MONOID EXTENSIONS ADMITTING COCYCLES

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ABSTRACT. We characterize those monoid extensions which are associated to a certain class of 2-cocycles. Algebraic, topological as well as involutive aspects are discussed. Applications to representation theory are given.

INTRODUCTION.

One purpose of this article is to describe the family of all semigroup extensions which can be associated to some "reasonable" class of 2-cocycles. We discuss both the algebraic and the topological aspects of the subject. For simplicity we shall assume all semigroups have an identity element, i.e., they are monoids. Since an identity element can be adjoined to any semigroup, a similar extension theory is valid for a certain class of semigroups (without identity) as well.

The article is organized as follows. In §1 we adapt the basic language of the Eilenberg-MacLane cohomology to the monoid setting. Next, we develop the algebraic rudiments of a noncommutative extension theory for a certain class of monoids (§2). To be more precise, we study monoids S possessing a normal submonoid N and a cross section $u : S/N \to S$ such that all left and right translation maps on N, defined by elements in $u(S/N)^2$, are injective. It turns out that this class of extensions admits a description in terms of cocycles with values in the maximal group of N, as described in §1. Conversely, any such cocycle defines a monoid S with the above properties.

In §3 we assume the monoids are involutive. Examples of such monoids can be obtained from groups G with an involutive automorphism τ (i.e. (G, τ) is a symmetric group) as follows. If $g \in G$, we

1991 *Mathematics Subject Classification.* Primary 20M10; Secondary 22A15, 20M30.

put $g^* = \tau(g)^{-1}$. Any submonoid S of G for which $S^* = S$, is an involutive monoid and, moreover, it is (bi-) cancellative. Of particular interest is the case where (G, τ) is a symmetric Lie group, [HN, §7.3]. The involutive extensions are described in terms of "cocycle triplets" consisting of a prerepresentation, a corresponding 2-cocycle, and also a 1-cochain (Prop. 3.1, and 3.2). In §4 we proceed to study the topological properties of monoid-extensions. By analogy to groups, a monoid extension is almost fibered (Def. 4.1) if and only if it admits a cross-section continuous at the identity element, Prop. 4.1. Moreover, if the cross-section is continuous at the identity, the corresponding extension can be organized as a topological monoid in a canonical way. As an application to the extension theory of the present article, we explain how to obtain a proof of an analogue to the Mackey extension theorem for multiplier (projective) representations of second countable locally compact groups in the setting of discrete cancellative monoids, §5. In the presence of a "Haar measure", Mackey's Theorem should also be valid for many locally compact cancellative monoids. As far as we know, the existence of such a measure remains an open question.

We are aware of very few articles in the literature devoted to cocycles on semigroups. The only notable exception we have found is [DE], where it is shown that a symmetric cocycle on an abelian cancellative semigroup into a divisible abelian group must be a coboundary. We remark also that although compact cancellative semigroups are groups, this fails already in the countably compact case, [RS].

1. Eilenberg-MacLane cohomology of monoids.

Let S be a (discrete) monoid. We denote by $\mathrm{Aut}(S)$ the automorphism group of S, that is the group of all invertible maps $\alpha\colon S \to S$ which preserve the monoid composition, $\alpha(s_1 s_2) = \alpha(s_1)\alpha(s_2)$, $s_1, s_2 \in S$, and leave the identity element $e = e_S$ of S fixed. The group $G(S)$ of all invertible elements in S, is called the maximal group of S. Each x in $G(S)$ defines an inner automorphism I_x of S by $I_x(s) = x^{-1}Sx$, $s \in S$. The group $\mathrm{Inn}(S)$ of all inner automorphisms, is a normal subgroup of $\mathrm{Aut}(S)$ since $\alpha I_x = I_{\alpha(x)}\alpha$ ($\alpha \in \mathrm{Aut}(S)$, $x \in G(S)$). The quotient group $\mathrm{Out}(S) = \mathrm{Aut}(S)/\mathrm{Inn}(S)$ is called the outer automorphism group of S.

Definition 1.1. Let Q and N be monoids, $q\colon \mathrm{Aut}(N) \to \mathrm{Out}(N)$ the canonical group epimorphism. A map $\theta\colon Q \to \mathrm{Aut}(N)$ is a *pre-representation* of Q in N if $\theta_{e_Q} = \iota$ (= the identity map of N) and its projection $\chi = q \circ \theta$ on $\mathrm{Out}(N)$ is a representation (that is a semigroup homomorphism of Q into $\mathrm{Out}(N)$).

Definition 1.2. Let Q and N be monoids. An *n-cochain* ($n \geq 1$) of Q with values in N is a map f from the direct product $Q^n = Q \times \ldots \times Q$ to $G(N)$ such that $f(x_1, x_2, \ldots, x_n) = e_N$ if $x_i = e_Q$ for some i ($1 \leq i \leq h$). By a 0-cochain of Q with values in N, we understand an element of the maximal group $G(N)$. An n-cochain f is said to be continuous at e_Q (resp. locally continuous) if f is continuous (resp. locally continuous) at $(e_Q, e_Q, \ldots, e_Q)$. A 0-cochain is always regarded as continuous. We denote by $C^n(Q, N)$ the set of all n-cochains of Q with values in N.

We remark that $C^n(Q, N)$ is a group with pointwise multiplication of functions as product.

Definition 1.3. Let f be an n-cochain ($n \geq 1$) of Q with values in N, and let θ be a pre-representation of Q in $\mathrm{Aut}(N)$. The $n+1$-cochain defined by

$$(1.1)\quad (\delta f)(x_1, x_2, \ldots, x_{n+1}) = f(x_1, x_2, \ldots, x_n)^{(-1)^{n+1}} \cdot \Big[\prod_{1}^{n} f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i}\Big] \cdot \theta_{x_1}\{f(x_2, x_3, \ldots, x_n)\},$$

is called *the coboundary* of f relative to θ. A cochain f for which the coboundary is constant (and hence equal to e_N) is called a *cocycle* relative to θ. In this case we say that the pair (f, θ) is a *cocycle of Q in N*. If $n = 0$ and $f = m$, $m \in G(N)$, $(\delta m) = m\theta_x(m^{-1})$.

The set of n-cocycles relative to θ is denoted by $Z^n(Q, N, \theta)$ and the set of n-cochains ($n \geq 1$) which are coboundaries relative to θ is denoted by $B^n(Q, N, \theta)$, $B^0(Q, N, \theta)$ consists by definition of the single 0-cochain $f = e_N$.

Definition 1.4. Assume the maximal group $G(N)$ of N is abelian, and let f and g be two n-cochains of Q with values in N. f is said to be *cohomologous* to g w.r.t. the representation θ of Q in $\operatorname{Aut}(N)$, if there exists an $(n-1)$-cochain h of Q with values in N such that

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= g(x_1, x_2, \ldots, x_n) h(x_1, x_2, \ldots, x_n)^{(-1)^{n-1}} \cdot \\ &\cdot \prod_1^{n-1} h(x_1, x_2, \ldots, x_i x_{i+1}, \ldots, x_n)^{(-1)^{i+1}} \theta_{x_1}\{h(x_1, x_2, \ldots, x_n)^{-1}\} \end{aligned} \tag{1.2}$$

If $G(N)$ is abelian, it is easy to verify that every coboundary is a cocycle and that $Z^n(Q, N, \theta)$ and $B^n(Q, N, \theta)$ are subgroups of $C^n(Q, N)$, which is then abelian. The quotient group

$$H^n(Q, N, \theta) = Z^n(Q, N, \theta)/B^n(Q, N, \theta)$$

is then called the n-th cohomology group of Q with values in N relative to θ (cf. [EM1]).

Lemma 1.1. *Let Q, Q', and N be three monoids, θ a pre-representation of Q' in $\operatorname{Aut}(N)$, ψ a representation of Q in Q'. The map $\psi^*: C^n(Q', N) \to C^n(Q, N)$ defined by*

$$(\psi^* f)(x_1, x_2, \ldots, x_n) = f(\psi(x_1), \psi(x_2), \ldots, \psi(x_n)), \qquad x_i \in Q,$$

is a representation which maps cocycles and coboundaries relative to θ to cocycles and coboundaries relative to $\theta\psi$, respectively.

Corollary 1.1. *If, in addition to the hypothesis of the above lemma, the maximal group $G(N)$ is abelian, we obtain from ψ^*, on forming quotients, a representation of $H^n(Q', N, \theta)$ in $H^n(Q, N, \theta\psi)$.*

It can be shown that this representation is an isomorphism (see [Ca, I §10]).

2. Discrete monoid extensions and cocycles.

Let us begin by recalling some basic facts about congruences on monoids, see also [H]. Assume S is a monoid and N a submonoid of

S. We say that N is *normal* in S if $xN = Nx$ for all $x \in S$. If N is normal in S, we may define an equivalence relation $\rho = \rho_N$ on S by

$$(\forall x, y \in S) \quad (x,y) \in \rho \Longleftrightarrow xN = yN$$

Since N is normal, ρ is even a congruence, i.e., it is compatible with the semigroup operation:

$$(\forall s,t,x,y \in S) \quad (s,t) \in \rho \text{ and } (x,y) \in \rho \Rightarrow (sx, ty) \in \rho$$

By ρ-compatibility, the family S/ρ of all congruence classes is also a semigroup with its natural operation, $(x\rho)(y\rho) = (xy)\rho$. Here

$$x\rho = \{y \in S : (x,y) \in \rho\} = \{y \in S : xN = yN\}.$$

We refer to S/ρ as the quotient (semigroup) of S by N (or by ρ). The quotient map $\rho^\natural = \{(x, x\rho) : x \in S\} \subset S \times S/\rho$, is a semigroup homomorphism. Its kernel is $\ker \rho^\natural = \rho = \{(x,y) \in S \times S : x\rho = y\rho\} = \{(x,y) : xN = yN\}$. Observe that $e\rho = N$. More generally, the kernel $\ker \phi = \phi \circ \phi^{-1}$ of any semigroup homomorphism $\phi : S \to Q$ is a congruence on S, and the quotient semigroup $S/\ker \phi$ is isomorphic to Q. In view of this, we make the following

Definition 2.1. Let S and Q be semigroups, and let ρ be a congruence on S. We say that S is an *extension* of Q by ρ, $S = S(Q, \rho)$, if there is a homomorphism $\phi : S \to Q$ such that $\rho = \ker \phi$.

Now if ρ is a congruence on S, the equivalence class $e\rho$ of the identity element e is a subsemigroup of S, but it need not be normal. Further, $x \cdot (e\rho) \subset x\rho \quad (\forall x \in S)$, but the two sets need not be equal.

Definition 2.2. Let ρ be a congruence on S. We say that ρ is *admissible* if for each x in S, $x(e\rho) = (e\rho)x = x\rho$. The congruence ρ is *left* (resp. *right*) *admissible* if $x(e\rho) = x\rho$ (resp. $(e\rho)x = x\rho$) $\quad (\forall x \in S)$.

If $\rho = \rho_N$ where N is normal in S, then ρ is always admissible. The converse follows readily.

Lemma 2.1. *A congruence ρ on S is admissible if and only if $\rho = \rho_N$ where $N = e\rho$.*

Definition 2.3. We write $S = S(Q, N)$ if $\rho = \rho_N$ is an admissible congruence on S and $Q = S/\rho$. In this case we say that S is an *extension of Q by N.*

The following lemma is also clear.

Lemma 2.2. *Let ρ be a congruence on S, and assume u is a cross-section for S/ρ in S with $u(e\rho) = e$. Then ρ is left (resp. right) admissible if and only if each s in S can be written $s = u(x)n$ (resp. $s = nu(x)$) where $n \in e\rho$ and $x = s\rho$.*

We remark that the decomposition $s = u(s\rho)n$ of the above lemma need not be unique. We will return to the question of uniqueness below.

Now if S carries an involution, $x \mapsto x^*$, the quotient S/ρ with any congruence ρ becomes an involutive semigroup with the operation $(x\rho)^* = (x^*\rho)$ $(x \in S)$. In §3 we shall assume the semigroups are involutive and (right) *cancellative*:

$$xy = xz \quad \Rightarrow \quad y = z \qquad (x, y, z \in S). \tag{2.0}$$

By the involutive property, this also implies a corresponding left cancellation, since

$$\begin{aligned} yx = zx \quad &\Rightarrow \quad (yx)^* = (zx)^* \quad \Rightarrow \quad x^*y^* = x^*z^* \\ &\Rightarrow \quad y^* = x^* \quad \Rightarrow \quad x = y \end{aligned} \tag{2.1}$$

On any semigroup S, we have left and right translation operators $x \mapsto L_x$, $x \mapsto R_x$,

$$L_x y = xy, \quad R_x y = yx \qquad (x, y \in S).$$

Assuming S is bi-cancellative, both L_x and R_x are injective for each $x \in S$, and we have operators $L_x^{\sim} \colon L_x S = x \cdot S \to S$ and $R_x^{\sim} \colon S \cdot x \to S$ which are left-inverses of L_x and R_x, respectively. Notice that if N is normal in S, then $L_x N = R_x N$, so that the map

$$\Theta(x) \colon N \to N, \quad \Theta(x)n = R_x^{\sim} L_x n = (L_x^{\sim} R_x)^{-1} n \qquad (x \in S,\ n \in N) \tag{2.2}$$

is a well-defined automorphism of N. Thus we have a semigroup homomorphism

$$\Theta \colon S \to \mathrm{Aut}(N),$$

whenever the restriction to N of all left and right translation operators of S are injective. In light of this we introduce

Definition 2.4. Let S be a monoid, N be a normal submonoid of S, and $H \subset S$ a subset.

(a) N is *H-cancellative* if, for each $h \in H$, the restricted translation maps $L_h|N$ and $R_h|N$ are injective.

(b) Assume r is a positive integer and put $Q = S/N$. We say that N is *Q^r-cancellative* if there is a cross section $u : Q \to S$ such that N is $u(Q)^r$-cancellative. In this case we shall also say that the *extension* $S = S(Q, N)$ is Q^r-cancellative.

Similarly, one can also formulate notions of left and right Q^r-cancellative monoid extensions. In the present paper we shall only be concerned with the cases $r = 1$ and 2.

Remark 2.1. It is not hard to see that if $S(Q, N)$ is Q^r-cancellative then the injectivity properties of the above definition is valid for all cross sections u associated to the extension. Notice also that an extension $S(Q, N)$ is Q^r-cancellative if and only if for each cross section $u : Q \to S$ associated to the extension left and right translations on N with every $y \in u(Q) \cdot Range(\omega)^{r-1}$ is injective. This follows from the identity $u(xy) = \omega(x, y)u(x)u(y)$, $x, y \in Q$ (ω denotes the cocycle corresponding to u).

If N is S-cancellative, we define $\mathrm{I}(N)$ as the subgroup of $\mathrm{Aut}(N)$ generated by all $\Theta(n)$ and $\Theta(n)^{-1}$ where n runs through N. $\mathrm{O}(N) = \mathrm{Aut}(N)/\mathrm{I}(N)$ will stand for he quotient group. Clearly the inner automorphism group $\mathrm{Inn}(N)$ is a normal subgroup of $\mathrm{I}(N)$. Θ is a representation of the monoid S in $\mathrm{Aut}(N)$ that maps N onto $\Theta(N) \subset \mathrm{I}(N)$. Furthermore, $\Theta(N)$ is a normal submonoid of $\mathrm{Aut}(N)$, as is a consequence of the identity

$$\alpha\Theta(n) = \Theta(\alpha(n))\alpha \quad (\forall \alpha \in \mathrm{Aut}(N), n \in N).$$

We denote by q_N the corresponding congruence, $q_N = \{(\alpha, \beta) \in \mathrm{Aut}(N) \times \mathrm{Aut}(N) : \alpha\Theta(N) = \beta\Theta(N)\}$. Passing to quotients, Θ projects to a representation χ of $Q = S/q_N$ in the monoid $\mathrm{O}(N)_+ = \mathrm{Aut}(N)/q_N$, defined by $\chi \circ q_N = q_N \circ \Theta$.

Definition 2.5. Let $S = S(Q, N)$ be an admissible extension of monoids, and assume N is Q^1-cancellative. Then the representation χ of Q in $\mathrm{O}(N)_+$ defined by the extension, is called *the character* of the extension.

Now let ρ be a left (resp. right) admissible congruence on S, and u be a fixed cross section of $Q = S/\rho$ in S with $u(e\rho) = e$. In the sequel, we shall assume all cross sections are enjoying this property. In view of Lemma 2.2, each element s of the extension $S = S(Q, \rho)$ can be written $s = nu(x)$ (resp. $s = u(x)n$), $n \in e\rho$, $x \in Q$. Put $N = e\rho$. We remark that $u(x)N = x$ (resp. $Nu(x) = x$) for all $x \in Q$.

Notice that in the representation $s = nu(x)$ (resp. $s = u(x)n$), the factor $u(x)$ is always uniquely given by s (the cross-section u being fixed), whereas n is unique in case N is right (resp. left) $u(Q)$-cancellative. Let $\phi : S \to Q$ denote the quotient map. Then

$$\phi[u(x_1)u(x_2)] = x_1x_2 = \phi u(x_1x_2),$$

Assuming ρ is left admissible, we can always write

$$u(x_1x_2) = \omega(x_1, x_2)u(x_1)u(x_2), \tag{2.3}$$

and

$$u(x_1)u(x_2) = \sigma(x_1, x_2)u(x_1x_2), \qquad (x_1, x_2 \in Q) \tag{2.4}$$

where

$$\omega(x_1, x_2), \sigma(x_1, x_2) \in N.$$

Similar relations follow if ρ is right admissible.

Lemma 2.3. *Let $S = S(Q, \rho)$ be a left admissible monoid extension and $u : Q \to S$ be a cross-section with $u(e\rho) = e$, and let ω and σ be as in (2.3) and (2.4).*

(a) *Assume the restricted right translation operator $R_{u(x)}|e\rho$ is injective for all x in Q (i.e., $e\rho$ is right Q^1-cancellative). Then $\sigma(x, y)$ is a left (resp. right) inverse to $\omega(x, y)$ $\quad(\forall x, y \in Q)$.*

(b) *Assume the restricted right translation operator $R_{u(x)u(y)}|e\rho$ is injective for all x, y in Q (i.e., $e\rho$ is right Q^2-cancellative). Then ω and σ are cochains mapping $Q \times Q$ into the maximal group $G(e\rho)$ of $e\rho$. In addition, $\sigma(x, y) = \omega(x, y)^{-1}$ $(\forall x, y \in Q)$.*

Proof. We have

$$\begin{aligned} u(xy) &= \omega(x,y)u(x)u(y) \\ &= \omega(x,y)\sigma(x,y)u(xy) \end{aligned}$$

Using cancellation to the right by $u(xy)$, we find that

$$\omega(x,y)\sigma(x,y) = e \qquad (x, y \in Q),$$

which proves (a). On the other hand,

$$\begin{aligned} u(x)u(y) &= \sigma(x,y)u(xy) \\ &= \sigma(x,y)\omega(x,y)u(x)u(y) \end{aligned}$$

Assuming $e\rho$ is Q^2-cancellative to the right, we find

$$\sigma(x,y)\omega(x,y) = e.$$

Consequently $\sigma = \omega^{-1}$. In addition, $u(e_Q) = e$, so that ω is a 2-cochain of $Q \times Q$ into $G(e\rho)$, and (b) follows. □

Remark 2.2. Replacing (2.3) and (2.4) with

$$\begin{aligned} &u(xy) = u(x)u(y)\tau(x,y), \\ &u(x)u(y) = u(xy)\mu(x,y), \quad x, y \in Q,\ \tau(x,y), \mu(x,y) \in N, \end{aligned} \tag{2.5}$$

we find that the above lemma holds true for τ and μ if we assume that each $L_{u(x)u(y)}|N$ $(x, y \in Q)$ is injective, $N = e\rho$.

If N is (left and right) Q^1-cancellative, the following relations follows readily,

$$\omega(x,y) = \theta_x\theta_y\tau(x,y)$$

and

$$\sigma(x,y) = \theta_{xy}\mu(x,y) \quad x, y \in Q.$$

Next we characterize those monoid extensions which are cancellative.

Proposition 2.1. *Let $S, Q,$ and N be monoids, and assume $S = S(Q,N)$ is an extension of Q by N. Then S is left (resp. right) cancellative if and only if*

(a) *both Q and N are left (resp. right) cancellative, and*

(b) *for any cross-section $u\colon Q \to S$ and each $x \in Q$, the translation operators $L_{u(x)}$ (resp. $R_{u(x)}$)$\colon N \to Nu(x)$ are injective.*

Proof. We shall verify the left property, the right one being similar. Let $p\colon S \to Q$ be the quotient map. Let Q be left cancellative and $u\colon Q \to S$ be a cross-section. Assume

$$mu(x)nu(y) = mu(x)n'u(y') \quad (m, n \in N; x, y, y' \in Q).$$

On applying p to both sides, we derive at once that $xy = xy'$. Hence $y = y'$ by the left cancellation law of Q. Consequently $u(y) = u(y')$, and

$$mu(x)nu(y) = mu(x)n'u(y).$$

Since N is normal, there are $n_1, n_1' \in N$ such that

$$u(x)n = n_1u(x), \quad u(x)n' = n_1'u(x).$$

Hence, if σ is as in (2.3), we find

$$mu(x)nu(y) = mn_1u(x)u(y) = mn_1\sigma(x, y)u(xy)$$

and

$$mu(x)n'u(y) = mn_1'\sigma(x, y)u(xy) \qquad (\sigma(x, y) \in N)$$

Now, if $R_{u(xy)}$ is injective on N, then

$$mn_1\sigma(x, y) = mn_1'\sigma(x, y),$$

which yields $n_1\sigma(x, y) = n_1'\sigma(x, y)$ if N is left cancellative. In view of Lemma 2.3 (a) we conclude $n_1 = n_1'$. Thus we have shown that $u(x)n = u(x)n'$. Hence, since $L_{u(x)}$ is injective, $n = n'$, and we have shown that S is left cancellative. Conversely, if S is left cancellative, it is obvious that N is so. Let us verify that Q is left cancellative. Assume $x, y, y' \in Q$ and $xy = xy'$. Then $u(xy) = u(xy')$, and if τ is as in Remark 2.2, we find $u(x)u(y) = u(x)u(y')\tau(x, y')$. Consequently, using injectivity of $L_{u(x)}$ on S, $u(y) = u(y')\tau(x, y')$. On applying the quotient map $S \to Q$, we obtain $y = y'$. Thus Q is left cancellative. Finally, statement (b) of the proposition is obvious. $\square$

Now assume $S = S(Q, N)$ is an extension of monoids. Let $u : Q \to S$ be a cross section. If for each x in Q the translations $L_{u(x)}$ and $R_{u(x)}$

are injective on N, then each x in Q defines an automorphism $\theta_x = R^{\sim}_{u(x)}L_{u(x)}$ of N. Expanding the cross section of products, we obtain:

$$\begin{aligned} u(x_1(x_2x_3)) &= \omega(x_1, x_2x_3)u(x_1)u(x_2x_3) \\ &= \omega(x_1, x_2x_3)u(x_1)\omega(x_2, x_3)u(x_2)u(x_3) \\ &= \omega(x_1, x_2x_3)\theta_{x_1}\omega(x_2, x_3)u(x_1u(x_2)u(x_3) \\ &= \omega(x_1, x_2x_3)\theta_{x_1}\omega(x_2, x_3)\sigma(x_1, x_2)u(x_1x_2)u(x_3) \\ &= \omega(x_1, x_2x_3)\theta_{x_1}\omega(x_2, x_3)\sigma(x_1, x_2)\sigma(x_1x_2, x_3)u((x_1x_2)x_3) \end{aligned}$$

Here ω and σ are as in (2.3) and (2.4). Further, using the associative law and the fact that $R_{u((x_1x_2)x_3)}$ is injective on N, we find

$$\omega(x_1, x_2x_3)\theta_{x_1}\omega(x_2, x_3)\sigma(x_1, x_2)\sigma(x_1x_2, x_3) = e \tag{2.6}$$

Consequently, if N is right Q^2-cancellative, we find

$$\omega(x_1, x_2x_3)\theta_{x_1}\omega(x_2, x_3) = \omega(x_1x_2, x_3)\omega(x_1, x_2), \tag{2.7}$$

which is called the cocycle identity, cf. Definition 1.3. In addition, the following representation property follows easily,

$$\theta_{x_1x_2} = I_{\omega(x_1,x_2)}\theta_{x_1}\theta_{x_2} \qquad (x_1, x_2 \in Q) \tag{2.8}$$

(where $I_x(n) = xnx^{-1}$ for each $x \in G(N)$, $n \in N$). Hence we have shown,

Lemma 2.4. *Let S be a monoid, N be a normal submonoid. Assume $u\colon Q = S/N \to S$ is a cross-section and each $R_{u(x)u(y)}$ $(x, y \in Q)$ is injective on N. Then the 2-cochain $\omega\colon Q \times Q \to G(N)$ defined by (2.3) (cf. Lemma 2.3 (b)) is a 2-cocycle for Q with values in N relative to the pre-representation $\theta = \Theta \circ u\colon Q \to Aut(N)$.*

The composition law on S can be expressed as follows,

$$\begin{aligned} mu(x)nu(y) &= m\theta_x(n)u(x)u(y) \\ &= m\theta_x(n)\omega(x, y)^{-1}u(xy) \end{aligned}$$

Another cross-section v of Q in S will give, in a similar way, a different cocycle α relative to another pre-representation ρ. Since $v(x) =$

$f(x)u(x)$, $u(x) = g(x)v(x)$ $(f(x), g(x) \in N)$, we find that $f(x) \in G(N)$. In addition, we derive at once the relations

$$\alpha(x,y) = f(x)\theta_x[f(y)]\omega(x,y)f(xy)^{-1} \tag{2.9}$$

and

$$\rho(x)(n) = f(x)\theta_x(n)f(x)^{-1} = I_{f(x)}\theta_x(n) \qquad (x,y \in Q,\ n \in N) \tag{2.10}$$

Definition 2.6. Let Q and N be monoids. Two cocycles (ω,θ) and (α,ρ) of Q with values in N are said to be *equivalent* if there exists a 1-cochain $f \in C^1(Q,N)$ such that the relations (2.9) and (2.10) are satisfied.

We remark that if the maximal group $G(N)$ is abelian, this equivalence relation reduces to the cohomology relation between cocycles.

Proposition 2.2. *Let $S = S(Q,N)$ be a Q^1-cancellative extension of monoids. Then the cocycles associated to the extension form an equivalence class.*

Proof. This is similar to the group case. □

We are now ready to characterize the class of all monoid extensions associated to cocycles.

Theorem 1. *Let Q and N be discrete monoids. An extension $S = S(Q,N)$ of Q by N is Q^2-cancellative if and only if $S(Q,N)$ is associated to a cocycle (ω,θ) of Q in N.*

Proof. In fact, if (ω,θ) is a cocycle of Q in N, we can define a semigroup operation on the direct product $N \times Q$ by letting

$$(m,x)(n,y) = (m\theta_x n\,\omega(x,y)^{-1}, xy). \tag{2.11}$$

Associativity follows from the cocycle-identity (2.7) and the representation property (2.8). That $S(Q,N)$ is Q^2-cancellative, follows from the fact that ω as well as each θ_x $(x \in Q)$ are invertible. Conversely, suppose $S = S(Q,N)$ is a Q^2-cancellative extension, and let $u : Q \to S$ be a cross section such that $R_{u(x)u(y)}$ and $L_{u(x)u(y)}$ are invertible on N for all x, y in Q. In light of Lemma 2.4, we can find a cocycle (ω,θ) associated to $S(Q,N)$. □

Let Q, N, and (ω, θ) be as in the above theorem. The extension $S(Q, N)$ given by 2.11 is called the *canonical (algebraic) extension* of Q by N associated to the cocycle (ω, θ).

Definition 2.7. Two extensions $S = S(Q, N)$ and $S' = S'(Q, N)$ of monoids are said to be *equivalent* if there is an isomorphism $f\colon S \to S'$ whose restriction to N and projection on Q (identified to S'/N) are the identity maps. f is called an *equivalence isomorphism of* $S(Q, N)$ *onto* $S'(Q, N)$.

Thus an equivalence isomorphism of $S = S(Q, N)$ onto $S' = S'(Q, N)$ maps each cross-section of Q in S onto a cross-section of Q in S'. Hence we have

Lemma 2.5. *The cocycles associated with two equivalent extensions are equivalent.*

Proposition 2.2, Theorem 1, and the above lemma are summarized in

Proposition 2.3. ("Schreier's theorem for monoids") *Let Q and N be two discrete monoids. Then there is a bijective correspondence between the equivalence classes of Q^2-cancellative extensions $S(Q, N)$ and the equivalence classes of cocycles (ω, θ) of Q with values in N.*

We also mention the following analogue to [EM2, Thm. 11.1] for monoids.

Proposition 2.4. *Assume Q and N are discrete monoids and $\chi\colon Q \to Out(N)$ is a representation. If there exist cocycles (ω, θ) of Q with values in N such that $q\theta = \chi$ ($\theta\colon Aut(N) \to Out(N)$ the canonical map), then their families of equivalence classes are in bijective correspondence with $H^2(Q, Z_N, \chi)$, where Z_N stands for the center of N in the maximal group $G(N)$, $Z_N = \mathrm{Cent}(N) \cap G(N)$.*

Proof. This is similar to the group case.

3. Extensions of *-monoids.

In this section S will denote a (discrete) monoid possessing an involution. Assume, in addition, S is an extension $S = S(Q, N)$ of monoids

Q and N in which N is Q^1-cancellative. Clearly, both N and Q inherit natural involutions from S. Let $u\colon Q \to S$ be a cross-section. Both $u(x^*)$ and $u(x)^*$ $(x \in Q)$ projects onto x^*, hence there are mappings $\mu, \nu\colon Q \to N$ such that

$$\begin{aligned} u(x)^* &= \nu(x)u(x^*), \\ u(x^*) &= u(x)^*\mu(x) \end{aligned} \tag{3.0}$$

Therefore,

$$u(x) = u(x^*)^*\nu(x)^* \quad \text{and} \quad u(x) = u(x^*)^*\mu(x^*),$$

so

$$\mu(x) = \nu(x^*)^*,$$

by the Q^1-cancellation property. (3.0) can then be rewritten as

$$\begin{aligned} u(x)^* &= \nu(x)u(x^*), \\ u(x^*) &= u(x)^*\nu(x^*)^* \end{aligned} \tag{3.1}$$

We can also write $u(x)^* = u(x^*)\nu_1(x)$, $\nu_1(x) \in N$, and combining this with (3.1), we derive

$$u(x) \overset{(3.1)}{=} u(x^*)^*\nu(x)^* = u(x)\nu_1(x^*)\nu(x)^*,$$

hence

$$\nu_1(x)\nu(x^*)^* = e_N,$$

and we conclude that $\nu(x)$ and $\nu_1(x)$ are invertible, hence are elements of $G(N)$. Moreover, $\nu_1(x) = \nu(x)^*$. In particular,

$$\begin{aligned} u(x)^* &= u(x^*)\nu(x^*) \\ u(x)^* &= \nu(x)u(x^*) \end{aligned} \tag{3.2}$$

From (3.2) we also derive that

$$u(x)\nu(x) = \nu(x^*)u(x),$$

and hence,

$$\nu(x^*) = \theta_x\nu(x) \tag{3.3}$$

Lemma 3.1. *Let $S = S(Q, N)$ be a monoid with involution. Assume N is Q^1-cancellative, and let $u\colon Q \to S$ be a cross-section. Then there is a one-cochain ν of Q into the maximal group $G(N)$ of $N, \nu \in C^1(Q, G(N))$, such that*

$$u(x)^* = u(x^*)\nu(x^*) = \nu(x)u(x^*) \qquad (x \in Q)$$

ν is uniquely given by u, and also satisfies the relation

$$\nu(x^*) = \theta_x[\nu(x)]$$

Combining the product property of the involution with the definition of θ, we easily find

$$\theta_x(n^*) = [\theta_x n]^* \qquad (n \in N,\ x \in Q) \tag{3.4}$$

Moreover, if $\theta_{x^*}n = m$, we derive $u(x^*)n = mu(x^*)$, so that $\nu(x)^{-1}u(x)^*n = m\nu(x)^{-1}u(x)^*$, or

$$n^*u(x)\nu(x) = u(x)\nu(x)m^*.$$

Hence

$$I_{\nu(x)^{-1}}[\theta_x^{-1}(n^*)]^* = m,$$

or by (3.4),

Lemma 3.2. *With notations as above, the pre-representation θ is enjoying the property*

$$\theta_{x^*} = I_{\nu(x)^{-1}}\theta_x{}^{-1} \qquad (x \in Q) \tag{3.5}$$

Applying (3.5) twice, we also derive,

$$I_{\nu(x^*)}\theta_x = \theta_x I_{\nu(x)} \tag{3.6}$$

On calculating the involution of a product $u(x)u(y)$, we obtain;

$$[u(x)u(y)]^* = [\omega(x,y)^{-1}u(xy)]^* = \nu(xy)\theta_{y^*x^*}[\omega(x,y)]u(y^*x^*)$$

and

$$\begin{aligned}[u(x)u(y)]^* &= u(y)^*u(x)^* = \nu(y)u(y^*)\nu(x)u(x^*) \\ &= \nu(y)\theta_{y^*}[\nu(x)]u(y^*)u(x^*) \qquad (x, y \in Q)\end{aligned}$$

Hence we have the following important relationship between $\nu(x), \nu(y)$, and $\nu(xy)$.

Lemma 3.3. *Let $S(Q,N)$ be a monoid with involution. Assume N is Q^1-cancellative and let $u : Q \to S$ be a cross section. Then*

$$\nu(y)\theta_{y^*}[\nu(x)] = \nu(xy)\theta_{y^*x^*}[\omega(x,y)]\omega(y^*,x^*) \qquad (x,y \in Q). \tag{3.7}$$

We remark that for central extensions (i.e. $\theta = 1$ and N abelian) (3.7) reduces to

$$\nu(y)\nu(x) = \nu(xy)\omega(x,y)\omega(y^*,x^*)$$

If in addition ω is normalized, this is simply anti-multiplicativity, $\nu(y)\nu(x) = \nu(xy)$.

Definition 3.1. We say that an extension $S(Q,N)$ of two monoids Q and N with involution is an *involutive extension*, if S has an involution which yields the involution of N and Q by restriction and projection, respectively.

Proposition 3.1. *A Q^1-cancellative extension $S(Q,N)$ of discrete monoids Q and N with involution is an involutive extension if and only if, for any cross-section $u : Q \to N$ and corresponding cocycle (ω,θ) associated to the extension, there exists a 1-cochain $\nu\colon Q \to G(N)$ such that (3.7) is satisfied. In this case the involution on $S(Q,N)$ is uniquely given by the relation*

$$(nu(x))^* = \nu(x)\theta_{x^*}(n^*)u(x^*) \qquad (n \in N,\ x \in Q). \tag{3.8}$$

Proof. First, if $S(Q,N)$ is involutive, we have already seen that (3.7) holds. Conversely, by (3.7)

$$\begin{aligned}
[mu(x)nu(y)]^* &= [m\theta_x(n)\omega(x,y)^{-1}u(xy)]^* \\
&= \nu(xy)\theta_{(xy)^*}[\omega(x,y)\theta_x(n^*)m^*]u((xy)^*) \\
&= \nu(y)\theta_{y^*}[\nu(x)]\omega(y^*,x^*)^{-1}\theta_{y^*x^*}\theta_x(n^*)\theta_{y^*x^*}(m^*)u((xy)^*)
\end{aligned}$$

and

$$\begin{aligned}
[nu(y)]^*[mu(x)]^* &= [\nu(y)\theta_{y^*}(n^*)u(y^*)][\nu(x)\theta_{x^*}(m^*)u(x^*)] \\
&= \nu(y)\theta_{y^*}(n^*)\theta_{y^*}[\nu(x)\theta_{x^*}(m^*)]\omega(y^*,x^*)^{-1}u(y^*x^*)
\end{aligned}$$

We calculate the expression

$$A = \nu(y)\theta_{y^*}[\nu(x)]\omega(y^*,x^*)^{-1}\theta_{y^*x^*}\theta_x(n^*)\theta_{y^*x^*}(m^*)$$

Here

$$\begin{aligned}\omega(y^*,x^*)^{-1}\theta_{y^*x^*}\theta_x(n^*) &= \theta_{y^*}\theta_{x^*}\theta_x(n^*)\omega(y^*,x^*)^{-1}\\ &= [\theta_{y^*}I_{\nu(x)^{-1}}(n^*)]\omega(y^*,x^*)^{-1},\end{aligned}$$

so we find

$$\begin{aligned}A &= \nu(y)\theta_{y^*}[\nu(x)]\theta_{y^*}[\nu(x)^{-1}]\theta_{y^*}[n^*\nu(x)]\omega(y^*,x^*)^{-1}\theta_{y^*x^*}(m^*)\\ &= \nu(y)\theta_{y^*}(n^*)\theta_{y^*}[\nu(x)]\theta_{y^*}\theta_{x^*}(m^*)\omega(y^*,x^*)^{-1}\\ &= \nu(y)\theta_{y^*}(n^*)\theta_{y^*}[\nu(x)\theta_{x^*}(m^*)]\omega(y^*,x^*)^{-1}.\end{aligned}$$

This yields readily the product property of the involution:

$$[mu(x)nu(y)]^* = [nu(y)]^*[mu(x)]^*. \tag{3.9}$$

As remarked above, the product property (3.9) implies (3.6). Hence

$$\begin{aligned}[mu(x)]^{**} &= [\nu(x)\theta_{x^*}(m^*)u(x^*)]^*\\ &= \nu(x^*)\theta_x[\theta_{x^*}(m)\nu(x)^{-1}]u(x)\\ &= \theta_x[\nu(x)\theta_{x^*}(m)\nu(x)^{-1}]u(x) &&\text{(by (3.3))}\\ &= \theta_x\theta_x^{-1}(m)u(x) &&\text{(by (3.6))}\\ &= mu(x).\end{aligned}$$

We have shown that (3.8) defines an involution on $S(Q,N)$. Clearly then, $S(Q,N)$ is involutive. □

Proposition 3.2. *If a monoid S is an involutive Q^2-cancellative extension, $S = S(Q,N)$, then we can find a triplet (ω,θ,ν) satisfying the cocycle identity (2.10), the pre-representation property (2.11), and the involutive relation (3.7).*

Definition 3.2. A triplet (ω,θ,ν) as in Proposition 3.2, determined by a cross-section $u\colon Q\to N$, is called an *(involutive) cocycle* associated to the extension $S(Q,N)$.

Assume $S(Q,N)$ is an involutive Q^2-cancellative extension with a cross-section $u\colon Q\to S$ and an associated cocycle (ω,θ,ν). Any other cross-section $v\colon Q\to S$ gives in a similar fashion a cocycle (σ,ρ,μ) associated to $S(Q,N)$. Since

$$v(x) = f(x)u(x), \quad u(x) = g(x)v(x) \qquad (x\in Q),$$

where $f(x), g(x) \in N$, we find $f(x) = g(x)^{-1}$ and hence $f(x) \in G(N)$. Consequently $f \in C^1(Q, N)$. In addition to the relations (2.9) and (2.10), we derive

$$\mu(x) = \nu(x)[\theta_{x^*} f(x)^{-1}] f(x^*)^{-1} \qquad (x \in Q). \tag{3.10}$$

In fact, this follows from the identities

$$v(x)^* = \mu(x) v(x^*) = \mu(x) f(x^*) u(x^*)$$

and

$$\begin{aligned} v(x)^* &= [f(x)u(x)]^* = u(x)^* f(x)^{-1} = \nu(x) u(x^*) f(x)^{-1} \\ &= \nu(x) \theta_{x^*}[f(x)^{-1}] u(x^*) \end{aligned}$$

Definition 3.3. Two involutive cocycles (ω, θ, ν) and (σ, ρ, μ) associated to an involutive Q^2-cancellative extension $S(Q, N)$ of monoids are said to be *equivalent* if there exists a 1-cochain $f \in C^1(Q, N)$ satisfying the relations (2.9), (2.10), and (3.10).

Proposition 3.3. *The involutive cocycles associated to an extension form an equivalence class.*

Proposition 3.4. *Assume Q and N are discrete monoids with involution. Each involutive cocycle (ω, θ, ν) of Q with values in N is associated to an involutive Q^2-cancellative extension $S(Q, N)$.*

Proof. Exactly as for ordinary monoids, the direct product $S = Q \times N$ is endowed with the composition law of (2.11):

$$(m, x)(n, y) = (m\theta_x(n)\omega(x, y)^{-1}, xy)$$

Next, the involution is defined by

$$(m, x)^* = (\nu(x)\theta_{x^*}(m^*), x^*) \qquad (m \in N,\ x \in Q). \tag{3.11}$$

Since the cocycle is involutive, the identity in (3.8) holds. Hence S is an involutive extension by Proposition 3.1. □

Equivalence between two involutive extensions is defined analogously to the case of ordinary monoid exensions, see Definition 2.4 (the isomorphism is now required to be involutive). An equivalence isomorphism of two involutive extensions $S(Q, N)$ and $S'(Q, N)$ maps each cross-section of Q in S onto a cross-section of Q in S'.

Lemma 3.4. *The cocycles associated with two equivalent involutive extensions are equivalent.*

We also have the following analogue to Schreier's theorem for involutive monoids:

Proposition 3.5. *Let Q and N be two discrete monoids with involution. Then there is a bijective correspondence between the equivalence classes of involutive Q^2-cancellative extensions $S(Q,N)$ and the equivalence classes of cocycles (ω,θ,ν) of Q with values in N.*

In Section 4 we shall extend this result to topological monoids. Let us mention that an analogue to Proposition 2.5 also holds for cancellative monoids with involution.

4. Extensions of topological monoids.

In this section we study the topological properties of monoid extensions. Because of the lack of an inverse operation, special care must be taken when developing an extension theory. A semigroup S is said to be *topological*, if S is equipped with a topology which makes the map

$$(x,y)\mapsto xy,\quad S\times S\to S \tag{4.0}$$

continuous (where $S\times S$ is endowed with the product topology). In the sequel we shall assume all topological semigroups are Hausdorff. Clearly, since translations need not be open maps, a neighborhood base at a single point does not always determine the entire topology on a semigroup. A simple example is provided by the nonnegative real numbers with addition and the relative topology from $\mathbb{R}$. Similar examples can be obtained by looking at monoids for which the identity element is contained in the topological boundary. If a topological semigroup S carries a continuous involution

$$x\mapsto x^*,\quad S\to S, \tag{4.1}$$

we say that S is an *involutive topological semigroup*, or simply a topological $*$-semigroup. By analogy to the case of topological groups, an extension $S(Q,N)$ of topological monoids is called *fibered* if there exists a continuous cross section $u: Q\to S$. We also make the following definition, cf.[C, Def. 3.2],

Definition 4.1. An extension $S(Q,N)$ of topological monoids is called *almost fibered* if it admits a cross-section $u\colon Q \to S$ such that the map $h\colon nu(x) \mapsto (n,x)$, of S onto the direct product semi-group $N \times Q$, is continuous at the point $e_N u(e_Q)$ and such that the inverse map h^{-1} is continuous at the point (e_N, e_Q)).

In the almost fibered case it is clear that the cross-section u is continuous at e_Q. For, the map $(e_N, x) \overset{h^{-1}}{\longmapsto} e_N u(x) = u(x)$ is continuous at (e_N, e_Q). We shall see next that for Q^1-cancellative extensions $S(Q,N)$, the converse statement also holds.

Proposition 4.1. *Let S be a topological monoid, and assume $S = S(Q,N)$ is a Q^1-cancellative extension of topological monoids Q and N. Then $S(Q,N)$ is almost fibered if and only if there exists a cross-section $u\colon Q \to S$ which is continuous at the identity e_Q.*

Proof. Assume $u\colon Q \to S$ is a cross-section continuous at e_Q, and let $p\colon S \to Q$ be the canonical map. Obviously $h^{-1}\colon (n,x) \mapsto nu(x)$, $N \times Q \to S$, is continuous at (e_N, e_Q). In order to verify the continuity property of h, let $U \times V$ be a neighborhood of (e_N, e_Q) in $N \times Q$. It suffices to find a neighborhood of $e = e_N u(e_Q)$ contained in $h^{-1}(U \times V) = U \cdot u(V)$. Let W be a neighborhood of e such that $W \cap N = U$ and $p(W) \subset V$. Since u is continuous at e_Q, we can find a neighborhood W_1 of e such that $W_1 \subset W$ and $u \circ p(W) \subset W_1$. For each x in $p(W_1)$, let $U'_x = p^{-1}(\{x\}) \cap W_1$. Then

$$U'_x = (Nu(x)) \cap W_1 \subset Uu(x),$$

and consequently

$$W_1 = \bigcup_{x \in p(W_1)} U'_x$$

On U'_x, $\rho(x)^{\sim} = R^{\sim}_{u(x)}$ is well-defined (since $Nu(x) = \rho(x)N \subset \operatorname{Range}\rho(x)$).

Put $U_x = \rho(x)^{\sim} U'_x$. Then, since $U'_x \subset Uu(x)$, we have $U_x \subset U$. Now

$$U_x = \rho(x)^{\sim} U'_x = \rho(x)^{\sim}(Nu(x) \cap W_1) \subset N,$$

and since $\rho(x)$ is continuous, its left inverse $\rho(x)^{\sim}$ is an open map, and we find that U_x is a neighborhood of e_N in N. Hence

$$W_1 = \bigcup_{x \in p(W_1)} \rho(x) U_x \subset Uu(V) = h^{-1}(V \times U).$$

The converse implication is clear. This completes the proof. □

In order to investigate further the connection between 2−cocycles and extensions of topological monoids, we shall need the following,

Proposition 4.2. *Assume S is a cancellative monoid, and let $\Theta : S \to Aut(S)$ be as in (2.2). Let $\mathcal{U}$ be a family of subsets of S which all contains the identity e of S. Assume the following properties hold:*

(i) For all U in $\mathcal{U}$, there is a V in $\mathcal{U}$ such that $V^2 \subset U$.

(ii) For all U in $\mathcal{U}$ and all x in U, there is a V in $\mathcal{U}$ such that $Vx \subset U$.

(iii) For all U in $\mathcal{U}$ and all x in S there is a V in $\mathcal{U}$ such that $\Theta(x)V \subset U$.

(iv) For all U and V in $\mathcal{U}$, there is a W in $\mathcal{U}$ such that $W \subset U \cap V$.
Then the family $\{Ux : x \in S, U \in \mathcal{U}\}$ is an open basis for a topology of S. With this topology, S is a topological monoid.
If, in addition, S is involutive and

(v) For all U in $\mathcal{U}$, there is a V in $\mathcal{U}$ such that $V^* \subset U$,
then S is an involutive topological monoid with the above topology.

Proof. S is a topological monoid: Let $a, b \in S$, $U \in \mathcal{U}$. By (i) there is a $V \in \mathcal{U}$ such that $V^2 \subset U$ and, by (iii) there is a $W \in \mathcal{U}$ with $\Theta(a)W \subset V$. Hence $VaWb = (V\Theta(a)W)ab \subset (VV)ab \subset Uab$, and the map $(a,b) \mapsto ab, \quad S \times S \to S$ is continuous.

Next, assume S is involutive and satisfies (v). If $a \in S$ and $U \in \mathcal{U}$, there is a $W \in \mathcal{U}$ with $\Theta(a^*)W \subset U$, using (iii). By (v) there is a $V \in \mathcal{U}$ such that $V^* \subset W$. Hence

$$(Va)^* = a^*V^* = (\Theta(a^*)V^*)a^* \subset \Theta(a^*)W)a^* \subset Ua^*,$$

and it follows that the involution $a \mapsto a^*$ is continuous on S. □

Definition 4.2. Let Q and N be topological monoids and let (ω, θ) be a cocycle of Q with values in N. (ω, θ) is said to be *continuous at the identity* e_Q of Q (resp. *continuous*) if

(a) the 2-cochain ω is continuous at (e_Q, e_Q) in $Q \times Q$ (resp. continuous on $Q \times Q$),

(b) for each $x \in Q$, the mapping $s \mapsto \omega(\Phi(s)x, s)^{-1}\omega(s,x)$ is continuous at e_Q (resp. on Q), in which $\Phi(s) = R_s^{\sim} L_s$, and

(c) the mapping $(n, x) \mapsto \theta_x(n)$, $N \times Q \to N$, is continuous at each point (n, e_Q), $n \in N$ (resp. on $N \times Q$).

Let Q and N be topological monoids. Assuming (ω, θ) is a cocycle of Q with values in N, we have seen above (Theorem 1 of §2) how to construct (algebraically) a monoid extension $S(Q, N, \omega, \theta)$ of Q by N, associated to the cocycle (ω, θ), by defining the composition rule

$$(m, x)(n, y) = (m\theta_x(n)\omega(x, y)^{-1}, xy) \tag{4.2}$$

on the direct product $N \times Q$. $S(Q, N, \omega, \theta)$ will be Q^2-cancellative in view of Theorem 1. If (ω, θ) is continuous at the identity, we can topologize this extension as follows. As a neighborhood base $\mathcal{U}$ at $e = (e_N, e_Q)$, we take the family of all sets $U \times V$ where U, resp. V, is a neighborhood of e_N, resp. e_Q. A neighborhood base at an arbitrary point (n, x) is obtained by translation. If the semigroup composition of (4.2) is compatible with this topology, then $S = S(Q, N, \omega, \theta)$ is by definition an almost fibered extension. Let us verify the conditions of Prop. 4.2. (i): Is similar to the group case. (ii) and (iv) are clear. (iii): Let $U \times V \in \mathcal{U}$ and $(n, x) \in S$. Since S is cancellative, the map $\Theta(m, x) = R^{\sim}_{(m,x)} L_{(m,x)}$ is well-defined for all $(m, x) \in S$. For $(m, x), (n, y) \in S$ arbitrary, we write

$$\Theta(m, x)(n, y) = (k, v)$$

This means

$$(m, x)(n, y) = (k, v)(m, x)$$

or

$$(m\theta_x(n)\omega(x, y)^{-1}, xy) = (k\theta_v(m)\omega(v, x)^{-1}, vx)$$

Equivalently,

$$v = \Theta(x)y$$

and

$$k\theta_v(m) = m\theta_x(n)\omega(x, y)^{-1}\omega(v, x)$$

Let $\{(n_\nu, y_\nu)\}_{\nu \in A}$ be a net in S that converges to the identity. Put

$$v_\nu = \theta_x(n_\nu), \ m_\nu = \theta_{v_\nu}(m),$$

and

$$\Theta(m, x)(n_\nu, y_\nu) = (k_\nu, v_\nu).$$

Then

$$k_\nu \theta_{v_\nu}(m) = m\theta_x(n_\nu)\omega(x, y_\nu)^{-1}\omega(\Theta(x)(y_\nu), x),$$

and since (ω, θ) is continuous at e_Q, we find

$$k_\nu m_\nu \to m, \quad m_\nu \to m$$

Hence, in view of the cancellation property, it is clear that $k_\nu \to e_N$. Indeed, assume instead $\{k_\nu\}_\nu$ does not converge to e_N, and let V_m be an arbitrary neighborhood of m. Then we can find a $\nu_0 \in A$ such that

$$k_\nu m_\nu \in V_m \textit{ and } m_\nu \in V_m, \textit{ for } \nu \succ \nu_0$$

If U_e is an arbitrary neighborhood of e_N, we can find a subnet of $\{k_\nu\}_\nu$, also denoted $\{k_\nu\}_\nu$, such that for all ν, $k_\nu \notin U_e$. Then, using cancellation,

$$k_\nu m_\nu \notin U_e m_\nu$$

By convergence of the nets $\{k_\nu m_\nu\}_\nu$ and $\{m_\nu\}_\nu$, we find that $m \notin U_e m$, clearly a contradiction. Thus $k_\nu \to e_N, n_\nu \to e_Q$, and (iii) follows.

Definition 4.3. Let Q, N, and (ω, θ) be as above. We assume the cocycle (ω, θ) is continuous at e_Q. The extension $S(Q, N, \omega, \theta)$, topologized as above, is called the *canonical (topological) extension* of Q by N *associated to* (ω, θ).

By its definition, a canonical extension $S(Q, N, \omega, \theta)$ is almost fibered. Conversely, assume $S(Q, N)$ is an almost fibered cancellative extension. If $u : Q \to S(Q, N)$ is a cross-section, continuous at e_Q, and (ω, θ) denotes the corresponding cocycle, then the map $nu(x) \mapsto (n, x)$ of $S(Q, N)$ onto $S(Q, N, \omega, \theta)$ is bicontinuous at the identiy, and is an equivalence isomorphism (cf. Def. 2.5). However, since the topology of $S(Q, N)$ can not in general be recovered from a neighborhood of the identity element, the canonical topological extension $S(Q, N, \omega, \theta)$ need not agree topologically with $S(Q, N)$. We summarize the above arguments in

Proposition 4.3. *A cancellative extension $S(Q, N)$ is almost fibered if $S(Q, N)$ is equivalent to the canonical extension of Q by N associated to a cocycle (ω, θ), via an equivalence isomorphism that is bicontinuous* *at the identity. The converse implication holds true if translations on* $S(Q,N)$ *are open maps.*

Let (ω,θ) and (σ,ρ) be two equivalent cocycles, both continuous at e_Q. If the two cocycles correspond to cross sections u and v, respectively, then there is a map $f : Q \to G(N)$ such that $v(x) = f(x)u(x)$ $(x \in Q)$ (cf. (2.9) and (2.10)). The map $(n,x) \mapsto (nf(x),x)$ is an equivalence isomorphism (in the algebraic sense) of $S(Q,N,\sigma,\rho)$ onto $S(Q,N,\omega,\theta)$. If the maximal group $G(N)$ is a topological group (i.e. if the inverse operation on $G(N)$ is continuous), this map is bi-continuous at the identity if and only if f is continuous at e_Q.

Definition 4.4. Two cocycles (ω,θ) and (σ,ρ) of Q with values in N are said to be *continuously equivalent at the identity* (resp. *continuously equivalent*) if there is an $f \in C^1(Q,N)$ which satisfies

$$\alpha(x,y) = f(x)\theta_x[f(y)]\omega(x,y)f(xy)^{-1} \tag{4.3}$$

and

$$\rho(x)(n) = f(x)\theta_x(n)f(x)^{-1} = I_{f(x)}\theta_x(n) \qquad (x,y \in Q,\ n \in N), \tag{4.4}$$

and such that the corresponding equivalence isomorphism $(n,x) \mapsto (nf(x),x)$; $S(Q,N,\sigma,\rho) \to S(Q,N,\omega,\theta)$ is continuous at the identity e_Q (respectively continuous).

According to this, each almost fibered extension is canonically associated to a class consisting of cocycles continuous at e_Q and continuously equivalent at the identity. In particular, if $G(N)$ is a topological group, this is the class of cocycles defined by cross-sections continuous at the point e_Q, Prop. 4.1. Now, two equivalent extensions are associated to the same class of cocycles, and we have:

Proposition 4.4. *Let Q and N be cancellative monoids. There is a bijective correspondence between the equivalence classes of almost fibered extensions $S(Q,N)$ and the classes of cocycles (ω,θ) of Q with values in N, continuous at e_Q, which are continuously equivalent at the identity. If $G(N)$ is a topological group, this is the class of cocycles defined by cross-sections continuous at the point e_Q.*

5. Applications to representation theory.

We shall now apply the above extension theory to the theory of representations of cancellative monoids. With a few exceptions we do not assume the monoids are involutive. There are several possible choices for representations, such as representations by bounded operators, by invertible operators, by isometries, or by unitary operators in Hilbert spaces. We shall confine ourself to the isometric and unitary cases. More specifically, we shall treat the problem of extending a unitary representation from a normal S-cancellative submonoid N to its stability submonoid in S. Just as for groups it becomes necessary to consider multipliers (cocycles) and multiplier representations, [Ma, Theorem 8.2]. Let ω be a fixed multiplier on N. As we shall see below (Cor. 5.1), the cancellation property induces an $\omega-$action of S on $\hat{N}$, the set of equivalence classes of irreducible unitary ω representations of N, by automorphisms. Although the following results are valid for many locally compact second countable monoids S possessing a Haar measure, we give here the details only in the discrete case. Notice that by the cancellation property, the left and right translation operators preserve cardinalities. Hence counting measure is translation invariant on a discrete cancellative monoid. As far as we know, existence of a Haar measure on a locally compact cancellative semigroup is an open problem. In view of the fact that right (and left) translations need not be surjective, the right regular and induced representations are generally isometric but not unitary, cf. Example 5.1.

We remark that all ray semigroups (i.e. semigroups generated by their one-parameter subsemigroups) are cancellative, [G]. On the other hand, all $C_s{}^\infty-$semigroups in the sense of [G] are locally cancellative, but need not be cancellative. Also, there exist cancellative semigroups which do not embed in a topological group.

Definition 5.1. Let S be a discrete monoid. By a *multiplier* on S we understand a 2-cocycle ω of S with values in the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers and with respect to the trivial pre-representation θ, i.e., $\omega : S \times S \to \mathbb{C}^*$ is enjoying the properties

$$(5.1) \qquad \begin{aligned} &(i) \quad \omega(x,e) = \omega(y,e) = 1 \\ &(ii) \quad \omega(xy,z)\omega(x,y) = \omega(x,yz)\omega(y,z) \qquad (x,y,z \in S) \end{aligned}$$

If in addition S has an involution $x \mapsto x^*$ and ω is enjoying the further property

$$\omega(x,x^*) = 1 \qquad (x \in S),$$

we shall say that ω is *normalized.* ω is *unitary* if $|\omega(x,y)| = 1 \quad (\forall x, y \in S)$.

Remark 5.1. *It is clear from the definition that multipliers correspond to central extensions of S by $\mathbb{C}^*$, or by the circle group $\mathbb{T}$ in the unitary case, cf. §§1 and 3. If S is involutive, one can prove that each unitary multiplier is similar to a normalized one. If ω is unitary and normalized one can also show that $\omega(x,y)^{-1} = \omega(y^*,x^*)$.*

Definition 5.2. Let S be a monoid and ω be a multiplier on S. A map T of S into the monoid $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H} = \mathcal{H}_T$ is called an ω *representation* of S if the following conditions hold true for all $x, y \in S$,

$$(5.2) \qquad \begin{aligned} &(i) \quad T_e = I \\ &(ii) \quad T_{xy} = \omega(x,y)T_xT_y. \end{aligned}$$

T is *isometric* if each operator T_x is an isometry, and T is *unitary* if each T_x is unitary. If in addition S carries an involution, it is also natural to require that T be a $*$-representation, i.e.,

$$(iii)\ {T_x}^* = \omega(x^*,x)T_{x^*}$$

(We remark that unitary *-representations are not extremely interesting in the present context, since their existence implies that S is a group with inverse operation $x^{-1} = x^*$.)

Example 5.1. Let S be a cancellative monoid and assume ω is a unitary multiplier on S.
(1) There exist isometric ω representations. The right regular ω representation of S on $\ell^2 = \ell^2(S)$ can be defined by

$$(R_xf)(y) = \omega(y,x)^{-1}f(yx) \qquad (\forall f \in \ell^2,\ x, y \in S).$$

The fact that counting measure on S is translation invariant implies that R is isometric. R is unitary if, in addition, right translations are surjective. In the present context, the left regular representation is defined (and is unitary) if left translations are surjective.
(2) There exist unitary representations. In fact, the "conjugation" representation C of S on ℓ^2 by inner automorphisms is given by

$$(C_x f)(y) = f(\Theta(x)^{-1}y) \qquad (\forall f \in \ell^2(S),\ x, y \in S).$$

Since the automorphisms $\Theta(x)$ preserve cardinalities, C is unitary. The kernel of C is the centre of S.
(3) Let $I(S)$ denote the subgroup of $Aut(S)$ generated by all $\Theta(s)$ and $\Theta(t)^{-1}$, $s, t \in S$. Assume ω is given by a multiplier ϕ on $I(S)$, i.e., $\omega(x, y) = \phi(\Theta(x), \Theta(y))$, $x, y \in S$. In this case we can define a unitary representation U of S on $\ell^2(I(S))$ by $U_x f(\zeta) = \omega(x, y)^{-1} f(\Theta(x)\zeta)$, $x \in S, \zeta \in I(S)$. U is the identity on the centre Z of S. The lifting of U to S/Z can be identified with the restriction to $S/Z \cong \Theta(S)$ of the right regular ω representation of $I(S)$. Via restriction, $\ell^2(S/Z)$ is embedded as an S/Z invariant subspace of $\ell^2(I(S))$. On this subspace U operates as the right regular ω representation R of S/Z.

When trying to transfer the "multiplier action" of Mackey [Ma, Lemma 4.2] from the group situation to the unitary dual of a normal submonoid of S, the first problem one encounters is that the expression

$$g_s(x) = \omega(sx, s^{-1})\omega(s, x)/\omega(s^{-1}, s)$$

does not make sense on monoids. However, this is solved relatively comfortably by noting that on groups the relation

$$\omega(sx, s^{-1})\omega(s^{-1}, s)^{-1} = \omega(sxs^{-1}, s)^{-1}$$

holds true. This is a consequence of the cocycle identity. Now, on cancellative monoids, the right hand side of this formula has the obvious analogue $\omega(\Theta(s)x, s)^{-1}$, suggesting the following result,

Lemma 5.1. *Let S be a monoid, K be a submonoid of S, and assume ω is a multiplier on S. Further, assume $x \in S$ normalizes K, $xK = Kx$, and both $L_x|K$ and $R_x|K$ are injective. Put*

$$\omega'(y, z) = \omega(\Theta(x)y, \Theta(x)z) \quad (\forall y, z \in S).$$

Then the multipliers ω' and ω are similar. Indeed,

$$\omega'(y,z)/\omega(y,z) = g_x(yz)/g_x(y)g_x(z) \tag{5.3}$$

where $g_x(y) = \omega(\Theta(x)y, x)^{-1}\omega(x,y)$.

Proof. On multiplying with denominators, the formula in (5.3) is seen to be equivalent to

$$\begin{aligned} &\omega(\Theta(x)y, \Theta(x)z)\omega(\Theta(x)(yz), x)\omega(x,y)\omega(x,z) \\ &= \omega(x,yz)\omega(y,z)\omega(\Theta(x)y,x)\omega(y,z)\omega(\Theta(x)z,x) \end{aligned} \tag{5.4}$$

Now, using the cocycle identity, we find

$$\begin{aligned} &\omega(\Theta(x)y, \Theta(x)z)\omega(\Theta(x)(yz), x) \\ &= \omega(\Theta(x)y, \Theta(x)z)\omega(\Theta(x)y\Theta(x)z, x) \\ &= \omega(\Theta(x)y, xz)\omega(\Theta(x)z, x) \end{aligned} \tag{5.5}$$

Further,

$$\begin{aligned} &\omega(\Theta(x)y, xz)\omega(x,z) = \omega([\Theta(x)y]x, z)\omega(\Theta(x)y, x) \\ &= \omega(xy, z)\omega(\Theta(x)y, x) \end{aligned} \tag{5.6}$$

and

$$\omega(xy,z)\omega(x,y) = \omega(x,yz)\omega(y,z) \tag{5.7}$$

Combining the equations (5.5)-(5.7), we obtain

$$\begin{aligned} &\omega(\Theta(x)y, \Theta(x)z)\omega(\Theta(x)(yz), x)\omega(x,y)\omega(x,y)\omega(x,z) \\ &= \omega(\Theta(x)y, xz)\omega(\Theta(x)z, x)\omega(x,z)\omega(x,y) \quad (\textit{by } (5.5)) \\ &= \omega(xy,z)\omega(\Theta(x)y,x)\omega(x,y)\omega(\Theta(x)z,x) \quad (\textit{by } (5.6)) \\ &= \omega(x,yz)\omega(y,z)\omega(\Theta(x)y,x)\omega(\Theta(x)z,x) \quad (\textit{by } (5.7)) \end{aligned} \tag{5.8}$$

which proves the lemma. □

Corollary 5.1. *Let S, K, x, and ω be as in the above lemma. If T is an ω representation of K then the mapping*

$$y \mapsto \omega(\Theta(x)y, x)\omega(x,y)^{-1}T_{\Theta(x)y}$$

is an ω representation of K.

Remark 5.2. *Let the hypothesis be as in Lemma 5.1. The ω representation defined in the corollary is denoted by ${}_\omega T^x$, or simply by T^x. Just as for groups we find $T^{xu} = (T^x)^u$ whenever x and u satisfy the hypothesis of Lemma 5.1. Moreover, for all $y \in K$,*

$$\begin{aligned} T_{\Theta(x)y}T_x &= \omega(\Theta(x)y,x)^{-1}T_{\Theta(x)(y)x} \\ &= \omega(\Theta(x)y,x)^{-1}T_{xy} = \omega(\Theta(x)y,x)^{-1}\omega(x,y)T_xT_y \end{aligned}$$

Hence, if T_x is invertible,

$$T_y^x = T_xT_yT_x^{-1} \quad (\forall y \in K)$$

If S is a cancellative monoid with a unitary multiplier ω, K is a submonoid of S, and T is an isometric ω representation of K, the induced ω representation $U = \omega\text{-}ind_K^S(T)$ can be defined exactly as for groups [Ma, §4], using right translations on the space of functions $f : S \to \mathcal{H}_T$ satisfying $f(kx) = \omega(k,x)T_kf(x)$ $(\forall k \in K, x \in S)$, and $\sum_{S/K} \|f(x)\|^2 < \infty$, and with the $l^2(S/K)$ inner product. Thus $(U_sf)(x) = \omega(x,s)f(xs)$. U is always isometric, but need not be unitary even if T is so.

Theorem 2. *(Mackey's extension theorem) Let S be a discrete monoid, K be a normal S-cancellative submonoid of S, and let σ be a unitary multiplier on S. Assume L is an irreducible unitary σ representation of K such that $L^x \cong L$ (unitary equivalent) for all x in S. Then there exists a unitary multiplier τ on S and a unitary τ representation M of S such that $L_z = M_z$ for all $z \in K$. τ may be chosen so as to be the product with σ of a multiplier of the form $1/\omega \circ p$ where $p : S \times S \to S/\phi_N \times S/\phi_N$ is the canonical homomorphism and ω is a multiplier on S/K. When τ is so chosen, ω is uniquely determined by σ and L up to multiplication by a trivial multiplier.*

Proof. In view of the above lemma and its corollary, the original proof of Mackey applies with only minor changes, when stripped of measure theoretical arguments. We remark only that in [Ma, p. 300], the Subgroup Theorem [Ma, Theorem 4.5] is used to prove that the induced representation $W = \omega\text{-}ind_K^S(I)$ is the identity on K. However, this can be verified directly, I being the one dimensional identity representation of K.

We include the proof for the sake of completeness:
Since $L^x \cong L$ for all x there exists a unitary M_x such that for all $z \in K$,

$$M_x L_z {M_x}^{-1} = L_z^x$$

Since L is irreducible, M_x is uniquely determined up to a multiplicative constant. We shall show that these constants may be chosen so that $x \mapsto M_x$ has the properties stated in the theorem. It will be convenient to do this in stages. Let $\mathcal{U}(\mathcal{H}_L)$ denote the unitary group of the Hilbert space $\mathcal{H}_L$. Since $L_z^x = L_x L_z {L_x}^{-1} \quad (\forall x, z \in K)$ and L is unitary, we may assume $L_x = M_x$ for each x in K. Hence we see that

$$L_z^x = M_x L_z {M_x}^{-1} \quad (\forall (z,x) \in K \times S)$$

Now, for all $(k, x, y) \in K \times S \times S$, we have

$$\begin{aligned} M_{xy} L_k {M_{xy}}^{-1} &= {L_k}^{xy} = (L^x)_k^y \\ (M_x L {M_x}^{-1})_k^y &= M_x {L_k}^y {M_x}^{-1} \\ &= M_x M_y L_k {M_y}^{-1} {M_x}^{-1} \end{aligned} \tag{5.9}$$

Thus for all $x, y \in S$ the operator $M_x M_y {M_{xy}}^{-1}$ commutes with L_k, for all $k \in K$. Since L is irreducible it follows that there exists a nonzero complex number of modulus one, $\tau'(x,y)$, such that

$$M_{xy} = \tau'(x,y) M_x M_y$$

Thus $x \mapsto M_x$ is a τ' representation of S and τ' is a multiplier on S. Let $\nu(x,y) = \sigma(x,y)/\tau'(x,y)$. Then ν is a multiplier for S which reduces to the identity on K. However ν need not be of the form $\omega \circ p$. As the final stage in the construction of M we show that we may change M so that the corresponding ν is of the desired form. Let I denote the one dimensional identity representation of K. Since ν is the identity on K it follows that I is a ν representation. Hence we may form the induced isometric ν representation $W = ind_K^S(I)$ of S. Now for all $(k,x) \in K \times S$ we have $W_{kx} = W_k W_x \nu(k,x)$. Moreover $(I^x)_k$ is multiplication by $\nu(\Theta(x)k, x)\nu(x,k)^{-1}$. But

$$\begin{aligned} L_{\Theta(x)k} M_x = M_{\Theta(x)k} M_x = \tau'(\Theta(x)k, x) M_{xk} \\ = \tau'(\Theta(x)k, x)\tau'(x,k)^{-1} M_x M_k \end{aligned} \tag{5.10}$$

and

$$M_x L_k M_x{}^{-1} = L_k^x = \sigma(\Theta(x)k, x)\sigma(x,k)^{-1} L_{\Theta(x)k} \tag{5.11}$$

Combining these two equations we deduce at once that

$$\nu(\Theta(x)k, x)\nu(x,k)^{-1} = 1 \quad (\forall (k,x) \in K \times S) \tag{5.12}$$

Thus I^x is the one dimensional identity for all $x \in S$. Now W can be realized on the space of all complex-valued functions f on S such that

$$f(kx) = \nu(k,x) I_k f(x) = \nu(k,x) f(x) \quad (\forall k \in K, x \in S)$$

and $W_x f(y) = \frac{1}{\nu(x,y)} f(yx) \quad (\forall x, y \in S)$. In particular, for all $k \in K, x \in S$,

$$\begin{aligned} W_k f(x) &= \tfrac{1}{\nu(k,x)} f(xk) = \tfrac{1}{\nu(k,x)} f(\Theta(x)kx) \\ &= \tfrac{1}{\nu(k,x)} \nu(\Theta(x)k, x) I_{\Theta(x)k} f(x) = f(x) \end{aligned} \tag{5.13}$$

Hence W_k is the identity for all $k \in K$, and therefore

$$W_{kx} = \nu(k,x) W_x \quad (\forall\, k, x \in K \times S).$$

Now let $u : S/K \to S$ be a cross-section with $u(K) = e$, and let $q : S \to S/K$ be the canonical map. For each $x \in S$, there is a unique $k(x) \in K$ such that $u(x) = k(x)x$. We put $W'_x = W_{k(x)x} \quad (\forall\, x \in S)$. Since W is a ν representation we find that W' is a ν' representation, in which

$$\nu'(x,y) = \nu(x,y) \cdot \frac{g(xy)}{g(x)g(y)} \quad (x, y \in S).$$

We now define A_x for all $x \in S$ as $\frac{1}{g(x)} M_x$. Since M is a $\frac{\sigma}{\nu} (= \tau')$ representation of S, it follows that A is a $\frac{\sigma}{\nu'}$ representation of S. But since W' is constant on the K cosets of S and W' is a ν' representation of S it follows at once that ν' is of the form $\omega \circ p$. That $A_k = L_k \quad (k \in K)$ follows from the fact that W_k is the identity. To complete the proof of the theorem we have now only to establish the essential uniqueness of ω. Let N be a $\sigma/\omega' \circ p$ representation of S which agrees on K with L. We compute at once that

$$N_x L_k N_x{}^{-1} = A_x L_k A_x{}^{-1} = L_k^x \quad (\forall k \in K, x \in S)$$

and hence that $N_x = \rho(x)A_x$ for all x, where $\rho(x)$ is a complex number of modulus one. Hence

$$N_{kx} = \rho(kx)M_{kx} \quad (\forall k \in K, x \in S).$$

Since $(\omega' \circ p)(k,x) = (\omega \circ p)(k,x) = 1$ we conclude that

$$\sigma(k,x)L_k N_x = \rho(kx)\sigma(k,x)L_k A_x.$$

Hence $\rho(kx) = \rho(x)$, and ρ is constant on the K cosets. Since

$$\omega \circ p(x,y) = (\omega' \circ p)(x,y)\rho(x,y)/\rho(x)\rho(y)$$

the desired result follows at once. □

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