be the **level n geometry** of the affine scheme R . For the subgeometry corresponding to the upper triangular matrices we use the notation

$$\mathcal{T}_R^n(-) = \text{Hom}_{\text{Alg}_k}(R, T_n(-))$$

If the argument is the ground ring k we simply write \mathcal{T}_R^n . We can consider $T_n(k)$ as the quiver algebra of the quiver

$$\bullet_1 \rightarrow \bullet_2 \rightarrow \dots \rightarrow \bullet_n$$

i.e. a system of n points with a directed tangent of interaction. The set \mathcal{T}_R^n parametrizes all such systems in the affine geometry \mathcal{X}_R . For any two-sided ideal in a associative k -algebra R we define $\text{rad}(I)$ to be the set of nilpotent elements in the quotient R/I , i.e. all elements $r \in R$ such that $r^m \in I$ for some $m \geq 1$.

THEOREM (1.10.). Let $k = \bar{k}$ be an algebraically closed field and let $R = k\langle x_1, \dots, x_n \rangle / I$ be some finitely generated, associative k -algebra, defined by some two-sided ideal I . Then

$$\bigcap_{\phi \in \mathcal{T}_R^n} \ker \phi = \text{rad}([R, R])^n$$

Proof. The inclusion

$$\text{rad}([R, R])^n \subset \bigcap_{\phi \in \mathcal{T}_R^n} \ker \phi$$

follows from well-known properties for upper triangular matrices.

We shall prove the other inclusion by induction on n . For $n = 1$ we have $T_R^1 = k$ and the result is precisely the Hilbert Nullstellensatz.

Assume the inclusion is valid for $m = n - 1$. There are two algebra maps $T_n(k) \rightarrow T_{n-1}(k)$, projections on the first, respectively on the last $n - 1$ -dimensional subspaces. Composing $\phi : R \rightarrow T_n(k)$ with these projections we obtain the inclusion

$$\bigcap_{\phi \in \mathcal{T}_R^n} \ker \phi \subset \text{rad}([R, R])^{n-1}$$

Let $r \in \bigcap_{\phi \in \mathcal{T}_R^n} \ker \phi$ and suppose $s = r^m \in ([R, R])^{n-1}$. Since the commutator operates as a derivation on each factor we can assume that s is a sum of expressions on the form

$$s = f^{(1)}[x_{i_1}, x_{j_1}]f^{(2)}[x_{i_2}, x_{j_2}] \cdots f^{(n-1)}[x_{i_{n-1}}, x_{j_{n-1}}]f^{(n)}$$

Thus $\phi(s)_{ij} = 0$ for all pairs (i, j) except for the upper rightmost corner $(1, n)$. This coefficient is given by

$$\begin{aligned} \phi(s)_{1n} = & \phi(f^{(1)})_{11}\phi([x_{i_1}, x_{j_1}])_{12}\phi(f^{(2)})_{22}\phi([x_{i_2}, x_{j_2}])_{23} \cdots \\ & \cdots \phi(f^{(n-1)})_{n-1, n-1}\phi([x_{i_{n-1}}, x_{j_{n-1}}])_{n-1, n}\phi(f^{(n)})_{nn} \end{aligned}$$

Assume $f^m \notin [R, R]$ for all $m = 1, 2, \dots, n$. Since k is an infinite set there are some $\phi \in \mathcal{T}_R^n$ such that $\phi(f^m)_{mm} \neq 0$ for all $m = 1, 2, \dots, n$. These expressions only involve diagonal elements of the images of the generators x_i and there are enough degrees of freedom to choose ϕ such that all $\phi([x_{i_m}, x_{j_m}])_{m, m+1} \neq 0$, obtaining a contradiction and the assumption must be wrong. Thus $s \in [R, R]^{n-1}$. \square

For any k -algebra R we define $R_{(n)}$ to be the quotient $R/\text{rad}([R, R])^n$. In particular we have $R_{(1)} = R_{ab}$. Let $\text{Alg}_k^{(n)}$ be the category of k -algebras of **commutation nilpotence degree** n , i.e. algebras R such that $\text{rad}([R, R])^n = 0$, and let $\sigma_{(n)}$ be the projection $\sigma_{(n)}(R) = R_{(n)}$.

COROLLARY (1.11.). *The pair $(T_n(k), \sigma_{(n)})$ is a geometry pair.*

Proof. An immediate consequence of the theorem. \square

2. Extensions and geometry @₂

2.1. Let $k = \bar{k}$ be a field and R some associative k -algebra. We shall denote by $M_2 = M_2(k)$ the algebra of two-by-two matrices with entries in the ground field k , and $T_2 = T_2(k)$, the subalgebra of upper triangular matrices. The algebra T_2 is the smallest possible non-commutative k -algebra. It is of dimension 3 over k and may be considered as the path-algebra of the path

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ s & p & m \end{array}$$

where $s^2 = s$, $m^2 = m$ and $sp = pm = p$. The rest of the products vanish. Observe that $[s, p] = [p, m] = p$. Thus for any one-dimensional representation $\rho : T_2(k) \rightarrow k$ we must have $p \mapsto 0$ and the only possible representations are the two projections on the diagonal. The two projections add up to a surjective map inducing a map

$$pr_* : \mathcal{T}_2(R) \rightarrow (X_R)^2$$

where $\mathcal{T}_2(R) = \text{Hom}_{\text{alg}}(R, T_2(k))$ and $X_R = \text{Hom}_{\text{alg}}(R, k)$ are the k -points of R . The automorphism group of M_2 is the special linear group SL_2 , acting by conjugation. For the group $\text{Aut}_{\text{Alg}_k}(T_2(k))$ of automorphisms of the k -algebra $T_2(k)$ we have the following.

PROPOSITION (2.2). *The automorphism group of $T_2(k)$ is given by*

$$\text{Aut}_{\text{Alg}_k}(T_2(k)) = k^* \times k$$

with group operation $(a, b)(c, d) = (ac, bc + d)$.

Proof. A simple computation, using the relations among k , m and p , shows that an automorphism $\phi : T_2(k) \rightarrow T_2(k)$ is given by $k \mapsto k + bp$, $m \mapsto m - bp$ and $p \mapsto ap$ where $a \neq 0$. The group law is given by composing two such automorphisms. \square

Since the field k is assumed to be algebraically closed the automorphisms can be identified with conjugations by upper triangular matrices with determinant equal to 1 via the identification

$$\begin{pmatrix} x & y \\ 0 & \frac{1}{x} \end{pmatrix} \mapsto (x^2, -xy)$$

Automorphisms of $T_2(k)$ induce automorphisms of the one-dimensional representations and it is easily seen that these are the identities.

2.3. The level 2 geometry of an algebra R is given by the affine scheme $\mathcal{X}_R^{(2)}(-)$, and in particular by the k -points $X_R^{(2)} = \mathcal{X}_R^{(2)}(k)$. The $M_2(k)$ -geometry of R is given by the quotient

$$X_R^{(2)} / \text{SL}_2$$

In the next example we shall describe this set for the affine quantum plane $R = k\langle x, y \rangle / (xy + yx)$.

Example (2.4). Let $R = k\langle x, y \rangle / (xy + yx)$ and let $\phi \in X_R^{(2)}$ be the map given by

$$x \mapsto A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad y \mapsto B = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

such that $AB + BA = 0$. Thus we have the following set of equations

$$\begin{aligned} x_{11}y_{11} + x_{12}y_{21} + x_{21}y_{12} + x_{22}y_{22} &= 0 \\ (x_{11} + x_{22})y_{12} + x_{12}(y_{11} + y_{22}) &= 0 \\ (x_{11} + x_{22})y_{21} + x_{21}(y_{11} + y_{22}) &= 0 \\ x_{11}y_{11} - x_{22}y_{22} &= 0 \end{aligned}$$

This map correspond to a R -module structure on $V \simeq k^2$, via the map $\phi : R \rightarrow \text{End}_k(V) \subseteq M_2(k)$. The equation $AB + BA = 0$ gives $AB - BA = 2AB$, hence

$$\det(AB - BA) = 4 \det A \det B$$

From [AS] we know that if $R = k\langle A, B \rangle$ is the k -algebra generated by two 2×2 -matrices A and B , then $R = M_2(k)$ if and only if $\det(AB - BA) \neq 0$. Thus the representation give by $x \mapsto A, y \mapsto B$ is irreducible if and only if $\det A, \det B \neq 0$.

Some straightforward, (but rather tedious) calculations shows that the set $x_R^{(2)} \subseteq \mathbf{A}^8(x_{ij}, y_{ij})$ has four components

$$X_R^{(2)} = Y \cup \mathcal{A} \cup \mathcal{B} \cup Z$$

of dimensions 5,4,4 and 4 respectively and sits inside the quadratic hypersurface $V(2x_{11}y_{11} + x_{12}y_{21} + x_{21}y_{12})$ with additional equations:

$$\begin{aligned} Y &= V(t_A, t_B) \quad \mathcal{A} = V(A) \quad \mathcal{B} = V(B) \\ Z &= V(d_A, d_B, x_{11}y_{11} - x_{22}y_{22}, t_A y_{12} + x_{12}t_B, t_A y_{21} + x_{21}t_B) \end{aligned}$$

Here we have used the notation $t_A = \text{Tr}(A)$ and $d_A = \det(A)$. The irreducible representations are parametrized by an open set of $Y - Z$.

The set $\mathcal{A} \subseteq \mathbf{A}^4$ has a discriminant $\mathcal{D} \subseteq \mathcal{A}$ given by the equation $\mathcal{D} = V((t_B)^2 - 4d_B)$. Outside of this set the R -module V splits, the discriminant parametrizes the indecomposable modules.

Dividing out by the automorphism group we get for each component a non-algebraic set. The sets \mathcal{A} and \mathcal{B} give rise to copies of the affine plane with an extra diagonal embedded, according to the Jordan normal form. The quotient of Z consists of pairs of points $(X_R)^2$ and all the irreducible modules sit as a two-dimensional torus inside Y .

2.5. For any pair $(\phi_1, \phi_2) \in (X_R)^2$ the fiber $pr_*^{-1}(\phi_1, \phi_2)$ is a subvariety of $\mathcal{T}_2(R)$. The automorphism group $k^* \times k$ of $\mathcal{T}_2(k)$ acts on $\mathcal{T}_2(R)$ fixing (ϕ_1, ϕ_2) . Consequently it acts on the fiber $pr_*^{-1}(\phi_1, \phi_2)$.

PROPOSITION (2.6). *Let $X(\phi_1, \phi_2) = pr_*^{-1}(\phi_1, \phi_2)$ be the fiber over (ϕ_1, ϕ_2) . Then we have*

$$X(\phi_1, \phi_2)/\text{Aut}(\mathcal{T}_2(k)) \simeq \text{Ext}_R^1(\phi_1, \phi_2)/k^*$$

Proof. Let $\Phi \in pr_*^{-1}(\phi_1, \phi_2)$ be given by

$$\Phi(r) = \begin{pmatrix} \phi_1(r) & \beta(r) \\ 0 & \phi_2(r) \end{pmatrix}$$

where the ring homomorphism condition $\Phi(rr') = \Phi(r)\Phi(r')$ forces $\beta : R \rightarrow k$ to be a derivation $\beta : R \rightarrow \text{Hom}(\phi_2, \phi_1)$ given by

$$\beta(rr') = \phi_1(r)\beta(r') + \beta(r)\phi_2(r')$$

The set $\text{Hom}(\phi_2, \phi_1)$ has an $\phi_1 - \phi_2$ R -bimodule structure. An automorphism (q, p) fixes ϕ_i and the action on β is given by

$$\beta \mapsto q^2\beta + pq(\phi_2 - \phi_1)$$

The map $p(\phi_2 - \phi_1)$ is an inner derivation and q gives a scaling. Thus Φ corresponds, up to automorphisms to an element of $\text{Ext}_R^1(\phi_1, \phi_2)/k^*$ and vice versa. \square

2.7. For any pair $(\phi_1, \phi_2) \in (X_R)^2$ of k -points we define the **order of interaction** as

$$d(\phi_1, \phi_2) = \dim_k(\text{Ext}_R^1(\phi_1, \phi_2))$$

The order of self-interaction is denoted by $d(\phi) = d(\phi, \phi)$. The embedding dimension of R is defined as the embedding dimension of R_{ab} , i.e $e(R) = e(R_{ab})$.

In the paper [CQ] J. Cuntz and D. Quillen define quasi-free (or formally smooth) algebras. An algebra R is **quasi-free** iff it satisfies the lifting property for nilpotent extensions, i.e. for any algebra B , a two-sided nilpotent ideal $I \subset B$, and for any algebra homomorphism $f : R \rightarrow B/I$, there exist a lifting $\tilde{f} : R \rightarrow B$. There are some easily proved consequences of this definition.

THEOREM (2.8). *If R is a quasi-free algebra, then R is hereditary, i.e. the category $R\text{mod}$ has homological dimension ≤ 1 .*

Proof. [CQ] \square

An equivalent formulation of this theorem is that if R is quasi-free, then every R -module admits a projective resolution of length 2.

PROPOSITION (2.9). *Let R be a quasi-free algebra. Then*

$$d(\phi_1, \phi_2) = \begin{cases} e(\phi_1) - 1 & \text{if } \phi_1 \neq \phi_2 \\ e(\phi_1) & \text{if } \phi_1 = \phi_2 \end{cases}$$

Proof. Since R is quasi-free it is hereditary. The complex

$$R^e \longrightarrow R \longrightarrow_{(\phi_1)} k$$

where $e = e(\phi_1)$ is therefore a resolution of ϕ_1 . Applying the functor $\text{Hom}_R(-, \phi_2)$ to this resolution gives the result. \square

2.10. Proposition 2.6. is easily generalized to the situation where the modules ϕ_i , $i = 1, 2$ are no more one-dimensional k -vector spaces. So let V, W be left R -modules. Then the ext-group $\text{Ext}_R^1(V, W)$ is given by the set of k -algebra homomorphisms

$$R \longrightarrow \begin{pmatrix} \text{End}_k(V) & \text{Hom}_k(V, W) \\ 0 & \text{End}_k(W) \end{pmatrix}$$

up to some automorphisms. Now suppose we have given to extensions

$$R \longrightarrow \begin{pmatrix} \text{End}_k(V) & \text{Hom}_k(V, W) \\ 0 & \text{End}_k(W) \end{pmatrix} \quad \text{and} \quad R \longrightarrow \begin{pmatrix} \text{End}_k(W) & \text{Hom}_k(W, Z) \\ 0 & \text{End}_k(Z) \end{pmatrix}$$

The existence of a k -algebra homomorphism

$$R \longrightarrow \begin{pmatrix} \text{End}_k(V) & \text{Hom}_k(V, W) & \text{Hom}_k(V, Z) \\ 0 & \text{End}_k(W) & \text{Hom}_k(W, Z) \\ 0 & 0 & \text{End}_k(Z) \end{pmatrix}$$

corresponds to vanishing of a certain element in $\text{Ext}_R^2(V, Z)$. Thus the geometrical interpretation of the group $\text{Ext}_R^2(V, Z)$ is closely connected to incusions of the geometry of the quiver algebra of the quiver

$$\bullet \rightarrow \bullet \rightarrow \bullet$$

2.11. Given a (non-commutative) polynomial F in the associative k -algebra $R = k\langle x_1, \dots, x_n \rangle$ and a closed point $\mathbf{p} = (a_1, \dots, a_n) \in k^n$, such that $F(\mathbf{p}) = 0$. Let f_1, \dots, f_n be polynomials of degree $d^\circ f_i \leq d^\circ F - 1$ such that

$$f_1(x_1 - a_1) + \dots + f_n(x_n - a_n) = F$$

By a constuction type argument, starting from the highest degree terms of F and using reversed induction and finally using the fact that $F(\mathbf{p}) = 0$ we see that such polynomials always exist. In the commutative case the right factors are easily calculated by Taylor series expansion at the point \mathbf{p} , but in general this does not work.

DEFINITION (2.12.). *The matrix $(f_1 \dots f_n)$, as defined above is called the **left factor** of F at $\mathbf{p} = (a_1, \dots, a_n) \in k^n$ and denoted $j_{\mathbf{p}}(F)$.*

The left factor $j_{\mathbf{p}}(F)$ has obviously rank 1 in the ring $R = k\langle x_1, \dots, x_n \rangle$. We are interested in the rank of the left factor evaluated in some point ϕ , i.e. the rank of the matrix $j_{\mathbf{p}}(F)_{\phi}$.

DEFINITION (2.13.). *Let F and \mathbf{p} be as above. We define the **left shadow** $S(\mathbf{p}, F)$ of the point \mathbf{p} with respect to F as the set*

$$S(\mathbf{p}, F) = \{\mathbf{q} \in V(F) \mid \text{rank } j_{\mathbf{p}}(F)_{\mathbf{q}} = 0\}$$

With some modifications in the definitions we see that in the commutative case $S(\mathbf{p}, F) = \{\mathbf{p}\}$ if \mathbf{p} is a singularity for F . In a regular point the shadow is the empty set. Notice also that if $F = 0$ then $S(\mathbf{p}, 0) = \mathbf{A}^2$.

REMARK (2.14.). There is of course a counterpart to the above text, substituting left by right. In the next chapter we shall see that in some special cases there are some nice connections between left and right shadows.

3. Geometry @₂ for conic sections

3.1. Let R be a non-commutative conic section, i.e.

$$R = k\langle x, y \rangle / (ax^2 + bxy + cyx + dy^2 + ex + fy + g)$$

The quadratic part of the equation can be written

$$\begin{aligned} K(x, y) &= (x \ y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x \ y) \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (x \ y) \begin{pmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

A linear shift of basis diagonalizes the first matrix and keeps the second one. Thus we can write the quadratic part as

$$K(x, y) = \lambda x^2 + \gamma y^2 + \delta[x, y]$$

So a general noncommutative plane conic section has the form

$$R = k\langle x, y \rangle / (\lambda x^2 + \gamma y^2 + \delta[x, y] + ex + fy + g)$$

Let $\phi : R \rightarrow M_2(k)$ given by $\phi(x) = A$, $\phi(y) = B$. Then we have the equation

$$\delta[A, B] + (\lambda t_A + e)A + (\gamma t_B + f)B - (\lambda d_A + \gamma d_B - g)I = 0$$

where we use the notation t_M for the trace and d_M for the determinant of M .

PROPOSITION (3.2.). *The k -algebra $R = k\langle x, y \rangle / I$ as given above ($I \neq 0$) has irreducible 2-dimensional representations if and only if the following conditions are satisfied.*

- i) $\delta = 0$
- ii) At least one of $\lambda, \gamma \neq 0$
- iii) If $\lambda = 0$ then $e = 0$ and $\gamma = 0$ implies $f = 0$.

Proof. Having 2-dimensional irreducible representations is equivalent to the existence of a surjective $\phi : R \rightarrow M_2(k)$ given by $\phi(x) = A$, $\phi(y) = B$ i.e. linear independence of the set

$$\{I, A, B, [A, B]\}$$

If $\delta \neq 0$ this set is obviously dependent. If $\delta = 0$, choose some M and N such that $\det[M, N] \neq 0$ and such that $M^2, N^2 \neq 0$. Then $\det[uM, N] \neq 0$ for $u \neq 0$ and $\det[M + vI, N] \neq 0$ for any $v \in k$. Thus there exist a solution $(A, B) \in M_2(k)^2$ with $\det[A, B] \neq 0$ and such that

$$\lambda t_A + e = \gamma t_B + f = \lambda d_A + \gamma d_B - g = 0$$

thereby satisfying the defining equation for R . \square

THEOREM (3.3.). *Up to isomorphism the following list gives a complete classification of all noncommutative plane conic sections:*

- 1) The pure quadratic ones $x^2 + y^2 + g + \delta[x, y]$ where either $g = 1$ (smooth case) or $g = 0$ (ordinary double point/quantum plane).
- 2) The parabolic case $x^2 + y + g + \delta[x, y]$
- 3) Two parallel lines $x^2 + g + \delta[x, y]$, including the degenerate case $g = 0$ (a double line)
- 4) A simple line $x + \delta[x, y]$
- 5) The first Weyl algebra $1 + \delta[x, y]$
- 6) The affine plane $[x, y]$

Proof. Just take care of all possibilities for the coefficients $\lambda, \gamma, \delta, e, f, g$. \square

3.4. Let $\phi = (a, b) \in k^2$ be some k -point of R , i.e.

$$\lambda a^2 + \gamma b^2 + ea + fb + g = 0$$

Then we have

$$(\lambda x - \delta y + e + \lambda a + \delta b)(x - a) + (\delta x + \gamma y + f - \delta a + \gamma b)(y - b) = \lambda x^2 + \gamma y^2 + \delta[x, y] + ex + fy + g$$

Consider the system of linear equations

$$\begin{aligned} \lambda x - \delta y + e + \lambda a + \delta b &= 0 \\ \delta x + \gamma y + f - \delta a + \gamma b &= 0 \end{aligned}$$

A solution ψ of this system is called a **left shadow** of ϕ in R . (see 2.13 for a more general definition) Using the notation from the previous chapter we see that in this situation ψ is a left shadow of ϕ if and only if $d(\phi, \psi) = 1 + \delta_{\phi, \psi}$. Notice that in the Weyl algebra case there are no k -points.

PROPOSITION (3.5.). *Assume that $\lambda\gamma + \delta^2 \neq 0$ and that R is not the first Weyl algebra. Then every point (a, b) in R has a unique left shadow $l(a, b)$. Furthermore we have $l(a, b) = (a, b)$ if and only if one of the following is true:*

- i) R is the affine plane
- ii) The point (a, b) is the origin of the quantum plane
- iii) The point (a, b) lies on the double line $x^2 + \delta[x, y]$.

Proof. The linear system defining $l(a, b)$ has determinant $\lambda\gamma + \delta^2$, proving the first statement of the proposition. The solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\lambda\gamma + \delta^2} \left(- \begin{pmatrix} \gamma & \delta \\ -\delta & \lambda \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} - \begin{pmatrix} \lambda\gamma - \delta^2 & 2\gamma\delta \\ -2\lambda\delta & \lambda\gamma - \delta^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right)$$

This gives (after some computations)

$$\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} = \frac{-1}{\lambda\gamma + \delta^2} \begin{pmatrix} \gamma & \delta \\ -\delta & \lambda \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} + 2 \begin{pmatrix} \lambda & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Thus $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ if and only if

$$e + 2\lambda a = f + 2\gamma b = 0$$

If $\lambda \neq 0$ we assume in our classification that $e = 0$ giving $a = 0$ as the only solution. Similarly for $\gamma \neq 0$. For $\gamma = 0$ we have infinitely many solutions, preassumed that $f = 0$. \square

3.6. If $\lambda\gamma + \delta^2 = 0$ we have a quite different situation. Then there are either no left shadows at all or $l(a, b) \simeq \mathbf{A}^1$. There are two cases where there are no solution at all, using an appropriate isomorphism they can be written in the form $xy - 1$ (correspond to $\lambda = \gamma = 1$ and $\delta = \pm i$ in the smooth case) and $y - x^2 - 1$. On the other hand, there is one case of infinite shadow, corresponding to the quantum plane $xy + qyx$ with $q = 0$. Then $l(0, b) = \mathbf{A}^1(y = 0)$ and $l(a, 0) = \mathbf{A}^1(x = 0)$.

3.7. Assume R belongs to case 1) in the classification 3.3, i.e. $\lambda, \gamma \neq 0$ and move the origin such that the linear terms vanish. A scaling of x and y allows us to assume $\lambda = \gamma = 1$. Then there are two possibilities, $x^2 + y^2 - 1 + \delta[x, y]$ or $x^2 + y^2 + \delta[x, y]$. Assume $\delta \neq 0$ and consider the inhomogenous case ($c = 1$). Then there are no irreducible 2-dimensional representations. The closed points of R are obtained by considering the quotient of R in the polynomial ring $k[x, y]$, i.e. let $[x, y] = 0$. Thus the closed points are given by the classical conic section $Q : x^2 + y^2 - 1 = 0$.

Pick a point $(1, 0) \in Q$ and consider the left R -module ${}_{\phi}k$ given by $\phi(x) = 1$, $\phi(y) = 0$. A resolution of ${}_{\phi}k$ as a left R -module is given by

$$0 \longrightarrow R \xrightarrow{M_1} R^2 \xrightarrow{M_0} R \longrightarrow {}_{\phi}k \longrightarrow 0$$

where

$$M_1 = \begin{pmatrix} x - \delta y + 1 & \delta x + y - \delta \end{pmatrix} \quad \text{and} \quad M_0 = \begin{pmatrix} x - 1 \\ y \end{pmatrix}$$

Apply the contravariant functor $\text{Hom}_{k\text{-alg}}(-, {}_{\psi}k)$ and compute cohomology to obtain the following result:

$$\text{Ext}_R^1({}_{\phi}k, {}_{\psi}k) = \begin{cases} k & \text{if } \phi = \psi \\ k & \text{if } \psi = (\cos v, \sin v) \text{ where } v = \arctg \frac{2\delta}{1-\delta^2} \\ 0 & \text{elsewhere} \end{cases}$$

The resolution above is constructed from the "factorization"

$$x^2 + y^2 - 1 + \delta[x, y] = (x - \delta y + 1)(x - 1) + (\delta x + y - \delta)y$$

Thus the left shadow is rotation by an angle v in the (x, y) -plane.

The homogenous case $x^2 + y^2 + \delta[x, y]$ is equivalent to the quantum plane $xy + qyx$ via a shift of basis, setting $q = \frac{1+\delta}{1-\delta}$. In this case the left shadow of the point (a, b) is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\lambda\gamma + \delta^2} \begin{pmatrix} 1 - \delta^2 & 2\delta \\ -2\delta & 1 - \delta^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

or if we put $\cos u = \frac{-1+\delta^2}{1+\delta^2}$, $\sin u = \frac{-2\delta}{1+\delta^2}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\cos u & -\sin u \\ \sin u & -\cos u \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Thus once again the left shadow is a rotation by an angle u in the complex 2-plane. Notice also that $\delta \rightarrow 0$ gives $u \rightarrow \pi$ and $\delta \rightarrow \infty$ corresponds to $u \rightarrow 0$.

3.8. It is also somewhat remarkable that for the double line $l(a, b) = (a, b)$, but for the simple line $x + \delta[x, y]$ we have $l(0, b) = (0, b + \frac{1}{\delta^2})$. For the case of two parallel lines $x^2 - 1 + \delta[x, y]$ we have $l(1, b) = (1, b + \frac{2}{\delta^2})$.

3.9. Notice that we have choosen to work with left structures. We could of course as well have choosen right structures. In the plane conic section case there is a nice connection between left and right shadows. Suppose that every point (a, b) has unique left and right shadows (e.g. in case $\lambda\gamma + \delta^2 \neq 0$). Use the notation $r(a, b)$ for the right-shadow of (a, b) .

LEMMA (3.10.) *Let R be a non-commutative plane conic section and suppose every point (a, b) has unique left and right shadows. Then l and r are inverse constructions, i.e. $l(r(a, b)) = (a, b)$ and vice versa.*

Proof. Let $R = k\langle x, y \rangle / (F)$ and suppose $F(a, b) = 0$. Let

$$f(x - a) + g(y - b) = F$$

be the left factorization of F in (a, b) . Then $(c, d) = l(a, b)$ is the unique common zero of f and g . Thus we can write

$$f = \lambda_1(x - c) + \lambda_2(y - d), \quad g = \gamma_1(x - c) + \gamma_2(y - d)$$

for some $\lambda_1, \lambda_2, \gamma_1, \gamma_2 \in k$. But then we have

$$(x - c)f' + (y - d)g' = F$$

where we can give f', g' explicetly and their commen zero is (a, b) . \square

3.11. In general the picture is slightly more complex. For a polynomial F and a zero point \mathbf{p} of F , \mathbf{p} is a right shadow for all its left shadows, but there is of course no uniqueness.

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