DECIDABILITY OF THE ISOMORPHISM PROBLEM FOR STATIONARY AF-ALGEBRAS

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ABSTRACT. The notion of isomorphism of stable AF-C^*-algebras is considered in this paper in the case when the corresponding Bratteli diagram is stationary, i.e., is associated with a single square primitive nonsingular incidence matrix. C^*-isomorphism induces an equivalence relation on these matrices, called C^*-equivalence. We show that the associated isomorphism equivalence problem is decidable, i.e., there is an algorithm that can be used to check in a finite number of steps whether two given primitive nonsingular matrices are C^*-equivalent or not.

INTRODUCTION

In [BJKR98] we studied isomorphism of the stable AF-algebras associated with constant square primitive nonsingular incidence matrices. This isomorphism is called C^*-equivalence of the matrices in [BJKR98] and weak equivalence of the (transposed) matrices in [SwVo99]. In this paper we prove that the isomorphism problem in this setting is decidable. This result was announced in [BJKR98], and the result is interesting in view of the fact that the corresponding problem for non-constant incidence matrices is undecidable [MuPa98]. That isomorphism is decidable means that there is an algorithm that can be used to decide, in a finite number of steps, whether two given primitive matrices are C^*-equivalent or not.

Bratteli diagrams were introduced in [Bra72] with a view to understanding the structure and the classification of those C^*-algebras which arise as inductive limits of finite-dimensional C^*-algebras, the so-called AF-algebras. In fact, the equivalence relation on Bratteli diagrams which is generated by the operation of telescoping is a complete C^*-isomorphism invariant for the AF-algebras; see [BJO99, Remark 5.6]. It is the decidability of this isomorphism problem in the case of stationary Bratteli diagrams which is our main result here. The diagrams are called *stationary* if the incidence matrix is constant; in the general case it is not constant, but varies from one level to the next. However, it was the stationary class of AF-algebras which came from the problem addressed in [BJO99], and while special, this subfamily is still general enough for the study of substitution dynamical systems, as noted below and in [DHS99]. These systems have significance in formal languages, quasi-crystals, aperiodic tilings of the plane [Rad96], and p-recognizable sets of numbers, see [DHS99] for more background. Hence the classification we address

here has some bearing not only on the original setting of AF-algebras, but also on recent developments in dynamical systems. For a survey of other dynamical system classifications related to more standard shifts than those considered in [DHS99], and the relation of our present classification to these, see [BJKR98]. In particular, it is explained in [BJKR98] that the notion of C^*-equivalence of two primitive nonsingular matrices is strictly weaker than shift equivalence, strong shift equivalence, or elementary shift equivalence. Specifically, formula (1.2) below shows that C^*-equivalence may be expressed also as a certain system of matrix factorizations, but these conditions for C^*-equivalence are less restrictive than those which define shift equivalence. This means that some techniques which are common in the study of shift equivalence, see, e.g., [BMT87], are also common in the study of isomorphism of C^*-algebras. The dimension group is one such tool, see [Ell76], [Eff81].

Our approach is partly based on studying isomorphism of ordered dimension groups (the order is essential!). We introduce those groups in (1.6)–(1.11), and we formulate the associated isomorphism problem. We then go on to prove that this problem is decidable, in Theorem 4.9. After decidability, the next question is a presentation of the answer in terms of numerical invariants. We take this up in Sections 5–8, which are a continuation of [BJO99]. Here the answers are not yet complete, so we present in Section 5 (Proposition 5.1 and Corollary 5.2) a subclass of incidence matrices for which the C^*-equivalence question is decided by the value of a numerical invariant. . The matrices A in the subclass allow a directsum decomposition, $A = A_0 \oplus (\lambda)$, such that A_0 is unimodular up to sign, and (λ) is multiplication by the Perron–Frobenius eigenvalue λ on the one-dimensional subspace spanned by the right Perron–Frobenius eigenvector. This property is equivalent to $|\det A| = \lambda$.
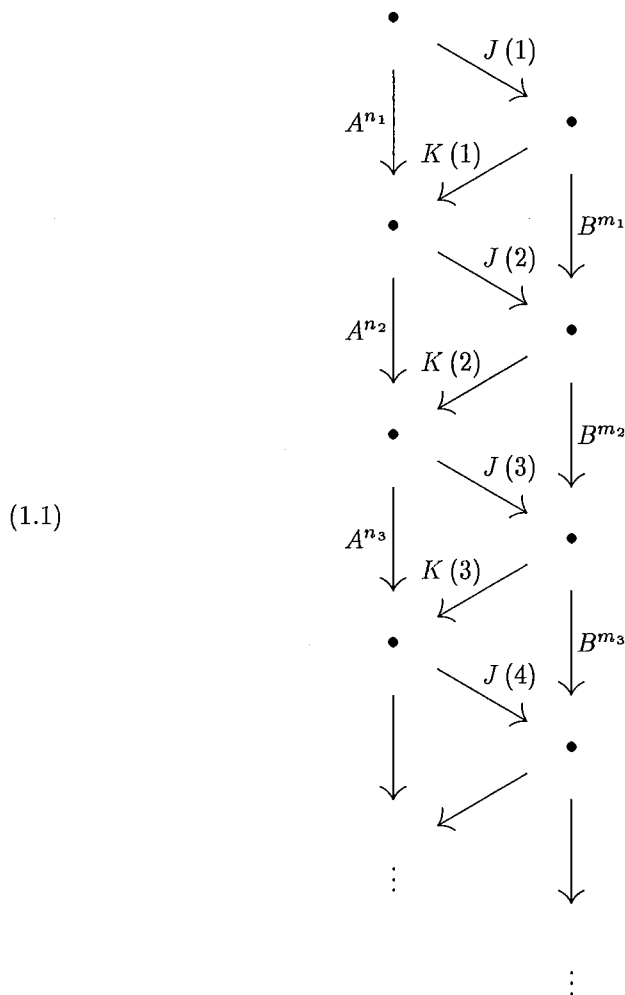
Section 7 and Section 10 address symmetry properties, pointing out that there are nonsymmetric primitive incidence matrices A which are C^*-equivalent to A^{tr}, the transposed matrix. But even in the 2-by-2 case, there are also examples where A and A^{tr} are not C^*-equivalent. The related symmetry question for shift equivalence comes from the issue of reversibility for topological Markov chains, which was studied in [PaTu82] and [CuKr80].

While the dimension group $G(A)$ associated with an incidence matrix A is torsion-free, it has a certain torsion group quotient $G(A)/L$ by a lattice L in $G(A)$. This quotient is natural in the sense that it is an invariant. It is well known that abelian torsion groups have explicitly computable and complete numerical invariants, and these invariants are thus also invariants for the dimension group (but not complete because they do not reflect order and some of the group structure). In this case they take an especially simple form, and they can be read off from the characteristic polynomial. This is proved in Section 8. Section 9 shows that the general case where A is primitive may be reduced to the more manageable one where a certain reduced version of A is both primitive and nonsingular. Section 10 presents a formulation of C^*-equivalence for matrices A, B in terms of a certain explicit matrix factorization $B = CAD$, where the two factors C, D are specified in the statement of the result, Theorem 10.2.

1. Equivalent isomorphism conditions

Recall from [BJKR98] that two matrices A, B with nonnegative integer matrix entries are said to be C^*-equivalent if there exist two sequences $n_1, n_2, \ldots$ and

$m_1, m_2, \ldots$ of natural numbers and two sequences of matrices $J(1), J(2), \ldots$ and $K(1), K(2), \ldots$ with nonnegative integer matrix entries such that the diagram (1.1) below commutes.

(1.1)

$$
\begin{array}{ccc}
\bullet & & \\
\Big\downarrow A^{n_1} & \searrow J(1) & \\
\bullet & & \bullet \\
 & K(1) \nearrow & \Big\downarrow B^{m_1} \\
\bullet & & \\
\Big\downarrow A^{n_2} & \searrow J(2) & \\
\bullet & & \bullet \\
 & K(2) \nearrow & \Big\downarrow B^{m_2} \\
\bullet & & \\
\Big\downarrow A^{n_3} & \searrow J(3) & \\
\bullet & & \bullet \\
 & K(3) \nearrow & \Big\downarrow B^{m_3} \\
\bullet & & \\
\Big\downarrow & \searrow J(4) & \\
 & \nearrow & \bullet \\
\vdots & & \Big\downarrow \\
 & & \vdots
\end{array}
$$

The diagram expresses the following two identities:

(1.2) $\qquad A^{n_k} = K(k) J(k), \qquad B^{m_k} = J(k+1) K(k),$

for $k = 1, 2, \ldots$. This corresponds to isomorphism of the associated stable AF-algebras [BJKR98, Bra72], and it corresponds to homeomorphism of one-dimensional connected orientable hyperbolic attractors of diffeomorphisms of manifolds by [Jac97]; see also [SwVo99]. We will assume throughout that A and B are primitive square matrices (i.e., sufficiently high powers have only strictly positive matrix entries) and that A and B are nonsingular, and hence C^*-equivalence implies that they have the same dimension N, because N is the rank of the associated dimension group [BJO99]. (We will show in Theorem 9.3 that the class of AF-algebras we

obtain in this manner will be exactly the same if A and B are merely required to be primitive but not necessarily nonsingular, but then N is no longer necessarily the rank, and some arguments become more complicated). In this case we note that $J\,(1)$ and the sequences $n_1, \ldots$ and $m_1, \ldots$ determine all other $K\,(k)$ and $J\,(j)$ from (1.1), i.e.,

$$K\,(1) = A^{n_1} J\,(1)^{-1},$$
$$J\,(2) = B^{m_1} J\,(1)\,A^{-n_1},$$
(1.3)
$$K\,(2) = A^{n_1+n_2} J\,(1)^{-1} B^{-m_1},$$
$$J\,(3) = B^{m_1+m_2} J\,(1)\,A^{-n_1-n_2},$$
$$\vdots$$

etc. If n is a nonzero integer, let $\mathrm{Prim}\,(n)$ denote the set of prime factors of n. Then (1.2) implies

(1.4) $$\mathrm{Prim}\,(\det\,(A)) = \mathrm{Prim}\,(\det\,(B)),$$

and thus (1.3) implies

(1.5) $$\mathrm{Prim}\,(\det\,(J\,(1))) \subseteq \mathrm{Prim}\,(\det\,(A)) = \mathrm{Prim}\,(\det\,(B)).$$

Thus a necessary and sufficient condition for C^*-equivalence of two primitive, nonsingular $N \times N$ matrices A, B with nonnegative integer matrix entries, is the existence of a (necessarily nonsingular) matrix $J\,(1)$ with nonnegative integer matrix entries and sequences $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ of natural numbers such that the matrices $K\,(1), J\,(2), \ldots$ defined by (1.3) have positive integer matrix entries.

Another way of formulating this is in terms of dimension groups (see [Bla86, Eff81] for details). Let $G\,(A)$ be the inductive limit of the sequence

(1.6) $$\mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \longrightarrow \cdots$$

of free abelian groups with order generated by the order defined on each $\mathbb{Z}^N$ by

(1.7) $$(m_1, \ldots, m_N) \geq 0 \iff m_i \geq 0, \qquad i = 1, \ldots, N.$$

Since we assume $\det A \neq 0$, we may realize $G\,(A)$ concretely as a subgroup of $\mathbb{Q}^N$ as follows: Put

(1.8) $$G_n\,(A) = A^{-n}\left(\mathbb{Z}^N\right),$$

and equip $G_n\,(A)$ with the order

(1.9) $$G_n^+\,(A) = A^{-n}\left(\left(\mathbb{Z}^N\right)^+\right).$$

Then $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ and we define

(1.10) $$G\,(A) = \bigcup_{n=0}^{\infty} G_n$$

with the order defined by

(1.11) $$g \geq 0 \qquad \text{if and only if } g \geq 0 \text{ in some } G_n\,(A).$$

Then one fundamental characterization of C^*-equivalence is that there exists a (necessarily nonsingular) matrix $J(1)$ in $M_N(\mathbb{Q})$ such that

$$(1.12) \qquad\qquad J(1)\,G(A) = G(B)$$

and

$$(1.13) \qquad\qquad J(1)\,G^+(A) = G^+(B)\,;$$

see [BJO99, Proposition 11.7]. The 1–1 correspondence between group isomorphism θ and matrix J referred to in [BJO99] is as follows: If a matrix $J = J(1)$ is specified as above, then $\theta\colon G(A) \to G(B)$, given by $\theta(g) = Jg$, $g \in G(A)$, will be an isomorphism. Here the product Jg is matrix multiplication, and each g is viewed as a column vector. Conversely, the observation in [BJO99] is that every isomorphism arises this way. This can also be formulated in other ways, as we shall presently do.

If A is a given primitive $N \times N$ matrix, let $\lambda_{(A)}$ denote its Perron–Frobenius eigenvalue, and let $v(A)$ denote a corresponding left (row) eigenvector with strictly positive components and $w(A)$ a corresponding right (column) eigenvector with strictly positive components, and in both cases use a normalization such that the components are contained in the field $\mathbb{Q}\left[\lambda_{(A)}\right]$. Define $V(A)$ as the orthogonal complement of $v(A)$. Then $V(A)$ is an $N-1$-dimensional vector space of column vectors which will sometimes be referred to as the linear span of the nonmaximal eigenvectors of A. Thus

$$(1.14) \quad v(A)\,A = \lambda_{(A)}v(A)\,,\ Aw(A) = \lambda_{(A)}w(A)\,,\ \text{and}$$

$$\langle v(A)\,|\,V(A)\rangle = \{0\}\,,\ \langle v(A)\,|\,w(A)\rangle \in \mathbb{Q}\left[\lambda_{(A)}\right] \cap (0,\infty)\,.$$

In particular, A leaves $V(A)$ invariant, for if $u \in V(A)$, then

$$(1.15) \qquad \langle v(A)\,|\,Au\rangle = \langle v(A)\,A\,|\,u\rangle = \lambda_{(A)}\langle v(A)\,|\,u\rangle = 0,$$

and it follows that $Au \in V(A)$. The same argument applies to the matrix J from (1.16) below. It shows that any J satisfying (1.16) must map $V(A)$ onto $V(B)$; i.e., $JV(A) = V(B)$. The number $\langle v(A)\,|\,w(A)\rangle$ from (1.14) plays an important role in the discussion of the isomorphism problem here (Section 5) and in [BJO99].

Let us mention an alternative form of the isomorphism criterion (1.12)–(1.13), formulated in [BJO99, Proposition 11.7]. Two primitive nonsingular $N \times N$ matrices A, B with positive integer matrix entries are C^*-equivalent if and only if there is a nonsingular $N \times N$ matrix $J = J(1)$ in $M_N(\mathbb{Z})$ satisfying the two conditions:

$$(1.16) \qquad v(B)\,J = \mu v(A) \qquad \text{for some } \mu \in (0,\infty)\,,$$

(1.17) for all $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that

$$B^m J A^{-n} \text{ and } A^m J^{-1} B^{-n} \text{ both have integer matrix entries;}$$

and then J^{-1} has matrix entries in $\mathbb{Z}\left[1/\det(A)\right] = \mathbb{Z}\left[1/\det(B)\right]$. It suffices to assume that $J \in \mathrm{GL}(N,\mathbb{R})$, but then (1.17) forces J, J^{-1} to lie in $M_N(\mathbb{Z}\left[1/\det A\right]) = M_N(\mathbb{Z}\left[1/\det B\right])$. So J is not unique: one may, for example, replace the given J with $B^m J A^{-n}$ for any $m, n \in \mathbb{N} \cup \{0\}$. By choosing m large enough, one may assure that $B^m J$ has integer matrix entries, and choosing it even larger one may

also assure that these entries are positive, and in fact (1.16) may be replaced by the condition

(1.16)′ J has positive matrix entries.

(But again, a given J may satisfy (1.16)–(1.17) without having positive or integer matrix entries.) The combined two conditions (1.16), (1.17) are equivalent to the two conditions (1.16)′, (1.17), and to (1.12), (1.13). For this one uses Perron–Frobenius theory: asymptotically when $m \to \infty$, B^m behaves like $\lambda_{(B)}^m$ times the projection onto $w\,(B)$, and $w\,(B)$ has strictly positive components.

In the two conditions (1.16)–(1.17) on J, positivity of the matrix entries is just hidden away in the first of the subconditions. However, from (1.1), one may merge the two conditions into the joint condition: There is a nonsingular $N \times N$ matrix $J = J\,(1)$ in $M_N\,(\mathbb{Z})$ such that,

(1.18)   for all $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that

$$B^m J A^{-n} \text{ and } A^m J^{-1} B^{-n} \text{ both have positive integer matrix entries.}$$

Thus the single condition (1.18) is equivalent to each of the three pairs of conditions (1.12)–(1.13), (1.16)′–(1.17), and (1.16)–(1.17).

Let us record a fact which was not mentioned in [BJO99], namely that the m in (1.18) can be taken to depend linearly on n:

Proposition 1.1. *Let A, B be nonsingular primitive $N \times N$ matrices with positive integer matrix entries, and assume that there is a nonsingular matrix $J \in \mathrm{GL}\,(N, \mathbb{R})$ such that (1.18) holds. It follows that there exists a positive integer k and an integer l such that*

(1.19)   *for all positive integers n, the matrices*

$$B^{kn+l} J A^{-n} \text{ and } A^{kn+l} J^{-1} B^{-n} \text{ both have positive integer matrix entries.}$$

Proof. To show the existence of k, l giving positivity we may modify the proof of Theorem 6 in [BJKR98] so as to make some specific estimates, i.e., we show that if a solution to (1.1) exists, then the sequences n_i, m_i may be taken to grow at most linearly. Let λ_1, λ_2 be the maximum eigenvalues of B, A, and λ_3 be the largest absolute value of any other eigenvalue, λ_4 the largest absolute value of the reciprocal of any eigenvalue. Consider $B^m J A^{-n}$. Using the above-mentioned (see (1.14)) two invariant complex vector-space (column vectors) decompositions

(1.20) $\mathbb{C}^N = V\,(A) \oplus \mathbb{C}w\,(A)$ and $\mathbb{C}^N = V\,(B) \oplus \mathbb{C}w\,(B)$,

we note that the contribution of the maximum eigenvector in $(J A^{-n})$ will be at least $C \lambda_2^{-n}$ for some positive C. When we multiply it by B^m we get $\lambda_1^m C \lambda_2^{-n}$. The largest magnitude of any other term will be some $\lambda_3^m C_1 \lambda_4^n$. We want the former terms to dominate the sum of all the others, say to be N^2 times the largest, where N is the dimension of the matrices. Take logarithms, and we want

(1.21) $m \log \lambda_1 + \log C - n \log \lambda_2 > m \log \lambda_3 + \log\left(C_1 N^2\right) + n \log \lambda_4$

or rearranged equivalently as

(1.22) $m\,(\log \lambda_1 - \log \lambda_3) > -\log C + \log\left(C_1 N^2\right) + n\,(\log \lambda_2 + \log \lambda_4)\,.$

Then some arithmetic progression where the ratio of m to n exceeds

$$(1.23) \qquad \frac{\log \lambda_2 + \log \lambda_4}{\log \lambda_1 - \log \lambda_3}$$

will give the domination.

Consider denominators which involve some algebraic prime p. For simplicity extend the coefficient field and assume we can diagonalize the matrices (the case of a standard Jordan form can be treated similarly). The maximum denominator in A^{-n} is p^{-kn} for some constant k, which for instance can be worked out from the determinant. Then consider the entries in $B^m J A^{-n}$. They will be sums of constants from the diagonalizing matrices times m powers of the eigenvalues of B, i.e., $\sum_i c_i \lambda_i^m$. The eigenvalues only involve nonnegative powers of p, since they are algebraic integers.

The terms in this sum for eigenvalues not divisible by p must add up to be an integer at the prime p: otherwise, no very large powers m could make the total an integer. For the other terms, as soon as m exceeds nk plus the degrees of constants arising from diagonalization process, we will have algebraic integers. $\qquad \square$

There is another general observation about solving for J and K in (1.2), with A and B given, which motivates the p-adic analysis to follow. The identities (1.2) are quadratic. Since the matrix entries on the left are all integral, solving for J and K is therefore a quadratic diophantine problem in the sense of [BoSh66, Ch. 1]: We thus have a system of quadratic equations in the respective matrix entries of J and K, and the result in [BoSh66] amounts to the assertion that the solution to a quadratic diophantine problem is equivalent to instead solving a finite system of related p-adic congruences, but for all p. Hence, in the following, we will be stating criteria for C^*-equivalence in terms of p-adic conditions. We will specify for which p we need the conditions, for example in Corollary 3.2, and we will show that there are finite algorithms for deciding the problem.

It seems to be difficult to convert Proposition 1.1 into an effective decision procedure for isomorphism, since J is not unique, and hence it is difficult to obtain *a priori* estimates on the norm of J and on the coefficients k and l. Instead we will turn to the completely different method developed in [KiRo88], which is described in the previous paragraph and in Section 4. Instead of starting with an explicit norm estimate on J, we reduce the problem to a collection of congruences and norm restrictions which are decidable by Lemma 4.1.

The simple-minded way of determining the dimension group from (1.8)–(1.11) is to take the algebraic extension of $\mathbb{Q}$ determined by all the roots of the characteristic equation of A, write A in Smith-Jordan form [New72], and then compute $\bigcup_m A^{-m} (\mathbb{Z}^N)$ in the new basis. Vestiges of this approach appear in our argument, but instead of using the complete Jordan form we merely use a reduction to block-diagonal form where the blocks correspond to eigenspaces, first when we determine the subspaces which a rational matrix J_0 has to preserve, and then in studying the matrix giving the difference of the actual matrix J from J_0.

The ordered dimension groups historically came from the AF-C^*-algebra classification problem [Bra72], but have now come to play a role also in dynamical systems, see, e.g., [Kit98]. Consider, for example, a substitution dynamical system σ (letters to words) derived from a given alphabet S of size N. For $i, j \in S$, let

a_{ij} count the number of occurrences of i in the word $\sigma(j)$, resulting from the substitution σ, and let A be the corresponding matrix with group $G(A)$ (see (1.10)). In [DHS99], the co-authors use $G(A)$ in their classification of these systems, which may also be realized as shift dynamical systems on the paths in the corresponding Bratteli diagrams.

There are some general blanket references which we will use throughout the paper without always citing them explicitly: [PoZa97] on algorithms of algebraic number theory, [New72] on integral matrices and their factorizations, and [Kit98], [Wag99] on symbolic dynamics. The references represent diverse areas of mathematics which are not always thought to be directly related. But the proofs to follow fall at the interface of techniques from these different subjects. For that reason, we include a bit more detail and discussion than is customary in a paper which does not cut across boundaries between fields.

2. SUBSPACE STRUCTURE AND LOCALIZATION

In this section we will analyze the structure of subspaces of $\mathbb{C}^N$ which are mapped into each other by a possible intertwiner matrix $J \in M_N(\mathbb{Z})$. One general idea is the following: Consider a certain subset D_A of G_A which is defined by a property which is invariant under group isomorphism. Then

$$(2.1) \quad \tilde{D}_A = \{g \in G \mid \exists\, n, n_1, \ldots, n_m \in \mathbb{Z},\, g_1, \ldots, g_m \in D_A \ni\!- ng = \textstyle\sum_{i=1}^m n_i g_i\}$$
$$= \text{"the subgroup of } G \text{ linearly spanned by } D_A\text{"}$$

is a subgroup of G. If G_B is another subgroup of $\mathbb{C}^N$, and $G_B = JG_A$, and D_B and $\tilde{D}_B$ are defined as above for G_B, we must have

$$(2.2) \qquad\qquad D_B = JD_A, \qquad \tilde{D}_B = J\tilde{D}_A,$$

and hence

$$(2.3) \qquad\qquad\qquad J\mathbb{R}D_A = \mathbb{R}D_B.$$

This idea was much exploited in [BJO99] on the subgroups

$$(2.4) \quad D_m(G_A) = \bigcap_i m^i G_A$$
$$= \text{the set of elements of } G_A \text{ which are infinitely divisible by } m,$$

and we will soon give an example of this in a more general setting than in [BJO99]. Note in particular that if m is a rational eigenvalue of A, then m is an integer since the characteristic equation of A is monic, and hence $D_m(G_A)$ is nonzero and gives nontrivial information about J. We would like to exploit this idea also when λ is an irrational eigenvalue of A, but since $G_A \subset \mathbb{Z}\left[1/\left|\det A\right|\right]^N$, G_A then clearly does not contain eigenvectors of A. To remedy this situation, we may augment or localize G_A and G_B by equipping them with coefficients outside $\mathbb{Z}$, i.e., by considering tensor products

$$(2.5) \qquad\qquad \tilde{G}_A = E \otimes G_A, \qquad \tilde{G}_B = E \otimes G_B,$$

where E is any $\mathbb{Z}$-module, and then J still defines an isomorphism between $\tilde{G}_A$ and $\tilde{G}_B$. One then tries to choose E to optimize the information about subspaces. In [BJO99] this remedy was used with E finite cyclic groups, but one may use p-adic numbers, or, as we will also do, various finite algebraic extensions of $\mathbb{Z}$. Which extension is used has to be fine-tuned to the problem. For example, if $E =$

$\mathbb{Z}\left[1/\left|\det A\right|\right]$, then $\tilde{G}_A = \mathbb{Z}\left[1/\left|\det A\right|\right]^N$, and all information about G_A disappears (except for its rank and the prime factors of $\left|\det A\right|$, which both are invariants). Similarly, if λ is an algebraic integer which is a unit, i.e., is such that the constant term in its minimal polynomial is ± 1, then $\lambda^{-1} \in \mathbb{Z}[\lambda]$, and hence *all* elements of $\mathbb{Z}[\lambda] \otimes G_A$ are divisible by λ, and no information on the subspace structure is obtained. One useful choice of E is based on Theorem 10 in [BJKR98]: If G_A and G_B are isomorphic and $\lambda_{(A)}$ and $\lambda_{(B)}$ are the Perron–Frobenius eigenvalues of A and B, then the fields $\mathbb{Q}\left[\lambda_{(A)}\right]$ and $\mathbb{Q}\left[\lambda_{(B)}\right]$ are the same, and $\lambda_{(A)}$ and $\lambda_{(B)}$ are the products of the same primes over this field. A prime in this context means a prime ideal in the associated subring $\mathbb{O}[\lambda]$ of algebraic integers, i.e., $\mathbb{O}[\lambda]$ is the ring of all elements of $\mathbb{Q}[\lambda]$ which satisfy equations in monic polynomials over $\mathbb{Z}$, so that

$$(2.6) \qquad \mathbb{Z}[\lambda] \subset \mathbb{O}[\lambda] \subset \mathbb{Q}[\lambda].$$

Recall that an ideal $\mathcal{I}$ in a ring is a prime ideal if whenever $\mathcal{I} = \mathcal{I}_1 \mathcal{I}_2$ for two ideals $\mathcal{I}_1, \mathcal{I}_2$, then $\mathcal{I} = \mathcal{I}_1$ or $\mathcal{I} = \mathcal{I}_2$. One useful choice for E is thus $\mathbb{O}\left[\lambda_{(A)}\right] = \mathbb{O}\left[\lambda_{(B)}\right]$. One other choice we shall use is

$$(2.7) \qquad \Omega = \mathbb{O}\left[\lambda_1, \ldots, \lambda_N, \mu_1, \ldots, \mu_N\right],$$

where $\lambda_1, \ldots, \lambda_N, \mu_1, \ldots, \mu_N$ are the respective roots in $\mathbb{C}$ of the characteristic equations of A and B:

$$(2.8) \qquad \det\left(\lambda \mathbb{1} - A\right) = 0, \qquad \det\left(\mu \mathbb{1} - B\right) = 0.$$

3. p-ADIC CHARACTERIZATION OF J

We have already given several characterizations of the intertwiner J more or less in terms of the dimension groups $G(A)$, $G(B)$, i.e., (1.1), ((1.16) & (1.17)), ((1.16)$'$ & (1.17)), and (1.18). Here G_A and G_B are defined in terms of asymptotic properties of A^{-n} and B^{-n} as $n \to \infty$. We will now give an exposition of another property of J given in terms of asymptotic properties of the *positive* powers A^n and B^n as $n \to \infty$. Since $n \to A^n \mathbb{Z}^N$ is decreasing, and

$$(3.1) \qquad \bigcap_n A^n \mathbb{Z}^N = \left\{ m \in \mathbb{Z}^N \mid q(A)\, m = 0 \right\}$$

by [BJO99, Proposition 12.1], where $q(t)$ is the product of those irreducible (over $\mathbb{Z}$) factors of $\det\left(t\mathbb{1} - A\right)$ which have constant term ± 1, the lattices $\bigcap_n A^n \mathbb{Z}^N$ give very little information except that J has to map $\bigcap_n A^n \mathbb{Z}^N$ onto $\bigcap_n B^n \mathbb{Z}^N$. However, if one replaces these intersections by p-adic limits, one can say much more. Recall that if $p \in \{2, 3, 5, 7, 11, \ldots\}$ is an ordinary prime, the ring of p-adic integers $\mathbb{Z}_{(p)}$ is the projective limit

$$(3.2) \qquad 0 \xleftarrow{\ p\ } \mathbb{Z}_p \xleftarrow{\ p\ } \mathbb{Z}_{p^2} \xleftarrow{\ p\ } \mathbb{Z}_{p^3} \longleftarrow \cdots \longleftarrow \mathbb{Z}_{(p)},$$

where the left maps are multiplication by p. It can be equipped with a topology making it into a compact totally disconnected ring. This is in fact the topology the additive group $\mathbb{Z}_{(p)}$ has as a dual group to $\mathbb{Z}_{p^\infty}$ viewed as the inductive limit of the discrete groups

$$(3.3) \qquad 0 \hookrightarrow \mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^2} \hookrightarrow \mathbb{Z}_{p^3} \hookrightarrow \cdots \hookrightarrow \mathbb{Z}_{p^\infty},$$

where the injections come from the standard realization of $\mathbb{Z}_{p^\infty} = \mathbb{Z}[1/p]/\mathbb{Z}$ as a subgroup of the circle group $\mathbb{T}$; see [Kob84], [Ser79], [Ser98]. Koblitz uses the

terminology $\mathbb{Z}_p$ for the p-adic integers, our $\mathbb{Z}_{(p)}$, while we reserve $\mathbb{Z}_p$ for $\mathbb{Z}/p\mathbb{Z}$. Other authors, e.g., [BoSh66], use O_p for the p-adic integers. In the duality consideration of the two groups $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{p^\infty}$ of (3.2)–(3.3), we use the duality notion of locally compact abelian *groups*, e.g., $\mathbb{Z}_{p^\infty}$ is realized as the group of continuous characters on $\mathbb{Z}_{(p)}$, and conversely, $\mathbb{Z}_{(p)}$ acts as the group of all characters on $\mathbb{Z}_{p^\infty}$. Now $\mathbb{Z}_{(p)}$ is a ring and thus a $\mathbb{Z}$-module, but it is not a field: If q is an integer, then $1/q \in \mathbb{Z}_{(p)}$ if and only if q is mutually prime with p. However, $\mathbb{Z}[1/p] \otimes \mathbb{Z}_{(p)} = \mathbb{Q}_{(p)}$ is a field called the p-adic numbers.

Now if $A \in M_N(\mathbb{Z})$ is a matrix, we may view A as a matrix with matrix entries in $\mathbb{Z}_{(p)}$, and we may associate a unique idempotent

$$(3.4) \qquad E_{(p)}(A) = E(A) \in M_N\left(\mathbb{Z}_{(p)}\right)$$

with A by using the following presumably known lemma (we did not find a reference).

Lemma 3.1. *If $A \in M_N(\mathbb{Z}/q\mathbb{Z})$ for a $q \in \mathbb{Z}$, then the semigroup $\left\{A, A^2, A^3, \dots\right\}$ contains an idempotent E. This idempotent is unique, and $\{n \mid A^n = E\}$ is a subsemigroup of $\mathbb{N}$.*

Proof. Since $M_N(\mathbb{Z}/q\mathbb{Z})$ is finite, there is an $m_0 \in \mathbb{N}$ and an $n_0 \in \mathbb{N}$ such that $A^{n_0+m_0} = A^{m_0}$. (Our convention throughout for the natural numbers is $\mathbb{N} = \{0, 1, 2, \dots\}$.) But then $A^{n_0+m} = A^m$ for all $m \geq m_0$ and thus $A^{kn_0+m} = A^m$ for all $k \in \mathbb{N}$. Choose k such that $kn_0 \geq m_0$ and put $m = kn_0$. This gives $\left(A^{kn_0}\right)^2 = A^{kn_0}$ so A^{kn_0} is idempotent.

If A^n and A^m are idempotents, then $A^n = (A^n)^m = (A^m)^n = A^m$, so the idempotent is unique. If it is called E, then if $A^n = A^m = E$, then $A^{n+m} = E \cdot E = E$, so $\{n \mid A^n = E\}$ is a semigroup. □

We now turn to some of the uses of the p-adic analysis.

Fix a prime p, and let $e(m)$ be an increasing sequence of integers such that $A^{e(m)}$ is an idempotent modulo p^m in M_N,

$$(3.5) \qquad \left(A^{e(m)}\right)^2 = A^{e(m)} \mod p^m M_N(\mathbb{Z}).$$

This sequence exists because of Lemma 3.1, and by thinning out the sequence, and using Lemma 3.1 again, we may also assume

$$(3.6) \qquad \left(B^{e(m)}\right)^2 = B^{e(m)} \mod p^m M_N(\mathbb{Z}).$$

But by the uniqueness of the idempotent, it follows that

$$(3.7) \qquad m' > m \implies A^{e(m')} = A^{e(m)} \mod p^m,$$

and hence

$$(3.8) \qquad E_{(p)}(A) = \lim_{m \to \infty} A^{e(m)}$$

exists in $M_N\left(\mathbb{Z}_{(p)}\right)$, and $E_{(p)}(A)$ is an idempotent matrix. Correspondingly, $E_{(p)}(B)$ is an idempotent matrix. Write (1.17) in the form

$$(3.9) \qquad B^{m_n} J = K_n A^n,$$

$$(3.10) \qquad A^{m_n} = L_n B^n J.$$

Since $\mathbb{Z}_{(p)}$ is compact (and metrizable), it follows that there is a subsequence of $n \to \infty$ such that $\lim_n K_n = K$ and $\lim_n L_n = L$ exist in $M_N\left(\mathbb{Z}_{(p)}\right)$, and we get from the relations above that

$$(3.11) \qquad\qquad E_{(p)}\left(B\right) J = K E_{(p)}\left(A\right),$$

$$(3.12) \qquad\qquad E_{(p)}\left(A\right) = L E_{(p)}\left(B\right) J.$$

Now define the $\mathbb{Z}_{(p)}$-eventual row space $G_{(p)}\left(A\right)$ of A as the linear combinations over $\mathbb{Z}_{(p)}$ of the row-vectors of $E_{(p)}\left(A\right)$, and similarly for $E_{(p)}\left(B\right)$. Then (3.11) and (3.12) together say that

$$(3.13) \qquad\qquad G_{(p)}\left(B\right) J = G_{(p)}\left(A\right).$$

Thus (3.13) holds for any prime p. But conversely, by taking p-adic limits as in the proof of Theorem 7 in [BJKR98], if (3.13) holds for all primes p in the set $\mathrm{Prim}\left(\det\left(A\right)\right) = \mathrm{Prim}\left(\det\left(B\right)\right)$, then we can recover (1.17). Thus

$$(3.13)' \quad G_{(p)}\left(B\right) J = G_{(p)}\left(A\right) \qquad \text{for all } p \in \mathrm{Prim}\left(\det\left(A\right)\right) = \mathrm{Prim}\left(\det\left(B\right)\right)$$

is equivalent to (1.17) (the equivalence of (1.17) and (3.13)$'$ is Theorem 7 in [BJKR98]). The details supplied above expand on the arguments from [BJKR98], which were somewhat terse.

What makes this particularly useful for the decidability problem is that any countably generated torsion-free module over the p-adic integers has a trivial structure: such a module is merely a direct sum of replicas of the p-adic numbers or the p-adic integers ([Pru25]; see also [KaMa51]). The total number of direct summands in $G_{(p)}\left(B\right)$ and $G_{(p)}\left(A\right)$ is bounded by the rank N of A or B. This makes it possible to decide whether or not J exists with the property (3.13) for each p, but the remaining problem is to find a joint J for all p in $\mathrm{Prim}\left(A\right)$ and to ensure the positivity property (1.16). Note that in our setting we have $G_{(p)}\left(A\right) \in \mathbb{Z}_{(p)}^{N}$ by construction as p-adic limits of integer vectors, and hence $G_{(p)}\left(A\right)$ cannot contain any element which is infinitely divisible by p, and thus $G_{(p)}\left(A\right)$ as a $\mathbb{Z}_{(p)}$-module is just a direct sum of at most N copies of $\mathbb{Z}_{(p)}$ (no direct summand $\mathbb{Q}_{(p)}$ can occur). However, be warned, since $\mathbb{Z}_{(p)}$ is not a field, this is not as useful as knowing that a vector space (over a field) has a certain dimension, since the usual operations of change of basis, etc., cannot be performed within the ring $\mathbb{Z}_{(p)}$. In particular, (3.13) says much more than that the p-adic row spaces have the same rank. To emphasize this, let us cast Theorem 7 in [BJKR98] in a somewhat different, but equivalent, form:

Corollary 3.2. *In order that the unordered dimension groups* $\bigcup_n A^{-n}\mathbb{Z}^N$ *and* $\bigcup_n B^{-n}\mathbb{Z}^N$ *associated with a pair of nonsingular matrices A, B be isomorphic, it is necessary and sufficient that* $\mathrm{Prim}\left(\det\left(A\right)\right) = \mathrm{Prim}\left(\det\left(B\right)\right)$, *and that there exists a nonsingular matrix* $J \in \mathrm{GL}\left(N, \mathbb{Z}\left[1/\det\left(A\right)\right]\right)$ *(i.e., the matrix entries of J are in $\mathbb{Z}\left[1/\det\left(J\right)\right]$ and $\det\left(J\right)$ is invertible in the ring $\mathbb{Z}\left[1/\det\left(J\right)\right]$) such that*

$$(3.14) \qquad\qquad G_{(p)}\left(B\right) J = G_{(p)}\left(A\right)$$

for each prime $p \in \mathrm{Prim}\left(\det\left(A\right)\right)$.

4. DECIDABILITY OF C^*-EQUIVALENCE

In this section we will prove that the problem of finding an integer matrix $J = J(1)$, satisfying any of the equivalent conditions (1.12)–(1.13), (1.16)–(1.17), (1.16)'–(1.17), (1.18), (1.19), (3.13)' together with positivity, is decidable. In these considerations, positivity will only play a minor role, but instead we will, as partially explained in Section 2, work in various algebraic extensions R of $\mathbb{Z}$. The idea is roughly that if J satisfies (1.12):

$$(4.1) \qquad\qquad J(1) G(A) = G(B),$$

then $J(1)$ also satisfies

$$(4.2) \qquad\qquad J(1)(R \otimes_{\mathbb{Z}} G(A)) = R \otimes_{\mathbb{Z}} (G(B)),$$

and, conversely, if (4.2) has no solution $J(1) \in M_N(R)$, then (4.1) certainly has no solution, and this can be used to decide absence of C^*-equivalence.

The operator J must preserve Galois conjugation on the maximal eigenspace. The conditions (4.1)–(4.2) amount to having a linear mapping which preserves a lattice of subspaces defined by a lattice of basis elements over an extension field, having only specified primes in its determinant, and satisfying congruences. In addition we can multiply the matrix $J(1)$ by powers of A, B which can automatically make it divisible by any power of π at the π-eigenspace. We will show these conditions are decidable.

By congruences, we mean that a finite set of vectors over a ring R has its image modulo some ideal I to lie in a specified finite set where R/I is finite: in particular any Boolean or logical combination of congruences is a set of congruences. We can test congruences by testing each element of this set of residue classes.

Over the integers, a matrix which preserves a sequence of rational subspaces in a direct sum decomposition can be conjugated into a block-triangular form, by taking bases over the integers corresponding to the sequence of subspaces [New72]. Every subgroup of a free abelian group is free, and a finitely generated subgroup is a summand if and only if it has no elements which are not divisible by a prime p in it but are divisible in the total group [Kap69]. However, an integer matrix which preserves a sequence of rational subspaces in a direct sum decomposition cannot always be conjugated further to be block-diagonal over the integers without introducing fractions.

In an algebraic number ring, some finite, computable power of any ideal (the order of the class group [Ser79]) will be principal. This means that congruences to a modulus which is an ideal, or fractions whose denominators lie in an ideal, can be restated as congruences to a modulus which is an element, or fractions whose denominators divide a power of some element. Thus we need only to consider ideals (m) generated by a single element $m \in \Omega$ in the following lemma.

Lemma 4.1. *Let Ω be an algebraic number ring with quotient field F, and let m_1, m_2 be relatively prime elements of Ω, i.e., $(m_1)+(m_2) = \Omega$. Let $f \in F$ be relatively prime to m_1 also. Let $\mathrm{CC}\,[m_1, m_2, f]$ be the following set of congruence classes of matrices M:*

$$(4.3) \quad \mathrm{CC}\,[m_1, m_2, f] = \{M \pmod{m_1} \mid M \in M_N(\Omega\,[1/m_2]) \text{ and}$$

$$\text{there exists an } x \in \Omega\,[1/m_2] \text{ such that } 1/x \in \Omega\,[1/m_2] \text{ and } \det(M) = fx\}.$$

In words, $CC[m_1, m_2, f]$ *is the set of modulo-m_1 reductions of matrices M over F whose entries m_{ij} can be expressed as fractions of elements of Ω whose denominators divide a power of m_2 and such that the determinant of M is a product of units and powers of primes dividing m_2 and f.*

It follows that there is a finite algorithm to determine the finite set $CC[m_1, m_2, f]$.

Remark 4.2. The set $CC[m_1, m_2, f]$ is finite since m_2 is invertible modulo m_1. It is a subset of $M_N(\Omega/(m_1))$, and the quotient ring $\Omega/(m_1)$ is finite. That an algorithm determines something means that the algorithm always gives the correct answer in a finite number of steps.

Proof of Lemma 4.1. We will first put the matrix M (mod m_1) into a standard diagonal form, where each main-diagonal entry divides the next, using elementary matrices (row and column operations). Note that we can assemble factorizations modulo separate primes using a form of the Chinese remainder theorem (making each elementary matrix off-diagonal to be divisible by a power of all but one prime of m_1). And moreover an elementary matrix over the quotient ring $\Omega/m_1\Omega$ lifts to an elementary matrix over Ω, since any element of the quotient ring lifts.

Products of elementary matrices will give all permutations, except that we may have to introduce a negative sign to allow for the determinant being 1. We may use row and column operations to produce the sequence of matrices below:

$$(4.4) \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now we use the standard procedures involved in determining torsion numbers, locally at each prime. We find some entry divisible by a minimal power of the prime and put it in the $(1,1)$ matrix entry location, and use it to clear out its row and column; and then work on the submatrix obtained by deleting these.

Thus by elementary operations on M, we can put the matrix M (mod m_1) into the prescribed diagonal form, say with main-diagonal entries d_{ii}, each dividing the next. Now we may state the condition, that a matrix modulo m_1 in this form is the reduction of a matrix over $\Omega[1/m_2]$: there must be elements of the latter ring r_{ii} reducing to d_{ii} modulo m_1, and such that the determinant has the correct form modulo $(m_1^N, m_1 \prod_1^{N-1} r_{ii})$. This criterion is decidable, since it depends only on congruences modulo m_1^N, including a knowledge of the multiplicative units of the ring to this modulus.

The criterion is necessary, since if a matrix has the given form, its determinant is determined modulo $(m_1^N, m_1 \prod_1^{N-1} r_{ii})$. We consider the fact that each entry can vary only by multiples of m_1, hence the effect of changing any entry will be to change the determinant by a multiple of the minor of this entry. At any prime p, the p-part of each term in a minor (times m_1) must be at least this great, and this amount is realized when the (N, N) entry is chosen, and the specific term in the minor is the product of the other diagonal entries.

Finally we will show sufficiency. Choose numbers m_{ii} with the given reductions. At each prime $p|m_1$ we may alter the (N, N) entry by a multiple of m_1 and high powers of other primes to alter the determinant by any multiple of $(m_1^N, m_1 \prod_1^{N-1} r_{ii})$ modulo m_1^N. For this, we want that $\det(M)$ is correct modulo m_1^N; this is linear in m_{NN}, and in the coefficient of m_{NN} the lowest powers of all primes in m_1 occur in the term $\prod_{i=1}^{N-1} m_{ii}$, so that we can adjust by any multiple of its greatest common

divisor with m_1^N. Now we have a diagonal matrix with the right determinant modulo m_1^N. Now if we make $m_{i\,i+1} = m_1$, $i = 1, 2, \ldots, N-1$, and make $m_{N1} = sm_1$ we alter the determinant by any multiple sm_1^N and hence make it to have exactly the right form. $\square$

We next describe the algebra of endomorphisms which preserves a collection of subspaces like the ones here.

Definition 4.3. Let A, B be matrices with rational matrix elements. Let K denote the field generated by the eigenvalues of A and B. Assume that A and B act on vector spaces V, W over $\mathbb{Q}$. Let R be a subring of K. Then A and B act in a natural manner on $V \otimes_{\mathbb{Z}} R$ and $W \otimes_{\mathbb{Z}} R$ respectively. Then $\mathrm{DGI}(A, B, R)$ denotes the additive group of R-homomorphisms $J\,(1)$ from $V \otimes R$ to $W \otimes R$ such that

- (α) the direct sum of all nonmaximal eigenspaces is preserved,
- (β) for each algebraic prime π of K which divides an eigenvalue, $J\,(1)$ preserves the span of the eigenspaces $E\,(\pi)$ whose eigenvalues are divisible by π,
- (γ) the 0-eigenspaces are preserved.

We shall use the abbreviation DGI, for "dimension group isomorphisms", although "dimension group pre-isomorphisms" would be a more accurate description.

Note that $\mathrm{DGI}(A, B, R)$ really depends on A, B and not merely on V, W, because the eigenspaces and eigenvalues of A and B occur in these conditions. Then our criterion for a dimension group isomorphism says that there is such a map $J\,(1)$ defined over $\mathbb{Z}$ with the following additional properties (we identify $J\,(1)$ with the map it defines on various sub- and quotient-modules):

- (i) $J\,(1)$ is nonzero modulo the nonmaximal eigenspaces (which can be ensured by congruences relatively prime to π),
- (ii) on the quotient $V/E\,(\pi)$ the determinant of $J\,(1)$ is relatively prime to π,
- (iii) the determinant of $J\,(1)$ is divisible only by primes $\pi |\det(A)\det(B)$, restricted to the nonzero eigenspaces.

Here (i), (ii) are congruence conditions and (iii) is a determinant condition; these will be transformed a little so that they become the basic criteria whose satisfiability we must decide. By linear algebra [New72] we find a nonsingular map J_0 over the rational numbers satisfying the first three conditions (α)–(γ), if it exists, from V to W, and then a general hypothetical map must differ from J_0 by a map J_a in $\mathrm{DGI}(A, A, \mathbb{Q})$: $J\,(1) = J_0 J_a$. Replace J_0 by some $c_0 J_0$, $c_0 \in \mathbb{Q}$, so that $J_0^{-1} \in M_N(\mathbb{Z})$, where $M_N(R)$ is the algebra of all $N \times N$ matrices over the ring R. Then $J_a \in M_N(\mathbb{Z})$. Write $J_0 = J_c/N_c$, $J_c \in M_N(\mathbb{Z})$, $N_c \in \mathbb{Z}$. Then (i), (ii), (iii) translate into congruence conditions and norm conditions on J_a:

- (i$_a$) $J_c J_a$ on a chosen maximal eigenvector of A is nonzero modulo p_a (a fixed prime relatively prime to $\det(A)$, $\det(J_c)$, N_c;
- (ii$_a$) $J_c J_a$ on the quotient $V/E(\pi)$ has determinant a multiple of N_c^N by a invertible number modulo $\pi(N_c, \pi)^N$;
- (iii$_a$) the determinant of J_a is $N_c^N/\det(J_c)$ times a number dividing some power of $\det(A)\det(B)$;
- (iv$_a$)

$$(4.5) \qquad\qquad J_c J_a \equiv 0 \pmod{N_c}.$$

The vector space $\mathrm{DGI}(A, A, \mathbb{Q})$ is in fact also an algebra, which we next describe. Let K denote the field generated by the eigenvalues of A. The next proposition is based on general principles of Galois theory, see, e.g., [Rot98] and [Jac75], as well as standard facts about linear resolutions, see [New72], [Ser77], [Ser98]. The proposition also extends a more primitive variant which appeared earlier in [BJO99, Corollary 9.5]. To understand the statement of the proposition, recall the following standard terminology: If $K \supset \mathbb{Q}$ is a number field, the Galois group $\Gamma = \mathrm{Gal}\,(K/\mathbb{Q})$ is the group of automorphisms γ of K which fix $\mathbb{Q}$ pointwise, i.e., $\gamma\,(x) = x$ for $\gamma \in \Gamma$ and $x \in \mathbb{Q}$. But we shall also consider Γ as a group of transformations of column vectors K^N. If $x = (x_i)_{i=1}^{N} \in K^N$, we set $\gamma\,(x) = (\gamma\,(x_i))_{i=1}^{N}$.

Proposition 4.4. *There is a filtration V_i, $i = 0, \ldots, s$, of the vector space on which A acts in which $\mathrm{DGI}(A, A, K)$ has a block-triangular structure. The ideal $J = \{M \in \mathrm{DGI}(A, A, K) \mid MV_i \subset V_{i+1}, \ i = 0, \ldots, s - 1\}$ is a nilpotent ideal and $\mathrm{DGI}(A, A, K)/J$ has a natural embedding by the block structure into $\bigoplus_i \mathrm{GL}(V_i/V_{i+1})$. This embedding is an isomorphism. There is a subfiltration $V_{s(i)}$ defined over $\mathbb{Q}$ such that $V_{s(i)}/V_{s(i+1)}$ is a direct sum of Galois conjugates of $V_{s(i+1)-1}/V_{s(i+1)}$. These structures can be finitely computed.*

Proof. We find the eigenvalues of A, diagonalize A over K, factoring ideals into primes, using standard algorithms, e.g., [PoZa97]. We consider matrices which preserve $E\,(\pi)$ (and preserve the sum of all non-Perron–Frobenius eigenspaces, to ensure positivity). These subspaces $E\,(\pi)$ will be direct sums of sets of eigenspaces, and these sets of eigenspaces will be permuted according to Galois action on π. The effect of Galois action and the families of intersections of these spaces can be considered by taking Galois-invariant bases B_π for $E\,(\pi)$. All intersections of subsets of the $E\,(\pi)$ must be invariant. Order the intersections I_j with bases B_j so that the number of sets being intersected is increasing, and put Galois conjugates adjacent to each other, and relabel them as the new V_j spaces.

Then the subspace generated by all bases succeeding any given basis is preserved, and we have a block-triangular structure corresponding to it, and a larger block-triangular structure, whose blocks are the sets of Galois-conjugate blocks of those from the former structure. The latter will be defined over $\mathbb{Q}$ as required. Since elements J increase filtration, any s-fold product is zero; they are the matrices in the algebra which are zero on the main-diagonal blocks, and so the quotient maps isomorphically into the sum of the general linear groups on V_j/V_{j+1} with basis $B_{0j} = B_j \setminus \bigcup_{k>j} B_k$. But we note that the general linear group on the span of B_{0j} will preserve all subspaces $E\,(\pi)$, hence the subquotient is precisely this general linear group. $\square$

It follows that all Galois-invariant linear maps on $V_{s(j)}/V_{s(j+1)}$ will also lift to $\mathrm{DGI}(A, A, \mathbb{Q})$.

Proposition 4.5. *Suppose a vector space V over $\mathbb{Q}$ is a direct sum over an extension field $K \supset \mathbb{Q}$ of Galois-conjugate subspaces V_i (with corresponding bases), transitively permuted by the Galois group of K. Then the algebra of automorphisms of V over $\mathbb{Q}$ which preserve each space V_i is isomorphic to the general linear group of V_1 over the minimal field K_1 required to define V_1, which corresponds to the subgroup N of the Galois group that sends V_1 to itself.*

Proof. If V_1 can be defined over a subfield of K, then the Galois group of that field must fix V_1; conversely if the Galois group fixes V_1, it will also fix the complementary sum of eigenspaces, hence it will fix a projection operator to the subspace whose kernel is the complementary sum of eigenspaces, and from the columns of a matrix for this operator, the subspace can be defined.

Given an endomorphism of V_1 over K which arises from a mapping over $\mathbb{Q}$, the endomorphisms of all other V_i are uniquely determined as its Galois conjugates. This means we have a one-to-one linear mapping from endomorphisms of V over $\mathbb{Q}$ fixing V_1 (and these by Galois conjugacy fix every V_i), into the general linear group of V_1 over K. In fact the image lies in the general linear group over K_1 since over it, we can define a projection operator to V_1. This mapping is also an epimorphism, since given any K_1-linear mapping h of V_1 to itself, there are Galois conjugates defined on the other V_i (the Galois operator is unique up to the subgroup fixing V_1, which also fixes h). We can take the sum of h and its Galois conjugates on the other V_i, and the sum will be a Galois-invariant mapping of V, and therefore defined over $\mathbb{Q}$. $\square$

Example 4.6. We might consider a case of a matrix with three eigenvalues p, q, pq with respective eigenspaces E_1, E_2, E_3, so that the two sum spaces $E_1 \oplus E_3$ and $E_2 \oplus E_3$ are preserved under the Galois action, as is their intersection E_3. Then the algebra of endomorphism has a block-triangular structure with three blocks and the main-diagonal blocks are isomorphic to the respective endomorphism algebras $\mathrm{End}(E_1)$, $\mathrm{End}(E_2)$, $\mathrm{End}(E_3)$. Suppose now that p and q are Galois conjugates so that the product pq is Galois-invariant. The larger block structure will then correspond to the two spaces $E_1 \oplus E_2$ and E_3. The group of endomorphisms of $E_1 \oplus E_2$ over the rational numbers will be isomorphic to the automorphisms of E_1 over a quadratic extension field corresponding to the Galois conjugation which interchanges p and q.

When we conjugate an algebra of matrices by a fixed matrix, this will modify congruences on the algebra in a computable way, where we may multiply the denominators involved in the matrix conjugation by the previous moduli. In particular, we apply this idea to a block diagonalization of $\mathrm{DGI}(A, A, \mathbb{Q})$. Take an integer matrix J_f, whose columns are bases, taken in order, for $V_{s(i)}/V_{s(i+1)}$, so that $J_g = \det(J_f)J_f^{-1}J_aJ_f$ puts J_a into block-triangular form. The previous congruences and determinant conditions translate to:

(i$_g$) $J_c J_f J_g \det(J_f)J_f^{-1}$ on a chosen maximal eigenvector of A is nonzero modulo p_a (which is a fixed prime relatively prime to $\det(A)$, $\det(J_c)$, N_c, $\det(J_f)$);

(ii$_g$) $J_c J_f J_g \det(J_f)J_f^{-1}$ on the quotient $V/E(\pi)$ has determinant a multiple of $\det(J_f)^{2N} N_c^N$ by an invertible number modulo $\pi(N_c^N \det(J_f)^{2N}, \pi)$;

(iii$_g$) the determinant of J_g is $\det(J_f)^N N_c^N / \det(J_c)$ times a number dividing some power of $\det(A)\det(B)$;

(iv$_g$)

$$(4.6) \qquad J_f J_g \det(J_f)J_f^{-1} \equiv 0 \quad (\mathrm{mod}\ \det(J_f)^2),$$

$$(4.7) \qquad J_c J_f J_g \det(J_f)J_f^{-1} \equiv 0 \quad (\mathrm{mod}\ N_c \det(J_f)^2)$$

(any further multiples by constant matrices could be treated in similar fashion; we are multiplying matrices by these quantities, so when we take determinants we multiply by Nth powers).

Recall some aspects of the theory of finite-dimensional algebras with unit over a field. Our background reference is [Jac75]. The Jacobson radical is the intersection of all maximal proper ideals, equivalently the maximal nilpotent ideal, equivalently in characteristic 0, the kernel of the Trace representation $\mathrm{Tr}(xy)$, where algebra elements are represented as matrices acting on a basis for the algebra. Modulo the Jacobson radical, the algebra is semisimple, which means it has no nilpotent ideals, and then that every element a is regular in the sense there exists x such that $axa = a$. A semisimple algebra is isomorphic to a direct sum of simple algebras; this decomposition is unique, and corresponds to the set of central idempotents of the algebra. We will not have to go into simple algebras here because they will be given as matrix algebras over algebraic number rings. However, simple finite-dimensional algebras must always be full matrix algebras over division rings.

We will apply the next proposition to integer matrices in $J_f^{-1} \, \mathrm{DGI}(A, A, \mathbb{Q}) J_f$ and the congruences ($\mathrm{i_g}$), ($\mathrm{ii_g}$), ($\mathrm{iv_g}$), and the determinant condition ($\mathrm{iii_g}$).

Note that we can write any Boolean combination of congruences on a single matrix variable x to various moduli in the form

(4.8) $\exists \, s \in S \ni x \equiv s \pmod{m}$

for a finite computable set S. In the application of Proposition 4.7, m can be taken as, say, the product of the $2N$th power of all denominators and determinants for A, B, M_c, M_f, p_a.

Proposition 4.7. *Let $\mathcal{A}$ be a finite-dimensional algebra of matrices over a commutative ring R in block-triangular form, and let J be its Jacobson radical consisting of matrices which have zero main-diagonal blocks. If we can solve any finite system of additive congruences on $\mathcal{A}/J$ subject to any restrictions on the determinant, then we can solve any finite system of additive congruences on $\mathcal{A}$ subject to any restrictions on the determinant. More generally we can restate the congruences on $\mathcal{A}$ as congruences on $\mathcal{A}/J$ and use the same determinant conditions.*

Proof. Note that for our matrix representation the norm conditions on $\mathcal{A}$ will give norm conditions on $\mathcal{A}/J$, since the latter gives the main-diagonal blocks in a block-triangular representation, and the product of their determinants is the determinant in $\mathcal{A}$. The condition that the determinant is a fixed algebraic integer F times products from a finite list of primes and units will translate into a finite list of similar conditions at each main diagonal block, based on the prime factorizations of F. Additively, write an element which is to have determinant involving certain primes, and satisfy congruences, as $x + j$ where j is in the Jacobson radical. The congruences will say, for some $j \in J$, a Boolean combination of congruences $x+j \equiv c \pmod{m}$ hold. If we take all possibilities j_0 for $j \pmod{m}$, this will be a Boolean combination of congruences $x \equiv c - j_0 \pmod{m}$. $\qquad\square$

Congruences on an element of a ring Ω modulo m will not be changed if we pass to an extension field (but require the element to belong in the original ring), and it will suffice to take congruences modulo the prime power factors of m in the new ring.

We make one further transformation of our congruences and determinant conditions. Since it is of the same nature as the previous changes except that we must use Lemma 4.1 and Proposition 4.7 in a way which is difficult to predict, we will not state the formulas explicitly but describe the changes. Using Proposition 4.7, we pass to congruences on the indecomposable blocks of the matrix representations. We use Lemma 4.1 and a further conjugation to pass to congruences over an algebraic number field on particular eigenspaces. This will result in congruences (i_h), (ii_h), (iv_h), and a determinant condition (iii_h). The conditions (ii), and so on, will bound the powers of all primes occurring in the determinant of $J(1)$, J_a, J_g at that eigenspace, except for those which divide the eigenvalue.

Proposition 4.8. *We may replace the congruences* (i_h), (ii_h), (iv_h) *by equivalent congruences in which the moduli for each eigenspace are relatively prime to the corresponding eigenvalue* π. *Moreover it suffices to find matrices satisfying these conditions which lie in* $\Omega[1/\pi]$.

Proof. We can eliminate this dependence and the denominators by multiplying by a power of the defining matrix A large enough to cancel off the denominators. That is, if we have a solution mapping J_c at a particular eigenspace which satisfies congruences for all primes except those which divide the eigenvalue λ, then $A^n J_a$ will produce a solution at all the other primes, which is congruent to zero modulo any set power of the primes in λ and exists over Ω. And if any solution does exist, multiplication by a large power of A must produce one which is congruent to zero modulo high powers of the primes in λ, hence one that can be found in this way.   □

Theorem 4.9. *There is an algorithm to decide isomorphism of stationary AF-algebras arising from primitive matrices.*

Proof. This result is a consequence of the preliminary discussion and the propositions above. That is, we first reduce the problem to one of finding a matrix $J(1)$ which preserves certain subspaces, has certain primes in its determinants, and satisfies congruences, going from A to B. We find such a matrix J_0 over the rational numbers; the proposed solution must differ from it by multiplying with a matrix J_a from A to B meeting corresponding conditions (we multiply by a constant N_c to arrange that J_a have integer entries). We find the Jacobson radical and the simple components of the quotient by it, and restate the congruences in terms of those simple components. They are determined in terms of certain combinations of eigenspaces, as general linear groups over algebraic number rings. Again we conjugate, and obtain a new set of finite congruences of the same general nature as the originals, (i_h), (ii_h), (iv_h), and a determinant condition (iii_h). We use Proposition 4.8 to ensure that the congruences involve moduli relatively prime to the eigenvalues and can allow these eigenvalues as denominators. By Lemma 4.1, we can solve them.   □

5. THE CASE $\lambda = |\det (A)|$

Theorem 4.9 gives, in principle, a finite algorithm to decide whether two square, nonsingular, integer primitive matrices A, B are C^*-equivalent or not. In special cases, like those considered in [BJO99], this algorithm can be substantially simplified. One nice feature of the algorithm is that it uses only "elementary" algebraic results, and avoids using the deep results on decidability from [GrSe80a, GrSe80b].

Nevertheless, the implementation of the algorithm for general pairs A, B may of course be complicated. Let us pick up and generalize one special case from [BJO99]. In Theorem 17.18 and Corollary 17.21 there, it was proved that if A, B had a special form, and $\lambda_{(A)} = |\det(A)|$ and $\lambda_{(B)} = |\det(B)|$, then the ideal generated by $\langle v(A) | w(A) \rangle$ in $\mathbb{Z}[1/\det(A)]$ is a complete invariant, if the left and right Perron–Frobenius eigenvectors are taken to have integer components, and $\gcd(v(A)) = 1$, $\gcd(w(A)) = 1$, where gcd denotes the greatest common divisor of the components. We now prove that this is also true for more general matrices A, B.

In stating this more general result, there is a technical complication. In picking extension fields F and an associated ring R of algebraic integers, it is not automatically true that the ideals in R are principal. But by a result in [Wei98] or [Ser79], there is always a finite extension E of F in which the associated ideals are automatically principal. We refer to this in the statement of the proposition. To further simplify the terminology in the statement of the proposition we denote the above-mentioned respective Perron–Frobenius column vectors w, w', i.e., $Aw = \lambda w$ and $Bw' = \lambda'w'$, and similarly v, v' for the two respective Perron–Frobenius row vectors.

Proposition 5.1. *Choose a finite extension E of the algebraic number field F of the eigenvalues of primitive nonsingular integer matrices A, B in which all ideals of F become principal and consider primes in it.*

(i) *An isomorphism J on ordered dimension groups from the dimension group of A to that of B sends the row Perron–Frobenius eigenvector v' (normalized so all entries are algebraic integers with $\gcd 1$) of B to a multiple c times the row Perron–Frobenius eigenvector v of A.*

(ii) *The two Perron–Frobenius eigenvalues generate the same algebraic number field and involve the same primes of that field.*

(iii) *If the Perron–Frobenius eigenvalue λ as compared with any other eigenvalue of either matrix A, B, is divisible by some algebraic prime not in the other, then the Perron–Frobenius column eigenvector w is mapped to a multiple ξ times the other Perron–Frobenius eigenvector w'.*

(iv) *The former coefficient c involves only primes dividing λ.*

(v) *The latter coefficient ξ involves only the primes in the Perron–Frobenius eigenvalue. If we ignore normalization and work directly with images of eigenvectors, then the inner products are equal: $v'Jw' = cvw' = (c/\xi)\,vJw$. Therefore the inner product of the normalized Perron–Frobenius eigenvectors is thus an invariant up to units in the algebraic number ring generated by $1/\lambda$.*

Proof. The first assertion is by [BJKR98, Theorem 6], and the second is by [BJKR98, Theorem 10].

The third assertion follows because the space of vectors in the dimension group such that some fixed multiple is arbitrarily divisible by a given prime is sent to the corresponding subspace of the other dimension group, and this set is the sum of the eigenspaces for all eigenvalues divisible by the prime. If these spaces are intersected over all primes dividing the Perron–Frobenius eigenvalue, we get, by our hypothesis, only the Perron–Frobenius eigenspace.

The fourth assertion follows because the eigenspace of v will consist precisely of those vectors in the dimension group which are divisible by arbitrary powers of primes occurring only in λ, so it must be preserved by any isomorphism of

dimension groups. In addition, vectors in this 1-dimensional space which are not divisible by primes other than those in λ will be unique up to multiplication by units and primes dividing λ, so they will be preserved by any isomorphism, up to such multiplication.

The fifth assertion follows by [BJKR98, Theorem 7], at all primes not in λ the isomorphism induces an isomorphism on eventual row spaces, and the image of the Perron–Frobenius eigenvector (times any power of λ) will be nonzero modulo such a prime. □

The following is a partial converse to Proposition 5.1.

Corollary 5.2. *Suppose A, B are nonsingular primitive integer matrices, their Perron–Frobenius eigenvalues are integers and that the inner products as above are equal, the primes dividing the Perron–Frobenius eigenvalues are equal, and the dimensions of the matrices are at least 3. Suppose the Perron–Frobenius eigenvalues are the determinants of A, B up to sign. Then there exists an isomorphism between the ordered dimension groups of A and B.*

Proof. By Lemma 17.19 of [BJO99], there is a unimodular matrix J sending the Perron-Frobenius row eigenvector of A to the Perron-Frobenius row eigenvector of B and the Perron-Frobenius column eigenvector of A to the Perron-Frobenius column eigenvector of B (and we can choose signs for positivity). By [BJKR98, Theorem 6] this gives a positive mapping on dimension groups. Since the row eigenvectors are perpendicular to the sum $V(A)$ of all non-Perron-Frobenius eigenspaces, $JV(A) = V(B)$, and also $v(B)J \subseteq \mathbb{Q}v(A)$ as noted in Section 1. Write any vector v as a direct sum according to (1.20), $v = x + y$. This splitting can introduce certain fixed primes p in the denominator.

Note that the matrix A is unimodular and integer restricted to the integer vectors in $V(A)$ (and similarly for the matrix B), because each determinant is the product of its determinant on this space and its determinant on the Perron–Frobenius eigenspace, and because it is an integer matrix preserving this subspace. Multiplication by A is multiplication by λ on x (see Figure 1), and the same is true for B.

$$
A \sim \begin{pmatrix}
* & \cdots & * & 0 \\
\vdots & \begin{array}{c} A|_{V(A)} \\ \text{with} \\ |\det(A|_{V(A)})| = 1 \end{array} & \vdots & \vdots \\
* & \cdots & * & 0 \\
\hline
0 & \cdots & 0 & \begin{array}{c} \lambda_{(A)} \\ = |\det A| \end{array}
\end{pmatrix}
\begin{array}{l} \\ V(A) \\ \\ \\ \mathbb{C}w(A) \end{array}
$$

FIGURE 1. The case $|\det A| = \lambda_{(A)}$: Decomposition relative to (1.20) and unimodular restriction.

For v to be in the dimension group means for all sufficiently large n, $A^n v$ has integer entries. Any prime p which does not divide λ will not occur in the denominator of the expression $B^m J A^{-n}(x + y)$.

Consider those primes p which divide λ. We claim that they cannot occur in denominators of y. Restricted to vectors y, the matrix A is unimodular, so modulo any powers of those primes it lies in a finite group, $\mathrm{GL}(N, \mathbb{Z}_{p^s})$. Thus we can choose arbitrarily large n so that A^n is congruent to the identity. But then in $A^n(x + y)$, the denominators in x have vanished, being multiplied by λ^n and those in y remain. So $x + y$ is not in the dimension group, a contradiction.

Therefore in

$$
(5.1) \qquad
\begin{aligned}
B^m J A^{-n}(x + y) &= B^m J A^{-n} x + B^m J A^{-n} y \\
&= J \lambda'^m \lambda^{-n} x + B^m J A^{-n} y
\end{aligned}
$$

both terms are integer for sufficiently large m (with a symmetrical argument the other way) which verifies the conditions in Section 1 for isomorphism of ordered dimension groups. □

6. The case of no infinitesimal elements and the case of rational eigenvalues

In this section we will consider the C^*-equivalence problem in two extreme cases. To describe these two cases, let us recall some facts from [Eff81], [BJO99]. We define a functional τ_A on $G(A)$ by the formula

$$
(6.1) \qquad \tau_A(g) = \langle v(A) \,|\, g \rangle, \qquad g \in G(A),
$$

where $v(A)$ is a left Perron–Frobenius eigenvector for A. This functional τ_A is called "the" trace since it defines a trace on the corresponding C^*-algebra. It follows from the eigenvalue equation that $v(A)$ can be taken to have components in the field $\mathbb{Q}[\lambda] = \mathbb{Q}[1/\lambda]$, where λ is the Perron–Frobenius eigenvalue. But multiplying $v(A)$ by a positive integer, we may assume that the components of $v(A)$ are contained in the ring $\mathbb{Z}[1/\lambda]$. It then follows from (1.8), and $v(A) A^{-n} = \lambda^{-n} v(A)$, that

$$
(6.2) \qquad \tau_A(G(A)) \subset \mathbb{Z}[1/\lambda].
$$

Furthermore, $\tau_A(G(A))$ is invariant under multiplication by elements of $\mathbb{Z}$ and by $1/\lambda$, so it is a $\mathbb{Z}[1/\lambda]$-module. In particular, $\tau_A(G(A))$ is an ideal in the ring $\mathbb{Z}[1/\lambda]$. We need only verify that $\frac{1}{\lambda}\tau_A(g)$ is in $\mathrm{ran}(\tau_A)$ for all $g \in G(A)$. Pick $g \in G(A)$, and set $g = A^{-n}m$, $n \in \mathbb{N}$, $m \in \mathbb{Z}^N$. Then $\frac{1}{\lambda}\tau_A(g) = \langle v(A) A^{-1} \,|\, g \rangle = \langle v(A) \,|\, A^{-1}g \rangle = \tau_A(A^{-(n+1)}m) \in \mathrm{ran}(\tau_A)$ as claimed. This is a very special feature of the constant-incidence-matrix situation which is not shared by the range of a trace on a general dimension group of general AF-algebras. This range is not even closed under multiplication in the general case when the incidence matrix is not assumed constant. We have the natural short exact sequence of groups

$$
(6.3) \qquad 0 \longrightarrow \ker(\tau_A) \lhook\joinrel\longrightarrow G(A) \xrightarrow{\tau_A} \tau_A(G(A)) \longrightarrow 0
$$

and the order isomorphism

$$
(6.4) \qquad G(A) / \ker(\tau_A) \xrightarrow[\approx]{\tau_A} \mathrm{ran}(\tau_A) \subset \mathbb{Z}[1/\lambda_{(A)}],
$$

where $\mathrm{ran}(\tau_A)$ inherits the natural order from $\mathbb{Z}[1/\lambda]$. Note that for the particular matrices we considered in [BJO99], we had

$$
(6.5) \qquad \mathrm{ran}(\tau_A) = \mathbb{Z}[1/\lambda]
$$

(see [BJO99, (5.21)–(5.22)]), but be warned that this is not a general feature. This will be discussed further in Remarks 7.5 and 7.7. Chapter 5 in [BMT87] also has a nice treatment of ran (τ_A) in the general case. Let us already at this point state and prove the remarkable fact that *any* subset I of $\mathbb{Q}[\lambda]$ which is an ideal over $\mathbb{Z}[1/\lambda]$ occurs as the image of the trace for a suitable primitive nonsingular matrix A (this is a version of [BMT87, Corollary 5.15] which is a consequence of results of Handelman, see [Han81]):

Proposition 6.1. *Let λ be a real algebraic integer larger than the absolute value of any of its conjugates, and let $I \subset \mathbb{Q}[\lambda]$ be an ideal over $\mathbb{Z}[1/\lambda]$. Then I can occur as the image of the trace for some matrix whose Perron–Frobenius eigenvalue is a power of λ (the size of the matrix will be the degree $\mathbb{Q}[\lambda]/\mathbb{Q}$).*

Proof. Let $I_1 = I \cap \mathbb{Z}[\lambda]$; it will be a $\mathbb{Z}[\lambda]$-ideal which spans I over $\mathbb{Z}[1/\lambda]$.

Now define an integer matrix M which expresses the action of λ on I_2, that is, form an additive basis w_i for I_1, let $\lambda w_i = \sum_j m_{ij} w_j$, $m_{ij} \in \mathbb{Z}$. This matrix will have an eigenvalue λ, and we claim that at the corresponding eigenspace, the image of the trace is isomorphic to I. This is because the action of M on $\mathbb{Z}^N$ has been forced to be that of λ on I_1, and because the trace reflects this module structure, by means of the short (nearly exact) sequence.

Finally we claim that we can conjugate M over $\mathrm{GL}(N, \mathbb{Z})$ to a matrix whose powers are eventually positive; then those powers will be nonnegative matrices whose image of trace is the same. To get eventual positivity, given that λ is the largest eigenvalue (the largest of its Galois conjugates), it is necessary and sufficient that its row and column eigenvectors for this eigenvalue be positive, by a limit argument somewhat like that in Proposition 1.1. Let v, w be row and column eigenvectors at the eigenvalue λ, with signs chosen so that their inner product is positive. Multiply each by a large integer, and then take relatively prime integers approximating its components. Such a pair of vectors can be mapped over $\mathrm{GL}(N, \mathbb{Z})$ to any vectors whose entries are relatively prime integers having the same inner product, by [BJO99, Lemma 17.19], in particular, to ones which are positive, if $N > 2$. If $N = 2$ we use the same result and get a congruence condition, but that is compatible with positivity.   □

Remark 6.2. The quotient of the ring $\mathbb{Z}[1/\lambda]$ by any of these ideals will be finite. The ideal can be lifted to an ideal inside the rank N additive group $\mathbb{Z}[\lambda]$, and the quotient of two rank-N free abelian groups is finite—its order is given by the determinant of the map expressing the inclusion.

Let us return to the two special cases of C^*-equivalence we shall discuss in this section. These are the following.

(i) The kernel $\ker(\tau_A)$ is 0, i.e., $G(A)$ has no infinitesimal elements, i.e., the characteristic polynomial of A is irreducible over $\mathbb{Z}$ (equivalent: over $\mathbb{Q}$).

(ii) All the eigenvalues of A are rational (thus integer), each of them is relatively prime to the rest, and none is equal to ± 1.

In Sections 7 and 10 we will apply this to many examples. See, for example, Example 7.9 for an application in the situation (ii) above.

Theorem 6.3. *Two primitive $N \times N$ matrices A, B over $\mathbb{Z}_+$ with irreducible characteristic polynomials are C^*-equivalent if and only if the following three conditions all hold:*

(i) *the roots of their characteristic polynomials generate the same field,*

(ii) *their Perron–Frobenius eigenvalues are divisible by the same algebraic primes, and*

(iii) *their dimension groups, as modules over $\mathbb{Z}[1/\lambda]$ (or a full-rank subring), are isomorphic. These modules are isomorphic to the fractional ideals given by the image of the trace τ.*

Moreover, these three conditions are equivalent to the one condition:

(iv) *the two ordered additive subgroups in $\mathbb{Z}[1/\lambda]$ defined by the ranges of the respective traces are isomorphic.*

If in addition the characteristic polynomials of A, B are equal, then C^*-equivalence (isomorphism of ordered dimension groups) is the same as shift equivalence.

Note that taking powers of the matrix will preserve the $\mathbb{Z}[1/\lambda]$-module mentioned in (iii), i.e., the ideal in $\mathbb{Z}[1/\lambda]$, and not replace it by its powers.

Remark 6.4. To say that the dimension groups $G(A)$ and $G(B)$ as modules over $\mathbb{Z}[1/\lambda]$ are isomorphic means that there is an isomorphism $\varphi \colon G(A) \to G(B)$ of abelian groups such that

$$(6.6) \qquad \varphi(\omega g) = \omega\varphi(g)$$

for all $g \in G(A)$, $\omega \in \mathbb{Z}[1/\lambda]$. This is not the same as saying that $G(A)$ is isomorphic to $G(B)$ as ideals in $\mathbb{Z}[1/\lambda]$. The latter concept means that there is an automorphism φ of the ring $\mathbb{Z}[1/\lambda]$ such that $\varphi(G(A)) = G(B)$. When we talk about equivalence of ideals it is the *first* concept we are thinking about, i.e., there is an element of the quotient field $\mathbb{Q}[1/\lambda] = \mathbb{Q}[\lambda]$ mapping the one ideal into the other by multiplication.

Proof of Theorem 6.3. The first statement follows from [BJKR98, Proposition 10]. The properties (1.10) and

$$(6.7) \qquad \tau_A \circ A^{-1} = \lambda^{-1}\tau_A$$

imply that the image of the trace is a module over $\mathbb{Z}[1/\lambda]$ and a subset of $\mathbb{Z}[1/\lambda]$ and that it is (using the standard basis for $\mathbb{Z}^N$) generated by $\langle v \,|\, e_i \rangle = v_i$ as a module over $\mathbb{Z}[1/\lambda]$. The trace mapping is an epimorphism over the rationals. Hence its kernel is zero and it maps between the dimension groups isomorphically. So it gives the same ideal considered in [BJKR98, Proposition 10]. To prove shift equivalence assuming equality of characteristic polynomials, it will suffice to show that the matrices themselves represent the same field element acting on this module. Because they are roots of the same irreducible characteristic polynomials, they must be Galois conjugates. But in terms of positivity, they both represent the Perron–Frobenius eigenvalue, so that we can say they are equal.

Equivalence to (iv): First assume (iv). Then the images of the traces generate the fields $\mathbb{Q}[\lambda_{(A)}] = \mathbb{Q}[\lambda_{(B)}]$. In case the rings $\mathbb{Z}[\lambda_{(A)}]$ and $\mathbb{Z}[\lambda_{(B)}]$ are different, the isomorphism of modules in Proposition 10 of [BJKR98] is to be understood in the sense of modules over the intersection ring $\mathbb{Z}[\lambda_{(A)}] \cap \mathbb{Z}[\lambda_{(B)}]$, which will have additive rank N since each of the two rings has rank N. Note that the module structure over such a subring determines the module structure over any larger subring of $\mathbb{Q}[\lambda_{(A)}]$ where a module structure exists, by linear extension. These images of the traces are isomorphic to the dimension groups $G(A)$, $G(B)$, and the ring $\mathbb{Z}[\lambda_{(A)}] \cap \mathbb{Z}[\lambda_{(B)}]$ will at least be a subring of $\{x \in \mathbb{Q}(\lambda) \mid xG(A) \subset G(A), \, xG(B) \subset G(B)\}$

so that the dimension groups are isomorphic as modules over this ring. Isomorphism implies the prime divisibility condition, as the primes will be those which can occur to arbitrary powers in the denominators.

Conversely, suppose we are given (i), (ii), (iii). The equality of fields asserted in Proposition 10 of [BJKR98], which could be clarified as "equality of $\mathbb{Q}\left[\lambda_{(A)}\right]$ and $\mathbb{Q}\left[\lambda_{(B)}\right]$ as subfields of the real numbers", gives embeddings of $\mathbb{Z}\left[\lambda_{(A)}\right]$ and $\mathbb{Z}\left[\lambda_{(B)}\right]$ into the real numbers. The modules are embedded in $\mathbb{Q}\left[\lambda_{(A)}\right]$ (they are isomorphic to ideals over subrings), so they have natural corresponding embeddings in the real numbers. The isomorphism of modules as additive groups acted on by subrings of $\mathbb{Q}\left[\lambda_{(A)}\right]$ having full rank additively will mean there is some element of the quotient field mapping one to the other: this element a ratio of corresponding elements under the isomorphism, which by the module isomorphism is the same for any two elements which correspond under the isomorphism. This is the same sense in which the trace images are isomorphic.   □

Note that this applies in particular to Example 7.8 below.

Theorem 6.5. *Let A, B be matrices over $\mathbb{Z}_+$, all of whose eigenvalues are rational, and each of which is relatively prime to the rest, and none is one. Assume A, B have the same characteristic polynomial. Let E_a, E_b be their matrices of column eigenvectors normalized to be integer vectors having greatest common divisor 1. Let D be a diagonal matrix whose entries involve only powers of primes in the respective eigenvalues, let D_s be a diagonal matrix consisting of precisely the diagonal eigenvalues. Then the following are equivalent:*

   (i) *A and B are C^*-equivalent;*
   (ii) *A and B are shift equivalent, as follows: for some choice of signs in E_a, E_b, and some choice of D, and for all sufficiently large n, $E_a D D_s^n E_b^{-1}$ and $E_b D^{-1} D_s^n E_a^{-1}$ are integer matrices. .*

Proof. Consider an isomorphism of dimension groups. The eigenvectors generate the 1-dimensional spaces of vectors such that some multiples of those vectors are in the dimension group and are divisible by arbitrary powers of the respective eigenvalues. Hence any dimension group isomorphism must preserve those subspaces. Moreover we claim a dimension group isomorphism must send normalized eigenvectors to one another, that the rational multiples of a normalized eigenvector v with eigenvalue η which lie in the dimension group are $\{(n/m)v \mid n \in \mathbb{Z},\ m|\eta^j,\ j \in \mathbb{Z}_+\}$. These will lie in it, obtained from integer vectors by negative powers of the matrix. And if $w \in A^{-n}\mathbb{Z}^N$ is in this subspace then $\eta^n w \in \mathbb{Z}^N$ so it is a multiple of v. It follows that dimension group isomorphism implies isomorphism of $\mathbb{Q}^N$ which sends each eigenvector to a multiple of the other eigenvector by a number which divides a power of η. Such a mapping must preserve the action of multiplication by A, given the characteristic polynomials are equal, because this multiplies each eigenvector by its eigenvalue, and the eigenvalues are the same. So it will be a shift equivalence. Let D be the diagonal matrix whose main diagonal entries are the multiples just mentioned. Then the isomorphism of dimension groups will be, specifically, $E_b(E_a D)^{-1}$ if it exists. It must be, if we multiply it and its inverse on the left by a large power of A, B respectively that they exist over the integers. And these multiples are the matrices stated in the theorem.   □

7. THE TRANSPOSE MAP AND C^*-SYMMETRY

In this section we will study the behavior of the dimension group $\left(G\left(A\right),G\left(A\right)_+\right)$ under the transpose map $A \to A^{\mathrm{tr}}$. In particular, we say that A is C^*-symmetric if A is C^*-equivalent to A^{tr}, i.e., $G\left(A\right)$ and $G\left(A^{\mathrm{tr}}\right)$ are isomorphic as ordered groups. We give several examples showing that A may be C^*-symmetric, (7.2), Remark 7.5, or not, Example 7.6 (2×2 matrices with rational eigenvalues), Example 7.8 (2×2 matrices with irreducible eigenvalues) and Example 7.9. An interesting feature with these particular examples is that when A is a 2×2 matrix, then C^*-symmetry is equivalent to shift-symmetry (i.e., A and A^{tr} are shift equivalent). For 2×2 matrices, symmetry seems to be more common than non-symmetry. Our first example, while very simple, illustrates both C^*-symmetry and a nontrivial Ext-element. It has $\lambda = \lambda_{(A)} = 2$. The Ext-group represents another contrast between the two cases, λ rational (and hence integral), and λ irrational. In the first case, we generally have $\ker\left(\tau_A\right) \neq 0$, and as we note in Remark 7.5, $\mathrm{ran}\left(\tau_A\right) = \mathbb{Z}\left[1/\lambda\right]$. Hence this extra extension structure for $G\left(A\right)$ arises only in the rational case: The corresponding short exact sequence

$$(7.1) \qquad 0 \longrightarrow \ker\left(\tau_A\right) \longrightarrow G\left(A\right) \xrightarrow{\tau_A} \mathbb{Z}\left[1/\lambda\right] \longrightarrow 0$$

may be non-split, which means that $G\left(A\right)$ is then not the direct sum of the two groups $\ker\left(\tau_A\right)$ and $\mathbb{Z}\left[1/\lambda\right]$.

Example 7.1. The dimension group defined by A may be order isomorphic to that defined by its transpose $B = A^{\mathrm{tr}}$. Hence an AF-C^*-algebra built on such a matrix A (i.e., from the corresponding stationary Bratteli diagram) has a nontrivial period-two symmetry corresponding to $A \mapsto A^{\mathrm{tr}}$. An example here is

$$(7.2) \qquad A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \qquad A^{\mathrm{tr}} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

In this case A and A^{tr} have eigenvalues 2 and -1, and both of the dimension groups G and G^{tr} are in $\mathrm{Ext}\left(\mathbb{Z}\left[1/2\right],\mathbb{Z}\right)$. It can be checked (by use of [BJO99, Corollary 11.28]) that this Ext-element is not zero. Here $\ker\left(\tau\right) = \mathbb{Z}$, $\mathrm{ran}\left(\tau\right) = \mathbb{Z}\left[1/2\right]$, and the corresponding short exact sequence

$$(7.3) \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow G\left(A\right) \xrightarrow{\tau} \mathbb{Z}\left[1/2\right] \longrightarrow 0$$

does not split. Equivalently, $G\left(A\right)$ is not $\mathbb{Z} \oplus \mathbb{Z}\left[1/2\right]$ as a group. If it were, we would get $\tau\left(w\right)^{-1} \in \mathbb{Z}\left[1/2\right]$ by [BJO99, Corollary 11.28]. But we computed $\tau\left(w\right) = 3$, and $1/3$ is not in $\mathbb{Z}\left[1/2\right]$. Since $\lambda_{(A)} = 2 = |\det A|$, it is tempting to apply Theorem 5.2. In fact the inner-product invariants are $\langle v\,|\,w\rangle = \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3$, and $\langle v'\,|\,w'\rangle = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3$. But since the dimension is 2 (< 3), Theorem 5.2 does not apply directly, and instead we will verify directly that A and A^{tr} are C^*-equivalent. Define matrices J, K by

$$(7.4) \qquad J = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

One verifies that

$$(7.5) \qquad A = KJ, \qquad A^{\mathrm{tr}} = JK.$$

Thus A and A^{tr} are elementary shift equivalent, and it follows that they are shift equivalent and C^*-equivalent (see the discussion in [BJKR98]).

However, we will see in Examples 7.9 and 10.4 that this is not a general feature of the transpose map.

We may analyze the C^*-symmetry question by dimension-group analysis: If we show that the ordered group $G(A)$ is order isomorphic to $G(A^{\mathrm{tr}})$, then A is C^*-equivalent to A^{tr}, i.e., A is C^*-symmetric. Clearly then the two groups $G(A)$ and $\operatorname{ran}(\tau_A)$ are order isomorphic whenever $\ker(\tau_A) = 0$, and we have the result:

Proposition 7.2. *Let $A \in M_N(\mathbb{Z})$ be nonsingular and primitive, and suppose its characteristic polynomial $p_{(A)}(x)$ is irreducible, and $\operatorname{ran}(\tau_A) = \operatorname{ran}(\tau_{A^{\mathrm{tr}}})$: then A is C^*-equivalent to A^{tr}. Note in particular that this holds if:*

(i) $N = 2$,
(ii) *the Perron–Frobenius eigenvalue $\lambda_{(A)}$ is irrational, and*
(iii) $\operatorname{ran}(\tau_A) = \operatorname{ran}(\tau_{A^{\mathrm{tr}}})$.

Proof. This follows directly from Theorem 6.3.   □

Remark 7.3. We saw that by scaling out denominators in the entries v_i of the left (row) Perron–Frobenius eigenvector $v(A) = (v_1, \ldots, v_N)$ we can arrange that $v_i \in \mathbb{Z}[1/\lambda]$ for all i. But then a further scaling with a power of λ we can get each v_i in the subring $\mathbb{Z}[\lambda] \subset \mathbb{Z}[1/\lambda]$. Suppose that the characteristic polynomial of A is irreducible. Note that, as a group, $\mathbb{Z}[\lambda]$ is then a copy of the lattice $\mathbb{Z}^N$ so the entries v_i may therefore be viewed as vectors in $\mathbb{Z}^N$. Then pick $v(A)$ such that $\gcd(v_i) = 1$ for each i. In this case the matrix V with the v_i's as rows is in $M_N(\mathbb{Z})$ and is nonsingular. The number $|\det V|$ is the index of $\tau(G(A))$ in $\mathbb{Z}[1/\lambda]$ where $\lambda = \lambda_{(A)}$. If we could define greatest common divisors in the ring $\mathbb{Z}[\lambda]$ then we could divide v by this greatest common divisor and obtain some new v defined over $\mathbb{Z}[\lambda]$ which has g.c.d. 1. Then the image of its trace would contain the span of its coordinates v_i over $\mathbb{Z}[\lambda]$, that is, the entire ring $\mathbb{Z}[\lambda]$. Moreover the image of the trace will be contained in this ring, so they are equal. In general, however, this ring will not be a principal ideal domain, so that the class of the ideal generated by the trace becomes an obstruction. In fact, the subgroup in $\mathbb{Z}[\lambda]$ which is generated by the v_i's is also an ideal in $\mathbb{Z}[\lambda]$. Indeed, for $m \in \mathbb{Z}^N$, $\sum_i m_i v_i = \tau(m) = \langle v \mid m \rangle$, so $\lambda \sum_i m_i v_i = \langle vA \mid m \rangle = \langle v \mid A^{\mathrm{tr}} m \rangle$, and $A^{\mathrm{tr}} m \in \mathbb{Z}^N$. As a consequence, we get that the special incidence matrices A which we considered in [BJO99] satisfy the condition $\operatorname{ran}(\tau_A) = \mathbb{Z}[1/\lambda_{(A)}]$. However, this fails for the matrix A from Example 7.8, and others.

Proposition 7.4. *Assume that the Perron–Frobenius row eigenvector v is chosen to lie in $\mathbb{Z}^N[\lambda]$. Then the inclusion map is an epimorphism and its kernel is those elements annihilated by λ^n for sufficiently large n. That is, the map $\lambda^n \mathbb{Z}[\lambda]/\tau(\mathbb{Z}^N) \to \mathbb{Z}[1/\lambda]/\tau(G)$ is an isomorphism.*

Proof. The inclusion gives a natural mapping. If we multiply any element in $\mathbb{Z}[1/\lambda]$ by a power of power of λ, we can get an element of $\mathbb{Z}[\lambda]$, so this mapping is an epimorphism. We also claim that if we multiply any element of $\tau(G)$, say $vA^{-n}x$, $x \in \mathbb{Z}^N$ by a power of λ, we will get an element of $\tau(\mathbb{Z}^N)$. This is because $vA^{-n}x = v\lambda^{-n}x$ using the left two factors.

Note that since $\tau(G)$ and $\mathbb{Z}[1/\lambda]$ are both torsion-free and λ-divisible, their quotient has no λ-torsion. Hence every element annihilated by a power of λ lies in the kernel.

Let y be in the kernel of this mapping. Then $y \in \tau(G)$, so that for some $n \in \mathbb{Z}_+$, $\lambda^n y \in \tau(\mathbb{Z}^N)$ and is zero in the original group. This identifies the quotient. The left hand group, the quotient of a free abelian group by a full rank subgroup, is finite, so some fixed n works for the whole kernel. $\square$

Remark 7.5 (Rational λ). Even if $N = 2$, the dimension group $G(A)$ is not yet completely understood [BJO99] (perhaps far from it!). If $\lambda = \lambda_{(A)}$ is rational, and therefore an integer, we can have nonisomorphic $G(A_1)$ and $G(A_2)$ even when A_1 and A_2 have the same characteristic polynomial and thus the same Perron–Frobenius eigenvalue λ, as different extensions, $i = 1, 2$,

$$(7.6) \qquad\qquad 0 \longrightarrow \mathbb{Z}[1/\mu] \longhookrightarrow G(A_i) \overset{\tau}{\longrightarrow} \mathbb{Z}[1/\lambda] \longrightarrow 0,$$

i.e., as different elements of the group $\mathrm{Ext}(\mathbb{Z}[1/\lambda], \mathbb{Z}[1/\mu])$. Here μ is the other root of the characteristic polynomial, so μ is a nonzero integer with $|\mu| < \lambda$. See (6.3) and (6.4). This may even happen when A_2 is the transpose of the matrix A_1, by Example 7.6 below. Since here λ is rational one may arrange that $\tau_A(G(A)) = \mathbb{Z}[1/\lambda]$ by choosing v with $\gcd(v) = 1$, and $\ker(\tau_A)$ is a rank-1 nonzero group isomorphic to $\mathbb{Z}[1/\mu]$; see also below. There are specimens of 2×2 primitive matrices A, even with integral Perron–Frobenius eigenvalue such that $\lambda_{(A)} < |\det A|$, and yet the two groups $G(A)$ and $G(A^{\mathrm{tr}})$ are order isomorphic. For example, $A = \left(\begin{smallmatrix} 1 & 5 \\ 3 & 3 \end{smallmatrix}\right)$ has that property. To see this, we may use (1.16)–(1.17). Since $J = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ satisfies $JA = A^{\mathrm{tr}}J$, the two conditions hold, and hence the matrix A is C^*-symmetric. So for this particular pair A, A^{tr}, the respective groups $G(A)$ and $G(A^{\mathrm{tr}})$ from the middle term in the diagram (7.6) will then in fact represent the same zero element of $\mathrm{Ext}(\mathbb{Z}[1/6], \mathbb{Z}[1/2])$. For this particular A,

$$(7.7) \qquad\qquad G(A) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/6] \qquad (\ker(\tau) \cong \mathbb{Z}[1/2])$$

as direct sum of abelian groups. For this, note that the integral column eigenvectors for A are $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 5 \\ -3 \end{smallmatrix}\right)$. Since $\det\left(\begin{smallmatrix} 1 & 5 \\ 1 & -3 \end{smallmatrix}\right) = -8 = -2^3$ and the eigenvalues of A are $6 = 2 \cdot 3$ and -2, we have $\mathbb{Z}[1/2]^2 \subset G(A)$. Thus $G(A) = \bigcup_{n=0}^{\infty} A^{-n}(\mathbb{Z}[1/2]^2)$, and (7.7) follows. Specifically, the representation (7.7) may be derived from (1.20), (6.1), and the two identities

$$(7.8) \qquad\qquad \ker(\tau_A) = V(A) \cap G(A) = \mathbb{Z}[1/2]\begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

and

$$(7.9) \qquad\qquad G(A) \cap \mathbb{C}w(A) = \mathbb{Z}[1/6]\, w(A),$$

where $w(A) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. The present computation of $G(A)$ is simplified by the fact that the orthogonal complement of the trace vector $v(A) = (3, 5)$ is spanned by the nonmaximal column eigenvector. Here $\lambda_{(A)} = 6$, and so $\mathbb{Z}[\lambda_{(A)}] = \mathbb{Z}$. That $\tau(G(A)) = \mathbb{Z}[1/6]$ in this case follows from Remark 7.3 and the general observation that with our choice of $v(A)$, we will have $\tau(G(A)) = \mathbb{Z}[1/\lambda_{(A)}]$ provided the ideal in $\mathbb{Z}[\lambda_{(A)}]$ generated by the $v_i(A)$ entries is principal. Ideals in $\mathbb{Z}$ are principal, of course. Here in this case the Ext-element corresponding to $G(A)$ is trivial. (Looking at prime factors in $\det A$, one could also get a $G(A)$ which is non-split. For example, taking $A = \left(\begin{smallmatrix} 1 & 6 \\ 2 & 2 \end{smallmatrix}\right)$, we get the spectrum $\{5, -2\}$ and that

the corresponding dimension group $G(A)$ is here represented by a nonzero element of $\mathrm{Ext}\,(\mathbb{Z}\,[1/5]\,,\mathbb{Z}\,[1/2])$. The analysis here is analogous to that presented above: We get $\ker(\tau_A) \cong \mathbb{Z}\,[1/2]$, $\mathrm{ran}\,(\tau_A) \cong \mathbb{Z}\,[1/5]$, and the corresponding short exact sequence

$$(7.10) \qquad\qquad 0 \longrightarrow \mathbb{Z}\,[1/2] \longrightarrow G(A) \longrightarrow \mathbb{Z}\,[1/5] \longrightarrow 0$$

is now non-split. The example $A = \left(\begin{smallmatrix} 1 & 6 \\ 2 & 2 \end{smallmatrix}\right)$ is C^*-symmetric, as A and A^{tr} are in fact shift equivalent: Take $R = \left(\begin{smallmatrix} -1 & 2 \\ 2 & -2 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 7 & 4 \\ 4 & 3 \end{smallmatrix}\right)$. Then $RS = A^{\mathrm{tr}}$ and $SR = A$. It follows from [BJO99] that $G(A)$, when represented in the Ext-group, is generally not the zero element.)

Example 7.6. Here we will exhibit a primitive nonsingular 2×2 matrix A with rational eigenvalues such that A is not C^*-equivalent to A^{tr} (and thus is not shift equivalent to A^{tr}). The respective dimension groups $G(A)$ and $G(A^{\mathrm{tr}})$ are not even isomorphic as groups, let alone order isomorphic, and hence this A in (7.11) is "more" nonsymmetric then the corresponding specimen (7.14) in Example 7.8. The example is

$$(7.11) \qquad\qquad A = \begin{pmatrix} 65 & 7 \\ 24 & 67 \end{pmatrix}.$$

Putting

$$(7.12) \qquad E_A = \begin{pmatrix} -7 & 1 \\ 12 & 2 \end{pmatrix}, \qquad E_B = \begin{pmatrix} -2 & 12 \\ 1 & 7 \end{pmatrix}, \qquad D = \begin{pmatrix} 53 & 0 \\ 0 & 79 \end{pmatrix},$$

we have

$$(7.13) \qquad\qquad A = E_A D E_A^{-1}, \qquad B = A^{\mathrm{tr}} = E_B D E_B^{-1}.$$

The eigenvalues of A and B are 53 and 79, which are both prime and congruent to $-1 \mod 13$. Using Theorem 6.5 it follows that if A and B were C^*-equivalent there would exist some diagonal matrix $D_0 = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ where x, y are congruent to ± 1 mod 13 such that $E_A D_0 E_B^{-1}$ would have integral entries. But the $(1,1)$ entry of this matrix is $(49x + y)/26$. If this is an integer, and $x = \varepsilon_1 + n_1 \cdot 13$, $y = \varepsilon_2 + n_2 \cdot 13$, where n_1, n_2 are integers and $\varepsilon_i = \pm 1$, then $\frac{1}{13}(49x + y) = \frac{1}{13}((4 \cdot 13 - 3)x + y) = -\varepsilon_1 \frac{3}{13} + \varepsilon_2 \frac{1}{13} \mod 1$, but this can never be an integer. Thus A is not C^*-symmetric.

Remark 7.7 (Irrational λ). The assumption in Proposition 7.2 that the range of the respective traces τ_A and $\tau_{A^{\mathrm{tr}}}$ be the same (viewed as subgroups of $\mathbb{Z}\,[1/\lambda_{(A)}]$) cannot be omitted. It is true in general that $\mathrm{ran}\,(\tau_A)$ is an ideal in $\mathbb{Z}\,[1/\lambda_{(A)}]$, but the ideal may be proper, and it may be different from one to the other. An example showing this to be the case can be found in [BMT87, p. 104], [PaTu82, pp. 79–83]. The example is a matrix A such that A and its transpose $B = A^{\mathrm{tr}}$ are not shift equivalent. We will give another example of this, and then apply Theorem 6.3 to show that they are not C^*-equivalent either:

Example 7.8. The example is $A = \left(\begin{smallmatrix} 19 & 5 \\ 4 & 1 \end{smallmatrix}\right)$. Here $\lambda = 10 + \sqrt{101}$, so the characteristic polynomial is irreducible and therefore $\ker(\tau_A) = 0$. Since $\det A = -1$, the unordered dimension groups $G(A)$ and $G(A^{\mathrm{tr}})$ are both $\mathbb{Z}^2$. However, we will show that they are not order isomorphic. We have

$$(7.14) \qquad\qquad A = \begin{pmatrix} 19 & 5 \\ 4 & 1 \end{pmatrix}, \qquad B = A^{\mathrm{tr}} = \begin{pmatrix} 19 & 4 \\ 5 & 1 \end{pmatrix}.$$

We prove that the two ideals $\operatorname{ran}(\tau_A)$ and $\operatorname{ran}(\tau_{A^{tr}})$ are nonisomorphic. The eigenvalues are $10 \pm \sqrt{101}$. Let $\omega = (1 + \sqrt{101})/2$ so that $1, \omega$ form a $\mathbb{Z}$-basis for the algebraic integers in $\mathbb{Q}(\sqrt{101})$. (The fact that all algebraic integers in a quadratic field have this form is [Wei98, Theorem 6-1-1, p. 234]. One can check that $1, \omega$ are algebraic integers, then that the trace must be an algebraic integer, and see what happens when we subtract some $a + b\omega$ to simplify, in terms of the norm being an algebraic integer.) The respective (column) eigenvectors for A, A^{tr} are

$$(7.15) \qquad \begin{pmatrix} 4 + \omega \\ 2 \end{pmatrix}, \begin{pmatrix} 5 - \omega \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -5 + \omega \end{pmatrix}, \begin{pmatrix} -2 \\ 4 + \omega \end{pmatrix}.$$

By transposing and interchanging the two, we get as Perron–Frobenius row eigenvectors for A, A^{tr}

$$(7.16) \qquad \begin{pmatrix} 2, & \omega - 5 \end{pmatrix}, \qquad \begin{pmatrix} \omega + 4, & 2 \end{pmatrix}.$$

Let I_1, I_2 denote the ideals they generate. We note that $\omega - 5 = \left(-9 + \sqrt{101}\right)/2$ and that the norm of this number is $(81 - 101)/4 = -5$. Hence over the algebraic number ring which is $\mathbb{Z}[\omega]$ and properly contains $Z[1/\lambda]$ both ideals would be the entire ring $(1) = \mathbb{Z}[\omega]$ if they were equivalent. We will now complete the proof.

If the ideals were equivalent, some element in the quotient field would multiply one ideal to the other, as additive groups, or modules over $\mathbb{Z}[1/\lambda] = \mathbb{Z}[\lambda] = \{a + b\sqrt{101} \mid a, b \in \mathbb{Z}\}$.

Note that the two generators listed in (7.16) will actually generate each ideal over $\mathbb{Z}$ additively, not just as modules over $\mathbb{Z}[\lambda]$, since multiplication by $\sqrt{101} = 2\omega - 1$ sends

$$(7.17) \qquad \begin{pmatrix} 1, & \omega \end{pmatrix} \longrightarrow \begin{pmatrix} 2\omega - 1, & 2(\omega + 25) - \omega \end{pmatrix} = \begin{pmatrix} 2\omega - 1, & \omega + 50 \end{pmatrix},$$

$$(7.18) \quad \begin{pmatrix} 2, & \omega - 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 4\omega - 2, & (\omega + 50) - 5(2\omega - 1) \end{pmatrix} = \begin{pmatrix} 4\omega - 2, & 55 - 9\omega \end{pmatrix},$$

$$(7.19) \quad \begin{pmatrix} 4 + \omega, & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} (\omega + 50) + 4(2\omega - 1), & 4\omega - 2 \end{pmatrix} = \begin{pmatrix} 9\omega + 46, & 4\omega - 2 \end{pmatrix},$$

which are still in the same additive subgroups.

The additive spans of the two pairs of generators in (7.16) are, respectively

$$(7.20) \qquad 2\mathbb{Z} + (\omega - 5)\mathbb{Z} = \{a + b\omega \mid a, b \in \mathbb{Z} \text{ such that } a - b \equiv 0 \pmod{2}\},$$

$$(7.21) \qquad 2\mathbb{Z} + (\omega + 4)\mathbb{Z} = \{a + b\omega \mid a, b \in \mathbb{Z} \text{ such that } a - b \equiv 1 \pmod{2}\}.$$

These are preserved by multiplication by $\sqrt{101} = 2\omega - 1 \equiv 1 \pmod{2}$, so that each span over $\mathbb{Z}$ is a $\mathbb{Z}[\lambda]$-module.

If the ideals were isomorphic under multiplication by some $f \in \mathbb{Q}[\lambda]$, then f cannot involve primes of the algebraic number ring, since both ideals span the complete algebraic number ring as modules over it. Therefore f is a unit. Thus f is up to a sign a power of $\lambda = 9 + 2\omega \equiv 1 \pmod{2}$. Hence multiplication by f preserves the congruence conditions defining the two additive spans, and thus it preserves each ideal separately. So it is impossible for a unit to send one ideal to the other.

Example 7.9. The following is an example of integer matrices A, B which have isomorphic dimension groups (unordered) but such that the corresponding transposed matrices A^{tr}, B^{tr} do not have isomorphic dimension groups. They are constructed as fairly typical block-triangular matrices, for which there will be differences between the extensions in dimension groups for the originals and transposes. A, B,

and most matrices in the construction will be block-triangular; they will be 6×6 matrices blocked into 2×2 blocks. Let the main-diagonal entries of A, B, their eigenvalues, be distinct primes $p_1, \ldots, p_6$, which are all congruent to 1 modulo the cube of some prime p, which is where the obstruction to isomorphism of dimension groups will be nontrivial. Let D denote the diagonal matrix whose main-diagonal entries are the p_i. We will write $A = E^{-1}DE$, $B = F^{-1}DF$ where E, F will be triangular matrices giving the column eigenvectors of A, B. We assume E, F have 1 on the main diagonal and their only denominators are p. Then, for instance, $A = E^{-1}(p_1 I + p^3 D_r)E = p_1 I + p^2 E^{-1} D_r p E$ is an integer matrix.

Since the eigenvalues of A, B are distinct primes, the dimension group over $\mathbb{Q}$ is a direct sum of the corresponding eigenspaces. The p_i eigenspaces consists of those vectors such that some multiple is divisible by an arbitrary value of p_i. An assumed isomorphism realized by a matrix over $\mathbb{Q}$ between the dimension groups must preserve the eigenspaces, and therefore will have the form FD_0E^{-1}, where D_0 is a diagonal matrix. Moreover the least integer vectors in these eigenspaces are unique up to sign and powers of p_i, so they must map to one another. We assume for each pair of corresponding eigenvectors, one is divisible by p if and only if the other is, so that either the originals, or p times them, are these least integer vectors. Then D_0 has as its (i,i) entry $\pm p_i^{k_i}$ for some $k_i \in \mathbb{Z}$.

Only the p_i's occur as denominators in the dimension group, and hence also in this matrix and its inverse. For transposes, we are working with the corresponding equation on the transposes of the row eigenvectors of the matrices. But the row eigenvectors of a matrix are given by the inverse of a matrix of column eigenvectors. So for the transposes, we have a similar problem but with a mapping $(F^{-1})^{\mathrm{tr}}D_1 E^{\mathrm{tr}}$, or equivalently ED_1F^{-1}.

To be specific, let

$$(7.22) \qquad E = \begin{pmatrix} I & 0 & 0 \\ A_1 & I & 0 \\ B_1 & C_1 & I \end{pmatrix}$$

$$(7.23) \qquad F = \begin{pmatrix} I & 0 & 0 \\ A_2 & I & 0 \\ B_2 & C_2 & I \end{pmatrix}.$$

Let the main-diagonal entries of D_0, D_1 be denoted as X_0, Y_0, Z_0; X_1, Y_1, Z_1. Then set $B^m J A^{-n} = E D_1 F^{-1}$ where

$$(7.24) \quad ED_1F^{-1} = \begin{pmatrix} X_1 & 0 & 0 \\ A_1X_1 - Y_1A_2 & Y_1 & 0 \\ B_1X_1 - C_1Y_1A_2 + Z_1A_2C_2 - Z_1B_2 & C_1Y_1 - Z_1C_2 & Z_1 \end{pmatrix}$$

$$(7.25) \quad FD_0E^{-1} = \begin{pmatrix} X_0 & 0 & 0 \\ A_2X_0 - Y_0A_1 & Y_0 & 0 \\ B_2X_0 - C_2Y_0A_1 + Z_0A_1C_1 - Z_0B_1 & C_2Y_0 - Z_0C_1 & Z_0 \end{pmatrix}.$$

These matrices will have fractions only having denominators powers of the prime p. Let A_1 and A_2 have the same fractional part F_a, that is $A_1 - A_2 \in M_2(\mathbb{Z})$, and C_1 and C_2 have the same fractional part F_c, where $F_aF_c = F_cF_a$. Let all these

fractional parts be

$$(7.26) \qquad F_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The congruences modulo p for the $(1,2)$ and $(2,1)$ matrix entries require that the signs in the diagonal matrices D_0, D_1 be all the same, so that after possibly multiplying these matrices by $-I$, which won't affect unordered isomorphisms, we have plus signs and $X_i, Z_i \equiv I \pmod{p^3}$. Choose B_1, B_2 so that the first matrix, $B^m J A^{-n}$ in (7.24), consists of integers; hence we have an isomorphism from one dimension group to the other. Choose the integer parts of A_2, C_2 (giving a difference in the two equations), so that the second equation, (7.25), cannot hold. Then the $(3,1)$ entry cannot be an integer at the prime p. This difference is congruent to

$$(7.27) \qquad -C_1 A_2 + A_2 C_2 + C_2 A_1 - A_1 C_1,$$

since the parts involving B_i are congruent. To do this, we need only adjust the integer W_{a2} part of A_2, for instance, by a matrix which does not commute with F_1 modulo p, since the effect of this change is solely reflected in $-F_1 W_{a2} + W_{a2} F_1$. Thus the transpose dimension groups are not isomorphic.

Remark 7.10. It follows from Theorem 3.1 of Boyle and Handelman [BoHa93] that there are nonnegative integer matrices that are shift equivalent to the pair in Example 7.9 and hence have the same dimension groups. It seems nearly certain that an example could also be constructed where the ordered dimension groups are isomorphic for two matrices, but for the transpose matrices, the ordered dimension groups are not isomorphic. However, this example would be even more complicated. In effect, we could add a large positive eigenvalue which is distinct, so that the sum of nonmaximal column eigenspaces is automatically preserved, and so that the direct sum of the new eigenspace and the old is either conjugate over $\mathbb{Z}$ to a nonnegative matrix, or can be converted to one by the techniques of Boyle and Handelman.

8. THE QUOTIENT $G/\mathbb{Z}^N$ IS AN INVARIANT

Recall that $G = G_A = \bigcup_{n=0}^{\infty} A^{-n} \mathbb{Z}^N$. In this section we will consider the quotient group $G/\mathbb{Z}^N$. Here $\mathbb{Z}^N$ can be replaced with any free abelian subgroup L of G such that

$$(8.1) \qquad G_A = \bigcup_{n=0}^{\infty} A^{-n} L$$

and

$$(8.2) \qquad AL \subset L.$$

We used the quotient group in [BJO99], but at the time we did not know if it was an invariant, and what the isomorphism classes were (in the category of abelian torsion groups). These issues are now resolved in the next proposition, which implies that the quotient is indeed an isomorphism invariant, i.e., that a given C^*-isomorphism implies that the corresponding two quotients are isomorphic groups.

Abelian torsion groups are classified in general by the so-called Ulm invariant [Kap69, pp. 26–27], [KaMa51, and references given there]. The Ulm invariant in general is a sequence of natural numbers suitably indexed by ordinals. These

numbers are calculated as dimensions of certain vector spaces over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. First, any given torsion group decomposes over its p-subgroups, and the Ulm dimensions are then calculated for each ordinal, when p is fixed. In the present application, the Ulm invariant is, as we show, very simple and concrete.

Example 8.1. *The quotient group $G/\mathbb{Z}^N$ for the special case of Section 5.* It is easy to understand concretely the torsion group quotient for the special case of Section 5 when it is assumed that $|\det A| = \lambda_{(A)}$. Of course then $\lambda_{(A)}$ is an integer, and we may therefore form $\mathbb{Z}\left[1/\lambda_{(A)}\right]$ the usual way as an inductive limit $\bigcup_{n=1}^{\infty} \mathbb{Z}_{\lambda_{(A)}^n}$ as described in (3.3) with natural embeddings $\mathbb{Z}_{\lambda_{(A)}^n} \hookrightarrow \mathbb{Z}_{\lambda_{(A)}^{n+1}}$, and it follows from the discussion in Section 3 and Section 5 (Figure 1) that there is then a natural isomorphism between the two groups $G\left(A\right)/\mathbb{Z}^N$ and $\mathbb{Z}\left[1/\lambda_{(A)}\right]/\mathbb{Z}$. Hence, in this very special case, $\mathrm{Prim}\left(\lambda_{(A)}\right)$ is a complete invariant for the corresponding torsion group quotient. See also [BJO99] for more details. It is the case of dimension groups $G\left(A\right)$ more general than that of Section 5 which requires a nontrivial localization. The next two propositions deal with the general case, and the appropriate localizations.

One method of localizing at a prime p is to take the tensor product of an abelian group with $\mathbb{Z}\left[1/2, 1/3, \ldots, \widehat{1/p}, \ldots\right]$, inverting all primes except p; another is to tensor with the p-adic integers. Both agree for all torsion groups; the latter localization factors through the former. These tensor products are exact functors of abelian groups A which are subgroups of $\mathbb{Q}^N$, that is, they preserve exact sequences; this follows from [CaEi56, Proposition 7.2, p. 138], since the group $D(A)$ is a direct sum of copies of $\mathbb{R}/\mathbb{Z}$ which has no nontrivial continuous homomorphisms into the totally disconnected p-adics. Thus $\mathrm{Tor}^1(A, C)$ is zero. Unless otherwise specified we will mean the former, smaller tensor product when we localize.

Proposition 8.2. *Let $G = \bigcup_{n=0}^{\infty} A^{-n}\mathbb{Z}^N$ be the dimension group of a primitive nonsingular integer matrix A. Then for all sufficiently large lattices (finitely generated free abelian subgroups) L in G, the group G/L is a torsion group and the order of any element is a product of primes dividing $\det A$. In particular, this is true for all lattices L such that $\bigcup_n A^{-n}L = G$. For each such prime p, its p-torsion part (the elements $G\left(p\right)$ whose order in G/L is a power of p, $G/L \cong \bigoplus_{p \text{ prime}} G\left(p\right)$) is p-divisible and the subgroup of order p is finite. Hence the rank at each such prime provides a complete invariant of G/L. This rank is an isomorphism invariant of G, and in fact, if we write the characteristic polynomial as $x^N + c_1 x^{N-1} + \cdots + c_{N-1}x + c_N$, it is the largest j such that p does not divide c_j. The extension of the free abelian group L by the torsion group G/L is computable; it will lie in a direct sum of copies of the p-adic integers for these primes p.*

Proof. G written as $\bigcup_{n=0}^{\infty} A^{-n}\left(\mathbb{Z}^N\right)$ will have as denominators only primes dividing $\det A$. If L includes $\mathbb{Z}^N$ then we have a torsion group whose torsion involves only primes in $\det A$.

We first argue that locally at each prime p in it, G consists of those vectors dual to the eventual p-adic row space $R(E_A)$ of A. That is,

$$(8.3) \qquad G \otimes \mathbb{Z}\left[1/2, \ldots, \widehat{1/p}, \ldots\right] = \left\{v \in \mathbb{Q}^N \mid \langle w \mid v \rangle \in \mathbb{Z}_{(p)} \; \forall\, w \in R(E_A)\right\}.$$

The dimension group is the group of vectors x such that for some n, $A^n x \in \mathbb{Z}^N$. This is the group of vectors such that $\exists\, n \in \mathbb{Z}^+$ such that for $\forall\, w \in \mathbb{Z}^N$, we have

$wA^n x \in \mathbb{Z}$. This is the group of rational vectors whose products with the row space of A^n is integer. This construction also goes through if we localize at any prime. To say that a vector has p-integer product with the row space of A^n for some n then implies that it has p-integer product with the idempotent p-adic limit E_A of powers of A, mentioned in Lemma 3.1 and in Theorem 7 of [BJKR98]. Conversely suppose it has p-integer product with the idempotent p-adic limit, then by p-adic continuity, it must have p-integer product with some finite power. This gives the claim.

Now to show that the quotient group at the prime p is p-divisible, take a p-adic dual basis to the row space of E_A, which like any p-adic torsion-free module, must be a free module (the p-adic integers are a principal ideal domain, and argue as with the ordinary integers). Approximate these vectors p-adically by rational vectors b_i using a p-adic approximation theorem such as [Wei98, 1-2-3, p. 8]. Take the free abelian group L_1 generated by b_i and add in the free abelian group $\mathbb{Z}^N$ and take a basis for the result. As soon as we have a lattice L including L_1, the p-adic dimension group consists of a sum of copies of the p-adic integers corresponding to L_1 and a sum of copies of the p-adic field corresponding to the remaining vectors (in the null space of E_A—we can take additional basis vectors for it). When we divide by L, we are dividing out by all the L_1 part p-adically, and by something isomorphic inside a p-adic field in the rest, and the result will be p-divisible.

In fact, for any lattice L such that $\bigcup_n A^{-n} L$ is the dimension group, the quotient will be isomorphic to this, since multiplication by A^{-1} gives an isomorphism of pairs $\left(\bigcup_n A^{-n} L, L \right) \to \left(\bigcup_n A^{-n} L, A^{-1} L \right)$, and eventually this lattice must be large enough.

Now consider the p-adic rank, in relation to the characteristic polynomial. By Newton's method [Wei98, 3-1-1, p. 74], if the characteristic polynomial has the given form, we can factor it over the p-adics as a product of two polynomials, one of which is x^{N-j} modulo p, and the other of which has invertible constant term over the p-adics. We can put the matrix into corresponding block form. The former part will be p-adically nilpotent, and the null space will be its row space. $\square$

Remark 8.3. As noted, our groups $G(A)$ are contained in $\mathbb{R}^N$ (even in $\mathbb{Q}^N$) where N is the rank of $G(A)$. But it is clear that general lattices L in $\mathbb{R}^N$ are given by a choice of basis in $\mathbb{R}^N$ as a vector space. Writing the vectors in a basis, equivalently the generators for L, as column vectors, we note that the lattices L may be viewed as, or identified with, nonsingular real matrices. Making this identification, and fixing the rank N, we further note that the containment $L \subset L'$, for two given lattices, holds if and only if there is some $C \in M_N(\mathbb{Z})$ such that we have the following matrix factorization:

$$(8.4) \qquad\qquad L = L'C.$$

There is a similar version of this for row spaces (or lattices defined from row vectors), as well as a p-adic variation, *mutatis mutandis;* and we have already seen an instance of the latter in (3.11)–(3.13).

Remark 8.4. We now show, using (8.4), that the conditions on L from Proposition 8.2 are all rationality conditions. There are three in all, and we proceed to spell them out. If A is given as usual, and if $G(A)$ is the corresponding group, i.e., $\bigcup_{n=0}^{\infty} A^{-n} \left(\mathbb{Z}^N \right)$, then a lattice L is a subgroup, i.e., $L \subset G(A)$, if and only if there

is a natural number n such that

$$(8.5) \qquad\qquad A^n L \in M_N\left(\mathbb{Z}\right).$$

Some given lattice L will satisfy the invariance property $A\left(L\right) \subset L$ if and only if the conjugate matrix $L^{-1}AL$ satisfies

$$(8.6) \qquad\qquad L^{-1}AL \in M_N\left(\mathbb{Z}\right).$$

The further condition on L that it is generating, i.e., that $\bigcup_{n=0}^{\infty} A^{-n}\left(L\right) = G\left(A\right)$, holds if and only if for some natural number n we have

$$(8.7) \qquad\qquad L^{-1}A^n \in M_N\left(\mathbb{Z}\right).$$

The three conditions should also be compared with (1.17) from Section 1.

Now Proposition 8.2 applies when $G\left(A\right)$ is given and some lattice satisfies all three conditions (8.5)–(8.7), and we get as a corollary that if two lattices L and L' both satisfy the conditions, then the two torsion groups $G\left(A\right)/L$ and $G\left(A\right)/L'$ are isomorphic groups.

In the study of dimension groups, it is convenient to explicitly compute certain extensions. Let $\mathbb{Z}_{p^\infty}$ denote the union of $\mathbb{Z}_{p^n}$ under inclusion, a divisible p-torsion group whose order p subgroup has rank 1. By standard theory [CaEi56], the extension group $\mathrm{Ext}(\mathbb{Z}_{p^\infty}, \mathbb{Z})$ can be computed using the exact sequence

$$(8.8) \qquad\qquad 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

as the cokernel of the map $\mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Q}/\mathbb{Z})$; the former group is zero and the latter group is $\mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty})$. Every p-adic integer gives a mapping in this group; we check this mapping is one-to-one and onto, so that the Ext group is the p-adic integers. (To check "onto", note that we get every mapping $\mathbb{Z}_{p^n} \to \mathbb{Z}_{p^n}$ and take limits.) In general, we are dealing with a direct sum of copies of these Ext groups.

Next we look at the problem of isomorphism of dimension groups in a somewhat different way, by showing that dimension groups can easily be computed as extensions. In some cases this leads to a quick decision about whether two dimension groups are isomorphic. However in the most general case, the problem of deciding isomorphism given this extension structure seems to still require the methods of Section 4. In view of Remark 8.3 we need only state the result for the case when the lattice L is $\mathbb{Z}^N$.

Corollary 8.5. *As in Proposition 8.2, consider an unordered dimension group as an extension of $\mathbb{Z}^N$ by a divisible torsion group $G/\mathbb{Z}^N$ whose structure, computed as in Proposition 8.2, is a direct sum of n_i copies of $\mathbb{Z}_{p_i^\infty}$. The extension class in $\mathrm{Ext}^1(G/\mathbb{Z}^N, \mathbb{Z}^N)$ is an element of $\bigoplus_i \bigoplus_{j=1}^{n_i} \mathbb{Z}_{p(i)}$. We write this as an $N \times \sum_i n_i$ matrix whose entries are p_i-adic integers. Its columns consist precisely of a basis for the null-space of the matrices E_A taken at each prime p_i. Two such matrices M_1, M_2 represent isomorphic unordered dimension groups if and only if there is a matrix $C \in \mathrm{GL}(N, \mathbb{Z}\left[1/\det(A)\right])$ and an invertible direct sum of p_i-adic integer matrices D such that $CM_1D = M_2$.*

Proof. Regarding the last statement, suppose C, D exist and the columns of M_1, M_2 form bases for summands of $Z_{(p_i)}$, then the extensions are isomorphic, by taking the corresponding maps on $\mathbb{Z}^N$, $G/\mathbb{Z}^N$. The action of the p-adic integer matrices

just gives an isomorphism on $G/\mathbb{Z}^N$, and hence it gives an isomorphism on the extensions. The same is true for a mapping $C \in \mathrm{GL}(N, \mathbb{Z})$.

The given structure is an isomorphism invariant because the p-adic row spaces of E_A are invariants. Theorem 7 of [BJKR98] showed that a rational matrix over $\mathbb{Z}\,[1/\det(A)]$ giving an isomorphism on dimension groups must give an isomorphism on the p-adic row spaces, hence the dual p-adic null spaces of $E(A)$. In fact [BJKR98, Theorem 7] gives as necessary and sufficient conditions for unordered dimension group isomorphism, in effect, existence of C, D: the p-adic symmetries just mean we are considering the row spaces up to isomorphism, and the $\mathrm{GL}(N, \mathbb{Z}\,[1/\det A])$ symmetry means that we have a rational map which is an isomorphism at all primes other than the ones considered here.

The extension class of any extension of $\mathbb{Z}^N$ by a group $G/\mathbb{Z}^N$ may be computed by extending the map $\mathbb{Z}^N \subset \mathbb{Q}^N$ to a mapping $G \to \mathbb{Q}^N$, and letting this give a map in $\mathrm{Hom}(G/\mathbb{Z}^N, (\mathbb{Q}/\mathbb{Z})^N) \cong \mathrm{Ext}^1(G/\mathbb{Z}^N, \mathbb{Z}^N)$. This is the remark of Cartan–Eilenberg [CaEi56, p. 292]. To identify this class it suffices to look at the p-torsion subgroup of $G/\mathbb{Z}^N$ for each prime p since the group is the direct sum of its p-torsion subgroups. To identify this class, take the tensor product of G with the p-adic integers, getting a localized extension of $\mathbb{Z}_{(p)}^N$ by the p-torsion subgroup of $G/\mathbb{Z}^N$, which is $G \otimes \mathbb{Z}_{(p)}^N$. But if we write all p-adic vectors as the direct sum $K \oplus R$ of the p-adic null space of R_A and a complementary space R, by Proposition 8.2,

$$(8.9) \qquad G \otimes \mathbb{Z}_{(p)}^N = (K \otimes \mathbb{Q}_{(p)}) \oplus (R \otimes \mathbb{Z}_{(p)}).$$

Thus the extension class is represented taking

$$(8.10) \qquad (K \otimes \mathbb{Q}_{(p)}) \oplus (R \otimes \mathbb{Z}_{(p)}) \to (K + R) \otimes \mathbb{Q}_{(p)}$$

and collapsing by $\mathbb{Z}_{(p)}^N$ to give the inclusion

$$(8.11) \qquad (K \otimes \mathbb{Q}_{(p)})/(K \otimes \mathbb{Z}_{(p)}) \to (K + R) \otimes (\mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}) = (\mathbb{Q}_{(p)}/\mathbb{Z}_{(p)})^N.$$

This map is induced by the map

$$(8.12) \qquad (K \otimes \mathbb{Q}_{(p)}) \to (K + R) \otimes \mathbb{Q}_{(p)} = \mathbb{Z}^N \otimes \mathbb{Q}_{(p)}$$

which can be taken to send the ith unit vector on the left to the ith vector in a basis for K on the right. This means taking basis vectors for the null space of E_A as forming the columns of the matrix giving the extension. $\square$

9. Reduction to the nonsingular case

Throughout this paper we have considered AF-algebras defined by nonsingular primitive matrices A, B, ... The purpose of the present section is to point out that even though the condition of nonsingularity is convenient in several arguments, it is not essential for the definition of the class of C^*-algebras we consider. We will prove in Theorem 9.3 below that the class of C^*-algebras remains exactly the same if the condition of nonsingularity of the matrix A is removed and A is merely assumed to be primitive, i.e., some power of A has only strictly positive matrix elements. In the case when A is not assumed invertible, we may introduce the eventual range of A,

$$(9.1) \qquad \mathcal{W}(A) := \bigcap_{i=0}^{\infty} A^i \mathbb{Q}^N = A^N \mathbb{Q}^N.$$

Note that A is bijective as a map $\mathcal{W}(A) \to \mathcal{W}(A)$. We may now introduce an additive group $G(A)$ by

$$(9.2) \qquad G(A) := \left\{ g \in \mathcal{W}(A) \mid A^k g \in \mathbb{Z}^N \text{ for some } k \in \mathbb{N} \right\},$$

and one notes that this group $G(A)$ identifies with the inductive limit of the sequence (1.6), i.e. $G(A)$ is the dimension group when it is equipped with the obvious order. Let us give some details. An element of the inductive limit (1.6) can be represented by a sequence $\{g_m, g_{m+1}, \dots\}$ in $\mathbb{Z}^N$ with $Ag_n = g_{n+1}$ for $n = m, m+1, \dots$. Two such sequences represent the same element if they coincide from a certain step n onward. Given such a sequence, there is a unique sequence $\{h_1, h_2, \dots\}$ in $\mathcal{W}(A)$ such that $Ah_n = h_{n+1}$ for $n = 1, 2, \dots$ and such that $h_n = g_n$ for all large n. Then h_1 is the element of $G(A) \subset \mathcal{W}(A)$ representing the dimension group element in (9.2), so this shows the equivalence between the two definitions (9.2) and (1.6) of $G(A)$. The definition (9.2) is the definition used in [BMT87, p. 49]. If A is nonsingular and, as in (9.2), $A^k g = m \in \mathbb{Z}^N$, then $g = A^{-k} m$ is a typical element of the (1.8)–(1.10) version of $G(A)$, and *vice versa*. If A is primitive, we still have the Perron–Frobenius data, and the order can be defined as before, *mutatis mutandis*.

Lemma 9.1. *Given a vector $u \in \mathbb{R}^r$ there exist r vectors $w_i \in \mathbb{Z}^r$ such that the convex cone generated by the w_i contains an open neighborhood of u, and the determinant of the matrix the w_i form is ± 1.*

Proof. The standard unit vectors do this for any vector in the half space of strictly positive integer vectors. We claim transforms of these by integer row and column operations, permutations, and reversals of sign, take any vector to the interior of this half space—then just reverse those operations on the standard basis vectors. In fact, we get all coordinates nonzero by certain linear combinations, then reverse their signs.   □

Remark 9.2. It is not in general possible to get a determinant-1 system of matrices which approximate multiples by some positive constant C of a given set of nonnegative vectors u_i. This is easiest to see when the vectors u_i are chosen diagonally dominant. But Lemma 9.1 can probably be strengthened a little.

Theorem 9.3. *Every ordered dimension group arising from any nonnegative primitive integer matrix A is order isomorphic to one arising from a nonsingular positive integer matrix B.*

Proof. Note that by results in [BoHa91], this is not true for shift equivalence, but the ability to replace matrices by powers of themselves gives much more flexibility here. Let the dimension of A be d and the rank of all sufficiently large powers A^s be r. By Lemma 9.1, we find a set of r vectors w_i in the eventual row space R, that is, the row space of A^N, or some specific higher power, a rank-r subspace of $\mathbb{Z}^d$ such that the cone over $\mathbb{Q}_+$ generated by this set includes a neighborhood of the maximum eigenvector v within R. This is sufficient to establish that all sufficiently large powers of A have their rows expressed as (unique) nonnegative linear combinations of w_i, since all rows of A^s divided by their lengths converge to fixed multiples of v and hence are eventually in the convex cone; but to be in the convex cone means that we have these convex combinations.

However, we also need that it can be chosen that these convex combinations are eventually integer. For that, it suffices that the determinant of the w_i expressed

as combinations of a basis for the integral vectors in the eventual row space, i.e., $R \cap \mathbb{Z}^d$, a rank-r free abelian group, is 1 or -1. This follows from the lemma.

Now let B be the matrix of A^s expressed as acting on the vectors w_i, which will be nonnegative, and positive. Then B is shift equivalent to A^s over the integers (maybe with negative entries), just by the inclusion mapping given by the vectors w_i. By a theorem of Parry and Williams [PaWi77] (reproved in our 1979 paper [KiRo79]), any shift equivalence over $\mathbb{Z}$ of primitive matrices can be realized by a shift equivalence over $\mathbb{Z}_+$. This shift equivalence will induce an isomorphism of ordered dimension groups.   □

10. STRONG LOCAL ISOMORPHISM

Definition 10.1. We will say that two dimension groups G, G' are strongly locally isomorphic at the prime p if and only if there is an isomorphism $G \otimes \mathbb{Z}_{(p)} \rightarrow G' \otimes \mathbb{Z}_{(p)}$ which is induced by a matrix of rational integers.

Theorem 10.2. *Strong local isomorphism is decidable and corresponds to the isomorphism condition of Corollary 8.5 if we take the submatrix corresponding to one specific prime. That is we form a matrix whose rows are a basis for the p-adic eventual row space, whose rank is n_p. Two dimension groups are strongly locally isomorphic if and only if the corresponding two matrices A, B for each p admit some matrices C and D where $C \in \mathrm{GL}(N, \mathbb{Q})$, $D \in \mathrm{GL}(n_p, \mathbb{Z}_{(p)})$, and the two matrices C and D satisfy $CAD = B$.*

Proof. The latter condition follows from Theorem 7 of [BJKR98], which is identical to Corollary 3.2 in the present paper. We may determine an algebraic number field over which the p-adic eventual row spaces of both matrices may be realized; the field generated by all their eigenvalues will suffice. Then the required p-adic matrix, if it exists must be a matrix over this field, having denominators relatively prime to p. It can be expanded as a larger matrix over $\mathbb{Z}[1/2, 1/3, \ldots, \widehat{1/p}, \ldots]$, using a basis for the p-adic algebraic integers of this field. The conditions that a rational matrix C exist are linear, corresponding to Galois invariance. Then in this linear space of matrices, we must have a matrix over $\mathbb{Z}[1/2, 1/3, \ldots, \widehat{1/p}, \ldots]$, which is p-adically invertible. We can determine a basis for this linear space, write the determinant as polynomial in variables representing an expansion of a given matrix in terms of this basis, and test each case of congruence classes for the variables to see if the determinant is nonzero modulo p.   □

Example 10.3. Let

(10.1)
$$A = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}.$$

The respective characteristic polynomials are $x^2 - 6x + 7$ and $x^2 - 8x + 7$, with determinant 7, and we consider the local dimension groups at 7. Since 7 does not divide 8, only one root of the former polynomial is divisible by 7. Thus only the identity element of the $\mathbb{Z}_2$ Galois group fixes the eigenvalue not divisible by 7. This implies that the 7-adic row space is irrational, and the minimal fields over which eventual row spaces are defined are respectively $\mathbb{Q}\left[\sqrt{2}\right]$, $\mathbb{Q}$, so the dimension groups are not locally isomorphic. Note that, even so, the two quotient groups $G\left(A\right)/\mathbb{Z}^2$ and $G\left(B\right)/\mathbb{Z}^2$ are isomorphic. This follows from Proposition 8.2: Recall, to verify this we need only compute the respective Ulm numbers from the characteristic

polynomials, and there is only the prime $p = 7$ to check. So the 7-reduced rank is 1 for each of the two quotient torsion groups calculated from A and B.

Example 10.4. Consider

$$(10.2) \qquad A = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$$

and its transpose $B = A^{\mathrm{tr}}$; for this particular choice of the pair A, B, the minimal fields over which eventual row spaces are defined are isomorphic. We can arbitrarily choose which root of the characteristic polynomial $x^2 - 6x + 7$, the same as for A in Example 10.3, is divisible by 7 (representing the unique p-adic root which is divisible by 7), say $3 - \sqrt{2}$. The eventual row eigenspaces are spanned by the corresponding row eigenvectors, which are $\begin{pmatrix} 1 & -\sqrt{2} \end{pmatrix}$ for B and $\begin{pmatrix} -\sqrt{2} & 1 \end{pmatrix}$ for A. A mapping of eigenspaces must map one to a multiple c times the other. If it commutes with the Galois action, then it must do the same for their conjugates, so that it has the form

$$(10.3) \qquad \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}.$$

The determinant restricted to the eventual 7-adic row space is c, so the congruences are $c \equiv 0 \pmod 7$, which are solvable. The dimension groups of this matrix and its transpose are locally isomorphic at the prime 7. Since 7 is the only prime involved, this implies global isomorphism of the unordered dimension groups.

Note also in this case that A and A^{tr} are conjugate by the unimodular matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; and hence it even follows directly as in Example 7.1 that A and A^{tr} are (elementary) shift equivalent.

Note finally that if one denotes the A matrices in (10.1) and (10.2) by A_1, A_2, respectively, and one defines

$$(10.4) \qquad J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then $A_2 J = J A_1$, and thus if $K = A_1 J^{-1}$ we have the system

$$(10.5) \qquad A_2 = JK, \qquad A_1 = KJ.$$

But K does not have positive matrix entries, so this does not imply elementary shift equivalence. However, if we redefine

$$(10.6) \qquad K = A_1^2 J^{-1} = \begin{pmatrix} 11 & 6 \\ 1 & 3 \end{pmatrix},$$

then we have the pair of shift relations for the squares,

$$(10.7) \qquad A_1^2 = KJ, \qquad A_2^2 = JK,$$

which is the assertion that A_1^2 and A_2^2 are elementary shift equivalent. In particular, A_1 and A_2 are C^*-equivalent. This latter conclusion and the one in Example 7.1 also follow the next general observation:

Observation 10.5. If A, B are nonsingular primitive $N \times N$ matrices and there exists a unimodular matrix J in $M_N\,(\mathbb{Z})$ such that

$$(10.8) \qquad v\,(B)\,J = \mu v\,(A)$$

for a positive number μ, and

$$(10.9) \qquad BJ = JA,$$

then A and B are C^*-equivalent.

Proof. Since J is unimodular, we have

$$(10.10) \qquad \begin{cases} B^n J A^{-n} = J A^n A^{-n} = J \in M_N\left(\mathbb{Z}\right), \\ A^n J^{-1} B^{-n} = A^n A^{-n} J^{-1} = J^{-1} \in M_N\left(\mathbb{Z}\right), \end{cases}$$

and the observation follows from (1.16)–(1.17). (The condition (10.8) may be replaced by the strictly stronger requirement that J and J^{-1} have only nonnegative matrix entries.) $\qquad\square$

Remark 10.6. In fact we have the "partial" implication (10.9) $\Rightarrow$ (10.8), but (10.8) for some real scalar μ, while the positivity restriction on μ is not a consequence of (10.9) alone. We further stress that (10.9)–(10.8) are more restrictive than C^*-equivalence, even more restrictive than shift equivalence: take, for example, $A = \left(\begin{smallmatrix} 2 & 1 \\ 4 & 4 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 4 \end{smallmatrix}\right)$, which are shift equivalent by [Bak83], but do not satisfy (10.9).

To summarize, the two examples have four matrices in all, and the first one in Example 10.3 is C^*-equivalent to the two in Example 10.4, but $\left(\begin{smallmatrix} 7 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ from Example 10.3 is not C^*-equivalent to the other three. The first one, $\left(\begin{smallmatrix} 4 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ in Example 10.3, is symmetric, and A from Example 10.4 is C^*-symmetric in that it is C^*-equivalent to its own transpose.

Remark 10.7. Note that the two matrices A_1, A_2 in (10.1), (10.2) considered above are elementary shift equivalent over $\mathbb{Z}$ since they are conjugate over $\mathbb{Z}$. But while A_1^2, A_2^2 are elementary shift equivalent over $\mathbb{Z}_+$, A_1 and A_2 are not! (These types of 2×2 examples have been considered earlier by Kirby Baker [Bak83, Bak87].) This is seen as follows: Suppose $A_1 = CD$ where C, D are nonnegative integer 2×2 matrices. Then C expresses the rows of A_1 as nonnegative integer combinations of the rows of D. The entries 1 in the rows of A_1 can only come from entries 1 in the rows of D. Moreover these 1's can only be in the same row. Furthermore, in the linear combinations these 1's can only be multiplied by 1's. So the product CD looks like, up to symmetry,

$$(10.11) \qquad \begin{pmatrix} 1 \cdot c_{12} & d_{11} \cdot 1 \\ c_{21} \cdot 1 & 1 \cdot d_{22} \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}.$$

But if we write out the equations, there are no solutions unless one of C, D is a permutation matrix, and thus DC cannot be equal to A_2.

11. CONCLUDING REMARKS

In the paper we addressed the interplay between the local and the global versions of the isomorphism problem. There are different, but related, decidability results in the literature. Ax and Kochen [AxKo65a, AxKo65b, AxKo66] and Grunewald and Segal [GrSe82] address decidability in a p-adic setting.

Richtiges Auffassen einer Sache und Missverstehen der gleichen Sache
schliessen einander nicht vollständig aus.

—FRANZ KAFKA

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