UiO : Department of Mathematics
University of Oslo

# Rational Curves in Positive Characteristic

Oliver E. Anderson
Master's thesis for the degree: Master of Mathematics

The Faculty of Mathematics and Natural Sciences

November 17th, 2014

# Preface

## Introduction

Rational curves is a central topic in algebraic geometry which has been extensively studied. These curves are very simple in many aspects, yet they turn out to be powerful tools used to answer many questions in algebraic geometry. For instance in the study of higher dimensional varieties, one is especially interested in free and very free rational curves, as these help us answer questions of both geometrical and numerical nature. As an example of this, consider a smooth projective variety $X$ over a field $k = \bar{k}$. It is known that if $X$ has a free rational curve, then $H^0(X, K_X^{\otimes m})$ vanishes for all positive integers $m$, moreover, there will also pass a rational curve through a general point of $X$.

Any smooth Fano variety over an algebraically closed field of characteristic zero contains a very free rational curve. In positive characteristic however, this is still an open question. This inspires us to study rational curves on Fano varieties over fields of positive characteristic. Furthermore on a Fano variety in positive characteristic over an algebraically closed field $k$, one can construct a rational curve of $(-K_X)$-degree at most $n + 1$ through any point. This construction uses the Frobenius morphism. Mori proved that one can do this in characteristic zero as well, by passing from the characteristic $p$ case. Moreover, one does not know of any proofs of this fact that do not reduce to positive characteristic, which further motivates the study of rational curves in characteristic $p$.

In this thesis we will first introduce the relevant background material regarding higher dimensional algebraic geometry. Then we will study the theory of free and very free rational curves on Fermat hypersurfaces in positive characteristic. The Fermat hypersurfaces that we shall consider are all Fano varieties. We shall try to work over non (algebraically) closed fields whenever this is possible.

## A description of each chapter

### Chapter 1

We define uniruled and rationally connected varieties, and state some results concerning such varieties to motivate the study of them.

### Chapter 2

This is an interlude from higher dimensional algebraic geometry. Here we introduce the concept of an ample vector bundle, and establish some theory related to this topic, which will be used throughout this thesis. Everything in this chapter is well known, but we mostly give our own proofs of the results in this chapter.

### Chapter 3

The concepts of free and very free rational curves are introduced. We establish some properties of such rational curves, and relate this to uniruled and rationally connected varieties. From this point on, we shall be very conscious of the field we are working over, and always try to have as few constraints on the field as possible. Our main reference to this chapter is chapter four in [Deb01], however we have deviated from this text at a few points in order to develop some of the theory over non closed fields [1].

### Chapter 4

The Fermat hypersurface is introduced, we prove when it is Fano, and we discuss its moduli space of rational curves of a given degree.

### Chapter 5

We here present Mingmin Shen's article "Rational curves on Fermat hypersurfaces", elaborate on the proofs in this article, and we do not make the assumption that we are working over an algebraically closed field.

[1] Some time after writing this chapter, the author of this thesis admittedly found some lecture notes by Debarre which also relax the conditions on the base field. Our exposition is however at many points more detailed than the aforementioned notes, hence we shall leave this chapter as it is.

### Chapter 6

The paper "Free and very free morphisms into a Fermat hypersurface" [Bri+13], gives among other things, some constraints on the degree of a free and a very free rational curve on the degree 5 Fermat hypersurface in $\mathbb{P}^5_{\bar{\mathbb{F}}_2}$. This is done by translating the problem to a question in commutative algebra, regarding the splitting types of two graded free $\bar{k}[x_0, x_1]$-modules. We will to some extent follow their approach, and give many of the same constraints in $X_{d,d}$ where $d = p^r + 1$ for any prime number $p$ which is the characteristic of the field we are working over (which we do not assume to be algebraically closed). We finish this chapter by giving a very concrete criterion on the coefficients of the homogeneous polynomials defining a rational curve of degree $2p^r + 1$ on $X_{d,d}$ for this rational curve to be very free.

### Chapter 7

In this last chapter we introduce some ideas, and discuss wether following them up will be fruitfull or not.

## Preliminaries and Conventions

### Preliminaries

We assume that the reader has some familiarity with basic algebraic geometry. It is hard to explain exactly what we mean by this, but roughly the reader should be able to read most pages of one of the three texts: [Vak13], [Liu02] , or [Har77].

We will try to refer to the three texts mentioned above, when we are using relatively advanced theory from either of them.

Basic category theory will be used from time to time, everything the reader needs to know about this is covered in Chapter 1 in [Vak13].

### Conventions

The texts we refer to all have different conventions regarding what is meant by a variety. For us a variety shall be a *separated integral scheme over a field $k$*. Whenever a scheme satisfies these conditions, it will qualify for a variety in [Deb01], [Vak13] and [Liu02].

If we only write "subscheme", we mean a closed subscheme. The following definition is essential throughout this entire thesis.

**Definition 0.0.1.** Let $X$ be a $k$-scheme. A *rational curve on* $X$ is a non-constant $k$-morphism $f : \mathbb{P}^1_k \to X$. If $X$ is a subscheme of $\mathbb{P}^n_k$, a rational curve on $X$ is determined by $n+1$ homogeneous polynomials of equal degree $e$ with coefficients in $k$, in this case we say that the degree of the rational curve is $e$.

### Notation

We denote the category of schemes by $\mathfrak{Sch}$ and the category of $S$-schemes by $\mathfrak{Sch}/S$ (the slice category of $\mathfrak{Sch}$ over $S$).

The notation $\mathfrak{Sch}/S(X, Y)$ denotes the set of $S$-morphisms from the scheme $X$ to $Y$, and we will drop the $/S$ whenever $S$ is $\operatorname{Spec} k$.

The convention above is only used to avoid confusion as we will also be considering a functor called $\operatorname{Hom}_S(X, Y)$ and its moduli space (just as above we will also here drop the $S$ sub index whenever $S = \operatorname{Spec} k$). In order to avoid any potential confusion caused from having three different objects all with the same name, we choose to call the scheme that represents $\operatorname{Hom}_S(X, Y)$, $\operatorname{Mor}_S(X, Y)$ , and we use relatively standard categorical notation for the sets of morphisms in the category of schemes. On the other hand we will follow the literature and still use $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ for the set of morphisms between two sheaves $\mathcal{F}$ and $\mathcal{G}$.

Both $H^0(X, \mathcal{F})$ and $\mathcal{F}(X)$ denote the set of global sections of the sheaf $\mathcal{F}$.

## Acknowledgements

First of all I want to thank my supervisor Ragni Piene, who has shown a keen interest in this project, since the very day I visited her office and asked her if she would be my supervisor. The guidance and inspiration she has given me, has been invaluable.

I would also like to thank Paul Arne Østvær, who was given the nearly impossible task of teaching us scheme theory in only two months. I believe it is fair to say that he was the perfect man for the job.

The fantastic coffee provided by Nikolai B. Hansen has proven to yield moments of great social and academic value. For this I thank him.

Bernt Ivar Nødland inspired me to learn as much as possible during our days as freshmen. I want to thank him for this and for being a good friend.

A special mention goes to the good people with whom I share a reading room, moreover I thank the entire Department of Mathematics at the University of Oslo for making this a pleasant place to study at.

Last but not least a massive thank you goes out to Martin Helsø, who applied his cunning and incredible diplomatic skills to provide a safe passage through the land of LaTeX. Moreover when the LaTeX authorities broke their promise and sent the evil compiler minions to attack us, Martin single handedly dodged all their filthy strikes and brought us to safety.

# Contents

# Chapter 1

# Motivation part 1: Higher dimensional analogues of rational curves

In this chapter we will motivate and introduce uniruled and rationally connected varieties. In these two motivational parts, we shall mainly base ourselves on two texts. The first being the article Rational curves on varities by Carolina Araujo and János Kollár, [AK03]. The second text we shall base ourselves on is the book Higher-dimensional Algebraic Geometry [Deb01], by Olivier Debarre, which shall also be a valuable reference in other parts of this thesis.

It is well established that the rational curves are in many aspects the simplest algebraic curves. From [Liu02, ch. 7, sec. 4, Prop.4.1] we have a numerical criterion which classifies rational curves:

**Proposition 1.0.2.** *Let $X$ be a geometrically integral projective curve over a field $k$, and assume that $X(k) \neq \emptyset$. Then we have that $X \cong \mathbb{P}^1_k$ if and only if $H^1(X, \mathcal{O}_X) = 0$.*

In dimension two, rational surfaces over algebraically closed fields satisfy a similar criterion:

**Theorem 1.0.3** ([Kol96, Ch. 3, Sec. 2, Thm. 2.4])**.** *(Castelnuovo-Zariski Rationality Criterion) Let $X$ be a smooth projective surface over an algebraically closed field $k$. Then $X$ is rational if and only if*

$$H^1(\mathcal{O}_X) = H^0(\mathcal{O}_X(2K_X)) = 0.$$

One might hope that rationality continues to be intrinsic to varieties that satisfy $H^1(X, \mathcal{O}_X) = 0$ or $H^0(X, \mathcal{O}(2K_X)) = 0$ or both, however already in dimension three, we find counter examples. For instance we have that smooth-cubic $3-$folds over $\mathbb{C}$ share many of the properties of rational varietes, one of them being $H^i(X, \mathcal{O}_X) = H^0(X, \mathcal{O}(mK_X)) = 0$ for every $i, m \geq 0$, but these are not rational.

As there are non-rational varieties that share most relevant properties of rational varieties, we see that rationality is not a good classification of the simplest algebraic varieties. A possible remedy was introduced in [KMM92b], where the notion of a rationally connected variety was introduced. The idea behind this definition was that $\mathbb{P}^N_k$ has many rational curves, thus one might expect that the sufficient and necessary condition for a variety to behave like $\mathbb{P}^N_k$ is that it contains plenty of rational curves.

In characteristic 0, rationally connected varieties have good analogues of properties which are enjoyed by rational surfaces, Theorem 1.0.3 being one of them. Hence in characteristic zero, rationally connected varieties seem to be the correct higher dimensional analogues of rational curves and rational surfaces. We will get back to these analogues later on in this chapter, moreover we shall adress the characteristic $p$ cases in Motivation part 2.

Before we define rationally connected varieties, we define the notion of a uniruled variety as in [Deb01], which is conjectured to be a weaker notion than that of rationally connectedness.

**Definition 1.0.4.** A variety $X$ of dimension $n$ is called *uniruled* if there exist a variety $Y$ of dimension $n-1$ and a dominant rational map over $k$, $\mathbb{P}^1_k \times Y \dashrightarrow X$.

It is explained in [Deb01] that if $X$ is an $n-$dimensional variety over an uncountable algebraically closed field $k$, then $X$ is uniruled if and only if there is a rational curve through every point of $X$.

We have the following analogue of Theorem 1.0.3 for uniruled varieties.

**Proposition 1.0.5.** *Assume that $X$ is a smooth projective uniruled variety over an algebraically closed field $k$ of characteristic* 0. *Then*

$$H^0(X, \mathcal{O}_X(mK_X)) = 0$$

*for all positive integers $m$.*

This will be a corollary of the more general statement that $H^0(X, K_X^{\otimes m})$ vanishes for any $k$-variety $X$ that has a free rational curve, which we will prove in Motivation part 2. The converse of the proposition above is conjectured to hold, and it does so in dimension 3 and below.

**Definition 1.0.6.** A variety $X$ is called *rationally connected* if it is proper and if there exists a variety $M$ and a rational map over $k$, $e : \mathbb{P}^1_k \times M \dashrightarrow X$ such that the rational map

$$\begin{aligned} \mathbb{P}^1_k \times \mathbb{P}^1_k \times M &\dashrightarrow X \times X \\ (t, t^{'}, z) &\mapsto ((e(t, z), e(t^{'}, z)) \end{aligned}$$

is dominant.

We also have a more geometric description of rationally connected varieties when the field $k$ is algebraically closed. In that case a general pair of points on a rationally connected variety can be joined by a rational curve. We will prove this in Motivation part 2. The converse of the aforementioned statement is also true when the field is uncountable. Thus we see that rationally connected Varieties resemble path connected topological spaces in some sense.

Earlier in this chapter we discussed Castelnuovo's criterion for smooth surfaces. We shall now bring forth two other properties of smooth rational surfaces.

**Theorem 1.0.7.** *(1) (Deformation invariance) Let $X \to S$ be a flat family of smooth projective surfaces over $k$ and let $S$ be irreducible. If $X_o$ is rational for some $o \in S$, there is a non-empty open subset $U$ of $S$, such that $X_s$ is rational for all $s \in U$.*

*(2) (Noether's Theorem) Let $k$ be an algebraically closed field and let $S$ be a surface. If there is a dominant rational map $S \dashrightarrow \mathbb{P}^1_k$ such that the generic fiber is a rational curve, then $S$ itself is rational.*

*Proof.* For(1): Since $X_o$ is rational, we have that

$$H^1(X_o, \mathcal{O}_{X_o}) = H^0(X_o, \mathcal{O}_{X_o}(2K_{X_o})) = 0,$$

by Castelnuovo's criterion. From The semicontinuity theorem ([Har77, Ch.3, Sec. 12, Thm. 12.8]), it follows that we can find a non-empty open set $U^{'}$ such that $H^1(X_s, \mathcal{O}_{X_s}) = 0$ for all $s \in U^{'}$ and we can find a non-empty open set $U^{''}$ such that $H^0(X_s, \mathcal{O}_{X_s}(2K_{X_s})) = 0$ for all $s \in U^{''}$. Taking $U = U^{'} \cap U^{''}$, we see that $H^1(X_s, \mathcal{O}_{X_s}) = H^0(X_s, \mathcal{O}_{X_s}(2K_{X_s})) = 0$ for all $s \in U$, hence $X_s$ is rational for all $s \in U$ by Castelnuovo's criterion.

For (2): It follows from [Bǎd01, (Noether-Tsen), Ch.11, Thm. 11.3], that $S$ is birational to $\mathbb{P}^1_k \times \mathbb{P}^1_k$, which is again birational to $\mathbb{P}^2_k$. □

To strengthen the statement that rationally connected varieties are the correct higher dimensional analogues of rational curves and rational surfaces, we now state rationally connected analogues of Castelnuovo's criterion and the two aforementioned properties.

**Theorem 1.0.8.** *Assume that $X$ is a smooth projective rationally connected variety over an algebraically closed field of characteristic zero.*

*(1) (Castelnuovo's Criterion) $H^0(X, (\Omega_X^p)^{\otimes m}) = 0$ for all positive integers $m$ and $p$. In particular $H^m(X, \mathcal{O}_X) = 0$, for all $m \geq 1$.*

*(2) (Deformation invariance)Let $\pi : X \to T$ be a proper smooth morphism, and assume that $T$ is connected. If $f^{-1}(s)$ is rationally connected for some $s \in S$, then $f^{-1}(s)$ is rationally connected for every $s \in S$.*

*(3) (Noether's Theorem) Let $f : Z \to Y$ be any dominant morphism of complex varieties. If $Y$ and the general fiber of $f$ are rationally connected, then $Z$ is rationally connected.*

*Proof. (1)* will be a corollary of the more general result that if a $k$-variety $X$ has a very free rational curve, then $H^0(X, (\Omega_X^p which)^{\otimes m})$ vanishes for all positive integers $m$ and $p$, which we shall prove in Motivation part 2. *(2)* is Corollary 2.4 of [KMM92b]. The last statement is Corollary 1.3 of [GHS03]. □

Just as in the uniruled case, the converse of (1) in the theorem above is conjectured to be true.

There is also a version of deformation invariance for uniruled varieties over fields of characteristic zero (see [Kol96, Ch. 4, Cor. 1.10]). However, the author of this thesis has yet to see a uniruled analogue of Noether's Theorem.

# Chapter 2

# Interlude to ample locally free sheaves

In this chapter we shall give some definitions and results about globally generated and ample vector bundles, mostly on $\mathbb{P}^1_A$, where $A$ is a Noetherian ring. These results are all well known, however it turns out to be hard to find references which suit our needs, thus we shall give our own proofs of the results in this chapter.

When we relate the notions of this chapter to rational curves on a variety later on, we shall only be interested in $k$-schemes, where $k$ is a field, however most of the definitions and results in this chapter make sense for Noetherian rings as well, thus we shall not make the assumption that we are working over a field yet.

**Definition 2.0.9.** Let $X$ be a scheme and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X$. We say that $\mathcal{F}$ is *generated by global sections* (or *globally generated*) if the canonical map:

$$\mathcal{F}(X) \otimes \mathcal{O}_{X,x} \to \mathcal{F}_x$$

is surjective for every point $x \in X$.

This is a standard definition and can be found in both [Har77] and [Liu02]. We will shortly give a criteria for when a locally free sheaf on $\mathbb{P}^1_k$ is generated by global sections. First we give a few lemmas.

**Lemma 2.0.10.** *Let $\mathcal{E} = \oplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^1_A}(a_i)$, be a locally free sheaf of rank $r$ on $\mathbb{P}^1_A$. Then $\mathcal{E}$ is generated by global sections if and only if each line bundle $\mathcal{O}_{\mathbb{P}^1_A}(a_i)$ is generated by global sections.*

*Proof.* In the name of clean notation we shall assume that the rank $r = 2$ (the general case is completely analogous, but will require slightly more confusing notation). As the stalk functor is a left adjoint functor it commutes with colimits, we thus have $(\mathcal{O}_{\mathbb{P}^1_A}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_A}(a_2))_x = \mathcal{O}_{\mathbb{P}^1_A}(a_1)_x \oplus \mathcal{O}_{\mathbb{P}^1_A}(a_2)_x$, where $x \in \mathbb{P}^1_A$. Our canonical map

$$\begin{aligned} &(\mathcal{O}_{\mathbb{P}^1_A}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_A}(a_2))(\mathbb{P}^1_A) \otimes \mathcal{O}_{\mathbb{P}^1_A,x} \\ = &(\mathcal{O}_{\mathbb{P}^1_A}(a_1)(\mathbb{P}^1_A) \oplus \mathcal{O}_{\mathbb{P}^1_A}(a_2)(\mathbb{P}^1_A)) \otimes \mathcal{O}_{\mathbb{P}^1_A,x} \to \mathcal{O}_{\mathbb{P}^1_A}(a_1)_x \oplus \mathcal{O}_{\mathbb{P}^1_A}(a_2)_x \end{aligned}$$

is given by $\sum(b_i, c_i) \otimes f_i \mapsto \sum(f_i(b_i)_x, f_i(c_i)_x)$. From this we see that the map is surjective if and only if the projection onto each factor is surjective and thus the lemma follows. □

**Lemma 2.0.11.** *Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_A}(a)$ be a line bundle on $\mathbb{P}^1_A$. Then $\mathcal{L}$ is generated by global sections if and only if $a \geq 0$.*

*Proof.* This basically follows from the fact $\mathcal{O}_{\mathbb{P}^1_A}(a)(\mathbb{P}^1_A) = B_a$ if $a \geq 0$ and $\mathcal{O}_{\mathbb{P}^1_A}(a)(\mathbb{P}^1_A) = 0$ if $a < 0$, where $B = A[x_0, x_1]$ with canonical grading. See for example [Liu02, Ch.5, Lem.1.22] for a proof. Assume first that $a < 0$ and let $x \in \mathbb{P}^1_A$ correspond to a prime ideal $\mathfrak{p} \in \mathrm{Proj}(B)$. Then we have $\mathcal{O}_{\mathbb{P}^1_A}(a)_x = B(a)_{(\mathfrak{p})} = \{\frac{b}{g} \mid b \in B_{deg(g)-|a|}\}$ which is not the trivial module. However $\mathcal{O}_{\mathbb{P}^1_A}(a)(\mathbb{P}^1_A) = 0$ thus we see that $\mathcal{O}_{\mathbb{P}^1_A}(a)$ is not generated by global sections when $a < 0$.

Conversely assume that $a \geq 0$ and let $x \in \mathbb{P}^1_A$ correspond to the homogeneous prime ideal $\mathfrak{p} \subset B = A[x_o, x_1]$. Then we have $\mathcal{O}_{\mathbb{P}^1_A}(a)(\mathbb{P}^1_A) = B_a$. Moreover we have $\mathcal{O}_{\mathbb{P}^1_A}(a)_x = B(a)_\mathfrak{p} = \{\frac{b}{g} \mid b \in B_{deg(g)+a}\}$. Let $\frac{b}{g} \in B(a)_\mathfrak{p}$. Then $deg(b) = a + deg(g)$. As any element in $B_{a+deg(g)}$ can be written as $\sum c_i h_i$, where the $c_i \in B_{deg(g)}$and the $h_i \in B_a$. We have that $\sum \frac{c_i}{g} h_i \in B_{(\mathfrak{p})} \otimes B_a = \mathcal{O}_{\mathbb{P}^1_A,x} \otimes \mathcal{O}_{\mathbb{P}^1_A}(a)(\mathbb{P}^1_A)$, and this will map to $\frac{b}{g}$ under the canonical map in (2.0.9). This finishes the proof. □

**Corollary 2.0.12.** *Let $\mathcal{E} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1_A}(a_i)$ be a locally free sheaf of rank $r$ on $\mathbb{P}^1_A$. Then $\mathcal{E}$ is globally generated if and only if each $a_i \geq 0$.*

*Proof.* This follows from the two previous lemmas. □

**Definition 2.0.13.** Let $\mathcal{E}$ be a locally free sheaf of rank $r$ on $\mathbb{P}^1_A$. If $\mathcal{E}$ has a (unique) splitting $\mathcal{E} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1_A}(a_i)$, where $a_1 \leq a_2 \leq \ldots \leq a_r$, then we say that $\mathcal{E}$ has splitting type $a_1, \ldots, a_n$.

**Remark 2.0.14.** Recall what is sometimes referred to as Grothendieck's theorem. It says that any locally free sheaf $\mathcal{E}$ of rank $n$ on $\mathbb{P}^1_k$, where $k$ is a field, can be written as a direct sum of line bundles, in other words we have that $\mathcal{E} \cong \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1_k}(a_i)$, where the integers $a_i$ are uniquely determined. See [Vak13, Thm.18.5.6]. Thus on $\mathbb{P}^1_k$ we always have such a splitting as we assumed in the previous corollary, and we also have a splitting type.

These results are interesting in their own right, however they will also help us give a criteria on when a locally free sheaf of rank $r$ on $\mathbb{P}^1_A$ is ample.

Recall now definition 16.6.1 in [Vak13]:

**Definition 2.0.15.** We say that an invertible sheaf $\mathcal{L}$ on a proper $A-$scheme $X$ is *ample over* $A$ (or relatively ample) if for all finite type quasi-coherent sheaves $\mathcal{F}$ there exists an $n_0$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$.

We will now define ampleness of a locally free sheaf as it is done in [Har66], however instead of requiring the scheme to be of finite type over an algebraically closed field, we require it to be proper over a Noetherian ring.

**Definition 2.0.16.** Let $X$ be a scheme proper over a Noetherian ring $A$, and let $\mathcal{E}$ be a locally free sheaf on $X$. We say that $\mathcal{E}$ is *ample* if for every coherent sheaf $\mathcal{F}$, there is an integer $n_0 > 0$, such that for every $n \geq n_0$, the sheaf $\mathcal{F} \otimes S^n(\mathcal{E})$ (where $S^n(\mathcal{E})$ is the n'th symmetric power of $\mathcal{E}$) is generated by global sections.

The following lemma will be useful, and it will show that the two definitions coincide for line bundles.

**Lemma 2.0.17.** *The following statements are true.*

*(1) Let $\mathcal{L}$ be a line bundle on $X$, then $S^n(\mathcal{L})) \cong \mathcal{L}^{\otimes n}$. In particular we have that definitions 2.0.15 and 2.0.16 coincide for line bundles on a scheme $X$ proper over a Noetherian ring $A$.*

*(2) If $\mathcal{E}$, $\mathcal{E}'$ are locally free sheaves of ranks $r$, $r'$ respectively. Then $S^n(\mathcal{E} \oplus \mathcal{E}') = \bigoplus_{p+q=n} S^p(\mathcal{E}) \otimes S^q(\mathcal{E}')$, where $p, q \geq 0$.*

*(3) If $\mathcal{E}$ is a locally free sheaf with rank $r$, then $S^n(\mathcal{E})$ is also locally free with rank $\binom{n+r-1}{r-1}$.*

*Proof.* We only prove (1) and omit the proof of (2) and (3). We have a canonical morphism of presheaves $T^n(\mathcal{L})_{pre} \to S^n(\mathcal{L})_{pre}$ and this induces a morphism $\alpha : T^n(\mathcal{L}) \to S^n(\mathcal{L})$. Let $x \in X$. As $T^n(\mathcal{L})$ and $S^n(\mathcal{L})$ are quasi-coherent, hence $T^n(\mathcal{L})_x = T^n(\mathcal{L}_x)$ and similarly $S^n(\mathcal{L}_x) = S^n(\mathcal{L})_x$, because $T^n$ and $S^n$ commute with localisation of modules. It remains to show that $T^n(\mathcal{L}_x) = S^n(\mathcal{L}_x)$, as then the stalk functor applied to $\alpha$ will be an isomorphism, hence also $\alpha$. We shall again in the name of clean notation assume that $n = 2$, the general case is completely analogous, but we will then have to deal with cumbersome notation. Let $f \otimes g \in T^2(L_x)$, if we can show that $f \otimes g = g \otimes f$, we will be done. Since $\mathcal{L}_x$ is a locally free $\mathcal{O}_{X,x}$ module of rank 1, we have some $h \in \mathcal{L}_x$ that generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$ module. Let $f^{'}$, $g^{'} \in \mathcal{O}_{X,x}$ be such that $f^{'}h = f$ and $g^{'}h = g$. We have $f \otimes g = f^{'}h \otimes g^{'}h = g'h \otimes f'h = g \otimes f$. The other statement of part (1) now easily follows from the first statment and the definitions. $\square$

**Proposition 2.0.18.** *Let $A$ be a Noetherian ring. On $\mathbb{P}^1_A$ a locally free sheaf $\mathcal{E} = \oplus \mathcal{O}_{\mathbb{P}^1_A}(a_i)$ is ample if and only if each $a_i > 0$.*

*Proof.* First assume that $a_i \leq 0$ for some $i$, say $i = 1$. We have $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \mathcal{E}^{'}$, where $\mathcal{E}^{'} = \oplus_{i \neq 1} \mathcal{O}_{\mathbb{P}^1_A}(a_i)$. As $S^n(\mathcal{E}) = \bigoplus_{p+q=n} S^p(\mathcal{O}_{\mathbb{P}^1_A}(a_1)) \otimes S^q(\mathcal{E}^{'})$, this is again equal to $\mathcal{O}_{\mathbb{P}^1_A}(na_1) \oplus \mathcal{O}_{\mathbb{P}^1_A}((n-1)a_1) \otimes S^1(\mathcal{E}^{'}) \oplus \ldots \oplus S^n(\mathcal{E}^{'})$. Tensoring this with the coherent sheaf $\mathcal{O}_{\mathbb{P}^1_A}(-1)$ yields: $\mathcal{O}_{\mathbb{P}^1_A}(na_1 - 1) \oplus \mathcal{O}_{\mathbb{P}^1_A}((n-1)a_1 - 1) \otimes S^1(\mathcal{E}^{'}) \oplus \ldots \oplus S^n(\mathcal{E}^{'}) \otimes \mathcal{O}_{\mathbb{P}^1_A}(-1)$. As $a_1 \leq 0$, we must have $na_1 - 1 < 0$ for all $n$ and thus it follows from (2.0.12) that this sheaf is not generated by global sections.

Now assume that all the $a_i > 0$, let $\mathcal{F}$ be any coherent sheaf on $\mathbb{P}^1_A$ and let $x \in \mathbb{P}^1_A$. By [Liu02, Ch.5,Cor.1.28] there exists an integer $m \in \mathbb{Z}$ and $r \geq 1$ such that $\mathcal{F}$ is a quotient sheaf of $\mathcal{O}_{\mathbb{P}^1_A}(m)^r$ (we could also have used [Har77, Ch.5,cor.5.18]). We have a commutative diagram:

$$\begin{array}{ccc}
(\mathcal{O}_{\mathbb{P}^1_A}(m)^r \otimes S^n(\mathcal{E}))(\mathbb{P}^1_A) \otimes \mathcal{O}_{\mathbb{P}^1_A,x} & \longrightarrow & (\mathcal{F} \otimes S^n(\mathcal{E}))(\mathbb{P}^1_A) \otimes \mathcal{O}_{\mathbb{P}^1_A,x} \\
\downarrow & & \downarrow \\
(\mathcal{O}_{\mathbb{P}^1_A}(m)^r \otimes S^n(\mathcal{E}))_x & \longrightarrow & (\mathcal{F} \otimes S^n(\mathcal{E}))_x
\end{array}$$

where the bottom arrow is a surjection. It follows from this that it is enough to show that $\mathcal{O}_{\mathbb{P}^1_A}(m)^r \otimes S^n(\mathcal{E})$ is generated by global sections for all $n > n_0$. Since this is obvious if $m \geq 0$, we may assume $m = -l$ where $l$ is a positive integer. We will now use induction to prove the following statement: if $\mathcal{E}$ is locally free of rank $k$, then $\mathcal{O}_{\mathbb{P}^1_A}(m)^r \otimes S^n(\mathcal{E})$ is generated by global sections whenever $n \geq kl+1$. The proof uses induction on $k$: For the base case $k=1$ we have that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_A}(a)$ where $a > 0$, thus $\mathcal{O}_{\mathbb{P}^1_A}(m)^r \otimes S^n(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^1_A}((n-l)a)^r$ which is generated by global sections by (2.0.11). Assume now that the statement holds true for $k-1$ and that $\mathcal{E}$ has rank $k$. We thus have $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_A}(a_1) \oplus \mathcal{E}'$ where $\mathcal{E}' = \oplus_{i \neq 1} \mathcal{O}_{\mathbb{P}^1_A}(a_i)$ is locally free of rank $k-1$. As we have $S^n(\mathcal{O}_{\mathbb{P}^1_A}(a_1) \oplus \mathcal{E}') = \bigoplus_{p+q=n} S^p(\mathcal{O}_{\mathbb{P}^1_A}(a_1)) \otimes S^q(\mathcal{E}')$. We see that

$$\begin{aligned}&\mathcal{O}_{\mathbb{P}^1_A}(m)^r \otimes S^n(\mathcal{E})\\ &= \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1_A}(na_1 - l) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1_A}(la_1) \otimes (S^q(\mathcal{E}') \otimes \mathcal{O}_{\mathbb{P}^1_A}(-l)) \oplus \mathcal{D}\end{aligned}$$

where $\mathcal{D} = \mathcal{O}_{\mathbb{P}^1_A}((l-1)a_1) \otimes S^{q+1}(\mathcal{E}') \otimes \mathcal{O}_{\mathbb{P}^1_A}(-l) \oplus \ldots \oplus S^n(\mathcal{E}') \otimes \mathcal{O}_{\mathbb{P}^1_A}(-l)$. As we have chosen $n \geq kl+1$ it follows from the induction hypothesis that $S^q(\mathcal{E}') \otimes \mathcal{O}_{\mathbb{P}^1_A}(-l)$ is generated by global sections and thus each linebundle in this direct sum is twisted positively. We also clearly have that all the $S^p(\mathcal{O}_{\mathbb{P}^1_A}(a_1))$ are generated by global sections and thus by (2.0.12) it follows that $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1_A}(-l) \otimes S^n(\mathcal{E})$ is generated by global sections. This finishes the proof. □

We conclude this chapter by giving a lemma which will be useful later on.

**Lemma 2.0.19.** *Let* $A \neq 0$ *be a ring. Then* $\mathrm{Hom}_{\mathbb{P}^1_A}(\mathcal{O}_{\mathbb{P}^1_A}(n), \mathcal{O}_{\mathbb{P}^1_A}(m)) = 0$ *if and only if* $n > m$.

*Proof.* It follows from [Vak13, Ch.13.,Ex.13.1.F] (or [Har77, Ch.2.,Ex.5.1]) that $\mathscr{H}om(\mathcal{O}_{\mathbb{P}^1_A}(n), \mathcal{O}_{\mathbb{P}^1_A}(m)) = \mathcal{O}_{\mathbb{P}^1_A}(m-n)$. Thus we have that

$$\mathrm{Hom}_{\mathbb{P}^1_A}(\mathcal{O}_{\mathbb{P}^1_A}(n), \mathcal{O}_{\mathbb{P}^1_A}(m)) = \mathcal{O}_{\mathbb{P}^1_A}(m-n)(\mathbb{P}^1_A) = A[x_0, x_1]_{(m-n)},$$

which is equal to 0 if and only if $m - n < 0$, where the last equality follows from [Liu02, Ch.5,Lemma 1.22]. □

# Chapter 3

# Motivation part 2: Free and very free rational curves

## 3.1 A moduli problem

One can study rational curves on a variety by studying a certain fine moduli space of a moduli problem, the $k$-points of this fine moduli space will correspond to rational curves. In this section we will define the moduli problem, and discuss some of the properties of the fine moduli space. Moreover we shall relate this space to uniruled and rationally connected varieties. Through this we shall justify some of the claims regarding the geometric notions of uniruled and rationally connected varieties from Motivation part 1.

**Proposition 3.1.1.** *Let $e$ be a positive integer and let $k$ be a field. There exists an open subscheme $\mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^N_k)$ of $\mathbb{P}^{Ne+N+e}_k$, such that there is a bijection between*

$$\mathfrak{Sch}(\mathrm{Spec}(k), \mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^N_k))$$

*and*

$$\{k\text{-morphisms of degree } e,\ f : \mathbb{P}^1_k \to \mathbb{P}^N_k\}$$

*Proof.* Giving a morphism $f : \mathbb{P}^1_k \to \mathbb{P}^N_k$ is the same as giving $N+1$ homogeneous polynomials in $k[x_0, x_1]$ of equal degree $e$ without nonconstant common factors (see [Vak13, Ch.16, sec.4] or [Har77, Ch.2,sec.7,Thm.7.1]). Let these $N+1$ homogeneous polynomials be denoted by $F_0, \ldots, F_N$. We claim that the $F_i$ have no nonconstant common factor in $k[x_0, x_1]$ if and only if they have no nontrivial zero in $\bar{k}$, the algebraic closure of $k$. Indeed, if the $F_i$ have a common nontrivial zero in the algebraic closure of $k$, say $(a, b) \in Z_+(F_i)_i$, where we assume that $a \neq 0$, which means that $(1, b/a)$ is also a common nontrivial zero of the $F_i$. From this it follows that the polynomials $F_i(1, y) \in \bar{k}[y]$ have a common nonconstant factor $g(y)$, thus the $F_i$ have $x_0^e g(x_1/x_0)$ as a common nonconstant factor. Conversely if the $F_i$ have a common nonconstant factor $g(x_0, x_1)$, then as the polynomial $g(1, y)$ has a zero in $\bar{k}$, it follows that the $F_i$ have a common nontrivial zero in $\bar{k}$. By the Nullstellensatz we have that the $F_i$ have no common nontrivial zero in $\bar{k}$ if and only if the ideal generated by $(F_0, \ldots, F_N)$ in $\bar{k}[x_0, x_1]$ contains some power of the irrelevant ideal $(x_0, x_1)$. This in turn can equivalently be phrased in terms of linear algebra as follows: There exists a surjective $k$-linear map $(\bar{k}[x_0, x_1])_{m-e}^{N+1} \to \bar{k}[x_0, x_1]_m$ which is given by $(G_0, \ldots, G_N) \mapsto \sum_{i=0}^N G_i F_i$. Thus we see that the $F_i$ have a common nonconstant factor if and only if for every $m$ all $m+1$ minors of the matrix to the above map vanish. Since all the minors are polynomials in the coefficients of the $F_i$, we can interpret them as polynomial in $k[y_{\{0,0\}}, y_{\{0,1\}}, \ldots, y_{\{0,e\}}, y_{\{1,0\}}, \ldots, \ldots, y_{\{N,e\}}]$, and the $k$-points not contained in the vanishing of these polynomials will uniquely determine morphisms $\mathbb{P}_k^1 \to \mathbb{P}_k^N$. Hence the degree $e$ morphisms are parametrized by a Zariski open subset of $\mathbb{P}_k^{Ne+N+e}$. We denote this quasi-projective variety $\mathrm{Mor}_e(\mathbb{P}_k^1, \mathbb{P}_k^N)$. □

If $X$ is a closed subscheme of $\mathbb{P}_k^N$ defined by homogeneous polynomials $(G_1, \ldots, G_m)$, then giving a morphism of degree $e$ from $\mathbb{P}_k^1 \to X$ is the same as giving $N+1$ homogeneous polynomials of equal degree $e$ in $k[x_0, x_1]$ such that $G_j(F_0, \ldots, F_N) = 0$ for $j = 1, \ldots, j = m$. Using this one can show the following:

**Proposition 3.1.2.** *Let $X$ be a closed subscheme of $\mathbb{P}_k^N$. There is a closed subscheme $\mathrm{Mor}_e(\mathbb{P}_k^1, X)$ of $\mathrm{Mor}_e(\mathbb{P}_k^1, \mathbb{P}_k^N)$ such that we have a bijection between*

$$\mathfrak{Sch}(\mathrm{Spec}(k), \mathrm{Mor}(\mathbb{P}_k^1, X))$$

*and*

$$\{k\text{-morphisms } f : \mathbb{P}_k^1 \to X\}$$

In the next chapter we will write more explicitly what $\mathrm{Mor}_e(\mathbb{P}_k^1, X)$ is when $X$ is a Fermat hypersurface in $\mathbb{P}_k^d$. We shall now define our moduli problem which we mentioned in the introduction to this section.

**Definition 3.1.3.** Let $X/S$ and $Y/S$ be $S$-schemes. $\mathrm{Hom}_S(X,Y)$ is the functor $\mathrm{Hom}_S(X,Y) : (\mathfrak{Sch}/S)^{op} \to \mathsf{sets}$ defined by

$$\mathrm{Hom}_S(X,Y)(T) = \{T\text{-morphisms } X \times_s T \to Y \times_S T\}$$

If $T' \to T$ is a morphism of $S$-schemes, then

$$\mathrm{Hom}_S(X,Y)(T) \to \mathrm{Hom}_S(X,Y)(T^{'})$$

is given as follows: Let $g \in \mathrm{Hom}_S(X,Y)(T)$. The map $T' \to T$ induces a map $X \times_s T' \to X \times_s T$, now the composition of the aformentioned map together with $g$ induces a map $h : X \times_S T' \to Y \times_S T'$, we let

$$\mathrm{Hom}_S(X,Y)(T' \to T)(g) = h.$$

Under the correct circumstances there is a fine module space. More precisely we have:

**Theorem 3.1.4** ([Kol96, Ch. I, Sec. 1, Thm. 1.10])**.** *Let $X/S$ and $Y/S$ be projective schemes over $S$. Assume that $X$ is flat over $S$. Then* $\mathrm{Hom}_S(X,Y)$ *is represented by an open subscheme*

$$\mathrm{Mor}_S(X,Y) \subset \mathrm{Hilb}(X \times_S Y/S)$$

One actually has that $\mathrm{Hom}_k(\mathbb{P}^1_k, \mathbb{P}^N_k)$ (resp.) $\mathrm{Hom}_k(\mathbb{P}^1_k, X)$ is represented by $\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^N_k) = \coprod_{e\geq 0} \mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^N_k)$ (resp.) $\mathrm{Mor}(\mathbb{P}^1_k, X) = \coprod_{e\geq 0} \mathrm{Mor}_e(\mathbb{P}^1_k, X)$, where $X$ is as in Proposition 3.1.2.

If $\mathrm{Hom}_S(X,Y)$ is represented by a scheme $\mathrm{Mor}_S(X,Y)$ with natural isomorphism $\eta : \mathrm{Hom}_S(X,Y) \to \mathrm{h}_{\mathrm{Mor}_S(X,Y)}$ we follow the litterature and call the morphism $f^{univ} = \eta^{-1}(id_{\mathrm{Mor}_S(X,Y)})$, the universal morphism. It has the following property: If $g : T \to \mathrm{Mor}_S(X,Y)$ is an element of $\mathrm{h}_{\mathrm{Mor}_S(X,Y)}(T)$, then $\eta^{-1}(g) = \mathrm{Hom}_S(X,Y)(g)(f^{univ})$. Further we let

$$ev^{(1)} : X \times \mathrm{Mor}_S(X,Y) \to Y$$

be the morphism $pr_1 \circ f^{univ}$, and call this morphism the evaluation map. When we specialize to the case $X = \mathbb{P}^1_k$ and $Y = \mathbb{P}^N_k$, and if $((u,v),f)$ is a $k$-point of $\mathbb{P}^1_k \times \mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^N_k)$, then we have $f^{univ}((u,v),f) = (f(u,v),f)$ and $ev^{(1)}((u,v),f) = f(u,v)$.

Now let $X$ be a projective variety over a field $k$ and assume that $X$ is uniruled. Let $e : \mathbb{P}^1_k \times Y \dashrightarrow X$ be a dominant rational map where we have $\dim Y = n-1$. By possibly shrinking $Y$ we can assume that $e$ is a morphism.

This induces a morphism $e' : \mathbb{P}^1_k \times Y \to X \times Y$ such that $pr_X \circ e' = e$. Thus $e' \in \mathrm{Hom}_k(\mathbb{P}^1_k, X)(Y)$, hence there is some morphism $g : Y \to \mathrm{Mor}(\mathbb{P}^1_k, X)$ such that $e' = \mathrm{Hom}_k(\mathbb{P}^1_k, X)(g)(f^{univ})$. Thus $e$ factors as

$$\mathbb{P}^1_k \times Y \to \mathbb{P}^1_k \times \mathrm{Mor}(\mathbb{P}^1_k, X) \overset{ev^{(1)}}{\to} X$$

with $ev^{(1)}$ is dominant[1]. From this we see that when $k$ is algebraically closed, there is a rational curve through a general point of $X$.

When $X$ is rationally connected, there is a similar situation. Let

$$ev^{(s)} : (\mathbb{P}^1_k)^s \times \mathrm{Mor}(\mathbb{P}^1_k, X) \to X^s$$

be the morphism induced by $ev^{(1)}$. Let $e : (\mathbb{P}^1_k)^2 \times M \dashrightarrow X \times X$, be a dominant rational map. After possibly shrinking $M$ we may assume that the dominant map

$$e : \mathbb{P}^1_k \times \mathbb{P}^1_k \times M \dashrightarrow X \times X$$

is a morphism. By arguments similar to those in the uniruled case this factors as:

$$\mathbb{P}^1_k \times \mathbb{P}^1_k \times M \to \mathbb{P}^1_k \times \mathbb{P}^1_k \times \mathrm{Mor}(\mathbb{P}^1_k, X) \overset{ev^{(2)}}{\to} X \times X. \tag{3.1.1}$$

With $ev^{(2)}$ dominant. Thus when the field $k$ is algebraically closed, we see that when $X$ is projective and rationally connected, then there passes a rational curve through a general pair of points of $X$. We also remark that by using the universal property of fibered products together with the universal property of representable functors, and the fact that surjectivity is preserved under base change, one can show that a variety $X$ over $k$, is uniruled respectively rationally connected if and only if $X_K$ is uniruled respectively rationally connected for some field extension $K$ of $k$. We now state a useful result conserning the tangent space of $\mathrm{Mor}(X, Y)$.

**Proposition 3.1.5** ([Deb01, Ch. 2, Sec. 2, Prop. 2.4]). *Let $X$ and $Y$ be varieties, with $Y$ quasi-projective and $X$ projective, and let $f : X \to Y$ be a morphism. One has*

$$T_{\mathrm{Mor}(X,Y)_{[f]}} \cong H^0(X, \mathscr{H}om(f^*\Omega^1_Y, \mathcal{O}_X))$$

*In particular when $Y$ is smooth along the image of $f$,*

$$T_{\mathrm{Mor}(X,Y)_{[f]}} \cong H^0(X, f^*T_Y).$$

[1] we may infact replace $\mathrm{Mor}(\mathbb{P}^1_k, X)$ with $\mathrm{Mor}_d(\mathbb{P}^1_k, X)$ for some positive integer $d$.

Let $X/S$ and $Y/S$ be schemes, $B \subset X$ a subscheme, proper over $S$ and $g : B \to Y$ a morphism. Just as one can study morphisms from $X$ to $Y$ by studying $\mathrm{Mor}_S(X, Y)$ one can study morphisms from $X$ to $Y$ that restrict to $g$ by studying a fine moduli space $\mathrm{Mor}_S(X, Y, g)$. More precisely we have:

**Definition 3.1.6.** (Notation as above) $\mathrm{Hom}_S(X, Y, g)$ is the functor

$$\mathrm{Hom}(X, Y, g)(T) = \left\{ \begin{array}{l} T\text{-morphisms } f : X \times_S T \to Y \times_S T \\ \text{such that } f|_{B \times_S T} = g \times_S id_T. \end{array} \right\}$$

When $X$ and $Y$ are projective, this functor is represented by a subsceme $\mathrm{Mor}_S(X, Y; g)$ of $\mathrm{Mor}_S(X, Y)$. When $X$ and $Y$ are projective varieties we have

$$T_{\mathrm{Mor}(X,Y;g),[f]} \cong H^0(X, f^*T_X \otimes I_B).$$

where $I_B$ is the ideal sheaf of $B$.

## 3.2 Free and very free rational curves

This section is the corner stone of this chapter. We will here define free and very free rational curves, and connect these notions to uniruled and rationally connected varieties. We shall also use the existence of free respectively very free rational curves on a uniruled respectively rationally connected variety over an algebraically closed field of characteristic zero, to prove Theorem 1.0.5 respectively Theorem 1.0.8 (1).

### 3.2.1 Important definitions and basic facts

**Definition 3.2.1.** Let $X$ be a smooth variety and let $f : \mathbb{P}^1_k \to X$ be a rational curve on $X$. We say that $f$ is *free* (respectively *very free*) if $f^*T_X$ is globally generated (respectively ample).

**Remark 3.2.2.** If the variety $X$ in the definition above is of dimension $n$. Then $f^*T_X$ is a locally free sheaf on $\mathbb{P}^1_k$ of rank $n$, hence we have a splitting

$$f^*T_X \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1_k}(a_i)$$

for unique integers $a_1, \ldots a_n$. It follows from Corollary 2.0.12 respectively Proposition 2.0.18 that $f$ is free respectively very free if and only if $a_i \geq 0$ respectively $a_i > 0$ for all $i$.

In light of the remark above we follow [Deb01] and give the following generalisation of Definition 3.2.1:

**Definition 3.2.3.** A rational curve $f : \mathbb{P}^1_k \to X$ on a smooth variety $X$ is $r$-free if $f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1_k}(-r)$ is globally generated.

Notice that $free$ (respectively very free), coincides with 0-free (respectively 1-free). We also give a numerical criteria for $r$-freeness, which we shall use later in this thesis.

**Proposition 3.2.4.** *Suppose $X$ is a smooth projective variety over a field $k$. Let $f : \mathbb{P}^1_k \to X$ be a rational curve on $X$. Then $f$ is $r$-free if and only if*

$$H^1(\mathbb{P}^1_k, f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1_k}(-r-1)) = 0$$

*Proof.* We have integers $a_i$ such that $f^*T_X = \bigoplus_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i)$, then we have $f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-r-1) = \mathcal{O}_{\mathbb{P}^1}(a_i - r - 1)$. Now

$$h^1(\mathbb{P}^1, f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-r-1)) = h^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1} \otimes \mathscr{H}om(\oplus \mathcal{O}_{\mathbb{P}^1}(a_i - r - 1), \mathcal{O}_{\mathbb{P}^1})$$

which is again equal to $h^0(\mathbb{P}^1, \mathscr{H}om(\oplus \mathcal{O}_{\mathbb{P}^1}(a_i - r - 1), \mathcal{O}(-2)))$, where we have used Serre duality in the computations. This number is 0 if and only if $a_i - r - 1 > -2$ which is the case if and only if $a_i - r \geq 0$. Since we have that $f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-r) = \oplus \mathcal{O}_{\mathbb{P}^1}(a_i - r)$ it follows from corollary 2.0.12 that $f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-r)$ is globally generated if and only if $a_i - r \geq 0$. This finishes the proof. □

### 3.2.2 Connections to uniruled and rationally connected varieties

In this subsection we will prove that a projective variety over a field $k$ that has a free (respectively very free) rational curve is uniruled (respectively rationally connected), where the converse holds if the characteristic of the field $k$ is 0 and the field $k$ is algebraically closed. We will later use this to prove the Castelnuovo's criterion analogues that we stated in Motivation part 1. The literature usually restricts itself to the algebraically closed case while developing this theory. We shall on the other hand do this over non closed fields. We will use Propositions 4.8 and 4.9 in [Deb01] which are stated and proved for varieties over an algebraically closed field in [Deb01], however if we alter the statements to become statements about $k$-rational points rather than arbitrary points, the results will be valid over any field $k$. We will shortly be giving our formulations of the aforementioned propositions and elaborate on their proofs, however we shall need some facts about smooth morphisms first.

Recall from Chapter 12 in [Vak13] that a morphism of schemes $\pi : X \to Y$ is smooth at a point $p \in X$ if there is an open neighborhood $U$ of $p$ such that $\pi|_U$ is smooth. Further we recall that the locus of $X$ where the morphism $\pi : X \to Y$ is smooth is open (this is [Vak13, Ch. 12, Sec. 6, Ex.12.6.F]). The proof of Proposition 4.8 in [Deb01] uses Proposition [Har77, Ch.3, Sec. 10, Prop. 10.4] which is formulated for non-singular varieties over algebraically closed fields. As the author of this thesis could not find a reference for an analogue of this result for arbitrary fields, we shall state and prove a version for perfect fields, and a "point version" for arbitrary fields.

**Proposition 3.2.5.** *Let $\pi : X \to Y$ be a morphism of smooth $k$-varieties. Let $n = \dim X - \dim Y$. Then the following statements are true:*

*(1) If $k$ is a perfect field (or $k(x)$ is separable over $k$ for any closed point $x \in X$) and the induced map of Zariski tangent spaces*

$$T_{\pi,x} : T_{X,x} \to T_{Y,y} \otimes_{k(y)} k(x)$$

*is surjective for every closed point $x \in X$. Then $\pi$ is a smooth morphism.*

*(2) Let $x \in X$ and $y = \pi(x)$. If $k(x)/k$ and $k(y)/k$ are separable, and if $T_{\pi,x}$ is surjective, then $\pi$ is smooth at $x$.*

*Proof.* For (1): By [Vak13, Ch. 25, Sec. 2, Thm. 25.2.2] and [Vak13, Ch.24. Sec.5, Ex. 24.5J] it is enough to prove that $\pi$ is flat and that $\Omega_{X/Y}$ is locally free of rank $n = \dim X - \dim Y$. By the open condition of flatness (Thm. 24.5.13 in [Vak13]) and the fact that every point of a $k$-variety has a closed point in its closure, it is enough to prove that $\pi$ is flat at every closed point, to prove that $\pi$ is flat.

Let $x \in X$ be a closed point of $X$, and let $y = \pi(x)$. Since $Y$ and $X$ are smooth $k$-varieties, we have that $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ are regular local rings. Let $t_1, \ldots, t_r$ be a system of parameters of $\mathcal{O}_{Y,y}$ (see Chapter 11 in [AM69]). Then as $T_{\pi,x}$ is surjective,the map $(\mathfrak{m}_y/\mathfrak{m}_y^2) \otimes_{k(y)} k(x) \to (\mathfrak{m}_x/\mathfrak{m}_x^2)$ is injective, hence the images of $t_1, \ldots, t_r$ in $\mathcal{O}_{X,x}$ form part of a system of parameters of $\mathcal{O}_{X,x}$. Now since $\mathcal{O}_{X,x}/(t_1, \ldots, t_r)$ is flat over $\mathcal{O}_{Y,y}/(t_1, \ldots, t_r) = k(y)$ (everything is flat over a field), we can now either apply [Vak13, Ch. 24, Sec. 6, Thm. 24.6.5] or [Har77, Ch.3, Sec. 10, Lem. 10.3A] together with descending induction on $i$ to show that $\mathcal{O}_{X,x}/(t_1, \ldots, t_i)$ is flat over

$\mathcal{O}_{Y,y}/(t_1,\ldots,t_i)$ for every $0 \leq i \leq r$. In particular $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$. This proves that $\pi$ is flat.

By [Vak13, Ch.13, Sec.7, Ex.13.7K], it is enough to prove that the rank of $\Omega_{X/Y}$ at $x$ given by

$$\dim_{k(x)} \Omega_{X/Y}|_x = \dim_{k(x)} \Omega_{X/Y\,x} \otimes_{\mathcal{O}_{X,x}} k(x),$$

is constant and equal to $n$ for every $x \in X$, in order to prove that $\Omega_{X/Y}$ is a locally free sheaf of rank $n$. Let $p \in X$ be a point in $X$. By upper semicontinuity of rank (see [Vak13, Ch.13, Sec. 7, Ex.13.7.J]), we have that $\dim_{k(p)} \Omega_{X/Y}|_p \leq \dim_{k(q)} \Omega_{X/Y}|_q$, for every point $q$ contained in the closure of $p$. From this we see that it is enough to show that $\dim_{k(\xi)} \Omega_{X/Y}|_\xi \geq n$, where $\xi$ is the generic point of $X$, and that $\dim_{k(x)} \Omega_{X/Y}|_x = n$ for every closed point $x \in X$, in order to prove that $\Omega_{X/Y}$ is a locally free sheaf of rank $n$.

Since $\pi$ is flat, $\pi$ is dominant, hence if $\xi$ is the generic point of $X$, then $\pi(\xi)$ is the generic point of $Y$. Thus $\Omega_{X/Y}|_\xi = \Omega_{K(X)/K(Y)}$. Now by [Har77, Ch.2, Sec.8, Thm.8.6A], we have that

$$\dim_{k(\xi)} \Omega_{K(X)/K(Y)} \geq \operatorname{tr.deg} K(X)/K(Y).$$

By additivity of transcendence degrees (see [Lan02, Ch.8, Ex.3]), we have that $\operatorname{tr.deg} K(X)/K(Y) = \operatorname{tr.deg} K(X)/k - \operatorname{tr.deg} K(Y)/k$, which is again by [Vak13, Ch.11, Sec.2, Thm 11.2.1] , equal to

$$\dim X - \dim Y = n.$$

Thus we have $\dim_{k(\xi)} \Omega_{X/Y}|_\xi \geq n$.

Now let $x \in X$ be a closed point of $X$ and consider the relative cotangent exact sequence:

$$\pi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

which gives the exact sequence:

$$(\pi^*\Omega_Y)|_x \to \Omega_X|_x \to \Omega_{X/Y}|_x \to 0$$

Since $k(x)$ is seperable over $k$, we have that $\Omega_X|_x \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)$, and since $(\pi^*\Omega_Y)|_x = \Omega_y|_{\pi(y)} \otimes_{k(y)} k(x)$, we have that

$$(\pi^*\Omega_Y)|_x \cong (\mathfrak{m}_y/\mathfrak{m}_y^2) \otimes_{k(y)} k(x).$$

From this and surjectivity of $T_{\pi,x}$ it follows that $(\pi^*\Omega_Y)|_x \to \Omega_X|_x$ is injective. Together with the rank theorem for finite dimensional vector spaces, this yields:

$$\dim_{k(x)} \Omega_{X/Y}|_x = \dim_{k(x)} \Omega_X|_x - \dim_{k(x)}(\pi^*\Omega_Y)|_x = \dim X - \dim Y = n$$

where we have used that since $X$ and $Y$ are smooth $k$-varieties, $\Omega_X$ and $\pi^*\Omega_Y$ are locally free sheaves of rank $\dim X$ respectively $\dim Y$. This completes the proof of (1).

For (2): This follows from upper semicontinuity of rank,Exercise 13.7K in [Vak13], the fact the locus of points where $\pi$ is flat is open in X, and arguments similar to those given in the proof of (1).

□

**Proposition 3.2.6.** *Let $X$ be a smooth quasi-projective variety over a field $k$, let $r$ be a nonnegative integer, let $f : \mathbb{P}^1_k \to X$ be an $r$-free rational curve and let $B$ be a finite subscheme of $\mathbb{P}^1_k$ of length $b$. Let $s$ be a positive integer such that $s + b \leq r + 1$. The evaluation map*

$$ev_M = ev^{(s)}|_{(\mathbb{P}^1_k)^s \times \mathrm{Mor}(\mathbb{P}^1_k, X; f|B)} : (\mathbb{P}^1_k)^s \times \mathrm{Mor}(\mathbb{P}^1_k, X; f|B) \to X^s$$

*is smooth at $(t_1, \ldots, t_s, f)$ whenever the points $t_1, \ldots, t_s$ are $k$-rational and $\{t_1, \ldots, t_s\} \cap B = \emptyset$.*

To proceed as in the proof given in [Deb01] we need that $k(t_i) = k$, and $k(f(t_i)) = k$ for all $i$. These conditions will be satisfied whenever the points are $k$-rational.

*Proof of the Proposition.* If $(t_1, \ldots, t_s, g)$ is a $k$-rational point, then we have $ev_M(t_1, \ldots, t_s, g) = (g(t_1), \ldots, g(t_s))$, further the tangent map to the morphism $ev_M$ at the point $(t_1, \ldots, t_s, f)$ is the map

$$\bigoplus_{i=1}^{s} T_{\mathbb{P}^1, t_i} \oplus H^0(\mathbb{P}^1, f^*T_X(-B)) \to \bigoplus_{i=1}^{s} T_{X, f(t_i)} \cong \bigoplus_{i=1}^{s} (f^*T_X)|t_i$$

where $f^*T_X(-B) = f^*T_X \otimes I_B$ and $(f^*T_X)|t_i = (f^*T_X)_{t_i} \otimes k(t_i)$, moreover this map is given by

$$(u_1, \ldots, u_s, \sigma) \mapsto (T_{t_1} f(u_1) + \sigma(t_1), \ldots, T_{t_s} f(u_s) + \sigma(t_s))$$

where $T_{t_i} f(u_i)$ is the image of $u_i$ in $(f^*T_X)|t_i$ under the composition of the tangent map $T_{\mathbb{P}^1, t_i} \to T_{X, f(t_i)}$ and the isomorphism $T_{X, f(t_i)} \cong (f^*T_X)|t_i$, and $\sigma(t_s)$ is the image of $\sigma$ in $(f^*T_X)|t_i$ under the following composition: $H^0(\mathbb{P}^1, f^*T_X(-B)) \to H^0(\mathbb{P}^1, f^*T_X) \to (f^*T_X)|t_i$. Thus we see that the tangent map to the morphism $ev_M$ will clearly be surjective if the evaluation map

$$H^0(\mathbb{P}^1, f^*T_X(-B)) \to H^0(\mathbb{P}^1, f^*T_X) \to \oplus_{i=1}^{s}(f^*T_X)|t_i \tag{3.2.1}$$

given by $\sigma \mapsto (\sigma(t_1), \ldots, \sigma(t_s))$ is surjective. Now we have integers $a_1, \ldots a_n$ such that $f^*T_X = \displaystyle\bigoplus_{j=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_j)$. Moreover since $B$ is a finite subscheme of length $b$ on $\mathbb{P}^1$, we have $I_B = \mathcal{O}_{\mathbb{P}^1}(-b)$. Thus we have

$$f^*T_X(-B) \cong \bigoplus_{j=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_j - b)$$

Now consider the following commutative diagram:

$$\begin{array}{ccccc}
\bigoplus_{j=1}^{n} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j - b)) & \longrightarrow & \bigoplus_{j=1}^{n} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j)) & \longrightarrow & \bigoplus_{j=1}^{n} \bigoplus_{i=1}^{s} k(t_i) \\
\uparrow & & & & \uparrow \\
H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j - b)) & \longrightarrow & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j) & \longrightarrow & \bigoplus_{i=1}^{s} k(t_i)
\end{array}$$

If the composition of arrows at the bottom of the diagram is surjective for each $j$, then the composition of arrows at the top will be surjective as well, which in turn yields surjectivity of the composition given in (3.2.1). Since $B \cap \{t_1, \ldots, t_s\} = \emptyset$, we have that

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j - b)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j)) \to \bigoplus_{i=1}^{s} k(t_i) \tag{3.2.2}$$

is not the zero map, in fact composing this with any projection onto $k(t_i)$ will not be the zero map. Further pick a basis element $e_i = (0, \ldots 1, 0 \ldots 0)$ in $\displaystyle\bigoplus_{i=1}^{s} k(t_i)$, we shall find an element in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j - b))$ which is mapped to $e_i$ under (3.2.2). For each $i$, let $h_i$ be the degree 1 polynomial which generates the prime ideal $t_i$, for example if $t_i = (x_0 - \frac{a_0}{a_1}x_1)$ then $h_i = x_0 - \frac{a_0}{a_1}x_1$. Since $a_j - b \geq s - 1$, we can find a polynomial $g$ of degree $a_j - b - (s-1) \geq 0$ which is not contained in the prime ideal $t_i$. Let $f(x_0, x_1)$ be the polynomial $g \prod_{j \neq i} h_j \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j - b))$. The polynomial $f$ is mapped to $e_i$ under

(3.2.2). From this we conclude that the tangent map to the evaluation map $ev_M$ is surjective. Since $H^1(\mathbb{P}^1, f^*T_X(-B))$ vanishes, we have that $\mathrm{Mor}(\mathbb{P}^1, X; f|B)$ is smooth at $[f]$ (this is explained in [Deb01, Ch.2, Sec.3]) thus $ev_M$ is a map between two smooth varieties and the tangent map of this morphism at the point $(t_1, \dots t_s, f)$ is surjective, hence $ev_M$ is smooth at this point by (2) in Proposition 3.2.5. □

**Corollary 3.2.7.** *Let $X$ be a smooth quasi-projective variety over $k$. If $X$ contains a free (respectively very free) rational curve, then $X$ is uniruled (respectively rationally connected).*

*Proof.* If $X$ has a $r$ free rational curve $f$, then the map

$$ev_M^{(r+1)} = ev^{(r+1)}|_{\mathbb{P}^1_k \times \mathrm{Mor}(\mathbb{P}^1_k, X))}$$

is smooth at a point $(t_1, \dots, t_s, f)$ by the proposition above. From this it follows that the restriction of this map to the unique component containing this point is dominant (otherwise it would factor through a proper closed subset of $X$ and the this would imply that the tangent map is not surjective). Hence if $r$ is 0, $f$ is free and $X$ is uniruled. If $r$ is 1 , $f$ is very free and $X$ is rationally connected. □

We now give a partial converse to Proposition 3.2.6:

**Proposition 3.2.8.** *Let $X$ be a smooth quasi-projective variety over a field $k$, let $f : \mathbb{P}^1_k \to X$ be a rational curve, and let $B$ be a finite subscheme of $\mathbb{P}^1_k$ of length $b$. If the tangent map to the evaluation map $ev_M$ , defined as in Proposition 3.2.6, is surjective at some $k$-rational point of $(\mathbb{P}^1_k)^s \times \{f\}$, the rational curve $f$ is $min(2, b+s-1)$-free.*

*Proof.* Upon possibly replacing $B$ with a subscheme and $s$ by a smaller integer, we may assume $b + s \leq 3$. Our assumption is that the map

$$T_{ev_M,p} : \bigoplus_{i=1}^{s} T_{\mathbb{P}^1,t_i} \oplus H^0(\mathbb{P}^1, f^*T_X(-B)) \to \bigoplus_{i=1}^{s} T_{X,f(t_i)} \cong \bigoplus_{i=1}^{s} (f^*T_X)|t_i$$

given by $(u_1, \dots, u_s, \sigma) \mapsto (T_{t_1}f(u_1)+\sigma(t_1), \dots, T_{t_s}f(u_s)+\sigma(t_s))$ is surjective for some $t_1, \dots, t_s$. This implies that the evaluation map

$$H^0(\mathbb{P}^1, f^*T_X(-B)) \to H^0(\mathbb{P}^1, f^*T_X) \to \bigoplus_{i=1}^{s} ((f^*T_X)|t_i/Im(T_{t_i}f)) \quad (3.2.3)$$

is surjective. We may assume that the dimension of $X$ is at least two. Under this assumption we have that $(f^*T_X)|t_i$ has dimension at least two, and since $T_{\mathbb{P}^1,t_i}$ has dimension one, it follows that $Im(T_{t_i}f)$ has at most dimension one, thus $(f^*T_X)|t_i/Im(T_{t_i}f)$ does not vanish for any $i$. From this and the surjectivity of the map given in (3.2.3) we conclude that no $t_i$ is contained in $B$. Now consider the commutative diagram:

$$\begin{array}{ccc} H^0(\mathbb{P}^1, f^*T_X(-B)) & \xrightarrow{e} & \bigoplus_{i=1}^s (f^*T_X)|t_i \\ \uparrow & & \uparrow{\scriptstyle (T_{t_1}f,\ldots,T_{t_s}f)} \\ H^0(\mathbb{P}^1, T_{\mathbb{P}^1}(-B)) & \xrightarrow{e'} & \bigoplus T_{\mathbb{P}^1,t_i} \end{array}$$

Since $b + s \leq 3$ we get that $2 - b \geq s - 1$, hence we have

$$\deg(T_{\mathbb{P}^1}(-B)) \geq s - 1$$

and $e'$ is surjective by arguments that are similar to those which proved surjectivity of the map given in (3.2.2) in the previous proof. Since $e'$ is surjective, we see that the image of the map $e$ in the commutative diagram above, contains $Im(T_{t_1}f, \ldots, T_{t_s}f)$. As the map given in (3.2.3) is surjective, we can now conclude that $e$ is surjective. With the notation of the previous proof, we see that each map

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j - b)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j)) \to \bigoplus_{i=1}^{s} k(t_i)$$

for $j = 1, \ldots, n$, is surjective, hence $a_j - b \geq s - 1$ and $f$ is $(b+s-1)$-free. $\square$

We have the following geometric interpretation of the Proposition above together with Proposition 3.2.6:

**Corollary 3.2.9.** *Let $X$ be a smooth projective variety over an algebraically closed field $k$. Assume that $X$ has a free (respectively very free) rational curve, then there passes a free (respectively very free) rational curve through a general point of $X$.*

*Proof.* We first fix some notation: Let $ev_M^{(s)}$ be $ev_M$ as in Proposition 3.2.6, so $s = 1$ if $f$ is free, and $s = 2$ if $f$ is very free. Moreover we let $Y$ be the source of $ev_M^{(s)}$.

By Proposition 3.2.6 there is an open subset $U$ of $Y$ such that the evaluation map $ev_M^{(s)}|_U$ is smooth, hence dominant. Since the closed points are dense in a scheme of finite type over $k$, we see that the image of the closed points of $U$ are dense in $X$. This proves that there is a rational curve through a general point of $X$. To see that this rational curve can be taken to be free (respectively very free), we have that the tangent map of $ev^{(s)}|_U$ is surjective at a closed point $(t_1, \ldots, t_s, g)$ of $U$ since it is smooth, thus by Proposition 3.2.8 the rational curve $g$ is $s-1$-free. □

When the characteristic of the field $k$ is zero we have a converse to Corollary 3.2.7:

**Proposition 3.2.10.** *Let $X$ be a projective variety over an algebraically closed field $k$ of characteristic zero. Assume that $X$ is uniruled (respectively rationally connected), then $X$ has a free (respectively very free) rational curve.*

*Proof.* We only prove the rationally connected case, as the uniruled case is completely analogous. Let $\pi : (\mathbb{P}^1_k)^2 \times M \dashrightarrow X \times X$ be a dominant rational map. The dominance of this implies that the morphism $ev^{(2)} : (\mathbb{P}^1_k)^2 \times \mathrm{Mor}(\mathbb{P}^1_k, X) \to X \times X$ is dominant (as in (3.1.1)). By generic smoothness [Vak13, Ch. 25, Sec.3, Thm. 25.3.3] we have a dense open subset $U$ of $X$ such that $ev^{(2)}|_{(ev^{(2)})^{-1}(U)}$ is a smooth morphism. The rest now follows by Proposition 3.2.8. □

**Remark 3.2.11.** The hypothesis of the field being algebraically closed in the Proposition above can be relaxed. Indeed we only need the existence of a $k$-rational point of $(ev^{(2)})^{-1}(U)$.

We will shortly adress Castelnuovo's criterion for uniruled and rationally connected varieties, however we will first apply the technique of flat base extension to obtain some results which will help us generalise Theorem 1.0.5 and 1.0.8 (1).

### 3.2.3 Lifting of free rational curves to the algebraic closure

If $X$ is a projective variety over a field $k$, and $X$ has a $r$-free rational curve $f$, then we want $X_{\bar{k}}$ to have a $r$-free rational curve as well, as we then can exploit the geometric result of Corollary 3.2.9. This will yield results about $X_{\bar{k}}$, and by using the technique of flat base extension again, we will see that these results are also true for $X$.

*Setup:* Let $X$ be a projective variety over a field $k$. Let $\mu_X : X_{\bar{k}} \to X$ be the projection onto $X$. This is a flat morphism, since flatness is preserved under base change. We let $\mu_{\mathbb{P}^1_k} : \mathbb{P}^1_{\bar{k}} \to \mathbb{P}^1_k$ be the projection onto $\mathbb{P}^1_k$.

**Lemma 3.2.12.** *(notation as above) We have that* $\mu_X^* T_X \cong T_{X_{\bar{k}}}$.

*Proof.* By [Vak13, Ch. 21, Sec.2, Thm. 21.2.27 (b)] we have that

$$\mu_X^* \Omega_X \cong \Omega_{X_{\bar{k}}},$$

From this we see that we will be done if

$$\mu_X^* \mathscr{H}om(\Omega_X, \mathcal{O}_X) \cong \mathscr{H}om(\mu_X^* \Omega_X, \mathcal{O}_{X_{\bar{k}}})$$

Since $u_X$ is a flat morphism, and $\Omega_X$ is finitely presented it follows from [Eis95, Ch.2, Sec. 2, Prop. 2.10] that this is the case. □

**Proposition 3.2.13.** *Let $X$ be a projective variety over a field $k$ and let $f$ be a rational curve on $X$. Let $\bar{f}$ denote the induced rational curve on $X_{\bar{k}}$. We have that $f$ is $r$-free if and only if $\bar{f}$ is.*

*Proof.* We have $\bar{f}^* T_{X_{\bar{k}}} = \bar{f}^*(\mu_X^* T_X)$ by the lemma above. Further we have that $\bar{f}^* \mu_X^* T_X = \mu_{\mathbb{P}^1_k}^*(f^* T_X)$. Hence

$$\bar{f}^* T_X \otimes \mathcal{O}_{\mathbb{P}^1_{\bar{k}}}(-r-1) \cong \mu_{\mathbb{P}^1_k}^*(f^* T_X \otimes \mathcal{O}_{\mathbb{P}^1_k}(-r-1)).$$

Hence by flat base extension [Liu02, Ch. 5, Sec. 2, Cor. 2.27]:

$$H^1(\mathbb{P}^1_k, f^* T_X \otimes \mathcal{O}_{\mathbb{P}^1_k}(-r-1)) \otimes_k \bar{k} \cong H^1(\mathbb{P}^1_{\bar{k}}, \bar{f}^* T_{X_{\bar{k}}} \otimes \mathcal{O}_{\mathbb{P}^1_{\bar{k}}}(-r-1))$$

Since $\bar{k}$ is faithfully flat over $k$ we have that

$$H^1(\mathbb{P}^1_k, f^* T_X \otimes \mathcal{O}_{\mathbb{P}^1_k}(-r-1)) = 0$$

if and only if

$$H^1(\mathbb{P}^1_{\bar{k}}, \bar{f}^* T_{X_{\bar{k}}} \otimes \mathcal{O}_{\mathbb{P}^1_{\bar{k}}}(-r-1)) = 0.$$

The proposition now follows from this and Proposition 3.2.4. □

We see from the proposition above that if $X$ has a very free rational curve, then so does $X_{\bar{k}}$. It is proved in [Kol99] that the converse of the aforementioned statement is true when the field $k$ is local [2].

**Theorem 3.2.14.** *Let $X$ be a smooth projective variety of dimension $n$ over a field $k$. Then:*

[2] By a local field we mean either $\mathbb{R}, \mathbb{C}, \mathbb{F}_q((t))$ or a finite degree extension of a $p$-adic field $\mathbb{Q}_p$.

(1) *If $X$ has a free curve, $H^0(X, K_X^{\otimes m}) = 0$ for all $m \geq 1$.*

(2) *If $X$ has a very free curve, $H^0(X, (\Omega_X^p)^{\otimes m}) = 0$ for all $m, p \geq 1$.*

*Proof.* Let $\mu_X : X_{\bar{k}} \to X$ be the projection onto $X$. We have $\mu_X^* \Omega_X \cong \Omega_{X_{\bar{k}}}$. Moreover since $u_X^*$ commutes with all tensor operations, we have $\mu_X^*(\Omega_X^p)^{\otimes m} \cong (\Omega_{X_{\bar{k}}}^p)^{\otimes m}$. From this and flat base extension, we see that we can assume that $k = \bar{k}$ in (1) and (2).

We now prove (1): Assume that $f : \mathbb{P}_k^1 \to X$ is a free rational curve on $X$. We have $f^* T_X = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}_k^1}(a_i)$. By the conormal exact sequence for smooth varieties (Theorem 21.3.8 in [Vak13]) we have the inclusion $T_{\mathbb{P}_k^1} \to f^* T_X$. From this and Lemma 2.0.19 we see that some $a_i \geq 2$. Since $f$ is free, all the other $a_i \geq 0$ and $f^* K_X$ must have negative degree. Thus any section of $K_X^{\otimes m}$ must vanish on $f(\mathbb{P}_k^1)$, since the pullback of the locus where a section vanishes is the locus where the pulled back section vanishes. By Corollary 3.2.9 there passes a free rational curve through a general point of $X$ hence a section of $K_X^{\otimes m}$ vanishes on a dense subset of $X$ hence on $X$.

For (2): This is very similar to (1), the reason we need very free is to ensure that all the $a_i$ in the splitting type of $f^*(\Omega_X^p)^{\otimes m}$ are negative. □

**Corollary 3.2.15.** *Let $X$ be a smooth projective variety over a field $k$ of characteristic zero.*

(1) *If $k$ is algebraically closed and $X$ is uniruled, then $H^0(X, K_X^{\otimes m}) = 0$ for all $m \geq 0$.*

(2) *If $X$ has a very free curve then $H^m(X, \mathcal{O}_X) = 0$ for all $m \geq 1$. In particular $H^m(X, \mathcal{O}_X) = 0$ if $k = \bar{k}$ and $X$ is rationally connected.*

*Proof.* (1) follows from Proposition 3.2.10 and Theorem 3.2.14. (2) By Hodge theory, we have that $h^{p,0} = h^0(X, \Omega_X^p) = h^{0,p} = h^p(X, \Omega_X^0) = h^p(X, \mathcal{O}_X)$ for all $p$ (see section 21.5.10 in [Vak13]). By Theorem 3.2.14 $h^{p,0} = 0$. □

### 3.2.4 Fano varieties and rationally connectedness

We shall now define Fano varieties and cite a theorem which tells us that they are rationally connected whenever they are over an algebraically closed field of characteristic zero.

**Definition 3.2.16.** Suppose $X$ is a smooth projective $k$-variety. Then $X$ is said to be *Fano* if $K_X^\vee$ is ample.

**Theorem 3.2.17** ([KMM92a]). *Let $X$ be a smooth Fano variety of dimension $n$ over an algebraically closed field $k$ of characteristic zero. Then $X$ is rationally connected. In particular $X$ has a very free curve.*

It is an open problem wether this is true in charactersitic $p$ as well. Some advances have been made however. Yi Zhu proved in [Zhu11] that a general Fano hypersurface in projective space over an algebraically closef field has a very free rational curve. Using methods from logarithmic geometry, Zhu and Qile Chen, also gave a positive answer for general Fano complete intersections in [CZ13].

We also expect and want $\mathbb{P}_k^N$, where $k$ is any field, to be rationally connected. By considering the Euler exact sequence ([Vak13, Ch. 21, Sec. 4, Thm. 21.4.6]), we have a surjection: $\mathcal{O}_{\mathbb{P}_k^N}(1)^{N+1} \to T_{\mathbb{P}_k^N}$. Now let $f$ be any rational curve on $\mathbb{P}_k^N$ of degree $e$. Then the map

$$f^*\mathcal{O}_{\mathbb{P}_k^N}(1)^{N+1} = \mathcal{O}_{\mathbb{P}_k^1}(e)^{N+1} \to f^*T_{\mathbb{P}_k^N}$$

is surjective, which implies that $f^*T_{\mathbb{P}_k^N}$ is ample[3]. Hence $\mathbb{P}_k^N$ has a very free rational curve, thus $\mathbb{P}_k^N$ is indeed rationally connected.

In the rest of this thesis we will mostly investigate the theory of free and very free rational curves on Fermat hypersurfaces. The Fermat hypersurfaces that we shall consider are very special Fano hypersurfaces, thus we cannot take advantage of the results in [Zhu11].

[3]The proof of this claim is similar to the proof which we shall give of Lemma 5.3.1 (1) in Chapter 5.

# Chapter 4

# The Fermat hypersurface

In this chapter some of the fundamental properties of the Fermat hypersurface will be discussed. We start by recalling the definition.

**Definition 4.0.18.** Let $k$ be a field of characteristic $p$. Let $N$, $d$ be positive integers where $p$ does not divide $d$ and $N \geq 2$. We define the degree $d$ Fermat hypersurface $X_{d,N}$ in $\mathbb{P}_k^N$ to be $V(G)$, where $G$ is the homogeneous polynomial of degree $d$ given by

$$G = \sum_{i=0}^{N} x_i^d \tag{4.0.1}$$

We want the Fermat hypersurface to be a reduced scheme, hence we required $d$ not to be divisible by $p$, moreover by the Jacobi criterion we see that this is also the sufficient and necessary condition for the Fermat to be smooth and hence irreducible.

We will almost exclusively be studying rational curves of relatively high degree on the Fermat hypersurface. The reader may however take a look at [Deb01, Ch.2,Sec.4] for theory regarding lines on the Fermat hypersurface. We also recommend the highly readable article [BC66] where the authors prove that any Hermitian variety in finite projective $N$-dimensional space over the Galois field of $q^2$ elements, is projectively equivalent to a Fermat hypersurface. Enumerative, respectively geometric, notions such as counting of points and linear subspaces, respectively tangent spaces at points are also discussed in the aforementioned article.

Recall that a smooth projective variety $X$ is Fano if the anticanonical bundle $K_X^{-1}$ is ample.

**Proposition 4.0.19.** *Let $X = X_{d,N}$ be a Fermat hypersurface. Then $X$ is Fano if and only if $d \leq N$.*

*Proof.* We compute $K_X$. By [Vak13, Ch.21,Sec.3,Thm.21.3.8] we have the conormal exact sequence:

$$0 \to \mathscr{N}^\vee \to i^*\Omega_{\mathbb{P}^N_k} \to \Omega_X \to 0$$

From [Har77, Ch.2,Sec.5,Ex.5.16] we thus have that

$$i^* \bigwedge^N \Omega_{\mathbb{P}^N_k} = \mathscr{N}^\vee \otimes \bigwedge^{N-1} \Omega_X.$$

We have $\mathscr{N}^\vee = \mathcal{O}_X(-d)$ and by the exact sequence given in [Har77, Ch.2, Sec.8, Thm. 8.13]:

$$0 \to \Omega_{\mathbb{P}^N_k} \to \mathcal{O}_{\mathbb{P}^N_k}(-1)^{N+1} \to \mathcal{O}_{\mathbb{P}^N_k} \to 0$$

we see that $\bigwedge^N \Omega_{\mathbb{P}^N_k} = \mathcal{O}(-(N+1))$. All in all we have that $K_X = \mathcal{O}_X(d - N - 1)$, thus $K_X^{-1} = \mathcal{O}_X(N+1-d)$. Now if $d \leq N$, we have that $N+1-d \geq 1$ and thus $K_X^{-1}$ is ample by [Har77, Ch.2,Sec.5,Ex.5.13] and [Vak13, Ch.16,Sec.6,Ex.16.6.G].

Conversely if $d > N$, then $N+1-d \leq 0$. In that case $K_X^{-1} \otimes \mathcal{O}_X(-1) = \mathcal{O}_X(-r)$, for some $r > 0$. We show that there are no non-trivial global sections of this sheaf, hence it is not globally generated, thus not ample.

Consider the equalizer exact sequence:

$$0 \to \mathcal{O}_X(-r)(X) \to \prod_i \mathcal{O}_X(-r)(D_+(x_i)) \rightrightarrows \prod_{i,j} \mathcal{O}_X(-r)(D_+(x_i x_j))$$

The elements in $\prod_i \mathcal{O}_X(-r)(D_+(x_i))$ are of the form $(f_i/x_i^{h_i})$, where $f_i$ is a homogeneous polynomial (considered as an element in the quotient $k[x_0, \dots x_n]/(G)$, where $G$ is the polynomial in (4.0.1) ) of degree $h_i - r$ for some integer $h_i \geq r$. We have that $(f_i)_i$ is mapped to the same element under the two maps if and only if we have $(f_i x_j^{h_i} - f_j x_i^{h_j}) \in (G)$ for all $i, j$. We see that this is the case if and only if $(f_i x_j^{h_i} - f_j x_i^{h_j}) = 0$ in $k[x_0, \dots, x_N]$, and as $\deg(h_i) > \deg(f_i)$ for all $i$ it follows that this is again equivalent to having all $f_i = 0$. Hence we see that $\mathcal{O}_X(-r)(X) = 0$. This concludes the proof. □

### 4.0.5 The space of degree $e$ rational curves on the Fermat hypersurface

We will here explicitly describe the scheme $\mathrm{Mor}_e(\mathbb{P}^1_k, X)$, for $X = X_{d,d} \subseteq \mathbb{P}^d_k$. The equations that define this space will be useful at several points in this thesis.

**Proposition 4.0.20.** *Let $d = p^r + 1$ for some positive integer $r$, and let $X = X_{d,d}$. There is a bijection between the degree $e$ morphisms from $\mathbb{P}^1_k \to X$ and the $k$-rational points of a quasi-projective subvariety $M_e$ of $\mathbb{P}_k^{de+d+e}$.*

*Proof.* Giving a degree $e$ morphism $f : \mathbb{P}^1_k \to X$ is the same as giving $n+1$ homogeneous polynomials $F_0, \dots F_d \in k[x_0, x_1]_e$ without common factors, such that $\sum_{i=0}^d F_i^d = 0$ (see [Vak13, Ch.16,sec.4]). We shall now let $F_i(x_0, x_1) = \sum_{j=0}^e a_{ij} x_o^{e-j} x_1^j$. We now compute:

$$\begin{aligned}
\sum_{i=0}^{d} F_i^d &= \sum_i (\sum_{j=0}^{e} a_{ij} x_0^{e-j} x_1^j)^d \\
&= \sum_{i=0}^{d} (\sum_{j=0}^{e} a_{ij}^{p^r} (x_0^{e-j} x_1^j)^{p^r})(\sum_{j=0}^{e} a_{ij} x_0^{e-j} x_1^j) \\
&= \sum_{i=0}^{d} (\sum_{k=0}^{ep^r+e} x_0^{ep^r+e-k} x_1^k (\sum_{\substack{jp^r+l=k \\ l \geq 0}} a_{ij}^{p^r} a_{il})) \\
&= \sum_{k=0}^{de} x_0^{de-k} x_1^k (\sum_{i=0}^{d} \sum_{\substack{jp^r+l=k \\ l \geq 0}} a_{ij}^{p^r} a_{il})
\end{aligned}$$

Hence

$$\sum_{i=0}^{d} F_i^d = \sum_{k=0}^{de} x_0^{de-k} x_1^k (\sum_{i=0}^{d} \sum_{\substack{jp^r+l=k, \\ l \geq 0}} a_{ij}^{p^r} a_{il}) \tag{4.0.2}$$

Now let $A = k[\{y_{ij}\}_{0 \leq i \leq d,\, 0 \leq j \leq e}]$, further let

$$c_k = \sum_{i=0}^{d} (\sum_{jp^r+l=k, l \geq 0} y_{ij}^{p^r} y_{il}) \in A.$$

From (4.0.2) we see that the $F_i$ give a morphism to $X$ if and only if we have $\{a_{ij}\}_{ij} \in \cap_{k=0}^{de} Z_+(c_k) = Z_+(c_0, \dots, c_{de})$ and the $F_i$ don't have a common nonconstant factor, that is the $\{a_{ij}\}_{i,j} \in Z_+(c_0, \dots, c_{de}) \cap \mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^d_k)$ where the intersection is taken in $\mathbb{P}_k^{de+d+e}$. We have that the $k$-rational points of $Y = \mathrm{Proj}(A/(c_0, \dots, c_{de}))$ are in bijection with points of $Z_+(c_0, \dots, c_{de})$, by [Liu02, Ch.2, Cor. 3.44.], thus we conclude that the $k$-rational points of $M_e = Y \cap \mathrm{Mor}_e(\mathbb{P}^1_k, \mathbb{P}^d_k)$, are in bijection with morphisms $\mathbb{P}^1_k \to X$. □

**Definition 4.0.21.** We shall call the scheme $M_e$ given in the proof of the previous proposition, the space of degree $e$ morphisms. As explained in the previous chapter, this space is $\mathrm{Mor}_e(\mathbb{P}^1_k, X)$.

**Remark 4.0.22.** As $M_e$ is the intersection of $de + 1$ hypersurfaces in $\mathbb{P}^{de+e+d}$, we expect the dimension of $M_e$ to be $de+d+e-(de+1) = e+d-1$.

# Chapter 5

# Proofs in the article "Rational curves on Fermat hypersurfaces" by Mingmin Shen

### 5.0.6 Introduction

We will here try to make the proofs in [She12] more accessible. Moreover we show that all the results in this article are true without the hypothesis that the field $k$ is algebraically closed.

## 5.1 What we are dealing with

Throughout this chapter $k$ is a field of characteristic $p$ (not necessarily algebraically closed),$r$ is a positive integer, $d = p^r + 1$ and $N$ is an integer such that $N \geq d$. We let $X_{d,N}$ denote the Fermat hypersurface in $\mathbb{P}^N_k$, given by $X_{d,N} = V(G)$, where $G = \sum x_i^d$. We will sometimes just write $X$ instead of $X_{d,N}$.

## 5.2 A helpful diagram and a useful computation

In the proofs that are to come we will consider a few exact sequences and a diagram in order to investigate the tangent sheaf of $X$. In this section we give these sequences and put them into a diagram. As $X \subseteq \mathbb{P}^N_k$ is non-singular, we have the exact sequence:

$$0 \to \mathcal{N}^\vee \to \Omega_{\mathbb{P}^N} \otimes \mathcal{O}_X \to \Omega_X \to 0$$

by [Vak13, Ch.21, Thm.21.3.8] . Here $\mathcal{I}$ denotes the ideal sheaf of $X$. Further we have that $\mathcal{N}^\vee = \mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_X(-(p^r+1))$. Now consider the following map:

$$\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_X(-1)^{N+1}$$

which is given by multiplication with $\begin{pmatrix} \frac{\partial G}{\partial x_0} \\ \vdots \\ \frac{\partial G}{\partial x_N} \end{pmatrix} = \begin{pmatrix} x_0^{p^r} \\ \vdots \\ x_N^{p^r} \end{pmatrix}$.

Dualizing this yields the map

$$\mathcal{O}_X(1)^{N+1} \xrightarrow{(x_0^{p^r},\dots,x_N^{p^r})} \mathcal{O}_X(p^r+1).$$

The kernel of this map is the dual of the sheaf of principal parts of $X$, $(\mathcal{P}^1_X)^\vee$. After dualizing, all of this fits in the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & T_X & \longrightarrow & T_{\mathbb{P}^N_k}|X & \longrightarrow & \mathcal{O}(p^r+1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & (\mathcal{P}^1_X)^\vee & \longrightarrow & \mathcal{O}_X(1)^{N+1} & \xrightarrow{(x_i^{p^r})} & \mathcal{O}(p^r+1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{5.2.1}$$

Now by [Vak13, Ch.21,Thm.21.4.6] we have the exact sequence:

$$0 \longrightarrow \Omega_{\mathbb{P}^N_k} \longrightarrow \mathcal{O}_{\mathbb{P}^N_k}(-1)^{N+1} \xrightarrow{(x_i)} \mathcal{O}_{\mathbb{P}^N_k} \longrightarrow 0,$$

where $\mathcal{O}_{\mathbb{P}^N_k}(-1)^{N+1}$ is isomorphic to the sheaf of principal parts of $\mathbb{P}^N_k$. Pulling this back to $X$ and applying a positive twist gives us:

$$0 \to \Omega_{\mathbb{P}^N_k} \otimes \mathcal{O}_X(1) \to \mathcal{O}_X^{N+1} \to \mathcal{O}_X(1) \to 0. \tag{5.2.2}$$

Let $F : X \to X$ be the Frobenius morphism. We apply $(F^*)^r$ to (5.2.2) and obtain:

$$0 \longrightarrow (F^*)^r \Omega_{\mathbb{P}^N_k} \otimes \mathcal{O}_X(p^r) \longrightarrow \mathcal{O}_X^{N+1} \xrightarrow{(x_i^{p^r})} \mathcal{O}_X(p^r) \longrightarrow 0,$$

hence it follows that

$$(\mathcal{P}^1_X)^\vee \cong (F^*)^r \Omega_{\mathbb{P}^N_k} \otimes \mathcal{O}_X(p^r + 1). \tag{5.2.3}$$

## 5.3 The proofs

We start by giving a lemma.

**Lemma 5.3.1.** *Let $f : \mathbb{P}^1_k \to X$ be a rational curve on $X$. Assume that $f^*(\mathcal{P}^1_X)^\vee \cong \oplus_{i=1}^N \mathcal{O}_{\mathbb{P}^1_k}(f_i)$ and $f^*T_X = \oplus_{i=1}^{N-1} \mathcal{O}_{\mathbb{P}^1_k}(a_i)$.*

*(1) If all the $f_i > 0$, then all the $a_i > 0$ as well, in particular $f$ will be very free.*

*(2) If one of the $f_i < 0$, then $f$ is not free.*

*(3) If two of the $f_i = 0$, then $f$ is not very free.*

*Proof. (1)* Assume that $a_i \leq 0$ for some $i$. By applying $f^*$ to the first column in 5.2.1 we have a surjection

$$h : f^*(\mathcal{P}^1_X)^\vee \twoheadrightarrow f^*T_X.$$

Let $h_i : \oplus_{i=1}^N \mathcal{O}_{\mathbb{P}^1_k}(f_i) = f^*(\mathcal{P}^1_X)^\vee \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1_k}(a_i)$ given by composing h with the canonical projection. As $f_j > 0$ for all $j$, we have that the only map $\mathcal{O}_{\mathbb{P}^1_k}(f_j) \to \mathcal{O}_{\mathbb{P}^1_k}(a_i)$ is the trivial map, by (2.0.19), for all j , thus the composition $\sigma_j : \mathcal{O}_{\mathbb{P}^1_k}(f_j) \longrightarrow f^*(\mathcal{P}^1_X)^\vee = \oplus \mathcal{O}_{\mathbb{P}^1_k}(f_k) \xrightarrow{h_i} \mathcal{O}_{\mathbb{P}^1_k}(a_i)$ must be the 0-map. Let $g_j$ denote the canonical map $\mathcal{O}_{\mathbb{P}^1_k}(f_j) \to \oplus \mathcal{O}_{\mathbb{P}^1_k}(f_k)$. We have $\sigma_j = 0 \circ g_j$ for all $j$, but we also have $\sigma_j = h_i \circ g_j$ for all $j$, by the universal property of the co-product we must have $h_i = 0$, which is a contradiction.

*(2)* Consider the short exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^1_k} \to f^*(\mathcal{P}^1_X)^\vee = \oplus_{i=1}^N \mathcal{O}_{\mathbb{P}^1_k}(f_i) \to f^*T_X = \oplus_{i=1}^{N-1} \mathcal{O}_{\mathbb{P}^1_k}(a_i) \to 0.$$

This gives us a long exact sequence in cohomology:

$$0 \to H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \to H^0(\mathbb{P}^1_k, \oplus_{i=1}^N \mathcal{O}_{\mathbb{P}^1_k}(f_i)) = \oplus_{i=1}^N H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(f_i))$$
$$\to \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a_i)) \to H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = 0 \to \dots$$

Thus we have a surjective map $\oplus_{l=1}^{N} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(f_i)) \to \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a_i))$. Assume that we hace $f_i < 0$ for some $i$ and we may further assume that we have $f_1 \leq f_2 \leq \ldots \leq f_N$ and $a_1 \leq a_2 \leq \ldots \leq a_{N-1}$. If we have $a_1 \geq 0$, then as we have $\sum_{i=1}^{N} f_i = \sum_{i=1}^{N-1} a_i$, we must have $f_N > a_{N-1} \geq \ldots a_1$. Then by this and (2.0.19) it follows that the surjective map $\oplus_{i=1}^{N} \mathcal{O}_{\mathbb{P}^1_k}(f_i) \to \oplus_{i=1}^{N-1} \mathcal{O}_{\mathbb{P}^1_k}(a_i)$ factors through $\oplus_{i=1}^{N-1} \mathcal{O}_{\mathbb{P}^1_k}(f_i)$, and by possibly repeating this procedure, we may further assume that $f_{N-1} \leq a_{N-1}$., and thus have $f_i \leq a_i$ for $1 \leq i \leq N-1$. It follows that we have a surjective map

$$\varphi : \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(f_i)) \to \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a_i)).$$

Now as $f_1 < 0 \leq a_1$ and we have $f_i \leq a_i$, it follows that we have

$$\dim_k \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(f_i)) < \dim_k \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a_i))$$

which contradicts surjectivity of $\varphi$.

*(3)* Assume that $f_1 = f_2 = 0 \leq f_3 \leq \ldots \leq f_N$, and that $0 < a_1 \leq a_2 \leq \ldots a_{N-1}$. By arguments similar to those given in *(2)*, we get that $f_N > a_{N-1}$. and we can again assume that $f_{N-1} \leq a_{N-1}$. Moreover we also get a surjective map $\varphi : \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(f_i)) \to \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a_i))$, but as $f_1 = f_2 = 0 < a_1$ and $f_i \leq a_i$ for all other $i \leq N-1$ it follows that $\dim_k \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(f_i)) < \dim_k \oplus_{i=1}^{N-1} H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a_i))$ which contradicts surjectivity of $\varphi$. □

## 5.4 Theorem 1.7 and Corollary 1.8 in [She12]

We shall now state and prove Theorem 1.7 in [She12]. This theorem gives us a criterion for when a curve of degree $e$ into the Fermat hypersurface $X = X_{d,d}$ is not very free, where as before $X_{d,d}$ denotes the Fermat hypersurface of degree $d$ in $\mathbb{P}^d_k$.

**Theorem 5.4.1.** *Let $X = X_{d,d}$ be the Fermat hypersurface of degree $d = p^r + 1$ in $\mathbb{P}^d_k$. Let $f : \mathbb{P}^1_k \to X$ be a rational curve of degree $e$. If there exists some $0 \leq m \leq d-3 = p^r - 2$, such that $md < e \leq (m+1)(d-1)$, then $f$ is not very free.*

*Proof.* As the degrees of $f^*(\mathcal{P}^1_X)^\vee$ and $f^*T_X$ are the same (i.e., the sum of their splitting types are the same), we conclude that if either two of the summands of $f^*(\mathcal{P}^1_X)^\vee$ have a trivial twist or one of the summands has a negative twist, then $f$ will not be very free (see Lemma 5.3.1). From this we can conclude that if we want to avoid this, the best possible situation will be when the twists of the summands are as close to each other as possible. By (5.2.3), we see that this will be the case when $f^*\Omega_{\mathbb{P}^d_k}$ is balanced, this means that the twists of this sheafs summands are as close to each other as possible. Now assume first that $f^*\Omega_{\mathbb{P}^d_k}$ is balanced. Under this assumption we shall now compute the splitting type of $f^*\Omega_{\mathbb{P}^d_k}$. We have the exact sequence

$$0 \to \Omega_{\mathbb{P}^d_k}|\mathbb{P}^1_k \to \mathcal{O}_{\mathbb{P}^1_k}(-e)^{d+1} \to \mathcal{O}_{\mathbb{P}^1_k} \to 0$$

Thus as $\mathcal{O}_{\mathbb{P}^1_k}(-e)^{d+1}$ has degree $-(d+1)e$, because of this exact sequence, we have that the degree of $f^*\Omega_{\mathbb{P}^d_k}$ is also $-(d+1)e$, this will let us find it's splitting type, under the assumption that it is balanced. We now let $a = \left\lfloor \frac{(d+1)e}{d} \right\rfloor = e + \left\lfloor \frac{e}{d} \right\rfloor$. Let $l' = e \pmod d$, and let $l = d - l'$, then we have that $f^*\Omega_{\mathbb{P}^d_k} = \mathcal{O}_{\mathbb{P}^1_k}(-a)^l \otimes \mathcal{O}_{\mathbb{P}^1_k}(-a-1)^{l'}$. A simple calculation yields $l' = (d+1)e - da$. Using this and (5.2.3) we can now compute:

$$f^*(\mathcal{P}^1_X)^\vee = f^*((F^*)^r\Omega_{\mathbb{P}^d_k} \otimes \mathcal{O}_X(p^r+1))$$

which we identify with

$$f^*(F^{*r}\Omega_{\mathbb{P}^d_k}) \otimes f^*\mathcal{O}_X(p^r+1)$$

which is equal to

$$\mathcal{O}_{\mathbb{P}^1_k}(-ap^r)^l \oplus \mathcal{O}_{\mathbb{P}^1_k}((a-1)p^r)^{l'} \otimes \mathcal{O}_{\mathbb{P}^1_k}((p^r+1)e)$$

finally this equals

$$\mathcal{O}_{\mathbb{P}^1_k}(-ap^r + (p^r+1)e)^l \oplus \mathcal{O}_{\mathbb{P}^1_k}((-a-1)p^r + (p^r+1)e)^{l'}.$$

Set $b_1 = -ap^r + (p^r+1)e$ and $b_2 = (-a-1)p^r + (p^r+1)e$. Observe also that $f^*(\mathcal{P}^1_X)^\vee$ is highly unbalanced unless $e \pmod d = 0$ as then $l' = 0$, however the assumption that there exists $m$, $0 \le m \le d-3$, such that we have $md < e \le (m+1)(d-1)$ implies that $e \pmod d \neq 0$, thus in this case $f^*(\mathcal{P}^1_X)^\vee$ will be heavily unbalanced. If $md < e < (m+1)d$, then $m$ will be the largest integer such that $md < e$ and $e = md + (e \pmod d)$. From this it follows that $a = e + m$. Thus we have:

$$b_2 = -(a+1)p^r + e(p^r+1) = e - (m+1)(d-1)$$

where we have used that $d = p^r + 1$. Hence if $e < -(m+1)(d-1)$ we will have $b_2 < 0$ thus $f$ is not very free. Notice that we still haven't used the assumption that $m \leq d-3$, this will be needed now when we consider the case where $e = (m+1)(d-1)$. In this case we will have $l^{'} = d-m-1$, and if $m \leq d-3$, then $l^{'} \geq 2$. Thus we have at least two summands with trivial twists, and again $f$ is not very free. □

We state the corollary as well.

**Corollary 5.4.2.** *Let $X$ be as above. If there exists $N_0$ such that for all $e$ satisfying $e > N_0$ , there is a very free rational curve of degree $e$ on $X$, then $N_0 > p^r(p^r-1)$.*

## 5.5 Proposition 1.10 in [She12]

In this section we are going to deal with proposition 1.10. The proof which we are going to give uses the fact that if $f : \mathbb{P}^1_k \to \mathbb{P}^N_k$ is a rational normal curve of degree $N$, then we have

$$f^*\Omega_{\mathbb{P}^N_k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-N-1)^N.$$

The proof which we shall give of this fact belongs naturally in the next chapter, and thus that is where the proof shall be given as Lemma 6.3.1.

**Proposition 5.5.1.** *Let $X = X_{d,N}$. If $f : \mathbb{P}^1_k \to X$ is a rational normal curve of degree $N$ (viewed as a rational curve on $\mathbb{P}^N_k$), then $f$ is very free on $X$.*

*Proof.* We have $f^*\Omega_{\mathbb{P}^N_k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-N-1)^N$, since $f : \mathbb{P}^1_k \to \mathbb{P}^N_k$ is a rational normal curve. Using (5.2.3) we thus obtain:

$$\begin{aligned} f^*(\mathcal{P}^1_X)^\vee &= f^*(F^{*r}\Omega_{\mathbb{P}^N_k} \otimes \mathcal{O}_X(p^r+1)) \\ &= f^*(F^{*r}\Omega_{\mathbb{P}^N_k}) \otimes f^*\mathcal{O}_X(p^r+1) \\ &= \mathcal{O}_{\mathbb{P}^1_k}(-(N+1)p^r)^N \otimes \mathcal{O}_{\mathbb{P}^1_k}((p^r+1)N) \\ &= \mathcal{O}_{\mathbb{P}^1_k}((-N-1)p^r + (p^r+1)N)^N \\ &= \mathcal{O}_{\mathbb{P}^1_k}(N-p^r)^N \end{aligned}$$

As $N \geq d = p^r+1$ we see that $f$ is very free. □

## 5.6 Lemma 1.5 in [She12]

In this section we shall prove [She12, Lemma 1.5]. In our proof we shall use a result from [Har66]. The setting in this article is that all schemes are over a base field $k$ which is assumed to be algebraically closed. Proposition 3.2.13 lets us pass to this setting. We shall also give an alternative proof using Proposition 3.2.4.

**Lemma 5.6.1.** *If $X_{d,N}$ contains a very free curve for $N = d = p^r + 1$, then $X_{d,N}$ contains a very free curve for all $N$.*

*Proof.* We realize $X_{d,N}$ as a hyperplane section of $X_{d,N+1}$, thus we have a closed embedding $i : X_{d,N} \to X_{d,N+1}$ whose image is $V(x_{N+1})$. Assume that $f : \mathbb{P}^1_k \to X_{d,N}$ is very free. We claim that $g = i \circ f : \mathbb{P}^1_k \to X_{d,N+1}$ is very free.

Let $\mathcal{J}$ be the ideal sheaf of $X_{d,N}$ in $X_{d,N+1}$. We see that $i^*(\mathcal{J}/\mathcal{J}^2) \cong \mathcal{O}_{X_{d,N}}(-1)$. We have an exact sequence:

$$0 \to \mathcal{O}_{X_{d,N}}(-1) \to \Omega_{X_{d,N+1}}|X_{d,N} \to \Omega_{X_{d,N}} \to 0.$$

We dualize the exact sequence above and pull it back along $f$ to obtain:

$$0 \to f^*T_{X_{d,N}} \to f^*T_{X_{d,N+1}}|X_{d,N} = g^*T_{X_{d,N+1}} \to \mathcal{O}_{\mathbb{P}^1_k}(e) \to 0.$$

Here $e$ is as usual the degree of the curve $f$. Let $\mu_{\mathbb{P}^1_k} : \mathbb{P}^1_{\bar{k}} \to \mathbb{P}^1_k$ and $\mu_X : X_{\bar{k}} \to X$ be the canonical projections. As $\mu^*_{\mathbb{P}^1_k} f^* = \bar{f}^* \mu^*_X$ (for any rational curve $f$) we get an exact sequence:

$$0 \to \bar{f}^*T_{X_{\bar{k}_{d,N}}} \to \bar{g}^*T_{X_{\bar{k}_{d,N+1}}} \to \mathcal{O}_{\mathbb{P}_{\bar{k}}1}(e) \to 0$$

Since $\mathcal{O}_{\mathbb{P}^1_k}(e)$ is ample and $f^*T_{X_{d,N}}$ is ample by assumption, this is still the case after pulling this sequence back to $\mathbb{P}^1_{\bar{k}}$ by Proposition 3.2.13. It follows that $\bar{g}^*T_{X_{\bar{k}_{d,N+1}}}$ is ample as well, by [Har66, Cor.3.4]. After applying Proposition 3.2.13 again, we are done. □

### 5.6.1 An alternative proof

We shall here apply Proposition 3.2.4 to prove Lemma 5.6.1 as follows: Let $f$ and $X_{d,N}$ be as in Lemma 5.6.1. Further let $i$ be as in the proof of Lemma 5.6.1 above. Consider the following exact sequence:

$$0 \to f^*T_{X_{d,N}} \otimes \mathcal{O}_{\mathbb{P}^1_k}(-2) \to (i \circ f)^*T_{X_{d,N+1}} \otimes \mathcal{O}_{\mathbb{P}^1_k}(-2) \to \mathcal{O}_{\mathbb{P}^1_k}(e-2) \to 0$$

From this we obtain the exact sequence:

$$H^1(\mathbb{P}^1_k, f^*T_{X_{d,N}} \otimes \mathcal{O}_{\mathbb{P}^1_k}(-2)) \to H^1(\mathbb{P}^1_k, (i \circ f)^*T_{X_{d,N+1}} \otimes \mathcal{O}_{\mathbb{P}^1_k}(-2))$$
$$\to H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(e-2)$$

We compute that $h^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(e-2)) = h^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-e)) = 0$, using Serre duality and the fact that $\Omega^1_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2)$. Thus $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(e-2)) = 0$, further it follows from the lemma above that $H^1(\mathbb{P}^1_k, f^*T_{X_{d,N}} \otimes \mathcal{O}_{\mathbb{P}^1_k}(-2)) = 0$ since we have assumed that $f$ is very free. Thus by exactness we have that $H^1(\mathbb{P}^1_k, (i \circ f)^*T_{X_{d,N+1}} \otimes \mathcal{O}_{\mathbb{P}^1_k}(-2)) = 0$, hence $(i \circ f) : \mathbb{P}^1_k \to X_{d,N+1}$ is a very free rational curve by the lemma above. This completes the proof.

**Remark 5.6.2.** Proofs that heavily use (co)homology often have the advantage that it is easy to see exactly what makes them work, and one may then relax or increase the assumptions in order to obtain similar results. In the above proof we saw that the essential ingredient was that $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(e-2)) = 0$, it is not hard to see that $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(e-r-1)) = 0$ if $e \geq r$. Hence if we require the degree of the curve $e$ to be greater than or equal to $r$, we obtain the result for $r$-free curves as well.

## 5.7 Proposition 1.6 in [She12]

Finally we have now come to Proposition 1.6 in [She12]. The proof that follows is perhaps the most intricate given thus far.

**Lemma 5.7.1.** *Let $f : \mathbb{P}^1_k \to X$ be a degree $e$ morphism. Then we have $\dim M_e \leq \dim_k H^0(\mathbb{P}^1_k, f^*T_X)$.*

*Proof.* It follows from Proposition 3.1.5 that we have

$$T_{M_e,[f]} = T_{\mathrm{Mor}(\mathbb{P}^1_k,X),[f]} = H^0(\mathbb{P}^1_k, f^*T_X).$$

The inequality now follows because we have $\dim M_e \leq \dim_k T_{M_e,[f]}$. □

We shall also need the following lemma:

**Lemma 5.7.2.** *Let $f : \mathbb{P}^1_k \to X$ be a rational curve of degree $e$, then $\deg f^*T_X = e(N - p^r)$.*

*Proof.* By (5.2.3) we have $f^*((\mathcal{P}^1_X)^\vee) = (F^*)^r \Omega_{\mathbb{P}^N_k} \otimes \mathcal{O}_X(p^r+1)$. Furthermore we have by [Vak13, Ch.21,Thm.21.4.6] that the degree of $\Omega_{\mathbb{P}^N_k}$ is $-(N+1)$. From this it follows that $f^*((\mathcal{P}^1_X)^\vee)$ has degree $e(N-p^r)$ and it follows from the first column of (5.2.1) that this is also the degree of $f^*T_X$. □

Our final two ingredients in our proof shall be the Riemann–Roch theorem for locally free sheaves, and Serre duality . We state the version we shall need of the Riemann–Roch theorem here.

**Theorem 5.7.3** (Riemann–Roch)**.** *Let $X$ be a normal projective curve over a field $k$. Let $\mathcal{F}$ be a locally free sheaf of rank $r$ on $X$. Then we have*

$$\chi(\mathcal{F}) = \deg(\det(\mathcal{F})) + r\chi(\mathcal{O}_X),$$

*where $\chi(\mathcal{F})$ is the Euler-Poincaré characteristic of the sheaf $\mathcal{F}$.*

*Proof.* This is [Liu02, Ch.7., Ex. 3.3b)]. One can also see [Ful98] for a more comprehensive and general theory regarding the Riemann–Roch theorem. □

We are now ready to prove Proposition 1.6 in [She12], without the assumption that $k = \bar{k}$.

**Proposition 5.7.4.** *Let $X = X_{d,d}$ be the Fermat hypersurface of degree $d = p^r + 1$ in $\mathbb{P}^d_k$. Let $M_e$ be the space of degree $e$ morphisms from $\mathbb{P}^1_k \to X$. Then for $M_e$ to have the expected dimension (see Remark 4.0.22), $e$ has to be at least $p^r - 1$. In particular, if $e < p^r - 1$, then there is no free rational curve of degree $e$.*

*Proof.* By Riemann–Roch we have

$$\begin{aligned}\chi(f^*T_X) &= h^0(\mathbb{P}^1_k, f^*T_X) - h^1(\mathbb{P}^1_k, f^*T_X)\\ &= \deg(\det(f^*T_X)) + \operatorname{rank}(f^*T_X)\\ &= e + d - 1.\end{aligned}$$

From this we have

$$h^0(\mathbb{P}^1_k, f^*T_X) = e + d - 1 + h^1(\mathbb{P}^1_k, f^*T_X).$$

Serre duality (see [Vak13, Ch.18., Thm.18.5.1]) yields:

$$\begin{aligned}h^1(\mathbb{P}^1_k, f^*T_X) &= h^0(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k} \otimes \mathscr{H}om(f^*T_X, \mathcal{O}_{\mathbb{P}^1})\\ &= h^0(\mathbb{P}^1_k, \mathscr{H}om(f^*T_X, \mathcal{O}_{\mathbb{P}^1_k}(-2)))\end{aligned}$$

where we have used [Har77, Ch.2., Ex.5.1b)] and that $\Omega^1_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2)$. Now by (4.0.2), we see that if $e < p^r - 1$, then $\dim M_e > d + e - 1$. This is because in that case we never have $jp^r + l = mp^r - 1$ for $m = 0, \ldots, e$. Thus some of the polynomials which define $M_e$ are 0. Hence it is given by intersecting strictly fewer that $de+1$ polynomials in $\mathbb{P}_k^{de+d+e}$. We thus have

$$h^0(\mathbb{P}_k^1, f^*T_X) = e+d-1+h^0(\mathbb{P}_k^1, \mathscr{H}om(f^*T_X, \mathcal{O}_{\mathbb{P}_k^1}(-2))) \geq M_e > e+d-1,$$

hence we must have some contribution from $h^0(\mathbb{P}_k^1, \mathscr{H}om(f^*T_X, \mathcal{O}_{\mathbb{P}_k^1}(-2)))$ so $f$ is not free. □

# Chapter 6

# Further constraints on the degree of a very free rational curve on the Fermat hypersurface obtained through algebraic methods

From (5.2.3) we obtain the following:

**Proposition 6.0.5.** *Let $X = X_{d,N}$ be the Fermat hypersurface of degree $d = p^r + 1$ in $\mathbb{P}^N_k$ and let $f : \mathbb{P}^1_k \to X$ be a rational curve of degree $e$. Let $e_1, \ldots, e_N$ be the splitting type of $f^*\Omega_{\mathbb{P}^N_k}$ and let $f_1, \ldots, f_N$ denote the splitting type of $f^*((\mathcal{P}^1_X)^\vee)$ then we have $f_i = p^r e_i + de$.*

*Proof.* By (5.2.3) we have $f^*((\mathcal{P}^1_X)^\vee) = (F^*)^r(f^*\Omega_{\mathbb{P}^N_k}) \otimes f^*\mathcal{O}_X(d)$. The proposition follows from this. □

The paper [Bri+13] considers the Fermat hypersurface of degree 5 in $\mathbb{P}^5_k$ where $k$ is an algebraically closed field of characteristic 2. In this paper the authors prove the proposition above by reducing this to a question concerning graded free $R$-modules. More specifically given a rational curve $\varphi$ on $X_{5,5}$ they study two modules which they denote $\Omega_X(\varphi)$ and $E_X(\varphi)$ such that we have $\widetilde{\Omega_X(\varphi)} = \varphi^*\Omega_{\mathbb{P}^5_k}$ and $\widetilde{E_X(\varphi)} = \varphi^*((\mathcal{P}^1_X)^\vee)$. The authors find a way of relating the bases of these modules and from this they deduce that if $f_i$ denotes the splitting type of $E_X(\varphi)$ and $e_i$ denotes the splitting type of $\Omega_X(\varphi)$ where $1 \leq i \leq 5$ then $f_i = 4e_i + 5e$. The first part of this chapter will be devoted to following [Bri+13] and obtain a similar result in the case where $X = X_{d,d}$, where $d = p^r + 1$ and $p$ is the characteristic of the field $k$ which we are working over, as an alternative to the way we proved the proposition above. In this setting we will have to assume that the field $k$ contains all $p^r$'th roots.

In the second part of this chapter we shall prove the claim from the previous chapter concerning how the pullback of the cotangent bundle splits when we are pulling it back along a rational normal curve of degree $N$ which is the dimension of the projective $N$-space containing our Fermat hypersurface $X$. After this we shall show that when $p^r > 3$, the degree $d = p^r + 1$ Fermat hypersurface does not contain a degree $e$ free curve for $e < 2p^r$. This is shown in the case $X = X_{5,5} \subset \mathbb{P}^5_k$ in [Bri+13], the arguments given by the authors work very nicely in our setting as well, thus we shall to some extent follow their approach again.

Finally we end the chapter by giving a criteria for when a degree $2p^r + 1$ rational curve on $X_{d,d}$ is very free, where $d = p^r + 1$.

## 6.1 Passing to commutative algebra

Throughout this section we shall let $X = X_{d,d} \subset \mathbb{P}^d_k$ be the degree $d = p^r + 1$ Fermat hypersurface over a field $k$ of characteristic $p$, unless we say otherwise. Moreover we shall assume that it contains a rational curve

$$f : \mathbb{P}^1_k \to X$$

Where $f$ is given by $d+1$ homogeneous polynomials $F_0, \dots F_d \in k[x_0, x_1]$ of equal degree $e \geq 1$, thus $f$ is a degree $e$ curve. Moreover we shall let $R = k[x_0, x_1]$.

In Chapter 2: Interlude, we defined the splitting type of a locally free sheaf of rank $r$ on $\mathbb{P}^1_k$. We shall now define the analogue for graded R-modules.

**Definition 6.1.1.** A graded free $R$-module is a finite direct sum of $R(e)$'s. Thus $M$ is said to be a graded free $R$-module if we have $M \cong \oplus_{i=1}^r R(e_i)$. In that case we say that the splitting type of $M$ is $e_1, \dots, e_r$.

**Lemma 6.1.2.** *The kernel of a graded morphism of graded free $R$-modules is graded free.*

*Proof.* We only sketch the proof. A more detailed account can be read in Tabes Bridges notes from an REU of the summer 2012 [1].

Let $\varphi : M \to N$ be a graded morphism of graded free $R$-modules. Let $L = \ker(\varphi) = \oplus \operatorname{Ker}(M_n \to N_n)$. We can use the snake lemma to obtain an exact sequence $L/x_1 L \to M/x_1 M \to N/x_1 N$ of $\overline{R} = k[x_0]$-modules. One can show that a submodule $U$ of a graded free $\overline{R}$-module is graded free , by using Noetherianess to argue that we can pick a minimal generating set of $U$ and using the fact that $U$ has no $x_0$-torsion to show that this minimal generating set is a basis. As $M/x_1M$, $N/x_1N$ are graded free $\overline{R}$-modules, it follows that $L/x_1L$ is also graded free, thus we can pick a basis $\overline{l_1}, \dots, \overline{l_r}$ for $L/x_1L$. Let $l_1, \dots, l_r$ be homogeneous elements of $L$ which have images $\overline{l_i}$ in $L/x_1L$. Now using induction on $n$ one can show that for each $n$ the $l_i$ span $L_n$. Thus it only remains to show linear independence of the $l_i$. As we can assume that $x_1$ doesn't divide $l_i$ for all $i$, we can prove linear independence the same way as one does when showing that a submodule of an $\overline{R}$-module is graded free.

□

**Definition 6.1.3.** Let $f : \mathbb{P}^1_k \to X_{d,N}$ be a rational curve of degree $e$ given by $N+1$ homogeneous polynomials $F_0, \dots, F_N \in k[x_0, x_1]_e$. We define $\Omega(f) = \ker(R(-e)^{N+1} \xrightarrow{(F_i)} R$ and $E_X(f) = \ker(R(e)^{N+1} \xrightarrow{(F_i^{p^r})} R(de))$ where the $(F_i)$, respectively $(F_i^{p^r})$, above the maps means that they are given by multiplication by $(F_0, \dots, F_N)^t$, respectively $(F_0^{p^r}, \dots, F_N^{p^r})^t)$.

It follows from Lemma 6.1.2 above that these are graded free modules, and they are in fact closely related to sheaves we have already studied.

**Lemma 6.1.4.** *Let $f : \mathbb{P}^1_k \to X_{d,N}$ be a rational curve on the Fermat hypersurface $X_{d,N}$. Then $\widetilde{\Omega(f)} \cong f^*\Omega_{\mathbb{P}^N_k}$ and $\widetilde{E_X(f)} \cong f^*(\mathcal{P}^1_X)^\vee$. Further the splitting type of $\Omega(f)$ (respectively $E_X(f)$) equals the splitting type of $f^*\Omega_{\mathbb{P}^N_k}$ (respectively $f^*(\mathcal{P}^1_X)^\vee$).*

*Proof.* Recall the exact sequences

$$0 \longrightarrow \Omega_{\mathbb{P}^N_k} \longrightarrow \mathcal{O}_{\mathbb{P}^N_k}(-1)^{N+1} \xrightarrow{(x_i)} \mathcal{O}_{\mathbb{P}^N_k} \longrightarrow 0$$
$$0 \longrightarrow (\mathcal{P}^1_X)^\vee \longrightarrow \mathcal{O}_X(1)^{N+1} \xrightarrow{(x_i^{p^r})} \mathcal{O}_X(p^r+1) \longrightarrow 0$$

[1] http://math.columbia.edu/ dejong/reu/lib/exe/fetch.php?media=lecture_notes.pdf

We obtain the pullback of the map in the middle of the first sequence by applying the $\widetilde{(\bullet)}-$ functor to $R(-e)^{N+1} \xrightarrow{(F_i)} R$ and the pullback of the middle map of the second sequence is similarly obtained by applying the $\widetilde{(\bullet)}$-functor to $R(e)^{N+1} \xrightarrow{(F_i^{p^r})} R(de)$. Since the $\widetilde{(\bullet)}$-functor is exact, it commutes with kernels, hence $f^*\Omega_{\mathbb{P}^d} = \widetilde{\Omega(f)}$ and $f^*(\mathcal{P}^1_X)^\vee = \widetilde{E_X(f)}$. The last statement of the lemma follows from uniqueness of splitting type and the fact that $\widetilde{(\bullet)}-$ is a left-adjoint functor, thus it commutes with colimits. □

## 6.2 Relating the bases

Throughout this section we let $f : \mathbb{P}^1_k \to X_{d,d}$ be a rational curve on the Fermat hypersurface $X_{d,d}$, where $d = p^r + 1$.

If $(A_0, \dots, A_d) \in \Omega(f)$, then $\sum_{i=0}^d A_i F_i = 0$ and by the Frobenius endomorphism it follows that $(A_0^{p^r}, \dots, A_d^{p^r}) \in E_X$. Further we shall let $\mathcal{T} = \{(A_0^{p^r}, \dots, A_d^{p^r}) \mid (A_0, \dots, A_d) \in \Omega(f)\}$. We denote the $R$-module generated by $\mathcal{T}$ as $R\langle\mathcal{T}\rangle$.

**Lemma 6.2.1.** *Let $X = X_{d,d} \subset \mathbb{P}^d_k$, where $k$ is a field of characteristic $p$ where every element $a \in k$ has a $p^r$'th root in $k$. Then $E_X(f) = R\langle\mathcal{T}\rangle$.*

*Proof.* We have already seen one of the inequalities. Now we show the other: Let $(B_0, \dots, B_d) \in E_X$, where we can assume that all the $B_i$ are homogeneous of the same degree $b$. Assume that $b \equiv m \pmod{p^r}$, then every monomial term of $B_i$ is either of the form $(c^{1/p^r} x_0^l x_1^k)^{p^r} x_0^{m-n} x_1^n$, where $c \in k$ and $0 \le n \le m$, or of the form $(c^{1/p^r} x_0^l x_1^k)^{p^r} x_0^{p^r+m-n} x_1^n$ where $c \in k$ and $m < n < p^r$. We obtain $B_i = a_{i0}^{p^r} x_0^m + a_{i1}^{p^r} x_0^{m-1} x_1 + \dots a_{i(p^r-1)}^{p^r} x_0^{m+1} x_1^{p^r-1}$, after collecting terms, where the $a_{ij}$ are elements of $R$. Since we have $\sum_{i=0}^d B_i F_i^{p^r} = 0$, it follows that

$$(\sum_{i=0}^d a_{i0} F_i)^{p^r} x_0^m + \dots + (\sum_{i=0}^d a_{i(p^r-1)} F_i)^{p^r} x_0^{m+1} x_1^{p^r-1} = 0.$$

Since the degree of $x_0$ in each term is distinct modulo $p^r$, we must have $(\sum_{i=0}^d a_{ij} F_i)^{p^r} = 0$ for each $j$ hence $(\sum_{i=0}^d a_{ij} F_i) = 0$ for all $j$, thus we see that $(a_{0j}, \dots, a_{dj}^{p^r}) \in \mathcal{T}$, hence $(B_0, \dots, B_d) \in R\langle\mathcal{T}\rangle$. □

**Remark 6.2.2.** There are many fields which satisfy the criteria of the lemma above. For instance let $q = p^r$, then as we have $x^q = x$ for every $x \in \mathbb{F}_q$, it follows that all elements of $\mathbb{F}_q$ has a $q$th root in $\mathbb{F}_q$, moreover as every finite field extension of $\mathbb{F}_q$ is isomorphic to $\mathbb{F}_{q^n}$ and as $x^{q^n} = x$ for all $x \in \mathbb{F}_{q^n}$, we see that $(x^{q^{n-1}})$ is a $q$'th root of $x$, hence every finite field extension of $\mathbb{F}_q$ also satisfies the criteria of the lemma.

**Proposition 6.2.3.** *Let $X$ be as in the lemma above. If $x_i = (x_{i0}, \ldots, x_{id})$, where $1 \leq i \leq p^r + 1 = d$ form a basis for $\Omega(f)$, then $y_i = (x_{i0}^{p^r}, \ldots, x_{id}^{p^r})$ form a basis for $E_X(f)$.*

*Proof.* We first show that the $y_i$ span $E_X(f)$. Let $y \in E_X(f)$, then by the previous lemma we have $y = \sum r_j(a_{j0}^{p^r}, \ldots a_{jd}^{p^r})$, $(a_{jo}, \ldots, a_{jd}) \in \Omega(f)$ and $r_j \in R$ for all $j$. As the $x_i$ form a basis for $\Omega(f)$, we can find $c_i \in R$ such that we have $(a_{j0}, \ldots, a_{jd}) = \sum c_i x_i$. The Frobenius endomorphism yields $(a_{jo}^{p^r}, \ldots, a_{jd}^{p^r}) = \sum c_i^{p^r}(x_{i0}^{p^r}, \ldots, x_{id}^{p^r})$. This shows generation.

Linear independence of the $y_i$ follows from the fact that $E_X(f)$ is a free module of rank $p^r$ over an integral domain. □

From this we obtain a slightly weaker version of the isomorphism given in (5.2.3), as we obtained this in Chapter 5 without needing all $p^r$th roots in the base field $k$.

## 6.3 Computing the pullback along the rational normal curve

In Proposition 5.5.1 we used the following:

**Lemma 6.3.1.** *Let $f : \mathbb{P}_k^1 \to \mathbb{P}_k^N$ be a rational normal curve in $\mathbb{P}_k^N$ of degree $N$, then $f^*\Omega_{\mathbb{P}_k^N} \cong \mathcal{O}(-N-1)^N$.*

*Proof.* We want to show that $\Omega(f) = \ker(R(-N)^{N+1} \xrightarrow{(F_i)} R)$ is isomorphic to $R(-N-1)^N$. We assume first that $f$ is given by the $N+1$ homogeneous polynomials $x_0^N, x_0^{N-1}x_1, \ldots, x_1^N \in k[x_0, x_1]_N$. Let

$$y_1 = (x_0, -x_1, 0, 0, \ldots, 0), y_2 = (0, x_0, -x_1, 0 \ldots, 0), \ldots,$$
$$y_n = (0, 0, 0, \ldots, x_0, -x_1).$$

We see that the $y_i$ are linearly independent and generate $\Omega(f)$. Let $M$ be the matrix whose rows are the $y_i$, By multiplying with $M$ we get an isomorphism $R(-N-1)^N \to \Omega(f)$.

We now show the general case. As the $F_i$ are a basis for $k[x_0, x_1]_N$ we have an invertible matrix $A$ such that $A(F_0, \ldots F_N)^t = (x_0^N, \ldots x_1^N)^t$. Then the rows of $MA$ form a basis for $\Omega(f)$ and the map $R(-N-1)^N \to \Omega(f)$ given by multiplication with $MA$ is an isomorphism. □

## 6.4 A lower bound on the degree of a rational free curve

We shall now show that in most cases there doesn't exist a free curve of degree less than $2p^r$ on the degree $d = p^r + 1$ Fermat hypersurface $X = X_{d,d}$. We shall start by giving a couple of lemmas and a corollary, which might be considered as mere observations.

**Lemma 6.4.1.** *Let $f$ be a degree $e$ rational curve on the Fermat hypersurface $X = X_{d,N}$. Then the degree of $f^*\Omega_{\mathbb{P}^N_k} = -(N+1)e$.*

*Proof.* We have the exact sequence

$$0 \to \Omega_{\mathbb{P}^N_k} \to \mathcal{O}_{\mathbb{P}^N_k}(-1)^{N+1} \to \mathcal{O}_{\mathbb{P}^N_k} \to 0$$

pulling this back along $f$ yields our result. □

**Lemma 6.4.2.** *Let $f$ be a degree $e$ rational curve on $X_{d,N}$. Let $e_i$, $f_i$ denote the splitting type of $f^*\Omega_{\mathbb{P}^N_k}$ and $f^*(\mathcal{P}^1_X)^\vee$ respectively. If $f$ is free then $e_i \geq -\frac{de}{p^r}$, with strict inequality if $f$ is very free.*

*Proof.* This immediately follows from Proposition 6.0.5. □

**Proposition 6.4.3.** *Let $X = X_{d,d} \subset \mathbb{P}^d_k$, $d = p^r + 1$. The following statements are true:*

*(1) There does not exist a free rational curve of degree $p^r + j$ for $1 < j < p^r$.*

*(2) Any free rational curve of degree $d$ must be very free, moreover if $f$ is a very free rational curve of degree $d$,then $f^*\Omega_{\mathbb{P}^d_k}$ has splitting type $e_i = -p^r - 2 = -d - 1$ for $1 \leq i \leq d$.*

*(3) There does not exist a very free rational curve of degree $p^r$.*

*(4) There does not exist a free curve of degree $p^r - 1$ on $X$.*

*(5) There does not exist a very free rational curve of degree $2p^r$.*

*Proof.* For (1) We denote the splitting type of $f^*\Omega_{\mathbb{P}^d_k}$ $e_i$. By (6.4.2) we must have $e_i \geq -\frac{de}{p^r} = -(p^r+1)(p^r+j)/p^r = -(p^r+j+1+j/p^r)$ in order for this to be an integer we must have $e_i \geq -(p^r+j+1)$. From this it follows that $\sum e_i \geq -(p^r+1)(p^r+j+1)$. On the other hand by (6.4.1) we must have $\sum e_i = (-d-1)(p^r+j) = -(p^r+2)(p^r+j)$ which will be less than $-(p^r+1)(p^r+j+1)$ unless $j = 1$.

For (2) We see that by (6.4.2) we must have $e_i > -\frac{d^2}{p^r} = -p^r - 2 - \frac{1}{p^r}$, for each $i$, thus we must have $e_i \geq -p^r - 2$. It follows that the $e_i$ must sum to $(-d-1)d$ by (6.4.1). These two constraints together yield that we must have $e_i = -p^r - 2 = -(d+1)$ for $i = 1, \ldots, d$.

For (3) We have $e_i > -(p^r+1)(p^r)/p^r = -(p^r+1)$, thus $e_i \geq -p^r$. However we must have $\sum e_i = -(p^r+2)p^r$, which isn't possible since the rank of $f^*\Omega_{\mathbb{P}^d_k}$ is $d = p^r + 1$, thus $\sum e_i \geq -(p^r+1)p^r$.

For (4) We must have $e_i \geq -(p^r+1)(p^r-1)/p^r = -p^r + 1/p^r$, and from this it follows that $e_i \geq -p^r + 1$. From this it follows that $\sum e_i \geq -p^r + 1$. However we must also have $\sum e_i = (-p^r - 2)(p^r - 1) = -p^{2r} - p^r + 2$, which is strictly less than $-p^r + 1$.

The last statement follows from arguments which we are now highly familiar with. □

**Lemma 6.4.4.** *Let $f : \mathbb{P}^1 \to X_{d,d}$ be a rational curve of degree $e$ defined by homogeneous polynomials $F_0, \ldots, F_d \in k[x_0, x_1]_e$. Let $-e_1, \ldots, -e_d$ be the splitting type of $f^*\Omega_{\mathbb{P}^d}$ Then:*

*(1) $e_i \geq e$ for all $i$.*

*(2) The $F_i$ are linearly independent over $k$ if and only if $e_i > e$ for all $i$.*

*Proof.* We first recall that the kernel of $\tilde{f} : R(-e)^{d+1} \to R$, which we denoted $\Omega(f)$ has the same splitting type as $f^*\Omega_{\mathbb{P}^d}$. Now as the morphism $\tilde{f}$ has trivial kernel in all degrees lower than $e$, we see that all the $e_i \geq e$, for all $i$. This proves (1).

For (2), we have that if the $F_i$ are linearly independent, then $\Omega(f)_e$ is trivial, hence we must have $e_i > e$ for all $i$.

Conversely if the $F_i$ are not linearly independent over $k$, then $\Omega(f)_e$ is not trivial, hence we must have some $e_i = e$. □

**Proposition 6.4.5.** *Assume that $p^r > 2$. Let $X = X_{d,d}$ be the degree $d = p^r+1$ Fermat hypersurface in $\mathbb{P}^d_k$. This Fermat hypersurface does not contain a very free rational curve of degree $d = p^r + 1$.*

*Proof.* Assume (for the sake of contradiction) that $X$ has a very free curve $f$ of degree $d$, given by $F_i = \sum a_{ij} x_0^{d-j} x_1^j$ where $0 \leq i \leq d$. By Lemma 6.4.1 and Lemma 6.4.2 we must have that $f^*\Omega_{\mathbb{P}^d_k} \cong \oplus_{i=1}^d \mathcal{O}_{\mathbb{P}^1_k}(-d-1)$.

Now let $M$ be the $d+1 \times d+1$ matrix whose $i, j$'th entry is $a_{ij}$. From Lemma 6.4.4 (2), we see that the matrix $M$ must have full rank. Now by setting $k = mp^r + 2$ in (4.0.2) for $m = 0, 1, \ldots, d$, we must have $l = 2, j = m$

in order to obtain the equality $jp^r + l = k$,since $p^r > 2$. From this we deduce that $\sum_{i=0}^{d} a_{ij}^{p^r} a_{i2} = 0$ for $0 \leq j \leq d$. Now let $\overline{M}$ be the matrix $(a_{ij}^{p^r})$, it follows that $\overline{M}$ has full rank, since $\det(M)^{p^r} = \det((a_{ij}^{p^r})$ by the Frobenius endomorphism and $M$ has full rank. This contradicts the fact that we must have

$$(a_{02}, \ldots, a_{d2})\overline{M} = (0, 0, \ldots 0).$$

□

It can also be shown that if $p^r > 3$ then $X_{d,d}$ does not contain a free rational curve of degree $p^r$. The proof is similar to the one above, and it is proved in detail for $X_{5,5}$ in [Bri+13]. We summarize our results in a theorem.

**Theorem 6.4.6.** *Let $X = X_{d,d}$ be the degree $d = p^r + 1$ Fermat hypersurface in $\mathbb{P}_k^d$. If $p^r > 3$ then $X_{d,d}$ does not contain a free rational curve of degree $e < 2p^r$. Moreover if $p^r > 2$, then $X$ does not contain a very free rational curve of degree $e \leq 2p^r$.*

*Proof.* The case $e < p^r - 1$ follows from (5.7.4). The other cases follow from Proposition 6.4.3, Proposition 6.4.5 and the comment above. □

From (5.6.1) we know that the most interesting question for us is whether $X_{d,d}$ contains a very free rational curve, as if it does then $X_{d,N}$ will also contain a very free rational curve for all $N \geq d$. Moreover (5.5.1) tells us that it is interesting to see whether $X_{d,N}$ contains a rational normal curve of degree $N$, because then it will contain a very free rational curve. However we have just seen that $X_{d,d}$ does not contain a very free curve of degree $d$, hence not a rational normal curve of degree $d$, thus we cannot hope to obtain a very free curve in $X_{d,d}$ by finding a rational normal curve of degree $d$ in $X$. We state this as a corollary.

**Corollary 6.4.7.** *Let $X = X_{d,d}$ be the degree $d = p^r + 1$ Fermat hypersurface. Then $X$ does not contain a rational normal curve of degree $d$.*

## 6.5 A criterion

**Lemma 6.5.1.** *Let $f$ be as in Lemma 6.4.4. Let $\Omega(f)$ be as in the proof of Lemma 6.4.4, so $\Omega(f) = \bigoplus_{i=1}^{d} R(-e_i)$. Assume that $e_1 \leq e_2 \ldots \leq e_d$. If the degree $e$ homogeneous polynomials that define $f$, $F_0, \ldots, F_d \in k[x_0, x_1]_e$ are linearly independent, and $h = \dim_k \Omega(f)_{e+1} \neq 0$, then for all $i \leq h$ we have $e_i = -(e+1)$.*

*Proof.* By Lemma 6.4.4 we have $e_i \geq e+1$ for all $i$, hence $\Omega(f) = R(-(e+1))^l \oplus_{i=s}^{d} R(-e_s)$ where $l$ is some non-negative integer possibly equal to 0, $s \geq 1$ where we set $s = d+1$ if all the $e_i = e+1$, and use the convention that indexing from an integer $s$ to an integer $d$ is the same as indexing through the empty set if $s > d$. From this we see that $\Omega(f)_{e+1} = R(-(e+1))^l_{e+1} = k^l$, thus $\dim_k \Omega(f)_{e+1} = l$, but then $l = h \neq 0$ and we must have $e_1 = \ldots = e_h = e+1$. □

We saw in Theorem 6.4.6 that a very free rational curve of degree less that $2p^r + 1$ does not exist on most degree $d = p^r + 1$ Fermat hypersurfaces of the form $X_{d,d}$. Hence it will mostly be of interest to us studying the rational curves on the Fermat hypersurface $X_{d,d}$ of degree greater than or equal to $2p^r + 1$. We shall now apply the results above to give a criteria for when a rational curve of degree $2p^r + 1$ on the Fermat hypersurface is very free.

**Proposition 6.5.2.** *Let $d = p^r + 1$. Let $X = X_{d,d}$ be the degree $d$ Fermat hypersurface in $\mathbb{P}^d_k$. Further let $f : \mathbb{P}^1_k \to X$ be a rational curve of degree $e = 2p^r + 1$, given by $d+1$ homogeneous polynomials of degree $e$ which we denote $F_0, \ldots, F_d \in k[x_0, x_1]_e$. Then $f$ is very free if and only if* $\dim_k \Omega(f)_{e+1} = 1$.

*Proof.* By Lemma 6.4.2 and Lemma 6.4.1 it follows that $f$ is very free if and only if $f^*\Omega_{\mathbb{P}^d_k}$ has splitting type

$$-(2p^r+2), -(2p^r+3), -(2p^r+3), \ldots, -(2p^r+3). \tag{6.5.1}$$

If $f^*\Omega_{\mathbb{P}^d_k}$ has splitting type $-e_1, \ldots - e_d$ where $e_1 < e_2 \leq \ldots e_d$, then if $e_1 = 2p^r + 2$, we see that all the other $e_i = 2p^r + 3$ by Lemma 6.4.1. Hence the necessary and sufficient condition for $f$ to be a very free rational curve is that the smallest integer in the splitting type is equal to $-(2p^r + 2)$ and all the other integers in the splitting type are less than this. By Lemma 6.4.4 we see that if $f$ is very free then, the $F_i$ must be linearly independent over $k$, further from Lemma 6.5.1 we must have $\dim_k \Omega(f)_{e+1} = 1$.

Conversely if $\dim_k \Omega(f)_{e+1} = 1$, then we must have that the $F_i$ are linearly independent, otherwise $\Omega(f)_e$ is non zero, hence we can find some element $m \in \Omega(f)_e$, and $x_0 m$, $x_1 m$ are two linearly independent vectors in $\Omega(f)_{e+1}$ contradicting dimension 1. It now follows from Lemma 6.5.1 that $\Omega(f) = R(-(e+1)) \oplus_{i=2}^{d} R(-e_i)$ where $e_i > 2p^r + 2$. Hence the necessary and sufficient condition on the $e_i$ is satisfied, hence $f$ is a very free rational curve. □

We will now translate the content of the Proposition above to the language of matrices.

Let $f$ be a rational curve of degree $e = 2p^r+1$ given by d+1 homogeneous polynomials of degree $e$, $F_0, \dots F_d[x_0, x_1]_e$, where $F_i = \sum_{j=0}^{2p^r+1} a_{ij} x_0^{2p^r+1-j} x_1^j$. Let $\tilde{f} : R(-(2p^r+1))^{d+1} \to R$ be given by multiplication with $(F_0, \dots, F_d)^t$. Then we have $\Omega(f)_{2p^r+2} = \ker(\tilde{f}_{2p^r+2})$, where $\tilde{f}_{2p^r+2}$ denotes the $2p^r + 2'$th graded piece of $\tilde{f}$. By the rank theorem we have that $\ker(\tilde{f}_{2p^r+2}) + rank(\tilde{f}_{2p^r+2}) = 2(d+1) = 2(p^r+2) = 2p^r+4$. Now as the target of this morphism is $R_{2p^r+2}$ which has dimension $2p^r+3$, we see that $\dim_k \Omega(f) = 1$ if and only if $\tilde{f}_{2p^r+2}$ is surjective. Hence $f$ is very free if and only if the matrix of $\tilde{f}_{2p^r+2}$ which we denote $A_f$ has full rank, that is if and only if $A_f$ has rank $2p^r+3$. We now explicitly write down the matrix:

$$A_f = \begin{pmatrix} a_{00} & 0 & a_{10} & 0 & \cdots & 0 \\ a_{01} & a_{00} & a_{11} & a_{10} & \cdots & a_{d0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{02p^r+1} & a_{02p^r} & a_{12p^r+1} & a_{12p^r} & \cdots & a_{d2p^r} \\ 0 & a_{02p^r+1} & 0 & a_{12p^r+1} & \cdots & a_{d2p^r+1} \end{pmatrix} \qquad (6.5.2)$$

We reformulate Proposition 6.5.2 in terms of the rank of $A_f$.

**Proposition 6.5.3.** *Let $X = X_{d,d}$ be the degree $d = p^r+1$ and let $f$ be a rational curve on $X$ given by $F_0, \dots, F_d \in k[x_0, x_1]$ where $F_i$ is of the form $\sum_{j=0}^{2p^r+1} a_{ij} x_0^{2p^r+1-j} x_1^j$. Then $f$ is very free if and only if the matrix $A_f$ given in* (6.5.2) *has full rank.*

We can translate this to a condition on $M_e$, the space of degree $e = 2p^r+1$ rational curves on $X$ as follows:

**Corollary 6.5.4.** *Let $\mathcal{S}$ be the set of $2p^r+3$ minors of a matrix of the form* (6.5.2). *Let $U' = \mathbb{P}_k^{de+d+e} \setminus V(\mathcal{S})$, and let $U = U' \cap M_e$. If $U$ has a $k$-rational point, then $X$ has a very free rational curve. If $U$ is non-empty, then $U$ contains a closed point, and there exists a field extension $K$ of $k$, such that $X_K$ has a very free rational curve.*

# Chapter 7

# Ideas, observations and after thoughts

## 7.1 Problems related to finding a very free rational curve

Consider the Fermat hypersurface $X_{5,5}$ over $\mathbb{F}_2$. By Theorem 6.4.6 the smallest degree a very free rational curve on $X_{5,5}$ can have is $2 \cdot 4 + 1 = 9$. In the paper [Bri+13], the authors actually give a very free rational curve on $X_{5,5}$ over $\overline{\mathbb{F}_2}$, by explicitly writing down the polynomials that define it. As these polynomials have their coefficients in $\mathbb{F}_2$, it follows that they also give a very free rational curve on $X_{5,5}$ over $\mathbb{F}_2$.

In the light of Proposition 6.5.3, one might hope that one could program a computer to check whether $X_{d,d}$ has a very free rational curve of degree $2p^r+1$ for a given $d = p^r+1$, by checking all possibilities. The problem is that there quickly become way too many possible matrices of the form (6.5.2). For instance if we want to naturally follow the $X_{5,5}$ case up, by letting $p = 3$ and $r = 2$, then $d = 10$ and $e = 19$ and there are approximately $\binom{3^{18}}{11}$ possible rational maps to $\mathbb{P}^{10}_k$, and one must check for each of these whether it gives a map to $X_{10,10}$ and whether the coefficients of the polynomials definining this map satisfy the matrix criteria of Proposition 6.5.3.

The author of this thesis used Sage to write a class which consists of the following methods:

- *colelts*: A recursive method which is used to create a list of all the coefficients a homogeneous polynomial of degree $e = 2p^r + 1$ can have (the coefficients of a homogeneous polynomial is given as a list).

- *makepolynomials*: This method takes a list of coefficients and returns a list of homogeneous polynomials of equal degree $e$ with the given coefficients.
- *listofpolynomials*: This is a recursive method which generates a list consisting of all lists $C$ of $d + 1 = p^r + 2$ elements, where $C$ the elements in $C$ are polynomials.
- *mapstoPN*: Takes a list of $d + 1$ polynomials and checks wheter these polynomials have $gcd = 1$.
- *isonfFermat*: Checks if a morphism to $\mathbb{P}^d_k$ also give a morphism to the $Fermathypersurface$
- *Containsveryfreerationalcurve*: Uses the above methods to check wheter the degree $d$ Fermat hypersurface contains a very free rational curve of degree $e$. This method finishes as soon as a very free rational curve is found and returns the polynomials defining it.

The author of this theses hoped that the program above could find a very free rational curve on $X_{10,10}$, within reasonable time, however on all attempts, the program timed out. Perhaps one can extract some information from the equations given in Propostion 4.0.20 and use this to vastly decrease the number of cases one has to check? The program was also not written with parallelization in mind, perhaps if one paralellized the methods and ran this on a super computer, we could get lucky and find a very free rational curve?

## 7.2 Other hypersurfaces with partial derivatives that are powers of linear forms

In the two previous chapters we use that

$$(\mathcal{P}^1_X)^\vee \cong (F^*)^r\Omega_{\mathbb{P}^N_k} \otimes \mathcal{O}_X(p^r + 1)$$

in several proofs, many of which follow directly from this isomorphism. Hence a lot of this theory should be valid for other varieties satisfying this isomorphism. We recall from the previous chapter that the aforementioned isomorphism was obtained by identifying the kernel of a map given by multiplying with the partial derivatives of the polynomial defining the Fermat hypersurface, with the kernel of the Euler exact sequence, after it had been pulled back by the Frobenius endomorphism $r$ times. From this we see that if another smooth hypersurface $Y = V(F)$ for some polynomial $F$ of degree $p^r + 1$ such that the partial derivatives are $p^r$ powers of linear forms, then $(\mathcal{P}^1_Y)^\vee \cong (F^*)^r\Omega_{\mathbb{P}^N_k} \otimes \mathcal{O}_Y(p^r + 1)$, hence we could generalize a lot of the theory developed in the two last chapters to any such hypersurface.

On the other hand a hypersurface whose derivates are $p^r$ powers of linear forms may be isomorphic to a Fermat hypersurface. Consider for instance the following example:

**Example 7.2.1.** Let $N$ be an odd integer, and let $q = p^r$, with $p$ a prime number different from 2. Consider the polynomial $F$ in $\mathbb{F}_q[x_0, \dots, x_N]$, given by

$$F = X_0X_N^q - X_NX_0^q + X_1X_{N-1}^q - X_{N-1}X_1^q + \dots + X^q_{\frac{N+1}{2}}X_{\frac{N-1}{2}} - X^q_{\frac{N-1}{2}}X_{\frac{N+1}{2}}$$

which can also be given as:

$$\begin{pmatrix} X_0 & X_1 & \dots & X_N \end{pmatrix} M \begin{pmatrix} X_0^q \\ X_1^q \\ \vdots \\ X_N^q \end{pmatrix}$$

where $M$ is the $N+1 \times N+1$ skew symmetric matrix $(m_{ij})$, where the only nonzero entries in $M$ are

$$m_{(N+1)-(j-1),j} = \begin{cases} -1 & \text{if } j \leq \frac{N+1}{2} \\ 1 & \text{if } j > \frac{N+1}{2} \end{cases}$$

Now let $y \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, and set $z = y - y^q$. By the choice of $y$, it follows that $z \neq 0$. Let $C = zM \in \mathrm{GL}(N + 1, \mathbb{F}_{q^2})$ (where we consider $M$ as a matrix in $\mathrm{GL}(N + 1, \mathbb{F}_{q^2})$).

Following [BC66] we make the following conventions/definitions: Let $H = (h_{i,j})$ be an invertible matrix with entries in $\mathbb{F}_{q^2}$.

(1) We let $H^{(q)}$ be $(h^q_{i,j})$.

(2) The matrix $H$ is Hermitian if $H^{(q)} = H^T$.

(3) Assume that $H$ is Hermitian, and let $G$ be an other invertible Hermitian matrix with entries in $\mathbb{F}_{q^2}$, we shall say that $H$ and $G$ are equivalent, if there exists an invertible matrix $A$ with entries in $\mathbb{F}_{q^2}$ such that $A^THA^{(q)} = G$.

Now looking back at our matrix $C$ we see that it is Hermitian. Indeed we have that $(\pm z)^q = \mp z$. Since $C$ is Herimitian it follows from Theorem 4.1 in [BC66], that $C$ is equivalent to the identity matrix $I_{N+1}$. From this we gather that the hypersurface $V(F) = V(zF)$ is isomorphic to the Fermat hypersurface $V(X_0^{q+1} + \ldots + X_N^{q+1})$ in $\mathbb{P}^N_{\mathbb{F}_{q^2}}$ [1]. It might be possible that the Heisenberg hypersurface $V(F)$ and the Fermat hypersurface $V(G)$, where $G$ is the homogeneous polynomial $G = X_0^{q+1} + \ldots + V_{X_N}^{q+1}$, aren't isomorphic over $\mathbb{F}_q$, as varieties can be in different isomorphism classes over some base field, but become isomorphic over a bigger field. An example of this is given in Remark 8.19 in [HKT08] where one considers the Fermat curve $V(X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + X_2^{\sqrt{q}+1})$, where $q = p^{2r}$, for a prime number $p$ and a positive number $r$, and the hypersurface $V(X_0^{\sqrt{q}}X_1 + X_1^{\sqrt{q}}X_2 + X_2^{\sqrt{q}}X_0)$ which are not isomorphic over $\mathbb{F}_q$ but become isomorphic over $\mathbb{F}_{q^3}$.

The example(s) given above suggest that the the degree $d = p^r+1$ Fermat hypersurface in $\mathbb{P}^N_k$, where $k$ is an algebraically closed field of characteristic $p$, might be isomorphic to any hypersurface of degree $d$ in $\mathbb{P}^N_k$ whose partial derivatives are $p^r$ powers of linear forms. This is actually the case and it is proved in [Bea90]. More precisely Beauville proves that if $Y$ is some hypersurface of some degree $d$ in $\mathbb{P}^N_k$, then $Y$ is isomorphic to the Fermat hypersurface $X_{d,N}$ if an only if $d-1$ is a power of $p$ and the partial derivatives of $Y$ are powers of linear forms. Even though we don't necessarily have this isomorphism over non closed fields, this still diminishes the appeal to pursue the study of rational curves on such hypersurfaces further.

[1] This example was explained by S. Kleiman in the case $N = 3, q = 2$ at the workshop: " Workshop II: Tools from Algebraic Geometry, April 7-11, 2014" which was hosted by IPAM UCLA.

# Bibliography


[AK03] Carolina Araujo and János Kollár. "Rational curves on varieties". In: *Higher dimensional varieties and rational points (Budapest, 2001)*. Vol. 12. Bolyai Soc. Math. Stud. Springer, Berlin, 2003, pp. 13–68. DOI: 10.1007/978-3-662-05123-8_3. URL: http://dx.doi.org/10.1007/978-3-662-05123-8_3.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, pp. ix+128.

[BC66] R. C. Bose and I. M. Chakravarti. "Hermitian varieties in a finite projective space $\mathrm{PG}(N, q^2)$". In: *Canad. J. Math.* 18 (1966), pp. 1161–1182. ISSN: 0008-414X.

[Bea90] Arnaud Beauville. "Sur les hypersurfaces dont les sections hyperplanes sont à module constant". In: *The Grothendieck Festschrift, Vol. I*. Vol. 86. Progr. Math. With an appendix by David Eisenbud and Craig Huneke. Birkhäuser Boston, Boston, MA, 1990, pp. 121–133.

[Bri+13] Tabes Bridges et al. "Free and very free morphisms into a Fermat hypersurface". In: *Involve* 6.4 (2013), pp. 437–445. ISSN: 1944-4176. DOI: 10.2140/involve.2013.6.437. URL: http://dx.doi.org/10.2140/involve.2013.6.437.

[Băd01] Lucian Bădescu. *Algebraic surfaces*. Universitext. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author. Springer-Verlag, New York, 2001, pp. xii+258. ISBN: 0-387-98668-5. DOI: 10.1007/978-1-4757-3512-3. URL: http://dx.doi.org/10.1007/978-1-4757-3512-3.

[CZ13] Q. Chen and Y. Zhu. “Very free curves on Fano Complete Intersections”. In: *ArXiv e-prints* (Nov. 2013). arXiv: 1311.7189 [math.AG].

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001, pp. xiv+233. ISBN: 0-387-95227-6.

[Eis95] David Eisenbud. *Commutative algebra, with a view toward algebraic geometry*. Vol. 150. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. xvi+785. ISBN: 0-387-94268-8; 0-387-94269-6. DOI: 10.1007/978-1-4612-5350-1. URL: http://dx.doi.org/10.1007/978-1-4612-5350-1.

[Ful98] William Fulton. *Intersection theory*. Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998, pp. xiv+470. ISBN: 3-540-62046-X; 0-387-98549-2. DOI: 10.1007/978-1-4612-1700-8. URL: http://dx.doi.org/10.1007/978-1-4612-1700-8.

[GHS03] Tom Graber, Joe Harris, and Jason Starr. “Families of rationally connected varieties”. In: *J. Amer. Math. Soc.* 16.1 (2003), 57–67 (electronic). ISSN: 0894-0347. DOI: 10.1090/S0894-0347-02-00402-2. URL: http://dx.doi.org/10.1090/S0894-0347-02-00402-2.

[Har66] Robin Hartshorne. “Ample vector bundles”. In: *Inst. Hautes Études Sci. Publ. Math.* 29 (1966), pp. 63–94. ISSN: 0073-8301.

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[HKT08] J. W. P. Hirschfeld, G. Korchmáros, and F. Torres. *Algebraic curves over a finite field*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2008, pp. xx+696. ISBN: 978-0-691-09679-7.

[KMM92a] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. “Rational connectedness and boundedness of Fano manifolds”. In: *J. Differential Geom.* 36.3 (1992), pp. 765–779. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214453188.

[KMM92b] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. "Rationally connected varieties". In: *J. Algebraic Geom.* 1.3 (1992), pp. 429–448. ISSN: 1056-3911.

[Kol96] János Kollár. *Rational curves on algebraic varieties*. Vol. 32. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1996, pp. viii+320. ISBN: 3-540-60168-6. DOI: 10.1007/978-3-662-03276-3. URL: http://dx.doi.org/10.1007/978-3-662-03276-3.

[Kol99] János Kollár. "Rationally connected varieties over local fields". In: *Ann. of Math. (2)* 150.1 (1999), pp. 357–367. ISSN: 0003-486X. DOI: 10.2307/121107. URL: http://dx.doi.org/10.2307/121107.

[Lan02] Serge Lang. *Algebra*. third. Vol. 211. Graduate Texts in Mathematics. Springer-Verlag, New York, 2002, pp. xvi+914. ISBN: 0-387-95385-X. DOI: 10.1007/978-1-4613-0041-0. URL: http://dx.doi.org/10.1007/978-1-4613-0041-0.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*. Vol. 6. Oxford Graduate Texts in Mathematics. Translated from the French by Reinie Erné, Oxford Science Publications. Oxford University Press, Oxford, 2002, pp. xvi+576. ISBN: 0-19-850284-2.

[She12] Mingmin Shen. "Rational curves on Fermat hypersurfaces". In: *C. R. Math. Acad. Sci. Paris* 350.15-16 (2012), pp. 781–784. ISSN: 1631-073X. DOI: 10.1016/j.crma.2012.09.015. URL: http://dx.doi.org/10.1016/j.crma.2012.09.015.

[Vak13] Ravi Vakil. *Foundations of algebraic geometry*. 2013. URL: -math216.wordpress.com.

[Zhu11] Y. Zhu. "Fano Hypersurfaces in Positive Characteristic". In: *ArXiv e-prints* (Nov. 2011). arXiv: 1111.2964 [math.AG].