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Classification of filiform solvable Lie algebras

by

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CLASSIFICATION OF FILIFORM SOLVABLE LIE ALGEBRAS.

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SUMMARY. Let $\mathfrak{g}$ be a real, completely solvable Lie algebra whose nilradical is filiform and of positive codimension in $\mathfrak{g}$. The family of all such Lie algebras is classified up to isomorphisms. Rigidity properties follow.

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1. Preliminaries.

1.1. We denote by $Lie_n(F)$ the variety of all $n-$dimensional Lie algebra structures over a field F. In recent years, a considerable amount of work has been devoted to the study of the structure of, as well as the classification of this variety; particularly for small dimensions, [CD], [KN], [N]. We present here a somewhat different point of view by classifying inductively (up to isomorphisms), with respect to dimension, the family FS_∞ of all completely solvable Lie algebras whose nilradical is filiform and of positive codimension, Def. 1.10. As an application, rigidity properties of the Lie algebras in FS_∞ are derived, Thm. 3 (see [C2] and [AG] for previous results in this direction). Many of the resulting families form projective systems. One particular example is the prosolvable algebra of polynomial vectorfields on the line, defined by the basis relations $[e_j, e_k] = (k-j)e_{j+k} \quad (j, k = 0, 1, 2, \dots)$. In our setting, this algebra is isomorphic to the projective limit of the family $\{\mathfrak{f}_{5,k}\}_k$ (with a suitable choice of parameters), see Ex. 4.6. This is just the first in a countable series of similar projective systems, to be denoted by $\{\mathfrak{f}_{n,k}\}_k$ $(n = 5, 6, ...)$, Thm. 2 and §§6, 7.
Our classification is complete up to the exact computation of the algebraic parameter domains $S_{n,k}$ and $S^1_{n,k}$ of the two series of Lie algebras $\mathfrak{f}_{n,k}$, resp. $\mathfrak{f}^1_{n,k}$ (cf. Prop. 4.5). We expect these domains to be finite for $k \geq 2n-2$.
Roughly speaking, our method consists in calculating orbits in the second cohomology space of a Lie algebra under its automorphism group. An adaption to the context of filiform extensions is outlined below.

1.2. Let $\mathfrak{g}$ and $\mathfrak{a}$ be Lie algebras over any field of characteristic 0. We say that $\tilde{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ by $\mathfrak{a}$ if $\mathfrak{a}$ is an ideal of $\tilde{\mathfrak{g}}$ and the factor Lie algebra $\tilde{\mathfrak{g}}/\mathfrak{a}$ is isomorphic to $\mathfrak{g}$, i.e., we have a short exact sequence $0 \to \mathfrak{a} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$. In other words, "extension" will usually stand for "left extension." In our context this is natural, partly because the class FS_∞ is closed under *admissible* left extensions (Def. 1.3). On the other hand, we were led not to consider "right" extensions of nilpotent Lie algebras by abelian algebras of derivations, as they would seem to require a rather detailed knowledge of the vast class of filiform nilpotent Lie algebras. It might be a somewhat surprising fact that FS_∞ is small enough to admit a reasonably simple classification. Of course, the filiform nilpotent Lie algebras admitting external derivations with only real eigenvalues, are exactly the nilradicals of the algebras in the class FS_∞. Now assume $\mathfrak{g}$ is solvable and

$\mathfrak{a}$ is abelian. Further, let $\mathfrak{n}$ be a fixed nilpotent ideal of $\mathfrak{g}$ containing the commutator subalgebra $[\mathfrak{g},\mathfrak{g}]$. If $\theta : \mathfrak{g} \to End\,\mathfrak{a}$ is a Lie representation we may consider the second cohomology space $H^2(\mathfrak{g},\theta)$ of $\mathfrak{g}$ with coefficients in θ, [CE], which we identify to a certain quotient of the linear space $C^2(\mathfrak{g},\theta)$ consisting of all alternating bilinear maps $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ satisfying the cocycle identity,

$$\partial_\theta B(x,y,z) = \sum (B(x,[y,z]) + \theta(x)B(y,z)) = 0 \quad (x,y,z \in \mathfrak{g}). \qquad (1.1)$$

Here the sum is with respect to cyclic permutation of the ordered triple $< x,y,z >$. The space $H^2(\mathfrak{g},\theta)$ is obtained by factoring out the space of all exact bilinear maps, i.e. bilinear maps of the form $\partial_\theta f$, where $f : \mathfrak{g} \to \mathfrak{a}$ is linear and

$$\partial_\theta f(x,y) = f[x,y] - \theta(x)f(y) + \theta(y)f(x) \quad (x,y \in \mathfrak{g}). \qquad (1.2)$$

We write $B_1 \sim B_2$ if $B_1 - B_2 = \partial_\theta f$, i.e., if B_1 is cohomologous to B_2. As in [S1,2] we let $H^2(\mathfrak{g},\mathfrak{g}/\mathfrak{n},\mathfrak{a}) = \bigcup_\theta H^2(\mathfrak{g},\theta)$ where the union is taken over the family of all representations θ of $\mathfrak{g}$ in $\mathfrak{a}$ enjoying the two additional properties,

$$Ker\,\theta \supseteq \mathfrak{n} \qquad (1.3)$$

and

For all x in the nilradical of $\mathfrak{g}$, $\theta(x)$ is a nilpotent endomorphism (if and) only if x lies in the ideal $\mathfrak{n}$. (1.4)

There is a canonical action of $Aut\,\mathfrak{g} \times Aut\,\mathfrak{a}$ in $\bigcup_\theta H^2(\mathfrak{g},\theta)$ (θ an arbitrary Lie representation of $\mathfrak{g}$ in $\mathfrak{a}$) given by

$$(\alpha,\psi,B) \mapsto \psi \circ B \circ \alpha \quad (\alpha \in Aut\,\mathfrak{g},\ \psi \in Aut\,\mathfrak{a},\ B \in \bigcup_\theta H^2(\mathfrak{g},\theta))$$

Here $B\circ\alpha : (x,y) \mapsto B(\alpha x, \alpha y) \quad (x,y \in \mathfrak{g})$.
If $B \in H^2(\mathfrak{g},\theta)$, let B^0 denote its restriction to $\mathfrak{n}$. We let $\mathfrak{n}(B) = \{x \in \mathfrak{n} : B(x,\mathfrak{n}) = (0)\}$ be the orthogonal complement of $\mathfrak{n}$ w.r.t. B^0. Alternatively, $\mathfrak{n}(B)$ is the radical of the form B^0. The zero space of the representation θ is denoted by $Z(\theta)$, thus $Z(\theta) = \{a \in \mathfrak{a} : \theta(x)a = 0,\ \text{all } x \in \mathfrak{g}\}$.

1.3. DEFINITION. *Let $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ be solvable Lie algebras, and assume $\tilde{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ by an abelian Lie Algebra $\mathfrak{a}$. Let $\tilde{\mathfrak{n}}$ denote the nilradical of $\tilde{\mathfrak{g}}$, and suppose $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{g}$ containing the commutator subalgebra $[\mathfrak{g},\mathfrak{g}]$. We say that the extension $\tilde{\mathfrak{g}}$ is $\mathfrak{n}$-admissible if the following conditions are satisfied*

(i) $\tilde{\mathfrak{g}}$ *contains no non-zero abelian direct factor.*
(ii) The quotient algebra $\tilde{\mathfrak{n}}/\mathfrak{a}$ *is isomorphic to* $\mathfrak{n}$.
(iii) $\mathfrak{a}$ *is equal to the center of* $\tilde{\mathfrak{n}}$.

In case $\mathfrak{n}$ is equal to the nilradical of $\mathfrak{g}$, we simply say that the extension $\tilde{\mathfrak{g}}$ is admissible.

A 2−cocycle on $\mathfrak{g}$ defining an ($\mathfrak{n}$−)admissible extension, is said to be ($\mathfrak{n}$−)admissible. We are now ready to restate [S1, Theorem 3.8].[1]

1.4. THEOREM. *Let $\mathfrak{g}$ and $\mathfrak{a}$ be solvable Lie algebras over a field of characteristic 0, $\mathfrak{a}$ abelian. Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$. The isomorphism classes of all $\mathfrak{n}$-admissible extensions $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{a}$ are in bijective correspondence with the family of all $Aut\,\mathfrak{g} \times Aut\,\mathfrak{a}$ orbits Ω in $\bigcup_\theta H^2(\mathfrak{g},\theta)$ which satisfy the following conditions,*

(i) $\Omega \cap H^2(\mathfrak{g},\mathfrak{g}/\mathfrak{n},\mathfrak{a})$ *is nonempty.*
(ii) If $B \in \Omega \cap H^2(\mathfrak{g},\theta)$ where θ satisfies (1.3) and (1.4) then $\mathfrak{a}$ can not be written as a direct sum of subspaces, $\mathfrak{a} = U \oplus V$, in which U is θ-invariant and contains the range space $B(\mathfrak{g},\mathfrak{g})$ of B and, in addition, $(0) \neq V \subseteq Z(\theta)$.
(iii) $\mathfrak{n}(B) \cap \mathfrak{z} = (0)$, *where $\mathfrak{z}$ denotes the center of $\mathfrak{n}$.*

1.5. We shall comment briefly on the various hypothesis of the above theorem. First, if $\tilde{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ by $\mathfrak{a}$ determined by a representation θ of $\mathfrak{g}$ in $\mathfrak{a}$ and a bilinear B in $C^2(\mathfrak{g},\theta)$, then $\tilde{\mathfrak{g}}$ may be realized on the vector space $\mathfrak{g}\oplus\mathfrak{a}$ with Lie product

$$[(g,a),(g',a')] = ([g,g'],\theta(g)a' - \theta(g')a + B(g,g')) \quad (g,g' \in \mathfrak{g},\ a,a' \in \mathfrak{a}), \tag{1.5}$$

in which $[g,g']$ denotes the Lie product of g and g' in $\mathfrak{g}$. Therefore

$$(0,\theta(n)a) = [(n,b),(0,a)] \quad (n \in \mathfrak{n},\ a,b \in \mathfrak{a}),$$

[1] Correction: Read " *Aut* $\mathfrak{g} \times$ *Aut* $\mathfrak{a}$ − *orbits* Ω *with* $H^2(\mathfrak{g},\mathfrak{g}/\mathfrak{n},\mathfrak{a}) \cap \Omega \neq \emptyset$", instead of " *Aut* $\mathfrak{g} \times$ *Aut* $\mathfrak{a}$ − *orbits* Ω *in* $H^2(\mathfrak{g},\mathfrak{g}/\mathfrak{n},\mathfrak{a})$" in [S1, Proposition 3.5 and Theorem 3.8]

from which it follows that (1.3) $Ker\,\theta \supseteq \mathfrak{n}$ holds if and only if $\mathfrak{a}$ is central in $\tilde{\mathfrak{n}}$. Hence (1.3) means $\tilde{\mathfrak{n}}$ is a central extension of $\mathfrak{n}$ by $\mathfrak{a}$ whenever $\tilde{\mathfrak{n}}/\mathfrak{a}$ is isomorphic to $\mathfrak{n}$, in particular this holds for $\mathfrak{n}$-admissible extensions $\tilde{\mathfrak{g}}$. Let us show that statement (1.4) is equivalent to $\tilde{\mathfrak{n}}/\mathfrak{a} \cong \mathfrak{n}$. In fact, assuming (1.4), if $(x,a) \in \tilde{\mathfrak{g}}$ then $(x,a) \in \tilde{\mathfrak{n}}$ if and only if $ad(x,a)$ is nilpotent, that is $\theta(x)$ is nilpotent (by eq. (1.5)), which is equivalent to $x \in \mathfrak{n}$ by (1.4). This means $\tilde{\mathfrak{n}}/\mathfrak{a} \cong \mathfrak{n}$. Conversely, if $\tilde{\mathfrak{n}}/\mathfrak{a} \ncong \mathfrak{n}$, assume x lies in the nilradical of $\mathfrak{g}$. Then $\theta(x)$ is nilpotent if and only if $ad(x,a)$ is nilpotent (all $a \in \mathfrak{a}$), that is $(x,a) \in \tilde{\mathfrak{n}}$ $(a \in \mathfrak{a})$, or equivalently, $x \in \mathfrak{n}$ because $\tilde{\mathfrak{n}}/\mathfrak{a} \cong \mathfrak{n}$. We have shown

1.6. LEMMA. *Let the notation be as above. If $B \in H^2(\mathfrak{g},\theta)$ where $\theta : \mathfrak{g} \to End\,\mathfrak{a}$ is a representation of $\mathfrak{g}$ in the abelian Lie algebra $\mathfrak{a}$, then the following are equivalent.*

(*a*) $B \in H^2(\mathfrak{g}/\mathfrak{n},\mathfrak{a})$

(*b*) *The nilradical $\tilde{\mathfrak{n}}$ of the extension $\tilde{\mathfrak{g}} = \mathfrak{g}(B,\theta)$ of $\mathfrak{g}$ by $\mathfrak{a}$ determined by θ and B, is a central extension of $\mathfrak{n}$ by $\mathfrak{a}$.*

Now, if the center $\tilde{\mathfrak{z}}$ of the nilradical $\tilde{\mathfrak{n}}$ of $\tilde{\mathfrak{g}}$ is not equal to $\mathfrak{a}$, then we may regard $\tilde{\mathfrak{g}}$ as an extension of $\tilde{\mathfrak{g}}/\tilde{\mathfrak{z}}$ by $\tilde{\mathfrak{z}}$ where $\tilde{\mathfrak{g}}/\tilde{\mathfrak{z}}$ is isomorphic to a factor Lie algebra of $\mathfrak{g}$. Hence we should focus on extensions $\tilde{\mathfrak{g}}$ for which $\tilde{\mathfrak{z}}$ is equal to $\mathfrak{a}$, which is easily seen to be equivalent with the condition $\mathfrak{n}(B) \cap \mathfrak{z} = (0)$ of Theorem 1.4. Let us show that condition (2) of Theorem 1.4 is equivalent to Def. 1.3(*i*). In fact, suppose D is an abelian direct factor in the extension $\tilde{\mathfrak{g}}$, $D \neq (0)$. Clearly, $V \subseteq Z(\theta)$ otherwise we could find $v \in V$, $x \in \mathfrak{g}$ such that $\theta(x)v \neq 0$ which gives $[(x,0),(0,v)] = (0,\theta(x)v) \neq 0$, contradicting that V is central in $\tilde{\mathfrak{g}}$. Further, we must have $B(\mathfrak{g},\mathfrak{g}) \cap V = (0)$ because $\tilde{\mathfrak{g}}$ may be written as a direct sum $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_1 \oplus V$ where $\tilde{\mathfrak{g}}_1$ is an ideal. In particular, $\tilde{\mathfrak{g}}_1 \supseteq \{(x,0) : x \in \mathfrak{g}\}$ so that $[(x,0),(y,0)] = ([x,y],B(x,y))$ $(x,y \in \mathfrak{g})$ which enforces $B(x,y) \notin V$ whenever $B(x,y) \neq 0$.
Thus the sum $B(\mathfrak{g},\mathfrak{g})+V$ is direct. Let U denote a subspace of $\tilde{\mathfrak{g}}_1$ satisfying $U \supseteq B(\mathfrak{g},\mathfrak{g})$ and $U \oplus V = \mathfrak{a}$. We show that $\theta(\mathfrak{g})U \subseteq U$. If, on the contrary, $x \in \mathfrak{g}$ and $u \in U$ satisfy $\theta(x)u \notin U$, then $[(x,0),(0,u)] = (0,\theta(x)u) \notin \tilde{\mathfrak{g}}_1$ which is impossible because $\tilde{\mathfrak{g}}_1 = \{(x,u) : x \in \mathfrak{g}, u \in U\}$ is a subalgebra. The above observations are summarized in

1.7. LEMMA. *Let $\tilde{\mathfrak{g}}$ be an extension of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ determined by a representation $\theta : \mathfrak{g} \to End\,\mathfrak{a}$ together with* *a cocycle B in $H^2(\mathfrak{g},\theta)$. The following conditions are equivalent.*

(i) $\tilde{\mathfrak{g}}$ *contains a nonzero abelian direct factor.*

(ii) $\mathfrak{a}$ *can be written as a direct sum* $\mathfrak{a}=U\oplus V$ *of nonzero subspaces* U *and* V *satisfying* $B(\mathfrak{g},\mathfrak{g})\subseteq U$, $\theta(\mathfrak{g})U\subseteq U$, *and* $V\subseteq Z(\theta)$.

1.8. For convenience we restate Corollary 2.6 of [S1], it will be used over and over again below.

PROPOSITION. *Let* $\mathfrak{g}$ *be a solvable Lie algebra,* $B\in C^2(\mathfrak{g},\theta)$, $\theta:\mathfrak{g}\to End\,\mathfrak{a}$. *Assume* $\mathfrak{a}$ *is the center of the nilradical of the extended Lie algebra* $\mathfrak{g}(B,\theta)$. *Then the automorphism group of* $\mathfrak{g}(B,\theta)$ *is isomorphic to the group of all matrices* $\begin{pmatrix}0&0\\ \phi&\psi\end{pmatrix}$, *where* $\alpha_0\in Aut\,\mathfrak{g}$, $\phi\in Hom(\mathfrak{g},\mathfrak{a})$, $\psi\in Aut\,\mathfrak{a}$, *and*

$$\begin{aligned}&B\circ\alpha_0=\psi\circ B+\partial_\theta\phi, \qquad \partial_\theta\phi\in B^2(\mathfrak{g},\theta)\\ &\psi\theta\psi^{-1}=\theta\circ\alpha_0\end{aligned}\tag{1.6}$$

1.9. NOTATION. Let $\mathfrak{g}$ be any Lie algebra. We fix a basis $E_n=<e_1,e_2,\cdots,e_n>$ for $\mathfrak{g}$. The dual space $\mathfrak{g}^*$ to $\mathfrak{g}$ consists of all real linear functionals of $\mathfrak{g}$. We denote by ω_i (or by e_i^*) the functional dual to e_i, and by $E_n^*=<\omega_i>_{i=1}^n$ the basis of $\mathfrak{g}^*$ dual to E_n. The *elementary* alternating 2−forms $B_{i,j}$ $(1\le i<j\le n)$ w.r.t. $E=E_n$ are defined as

$$B_{i,j}(x,y)=\omega_i\wedge\omega_j(x,y)=x_iy_j-x_jy_i \quad (x=\sum_{i=1}^n x_ie_i,\quad y=\sum_{i=1}^n y_ie_i\in\mathfrak{g}).$$

They constitute a basis for the space $\wedge_2(\mathfrak{g})$ of all skew-symmetric bilinear forms on $\mathfrak{g}$. The structure constants of $\mathfrak{g}$ are denoted by c_{kl}^j $(1\le j,k,l\le n)$, thus

$$[e_k,e_l]=\sum_{j=1}^n c_{kl}^j e_j,$$

and

$$[x,y]=\sum_{1\le k<l\le n} B_{k,l}(x,y)[e_k,e_l]=\sum_{1\le k<l\le n}\sum_{j=1}^n c_{kl}^jB_{k,l}(x,y)e_j,\ (x,y\in\mathfrak{g}).$$

It follows for $x,y,z\in\mathfrak{g}$ that

$$B_{i,j}(x,[y,z])=x_i\sum_{k<l}c_{kl}^jB_{k,l}(y,z)-x_j\sum_{k<l}c_{kl}^iB_{k,l}(y,z)$$

We put

$$G_{i,k,l}(x,y,z)=\sum_{<x,y,z>} x_i B_{k,l}(y,z) \quad (x,y,z\in\mathfrak{g}), \tag{1.7}$$

in which the sum is extended over cyclic permutation of the ordered set $<x,y,z>$. Hence

$$\begin{aligned} G_{i,k,l}(x,y,z) &= \sum_{<x,y,z>} x_i(y_k z_l - y_l z_k) \\ &= \begin{vmatrix} x_i & y_i & z_i \\ x_k & y_k & z_k \\ x_l & y_l & z_l \end{vmatrix} = \omega_i\wedge\omega_j\wedge\omega_l(x,y,z) \end{aligned} \tag{1.8}$$

Moreover, the form $B_{i,j}$ is a (central) $2-$cocycle on $\mathfrak{g}$ if and only if

$$\sum_{<x,y,z>} B_{i,j}(x,[y,z]) = \sum_{k<l}(c^j_{kl}G_{i,k,l} - c^i_{kl}G_{j,k,l})(x,y,z) = 0 \quad (\forall x,y,z\in\mathfrak{g}), \tag{1.9}$$

We see from eq. (1.8) that the set of all $G_{i,k,l}$ $(1\le i<k\le n)$ constitutes a basis for the space $\wedge_3(\mathfrak{g})$ of all alternating $3-$forms on $\mathfrak{g}$. Given a basis $<e_i>_{i=1}^n$ for $\mathfrak{g}$, all expansions of forms in $\wedge_3(\mathfrak{g})$, will be relative to the basis $<G_{i,k,l}>$.

1.10. Let $\mathfrak{g}$ be a Lie algebra. The lower (descending) central series $\{\mathcal{C}^i\mathfrak{g}\}$ of $\mathfrak{g}$ is defined as

$$\mathcal{C}^0\mathfrak{g}=\mathfrak{g},\ \mathcal{C}^i\mathfrak{g}=[\mathfrak{g},\mathcal{C}^{i-1}\mathfrak{g}] \quad (i\geqq 1).$$

Now if $\mathfrak{n}$ is nilpotent, let r be the smallest integer for which $\mathcal{C}^r\mathfrak{n}=(0)$. We say that a nilpotent Lie algebra $\mathfrak{n}$ is of type $\{p_1,p_2,\cdots,p_r\}$ if $\dim(\mathcal{C}^{i-1}\mathfrak{n}/\mathcal{C}^i\mathfrak{n}) = p_i$ $(1\le i\le r)$. The *filiform* (nilpotent) Lie algebras are the ones of type $\{2,1,1,\cdots,1\}$. We shall say that a solvable Lie algebra is filiform if its nilradical is filiform and has codimension at most one. More generally, we make the following

DEFINITION. *We say that $\mathfrak{g}$ belongs to the class FS_k if $\mathfrak{g}$ is completely solvable and its nilradical is filiform and has codimension k in $\mathfrak{g}$. The subset of FS_k consisting of all $n-$dimensional algebras is denoted by $FS_{k,n}$. Furthermore, we let $FS_\infty=\bigcup_{k=1}^\infty FS_k$.*

We remark that any $\mathfrak{g}\in FS_\infty$ contains an ideal $\mathfrak{l}\in FS_1$ whose factor algebra $\mathfrak{g}/\mathfrak{l}$ is abelian. The classification of FS_∞ can be reduced to that of FS_1 (§5). Consequently, we start with the family FS_1 (§§2-4).

1.11. For any $\mathfrak{g}$ in $FS_{1,n}$ possessing nilradical $\mathfrak{n}$, we may consider the descending ideal sequence

$$\mathfrak{g} \supseteq \mathfrak{n} \supseteq \mathcal{C}^1\mathfrak{n} \supseteq \cdots \supseteq \mathcal{C}^{n-3}\mathfrak{n} \supseteq \mathcal{C}^{n-2}\mathfrak{n} = (0) \tag{1.10}$$

Hence we can choose a basis $E_n =< e_i >_{i=1}^n$ such that $e_1 \in \mathfrak{g}\setminus\mathfrak{n}$, $e_2, e_3 \in \mathfrak{n}\setminus\mathcal{C}^1\mathfrak{n}$, and $e_{j+3} \in \mathcal{C}^j\mathfrak{n}\setminus\mathcal{C}^{j+1}\mathfrak{n}$ $(1 \leq j \leq n-3)$. Any such basis E_n will be referred to as canonical.

Now, assume $\mathfrak{g}$ is in $FS_{1,n}$ with canonical basis $E = E_n$ and, in addition, $\tilde{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ by $\mathfrak{a} = \mathbb{R}e_{n+1}$ determined by a character $\theta : \mathfrak{g} \to End\,\mathfrak{a}$, together with a cocycle B in $H^2(\mathfrak{g}, \theta)$. Under these circumstances $\theta = \theta_t$ is uniquely given by a real t such that

$$\theta(e_1)e_{n+1} = te_{n+1}. \tag{1.11}$$

We say that the extension $\tilde{\mathfrak{g}} = \mathfrak{g}(B, \theta_t)$ is defined by the data (B, θ_t) or simply by (B, t). Note that

$$\begin{aligned}\sum_{<x,y,z>} \theta(x)B_{i,j}(y,z) &= \sum_{<x,y,z>} tx_1 B_{i,j}(y,z) \\ &= t\omega_1 \wedge \omega_i \wedge \omega_j(x,y,z) \qquad (x,y,z \in \mathfrak{g}).\end{aligned} \tag{1.12}$$

Next, we define

$$\begin{aligned}\partial_t B_{i,j}(x,y,z) &= \partial_\theta B_{i,j}(x,y,z) \\ &= \sum_{<x,y,z>} (B_{i,j}(x,[y,z]) + \theta(x)B_{i,j}(y,z)) \quad (x,y,z \in \mathfrak{g}).\end{aligned} \tag{1.13}$$

In particular, $\partial_t B_{i,j} = 0$ if and only if $B_{i,j} \in C^2(\mathfrak{g}, \theta_t)$. We shall often write ∂B instead of ∂B_θ or ∂B_t.

Combining eq. (1.9) with eq. (1.12) we derive the identity

LEMMA.

$$\begin{aligned}\partial_t B_{i,j} =&(t - c_{1j}^j - c_{1i}^i)\omega_1 \wedge \omega_i \wedge \omega_j + \\ &\sum_{k<l,\,(k,l)\notin\{(1,i),(1,j)\}} (c_{kl}^j \omega_i \wedge \omega_k \wedge \omega_l - c_{kl}^i \omega_j \wedge \omega_k \wedge \omega_l),\end{aligned} \tag{1.14}$$

which will be used frequently in our subsequent calculations of cocycles.

2. Some basic results.

In what follows we shall apply Theorem 1.4 inductively to the class of all FS_1 over $\mathbb{R}$ (the classification is essentially the same over $\mathbb{C}$). Because of the uniform structure of such Lie algebras, several simplifications are possible. First of all, observe that no $\tilde{\mathfrak{g}}$ in FS_1 can be an extension of a filiform nilpotent $\mathfrak{g}$. In fact, if $\tilde{\mathfrak{g}}/\mathfrak{a} = \mathfrak{g}$ in which $\mathfrak{a}$ is abelian and $dim\, \mathfrak{g} > 1$, we have $\tilde{\mathfrak{n}} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ is nilpotent of codimension one in $\tilde{\mathfrak{g}}$. Consequently, $[\mathfrak{g}, \mathfrak{g}] \cong \tilde{\mathfrak{n}}/\mathfrak{a}$ has codimension one in $\mathfrak{g}$, and $\mathfrak{g}$ can not be nilpotent. We state this simple, but basic observation as,

2.1. PROPOSITION. *Any element in $FS_{1,n}$ of dimension greater than three is a one-dimensional extension of an element in $FS_{1,n-1}$.*

The above result shows that any $\mathfrak{g}$ in $FS_{1,n}$ can be obtained recursively from one of the three-dimensional Lie algebras listed below by performing repeatedly $n-3$ one-dimensional extensions. In view of this, Prop. 2.1 forms the basis for the classification of all such Lie algebras, as given in the following sections of this work.

2.2. Next, we list all elements in FS_1 of dimension three and four. As usual, we give a minimal number of nonzero Lie relations relative to a basis $< e_i >_{i=1}^{n}$ (those relations which follows by anti-symmetry are not listed). We shall find it convenient to denote these Lie algebras by new symbols, the notation of [BC, p. 180] is given in parenthesis.

(1) $\mathfrak{g}_2 \times \mathfrak{g}_1 :\quad [e_1, e_2] = e_2$

(2) $\mathfrak{f}_3(\alpha)$ $(\alpha\in[0,2], \alpha\neq 1) :\ [e_1, e_2] = (\alpha - 1)e_2,\ [e_1, e_3] = e_3\ (\cong \mathfrak{g}_{3,2}(\alpha - 1))$

(3) $\mathfrak{h}_3 :\quad [e_1, e_2] = e_2 + e_3,\ [e_1, e_3] = e_3 \quad (\cong \mathfrak{g}_{3,3})$

(4) $\mathfrak{e}_3(\beta)\ (\beta \geq 0) :\quad [e_1, e_2] = \beta e_2 - e_3,\ [e_1, e_3] = e_2 + \beta e_3 \quad (\cong \mathfrak{g}_{3,4}(\beta))$

In dimension four we have the following list. Only the Lie relations which must be added to the corresponding three dimensional relations above, are given. For instance, $\mathfrak{a}_4$ is an extension of $\mathfrak{a}_3$ by $\mathbb{R}e_4$, and is as such defined by the data $(B, t) = (B_{2,3}, 1)$ which give the additional Lie relations $[e_2, e_3] =$

e_4, $[e_1, e_4] = e_4$.

(1) $\mathfrak{a}_4$: $[e_2, e_3] = e_4$, $[e_1, e_4] = e_4$ $(\mathfrak{g}_{4,1})$

(2) $\mathfrak{f}_4(\alpha)$ ($\alpha \in [0,2]$, $\alpha \neq 1$) : $[e_2, e_3] = e_4$, $[e_1, e_4] = \alpha e_4$ $(= \mathfrak{g}_{4,9}(\alpha))$

(3) $\mathfrak{h}_4$: $[e_2, e_3] = e_4$, $[e_1, e_4] = 2e_4$ $(= \mathfrak{g}_{4,10})$

(4) $\mathfrak{e}_4(\beta)$ ($\beta \geq 0$) : $[e_2, e_3] = e_4$, $[e_1, e_4] = 2\beta e_4$ $(= \mathfrak{g}_{4,11}(\beta))$

We give in Prop. 2.11 below some "principal" families of filiform extensions of the above Lie algebras in arbitrary dimension n ($n \geq 4$). A complete list will be given at the end of the paper (cf. §§5,6, and 7).

2.3. In general the space $H^2(\mathfrak{g}, \mathfrak{g}/\mathfrak{n}, \mathfrak{a})$ need not be $Aut\,\mathfrak{g} \times Aut\,\mathfrak{a}$ stable. In fact, $\mathfrak{n}$ need not be $Aut\,\mathfrak{g}$-invariant. However, if $\mathfrak{g}$ belongs to FS_1 then $\mathfrak{n}$ must always be equal to the nilradical, which is invariant under all of $Aut\,\mathfrak{g}$. Moreover, $\theta = \theta_t$ is $Aut\,\mathfrak{a}$-fixed whenever the dimension of $\mathfrak{a}$ equals one: $\psi\theta_t\psi^{-1} = \theta_t$ $(0 \neq \psi \in \mathbb{R})$. We proceed to study the action of $Aut\,\mathfrak{g}$ on θ_t. Put, for $e_1 \in \mathfrak{g} \setminus \mathfrak{n}$, $s \in \mathbb{R}$,

$$Aut_s\,\mathfrak{g} = \{\alpha \in Aut\,\mathfrak{g} : \alpha(e_1) = se_1 \text{ modulo } \mathfrak{n}\} \tag{2.1}$$

Note that $Aut_s\,\mathfrak{g}$ is independent of the choice of e_1. $Aut_1\,\mathfrak{g}$ is equal to the common fix-point group under $Aut\,\mathfrak{g}$ of all the representations θ_t $(t \neq 0)$. We shall prove below (Prop. 2.7) that $Aut_1\,\mathfrak{g} = Aut\,\mathfrak{g}$ for all $\mathfrak{g}$ in FS_1 of dimension greater than 4.

First, let $\mathfrak{e}_4(\beta)$ and $\mathfrak{f}_4(\alpha)$ ($\alpha \in [0,2]$, $\alpha \neq 1$) be as in 2.2. It is not hard to verify that $\mathfrak{e}_4(\beta)$ admits no filiform extensions. Moreover, the automorphism group of $\mathfrak{f}_4(\alpha)$ is given as follows (a proof can be found in [S2]).

2.4. LEMMA. *The automorphism group $Aut\,\mathfrak{f}_4(\alpha)$ has the following matrix representation relative to the basis $< e_i >_{i=1}^4$.*

$$\alpha \in (0,2) \setminus \{1\} : \begin{pmatrix} 1 & 0 & 0 & 0 \\ r & a & 0 & 0 \\ s & 0 & d & 0 \\ t & as & u & ad \end{pmatrix}, \quad ad \neq 0, \quad u = (1-\alpha)^{-1} rd,$$

$$\alpha = 0 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ r & a & 0 & 0 \\ s & 0 & d & 0 \\ t & as & rd & ad \end{pmatrix}, \quad ad \neq 0; \begin{pmatrix} -1 & 0 & 0 & 0 \\ r & 0 & b & 0 \\ s & c & 0 & 0 \\ t & -cr & -bs & -bc \end{pmatrix}, \quad bc \neq 0$$

$$\alpha = 2: \begin{pmatrix} 1 & 0 & 0 & 0 \\ r & a & b & 0 \\ s & c & d & 0 \\ t & u & v & c_0 \end{pmatrix}, \quad c_0 = ad - bc \neq 0, \quad u = as - cr, \quad v = bs - dr.$$

In particular, $Aut_1\, \mathfrak{f}_4(\alpha) = Aut\, \mathfrak{f}_4(\alpha)$ *whenever* $\alpha \neq 0$, *and* $Aut_1\, \mathfrak{f}_4(0) \cup Aut_{-1}\, \mathfrak{f}_4(0) = Aut\, \mathfrak{f}_4(0)$.

2.5. It turns out that the only five dimensional filiform extensions of $\mathfrak{g} = \mathfrak{f}_4(\alpha)$ are (within isomorphisms) the Lie algebras $\tilde{\mathfrak{g}} = \mathfrak{f}_5(\alpha)$ ($\alpha \in [0,2] \setminus \{1\}$) and $\tilde{\mathfrak{g}} = \mathfrak{g}_5(\alpha)$ ($\alpha \in (0,2) \setminus \{1\}$). They are given by the extension data (θ_t, B) in which

$$t = 2\alpha - 1, \quad B = B_{2,4}, \quad \text{for } \mathfrak{f}_5(\alpha)$$

and

$$t = \alpha + 1, \quad B = B_{3,4}, \quad \text{for } \mathfrak{g}_5(\alpha).$$

More details will be given in Sec. 4. According to Prop. 1.8, the automorphisms of these Lie algebras have matrix representations

$$A = \begin{pmatrix} A_0 & 0 \\ \phi & \psi \end{pmatrix}, \quad A_0 \in Aut\, \mathfrak{g}, \ \phi \in Hom(\mathfrak{g}, \mathbb{R}), \ 0 \neq \psi \in \mathbb{R}, \tag{2.2}$$

in which

$$B \circ A_0 = \psi \cdot B + \partial_t \phi, \quad \theta = \theta \circ A_0. \tag{2.3}$$

It follows from this and Lemma 2.4 that $Aut\, \tilde{\mathfrak{g}} = Aut_1\, \tilde{\mathfrak{g}}$ except possibly for $\tilde{\mathfrak{g}} = \mathfrak{f}_5(0)$. An easy calculation shows that $Aut\, \mathfrak{f}_5(0)$ consists of all matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ r & a & 0 & 0 & 0 \\ s & 0 & d & 0 & 0 \\ rs & as & rd & ad & 0 \\ t & u & \frac{1}{2}r^2 d & rad & a^2 d \end{pmatrix}, \quad ad \neq 0,$$

which shows $Aut\, \mathfrak{f}_5(0)_1 = Aut\, \mathfrak{f}_5(0)$. In fact, $A_0 \in Aut_{-1}\, \mathfrak{g}$ admits no solution ϕ, ψ of eq. (2.3) since $\theta_1 \circ A_0 = \theta_{-1}$.

Using the above lemma together with an inductive argument based on eq. (2.2) and (2.3), we have the following result on $Aut\, \mathfrak{f}_n(\alpha)$ which will be needed in Sec. 4.

COROLLARY. *Let $\mathfrak{f}_n(\alpha)$ in $FS_{1,n}$ be realized with the basis relations,*

$$[e_1, e_2] = (\alpha - 1)e_2, \quad [e_1, e_i] = ((i-3)\alpha - (i-4))e_i \quad (3 \le i \le n),$$
$$[e_2, e_i] = e_{i+1} \quad (3 \le i \le n-1), \quad \alpha \in [0,2] \setminus \{1\}.$$

Then $Aut\, \mathfrak{f}_n(\alpha)$ has the following matrix representation relative to the basis $< e_i >_{i=1}^n$, $n \ge 5$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ r & a & 0 & 0 & \cdots & 0 \\ s & 0 & d & 0 & \cdots & 0 \\ & & & ad & \cdots & 0 \\ & * & & & \ddots & \vdots \\ & & & & & a^{n-3}d \end{pmatrix}, \quad \alpha \in [0,2) \setminus \{1\}. \tag{2.4}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ r & a & 0 & 0 & \cdots & 0 \\ s & c & d & 0 & \cdots & 0 \\ & & & ad & \cdots & 0 \\ & & * & & \ddots & \vdots \\ & & & & & a^{n-3}d \end{pmatrix}, \quad ad - bc \ne 0, \quad \alpha = 2. \tag{2.5}$$

2.6. In dimension four, the following result holds, as is readily verified (cf. Lemma 2.4).

LEMMA. *Let $\mathfrak{g}$ be in $FS_{1,4}$.*

(*a*) *Suppose $\mathfrak{g}$ is non-isomorphic to $\mathfrak{e}_4(0)$ and $\mathfrak{f}_4(0)$. Then $Aut_1\, \mathfrak{g} = Aut\, \mathfrak{g}$.*

(*b*) *If $\mathfrak{g}$ is isomorphic to one of the algebras $\mathfrak{e}_4(0)$ and $\mathfrak{f}_4(0)$ then $Aut\, \mathfrak{g} = Aut_1\, \mathfrak{g} \cup Aut_{-1}\, \mathfrak{g}$.*

(*c*) *$\mathfrak{e}_4(\beta)$ admits no filiform extensions ($\beta \ge 0$).*

We are ready to prove,

2.7. PROPOSITION. *Let $\tilde{\mathfrak{g}}$ be an element in FS_1 of dimension greater than four. Then $Aut_1\, \tilde{\mathfrak{g}} = Aut\, \tilde{\mathfrak{g}}$.*

Proof. We argue by induction on the dimension $\dim \tilde{\mathfrak{g}}$ of $\tilde{\mathfrak{g}}$. As we have seen above, the result is valid for $dim\, \tilde{\mathfrak{g}} = 5$. Now, if $dim\, \tilde{\mathfrak{g}} = n > 5$ and $\tilde{\mathfrak{g}}$ belongs to the class $FS_{1,n}$, then $\tilde{\mathfrak{g}}$ is an extension $\mathfrak{g}(B, \theta)$ of a $\mathfrak{g} \in FS_{1,n-1}$

by an abelian Lie algebra $\mathfrak{a}$ of dimension one. Consequently each A in $Aut\,\tilde{\mathfrak{g}}$ is of the form

$$A : (a, x) \mapsto (\psi \cdot a + \phi(x), A_0(x)) \quad (a \in \mathfrak{a},\ x \in \mathfrak{g}),$$

where

$$A_0 \in Aut\,\mathfrak{g}, \phi \in Hom(\mathfrak{g}, \mathfrak{a}),\ \psi \in \mathbf{R},\ \psi \neq 0,$$

and

$$B \circ A_0 = \psi \cdot B + \partial_\theta \phi,\ \theta \circ A_0 = \theta,$$

Prop. 1.8. Let $< e_i >_{i=1}^{n}$ denote a canonical basis for the algebra $\tilde{\mathfrak{g}}$. in $FS_{1,n}$. In light of the above and the inductive hypothesis,

$$Ae_1 = A_0 e_1 + \phi(e_1) e_n = A_0 e_1 \quad (\mathrm{mod}\ \tilde{\mathfrak{n}}),$$

so that $A \in Aut_1\,\mathfrak{g}$. The result follows by induction. **Q.E.D.**

2.8. If V is a vector space we let $G_k V$ denote the space of all k-dimensional linear subspaces of V, i.e., the Grassmannian. Let $H^2(\mathfrak{g})$ denote the linear space of all central 2-cocycles of a Lie algebra $\mathfrak{g}$, taking values in the field of $\mathfrak{g}$. We restate [S1; Corollary 3.6] as

PROPOSITION. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{z}$ its center. The isomorphism classes of all Lie algebras $\tilde{\mathfrak{g}}$ with center $\tilde{\mathfrak{z}}$ of dimension $k, \tilde{\mathfrak{g}}/\tilde{\mathfrak{z}} \cong \mathfrak{g}$, and without nonzero abelian direct factors, are in bijective correspondence with those $Aut\,\mathfrak{g}$-orbits Ω in $G_k H^2(\mathfrak{g})$ enjoying the property that $\bigcap_{B \in V} \mathfrak{g}(B) \cap \mathfrak{z} = (0)$, for all V in Ω. Here $\mathfrak{g}(B) = \{x \in \mathfrak{g} : B(x, \mathfrak{g}) = (0)\}$.*

2.9. Next, we give a simplified version of Theorem 1.4 applicable to the class FS_1. Let $\theta = \theta_t$ be a character of $\mathfrak{g} \in FS_1$ $(t \neq 0)$. By Prop. 2.7, the fix-point group of θ is all of $Aut\,\mathfrak{g}$. Therefore, the $Aut\,\mathfrak{g} \times Aut\,\mathfrak{a}$ orbit of any element B of $H^2(\mathfrak{g}, \theta)$ is contained in $H^2(\mathfrak{g}, \theta)$. Moreover, the $Aut\,\mathfrak{a}$-orbit of B lies in $H^2(\mathfrak{g}, \theta)$ since dim $\mathfrak{a} = 1$. Consequently it suffices to consider all $Aut\,\mathfrak{g}$-orbits in $H^2(\mathfrak{g}, \theta)$. We recall that $\mathfrak{n}(B)$ denotes the radical of the restriction of a bilinear form B to the nilradical $\mathfrak{n}$, and $\mathfrak{z}$ denotes the center of $\mathfrak{n}$.

2.10. THEOREM. *Let* $\mathfrak{g} \in FS_{1,n}$ *over* $\mathbf{R}$, *assume that* $n > 4$. *The isomorphism classes of all filiform extensions of* $\mathfrak{g}$ *by a one-dimensional Lie algebra, are in bijective correspondence with the set of all* $Aut\,\mathfrak{g}$*-orbits* Ω *in* $\bigcup_{t\in\mathbf{R}} H^2(\mathfrak{g},\theta_t)$ *with* $\mathfrak{n}(B) \cap \mathfrak{z} = (0)$ *for all* $B \in \Omega$. *(Here we let* $H^2(\mathfrak{g},0) = H^2(\mathfrak{g})$.*) Moreover,* $H^2(\mathfrak{g},\theta_t)$ *is* $Aut\,\mathfrak{g}$*-stable for each real number* t.

COROLLARY. *Let* $\mathfrak{g}$ *be in* $FS_{1,n}$ $(n > 4)$. *If* $B_1 \in H^2(\mathfrak{g},\theta_r)$ *and* $B_2 \in H^2(\mathfrak{g},\theta_s)$ *with* $r \neq s$, *then the corresponding extensions* $\mathfrak{g}(B_1,\theta_r)$ *and* $\mathfrak{g}(B_2,\theta_s)$ *are nonisomorphic.*

2.11. For convenience we list at this point certain "principal" families in FS_1. From these all other elements of FS_1 are obtained as extensions in a way to be described explicitly in the following sections. As usual the Lie algebras are defined in terms of all nonvanishing Lie products between the elements of a fixed canonical basis $E_n =< e_i >_{i=1}^{n}$. Relations obtained by antisymmetry of the Lie product are omitted. The next proposition will follow from the results in sections 3 and 4.[2]

PROPOSITION. *The Lie algebras* $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n(\alpha)$, $\mathfrak{d}_n(\alpha)$, $\mathfrak{e}_4(\beta)$, $\mathfrak{f}_n(\beta)$, $\mathfrak{g}_n(\beta)$, $\mathfrak{h}_n$, *described below are well-defined, are pairwise nonisomorphic and belong to* $FS_{1,n}$.

(*a*) $\mathfrak{a}_n$ $(n \geq 4)$:

$$[e_1,e_2] = e_2,\ [e_1,e_i] = (i-3)e_i,\ [e_2,e_{i-1}] = e_i \quad (4 \leq i \leq n)$$

(*b*) $\mathfrak{b}_n$ $(n \geq 5)$:

$$[e_1,e_2] = e_2,\ [e_2,e_3] = e_4,\ [e_1,e_i] = e_i\ (4 \leq i \leq n),\ [e_3,e_i] = e_{i+1}\ (4 \leq i \leq n-1)$$

(*c*) $\mathfrak{c}_5$:

$$[e_1,e_2] = e_2 + e_5,\ [e_1,e_4] = e_4,\ [e_1,e_5] = e_5, [e_2,e_3] = e_4, [e_3,e_4] = e_5.$$

$\mathfrak{c}_n(\alpha)$ $(n > 5,\ \alpha \geq 0)$:

$$[e_1,e_2] = e_2 + e_5 + \alpha e_n,\ [e_1,e_i] = e_i - e_{i+2} \quad (4 \leq i \leq n-2),$$
$$[e_1,e_{n-1}] = e_{n-1},\ [e_1,e_n] = e_n,$$
$$[e_2,e_3] = e_4,\ [e_3,e_i] = e_{i+1} \quad (4 \leq i \leq n-1)$$

[2] The Lie relations of $\mathfrak{c}_n(\alpha)$ and $\mathfrak{d}_n(\alpha)$ $(n \geq 7)$ were incorrectly announced in [S3] (put $\alpha_1 = \alpha_2 = ...\alpha_{n-6} = 0$, $\alpha_{n-5} \geq 0$).

(d) $\mathfrak{d}_5$:

$$[e_1, e_2] = e_2 - e_5,\ [e_1, e_4] = e_4,\ [e_1, e_5] = e_5,\ [e_2, e_3] = e_4,\ [e_3, e_4] = e_5,$$

$\mathfrak{d}_n(\alpha)\quad (n > 5,\ \alpha \geq 0):$

$$[e_1, e_2] = e_2 - e_5 + \alpha e_n,\ [e_1, e_i] = e_i + e_{i+2} \quad (4 \leq i \leq n-2),$$
$$[e_1, e_{n-1}] = e_{n-1}, [e_1, e_n] = e_n,$$
$$[e_2, e_3] = e_4, [e_3, e_i] = e_{i+1} \quad (4 \leq i \leq n-1)$$

(e) $\mathfrak{e}_4(\beta)\quad (\beta \geq 0)$:

$$[e_1, e_2] = \beta e_2 - e_3, [e_1, e_3] = e_2 + \beta e_3, [e_1, e_4] = 2\beta e_4, [e_2, e_3] = e_4.$$

$\mathfrak{e}_4(\beta)$ has no filiform extension $(\beta \geq 0)$.

(f) $\mathfrak{f}_n(\beta)\quad (n \geq 4,\ \beta \in [0,2],\ \beta \neq 1)$:

$$[e_1, e_2] = (\beta - 1)e_2, [e_1, e_i] = ((i-3)\beta - (i-4))e_i \quad (3 \leq i \leq n),$$
$$[e_2, e_i] = e_{i+1} \quad (3 \leq i \leq n-1).$$

(g) $\mathfrak{g}_n(\beta)\quad (n \geq 5,\ \beta \in (0,2),\ \beta \neq 1)$:

$$[e_1, e_2] = e_2, [e_1, e_i] = (\beta + (i-4))e_i\ (3 \leq i \leq n),$$
$$[e_2, e_3] = -e_4, [e_2, e_i] = e_{i+1} \quad (4 \leq i \leq n-1).$$

(h) $\mathfrak{h}_n\quad (n \geq 5)$:

$$[e_1, e_2] = e_2 + e_3, [e_1, e_i] = (i-2)e_i \quad (3 \leq i \leq n),$$
$$[e_2, e_3] = e_4, [e_3, e_i] = e_{i+1} \quad (4 \leq i \leq n-1).$$

2.12. REMARK.

(a) Let $\mathfrak{n}_k$ $(k \geq 3)$ denote the k-dimensional nilpotent Lie algebra whose nonzero basis relations are given by $[e_2, e_i] = e_{i+1}$ $(2 < i < k)$. Observe that all the Lie algebras listed above have nilradical isomorphic to $\mathfrak{n}_k$ for some k.

(b) We shall find it convenient to extend the parameter domain of the Lie algebras $\mathfrak{f}_n(\beta)$ and $\mathfrak{g}_n(\beta)$ to all of $\mathbb{R} \setminus \{1\}$. Note that $T : \mathfrak{f}_n(\beta) \mapsto \mathfrak{g}_n(\frac{\beta}{\beta-1})\ :\ e_i \mapsto u_i\ (4 \leq i \leq n), e_1 \mapsto (\beta-1)u_1,\ e_2 \mapsto u_3,\ e_3 \mapsto u_2,$ defines an isomorphism. Consequently, the family $\mathfrak{g}_n(\beta)$ can be included in $\mathfrak{f}_n(\beta)$, and vice versa. In the sequel we shall focus on the algebras $\mathfrak{f}_n(\beta)$, cf. §4. We remark that an isomorphic copy of the solvable subalgebra $t\mathbb{R}[t]\frac{d}{dt}$ of the Lie algebra of derivations with polynomial coefficients, is obtained by forming repeated filiform extensions of $\mathfrak{g}_5(3)$ (or of $\mathfrak{f}_5(\frac{3}{2})$).

2.13. NOTATION. (a) Let V be a vector space, $< v_i >_{i=1}^{n}$ a fixed basis for V. Let $v = \sum_i \alpha_i v_i \in V$. We say that $v \succ v_i$ if $\alpha_i \neq 0$.
(b) Let $\mathfrak{g}$ be in $FS_{1,n}$ with canonical basis $< e_i >_{i=1}^{n}$. Assume $B = \sum_{i<j} \alpha_{i,j} B_{i,j}$ is an alternating $2-$form on $\mathfrak{g}$. We say that B is based on the form $B_{i,n}$ if $B \succ B_{i,n}$ (i.e., $\alpha_{i,n} \neq 0$).

With notation as above, the filiform extensions of $\mathfrak{g}$ are the ones obtained from cocycles B based on $B_{i,n}$ for some i $(1 < i < n)$.

2.14. LEMMA. *Let the notation be as in 2.13. Assume $[e_i, e_j] \succ e_m$ for some i and j. Then no cocycle on $\mathfrak{g}$ is based on $B_{m,n}$.*

Proof. We observe that $\partial_\theta B_{m,n} \succ \omega_i \wedge \omega_j \wedge \omega_n$ (recall $\omega_i = e_i^*$). Further, $\partial_\theta B_{a,b} \not\succ \omega_i \wedge \omega_j \wedge \omega_n$ for $(a,b) \neq (i,j)$ since $< e_i >_i$ is canonical. Let B be any alternating $2-$form based on $B_{m,n}$. In view of the previous remarks, $\partial_\theta B \succ \omega_i \wedge \omega_j \wedge \omega_n$, and B is no cocycle. **Q.E.D.**

The following result reduces the extension problem (for the algebras listed in Prop. 2.11) to the study of cocycles based on $B_{2,n}$ and $B_{3,n}$.

2.15. PROPOSITION. *Let $\mathfrak{g}$ be any of the Lie algebras in Prop. 2.11 $(a)-(h)$ with its given basis $< e_i >_{i=1}^{n}$. Then every admissible cocycle is based on $B_{2,n}$ or $B_{3,n}$. Moreover, no cocycle is based on $B_{i,n}$ where $i > 3$. If, in addition, $\mathfrak{g} \neq \mathfrak{h}_n$ then every cocycle is cohomologous to a cocycle not based on $B_{1,n}$.*

Proof. The Lie algebras of Prop. 2.11 obey one of the basis relations $[e_3, e_i] = e_{i+1}$, $[e_2, e_i] = e_{i+1}$ $(3 \leq i \leq n-1)$. Consequently, no cocycle is based on $B_{i+1,n}$, $i+1 \geq 4$, Lemma 2.14. It follows that each admissible cocycle B must be based on $B_{i,n}$ for some $i \in \{1,2,3\}$. Assume B is based on $B_{1,n}$ and on no other $B_{i,n}$. Then the restriction of B to the nilradical $\mathfrak{n}$ has a radical containing the center $\mathbb{R}e_n$ of $\mathfrak{n}$. Hence B is nonadmissible.
Finally, assume $\mathfrak{g} \neq \mathfrak{h}_n$. We prove by induction on the dimension n of $\mathfrak{g}$ that each cocycle is cohomologous to a cocycle not based on $B_{1,n}$. The assertion is readily verified for $n = 5$. Assume it holds in all dimensions less than n. Now $\mathfrak{g}$ is an extension of a filiform algebra $\mathfrak{h}$ by $\mathbb{R}e_n$. We denote the corresponding extension data by (C, t_0). Let $p : \mathfrak{g} \rightarrow \mathfrak{h}$ be the canonical map. On $\mathfrak{g}$ we have $0 \sim \partial e_n^* = (t - t_0)B_{1,n} - C \circ p$. If $t - t_0 \neq 0$, we can assume the coefficient of $B_{1,n}$ in B is equal to $t - t_0$ (multiplying B by a nonzero constant). Hence B is cohomologous to the cocycle $B - \partial e_n^*$, which

is clearly not based on $B_{1,n}$.
Assume $t - t_0 = 0$. We write $C = \sum_{(i,j)\in I} \alpha_{i,j} B_{i,j}$. Then we have $\partial B_{1,n} = \sum_{(i,j)\in I} \alpha_{i,j} \omega_1 \wedge \omega_i \wedge \omega_j$. Now the cocycle C is admissible so the inductive hypothesis implies that $\alpha_{i,n-1} \neq 0$ for $i = 2$ or 3. Hence $\partial B_{1,n} \succ \omega_1 \wedge \omega_i \wedge \omega_{n-1}$ $(i = 2 \text{ or } 3)$. We proceed to show that $B_{k,l} \not\succ \omega_1 \wedge \omega_i \wedge \omega_{n-1}$ whenever $(i,j) \neq (1,n)$. For, suppose $B_{k,l} \succ \omega_1 \wedge \omega_i \wedge \omega_{n-1}$, where $1 < k < l$. If $l = n-1$, the relation $[e_1, e_i] \succ e_k$ must hold. However, this relation does not occur in Prop. 2.11 $(a) - (g)$ for $i = 2,3$ (contrary to this, $\mathfrak{h}_n$ does admit the relation $[e_1, e_2] \succ e_3$). If, on the other hand, $l \neq n-1$, we must have $l = n$ and $[e_i, e_{n-1}] \succ e_n$ or $[e_1, e_{n-1}] \succ e_n$. The latter relation does not occur in our list, whereas the first relation implies $k = 1$, contrary to what we have assumed. It follows that for any form B based on $B_{1,n}$, we have $\partial B \succ \omega_1 \wedge \omega_i \wedge \omega_n$ $(i = 2$ or $3)$, and B is no cocycle. This completes our inductive argument. **Q.E.D.**

2.16. REMARK. As we shall see below (Lemma 3.11), $\mathfrak{h}_n$ does in fact admit a nontrivial cocycle based on $B_{1,n}$.

3. The $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ and $\mathfrak{h}$–series.

We are now ready to begin our classification of FS_1. See Prop. 2.11 for notation. Notice that, throughout this article, the dimension of a given Lie algebra will always be the sum of its indices without braces (whenever such occur). For example, the Lie algebras denoted by $\mathfrak{b}_n^k$ and $\mathfrak{f}_{n,2}^{(r)}$ have dimensions $n + k$ and $n + 2$, respectively.

3.1. LEMMA. *Let $\mathfrak{g}$ be a filiform extension of $\mathfrak{a}_4$ of dimension 5. Then $\mathfrak{g}$ is isomorphic to exactly one of the Lie algebras $\mathfrak{a}_5, \mathfrak{b}_5, \mathfrak{c}_5, \mathfrak{d}_5$. Moreover, if $\theta_t(e_1) = te_5$, $Ker\, \theta_t = \mathfrak{n}(\mathfrak{a}_4)$ (the nilradical of $\mathfrak{a}_4$), then $H^2(\mathfrak{a}_4, \theta_t)$ is equal to: $(B_{1,3})$ if $t = 0$, $(B_{1,2}, B_{3,4})$ if $t = 1$, $(B_{2,4})$ if $t = 2$, and (0) otherwise.*

Proof. Let $f = \sum_{i=1}^{4} f_i e_i^* \in \mathfrak{a}_4^*$. Using the defining relations of $\mathfrak{a}_4$ (cf. Sec 2.2)

$$[e_1, e_2] = e_2,\ [e_2, e_3] = e_4,\ [e_1, e_4] = e_4,$$

we find by means of eq. (1.2)

$$\begin{aligned} \partial_t f &= -t(f_2 B_{1,2} + f_3 B_{1,3} + f_4 B_{1,4}) + f_2 B_{1,2} + f_4(B_{1,4} + B_{2,3}) \\ &= f_2(1-t)B_{1,2} - f_3 t B_{1,3} + f_4((1-t)B_{1,4} + B_{2,3}). \end{aligned}$$

Hence the trivial cocycles are,

$$B^2(\mathfrak{a}_4, \theta_t) = ((t-1)B_{1,2},\ tB_{1,3},\ (t-1)B_{1,4} - B_{2,3}) \tag{3.1}$$

Further, easy calculations using eq. (1.13) or (1.14) show (as usual $\omega_i = e_i^*$),

$$\begin{aligned}
&\partial B_{1,2} = \partial B_{1,3} = 0,\\
&\partial B_{1,4} = \omega_1 \wedge \omega_2 \wedge \omega_3,\ \partial B_{2,3} = (t-1)\omega_1 \wedge \omega_2 \wedge \omega_3\\
&\partial B_{2,4} = (t-2)\omega_1 \wedge \omega_2 \wedge \omega_4\\
&\partial B_{3,4} = (t-1)\omega_1 \wedge \omega_3 \wedge \omega_4
\end{aligned} \tag{3.2}$$

Combining eq. (3.1) and (3.2) we derive the statements about $H^2(\mathfrak{a}_4, \theta)$. Now, as straight forward calculations show, the automorphism group $Aut\,\mathfrak{a}_4$ can be represented as the group of all matrices

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & b & 0 \\ d & e & -bc & ab \end{pmatrix} \qquad (ab \neq 0) \tag{3.3}$$

relative to the basis $< e_i >_{i=1}^4$ (see Prop. 1.8). Hence we find

$$\begin{aligned}
&B_{2,4} \circ U = ab^2 B_{2,4}\\
&(B_{1,2} + B_{3,4}) \circ U = a(B_{1,2} + b^2 B_{3,4})\\
&(B_{1,2} - B_{3,4}) \circ U = a(B_{1,2} - b^2 B_{3,4})\\
&B_{3,4} \circ U = ab^2 B_{3,4},
\end{aligned}$$

where the calculations are carried out modulo coboundaries. Except for the "degenerate" orbit of $B_{1,2}$ (corresponding to an extension whose nilradical has two-dimensional center), we obtain three $Aut\,\mathfrak{a}_4 \times Aut\,\mathbb{R}e_5$ orbits in $H^2(\mathfrak{a}_4, 1)$. Thus there are exactly three nonisomorphic extensions of $\mathfrak{a}_4$ in this case, and they can be realized by the cocycles $B_{3,4}, B_{3,4} + B_{1,2}$, and $B_{3,4} - B_{1,2}$. Further, $H^2(\mathfrak{a}_4, 2)$ consists of the single orbit $(B_{2,4})$. Altogether we obtain (within isomorphisms) exactly the filiform extensions $\mathfrak{b}_5, \mathfrak{c}_5, \mathfrak{d}_5$, and $\mathfrak{a}_5$. In addition, the orbit of $B_{1,3}$ corresponds to a (non-filiform) central extension of $\mathfrak{a}_4$ whose nilradical has two-dimensional center. This completes the proof. **Q.E.D.**

3.2. LEMMA. *The only filiform extension of the Lie algebra* $\mathfrak{c}_5$ *(resp.* $\mathfrak{d}_5$*) of dimension six, is* $\mathfrak{c}_6(\alpha)$ *(resp.* $\mathfrak{d}_6(\alpha)$*),* $\alpha \geq 0$.
$H^2(\mathfrak{c}_5, \theta_t)$ *is equal to:* $(B_{1,3})$ *for* $t=0$, $(B_{1,2}, B_{3,5}-B_{1,4})$ *for* $t=1$, $(B_{2,4})$ *for* $t=2$.
$H^2(\mathfrak{d}_5, \theta_t) = (B_{1,2}, B_{3,5}+B_{1,4})$, *for* $t=1$, $(B_{1,3})$ *for* $t=0$, *and* $(B_{2,4})$ *for* $t=2$.

Proof. We consider $\mathfrak{c}_5$, the argument being similar for $\mathfrak{d}_5$. Recalling the basis relations,

$$[e_1, e_2] = e_2 + e_5,\ [e_1, e_4] = e_4,\ [e_1, e_5] = e_5, [e_2, e_3] = e_4, [e_3, e_4] = e_5,$$

one finds easily, using eq. (1.2),

$$B^2(\mathfrak{c}_5, \theta_t) = ((t-1)B_{1,2},\ tB_{1,3},\ (t-1)B_{1,4} - B_{2,3},\ (t-1)B_{1,5} - B_{1,2} - B_{3,4}).$$

Further, by eq. (1.14) we calculate (letting $\omega_i = e_i^*$),

$$\partial B_{1,2} = \partial B_{1,3} = 0,\ \partial B_{1,4} = \omega_1 \wedge \omega_2 \wedge \omega_3,\ \partial B_{1,5} = \omega_1 \wedge \omega_3 \wedge \omega_4,$$
$$\partial B_{2,3} = (t-1)\omega_1 \wedge \omega_2 \wedge \omega_3$$
$$\partial B_{2,4} = (t-2)\omega_1 \wedge \omega_2 \wedge \omega_4$$
$$\partial B_{2,5} = (t-2)\omega_1 \wedge \omega_2 \wedge \omega_5 + \omega_2 \wedge \omega_3 \wedge \omega_4$$
$$\partial B_{3,4} = (t-1)\omega_1 \wedge \omega_3 \wedge \omega_4$$
$$\partial B_{3,5} = (t-1)\omega_1 \wedge \omega_3 \wedge \omega_5 + \omega_1 \wedge \omega_2 \wedge \omega_3$$
$$\partial B_{4,5} = (t-2)\omega_1 \wedge \omega_4 \wedge \omega_5 + \omega_1 \wedge \omega_2 \wedge \omega_4 + \omega_1 \wedge \omega_2 \wedge \omega_5 - \omega_2 \wedge \omega_3 \wedge \omega_5$$

Hence no linear combination $\sum_{1 \leq i < j \leq 5} \alpha_{i,j} B_{i,j}$ can be a nontrivial cocycle unless $\alpha_{1,4} = -\alpha_{3,5}$. It follows that $\mathfrak{c}_5$ has an extension by $\mathbb{R}e_6$ for which the nilradical has center of dimension one, if and only if $t=1$. Further, $B_{2,3}$ and $B_{1,2} + B_{3,4}$ are trivial cocycles for $t=1$, so that $B_{1,2}$ and $-B_{3,4}$ are cohomologous (and nontrivial). Moreover, $B_{1,3}$ is nontrivial iff $t=0$, and $B_{2,4}$ is a cocycle iff $t=2$, and is nontrivial. Hence the statement about $H^2(\mathfrak{c}_5, \theta)$ follows. Next, we check the various extensions $\mathfrak{c}_5(B_{3,5} - B_{1,4} + sB_{1,2})$, $s \in \mathbb{R}$, for isomorphisms. Recall that $\mathfrak{c}_5$ is an extension of $\mathfrak{a}_4$ determined by the character $\theta_1(e_1)e_5 = e_5, Ker\,\theta_1 = \mathfrak{n}(\mathfrak{a}_4)$, and the cocycle $B = B_{1,2} + B_{3,4} \in H^2(\mathfrak{a}_4, \theta_1)$, Lemma 3.1.

In view of Prop. 1.8, the automorphisms of $\mathfrak{c}_5$ are of the form $A = \begin{pmatrix} A_0 & 0 \\ \phi & \psi \end{pmatrix}$, where $A_0 \in Aut\,\mathfrak{a}_4$, $\phi = (\phi_i)_{i=1}^4 \in \mathfrak{a}_4^*$, $0 \neq \psi \in \mathbb{R}$, and $B \circ A_0 = \psi B +$

$\partial_1\phi$. This gives $\phi_3 = bd$, $\phi_4 = -be$, $\psi = a$, and $b^2 = 1$. Hence $\phi = (\phi_1, \phi_2, \epsilon d, -\epsilon e)$, $b = \epsilon$, $\psi = a$ in which $\epsilon = \pm 1$, and we are using the notation of eq. (3.3) in the proof of the preceeding lemma for the matrix of $A_0 \in Aut\, \mathfrak{a}_4$. Therefore

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ c & a & 0 & 0 & 0 \\ 0 & 0 & \epsilon 1 & 0 & 0 \\ d & e & -\epsilon c & \epsilon a & 0 \\ u & v & \epsilon d & -\epsilon e & a \end{pmatrix} \qquad (a \neq 0;\ c, d, e, u, v \in \mathbb{R}). \tag{3.4}$$

(In particular, $Aut\, \mathfrak{c}_5$ has four connected components.) Using this we derive,

$$(B_{3,5} - B_{1,4} + \alpha B_{1,2})\circ A = \pm a(B_{3,5} - B_{1,4} \pm \alpha B_{1,2}) - eB_{1,2} \pm (c-u)B_{1,3}$$
$$- eB_{3,4} \mp vB_{2,3} \sim \pm a(B_{3,5} - B_{1,4} \pm \alpha B_{1,2})$$

since $-B_{3,4} \sim B_{1,2}, B_{1,3} \sim 0$, and $B_{2,3} \sim 0$. It follows that the $Aut\, \mathfrak{c}_5 \times Aut\, \mathbb{R}$-orbit of $B_\alpha = B_{3,5} - B_{1,4} + \alpha B_{1,2}$ consists of all tB_α and all $tB_{-\alpha}$, $t \neq 0$. Thus we have precisely a one-parameter family of (pairwise nonisomorphic) filiform extensions of $\mathfrak{c}_5$ by $\mathbb{R}e_6$, that is $\mathfrak{c}_6(\alpha)$, $\alpha \geq 0$. This completes the proof. **Q.E.D.**

3.3. The Lie algebras $\mathfrak{c}_n(\alpha)$ and $\mathfrak{d}_n(\alpha)$, $\alpha \geq 0$, are defined inductively by letting $\mathfrak{c}_{n+1}(\beta) = \mathfrak{c}_n(0)(\beta B_{1,2} + B_{3,n} - B_{1,n+1}, \theta_1)$ and $\mathfrak{d}_{n+1}(\beta) = \mathfrak{d}_n(0)(\beta B_{1,2} + B_{3,n} + B_{1,n-1}, \theta_1)$, respectively ($n > 5$, $\beta \geq 0$). Accordingly, a set of basis relations of $\mathfrak{c}_n(\alpha)$ can be specified as,

$$\begin{aligned} &[e_1, e_2] = e_2 + e_5 + \alpha e_n, \\ &[e_1, e_i] = e_i - e_{i+2} \quad (4 \leq i \leq n-2), \\ &[e_1, e_{n-1}] = e_{n-1},\ [e_1, e_n] = e_n, \\ &[e_2, e_3] = e_4,\ [e_3, e_i] = e_{i+1} \quad (4 \leq i \leq n-1), \end{aligned} \tag{3.5}$$

cf. Prop. 2.11 (c)-(d). In order to classify all filiform extensions of $\mathfrak{c}_n(\alpha)$ we shall need the following result on the automorphisms of such algebras. A similar result is valid for $\mathfrak{d}_n(\alpha)$.

LEMMA. *Let $\mathfrak{c}_\beta$ ($\beta \geq 0$) be the extension of $\mathfrak{c}_n(0)$ by $\mathbb{R}e_{n+1}$ with cocycle $B_\beta = \beta B_{1,2} + B_{3,n} - B_{1,n-1} \in H^2(\mathfrak{c}_n(0), \theta_1)$, $n > 5$.*

(*a*) *If $A = (a_{i,j}) \in Aut\, \mathfrak{c}_n(\alpha)$ then $a_{1,1} = 1$, $a_{2,2} = a_{n,n} = a \neq 0$, $a_{3,1} = a_{3,2} = 0$, $a_{3,3} = \pm 1$, $a_{4,4} = a_{3,3}a$, and $a_{i,j} = 0$ for $i < j$.*

(b) *Let* $A_\beta \in Aut\, \mathfrak{c}_\beta$,

$$A_\beta = \begin{pmatrix} A & 0 \\ \phi & \psi \end{pmatrix}, \ \phi = \sum_{i=1}^{n} \phi_i e_i^* \in \mathfrak{c}_n(0)^*, \ \psi \neq 0, \ A \in Aut\, \mathfrak{c}_n(0), \ A = (a_{i,j})_{i,j}.$$

Then $\psi = a, \ \phi_4 + a_{n,2} = 0$ *if* $a_{3,3} = 1, \ \phi_4 - a_{n,2} = 0$ *if* $a_{3,3} = -1$.

Proof. Since $\alpha \geq 0$), it follows readily that B_β satisfies the cocycle identity, cf. (3.7) below.
(a) is shown by finite induction on the basis of $Aut\, \mathfrak{c}_5$ (*eq.* (3.4)). The inductive step is taken care of using Prop. 1.8 and the fact that $\mathfrak{c}_{k+1}(\alpha)$ $(5 \leq k \leq n-1)$ is the $(B_{3,k} - B_{1,k-1} + \alpha B_{1,2}, \theta_1)$-extension of $\mathfrak{c}_k(0)$. We omit the straight forward details.
(b) : Using the defining relations on $\mathfrak{c}_n(\alpha)$ we calculate,

$$\begin{aligned} &\partial_1 \phi + \psi B_\beta = \\ &\quad - \phi_3 B_{1,3} + \phi_4 B_{2,3} + \phi_5(B_{1,2} + B_{3,4}) + \phi_6(B_{3,5} - B_{1,4}) + \\ &\quad \cdots + \phi_n(\alpha B_{1,2} + B_{3,n-1} - B_{1,n-2}) + \psi(\beta B_{1,2} + B_{3,n} - B_{1,n-1}). \end{aligned}$$

Further, for $A \in Aut\, \mathfrak{c}_n(\alpha)$ we find,

$$\begin{aligned} B_\beta \circ A &= a_{3,3}(\pm \beta a B_{1,2} - a_{n,1} B_{1,3} - a_{n,2} B_{2,3} + a_{n,4} B_{3,4} + \cdots + a_{n,n} B_{3,n}) \\ &- (a_{n-1,2} B_{1,2} + a_{n-1,3} B_{1,3} + \cdots + a_{n-1,n} B_{1,n}) \end{aligned}$$

Now $B_\beta \circ A = \psi \circ B_\beta + \partial_1 \phi$, Prop. 1.8, hence $(\psi - a_{n,n}) B_{3,n} = 0$ and $(\phi_4 + a_{3,3} a_{n,2}) B_{2,3} = 0$, the forms $B_{i,j}$ being linearly independent. This clearly implies $\psi = a_{n,n} = a, \ \phi_4 + a_{3,3} a_{n,2} = 0$. **Q.E.D.**

3.4. LEMMA. *The only filiform extensions of the Lie algebras* , $\mathfrak{c}_n(\alpha)$ *(resp.* $\mathfrak{d}_n(\alpha)$*),* $n > 5$, *of dimension* $n+1$ *occurs for* $\alpha = 0$ *and are the algebras* $\mathfrak{c}_{n+1}(\alpha)$ *(respectively* $\mathfrak{d}_{n+1}(\alpha)$*)* $(\alpha \geq 0)$.
$H^2(\mathfrak{c}_n(\alpha), \theta_t)$ *is equal to:*
$(B_{1,3})$, $t = 0$; $(B_{1,2}, B_{3,n} - B_{1,n-1})$, *if* $\alpha = 0$ *and* $t = 1$; $(B_{1,2})$ *if* $\alpha > 0$ *and* $t = 1$; $(B_{2,4})$, $t = 2$; (0) *otherwise.*
$H^2(\mathfrak{d}_n(\alpha), \theta_t)$ *is equal to:*
$(B_{1,3})$, *if* $t = 0$; $(B_{1,2}, B_{3,n} + B_{1,n-1})$, *if* $\alpha = 0$ *and* $t = 1$; $(B_{1,2})$ *if* $\alpha > 0$; $(B_{2,4})$, $t = 2$; (0) *otherwise.*
Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-5}), \alpha_i \geq 0, \ 1 \leq i \leq n-5$.

Proof. We consider $\mathfrak{c}_n(\alpha)$, the argument being similar for $\mathfrak{d}_n(\alpha)$. On letting $\mathfrak{c}_n(\alpha) = \mathfrak{c}_n$, $\alpha \geq 0$, the statement of our lemma becomes valid for $n = 5$

by Lemma 3.2. We argue by induction on the dimension n, assuming the result for Lie algebras of dimension less than n, where $n \geq 5$. Recalling the defining basis relations of $\mathfrak{c}_n(\alpha)$, (3.5), we calculate easily,

$$
\begin{aligned}
&B^2(\mathfrak{c}_n(\alpha), \theta_t) = \\
&((1-t)B_{1,2}, tB_{1,3}, (1-t)B_{1,4} + B_{2,3}, (1-t)B_{1,5} + B_{1,2} + B_{3,4}, \\
&(1-t)B_{1,6} - B_{1,4} + B_{3,5}, \cdots, \\
&(1-t)B_{1,n} - B_{1,n-2} + B_{3,n-1})
\end{aligned} \tag{3.6}
$$

Further,

$$
\begin{aligned}
&\partial B_{1,n} = \omega_1 \wedge \omega_3 \wedge \omega_{n-1}, \quad \partial B_{1,n-2} = \omega_1 \wedge \omega_3 \wedge \omega_{n-3}, \\
&\partial B_{3,n-1} = (t-1)\omega_1 \wedge \omega_3 \wedge \omega_{n-1} + \omega_1 \wedge \omega_3 \wedge \omega_{n-3}, \\
&\partial B_{2,n} = (t-2)\omega_1 \wedge \omega_2 \wedge \omega_n + \omega_2 \wedge \omega_3 \wedge \omega_{n-1} + \omega_1 \wedge \omega_2 \wedge \omega_{n-2} \\
&\partial B_{3,n} = (t-1)\omega_1 \wedge \omega_3 \wedge \omega_n + \alpha\omega_1 \wedge \omega_2 \wedge \omega_3 + \omega_1 \wedge \omega_3 \wedge \omega_{n-2} \\
&\partial B_{1,n-1} = \omega_1 \wedge \omega_3 \wedge \omega_{n-2} \\
&\partial B_{4,n} = (t-1)\omega_1 \wedge \omega_4 \wedge \omega_n + \omega_1 \wedge \omega_4 \wedge \omega_{n-2} - \omega_3 \wedge \omega_4 \wedge \omega_{n-1} \\
&\partial B_{i,n} = (t-1)\omega_1 \wedge \omega_i \wedge \omega_n + \omega_1 \wedge \omega_i \wedge \omega_{n-2} - \omega_3 \wedge \omega_i \wedge \omega_{n-1} - \\
&\omega_3 \wedge \omega_{i-1} \wedge \omega_n + \alpha\omega_1 \wedge \omega_2 \wedge \omega_i \ {\scriptstyle (5 \leq i \leq n-2)} \\
&\partial B_{4,5} = (t-2)\omega_1 \wedge \omega_4 \wedge \omega_5 - \omega_1 \wedge \omega_2 \wedge \omega_4 - \omega_2 \wedge \omega_3 \wedge \omega_5, \\
&\partial B_{4,j} = (t-2)\omega_1 \wedge \omega_4 \wedge \omega_j - \omega_1 \wedge \omega_4 \wedge \omega_{j-2} - \omega_3 \wedge \omega_4 \wedge \omega_{j-1} \\
&\qquad (6 \leq j \leq n-2) \\
&\partial B_{5,j} = (t-2)\omega_1 \wedge \omega_5 \wedge \omega_j - \omega_1 \wedge \omega_5 \wedge \omega_{j-2} - \omega_3 \wedge \omega_5 \wedge \omega_{j-1} \\
&\ - \omega_1 \wedge \omega_2 \wedge \omega_j \quad (6 < j \leq n-2) \\
&\partial B_{5,6} = (t-2)\omega_1 \wedge \omega_5 \wedge \omega_6 - \omega_1 \wedge \omega_2 \wedge \omega_6 \\
&\partial B_{i,j} = (t-2)\omega_1 \wedge \omega_i \wedge \omega_j + \omega_1 \wedge \omega_i \wedge \omega_{j-2} + \omega_1 \wedge \omega_{i-2} \wedge \omega_j \\
&\ - \omega_3 \wedge \omega_i \wedge \omega_{j-1} - \omega_3 \wedge \omega_{i-1} \wedge \omega_j \quad (6 \leq i < j \leq n-1),
\end{aligned} \tag{3.7}
$$

where as usual $\omega_i = e_i^*$. Let B be any cocycle on $\mathfrak{g}$. In view of Prop. 2.15 no cocycle is based on $B_{i,n}$ unless $i = 2$ or 3. Further, the only elementary form $B_{i,j}$ satisfying $\partial_t B_{i,j} \succ \omega_1 \wedge \omega_3 \wedge \omega_n$ is $B_{3,n}$ $(t \neq 1)$. Hence B is not based on $B_{3,n}$ for $t \neq 1$. If, on the other hand, $\alpha = 0$ and $t = 1$ then $B_{3,n} - B_{1,n-1}$ is a nontrivial cocycle by the fourth line of (3.7). Next, the 3−form $\omega_2 \wedge \omega_3 \wedge \omega_{n-1}$ occurs with nonzero coefficient only in the expansion of $\partial_t B_{2,n}$, so that B is not based on $B_{2,n}$. It remains only to study cocycles

B for which $\alpha_{i,n} = 0 \quad (1 \le i < n)$. In this case, B lifts to a cocycle $B \circ p$ on the factor Lie algebra $\mathfrak{c}_n(\alpha)/\mathbb{R}e_n \cong \mathfrak{c}_{n-1}(0)$ via the canonical map p. By our inductive hypothesis then,

$$B \circ p \sim \alpha_{1,2}B_{1,2} + \alpha_{1,3}B_{1,3} + \alpha_{2,4}B_{2,4} + \alpha_{3,n-1}(B_{3,n-1} - B_{1,n-2}).$$

Here $\alpha_{3,n-1} = 0$ if $t \neq 1$, whereas $B_{3,n-1} - B_{1,n-2}$ is a nontrivial cocycle whenever $t = 1$. Further, $B_{1,2} \sim 0$ unless $t = 1$, $B_{1,3} \sim 0$ unless $t = 0$, $\alpha_{2,4} = 0$ unless $t = 2$, and $B_{2,4}$ is a nontrivial cocycle if $t = 2$. Finally, for $t = 1$, the form $B_{3,n-1} - B_{1,n-2}$ is cohomologous to 0 on $\mathfrak{c}_n(\alpha)$, as is seen from $B^2(\mathfrak{c}_n(\alpha), \theta)$, (3.6). Hence the statement about $H^2(\mathfrak{c}_n(\alpha), \theta)$ follows.

It remains to show that all the cocycles $B_\beta = \beta B_{1,2} + B_{3,n} - B_{1,n-1}$ $(\beta \ge 0)$ give pairwise nonisomorphic extensions of $\mathfrak{c}_n(0)$ for $t = 1$, and that the Lie algebra given by B_β is isomorphic to $\mathfrak{c}_{n+1}(\beta)$. To this end, we calculate the orbit of B_β under $Aut\ \mathfrak{c}_n(0)$. Thus let $(a_{i,j})$ denote the matrix of $A \in Aut\ \mathfrak{c}_n(0)$, relative to $< e_i >$. From the basis relations, $\mathfrak{c}_n(0)$ is an extension of $\mathfrak{c}_{n-1}(0)$ by $\mathbb{R}e_n$, and Lemma 3.3 applies. In particular, $a_{i,j} = 0$ for $i < j$. Hence, using that $H^2(\mathfrak{c}_n(0), 1) = (B_{1,2}, B_{3,n} - B_{1,n-1})$, we find $B_\beta \circ A$ cohomologous to:

$$\begin{aligned}
&\pm(\pm\beta a_{2,2}B_{1,2} - \phi_1 B_{1,3} - \phi_2 B_{2,3} + \phi_4 B_{3,4} + \cdots + \phi_{n-1}B_{3,n-1} + \psi B_{3,n}) \\
&- (a_{n-1,2}B_{1,2} + a_{n-1,3}B_{1,3} + \cdots + a_{n-1,n-1}B_{1,n-1}) \\
&= a(B_{3,n} - B_{1,n-1} + \beta B_{1,2}) + (a_{n-1,2}B_{1,2} + \phi_4 B_{3,4}) + T \sim \pm a B_{\pm\beta} + T,
\end{aligned}$$

in which the projection of T on the subspace of $H^2(\mathfrak{c}_n(0), \theta_1)$ generated by $B_{1,2}$ and B_β is zero.

Now $H^2(\mathfrak{c}_n(0), \theta_1) = (B_{1,2}, B_{3,n} - B_{1,n-1})$, as shown above, hence T must be cohomologous to 0, and

$$B_\beta \circ A \sim \pm a B_{\pm\beta}.$$

This completes our proof. **Q.E.D.**

3.5. We summarize the above results in the following extension graph, in which arrows indicate direction of extensions. Accordingly, quotient maps run in the opposite directions.

$$
\begin{array}{lcccccccccc}
\mathfrak{a}_5 & & & & & & & & & & \\
\uparrow \nearrow \mathfrak{c}_5 & \rightarrow & \mathfrak{c}_6(\alpha) & \overset{(\alpha=0)}{\rightarrow} & \dots & \overset{(\alpha=0)}{\rightarrow} & \mathfrak{c}_n(\alpha) & \overset{(\alpha=0)}{\rightarrow} & \dots & & \\
\mathfrak{a}_4 & \rightarrow & \mathfrak{d}_5 & \rightarrow & \mathfrak{d}_6(\alpha) & \overset{(\alpha=0)}{\rightarrow} & \dots & \rightarrow & \mathfrak{d}_n(\alpha) & \overset{(\alpha=0)}{\rightarrow} \dots & (\alpha\geq 0) \\
\searrow & & & & & & & & & & \\
\quad \mathfrak{b}_5 & & & & & & & & & &
\end{array}
$$

3.6. The Lie algebras $\mathfrak{a}_n$ in FS_1 are defined inductively by letting $\mathfrak{a}_{n+1} = \mathfrak{a}_n(B_{2,n}, \theta_{n-2})$,
$(n \geq 4)$. The corresponding basis relations of $\mathfrak{a}_n$ are,

$$[e_1, e_2] = e_2,\ [e_1, e_i] = (i-3)e_i,\ [e_2, e_{i-1}] = e_i \quad (4 \leq i \leq n)$$

LEMMA. *Let $n \geq 5$. The only filiform extensions of $\mathfrak{a}_n$ of dimension $n+1$ are $\mathfrak{a}_{n+1} = \mathfrak{a}_n(B_{2,n}, n-2)$ and $\mathfrak{a}_{n,1} = \mathfrak{a}_n(\sum_{i=0}^{r}(-1)^i B_{3+i,n-i}, n-3)$ $(n = 2r+4)$. Furthermore, the second cohomology spaces of $\mathfrak{a}_n$ are as follows,*
$H^2(\mathfrak{a}_n,\ \theta_t) = (B_{1,3}),\ t = 0;\ (B_{1,2}),\ t = 1;\ (B_{2,n}),\ t = n-2;$ $(\sum_{i=0}^{r}(-1)^i B_{3+i,k-i}),\ t = k-3\ (k = 2r+4,\ 3 < k \leq n);\ (0)$ otherwise.

Proof. We calculate easily for the Lie algebra $\mathfrak{a}_n$,

$$
\begin{aligned}
B^2(\mathfrak{a}_n, \theta_t) =&((1-t)B_{1,2}, tB_{1,3}, (1-t)B_{1,4} + B_{2,3}, \\
&(2-t)B_{1,5} + B_{2,4}, \dots, (n-3-t)B_{1,n} + B_{2,n-1})
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
\partial B_{1,i} &= \omega_1 \wedge \omega_2 \wedge \omega_{i-1} \quad (2 \leq i \leq n), \\
\partial B_{2,i} &= (t-(i-2))\omega_1 \wedge \omega_2 \wedge \omega_i, \quad (3 \leq i \leq n), \\
\partial B_{i,j} &= (t-i-j+6)\omega_1 \wedge \omega_i \wedge \omega_j - \omega_2 \wedge \omega_i \wedge \omega_{j-1} - \omega_2 \wedge \omega_{i-1} \wedge \omega_j \\
&\quad (3 \leq i < j \leq n).
\end{aligned} \tag{3.9}
$$

We argue by induction on the dimension n, assuming the result for the algebras $\mathfrak{a}_j$, $5 \leq j < n$. Now, in view of (3.8) and (3.9) $B_{1,3}$ is a nontrivial cocycle iff $t = 0$, $B_{1,2}$ is nontrivial iff $t = 1$, and $B_{2,n}$ is a (nontrivial) cocycle iff $t = n-2$. For $t \neq n-2$ the form $\omega_1 \wedge \omega_2 \wedge \omega_n$ occurs with nonzero coefficient $t-(n-2)$ in the expansion of $\partial_t B_{2,n}$, and the coefficient of $\omega_1 \wedge \omega_2 \wedge \omega_n$ is seen to vanish in all other $\partial_t B_{i,j}$ $(1 \leq i < j \leq n)$. Hence if $B = \sum \alpha_{i,j} B_{i,j}$ is a nontrivial cocycle on $\mathfrak{a}_n$, we have $\alpha_{2,n} = 0$ if $t \neq n-2$.

Now, by Prop. 2.15 no admissible cocycle can be based on $B_{i,n}$ unless $i = 2$ or 3.

Consequently it remains only to study the coefficient $\alpha_{3,n}$. If $t \neq n-3$ the projection on the line $(\omega_1 \wedge \omega_3 \wedge \omega_n)$ is nonzero only for $\partial_t B_{3,n}$, hence $\alpha_{3,n} = 0$. For $t = n-3$, the projection on the line $(\omega_2 \wedge \omega_3 \wedge \omega_{n-1})$ is nonzero only for $\partial_t B_{3,n}$ and $\partial_t B_{4,n-1} = -\omega_2 \wedge \omega_4 \wedge \omega_{n-2} - \omega_2 \wedge \omega_3 \wedge \omega_{n-1}$. Similarly, the coefficient of $\omega_2 \wedge \omega_4 \wedge \omega_{n-2}$ is nonzero only in $\partial_t B_{4,n-1}$ and $\partial_t B_{5,n-2} = -\omega_2 \wedge \omega_5 \wedge \omega_{n-3} - \omega_2 \wedge \omega_4 \wedge \omega_{n-2}$. Proceeding recursively, the last step to consider is $\partial_t B_{3+r,4+r} = -\omega_2 \wedge \omega_{3+r} \wedge \omega_{3+r} - \omega_2 \wedge \omega_{2+r} \wedge \omega_{4+r}$, $n = 2r+4$. This yields the cocycle $\sum_{i=0}^{r}(-1)^i B_{3+i,n-i}$ and the corresponding extension is $\mathfrak{a}_n^1$. Observe that for $n = 2r+5$ we do not obtain any cocycle, as the term $-\omega_2 \wedge \omega_{3+r} \wedge \omega_{4+r}$ of $\partial_t B_{3+r,5+r}$ can not be cancelled.

More generally, a similar argument gives the cocycles $\sum_{i=0}^{r}(-1)^i B_{3+i,k-i}$ $(k = 2r+4,\ 3 < k \leq n)$. They are non-admissible unless $k = n$.

Let $p : \mathfrak{a}_n \to \mathfrak{a}_n/\mathbb{R}e_n \cong \mathfrak{a}_{n-1}$ denote the quotient map. For any nontrivial cocycle $B = \sum \alpha_{i,j} B_{i,j}$, $B - \alpha_{2,n} B_{2,n}$ is a cocycle for all t because $B_{2,n}$ is a cocycle if $t = n-2$, and $\alpha_{2,n} = 0$ otherwise. Similarly, $B' = B - \alpha_{2,n} B_{2,n} - \alpha_{3,n} \sum_{i=0}^{r}(-1)^i B_{3+i,n-i}$ is a cocycle for each t (if n is odd or $t \neq n-3$ then $\alpha_{3,n} = 0$). Moreover, $B' = \tilde{B} \circ p$ where $\tilde{B}$ is a cocycle on $\mathfrak{a}_{n-1}$ since $\alpha_{i,n} = 0$, $i \neq 2, 3$. By our inductive hypothesis, $H^2(\mathfrak{a}_{n-1}, \theta_t)$ is equal to $(B_{1,3}), t = 0$; $(B_{1,2})$, $t = 1$; $(B_{2,n-1})$, $t = n-3$, and $(\sum_{i=0}^{r}(-1)^i B_{3+i,k-i})$, $t = k - 3$ $(k = 2r+4,\ 3 < k \leq n-1)$. Hence we can write

$$B' \sim \alpha_{1,3} B_{1,3} + \alpha_{1,2} B_{1,2} + \alpha_{2,n-1} B_{2,n-1} + \alpha_{3,k} \sum_{i=0}^{r}(-1)^i B_{3+i,k-i},$$

where $\alpha_{2,n-1} = 0$ if $t \neq n-3$. We observe, for $t = n-3$, that $B_{2,n-1}$ is cohomologous to zero on $\mathfrak{a}_n$. Therefore B is cohomologous to

$$\alpha_{1,3} B_{1,3} + \alpha_{1,2} B_{1,2} + \alpha_{2,n} B_{2,n} + \alpha_{3,k} \sum_{i=0}^{r}(-1)^i B_{3+i,k-i} \ (k = 2r+4,\ 3 < k \leq n),$$

in which at most one of the coefficients $\alpha_{i,j}$ is nonzero. The statements about $H^2(\mathfrak{a}_n, \theta)$ follows.

Finally, as we have seen above, only $t = n-2$ or $t = n-3$ can give filiform extensions of $\mathfrak{a}_n$, and in this case the extensions are all isomorphic to $\mathfrak{a}_{n+1}$ or $\mathfrak{a}_n^1$, because the cocycles are of the form $sB_{2,n}$ or $s\sum_{i=0}^{r}(-1)^i B_{3+i,n-i}$, $s \neq 0$. **Q.E.D.**

The following is readily verified.

3.7. LEMMA. *The Lie algebras* $\mathfrak{a}_{n,1}$ $(n$ *even,* $n \geq 6)$ *admit no filiform extensions.*

Proof. As usual, forms $B_{i,n+1}$ $(i > 3)$ need not be considered since $\omega_2 \wedge \omega_{i-1} \wedge \omega_{n+1}$ does not cancel (cf. Prop. 2.15). Further, the coefficient matrix of the system $\partial B_{2,n+1}, \partial B_{4,n}, \partial B_{5,n-1}, \ldots, \partial B_{r+1,r+3}$ has a determinant,

$$\Delta_{r-1}=\begin{vmatrix} 1 & -1 & 1 & \ldots & (-1)^{r-1} & (-1)^r \\ -1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & -1 \end{vmatrix}=-\Delta_{r-2}+(-1)^r=(-1)^r(r-1),$$

which is nonzero, so we obtain no cocycle using $B_{i,n+1}$. Finally, The linear span of $B_{3,n+1}, B_{4,n-1}, \ldots$ contains no admissible cocycle since the term $-\omega_2 \wedge \omega_3 \wedge \omega_{n-1}$ in the expansion of $\partial_t B_{4,n-1}$ can not be cancelled. **Q.E.D.**

3.8. In view of the above results, the tree structure of the $\mathfrak{a}-$series is as follows.

$$\begin{array}{ccccccc} & & & & \mathfrak{a}_{n,1} \quad (n \text{ even}) & & \\ & & & & \nearrow & & \\ \mathfrak{a}_4 & \rightarrow & \mathfrak{a}_5 & \rightarrow \ldots \rightarrow & \mathfrak{a}_n & \rightarrow & \ldots \end{array}$$

3.9. We define the subfamily $\{\mathfrak{b}_n\}$ $(n \geq 5)$ of FS_1 inductively by letting $\mathfrak{b}_{n+1} = \mathfrak{b}_n(B_{3,n}, \theta_1)$. The corresponding basis relations of $\mathfrak{b}_n$ become,

$$\begin{aligned} &[e_1, e_2] = e_2,\ [e_2, e_3] = e_4,\ [e_1, e_i] = e_i\ (4 \leq i \leq n), \\ &[e_3, e_i] = e_{i+1}\ (4 \leq i \leq n-1) \end{aligned} \tag{3.10}$$

Further, one readily checks that $\mathfrak{b}_n$ admits the cocycles $B_{3,n} + B_{1,2}$ and $B_{3,n} - B_{1,2}$ in $C^2(\mathfrak{b}_n, \theta_1)$. We denote the corresponding extensions of $\mathfrak{b}_n$ by $\mathfrak{b}_{n,1}$ and $\mathfrak{b}_n^1$, respectively. Now, as we shall see below, these two Lie algebras are isomorphic if and only if n is odd. Next, we define inductively $\mathfrak{b}_{n,k+1}$ (resp. $\mathfrak{b}_n^{k+1}$, n even), as the extension $\mathfrak{b}_{n,k+1} = \mathfrak{b}_{n,k}(B_{3,n+k} - B_{1,k+3}, \theta_1)$ (resp. $\mathfrak{b}_n^{k+1} = \mathfrak{b}_n^k(B_{3,n+k} - B_{1,k+3}, \theta_1)$). Again, existence is readily verified. For $t = 2$ there are some less obvious cocycles B_{2m} on $\mathfrak{b}_n$, defined by $B_{2m} = B_{2,2m} + \sum_{i=1}^{m-2}(-1)^{i-1} B_{3+i,2m-i}$. If n is even this yields admissible cocycles $B = B_n + \sum_{j=3}^{\frac{n-2}{2}} \beta_{2j} B_{2j}$ $(\beta_{2j} \in \mathbb{R})$. Calculating the $Aut\, \mathfrak{b}_n-$orbit of B, we obtain the pairwise nonisomorphic Lie algebras ${}^1\mathfrak{b}_n(\alpha_0, \alpha_1, \ldots, \alpha_{\frac{n}{2}-4})$ $(\alpha_i \in \mathbb{R}, \alpha_0 \neq 0)$ in which the first nonzero α_i $(i \geq 1)$ following α_0 is taken equal to ± 1. See the proof of the following lemma for more details.
We have the following classification of the "$\mathfrak{b}-$series" of FS_1.

LEMMA. *(a) Within isomorphisms, the only filiform extension of* $\mathfrak{b}_n$ *of dimension* $n+1$ *are:* $\mathfrak{b}_{n+1}$ *and* $\mathfrak{b}_{n,1}$ $(n = 5,6,7,\ldots)$, $\mathfrak{b}_n^1$ *(n even,* $n \geq 6$*) and, in addition,* ${}^1\mathfrak{b}_6$, ${}^1\mathfrak{b}_n(\alpha_0, \alpha_1, \ldots, \alpha_{\frac{n}{2}-4})$ $(n > 6$, *n even) where* $(\alpha_0, \alpha_1, \ldots, \alpha_{\frac{n}{2}-4}) \in D_n$. *Here the parameter domain* D_n *is given by :* $\alpha_i \in \mathbf{R}$, $\alpha_0 \neq 0$, *and the first nonzero* α_i $(i \geq 1)$ *following* α_0 *is taken equal to* 1 *or* -1.
(b) The only such extension of $\mathfrak{b}_{n,k}$ *(resp* $\mathfrak{b}_n^k$*) of dimension* $n+k+1$ *is* $\mathfrak{b}_{n,k+1}$ *(resp.* $\mathfrak{b}_n^{k+1}$, *n even),* $k > 0$.
(c) The Lie algebras ${}^1\mathfrak{b}_n$ *admits no filiform extensions.*
The second cohomology spaces of these Lie algebras are as follows:

$$H^2(\mathfrak{b}_n, \theta_t) = \begin{cases} (B_{1,3}),\ t=0;\ (B_{1,2}, B_{3,n}),\ t=1; \\ (B_{2,k} + \sum_{i=1}^{r-3}(-1)^{i-1}B_{3+i,k-i})_{6\leq k\leq n},\ (t=2,\ k=2r-2); \end{cases}$$

$$H^2(\mathfrak{b}_{n,k}, \theta_t) = (B_{1,3}),\ t=0;\ (B_{1,2}, B_{3,n+k} - B_{1,k+3}),\ t=1;$$
$$H^2(\mathfrak{b}_n^k, \theta_t) = (B_{1,3}),\ t=0;\ (B_{1,2}, B_{3,n+k} - B_{1,k+3}),\ t=1\ (n \text{ even}).$$

All the above cohomology spaces are (0) *for* $t \notin \{0,1,2\}$.

Proof. In view of the above basis relations, (3.10), straight forward calculations show,

$$\begin{aligned} &B^2(\mathfrak{b}_n, \theta_t) = \\ &\quad ((1-t)B_{1,2}, tB_{1,3}, (1-t)B_{1,4} + B_{2,3}, (1-t)B_{1,5} + B_{3,4}, \\ &\quad \cdots, (1-t)B_{1,n} + B_{3,n-1}) \end{aligned} \tag{3.11}$$

and (letting $\omega_i = e_i^*$)

$$\begin{aligned} &\partial B_{1,2} = \partial B_{1,3} = 0,\ \partial B_{1,4} = \omega_1 \wedge \omega_2 \wedge \omega_3, \partial B_{2,3} = (t-1)\omega_1 \wedge \omega_2 \wedge \omega_3, \\ &\partial B_{1,i} = \omega_1 \wedge \omega_3 \wedge \omega_{i-1} \quad (5 \leq i \leq n) \\ &\partial B_{2,i} = (t-2)\omega_1 \wedge \omega_2 \wedge \omega_i + \omega_2 \wedge \omega_3 \wedge \omega_{i-1} \quad (4 \leq i \leq n) \\ &\partial B_{3,i} = (t-1)\omega_1 \wedge \omega_3 \wedge \omega_i \quad (4 \leq i \leq n) \\ &\partial B_{4,i} = (t-2)\omega_1 \wedge \omega_4 \wedge \omega_i - \omega_3 \wedge \omega_4 \wedge \omega_{i-1} - \omega_2 \wedge \omega_3 \wedge \omega_i \quad (5 \leq i \leq n) \\ &\partial B_{i,j} = (t-2)\omega_1 \wedge \omega_i \wedge \omega_j - \omega_3 \wedge \omega_i \wedge \omega_{j-1} - \omega_3 \wedge \omega_{i-1} \wedge \omega_j \\ &\quad (5 \leq i < j \leq n) \end{aligned} \tag{3.12}$$

Now, if $B = \sum \alpha_{i,j} B_{i,j}$ $(1 \leq i < j \leq n)$ is a nontrivial cocycle, it follows from eq. (3.12) that $t = 1$ or $t = 2$. Further, in view of Prop. 2.15, we may assume that $\alpha_{i,n} = 0$ unless $i \in \{2,3\}$. Hence the statement about $H^2(\mathfrak{b}_n, \theta_t)$ follows by induction as above. Next, we discuss the question of isomorphisms between extensions.

We note first that the automorphism group $Aut\,\mathfrak{b}_5$ may be realized as the group of all matrices

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ c & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ d & e & -bc & ab & 0 \\ u & v & bd & -b & ab^2 \end{pmatrix}, \quad ab \neq 0.$$

This follows from the fact that $\mathfrak{b}_5/\mathbf{R}e_5 \cong \mathfrak{a}_4$, and Prop. 1.8. Hence, for $t = 1$,

$$\begin{aligned}(B_{3,5} \pm B_{1,2})\circ A_0 &= a(b^3 B_{3,5} \pm B_{1,2}) - ubB_{1,3} - vbB_{2,3} - b^2 B_{3,4} \\ &\sim a(b^3 B_{3,5} \pm B_{1,2}).\end{aligned}$$

Since b occurs with odd power in the coefficient of $B_{3,5}$, we obtain exactly two $Aut\,\mathfrak{b}_5$ orbits. The corresponding filiform extensions of $\mathfrak{b}_5$, are $\mathfrak{b}_6$ (cocycle $B_{3,5}$) and $\mathfrak{b}_{5,1}$ (cocycle $B_{3,5} + B_{1,2}$). Next, let $A \in Aut\,\mathfrak{b}_6$, $A = \begin{pmatrix} A_0 & 0 \\ \phi & \psi \end{pmatrix}$, $\phi \in \mathfrak{b}_5^*$, $\psi \neq 0$, where $B_{3,5}\circ A_0 = \psi B_{3,5} + \partial_t \phi$, Prop. 1.8. Easy calculations show that $\psi = ab^3$. We may proceed inductively to prove $a_{n,n} = ab^{n-3}$, whenever $A = (a_{i,j}) \in Aut(\mathfrak{b}_n)$, $n \geq 5$.

This enables us to compute the $Aut\,\mathfrak{b}_n$-orbits of the cocycles $B = B_{3,n} \pm B_{1,2} \in H^2(\mathfrak{b}_n, 1)$ as follows $(n \geq 5)$:

$$\begin{aligned}B\circ A = a(b^{n-2}B_{3,n} \pm B_{1,2}) - a_{n,1}bB_{1,3} - a_{n,2}bB_{2,3} + a_{n,4}bB_{3,4} + \cdots + \\ + a_{n,n-1}bB_{3,n-1} \sim a(b^{n-2}B_{3,n} \pm B_{1,2}),\end{aligned}$$

because $a_{3,i} = 0\ (i \neq 3)$, $a_{3,3} = b$, and $a_{n,n} = ab^{n-3}$.

It follows that all the cocycles $B_{3,n} + tB_{1,2}$ $(t \neq 0)$ give isomorphic extensions of $\mathfrak{b}_n$, if n is odd. For n even we obtain exactly two nonisomorphic extensions, $\mathfrak{b}_{n,1}$ and $\mathfrak{b}_n^1$, given by the cocycles $B_{3,n} + B_{1,2}$ and $B_{3,n} - B_{1,2}$, respectively. In addition, we always have the extension $\mathfrak{b}_{n+1}$ (cocycle $B_{3,n}$). It remains to discuss the case $t = 2$, n even. To this end, let $B_{2m} = B_{2,2m} + \sum_{i=1}^{m-2}(-1)^{i-1}B_{3+i,2m-i}$. It is readily seen that, within cohomology classes, the $Aut\,\mathfrak{b}_n$−orbit of B_{2m} consists of all cocycles of the form

$a^2b^{2m-3}B_{2m}$. Hence, taking linear combinations, the orbit of $B = B_n + \sum_{j=3}^{\frac{n-2}{2}} \beta_{2j}B_{2j} \quad (\beta_{2j} \in \mathbb{R})$, consists of all

$$a^2b^3(\beta_6B_6 + b^2\beta_8B_8 + \cdots + b^{n-8}\beta_{n-2}B_{n-2} + b^{n-6}B_n) \quad (ab \neq 0) \qquad (3.13)$$

Accordingly, in the space of all lines in $H^2(\mathfrak{b}_n, \theta_2)$, orbits are curves of degree $\frac{n}{2}-4$. They can be parametrized in $\mathbb{R}^{\frac{n}{2}-3}$ by coordinates $(\alpha_0, \alpha_1, \ldots, \alpha_{\frac{n}{2}-4})$, in which the first nonzero α_i $(i \geq 1)$ following α_0, is taken equal to ± 1. For example, the case $\alpha_1 = \pm 1$ corresponds to curves of the form $(\alpha_0, \pm t, \alpha_2t^2, \ldots, \alpha_{\frac{n}{2}-4}t^{\frac{n}{2}-4})$ $(t > 0)$. This representation is derived from eq. (3.13) by letting $\beta_6 = \alpha_0$, $t = b^2|\beta_8|, \alpha_i = \frac{\beta_{2i+4}}{\beta_8^{2i+4}}$ $(2 \leq i \leq \frac{n}{2}-3;\ n \geq 10)$ (take $\beta_n = 1$). In particular, $\alpha_1 = sign(\beta_8)$. We denote the corresponding Lie algebras by ${}^1\mathfrak{b}_6$, ${}^1\mathfrak{b}_n(\alpha_0, \alpha_1, \ldots, \alpha_{\frac{n}{2}-4})$, n even, $n > 6$.
Hence all the filiform extensions of $\mathfrak{b}_n$ of dimension $n+1$ have been classified.

Let us show that ${}^1\mathfrak{b}_n$ admits no filiform extensions. First, the occurrence of the non-cancellable term $-\omega_3 \wedge \omega_{i-1} \wedge \omega_{n+1}$ $(-\omega_2 \wedge \omega_3 \wedge \omega_{n+1}$ if $i = 4)$ in the expansion of $\partial_t B_{i,n+1}$, shows that no admissible cocycle can be based on $B_{i,n+1}$ $(i \geq 4)$ (see also Lemma 2.14). Now, for any cocycle $B \in H^2({}^1\mathfrak{b}_n(\alpha_0, \alpha_1, \ldots, \alpha_{\frac{n}{2}-4}), \theta_t)$ based on $B_{2,n+1}$ we must have $t = 3$. However, this is clearly impossible since none of the remaining elementary forms $B_{i,j}$ has $t-3$ as coefficient of $\omega_1 \wedge \omega_i \wedge \omega_j$ (it is readily seen that $B_{2,n+1}$ is no cocycle). Accordingly, any admissible cocycle must be based on $B_{3,n+1}$. Combining $B_{i,j}$ linearly with other $B_{i,j}-$s, we are lead to a "diagonal block" system as indicated below.

$$\begin{aligned}
\partial B_{3,n+1} &= (t-2)\omega_1 \wedge \omega_3 \wedge \omega_{n+1} - \omega_2 \wedge \omega_3 \wedge \omega_n + \cdots + \\
&\quad + (-1)^{\frac{n}{2}-3}\omega_3 \wedge \omega_{\frac{n+2}{2}} \wedge \omega_{\frac{n+3}{2}} + \text{(terms linearly independent of the previous ones)} \\
\partial B_{4,n} &= (t-2)\omega_1 \wedge \omega_4 \wedge \omega_n - \omega_2 \wedge \omega_3 \wedge \omega_n + \omega_3 \wedge \omega_4 \wedge \omega_{n-1} \\
\partial B_{5,n-1} &= (t-2)\omega_1 \wedge \omega_4 \wedge \omega_n - \omega_3 \wedge \omega_4 \wedge \omega_{n-1} + \omega_3 \wedge \omega_5 \wedge \omega_{n-2} \\
&\ \ \vdots \\
\partial B_{\frac{n+2}{2},\frac{n+6}{2}} &= (t-2)\omega_1 \wedge \omega_{\frac{n+2}{2}} \wedge \omega_{\frac{n+6}{2}} - \omega_3 \wedge \omega_{\frac{n}{2}} \wedge \omega_{\frac{n+6}{2}} + \omega_3 \wedge \omega_{\frac{n}{2}} \wedge \omega_{\frac{n+4}{2}} \\
&\ \ \vdots
\end{aligned} \qquad (3.14)$$

The cocycle condition is satisfied if and only if each diagonal block matrix of this system is row dependent. However, the first block matrix of (3.14) is

quadratic $\frac{n-2}{2} \times \frac{n-2}{2}$ whose determinant is equal to the $\Delta_{\frac{n-2}{2}} = (-1)^{\frac{n-2}{2}} \frac{n-2}{2}$ of § 3.7. Consequently we have shown that ${}^1\mathfrak{b}_n(\alpha_0, \alpha_1, \ldots, \alpha_{\frac{n}{2}-4})$ admits no filiform extensions.
We turn next to the possible extensions of $\mathfrak{b}_{n-1,1}$ and $\mathfrak{b}^1_{n-1}$ $(n \geq 6)$. Now, for $\mathfrak{b}_{n-1,1}$ the $\partial B_{i,j}$-s are the same as for $\mathfrak{b}_n$, except for

$$\begin{aligned}
\partial B_{3,n} &= (t-1)\omega_1 \wedge \omega_3 \wedge \omega_n - \omega_1 \wedge \omega_2 \wedge \omega_3, \\
\partial B_{4,n} &= (t-2)\omega_1 \wedge \omega_4 \wedge \omega_n - \omega_3 \wedge \omega_4 \wedge \omega_{n-1} - \omega_2 \wedge \omega_3 \wedge \omega_n + \omega_1 \wedge \omega_2 \wedge \omega_4, \\
\partial B_{i,n} &= (t-2)\omega_1 \wedge \omega_i \wedge \omega_n - \omega_3 \wedge \omega_i \wedge \omega_{n-1} - \omega_3 \wedge \omega_{i-1} \wedge \omega_n + \omega_1 \wedge \omega_2 \wedge \omega_i \\
&\quad (4 < i \leq n-1).
\end{aligned} \tag{3.15}$$

Further,

$$\begin{aligned}
B^2(\mathfrak{b}_{n-1,1}, \theta) = ((1-t)B_{1,2}, tB_{1,3}, (1-t)B_{1,4} + B_{2,3}, \ldots, \\
(1-t)B_{1,n-1} + B_{3,n-2}, (1-t)B_{1,n} + B_{3,n-1} + B_{1,2}).
\end{aligned} \tag{3.16}$$

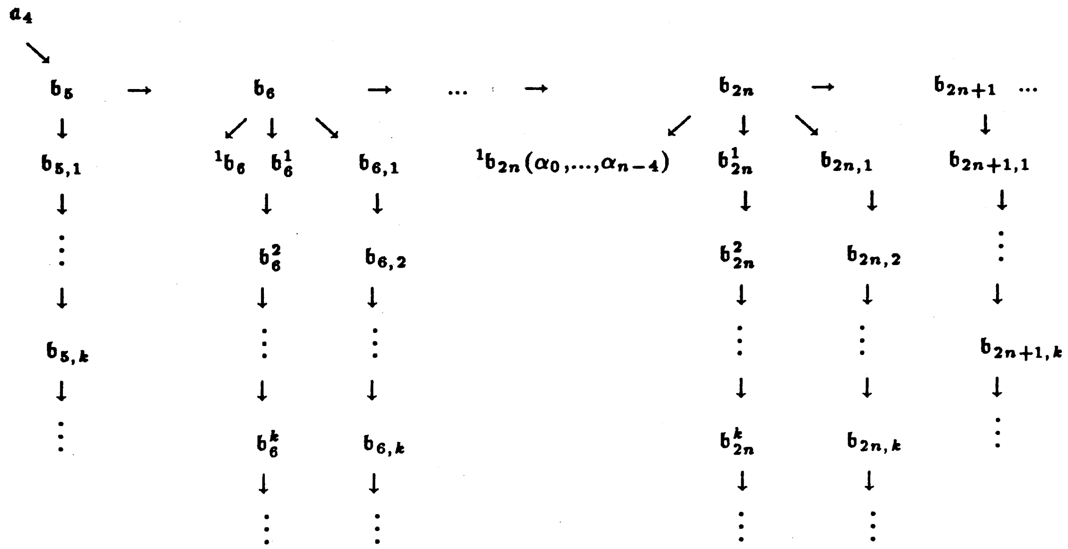
In particular $B_{3,n-1} \sim -B_{1,2}$, for $t = 1$. Let $B = \sum \alpha_{i,j} B_{i,j}$ $(1 \leq i < j \leq n)$ be a nontrivial cocycle on $\mathfrak{b}_{n-1,1}$. We show $\alpha_{i,n}$ may be assumed equal to 0, $i \neq 3$. The elementary 3−form $\omega_1 \wedge \omega_3 \wedge \omega_n$ occurs with nonzero coefficient only in $\partial_t B_{3,n}$ $(t \neq 1)$ so that $\alpha_{3,n} = 0$ $(t \neq 1)$. On the other hand, if $t = 1$, $\omega_1 \wedge \omega_2 \wedge \omega_3$ occurs with nonzero coefficient only in $\partial_t B_{3,n}$ and $\partial_t B_{1,4} = \omega_1 \wedge \omega_2 \wedge \omega_3$, which gives the cocycle $B_{3,n} - B_{1,4}$. Similarly, only $\partial_t B_{1,n}$ and $\partial_t B_{3,n-1}$ $(t \neq 1)$, have nonzero $\omega_1 \wedge \omega_3 \wedge \omega_{n-1}$−components. However, $(1-t)B_{1,n} + B_{3,n-1}$ is cohomologous to the trivial cocycle $-B_{1,2}$ $(t \neq 1)$, as is readily seen from (3.12). If $t = 1$, only $\partial_t B_{1,n}$ has a nonzero $\omega_1 \wedge \omega_3 \wedge \omega_{n-1}$−component. In view of all this, we may assume $\alpha_{1,n} = 0$ for all t.
Further, $\omega_1 \wedge \omega_2 \wedge \omega_n$ occurs with nonzero coefficient only in $\partial_t B_{2,n}$, $t \neq 2$. If $t = 2$, only $\partial_t B_{2,n}$ and $\partial_t B_{4,n-1} = -\omega_3 \wedge \omega_4 \wedge \omega_{n-1} - \omega_2 \wedge \omega_3 \wedge \omega_{n-1}$ have nonzero $\omega_2 \wedge \omega_3 \wedge \omega_{n-1}$−components. In addition, $\omega_3 \wedge \omega_4 \wedge \omega_{n-1}$ occurs with nonzero coefficient only in $\partial_t B_{4,n-1}$ and in $\partial_t B_{4,n} = -\omega_3 \wedge \omega_4 \wedge \omega_{n-1} - \omega_2 \wedge \omega_3 \wedge \omega_n + \omega_1 \wedge \omega_2 \wedge \omega_4$. However, $\omega_2 \wedge \omega_3 \wedge \omega_n$ occurs with nonzero coefficient only in $\partial_t B_{4,n}$, so that $\alpha_{2,n} = 0 = \alpha_{4,n}$. Similarly, $\alpha_{i,n} = 0$ $(i > 4)$ because $\omega_3 \wedge \omega_{i-1} \wedge \omega_n$ occurs with nonzero coefficient only in $\partial_t B_{i,n}$. Consequently, the statements about $H^2(\mathfrak{b}_{n-1,2}, \theta_t)$ follows by induction on the dimension n, passing to the quotient algebra $\mathfrak{b}_{n-1,1}/\mathbf{R}e_n \cong \mathfrak{b}_{n-1}$. The argument is analogous for $\mathfrak{b}^1_{n-1}$ $(n-1$ even).

Finally, we discuss briefly extensions of $\mathfrak{b}_{n,k}$ (and $\mathfrak{b}_n^k$). The basis relations on $\mathfrak{b}_{n,k}$ $(k > 1)$, associated to their inductive definitions, are:

$$\begin{aligned} &[e_1, e_2] = e_2 + e_{n+1},\ [e_1, e_i] = e_i - e_{n+i-2} \quad (4 \leq i \leq k+2), \\ &[e_2, e_3] = e_4,\ [e_3, e_i] = e_{i+1} \quad (4 \leq i < n) \\ &[e_1, e_i] = e_i \quad (k+2 < i \leq n+k). \end{aligned} \tag{3.17}$$

It follows readily that $B_{3,n+k} - B_{1,k+3} \in H^2(\mathfrak{b}_{n,k}, \theta_1)$. Arguing as above, it further follows that this yields the only filiform extension of $\mathfrak{b}_{n,k}$, i.e. $\mathfrak{b}_{n,k+1}$. The proof for $\mathfrak{b}_n^k$ is similar. **Q.E.D.**

3.10. In light of the above, we have the following picture of the $\mathfrak{b}$-series.



3.11. The Lie algebras $\mathfrak{h}_n$ $(n \geq 4)$ are defined inductively by letting $\mathfrak{h}_{n+1} = \mathfrak{h}_n(B_{2,n}, \theta_{n-1})$. The corresponding basis relations for the algebras $\mathfrak{h}_n$ $(n \geq 4)$ are:

$$\begin{aligned} &[e_1, e_2] = e_2 + e_3,\ [e_1, e_i] = (i-2)e_i \quad (3 \leq i \leq n), \\ &[e_2, e_i] = e_{i+1}\ (3 \leq i < n). \end{aligned}$$

LEMMA. *Every $n+1$-dimensional filiform extension of $\mathfrak{h}_n$ $(n \geq 4)$ is isomorphic to $\mathfrak{h}_{n+1}$. Further, the second cohomology spaces of $\mathfrak{h}_n$ in $\mathfrak{g}_1$ are*

$$H^2(\mathfrak{h}_n, \theta_t) = \begin{cases} (B_{1,3}), & t = 1, \\ (B_{2,n}), & t = n-1, \\ (B_{1,2k+1} + \sum_{i=3}^{k+1} (-1)^{i+1} B_{i,2k-i+3}), & t = 2k-1 \leq n-2,\ k \geq 2. \end{cases}$$

Proof. The result is easily verified for $n = 4$. We proceed by induction on the dimension, assuming the statements made on the second cohomology space, for all algebras $\mathfrak{h}_m$ with $m < n$. From the basis relations the space of coboundaries is seen to be,

$$B^2(\mathfrak{h}_n, \theta_t)$$
$$= (B_{1,2}, (t-1)B_{1,3}, (t-2)B_{1,4} - B_{2,3}, \cdots, (t-n+2)B_{1,n} - B_{2,n-1}) \tag{3.18}$$

Further (letting $\omega_i = e_i^*$),

$$\partial B_{1,2} = \partial B_{1,3} = 0, \ \partial B_{1,i} = \omega_1 \wedge \omega_2 \wedge \omega_{i-1} \ (4 \le i < n)$$
$$\partial B_{2,i} = (t-i+1)\omega_1 \wedge \omega_2 \wedge \omega_i \quad (3 \le i \le n)$$
$$\partial B_{3,i} = (t-i+1)\omega_1 \wedge \omega_3 \wedge \omega_i - \omega_1 \wedge \omega_2 \wedge \omega_i - \omega_2 \wedge \omega_3 \wedge \omega_{i-1} \ (4 \le i \le n)$$
$$\partial B_{i,j} = (t-i-j+4)\omega_1 \wedge \omega_i \wedge \omega_j - \omega_2 \wedge \omega_{i-1} \wedge \omega_j - \omega_2 \wedge \omega_i \wedge \omega_{j-1}$$
$$(2 < i < j \le n) \tag{3.19}$$

Let $B = \sum \alpha_{i,j} B_{i,j}$ be a nontrivial cocycle. We show that $\alpha_{i,n}$ may be assumed equal to zero, except possibly for $i = 1$ and $i = 2$. First, the form $\omega_1 \wedge \omega_2 \wedge \omega_{n-1}$ occurs with nonzero coefficient only in the following expansions

$$\partial_t B_{1,n} = \omega_1 \wedge \omega_2 \wedge \omega_{n-1}, \quad \partial_t B_{2,n} = (t-n+2)\omega_1 \wedge \omega_2 \wedge \omega_{n-1},$$
$$\partial_t B_{3,n-1} = (t-n+2)\omega_1 \wedge \omega_3 \wedge \omega_{n-1} - \omega_1 \wedge \omega_2 \wedge \omega_{n-1} - \omega_2 \wedge \omega_3 \wedge \omega_{n-2}.$$

If $t \neq n-2$ then $(t-n+2)B_{1,n} - B_{2,n-1}$ is a trivial cocycle and we may assume $\alpha_{1,n} = \alpha_{2,n-1} = 0$. In case $t = n-2$, $\omega_1 \wedge \omega_2 \wedge \omega_{n-1}$ occurs with nonzero coefficient only in $\partial_t B_{1,n}$ and $\partial_t B_{3,n-1} = -\omega_1 \wedge \omega_2 \wedge \omega_{n-1} - \omega_2 \wedge \omega_3 \wedge \omega_{n-2}$. Furthermore, $\omega_2 \wedge \omega_3 \wedge \omega_{n-2}$ occurs with nonzero coefficient only in $\partial_t B_{3,n-1}$ and in $\partial_t B_{4,n-2} = -\omega_2 \wedge \omega_3 \wedge \omega_{n-2} - \omega_2 \wedge \omega_4 \wedge \omega_{n-3}$, and so on. At an arbitrary step we find that $\partial_t B_{i+1,n-i+1} = -\omega_2 \wedge \omega_i \wedge \omega_{n-i+1} - \omega_2 \wedge \omega_{i+1} \wedge \omega_{n-i}$ $(2i < n)$. Now, for $n = 2k$ even, we find at the very last step, $i = k-1$, that the form $\omega_2 \wedge \omega_k \wedge \omega_{k+1}$ occurs with nonzero coefficient only in the basis expansion of $\partial_t B_{k,k+2} = -\omega_2 \wedge \omega_{k-1} \wedge \omega_{k+2} - \omega_2 \wedge \omega_k \wedge \omega_{k+1}$. Clearly, this implies $\alpha_{1,n} = 0$ (the term $-\omega_2 \wedge \omega_k \wedge \omega_{k+1}$ can not be cancelled). Similarly, assuming $n = 2k+1$ odd, $\omega_2 \wedge \omega_k \wedge \omega_{k+2}$ occurs only in the expansions of $\partial_t B_{k,k+3} = -\omega_2 \wedge \omega_{k-1} \wedge \omega_{k+3} - \omega_2 \wedge \omega_k \wedge \omega_{k+2}$ $(i = k-1)$ and $\partial_t B_{k+1,k+2} = -\omega_2 \wedge \omega_k \wedge \omega_{k+2} - \omega_2 \wedge \omega_{k+1} \wedge \omega_{k+1} = -\omega_2 \wedge \omega_k \wedge \omega_{k+2}$ $(i = k)$

Hence we arrive at the nonadmissible cocycle (it is nonadmissible since the radical of its restriction to $\mathfrak{n}$ contains the center of $\mathfrak{n}$)

$$B_{1,n} + \sum_{i=3}^{k+1}(-1)^{i+1}B_{i,n-i+2} \ (n = 2k+1,\ t = n-2),$$

which is nontrivial in view of the formulas in (3.18) describing $B^2(\mathfrak{h}_n, \theta_t)$. As a conclusion, we may assume $\alpha_{1,n} = 0$ unless $t = n-2$ and n is odd. Similarly, we find the (nonadmissible) cocycles

$$B_{1,2k+1} + \sum_{i=3}^{k+1}(-1)^{i+1}B_{i,2k-i+3}, \quad t = 2k-1 < n-2,\ k \geq 2,$$

for which $\alpha_{i,n} = 0$ (their existence is also part of our inductive hypothesis). Next, $\partial_t B_{2,n} = (t-n+1)\omega_1 \wedge \omega_2 \wedge \omega_n$. Hence $B_{2,n}$ is a nontrivial cocycle if and only if $t = n-1$. Moreover, the only $\partial_t B_{i,j}$ in which the elementary form $\omega_1 \wedge \omega_2 \wedge \omega_n$ occurs with nonzero coefficient is $\partial_t B_{2,n}$. Consequently, $\alpha_{2,n} = 0$ whenever $t \neq n-1$.

Furthermore, regarding $\alpha_{3,n}$, we observe that $\omega_1 \wedge \omega_3 \wedge \omega_n$ can occur with nonzero coefficient only in $\partial_t B_{3,n}$, so that $\alpha_{3,n} = 0$ in case $t \neq n-1$. If $t = n-1$, $\partial_t B_{3,n}$ is the only $\partial_t B_{i,j}$ possessing a nonzero projection on the line $(\omega_1 \wedge \omega_2 \wedge \omega_n)$ (since $\partial_t B_{2,n} = 0$). Hence $\alpha_{3,n} = 0$ for all t.

Finally, we have $\alpha_{i,n} = 0$, $3 < i < n$, by Prop. 2.15.

Summarising the above, we have $B = B_1 + B_2$ in which $B_1 = \sum_{1 \leq i < j < n} \alpha_{i,j} B_{i,j}$ and $B_2 = \alpha_{2,n} B_{2,n}$ $(t = n-1)$ or $B_2 = \alpha_{1,n}(B_{1,n} + \sum_{i=3}^{k+1}(-1)^{i+1}B_{i,n-i+2})$ $(n = 2k+1 \text{ odd}, t = n-2)$. Now, by the inductive hypothesis we must have $B_1 \sim 0$ (since $t = n-1$ or $t = n-2$). Hence the statement about cohomology follows by induction.

Considering next the question of filiform extensions, we may clearly assume $B \sim \alpha_{2,n} B_{2,n}$ and $t = n-1$ since $t = n-2$ gives a nonadmissible cocycle. It follows that all filiform extensions are isomorphic to $\mathfrak{h}_{n+1}$. **Q.E.D.**

3.12. In view of the above, the $\mathfrak{h}$-series admits the following simple extension graph.

$$\mathfrak{h}_4 \longrightarrow \mathfrak{h}_5 \longrightarrow \cdots \longrightarrow \mathfrak{h}_n \longrightarrow \cdots$$

4. The $\mathfrak{f}$-series.

We proceed to study all filiform extensions of the family $\mathfrak{f}_n(\alpha)$.

4.1. LEMMA. *The only filiform extensions of $\mathfrak{f}_n(\alpha)$ $(n \geq 5)$, $\alpha \in \mathbf{R} \setminus \{1\}$, of dimension $n+1$ are the families listed below. These Lie algebras are determined by the extension data $(\theta_t, B), B \in H^2(\mathfrak{f}_n(\alpha), \theta_t)$, $t \in \mathbf{R}$, as indicated.*

(1) $\mathfrak{f}_{n+1}(\alpha)$: $t = (n-2)\alpha - (n-3)$, $B = B_{2,n}$.

(2) $\mathfrak{f}_{n,1}^{(r-1)}$: $t = \frac{2n-(2r+1)}{n-(2r-1)}$, $B = B_{2,n} + B_{3,2r} - B_{4,2r-1} + \cdots + (-1)^{r+1} B_{r+1,r+2}$, $\alpha = \frac{n-(2r-2)}{n-(2r-1)}$, $4 \leq 2r \leq n$. *(We shall often use the notation* $\mathfrak{f}_{n,1} = \mathfrak{f}_{n,1}^{(1)}$.*)*

(3) ${}^1\mathfrak{f}_n(\alpha)$: $t = (n-3)\alpha - (n-5)$, $B = B_{3,n} - B_{4,n-1} + \cdots + (-1)^r B_{r+1,r+2}$, $n = 2r$ $(r \geq 3)$.

The Lie algebras given above are pairwise nonisomorphic, except that ${}^1\mathfrak{f}_n(2) \cong \mathfrak{f}_{n,1}^{(\frac{n-2}{2})}$ *(n even).*

Proof. First we calculate the trivial cocycles of $\mathfrak{f}_n(\alpha)$. Let $f = \sum_{i=1}^{n} f_i e_i^* :$ $\mathfrak{f}_n(\alpha) \to \mathbf{R}e_{n+1}$ be linear, and assume $\theta(e_1)e_{n+1} = te_{n+1}$, $Ker\,\theta =$ $(e_2, e_3, \cdots, e_n)$. Recall the basis relations on $\mathfrak{f}_n(\alpha)$ $(\alpha \in \mathbf{R} \setminus \{1\})$:

$$[e_1, e_2] = (\alpha - 1)e_2,\ [e_1, e_i] = ((i-3)\alpha - (i-4))e_i \quad (3 \leq i \leq n),$$
$$[e_2, e_j] = e_{j+1} \quad (3 \leq j \leq n-1).$$

Consequently

$$\begin{aligned}\partial_t f(x,y) &= \theta(x)f(y) - \theta(y)f(x) - f[x,y] \\ &= (t(f_2B_{1,2} + f_3B_{1,3} + \cdots + f_nB_{1,n}) - (\alpha-1)f_2B_{1,2} - s_3f_3B_{1,3} - f_4s_4B_{1,4} \\ &\quad - \cdots - s_nf_nB_{1,n} - f_4B_{2,3} - f_5B_{2,4} - \cdots - f_nB_{2,n-1})(x,y), \quad x, y \in \mathfrak{f}_n(\alpha).\end{aligned}$$

Here $s_j = (j-3)\alpha - (j-4)$, $3 \leq j \leq n+1$. Therefore the space of coboundaries is

$$\begin{aligned}B^2(\mathfrak{f}_n(\alpha),\theta) = ((t-(\alpha-1))B_{1,2}, (t-s_3)B_{1,3}, \\ (t-s_4)B_{1,4} - B_{2,3}, \cdots, (t-s_n)B_{1,n} - B_{2,n-1}).\end{aligned} \tag{4.1}$$

In particular, no linear combination of the forms $B_{k,j}$ $(3 \leq k < j \leq n)$ can be a trivial cocycle. Next we calculate the sums $\partial_t B_{i,j}$ $(1 \leq i < j \leq n)$, using eq.(1.12). First,

$$\begin{aligned}\partial_t B_{2,j} &= (t - s_j - (\alpha - 1))\omega_1 \wedge \omega_2 \wedge \omega_j + \omega_2 \wedge \omega_2 \wedge \omega_{j-1} \\ &= (t - s_{j+1})\omega_1 \wedge \omega_2 \wedge \omega_j \quad (3 \leq j \leq n)\end{aligned} \tag{4.2}$$

since

$$\begin{aligned}s_j + (\alpha - 1) &= (j-3)\alpha - (j-4) + \alpha - 1 \\ &= (j-2)\alpha - (j-3) = s_{j+1}.\end{aligned}$$

This shows that $B_{2,j}$ is a cocycle iff $t = s_{j+1}$. From (4.1) it follows that, for such t, $B_{2,j}$ is trivial whenever $3 \leq j < n$. Hence $B_{2,n}$ $(t = s_{n+1})$ is the only nontrivial cocycle of this type. Furthermore (letting $\omega_i = e_i^*$),

$$\partial_t B_{1,j} = \omega_1 \wedge \omega_2 \wedge \omega_{j-1} + ((j-3)\alpha - (j-4))\omega_1 \wedge \omega_1 \wedge \omega_j + t\omega_1 \wedge \omega_1 \wedge \omega_j$$

so that

$$\partial_t B_{1,j} = \omega_1 \wedge \omega_2 \wedge \omega_{j-1} \quad (3 \leq j \leq n). \tag{4.3}$$

Consequently $B_{1,3}$ is always a cocycle, however, it is trivial unless $t = s_3$ by (4.1). In addition we derive from (4.2) and (4.3) that

$$B_{2,j} - (t - s_{j+1})B_{1,j+1} \quad (4 \leq j \leq n-1)$$

satisfies the cocycle identity, giving coboundaries by (4.1). We proceed calculating

$$\partial_t B_{i,j} =$$
$$(t - (i+j-6)\alpha + (i+j-8))\omega_1 \wedge \omega_i \wedge \omega_j - \omega_2 \wedge \omega_i \wedge \omega_{j-1} - \omega_2 \wedge \omega_{i-1} \wedge \omega_j,$$

or

$$\begin{aligned}\partial_t B_{i,j} =&(t - (s_i + s_j))\omega_1 \wedge \omega_i \wedge \omega_j - \omega_2 \wedge \omega_i \wedge \omega_{j-1} - \omega_2 \wedge \omega_{i-1} \wedge \omega_j \\ &(3 \leq i < j \leq n)\end{aligned} \tag{4.4}$$

We conclude from (4.4) that $B_{3,4}$ is a nontrivial cocycle iff $t = \alpha + 1$, and $B_{i,j}$ $(3 \leq i < j \leq n)$ is never a cocycle for $(i,j) \neq (3,4)$.
Consider next linear combinations $B = \sum_{i<j} a_{i,j} B_{i,j}$, in which $a_{2,n} \neq 0$. In view of (4.2), our first choice is to search for sums $\partial_t B_{i,j}$ in which terms containing the polynomial $\omega_1 \wedge \omega_2 \wedge \omega_n$ occur. Now (4.4) reveals that $\omega_1 \wedge$

$\omega_2 \wedge \omega_n$ only occurs in the expansion of $\partial_t B_{i,j}$ for $(i,j) = (2,n)$. Accordingly the only possibility is

$$(i)\ B = B_{2,n}, \quad t = s_{n+1} = (n-2)\alpha - (n-3),$$

which leads to the extension $\mathfrak{f}_{n+1}(\alpha)$.

As our second alternative, we search for cocycles B_1 in the linear span of $\{B_{i,j}\}_{(i,j)\neq(2,n)}$, assuming $t = s_{n+1}$ for some α. Any such B_1 will combine with $B_{2,n}$ to give admissible extensions of $\mathfrak{f}_n(\alpha)$. Observe that $B_{3,4}$ is the only *elementary* 2−form solving this problem. As we have seen above, it occurs for $t = \alpha + 1$. Thus $t = \frac{2n-5}{n-3}$, $\alpha = \frac{n-2}{n-3}$, and the corresponding extensions of $\mathfrak{f}_n(\frac{n-2}{n-3})$ are described by the data,

$$(ii) \quad B = aB_{3,4} + B_{2,n},\ t = \frac{2n-5}{n-3},\ a \neq 0.$$

Letting $a = 1$ in (ii) we find the Lie algebra

$$\mathfrak{f}^{(1)}_{n,1} : \quad (B_{2,n} + B_{3,4}, \frac{2n-5}{n-3}).$$

We show all the extensions $F(a)$ of $\mathfrak{f}_n(\frac{n-2}{n-3})$ given by (ii) are isomorphic to $\mathfrak{f}^{(1)}_{n,1}$. To this end let T denote the linear map defined by

$$T(e_4) = ae_4,\ T(e_n) = ae_n,\ T(e_{n+1}) = ae_{n+1},$$
$$T(e_i) = e_i, \qquad i \notin \{4, n, n+1\},$$

where a is fixed. It is easily verified that T sets up a Lie algebra isomorphism between $F(a)$ and $\mathfrak{f}^{(1)}_{n,1}$ (both algebras being equipped with the standard basis $< e_1, e_2, \cdots, e_{n+1} >$). (Another way of seeing this would be by calculating $(B_{2,n} + B_{3,4})\circ A$ for $A = (a_{i,j}) \in Aut\, \mathfrak{f}_n(\frac{n-2}{n-3})$, using Cor. 2.5. In fact, one finds that the projection of $(B_{2,n} + B_{3,4})\circ A$ on the subspace generated by $B_{2,n}$ and $B_{3,4}$ takes the form $a^{n-2}dB_{2,n} + ad^2 B_{3,4} \quad (ad \neq 0)$.) It remains to show that $B_{2,n} + B_{3,4}$ and $B_{2,n}$ give non-isomorphic extensions of $\mathfrak{f}_n(\frac{n-2}{n-3})$, i.e., $\mathfrak{f}^{(1)}_{n,1} \not\cong \mathfrak{f}_{n+1}(\frac{n-2}{n-3})$. However, this is a consequence of Cor. 2.10. Hence our discussion concerning linear combinations of $B_{3,4}$ and $B_{2,n}$ is complete. Similarly, $\mathfrak{f}^{(2)}_{n,1}$ and $\mathfrak{f}_{n+1}(\frac{n-4}{n-5})$ are seen to be nonisomorphic (cf. the following discussion concerning $\mathfrak{f}^{(2)}_{n,1}$).

We wish to explore if any other linear combinations $\sum a_{i,j}B_{i,j}$ in which $a_{2,n} \neq 0$, can possibly be a cocycle. Consider first linear combinations of $B_{2,n}, B_{i,j}$ and $B_{k,l}$. By (4.2) and (4.4) the only interesting solution is the triple

$$B_{2,n},\ B_{4,5},\ B_{3,6} \quad (n \geq 6).$$

In this case we must have

$$t = (i+j-6)\alpha - (i+j-8) = 3\alpha - 1$$

and

$$t = (n-2)\alpha - (n-3),$$

using (4.4) again. Hence

$$\alpha = \frac{n-4}{n-5},\ t = \frac{2n-7}{n-5},$$

giving the cocycles

$$B_s = B_{2,n} + s(B_{4,5} - B_{3,6}) \quad (s \neq 0,\ n > 5)$$

For $n = 6$ we obtain, as a particular case, the admissible cocycle

$$B_6^{(1)} = B_{3,6} - B_{4,5} \quad (t = 3\alpha - 1).$$

This gives a one-parameter family of Lie algebras (cf. $\mathfrak{f}_{6,1}^{(2)}$ below),

$${}^1\mathfrak{f}_6(\alpha):\ t = 3\alpha - 1,\ B = B_{3,6} - B_{4,5}.$$

Now let $F(s)$ denote the extension of $\mathfrak{f}_n(\frac{n-4}{n-5})$ corresponding to B_s and $t = \frac{2n-7}{n-5}$. We claim the algebras $F(s)$ are pairwise isomorphic ($s \neq 0$). In fact, the linear map T defined as

$$T(e_i) = se_i \quad (3 \leq i \leq n+1),\ T(e_1) = e_1,\ T(e_2) = e_2$$

is a Lie isomorphism $F(s) \to F(1)$. From now on, we shall use the notation $\mathfrak{f}_{n,1}^{(2)} = F(1)$.

We wish to remark at this point that $\mathfrak{f}_{6,1}^{(2)} \cong {}^1\mathfrak{f}_6(2)$. In fact, the $Aut\,\mathfrak{f}_6(2)$-orbit of $B_{3,6} - B_{4,5}$ in $G_1H^2(\mathfrak{f}_6(2), \theta_5)$ consists of all 1-dimensional subspaces generated by the cocycles $a^3d(cB_{2,6} + d(B_{3,6} - B_{4,5}))$; $c \in \mathbf{R}$, $ad \neq 0$.

As usual, this follows on calculating $(B_{3,6}-B_{4,5})\circ A$, where $A \in Aut\,\mathfrak{f}_6(2)$ is arbitrary. We omit the straight forward details, giving only the standard matrix representation of an arbitrary $A \in Aut\,\mathfrak{f}_6(2)$ (cf. eq. (2.5)):

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ r & a & 0 & 0 & 0 & 0 \\ s & c & d & 0 & 0 & 0 \\ t & as-cr & -rd & ad & 0 & 0 \\ u & v & \frac{1}{2}r^2d & -rad & a^2d & 0 \\ w & -\frac{r}{3}(ad+v) & -\frac{1}{6}r^3d & \frac{1}{2}r^2ad & -ra^2d & a^3d \end{pmatrix},$$

where $ad \neq 0$, $v = \frac{1}{2}(at - r(as-cr))$, and r, s, t, u, w are arbitrary real numbers.

The more general problem of determining all admissible cocycles of the form $B_{2,n} + B_0$ is solved using arguments similar to the above. We find the following series of extensions, in which the first two terms are the algebras $\mathfrak{f}_{n,1}^{(1)}$ and $\mathfrak{f}_{n,1}^{(2)}$ discussed above.

$$\mathfrak{f}_{n,1}^{(r-1)} : t = \frac{2n-(2r+1)}{n-(2r-1)}, B = B_{2,n}+B_{3,2r} + ... + (-1)^{r+1}B_{r+1,r+2}$$

$$(n \geq 2r \geq 4,\ \alpha = \frac{n-(2r-2)}{n-(2r-1)}).$$

We proceed to determine all cocycles of the form $B = B_{m,n}+B_0$ in which B_0 denotes a linear combination of elementary forms $B_{i,j}$, $(i,j) \neq (m,n)$, $(i,j) \neq (2,n)$. First, in view of Prop. 2.15, we have $m = 3$.

Using arguments similar to the above, we find as the only solutions the one-parameter family ${}^1\mathfrak{f}_n(\alpha)$ $(B = B_{3,n} - B_{4,n-1} + \cdots + (-1)^r B_{r+1,r+2}$ $(n = 2r,\ r \geq 3)$, $t = (n-3)\alpha-(n-5))$ as described in statement (3) of the lemma. For odd n, the algebras ${}^1\mathfrak{f}_n(\alpha)$ do not exist since the term $\omega_2 \wedge \omega_3 \wedge \omega_{n-1}$ occurs in $\partial_t B_{3,n}$. The Lie algebras ${}^1\mathfrak{f}_n(\alpha)$ are readily seen to be pairwise nonisomorphic (Cor. 2.10). Finally, as in the case $n = 6$ discussed above, we find ${}^1\mathfrak{f}_n(2) \cong \mathfrak{f}_{n,1}^{(\frac{n}{2}-1)}$. **Q.E.D.**

The two following results are readily verified.

4.2. LEMMA. *The only filiform extensions of the algebras* $\mathfrak{f}_{n,1}^{(r)}$ $(r \neq 1)$ *occur for* $r = [\frac{n-3}{2}]$. *They are the algebras* $\mathfrak{f}_{n,2}^{(r)}$ *given by the following extension data,*

$$B_{2,n+1} + rB_{3,n-1} - (r-1)B_{4,n-2} + \cdots + (-1)^{r-1}B_{r+2,r+3},\ t = \frac{n+2}{3},\ (n \text{ even})$$

$$B_{2,n+1} + rB_{3,n} - (r-1)B_{4,n-1} + \cdots + (-1)^{r-1}B_{r+2,r+4},\ t = \frac{n+1}{2},\ (n \text{ odd}),$$

and its extensions

$\mathfrak{f}^{(r)}_{n,3}(\beta)$ (n *even*, $\beta \in \mathbf{R}$) :

$$B_{2,n+2} + (\frac{r(r+1)}{2} - \beta)B_{3,n} - (\frac{(r-1)r}{2} - \beta)B_{4,n-1} + \cdots +$$

$$(-1)^{r-1}\beta B_{3+r,4+r}, \quad t = \frac{n+2}{3}$$

$$\mathfrak{f}^{(r)}_{n,3} \quad (n \text{ odd}) : B_{2,n+2} + \sum_{i=1}^{r}(-1)^{i+1}(ri - \frac{i(i-1)}{2})B_{2+i,n+2-i}, \ t = \frac{n+2}{2}.$$

Proof. As in Prop. 2.15, one needs only look for cocycles based on $B_{2,n+3}$ and $B_{3,n+3}$.

$\mathfrak{f}^{(r)}_{n,1}$: The only admissible cocycle is seen to occur for $r = [\frac{n-3}{2}]$ and $t = \frac{n+2}{3}$ (n even) and $t = \frac{n+1}{2}$ (n odd); and this does indeed give the extensions $\mathfrak{f}^{(r)}_{n,2}$ of the lemma.

$\mathfrak{f}^{(r)}_{n,2}$: The argument is similar. For odd n the coefficient matrix of the cocycle equation $\partial_t B = 0$ with $B = \sum_{i=1}^{r+1} x_i B_{i+1,n+3-i}$ and $t = \frac{n+2}{2}$ has dimension $r \times (r+1)$, and is as folows,

$$\begin{pmatrix} r & -1 & 0 & 0 & 0 & \dots & & & \dots & 0 & 0 \\ r-1 & 1 & 1 & 0 & 0 & & & & & \vdots & \vdots \\ r-2 & 0 & -1 & -1 & 0 & & & & & & \\ \vdots & & & & & & & & & \vdots & \vdots \\ r-i & 0 & 0 & 0 & 0 & \dots & 0 & (-1)^{i+1} & (-1)^{i+1} \ \dots & 0 & 0 \\ \vdots & & & & & & & & & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \dots & & & \dots & (-1)^r & (-1)^r \end{pmatrix}.$$

This system has the solutions $x_i = (-1)^i((i-1)r - \frac{(i-1)(i-2)}{2})\tau$ ($i = 1, 2, \ldots, r+1$) in which τ can be taken arbitrary. Consequently, the cocycle listed under $\mathfrak{f}^{(r)}_{n,3}$ (n odd) in our lemma is a representative for the $Aut(\mathfrak{f}^{(r)}_{n,2})$-orbit in $H^2(\mathfrak{f}^{(r)}_{n,2}, \frac{n+2}{2})$.

For n even, the corresponding matrix has rank $r-1$, and we get the one-parameter family of extensions $\mathfrak{f}^{(\beta)}_{n,3}$ ($\beta \in \mathbf{R}$).

$\mathfrak{f}^{(r)}_{n,3}(\beta)$ (n even): Here, $B_{2,n+3}$ (resp. $B_{3,n+3}$) fails since the non-cancellable three-form $\omega_3 \wedge \omega_{r+2} \wedge \omega_{r+3}$ (resp. $\omega_2 \wedge \omega_{r+4} \wedge \omega_{r+5}$) occurs in $\partial_t B_{3,n+1}$ (resp. $\partial_t B_{r+4,r+6}$).

$\mathfrak{f}^{(r)}_{n,3}$ (n odd): The equations $\partial_t B = 0$ for cocycles based on $B_{2,n+3}$ has only the nonadmissible solution $B_{3,n+3} - B_{4,n+1} + \cdots + (-1)^r B_{r+3,r+4}$ ($t = \frac{n+3}{2}$). Finally, no nonzero cocycle can be based on $B_{3,n+3}$. Q.E.D.

4.3. LEMMA. *The Lie algebras* ${}^1\mathfrak{f}_n(\alpha)$ $(\alpha \in \mathbf{R} \setminus \{1\})$ *admit no filiform extensions.*

4.4. In order to complete our classification of the $\mathfrak{f}-$series of filiform solvable Lie algebras we shall need the following family of recursively defined polynomials, $\{p_{n,i}\}$ $(n \geq 2, i \geq 0)$, which appears naturally when forming extension cocycles on the algebras $\mathfrak{f}_{n,1} = \mathfrak{f}_{n,1}^{(1)}$ of Lemma 4.1.

$$p_{1,0}(\alpha_0) = 1,\ p_{2,0}(\alpha_0) = 1 \qquad \text{(all } \alpha_0 \in \mathbf{R})$$
$$p_{2r+1,0}(\alpha_0, ..., \alpha_r) = \alpha_r \qquad (r \geq 1)$$
$$p_{2r+1,i}(\alpha_0, ..., \alpha_r) = \sum_{j=0}^{i-1} (-1)^{i-j+1} p_{2r,j}(\alpha_0, ..., \alpha_{r-1}) + (-1)^i \alpha_r \quad (1 \leq i \leq r,)$$
$$p_{2r,i}(\alpha_0, ..., \alpha_{r-1}) = \sum_{j=0}^{r-i-1} (-1)^{j+r-i-1} p_{2r-1,r-j-1}(\alpha_0, ..., \alpha_{r-1})$$
$$(0 \leq i \leq r-1,\ r \geq 2) \tag{4.5}$$

We let $p_{n,i} = 0$ if $i \geq [\frac{n}{2}]$ ($[t]$ denotes the greatest integer in t).
Note that the variable α_0 is used only for formal reasons; all the polynomials $p_{n,i}$ are constant in α_0. Hence we shall often supress α_0. We have the relation

$$p_{n,i+1} = p_{n-1,i} - p_{n,i}, \tag{4.6}$$

which can also be used to define the polynomials (together with the above "boundary" conditions on $p_{n,0}$).
As we shall see, the algebras $\mathfrak{f}_{n,k}(\alpha_1, \ldots, \alpha_{[\frac{k-4}{2}]})$ $(k - n \geq 0)$ are defined on algebraic parameter domains $S_{n,k}$ depending on the polynomials $p_{n,i}$. We notice that algebras $\mathfrak{f}_{n,k}$ do exist for each integer $k \geq n$ since we have $(1, \ldots, 1) \in S_{n,k}$. To determine exactly the sets $S_{n,k}$ is a nontrivial problem. However, the number of parameters of the algebras $\mathfrak{f}_{n,k}$ is $[\frac{k-1}{2}]$, whereas the number of linearly independent equations (of second degree) seems to be at least $k - n$. In fact, calculations for small n, indicate that $S_{n,k}$ is finite for $k - n \geq n - 2$ (i.e. $k - n \geq [\frac{k-1}{2}]$). For instance, $S_{5,k}$ $(k \geq 8)$ consists of two and only two elements, corresponding to $\alpha_1 = 1$ and $\alpha_1 = \frac{9}{10}$, see Ex. 4.6.

4.5. PROPOSITION. *The only filiform extensions of* $\mathfrak{f}_{n,1}\,(=\mathfrak{f}_{n,1}^{(1)})$ $(n \geq 5)$ *are the following pairwise nonisomorphic Lie algebras* $(\alpha_i \in \mathbf{R})$,

(1) $\mathfrak{f}_{n,k}(\alpha_1,\dots,\alpha_{[\frac{k-1}{2}]})$, $\quad ((\alpha_1,\dots,\alpha_{[\frac{k-1}{2}]}) \in \mathbf{R}^{[\frac{k-1}{2}]},\ 2 \leq k < n)$
$\mathfrak{f}_{n,n+i}(\alpha_1,\dots,\alpha_{[\frac{n+i-1}{2}]})$, $\quad (\alpha_1,\alpha_2,\dots,\alpha_{[\frac{n+i-1}{2}]}) \in S_{n,n+i},\dots$ $(n \geq 5,\ i \geq 0)$.

Here, the parameter domains $S_{n,n+i}$ *are nonempty algebraic sets and* $(1,1,\dots,1) \in S_{n,n+i}$.

(2) (*a*) $n = 2m+1,\ m > 1:$
$\mathfrak{f}_{n,1}^1,\ \mathfrak{f}_{n,3}^1(\alpha_1),\ \mathfrak{f}_{n,5}^1(\alpha_1,\alpha_2),\dots,\mathfrak{f}_{n,2r-1}^1(\alpha_1,\alpha_2,\dots,\alpha_{r-1}),\dots$

(*b*) $n = 2m,\ m > 2:$
$\mathfrak{f}_{n,2}^1,\ \mathfrak{f}_{n,4}^1(\alpha_1),\ \mathfrak{f}_{n,6}^1(\alpha_1,\alpha_2),\dots,\mathfrak{f}_{n,2r}^1(\alpha_1,\alpha_2,\dots,\alpha_{r-1}),\dots$

In (1), $\mathfrak{f}_{n,k}(\alpha_1,\dots,\alpha_{[\frac{k-1}{2}]})$ *is defined inductively as the extension of* $\mathfrak{f}_{n,k-1}(\alpha_1,\dots,\alpha_{[\frac{k-2}{2}]})$ *given by the data*

$$t = 2(n+k-6)(n-3)^{-1},$$

$$B_{2,n}^{k-1} = B_{2,n+k-1} + \sum_{i=0}^{[\frac{k-1}{2}]} p_{k,i}(\alpha_0,\alpha_1,\dots,\alpha_{[\frac{k-1}{2}]})B_{3+i,k+3-i} \quad (k>1)$$

In (2*a*) *and* (2*b*) $\mathfrak{f}_{n,k}^1(\alpha_1,\dots,\alpha_{[\frac{k-1}{2}]})$ $\ (n+k$ *even) is the extension of* $\mathfrak{f}_{n,k}$ *defined on an algebraic (possibly empty) parameter domain* $S_{n,k}^1$ *by the data*

$$t = (3n-9+k)(n-3)^{-1}\ (1 \leq k \leq n-2), \quad B_{3,n}^k = \sum_{i=0}^{[\frac{n-1}{2}]} (-1)^i B_{3+i,n+k-i}.$$

Proof. We recall the basis relations of $\mathfrak{f}_{n,1}$ $(n \geq 5):$

$$[e_1,e_2] = (n-3)^{-1}e_2, \quad [e_1,e_i] = (n+i-6)(n-3)^{-1}e_i \quad (3 \leq i \leq n+1)$$
$$[e_2,e_j] = e_{j+1} \quad (3 \leq j \leq n), \quad [e_3,e_4] = e_{n+1}$$

We find first all extensions of dimension $n+2$. This will be the first step in our inductive argument. Let $t_{i,j}$ be the real number such that $t - t_{i,j}$ is the coefficient of $\omega_1 \wedge \omega_i \wedge \omega_j$ in the basis expansion of $\partial_t B_{i,j}$ (relative to $<\omega_k \wedge \omega_i \wedge \omega_j>_{1 \leq k<i<j \leq n+1}$, $\omega_i = e_i^*$). The following relations hold:

$$\partial_t B_{2,n+1} = (t - t_{2,n+1})\omega_1 \wedge \omega_2 \wedge \omega_{n+1} + \omega_2 \wedge \omega_3 \wedge \omega_4$$
$$\partial_t B_{3,5} = (t - t_{3,5})\omega_1 \wedge \omega_3 \wedge \omega_5 - \omega_2 \wedge \omega_3 \wedge \omega_4,$$

in which $t_{2,n+1} = t_{3,5} = (2n-4)(n-3)^{-1}$. Consequently, $B_{2,n+1} + B_{3,5}$ is a cocyle if and only if $t = (2n-4)(n-3)^{-1}$. This gives the Lie algebra $\mathfrak{f}_{n,2}$. Further, if $n = 2r+3$ is odd we have,

$$\begin{aligned}
\partial B_{3,n+1} &= (t - t_{3,n+1})\omega_1 \wedge \omega_3 \wedge \omega_{n+1} - \omega_2 \wedge \omega_3 \wedge \omega_n \\
\partial B_{4,n} &= (t - t_{4,n})\omega_1 \wedge \omega_4 \wedge \omega_n - \omega_2 \wedge \omega_3 \wedge \omega_n - \omega_2 \wedge \omega_4 \wedge \omega_{n-1} \\
\partial B_{5,n-1} &= (t - t_{5,n-1})\omega_1 \wedge \omega_5 \wedge \omega_{n-1} - \omega_2 \wedge \omega_4 \wedge \omega_{n-1} - \omega_2 \wedge \omega_5 \wedge \omega_{n-2} \\
&\vdots \\
\partial B_{3+r,4+r} &= (t - t_{3+r,4+r})\omega_1 \wedge \omega_{3+r} \wedge \omega_{4+r} - \omega_2 \wedge \omega_{2+r} \wedge \omega_{4+r}
\end{aligned}$$

If instead n is even, say $n = 2r+4$, the last equation above becomes

$$\partial B_{3+r,5+r} = (t - t_{3+r,5+r})\omega_1 \wedge \omega_{3+r} \wedge \omega_{5+r} - \omega_2 \wedge \omega_{3+r} \wedge \omega_{4+r}$$

In the first case $(n = 2r+3)$ the linear combination

$$B_{3,n}^{(0)} = B_{3,n+1} - B_{4,n} + B_{5,n-1} + \ldots + (-1)^r B_{r+3,r+4}$$

becomes a cocycle if $t = t_{3,n+1}$, since

$$t_{3,n+1} = t_{4,n} = \ldots = t_{r+3,r+4} = (3n-8)(n-3)^{-1}$$

However, the second case $(n = 2r+4)$ does not yield any cocycle since the term $\omega_2 \wedge \omega_{r+3} \wedge \omega_{r+4}$ can not be cancelled. On the other hand, the above is the only way $B_{3,n+1}$ can be combined with other forms to yield a cocycle, the term $-\omega_2 \wedge \omega_3 \wedge \omega_n$ occurring only in $\partial_t B_{3,n+1}$ and $\partial_t B_{4,n}$. The above gives the extension

$$\mathfrak{f}_{n,1}^1 : \quad t = (3n-8)(n-3)^{-1}, \quad B = B_{3,n}^{(0)} \quad (n = 2r+3)$$

of $\mathfrak{f}_{n,1}$.
Next, observe that, for $i > 3$,

$$\partial_t B_{i,n+1} =$$
$$(t - t_{i,n+1})\omega_1 \wedge \omega_i \wedge \omega_{n+1} - \omega_2 \wedge \omega_{i-1} \wedge \omega_{n+1} - \omega_2 \wedge \omega_i \wedge \omega_n + \omega_3 \wedge \omega_4 \wedge \omega_i,$$

in which $-\omega_2 \wedge \omega_{i-1} \wedge \omega_{n+1}$ occurs in no other $\partial_t B_{i,j}$. Consequently, we get no cocycles involving $B_{i,n+1}$ $(i > 3)$.
Finally, we must check if any other forms $B_{i,j}$ may combine so as to give

cocycles ($i < j < n+1$) whenever $t = t_{3,n+1}$ or $t = t_{2,n+1}$. Now the condition

$$t_{i,j} = (2n+(i+j)-12)(n-3)^{-1} = t_{2,n+1} = (2n-4)(n-3)^{-1} \quad (3 \le i < j \le n)$$

yields $i+j = 8$, which is easily seen to give no cocycle. Finally, $t_{i,j} = t_{3,n+1}$ implies $i + j = n + 4$, and we are reduced to $B_{4,n}, B_{5,n-1}, ..., B_{r+3,r+4}$ as treated above. Observe also there is exactly one n, $n = 4$, for which $(3n-8)(n-3)^{-1} = (2n-4)(n-3)^{-1}$, however this n is not permitted. We conclude that $\mathfrak{f}_{n,2}$ and $\mathfrak{f}^1_{n,1}$ are the only filiform extensions of $\mathfrak{f}_{n,1}$ having dimension $n+2$.

(I) : We prove next the inductive step for algebras $\mathfrak{f}_{n,k}$ satisfying $k < n$. The inductive step consists of two parts according to the parity of k.

(a) : Consider first extensions of $\mathfrak{f}_{n,2r+1}(\alpha_1, \alpha_2, ..., \alpha_r)$ $(3 \le 2r+1 \le n-3)$. We find, letting $\alpha = (\alpha_1, ..., \alpha_r)$,

$$\begin{aligned}
\partial B_{2,n+2r+1} &= (t - t_{2,n+2r-1})\omega_1 \wedge \omega_2 \wedge \omega_{n+2r+1} + \\
&\quad \sum_{i=0}^{r} p_{2r+1,i}(\alpha)\omega_2 \wedge \omega_{3+i} \wedge \omega_{2r+4-i} \\
\partial B_{3,2r+5} &= (t - t_{3,2r+5})\omega_1 \wedge \omega_3 \wedge \omega_{2r+5} - \omega_2 \wedge \omega_3 \wedge \omega_{2r+4} \\
\partial B_{4,2r+4} &= (t - t_{4,2r+4})\omega_1 \wedge \omega_4 \wedge \omega_{2r+4} - \omega_2 \wedge \omega_3 \wedge \omega_{2r+4} - \omega_2 \wedge \omega_4 \wedge \omega_{2r+3} \\
&\ \vdots \\
\partial B_{r+2,r+6} &= \\
&(t - t_{r+2,r+6})\omega_1 \wedge \omega_{r+2} \wedge \omega_{r+6} - \omega_2 \wedge \omega_{r+1} \wedge \omega_{r+6} - \omega_2 \wedge \omega_{r+2} \wedge \omega_{r+5} \\
\partial B_{r+3,r+5} &= \\
&(t - t_{r+3,r+5})\omega_1 \wedge \omega_{r+3} \wedge \omega_{r+4} - \omega_2 \wedge \omega_{r+2} \wedge \omega_{r+5} - \omega_2 \wedge \omega_{r+3} \wedge \omega_{r+4}
\end{aligned} \tag{4.7}$$

Taking linear combinations, $B_{2,n+2r+1} + \sum_{i=0}^{r} \alpha_{r+i+1} B_{3+i,2r+5-i}$ $(\alpha_{r+i+1} \in \mathbf{R})$, the cocycle condition gives the following linear system of equations for the coefficients α_i,

$$\begin{aligned}
\alpha_{r+i+1} + \alpha_{r+i+2} &= p_{2r+1,i}(\alpha) \qquad (0 \le i \le r-1), \\
\alpha_{2r+1} &= p_{2r+1,r}(\alpha)
\end{aligned} \tag{4.8}$$

which has the solution

$$\alpha_{r+i} = \sum_{j=0}^{r-i+1} (-1)^{j+r-i+1} p_{2r+1,r-j}(\alpha_1, \alpha_2, \ldots, \alpha_r) \quad (1 \le i \le r+1).$$

Hence each α_{r+i} $(1 \leq i \leq r+1)$ has been expressed as a polynomial, say $p_{2r+2,i-1}(\alpha_1, \alpha_2, ..., \alpha_r)$. Now *eq.* (4.8) may be reformulated as

$$p_{2r+2,i+1}(\alpha) = p_{2r+1,i}(\alpha) - p_{2r+2,i}(\alpha) \qquad (0 \leq i \leq r-1)$$
$$p_{2r+2,0}(\alpha) = \sum_{j=0}^{r}(-1)^{j+1}p_{2r+1,r-j}(\alpha) \tag{4.9}$$

We have found the extension

$$\mathfrak{f}_{n,2r+2}(\alpha_1, ..., \alpha_r): \quad t = (n-3)^{-1}(2n+2r-5), \quad B = B^r_{2,n}(\alpha_1, ..., \alpha_r),$$

in which

$$B^r_{2,n}(\alpha_1, ..., \alpha_r) = B_{2,n+2r+1} + \sum_{i=0}^{r} p_{2r+2,i}(\alpha_1, ..., \alpha_r)B_{3+i,2r+5-i}.$$

These Lie agebras are pairwise nonisomorphic for $(\alpha_1, ..., \alpha_r) \in \mathbf{R}^r$, since the corresponding quotients $\mathfrak{f}_{n,2r+1}(\alpha_1, ..., \alpha_r)$ are nonisomorphic.

(*b*) : Assume next the statement of the lemma holds for extensions of $\mathfrak{f}_{n,1}$ of dimension $\leq n+2r$ $(r \geq 2)$. Accordingly, the basis relations for $\mathfrak{f}_{n,2r}(\alpha_1, \alpha_2, ..., \alpha_{r-1})$ are as follows:

$$\begin{aligned}
&[e_1, e_2] = (n-3)^{-1}e_2,\\
&[e_1, e_i] = (n+i-6)(n-3)^{-1}e_i \; (3 \leq i \leq n+2r).\\
&[e_2, e_j] = e_{j+1} \; (3 \leq j \leq n+2r-1), \; [e_3, e_4] = e_{n+1},\\
&[e_3, e_j] = p_{j-3,0}(\alpha_1, .., \alpha_{[\frac{j}{2}]-2})e_{n+j-3} \; (4 < j \leq 2r+3)\\
&[e_4, e_j] = p_{j-2,1}(\alpha_1, ..., \alpha_{[\frac{j}{2}]-1})e_{n+j-2} \; (5 \leq j \leq 2r+2; r \geq 2)\\
&[e_5, e_j] = p_{j-1,2}(\alpha_1, ..., \alpha_{[\frac{j}{2}]-1})e_{n+j-1} \; (6 \leq j \leq 2r+1; r \geq 3)\\
&\quad\vdots\\
&[e_{r+1}, e_{r+2}] = p_{2r-1,r-2}(\alpha_1, ..., \alpha_{r-1})e_{n+2r-1},\\
&[e_{r+1}, e_{r+3}] = p_{2r,r-2}(\alpha_1, ..., \alpha_r)e_{n+2r}\\
&[e_{r+2}, e_{r+3}] = p_{2r-1,r-1}(\alpha_1, ..., \alpha_{r-1})e_{n+2r-1},\\
&[e_{r+2}, e_{r+4}] = p_{2r,r-1}(\alpha_1, ..., \alpha_r)e_{n+2r}
\end{aligned} \tag{4.10}$$

As usual, $[t]$ denotes the greatest integer less than or equal to t. We must determine the (filiform) extensions of $\mathfrak{f}_{n,2r}(\alpha_1, \alpha_2, ..., \alpha_{r-1})$. First, if $2r \leq$

$n-2$ (so that $[e_4, e_j] = 0$ if $n+j-2 = 2r+4$, $j \geq 4$), we find

$$\partial B_{2,n+2r} = (t - t_{2,n+2r})\omega_1 \wedge \omega_2 \wedge \omega_{n+2r} +$$
$$\sum_{i=0}^{r-1} p_{2r,i}(\alpha_1, ..., \alpha_{r-1})\omega_2 \wedge \omega_{3+i} \wedge \omega_{2r+3-i}$$
$$\partial B_{3,2r+4} = (t - t_{3,2n+4})\omega_1 \wedge \omega_3 \wedge \omega_{2n+4} - \omega_2 \wedge \omega_3 \wedge \omega_{2r+3}$$
$$\partial B_{4,2r+3} = (t - t_{4,2n+3})\omega_1 \wedge \omega_4 \wedge \omega_{2r+3} - \omega_2 \wedge \omega_3 \wedge \omega_{2r+3} - \omega_2 \wedge \omega_4 \wedge \omega_{2r+2}$$
$$\partial B_{5,2r+2} = (t - t_{5,2n+2})\omega_1 \wedge \omega_5 \wedge \omega_{2r+2} - \omega_2 \wedge \omega_4 \wedge \omega_{2r+2} - \omega_2 \wedge \omega_5 \wedge \omega_{2r+1}$$
$$\vdots$$
$$\partial B_{r+2,r+5} =$$
$$(t - t_{r+1,r+4})\omega_1 \wedge \omega_{r+1} \wedge \omega_{r+4} - \omega_2 \wedge \omega_{r+1} \wedge \omega_{r+5} - \omega_2 \wedge \omega_{r+2} \wedge \omega_{r+4}$$
$$\partial B_{r+3,r+4} =$$
$$(t - t_{r+2,r+3})\omega_1 \wedge \omega_{r+3} \wedge \omega_{r+4} - \omega_2 \wedge \omega_{r+2} \wedge \omega_{r+4} - \omega_2 \wedge \omega_{r+3} \wedge \omega_{r+3} \tag{4.11}$$

Hence, forming the linear combination,

$$B = B_{2,n+2r} + \sum_{i=1}^{r} \alpha_{r+i-1} B_{3+i,2r+4-i} \qquad (\alpha_{r+i-1} \in \mathbf{R}),$$

we conclude that B is a cocycle if and only if

$$\alpha_{r+i+1} + \alpha_{r+i} = p_{2r,i}(\alpha_1, ..., \alpha_{r-1}) \qquad (0 \leq i \leq r-1). \tag{4.12}$$

This linear system has the following solution for α_{r+i} $(1 \leq i \leq r)$, as a polynomial $p_{2r+1,i}$ in the r first parameters $(\alpha_1, ..., \alpha_r)$ $(1 \leq i \leq r)$,

$$\alpha_{r+i} = \sum_{j=0}^{i-1} (-1)^{i-j+1} p_{2r,j}(\alpha_1, ..., \alpha_{r-1}) + (-1)^i \alpha_r = p_{2r+1,i}(\alpha_1, ..., \alpha_r)$$

Thus, letting $p_{2r+1,0}(\alpha_1, .., \alpha_r) = \alpha_r$, we have the cocyle

$$B_{2,n+2r} + p_{2r+1,0}(\alpha_1, .., \alpha_r) B_{3,2r+4} + \sum_{i=1}^{r-1} p_{2r+1,i}(\alpha_1, \alpha_2, ..., \alpha_r) B_{i+4,2r+4-i},$$

$(2r \leq n-2)$ whenever $t = t_{2,n+2r} = t_{3,2r+4} = ... = t_{r+2,r+3} =$ $(2n+2r-5)(n-3)^{-1}$. We have proved the existence of the Lie algebra $\mathfrak{f}_{n,2r+1}(\alpha_1, \alpha_2, ..., \alpha_r)$

$(2r+1=1,3,\ldots,n-1)$.
It is readily seen that this is the only way of forming cocycles with nonzero $B_{2,n+k}$ component.
(II) : Now, if $k=2r\geq n$, $\partial_t B_{3,2r+4}$ contains the term $p_{2r-n+4,1}(\alpha_1,\ldots,\alpha_{r+2-[\frac{n}{2}]})\omega_3\wedge\omega_4\wedge\omega_{2r-n+6}$ which cannot be cancelled for all values of the parameters α_i. A similar observation applies to the case $k=2r+1\geq n$. However, we may still form extensions $\mathfrak{f}_{n,k+1}$ as above, with domains of definition restricted to the solution set $S_{n,k}$ of a certain system of polynomial equations given by the $p_{i,j}-$s. Indeed, the form $B_{2,n}^{k-1}$ of our proposition will satisfy the cocycle identity if and only if the coefficient matrix of the system of $\partial_t B_{i,j}-$s (compare (4.7) and (4.11) above) is row dependent. This yields a system of (2-nd degree in the parameters α_i) polynomial equations defined in terms of sums of products of pairs $\pm p_{i,j}$. (For example, one such equation is $q_i^n=p_{i+3,1}p_{n+i,0}-p_{i+2,0}p_{n+i,1}+p_{n+i,i+2}=0$ $(n\geq i+4,\ k=n+i)$.) The sets $S_{n,k}$ are never empty. In fact, using that $p_{2r+1,0}(1,\ldots,1)=1$ and $p_{i,j}(1,\ldots,1)=0$ if $j\neq 0$, we find that $(\alpha_1,\ldots,\alpha_{[\frac{k-1}{2}]})=(1,\ldots,1)$ always gives the cocycle, $B_{2,n+2r}+B_{3,2r+4}$. Similar arguments apply to the algebras $\mathfrak{f}^1_{n,k}$, however, in this case it is not clear that the corresponding algebraic parameter domains $S^1_{n,k}$ are always nonempty (in fact, one can show that $\mathfrak{f}^1_{5,k}$ does not exist for $k>7$).
Next if $i>3$, we have

$$\partial_t B_{i,n+2r}=(t-t_{i,n+2r})\omega_1\wedge\omega_i\wedge\omega_{n+2r}-\omega_2\wedge\omega_{i-1}\wedge\omega_{n+2r}-$$
$$-\omega_2\wedge\omega_i\wedge\omega_{n+2r-1}+\text{"other terms"}.$$

Observe that $\omega_2\wedge\omega_{i-1}\wedge\omega_{n+2r}$ occurs in no other $\partial_t B_{i,j}$, hence no cocycle possesses a nonzero component along $B_{i,n+2r}$. (The argument is similar for $B_{i,n+2r+1}$.)
Finally, we must check if any other forms $B_{i,j}$ may combine to give cocycles $(1<j<n+2r)$ whenever $t=t_{3,n+2r}$ or $t=t_{2,n+2r}$. Now

$$t_{i,j}=(2n+(i+j)-12)(n-3)^{-1}=t_{3,n+2r}=(3n+2r-9)(n-3)^{-1}$$

yields

$$i+j=n+2r+3,$$

and we are reduced to the case of $B_{3,n}^{2r}$ above.
Similarly,

$$t_{i,j}=t_{2,n+2r}=(2n+2r-5)(n-3)^{-1}\quad(3\leq i<j\leq n)$$

implies

$$i + j = 2r + 7 \qquad (2r < n - 2).$$

Thus we get the forms

$$B_{3,2r+4}, \dots, \quad B_{r+3,r+4}.$$

However,this yields only the admissible cocycle $B^{2r}_{2,n}$, as treated above ($\mathfrak{f}_{n,k+1}$). **Q.E.D.**

4.6. EXAMPLE. We conclude this section indicating the extension graph of the family $\mathfrak{f}_{5,k}$. The arguments are similar for the algebras $\mathfrak{f}_{n,k}$ with $n > 5$, however, in general the equations are of course harder to treat. First, $S_{5,5} = S_{5,6} = Z(q_{1,1})$, $q_{1,1} = p_{3,1}p_{5,0} - p_{5,1} + p_{5,2}$ ($Z(p)$ stands for the zero-set of a polynomial p). Next, one readily verifies that $S_{5,7} = S_{5,6} \cap Z(q_{2,1}, q_{2,2}) = S_{5,6} \cap Z(q_{2,1})$, since $q_{2,1} = p_{5,1}p_{7,0} - p_{4,0}p_{7,1} - p_{7,3}$, $q_{2,2} = p_{5,2}p_{7,0} - p_{3,0}p_{7,2} + p_{7,3}$ and $q_{2,1} + q_{2,2} = q_{1,1}$. Now, calculating $\mathfrak{f}_{5,8}$ as an extension of $\mathfrak{f}_{5,7}$, the 3-forms $\partial_t B_{2,12}, \partial_t B_{3,15}, \partial_t B_{4,14}, \dots, \partial_t B_{8,10}$ must be linearly dependent, so that the corresponding coefficient matrix

$$\begin{pmatrix} p_{7,0} & p_{7,1} & p_{7,2} & p_{7,3} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & p_{6,1} & p_{6,2} & 0 \\ -1 & -1 & 0 & 0 & -p_{5,0} & 0 & p_{5,2} \\ 0 & -1 & -1 & 0 & 0 & -p_{4,0} & -p_{4,1} \\ 0 & 0 & -1 & -1 & -1 & 0 & p_{3,1} \end{pmatrix} \tag{4.13}$$

must be row dependent. Multiplying the i-th row by $p_{8,i-2}$ $(2 \le i \le 5)$ and performing row operations using eq. (4.6), the first row can be transformed into

$$(0 \;\; 0 \;\; 0 \;\; 0 \;\; q_{3,1} \;\; q_{3,2} \;\; q_{3,3}), \tag{4.14}$$

in which $q_{3,1} = p_{6,1}p_{8,0} - p_{5,0}p_{8,1}$, $q_{3,2} = p_{3,1}p_{8,3} - p_{4,1}p_{8,2} + p_{5,2}p_{8,1}$, and $q_{3,3} = -p_{4,0}p_{8,2} + p_{6,2}p_{8,0}$. Invoking eq. (4.6) again we find,

$$q_{3,1} + q_{3,2} = q_{2,1} \text{ and } q_{3,2} + q_{3,3} = q_{2,2}.$$

Hence it follows that $S_{5,8} = S_{5,7} \cap Z(q_{3,1}) = Z(q_{1,1}, q_{2,1}, q_{3,1})$. We observe that because of (4.14), it would have sufficed to discuss row dependence of the matrix

$$M(2) = \begin{pmatrix} q_{3,1} & q_{3,2} & q_{3,3} \\ p_{6,1} & p_{6,2} & 0 \\ -p_{5,0} & 0 & p_{5,2} \\ 0 & -p_{4,0} & -p_{4,1} \\ -1 & 0 & p_{3,1} \end{pmatrix} \tag{4.15}$$

in place of (4.13). The argument is similar for $\mathfrak{f}_{n,k}$, $n > 5$, moreover, the corresponding matrix M turns out to depend only on $k-n$, rather than on n and k; we write $M = M(k-n)$.
Eliminating the parameters α_2 and α_3 from the equations $q_{1,1} = 0$, $q_{2,1} = 0$, and $q_{3,1} = 0$, we end up with the equation

$$10\alpha_1^6 - 59\alpha_1^5 + 145\alpha_1^4 - 190\alpha_1^3 + 140\alpha_1^2 - 55\alpha_1 + 9 = (10\alpha_1 - 9)(\alpha_1 - 1)^5 = 0. \tag{4.16}$$

Consequently, $S_{5,8} = \{(1,1,1), (\frac{9}{10}, \frac{5}{7}, \frac{7}{12})\}$. The extension tree of $\mathfrak{f}_{5,k}$ is as follows,

$\mathfrak{f}_5(\alpha)$

$\downarrow(\alpha = \frac{3}{2})$

$\mathfrak{f}_{5,1} \to \mathfrak{f}_{5,2} \to \mathfrak{f}_{5,3}(\alpha_1) \to \mathfrak{f}_{5,4}(\alpha_1) \to \mathfrak{f}_{5,5}(\alpha_1,\alpha_2) \to \mathfrak{f}_{5,6}(\alpha_1,\alpha_2) \to \mathfrak{f}_{5,7}(\alpha_1,\alpha_2,\alpha_3) \to \mathfrak{f}_{5,8} \to \mathfrak{f}_{5,9} \to \dots$

$S_{5,5}=Z(q_{1,1})$ $\quad S_{5,6}=Z(q_{1,1})$ $\quad S_{5,7}=Z(q_{1,1},q_{2,1})$ $\quad S_{5,8}$ $\quad S_{5,9}$

$\searrow$ $\quad \searrow(\alpha_1=3)$ $\quad \searrow$ $\quad \searrow$ $\quad \searrow$

$\mathfrak{f}_{5,1}^1$ $\quad \mathfrak{f}_{5,3}^1$ $\quad \mathfrak{f}_{5,5}^1(\alpha_1 \in \{0,2\})$ $\quad \mathfrak{f}_{5,7}^1(\alpha_1 = \frac{5 \pm \sqrt{10}}{5})$ *No extension*

$p_{3,1}+p_{2,0}+1=0$ $\quad p_{5,1}+p_{3,0}+1=0$ $\quad p_{7,1}+p_{6,0}+1=0$

We note that the solution $\alpha_1 = \frac{9}{10}$ yields a Lie algebra isomorphic to the prosolvable subalgebra $t\mathbf{R}[t]\frac{d}{dt}$ of the polynomial vectorfields. Further, the two real algebras $\mathfrak{f}_{5,7}^1(S_{5,7}^1)$ of dimension 13 are rigid, possessing *nonrational* structure constants. Here, $S_{5,7}^1 = S_{5,7} \cap Z(p_{7,1} + p_{6,0} + 1)$ is determined by the solutions of the equation $15\alpha_1^4 - 40\alpha_1^3 + 44\alpha_1^2 - 36\alpha_1^1 + 9 = 0$, whose real solutions are the irrational numbers $\alpha_1 = \frac{5 \pm \sqrt{10}}{5}$ (the nonreal solutions are $\frac{1 \pm i2\sqrt{2}}{3}$).

5. Main classification theorems.

5.1. We summarize our previous results in the following classification theorem.

THEOREM 1. *Let $\mathfrak{g}$ be a real Lie algebra of dimension greater than three, and assume $\mathfrak{g} \in FS_1$. Then $\mathfrak{g}$ is isomorphic to one of the following pairwise* *nonisomorphic Lie algebras (for some integers n, k)*

$\mathfrak{a}_n$, $\mathfrak{a}_{n,1}$ (n even), $\mathfrak{b}_n$, ${}^1\mathfrak{b}_n(\alpha_0, \ldots, \alpha_{\frac{n}{2}-4})$ (n even, $\vec{\alpha} \in D_n$), $\mathfrak{b}_{n,k}$, $\mathfrak{b}^k_{2n}$, $\mathfrak{c}_n(\alpha)$, $\mathfrak{d}_n(\alpha)$ $(\alpha \geq 0)$, $\mathfrak{e}_4(\beta)$ $(\beta \geq 0)$, $\mathfrak{f}_n(\beta)$ $(\beta \in \mathbf{R} \setminus \{1\})$, $\mathfrak{f}^{(r)}_{n,1}$ $(2 \leq 2r \leq n-4)$, ${}^1\mathfrak{f}_n(\beta)$ (n even, $\beta \in \mathbf{R} \setminus \{1\}$), $\mathfrak{f}^1_{n,1}$ (n odd), $\mathfrak{f}^{([\frac{n-3}{2}])}_{n,2}$ $\mathfrak{f}^{(\frac{n-4}{2})}_{n,3}(\gamma)$ (n even, $\gamma \in \mathbf{R}$), $\mathfrak{f}^{(\frac{n-3}{2})}_{n,3}$ (n odd), $\mathfrak{f}_{n,k}(\beta_1, \ldots, \beta_{[\frac{k-1}{2}]})$ $(1 < k)$, $\mathfrak{f}^1_{n,k}(\beta_1, \ldots, \beta_{[\frac{k-1}{2}]})$ ($k \geq 1$, $n+k$ even), $\mathfrak{h}_n$.

If $k < n$, the family $\mathfrak{f}_{n,k}$ is defined on $S_{n,k} = \mathbf{R}^{[\frac{k-1}{2}]}$. If $k \geq n$, the parameter domains $S_{n,k}$ of the Lie algebras $\mathfrak{f}_{n,k}(\beta_1, \cdots, \beta_{[\frac{k-1}{2}]})$ are nonempty algebraic sets, given as the solution sets of finitely many second order equations defined by sums of polynomials of the form $\pm p_{i,j}p_{l,m}$. Finally, the parameter domains $S^1_{n,k}$ of the Lie algebras $\mathfrak{f}^1_{n,k}(\beta_1, \cdots, \beta_{[\frac{k-1}{2}]})$ are algebraic, possibly empty, sets.

5.2. COROLLARY. *Let $\mathfrak{g} \in FS_{1,n}$ with $n > 4$. If the center of $\mathfrak{g}$ is nonzero then $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{f}_n(\frac{n-4}{n-3})$.*

5.3. Let U_n $(n > 4)$ be the subclass of $FS_{1,n}$ consisting of all Lie algebras given uniquely within isomorphisms by their nilradical.

COROLLARY. (*a*) *$FS_{1,n} \setminus U_n$ consists of all Lie algebras isomorphic to one of the following, $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{c}_n(\alpha), \mathfrak{d}_n(\alpha)$ $(\alpha \geq 0), \mathfrak{f}_n(\beta)$ $(\beta \in \mathbf{R} \setminus \{1\})$, and $\mathfrak{h}_n$.*

(*b*) *The nilradical of each $\mathfrak{g} \in U_n$ has a one dimensional maximal torus (cf. [Br]).*

(*c*) *The algebras $\mathfrak{g} = \mathbf{R}e_1 \oplus \mathfrak{n}$ ($\mathfrak{n}$ = the nilradical) of $FS_{1,n}$ for which e_1 is $ad_{\mathfrak{g}}$-semisimple, are (within isomorphisms) $\mathfrak{c}_n(\alpha), \mathfrak{d}_n(\alpha)$, and $\mathfrak{h}_n$.*

5.4. In light of Thm.1, the algebras in $FS_{1,n+1} \setminus U_{n+1}$ all have nilradicals isomorphic to $\mathfrak{n}_n$ whose nonvanishing Lie relations are $[e_2, e_i] = e_{i+1}$ $(i = 3, 4, ..., n)$. Now consider the class $D_{n,k}$ of all completely solvable real Lie algebras $\mathfrak{g} = \mathfrak{a} \oplus_\theta \mathfrak{n}$ (semidirect product), whose n-dimensional nilradical $\mathfrak{n}$ is filiform of codimension $k > 0$. We note that $D_{n,1}$ coincides with the class $FS_{1,n+1}$. If $\mathfrak{g} \in D_{n,k}$ with $k > 1$, it follows from Cor. 5.3 (*a*) that $\mathfrak{n}$ is isomorphic to $\mathfrak{n}_n$ and the eigenvalues of each $ad(a)$ must be of "type" $\mathfrak{a}_{n+1}, \mathfrak{b}_{n+1}, \mathfrak{c}_{n+1}(\alpha), \mathfrak{d}_{n+1}(\alpha)$ $(\alpha \geq 0), \mathfrak{f}_{n+1}(\beta)$ $(\beta \in \mathbf{R} \setminus \{1\})$, or $\mathfrak{h}_{n+1}$. Consequently, we arrive at the following classification of $D_{n,k}$:

COROLLARY. *Let $\mathfrak{g} \in D_{n,k}$, $k > 1$, and write $\mathfrak{g} = \mathfrak{a} \oplus_\theta \mathfrak{n}$ as above. Then we can choose a basis $< e_i >_{i=1}^k$ for $\mathfrak{a}$ such that the eigenvalues of each $ad(e_i)|\mathfrak{n}$ are (within isomorphisms) among the following types:*

$\mathfrak{a}_n : (1,0,1,2,...,n-3)$, $\mathfrak{b}_n$, $\mathfrak{c}_n(\alpha)$ *and* $\mathfrak{d}_n(\alpha) : (1,0,1,1,...,1)$,

$\mathfrak{f}_n(\beta) : (\beta-1,1,\beta,2\beta-1,...,(n-3)\beta-(n-4))$, $\mathfrak{h}_n : (1,1,2,3,...,n-4)$.

If, in addition, $\mathfrak{a}$ is $ad_\mathfrak{g}$-semisimple, only the types $\mathfrak{a}_n, \mathfrak{b}_n$, and $\mathfrak{f}_n(\beta)$ can occur.

5.5. LEMMA. *Let $\mathfrak{a}$ and $\mathfrak{n}$ be nilpotent Lie algebras, $\mathfrak{a}$ abelian. Assume we can form the Lie algebra extension of $\mathfrak{a}$ by $\mathfrak{n}$ with the corresponding extension data $\tilde{\theta} : \mathfrak{a} \to Der(\mathfrak{n})$, a representation, and $C \in H^2(\mathfrak{a},\mathfrak{n};\tilde{\theta})$ a two-cocycle. Assume further we can find a basis $< e_i >_{i=1}^n$ for $\mathfrak{n}$ satisfying*

(i) $[e_2, e_i] = e_{i+1} \quad (2 < i < n)$,

(ii) there is an $r \in \{3,4,...,n-1\}$ such that $[e_r, e_i] = 0 \quad (i \neq 2)$.

Then C is center valued.
Assume in addition to the above:

(iii) there is an $a \in \mathfrak{a}$ such that $\tilde{\theta}(a)e_n = te_n, \quad (t \neq 0)$.

Then C vanishes.

Proof. We form the solvable Lie algebra extension $\mathfrak{g}$ of $\mathfrak{a}$ by $\mathfrak{n}$ associated to the data (θ, C) by identifying the space of $\mathfrak{g}$ to the direct sum $\mathfrak{a} \oplus \mathfrak{n}$ and defining the Lie product as follows,

$$[(v,m),(w,n)] = (0, \theta(v)n - \theta(w)m + C(v,w) + [m,n]) \quad (v,w \in \mathfrak{a},\ m,n \in \mathfrak{n}).$$

We note that, in this realization, we have $\tilde{\theta}(u,0)(0,n) = (0,\theta(u)n)$ and $[\theta(u),\theta(v)] = ad\,C(u,v)$ $(u,v \in \mathfrak{a},\ n \in \mathfrak{n})$. The Jacobi identity and the fact that each $\theta(u)$ $(u \in \mathfrak{a})$ is a derivation of $\mathfrak{n}$, gives the following cocycle identity for $C :$ $(u,v,w \in \mathfrak{a}; k,m,n \in \mathfrak{n})$,

$$0 = \sum\{\theta(u)C(v,w) + [k, C(v,w)]\}$$

where the sum is taken over cyclic permutation of the triple (u,k), (v,m), (w,n). Now, letting $k = m = n = 0$ we find $\sum \theta(u)C(v,w) = 0$. Next, write $C = \sum_{i=1}^n C_i e_i$. Letting $k = m = n = e_2$, condition (i) yields

$$0 = \sum_{<u,v,w>} [e_2, C(v,w)] = \sum_{<u,v,w>} \sum_{j=3}^{n-1} C_j(v,w)e_{j+1}.$$

Consequently, $C_j = 0, \quad j = 3, 4, ..., n-1$. Similarly, taking $k = m = n = e_r$ and applying condition (ii), we find $C_2 = 0$. Accordingly $C = C_n e_n$ is center valued. Finally, using condition (iii) we have

$$0 = \sum_{<u,v,w>} \theta(a)C(v,w) = t(C(v,w) + C(u,v) + C(w,u)) \quad (u,v,w \in \mathfrak{a}),$$

which clearly implies $C = 0$. **Q.E.D.**

5.6. THEOREM 2. *Let $\mathfrak{g}$ be a completely solvable non-nilpotent Lie algebra whose nilradical $\mathfrak{n}$ is filiform. Then $\mathfrak{g}$ decomposes into a semidirect product $\mathfrak{a} \oplus_\theta \mathfrak{n}$, where $\mathfrak{a}$ is an abelian subalgebra.*

Proof. If $\mathfrak{g}$ has trivial center, we can find an $a \in \mathfrak{g} - \mathfrak{n}$ such that $ad(a)$ does not commute with the one-dimensional center $\mathbb{R}z$ of $\mathfrak{n}$. Consequently, $ad(a)z = tz$ for some nonzero real t. Since the ideal generated by a and $\mathfrak{n}$ belongs to FS_1, $\mathfrak{n}$ satisfies the first two conditions of Lemma 5.5, and we conclude that $\mathfrak{g}$ is a semidirect product (the cocycle vanishes). Finally, if $\mathfrak{g}$ has nontrivial center, the ideal generated by $\mathfrak{n}$ and $\mathfrak{a}$ is either isomorphic to $\mathfrak{f}_{n+1}(\frac{n-3}{n-2})$ (Cor. 5.2), or else it is a direct product of $\mathbb{R}$ and $\mathfrak{n}$. Observe that $\theta \neq 0$, since otherwise $\mathfrak{g}$ would be nilpotent. Hence we can find a basis $< e_i >_{i=1}^{k}$ for $\mathfrak{a}$ such that $\theta(e_i) = 0$ $(i = 2, 3, ..., k)$ and the eigenvalues of $\theta(e_i)|\mathfrak{n}$ are $1, n-3, , n-4, ..., 0$. But then $\mathfrak{n} \oplus \sum_{i=2}^{k} \mathbb{R}e_i$ is a nilpotent ideal, contrary to our assumptions. The theorem follows. **Q.E.D.**

5.7. On the basis of Thms.1 and 2, rigidity properties of the class FS_1 can be worked out. In fact, we have the following more general result:

THEOREM 3. *Let $\mathfrak{g}$ be a completely solvable non-nilpotent Lie algebra whose nilradical is filiform and of dimension > 4. If $\mathfrak{g}$ is rigid, then either $\mathfrak{g}$ is the unique element in $D_{n,2}$ $(n > 2)$ of mixed type $\mathfrak{b}_{n+1}, \mathfrak{f}_{n+1}(\beta)$ for any $\beta \neq 1$ (they are all isomorphic), or else $\mathfrak{g}$ belongs to the class FS_1 and is isomorphic to one of the following Lie algebras (for some $n > 4$) :*

$$\mathfrak{f}_{n,1}^{(r)} \; (2 \leq 2r < n-2), \mathfrak{f}_{n,2}, \mathfrak{f}_{n,2}^{([\frac{n-3}{2}])}, \mathfrak{f}_{n,3}^{(\frac{n-3}{2})} \; (n \textit{ odd}), \mathfrak{f}_{n,k}^{1} \; (k = 1, 2)$$

and, in addition, all elements of the families

$$\mathfrak{f}_{n,k}^{1}(\beta_1, \ldots, \beta_{[\frac{k-1}{2}]}) \quad (n + k \;\; \textit{even}) \textit{ and } \mathfrak{f}_{n,k}(\beta_1, \ldots, \beta_{[\frac{k-1}{2}]}),$$

whose parameter domains $S^1_{n,k}$ and $S_{n,k}$ (resp.) are finite.

Proof. Assume first that $\mathfrak{g}$ does not belong to FS_1. If $\mathfrak{g} \notin D_{n,k}$ non-rigidity follows from [C1, Prop. 3.1]. If, on the other hand, $\mathfrak{g} \in D_{n,k}$ then $k = 1$ or 2 by [AG, II.2]. If $k = 2$ the result follows from the theorem in [AG,II.3].

Finally, if $\mathfrak{g} \in FS_1$ is rigid, let e_1 be any nonzero element of $\mathfrak{g} - \mathfrak{n}$. We can find a basis $\{e_2, ..., e_n\}$ of the nilradical $\mathfrak{n}$ consisting of eigenvectors of $ad(e_1)$ and which satisfies the relations $[e_2, e_i] = e_{i+1}$ $(2 < i < n)$, [AG, Lemme II.1.1]. Using this and the rank theorem in [AG], we derive quickly by inspection of the algebras listed in Theorem 1 that $\mathfrak{g}$ must be an extension of $\mathfrak{f}_m(\beta)$ for some $m \geq 5$. Further, excluding parameter families, we end up with the Lie algebras of Thm.3. Finally, rigidity follows along the lines of [C2, Prop.(4.3)]: Note first that all the Lie algebras listed in the second part of Thm. 3 decomposes as semidirect products $\mathfrak{t} \oplus_\theta \mathfrak{n}$ of the ($ad_\mathfrak{g}$-semisimple) torus $\mathfrak{t} = \mathbb{R}e_1$ and the nilradical $\mathfrak{n}$. Moreover, the eigenvalues of $ad(e_1)$ are of the form $0, \alpha, 2\alpha, ..., n\alpha$ for a positive α, depending on $\mathfrak{g}$. Replacing e_1 by $\alpha^{-1}e_1$, we can assume $\alpha = 1$. Furthermore, by Cor. 5.3 (ii), $\mathfrak{t}$ is a maximal external torus for $\mathfrak{n}$. In fact, any such $\mathfrak{g}$ is uniquely given (within isomorphisms) by its nilradical. It follows that $H^1(\mathfrak{g}, \mathfrak{g}) = (0)$ (all the derivations of $\mathfrak{g}$ are in the conjugacy class of $\mathfrak{t}$).

We denote by V_ϵ the intersection of the following subsets of the variety $Lie_{n+1}(\mathbb{R})$:

$V = \{\mathfrak{g} : H^0(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = (0)\}$ (i.e. $\mathfrak{g}$ is complete).

V_1 consists of all $\mathfrak{g}$ whose maximal torus in $Aut(\mathfrak{g})$ has dimension less than 2.

U_ϵ consists of all $\mathfrak{g}$ for which there exists an x such that the spectrum of $ad_\mathfrak{g}x$ (within permutations) is contained in the open ball with radius ϵ in $\mathbb{R}^{n+1}$, centered at $(0, 1, 2, ..., n)$.

Observe that the above sets are open in $Lie_{n+1}(\mathbb{R})$ with respect to the metric topology, hence they are also Zariski open. Now, for ϵ sufficiently small, the Levi factor must vanish for any $\mathfrak{g} \in V_\epsilon$ (since $\mathfrak{g} \in U_\epsilon$). It follows that $\mathfrak{g}$ can be written as a semidirect product $\mathfrak{t} \oplus \mathfrak{n}$ of its one dimensional maximal torus $\mathfrak{t}$ and its nilradical $\mathfrak{n}$. Next, for Lie algebras of a fixed dimension there is only a finite number of conjugacy classes of maximal tori, [Br]. Consequently, we can choose ϵ so small that the weights of $\mathfrak{t}$ are of the form $(0, \alpha, 2\alpha, ..., n\alpha)$ for some $\alpha > 0$. The remaining part of the proof

is partitioned into several subcases.
First, let W_ϵ be the Zariski open subset of V_ϵ consisting of those algebras whose structure constants satisfies

(a): $c_{2,j}^{j+1} \neq 0\ (3 < j < n+1)$ and $c_{3+i,2r+2-i}^{n+1} \neq 0\ (2 \leq 2r \leq n-4,\ i = 0,1,...,r-1)$, where $r \in \{1,2,...,[\frac{n-3}{2}]\}$.

An inductive argument similar to the proof of Lemma 4.1, shows that $\mathfrak{g}/\mathbb{R}e_{n+1}$ is isomorphic to $\mathfrak{f}_n(\beta)$ for some β. By Thm.1 and the second condition in (a), $\mathfrak{g}$ must then be isomorphic to $\mathfrak{f}_{n,1}^{(r)}$ for some r. We have shown that W_ϵ is contained in the $GL_{n+1}(\mathbb{R})$ orbit of $\mathfrak{f}_{n,1}^{(r)}$. The remaining cases are similar, the conditions on the structure constants to consider are as follows:

If in addition to (a),
(b): $c_{3+i,n-2-i}^{n+1} \neq 0\ (i = 0,1,...,r-1)$, $r = [\frac{n-4}{2}]$,
then $\mathfrak{g} \cong \mathfrak{f}_{n-1,2}^{(r)}$, $r = [\frac{n-4}{2}]$.

If in addition to (a),
(c): $c_{3+i,n-1-i}^{n+1} \neq 0\ (i = 0,1,...,r-1)$, $r = [\frac{n-5}{2}]$
then $\mathfrak{g} \cong \mathfrak{f}_{n-2,3}^{(r)}$, $r = [\frac{n-5}{2}]$.

Suppose (a) holds with $r = 1$ and
(d): $c_{3,5}^{n+1} \neq 0$,
then $\mathfrak{g} \cong \mathfrak{f}_{n-1,2}$.

Let $\mathfrak{g}$ be a Lie algebra with basis $\{e_1, e_2, ..., e_{n+k+1}\}$, $(n > 4, k > 2)$ whose structure constants are $c_{i,j}^k$. Assume $c_{2,j}^{j+1} \neq 0\ (3 < j < n+1)$. If
(e): $c_{3+i,4+k-i}^{n+k+1} \neq 0\ (i = 0,1,...,[\frac{k-1}{2}])$,
then $\mathfrak{g} \cong \mathfrak{f}_{n,k+1}(\alpha_1,...,\alpha_{[\frac{k-1}{2}]})$ for some $\vec{\alpha} \in S_{n,k+1}$.
(Accordingly, these algebras are rigid whenever the corresponding parameter domains $S_{n,k+1}$ are finite, cf. Ex. 4.6).

If $n+k$ is even and
(f): $c_{3+i,n+k-i}^{n+k+1} \neq 0\ (i = 0,1,...,\frac{n-k}{2}-2)$,
then $\mathfrak{g} \cong \mathfrak{f}_{n,k}^1(\alpha_1,...,\alpha_{[\frac{k-1}{2}]})$, for some $(\alpha_1,...,\alpha_{[\frac{k-1}{2}]}) \in S_{n,k}^1$.
(As a consequence, these algebras are rigid whenever the parameter domains $S_{n,k}^1$ are finite). **Q.E.D.**

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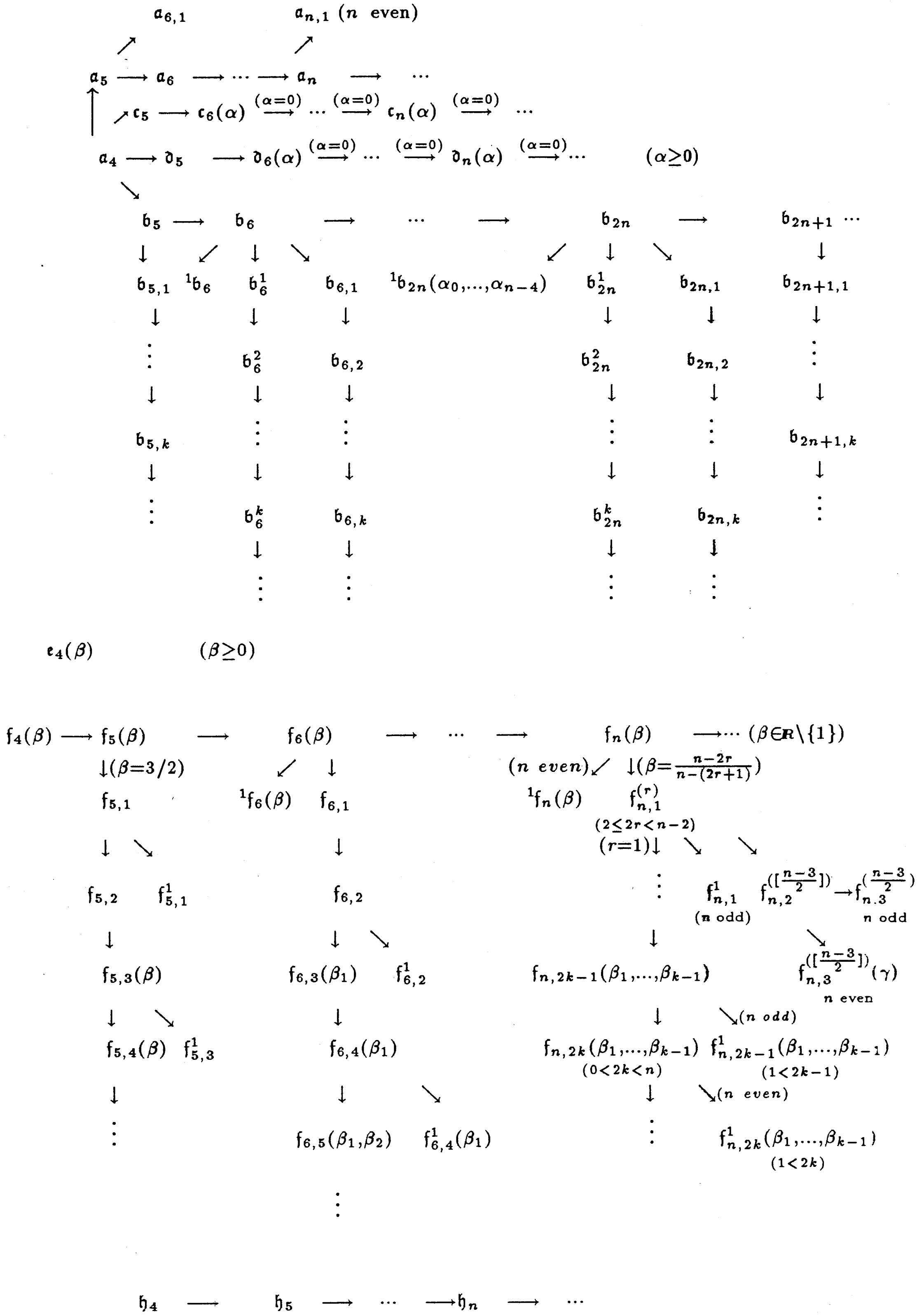
6. Extension graphs of the class FS_1.

Note: The arrows indicate direction of extensions; quotient maps run in the opposite directions.



7. Tables of filiform solvable Lie algebras.

TABLE 1

$\mathfrak{g}$	*Param.* *domain*	$\theta(e_1)$	*Cocycle B*	$\tilde{\mathfrak{g}} = \mathfrak{g}(B,\theta)$
$\mathfrak{a}_n$		$n-2$	$B_{2,n}$	$\mathfrak{a}_{n+1}$
		$n-3$	$B_{3,n} - B_{4,n-1} + \cdots + (-1)^r B_{r+1,r+2}$ $(n=2r)$	$\mathfrak{a}_{n,1}$ (n even)
$\mathfrak{b}_n$		1	$B_{3,n}$	$\mathfrak{b}_{n+1}$
		1	$B_{3,n} + B_{1,2}$	$\mathfrak{b}_{n,1}$
		1	$B_{3,n} - B_{1,2}$	$\mathfrak{b}_n^1$ (n even)
		2	$\alpha_0 B_6 + \alpha_1 B_8 + \cdots + \alpha_{\frac{n}{2}-4} B_n$ $B_k = B_{2,k} + \sum_{i=1}^{r-3} (-1)^{i-1} B_{3+i,k-i}$ $(k=2r-2)$	${}^1\mathfrak{b}_n(\alpha_0, \ldots, \alpha_{\frac{n}{2}-3})$ (n even, $\vec{\alpha} \in D_n$)
$\mathfrak{c}_n(0)$		1	$\alpha B_{1,2} + B_{3,n} - B_{1,n-1}$	$\mathfrak{c}_{n+1}(\alpha)$ $\alpha \geq 0$
$\mathfrak{d}_n(0)$		1	$\alpha B_{1,2} + B_{3,n} + B_{1,n-1}$	$\mathfrak{d}_{n+1}(\alpha)$ $\alpha \geq 0$
$\mathfrak{f}_n(\beta)$	$\mathbb{R} \setminus \{1\}$	$(n-2)\beta - (n-3)$	$B_{2,n}$	$\mathfrak{f}_{n+1}(\beta)$
	$\frac{n-2r}{n-(2r+1)}$	$\frac{2n-(2r+3)}{n-(2r+1)}$	$B_{2,n} + \sum_{i=0}^{r-1} (-1)^i B_{3+i,2r+2-i}$	$\mathfrak{f}_{n,1}^{(r)}$ $(r=1,2,\ldots,[\frac{n-2}{2}])$
	$\mathbb{R} \setminus \{1\}$	$(n-3)\beta - (n-5)$	$B_{3,n} - B_{4,n-1} + \cdots + (-1)^r B_{r+1,r+2}$ $(n=2r,\ r \geq 3)$	${}^1\mathfrak{f}_n(\beta)$ (n even)
$\mathfrak{h}_n$		$n-1$	$B_{3,n}$	$\mathfrak{h}_{n+1}$

TABLE 2

$\mathfrak{g}$	$\theta(e_1)$	Cocycle B	$\tilde{\mathfrak{g}} = \mathfrak{g}(B,\theta)$
$\mathfrak{b}_{n,k}$	1	$B_{3,n+k}-B_{1,k+3}$	$\mathfrak{b}_{n,k+1}$
$\mathfrak{b}_n^k$ (n even)	1	$B_{3,n+k}+B_{1,k+3}$	$\mathfrak{b}_n^{k+1}$
$\mathfrak{f}_{n,k}(\alpha_1,\ldots,\alpha_{[\frac{k}{2}]})$	$\frac{2n+k-5}{n-3}$	$B_{2,n+k}+\sum_{i=0}^{[\frac{k}{2}]} p_{k+1,i}(\alpha_1,\ldots,\alpha_{[\frac{k}{2}]})B_{3+i,k+4-i}$	$\mathfrak{f}_{n,k+1}(\alpha_1,\ldots,\alpha_{[\frac{k}{2}]})$ $\vec{\alpha}\in S_{n,k+1},\ k\geq 0$
	$\frac{3n+k-9}{n-3}$	$B_{3,n+k}+\sum_{i=1}^{\frac{n+k}{2}-2}(-1)^{i+1}B_{3+i,n+k-i}$	$\mathfrak{f}^1_{n,k}(\alpha_1,\ldots,\alpha_{[\frac{k-1}{2}]})$ $n+k$ even, $(\alpha_1,\ldots,\alpha_{[\frac{k-1}{2}]})\in S^1_{n,k}$
$\mathfrak{f}^{(r)}_{n,1}$ $r=[\frac{n-3}{2}]$	$\frac{n+2}{3}$	$B_{2,n+1}+\sum_{i=0}^{r-1}(-1)^i(r-i)B_{3+i,n-1-i}$	$\mathfrak{f}^{(r)}_{n,2}$ (n even)
	$\frac{n+1}{2}$	$B_{2,n+1}+\sum_{i=0}^{r-1}(-1)^i(r-i)B_{3+i,n-i}$	$\mathfrak{f}^{(r)}_{n,2}$ (n odd)
$\mathfrak{f}^{(r)}_{n,2}$ $r=[\frac{n-3}{2}]$	$\frac{n+3}{3}$	$B_{2,n+2}+\sum_{i=0}^{r-1}(-1)^i(\frac{(r-i)(r+1-i)}{2}-\gamma)B_{3+i,n-i}$	$\mathfrak{f}^{(r)}_{n,3}(\gamma)$ $(\gamma\in\mathbb{R})$ (n even)
	$\frac{n+2}{2}$	$B_{2,n+2}+\sum_{i=1}^{r}(-1)^{i+1}(ri-\frac{i(i-1)}{2})B_{3+i,n+1-i}$	$\mathfrak{f}^{(r)}_{n,3}$ (n odd)

REFERENCES

[AG] J. M. Ancochea Bermudez, M. Goze, *Le rang du système linéaire des racines d'une algèbre de Lie résoluble complexe*, Comm. Algebra **20 (3)** (1992), 875-887.

[BC] P. Bernat, N. Conze, et al., *Représentations des groupes de Lie résolubles*, Dunod, Paris (1972).

[Br] F. Bratzlavsky, *Sur les Algèbres Admettant un Tore d'Automorphismes Donné*, J. Algebra **30** (1974), 305-316.

[C1] R. Carles, *Sur la structure des algèbres de Lie rigides*, Ann. Inst. Fourier, Grenoble **33, 3** (1984), 65-82.

[C2] ______ , *Sur certaines classes d'algèbres de Lie rigides*, Math. Ann. **272** (1985), 477-488.

[CD] R. Carles, Y. Diakité, *Sur les variétés d'Algèbres de Lie de dimension* ≤ 7, J. Algebra **91** (1984), 53-63.

[CE] C. Chevalley, S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85-124.

[F] D. B. Fuks, *Cohomology of Infinite Dimensional Lie Algebras*, Contemporary Soviet Mathematics, Consultants Bureau, New York (1986).

[KN] A.A. Kirillov, Yu.A. Neretin, *The Variety A_n of n-Dimensional Lie Algebra Structures*, AMS Transl. **137** (1987), 21-30.

[N] Yu. A. Neretin, *An estimate of the number of parameters defining an n-dimensional algebra*, Math. of the USSR Izvestiya **30** (1988), 283-294.

[SS1] T. Skjelbred, T. Sund, *On the classification of nilpotent Lie algebras*, Oslo Preprint **no. 8** (1977).

[SS2] ______________, *Sur la classification des algèbres de Lie nilpotentes*, C.R. Acad. Sc. Paris **t.286** (1978), 241-242.

[S1] T. Sund, *On the structure of solvable Lie algebras*, Math. Scand. **44** (1979), 232-242.

[S2] ______ , *Solvable Lie algebras of dimension five*, Oslo preprint no. **18** (1981) (Revised edition).

[S3] ______, *Classification des algèbres de Lie filiformes résolubles*, C.R. Acad. Sc. Paris **t.317** (1993), 145-148.