ISBN 82–553–1209–9
ISSN 0806–2439
Pure Mathematics

No. 26
December 1999

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by

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PREPRINT SERIES

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSLO

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On infinite tensor products of projective unitary representations

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December 13, 1999

Abstract

We initiate a study of infinite tensor products of projective unitary representations of a discrete group G. Special attention is given to regular representations twisted by 2-cocycles and to projective representations associated with CCR-representations of bilinear maps. Detailed computations are presented in the case where G is a finitely generated free abelian group. We also apply our results to discuss an extension problem about product type actions of G, where the projective representation theory of G plays a central role.

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* partially supported by the Norwegian Research Council
** supported by the EU TMR network "Non Commutative Geometry"

1 Introduction

The theory of infinite tensor products of Hilbert spaces started with the seminal paper by von Neumann [19]. Later on, Guichardet [14, 15] approached this matter from a slightly different point of view and developed a unified framework for treating several related concepts involving operators, algebras and functionals. The notion of infinite tensor product has been mainly used in this form in operator algebras and quantum field theory over the last three decades (see e. g. [11] for a recent overview).

The existence of some infinite tensor product of representations of a group has been established and used in some recent works. For example, it was shown in [1] that a locally compact group is σ-amenable if and only if there exists an infinite tensor power of its regular representation. Such an infinite tensor power construction was then a useful tool for studying covariance of certain (induced) product-type representations of generalized Cuntz algebras with respect to natural product-type actions. This circle of ideas has been generalized and thoroughly investigated in [6]. In another direction, the infinite tensor product of certain unitary representations of some group of diffeomorphisms was shown to exist under suitable assumptions in [16], and a relation between such representations and some unitary representations of (infinite) permutation groups was obtained and used to characterize their irreducibility.

In this paper we start a discussion of infinite tensor products of *projective* unitary representations of a group. To avoid technicalities, we stick to the case of a *discrete* group, although it could be of interest in the future to consider a locally compact (or even just a topological) group and strongly continuous projective unitary representations of such a group. Now it is quite easy to realize that it is impossible to form the infinite tensor power of a single projective unitary representation unless the associated 2-cocycle vanishes. However one still has the possibility to form the infinite tensor product of different projective unitary representations. Somewhat surprisingly, this idea (which is due to the second author) leads to some potentially interesting results which we present below.

Our analysis, which uses standard techniques in representation theory and infinite tensor products, elucidates on quite general grounds the crucial points on which the whole construction relies. Besides its intrinsic interest, this new generality has also the potential advantage to allow for extensions of the analysis given in [1, 6] to a broader class of product-type actions on the 0^{th}-degree part of extended Cuntz algebras. It is also relevant when studying extensions of product-type actions from the algebraic to the von Neumann algebra level. Moreover many familiar (nuclear) C^*-algebras like the noncommutative tori may be presented as twisted group C^*-algebras of amenable groups. Our treatment can be successfully employed to obtain new faithful representations of such algebras on infinite tensor product spaces. We illustrate this for noncommutative tori.

The paper is organized as follows. Section 2 is devoted to some preliminaries on projective unitary representations, product sequences of 2-cocycles and infinite tensor products. Section 3 contains our main results concerning necessary conditions and sufficient conditions for the existence of infinite tensor products of projective unitary representations. We especially display some sufficient conditions for countable amenable groups in the case of projective regular representations and in the case of projective representations associated with CCR-representations of bilinear maps. In order to illustrate our work with some concrete examples we present in section 4 some explicit computations concerning finitely generated free abelian groups. The next section deals with some applications to the existence problem for infinite tensor products of actions of a group G on von Neumann algebras. One of our result exhibits an obstruction for extending some algebraic tensor power action of G to the weak closure that lies in the second cohomology group $H^2(G,\mathbb{T})$. In another result, the obstruction lies in the non-amenability of G. In the final section we collect some related remarks about projective unitary representations of restricted direct products of groups and associated operator algebras.

2 Preliminaries

Throughout this note G denotes a non-trivial discrete group, with neutral element e, while λ denotes the (left) regular representation of G acting on $\ell^2(G)$.

A *2-cocycle* (or multiplier) on G with values in the circle group $\mathbb{T}$ is a map $u : G \times G \to \mathbb{T}$ such that

$$u(x,y)u(xy,z) = u(y,z)u(x,yz) \qquad (x,y,z \in G),$$

see e.g. [7, Chapter IV]. We will consider only *normalized* 2-cocycles, satisfying

$$u(x,e) = u(e,x) = 1 \qquad (x \in G).$$

The set of all such 2-cocycles, which is denoted by $Z^2(G,\mathbb{T})$, becomes an abelian group under pointwise product. We equip $Z^2(G,\mathbb{T})$ with the topology of pointwise convergence.

A 2-cocycle v on G is called a *coboundary* whenever $v(x,y) = \rho(x)\rho(y)\overline{\rho(xy)}$ $(x,y \in G)$ for some $\rho : G \to \mathbb{T}$, $\rho(e) = 1$, in which case we write $v = \mathrm{d}\rho$ (such a ρ is uniquely determined up to multiplication by a character). The set of all coboundaries, which is denoted by $B^2(G,\mathbb{T})$, is a subgroup of $Z^2(G,\mathbb{T})$, which is easily seen to be closed (using Tychonov's theorem). The quotient group $H^2(G,\mathbb{T}) := Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$ is called the *second cohomology group* of G with values in $\mathbb{T}$. We denote elements in $H^2(G,\mathbb{T})$ by $[u]$ and write $v \sim u$ when $[v] = [u]$ $(u, v \in Z^2(G,\mathbb{T}))$. We also write $v \sim_\rho u$ when we have $v = (\mathrm{d}\rho)u$ for some coboundary $\mathrm{d}\rho$.

We shall be interested in *product* sequences in $Z^2(G,\mathbb{T})$: we call a sequence (u_i) in $Z^2(G,\mathbb{T})$ for a *product* sequence whenever the (pointwise) infinite product $u = \prod_i u_i$ exists on $G \times G$ (u being then obviously a 2-cocycle itself). Such a product sequence will occasionally be called *1-free* if $u_i \neq 1$ for every i. Notice that 1-free product sequences are easily seen to exist since we allow the u_i's to be coboundaries (and G is assumed to be non-trivial). One could also consider product sequences satisfying the stronger requirement that all u_i's are cohomogically non-trivial, but this would then be meaningful only when $H^2(G, \mathbb{T})$ is non-trivial, and in fact only when $H^2(G, \mathbb{T})$ is infinite (as may be deduced from the closedness of $B^2(G, \mathbb{T})$). However we have chosen not to make any cohomological assumption on G in this paper.

Another cohomological problem concerning product sequences is that perturbing a product sequence (by a coboundary in each component) does not necessarily lead to a product sequence, as may be illustrated by taking all u_i's to be 1 and perturbing by the same coboundary $v \neq 1$ in each component. (One possible way to avoid this problem could be to allow only for perturbation in finitely many components). The following lemma somewhat clarifies this problem.

Lemma 2.1. *Let (u_i) and (v_i) be two sequences in $Z^2(G, \mathbb{T})$ satisfying $v_i \sim_{\rho_i} u_i$ for every i.*

i) Assume that $\rho := \prod_i \rho_i$ exists. Then (v_i) is a product sequence if and only if (u_i) is a product sequence, in which case we have $\prod_i v_i \sim_\rho \prod_i u_i$.

ii) Assume that (u_i) and (v_i) are both product sequences. Then $\prod_i v_i \sim \prod_i u_i$ (even if $\prod_i \rho_i$ does not necessarily exist).

Proof. As i) is straightforward, we only show ii). So we assume that $u = \prod_i u_i$ and $v = \prod_i v_i$ both exist. Then $w := \prod_i \mathrm{d}\rho_i = \prod_i \overline{u_i} v_i$ also exists and is the limit of a net of 2-coboundaries. As $B^2(G, \mathbb{T})$ is closed, this implies that $w \in B^2(G, \mathbb{T})$. Since $v = wu$, this shows that $v \sim u$, as asserted. □

A *projective unitary representation* U of G on a Hilbert space $\mathcal{H}$ associated with some $u \in Z^2(G, \mathbb{T})$ is a map from G into the group of unitaries on $\mathcal{H}$ such that

$$U(x)U(y) = u(x, y)U(xy) \qquad (x, y \in G).$$

If we pick a $\rho : G \to \mathbb{T}$ satisfying $\rho(e) = 1$ and set $V = \rho U$, then V is also a projective unitary representation of G on $\mathcal{H}$ associated with a 2-cocycle v satisfying $v \sim_\rho u$. Such a V is called a *perturbation* of U.

To each $u \in Z^2(G, \mathbb{T})$ one may associate the (left) u-regular projective unitary representation λ_u of G on $\ell^2(G)$ defined by

$$(\lambda_u(x)f)(y) = u(y^{-1}, x)f(x^{-1}y) \qquad (f \in \ell^2(G),\ x, y \in G).$$

The (twisted) reduced group C*-algebra $C_r^*(G, u)$ (resp. group von Neumann algebra $VN(G, u)$) is then defined as the C*-algebra (resp. von Neumann algebra) acting on $\ell^2(G)$ generated by $\lambda_u(G)$. It is well known (and easy to see) that if $v \sim_\rho u$, then λ_v is unitarily equivalent to $\rho\lambda_u$. This implies that $C_r^*(G, v)$ (resp. $VN(G, v)$) is (spatially) isomorphic to $C_r^*(G, u)$ (resp. $VN(G, u)$) whenever $v \sim u$. One can also define the (twisted) *full* group C*-algebra $C^*(G, u)$. In the case where G is amenable ([10, 12, 22, 23]) it is known that $C^*(G, u)$ and $C_r^*(G, u)$ are (canonically) isomorphic, see [27].

For $i = 1, 2$, let U_i be a projective unitary representation of G on a Hilbert space $\mathcal{H}_i$ associated with $u_i \in Z^2(G, \mathbb{T})$. Then the naturally defined tensor product representation $U_1 \otimes U_2$ is easily seen to be a projective unitary representation of G on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ associated with the product cocycle $u_1 u_2$. In the case of ordinary unitary representations of a group, it is a classical result of Fell (cf. [10], 13.11.3) that the (left) regular representation acts in an absorbing way with respect to tensoring (up to multiplicity and equivalence). In the projective case we have the following analogue.

Proposition 2.2. *Let u, v be elements in $Z^2(G, \mathbb{T})$ and let V be any projective unitary representation of G on a Hilbert space $\mathcal{H}$ associated with v. Then the tensor product representation $\lambda_u \otimes V$ is unitarily equivalent to $\lambda_{uv} \otimes I_{\mathcal{H}}$, i.e. to $(\dim V) \cdot \lambda_{uv}$.*

Proof. We leave to the reader to check that the same unitary operator W as in the non-projective case (which is determined on $\ell^2(G) \otimes \mathcal{H}$ ($\cong \ell^2(G, \mathcal{H})$) by $(W(f \otimes \psi))(x) = f(x)V(x^{-1})\psi$) implements the asserted equivalence. $\square$

We conclude this section by recalling a few facts and some notation concerning infinite tensor products.

Let $\mathcal{H} = \{\mathcal{H}_i\}$ denote a sequence of Hilbert spaces and $\phi = \{\phi_i\}$ be a sequence of unit vectors where $\phi_i \in \mathcal{H}_i$ for each $i \geq 1$. We denote by $\mathcal{H}^\phi$ or by $\bigotimes_i^\phi \mathcal{H}_i$ the associated infinite tensor product Hilbert space of the $\mathcal{H}_i$'s along the sequence ϕ (sometimes called the incomplete direct product space detemined by ϕ), whose construction goes back to von Neumann [19]. We will follow the slightly different approach given by Guichardet in [14, 15]. We give here only a short account, and the reader should consult these papers for full details on this matter.

For any sequence $\psi_i \in \mathcal{H}_i$ such that

$$\sum_i |\, 1 - \|\psi_i\| \,| < \infty \text{ and } \sum_i |\, 1 - (\psi_i, \phi_i) \,| < \infty$$

there corresponds a so-called decomposable vector

$$\otimes_i \psi_i = \lim_n \psi_1 \otimes \ldots \otimes \psi_n \otimes \phi_{n+1} \otimes \phi_{n+2} \otimes \ldots \in \mathcal{H}^\phi$$

depending linearly on each ψ_i (in fact one gets here convergence over the net consisting of nonempty finite subsets of $\mathbb{N}$ ordered by inclusion, cf. [15, Part II, Proposition 5]). If $\otimes_i \xi_i$ is another decomposable vector in $\mathcal{H}^\phi$, then

$$(\otimes_i \psi_i, \otimes_i \xi_i) = \prod_i (\psi_i, \xi_i)$$

(where the infinite product above is convergent in the unordered sense, cf. [19, §2]). Each finite tensor product $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_k$ is embedded in $\mathcal{H}^\phi$ by extending the map identifying a simple tensor of the form $\psi_1 \otimes \ldots \otimes \psi_k$ with the elementary decomposable vector $\psi_1 \otimes \ldots \otimes \psi_k \otimes \phi_{k+1} \otimes \phi_{k+2} \otimes \ldots \in \mathcal{H}^\phi$. The set of all elementary decomposable vectors in $\mathcal{H}^\phi$ obtained by letting k vary in $\mathbb{N}$ is total in $\mathcal{H}^\phi$.

Let $T_1, T_2, \ldots$ be a sequence of bounded linear operators where each T_i acts on $\mathcal{H}_i$. For each fixed $n \in \mathbb{N}$ there exists a unique bounded linear operator $\tilde{T}_n$ acting on $\mathcal{H}^\phi$ which is determined by

$$\tilde{T}_n(\otimes_i \psi_i) = T_1\psi_1 \otimes \ldots \otimes T_n\psi_n \otimes \psi_{n+1} \otimes \psi_{n+2} \otimes \ldots$$

for each decomposable vector $\otimes_i \psi_i$. Similarly, one may define $\tilde{T}_J$ for each (nonempty) finite $J \subset \mathbb{N}$. Under certain assumptions, the net $\{\tilde{T}_J\}$ converges in the strong operator topology to a bounded linear operator on $\mathcal{H}^\phi$ which is then denoted by $\otimes_i T_i$.

By [15, Part II, Proposition 6]), a sufficient condition for $\otimes_i T_i$ to exist is that

$$\prod_i ||T_i|| \text{ exists}, \sum_i |1 - ||T_i\phi_i||\,| < \infty \text{ and } \sum_i |1 - (T_i\phi_i, \phi_i)\,| < \infty,$$

in which case we have $(\otimes_i T_i)\,(\otimes_i \psi_i) = \otimes_i T_i \psi_i$ for all elementary decomposable vectors $\otimes_i \psi_i$.

When all T_i's are unitaries (which is the case of interest in this paper) we have the following useful result will be used several times in the sequel.

Proposition 2.3. *Let (T_i) be a sequence of unitaries where each T_i acts on $\mathcal{H}_i$. Then $\otimes_i T_i$ exists on $\mathcal{H}^\phi$ if and only if*

$$(*) \quad \sum_i |1 - (T_i\phi_i, \phi_i)| < \infty,$$

in which case $\otimes_i T_i$ is a unitary on $\mathcal{H}^\phi$ satisfying $(\otimes_i T_i)^* = \otimes_i T_i^*$.

Proof. Assume first that $(*)$ holds. It is then quite elementary to deduce from Guichardet's result mentioned above that $\otimes_i T_i$ and $\otimes_i T_i^*$ both exist. Moreover, these two operators are then isometries, being the strong limit of a net of unitaries, and they are easily seen to be the inverse of each other. So both are unitaries satisfying $(\otimes_i T_i)^* = \otimes_i T_i^*$.

Assume now that $T := \otimes_i T_i$ exists on $\mathcal{H}^\phi$. Then T is non-zero (being an isometry), so there are elementary decomposable vectors $\otimes_i \psi_i$ and $\otimes_i \xi_i$ such that

$$0 \neq c := (T \otimes_i \psi_i, \otimes_i \xi_i).$$

Let J be any finite subset of $\mathbb{N}$ large enough so that $\psi_i = \xi_i = \phi_i$ for all $i \notin J$. Then we have

$$(\tilde{T}_J \otimes_i \psi_i, \otimes_i \xi_i) = \prod_{i \in J} (T_i\, \psi_i, \xi_i).$$

Since $T = \lim_J \tilde{T}_J$, we get $c = \lim_J \prod_{i\in J}(T_i\,\psi_i, \xi_i)$, i.e. $\prod_{i\in\mathbb{N}}(T_i\,\psi_i, \xi_i)$ converges to a non-zero value. Thus we get $\sum_i |1-(T_i\,\psi_i, \xi_i)| < \infty$ (see [19, Lemma 2.5.1]) and therefore $\sum_i |1-(T_i\,\phi_i, \phi_i)| < \infty$ since $\psi_i = \xi_i = \phi_i$ for all but finitely many i's. $\square$

3 Infinite tensor products of projective unitary representations

In this section we shall discuss the following (loosely formulated) problem: If U_i is a sequence of projective unitary representations of a group G, when is it possible to form the infinite tensor product $\otimes_i U_i$?

The most elementary case to consider consists obviously of picking just one projective unitary representation U of G on a Hilbert space $\mathcal{H}$ with associated 2-cocycle u and trying to form the infinite tensor product of U with itself infinitely many times, i.e. its infinite tensor power. For each $i \in \mathbb{N}$, put then $U_i = U$, $\mathcal{H}_i = \mathcal{H}$ and let $\phi = \{\phi_i\}$ be a sequence of unit vectors in $\mathcal{H}$.

Now, if we assume that $U^{\otimes\infty}(x) := \otimes_i U_i(x)$ exists on $\bigotimes_i^\phi \mathcal{H}_i$ for all $x \in G$, then Proposition 2.3 gives

$$\sum_i |\, 1 - (U(x)\phi_i, \phi_i)\,| < \infty,$$

and especially

$$\lim_i\, (U(x)\phi_i, \phi_i) = 1$$

for all $x \in G$. Letting then ω be any weak* limit point of the sequence (ω_{ϕ_i}) of vector states of $\mathcal{B}(\mathcal{H})$, we have $\omega(U(x)) = 1$ for all $x \in G$, from which it follows (using the Cauchy-Schwarz inequality for states, as in [4]) that ω is multiplicative at each $U(x)$. This implies that

$$1 = \omega(\, U(x)\,)\,\omega(\, U(y)\,) = \omega(\, U(x)U(y)\,)$$

$$= \omega(\, u(x,y)U(xy)\,) = u(x,y)\omega(\, U(xy)\,) = u(x,y)$$

for all $x, y \in G$, i.e. u is the trivial 2-cocycle on G and consequently, U is an ordinary unitary representation of G.

Concerning infinite tensor powers of unitary representations, we refer to [1] in the case of the regular representation and [6] in the case of the adjoint representation. More generally, we have (cf. [9]):

Proposition 3.1. *Let U be a unitary representation of G on $\mathcal{H}$.*

1) If $U^{\otimes\infty}$ exists (i.e. $U^{\otimes\infty}(x)$ exists for all $x \in G$) on $\mathcal{H}^\phi = \bigotimes_i^\phi \mathcal{H}_i$ for some sequence $\phi = \{\phi_i\}$ of unit vectors in $\mathcal{H}$, then 1 (= the trivial one-dimensional representation of G) is weakly contained in U in the sense of Fell ([10]).(As usual, we denote this by $1 \prec U$).

2) If G is countable and $1 \prec U$, then there exists a sequence $\phi = (\phi_i)$ of unit vectors in $\mathcal{H}$ such that $U^{\otimes\infty}$ exists on $\mathcal{H}^\phi$ (and $U^{\otimes\infty}$ is then a unitary representation of G).

Proof. For completeness, we sketch the proof.

1) When $U^{\otimes\infty}$ exists on $\mathcal{H}^\phi$, we have $\lim_i (U(x)\phi_i, \phi_i) = 1$ for all $x \in G$, so $1 \prec U$ by [10], 18.1.4.

2) Assume $G = \{g_1, g_2, \dots\}, 1 \prec U$ and set $G_n = \{g_1, \dots, g_n\}$. By a straightforward adaptation of the proof given by Eymard in [12, p. 48-49], to the present context, we obtain the following.

For each $n \in \mathbb{N}$, there exists a unit vector ϕ_n in $\mathcal{H}$ satisfying

$$|\,1 - (U(g)\phi_n, \phi_n)\,| < \frac{1}{n^2}$$

for all $g \in G_n$.

Let now $x \in G$ and choose $N \in \mathbb{N}$ such that $x \in G_N$ ($\subseteq G_{N+1} \subseteq \dots$).

Then we have

$$\begin{aligned}\sum_i |\,1 - (U(x)\phi_i, \phi_i)\,| &= \sum_{i<N} |\,1 - (U(x)\phi_i, \phi_i)\,| + \sum_{i\geq N} |\,1 - (U(x)\phi_i, \phi_i)\,| \\ &\leq \sum_{i<N} |\,1 - (U(x)\phi_i, \phi_i)\,| + \sum_{i\geq N} \frac{1}{i^2} \\ &< \infty.\end{aligned}$$

□

This proposition has some bearing on the concept of amenability for groups and on the concept of amenability of representations.

Concerning group-amenability, the celebrated Hulanicki-Reiter theorem asserts that G is amenable if and only if $1 \prec \lambda$. Using the above proposition, it is not difficult to deduce that G is countable and amenable if and only if $\lambda^{\otimes\infty}$ exists along some sequence of unit vectors in $\ell^2(G)$, as first shown in [1]. Moreover, the sequence of unit vectors may then be chosen in a specific way (cf. [8]) which we now review for later use.

We first introduce some terminology. A sequence (F_i) of non-empty, finite subsets of G will be called a F-sequence (resp. σF-sequence) for G whenever

$$\lim_i \frac{\#(F_i \cap xF_i)}{\#F_i} = 1 \text{ for all } x \in G,$$

$$(\text{resp.} \sum_i |1 - \frac{\#(F_i \cap xF_i)}{\#F_i}| < \infty \text{ for all} x \ \in G).$$

A F-sequence (F_i) for G is often called a *Fölner* sequence in the literature. We remark that the definition is usually phrased in a slightly different, but equivalent, way (involving the symmetric difference of sets) and that some authors also require that $F_i \subseteq F_{i+1}$ for every i. Anyhow, thanks to Fölner, one knows that G is countable and amenable if and ony if G has a F-sequence. We will call G for *σ-amenable* whenever G is countable and amenable. Now, obviously, a σF-sequence for G is also a F-sequence. Moreover, when G is σ-amenable, any F-sequence has some subsequence which is a σF-sequence, as is easily checked. Hence we can also conclude that a G is σ-amenable if and only if G has a

σF-sequence. Finally, when (F_i) is a σF-sequence for G, χ_{F_i} denotes the characteristic function of F_i and we set $\phi_i := \chi_{F_i}/\#(F_i)^{1/2}$, then $\lambda^{\otimes\infty}$ exists along the sequence (ϕ_i) of unit vectors in $\ell^2(G)$: this readily follows from the equation

$$(\lambda(x)\chi_F, \chi_F) = \#(F \cap xF)$$

which holds for every $x \in G$ and every non-empty, finite subset F of G.

Concerning amenability of representations, we recall that a unitary representation π of G on a Hilbert space $\mathcal{H}$ is called *amenable* (in the sense of Bekka, cf. [5]) whenever there exists a state ω on $\mathcal{B}(\mathcal{H})$ satisfying $\omega(\pi(x)T\pi(x)^*) = \omega(T)$ for all $x \in G$ and all $T \in \mathcal{B}(\mathcal{H})$. It is easy to see that π is amenable whenever $1 \prec \pi$, while the converse implication is not necessarily true. Bekka has shown that π is amenable if and only if $1 \prec \pi \otimes \overline{\pi}$, and that amenability of a group is characterized by the fact that all its unitary representations are amenable (and it suffices to check this for its regular representation). However, many non-amenable groups, such as non-abelian free groups, do have amenable representations. Hence, using the above proposition, it is clear that one may produce examples of unitary representations of non-amenable groups for which the associated infinite tensor power representation exists.

The next natural step now is to try to form the infinite tensor product of a sequence of (possibly) different unitary representations of G. In the simple case where $G = \mathbb{Z}$, this boils down to the question of existence of the infinite tensor product of a sequence of unitary operators, and we have no more conceptual answer to this question than the one provided by Proposition 2.3. In the case where G is a group acting on some standard Borel space S with a quasi-invariant measure, one may construct sequences of unitary representations of G associated with suitably chosen Borel 1-cocycles on $S \times G$ and study the existence of the resulting infinite tensor products, essentially along the same lines as in [16]. Since this would lead us too far away from our main task, we don't elaborate on this matter here. Therefore, we arrive to the final step of generality, which is to consider a sequence of *projective* unitary representations. Before attacking the (main) problem whether it is possible to form an infinite tensor product of such a sequence in some cases, we first show that this construction, when possible, produces a new projective unitary representation of G, and also make some general observations.

Theorem 3.2. *Let U_i be a sequence of projective unitary representations of G acting respectively on a Hilbert space $\mathcal{H}_i$ and with associated $u_i \in Z^2(G, \mathbb{T})$. Let $\phi = (\phi_i)$ be a sequence of unit vectors where each $\phi_i \in \mathcal{H}_i$. Assume that $\otimes_i U_i(x)$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \mathcal{H}_i$ for each $x \in G$. Then we have*

i) (u_i) is a product sequence in $Z^2(G, \mathbb{T})$.

ii) The map $x \to U^\phi(x) := \otimes_i U_i(x)$ is a projective unitary representation of G on $\mathcal{H}^\phi$ with $u = \prod_i u_i$ as its associated 2-cocycle.

Proof. We notice first that Proposition 2.3 implies that each $U^\phi(x) := \otimes_i U_i(x)$ is a unitary.

i) Let $g, h \in G$. We must show that $\prod_i u_i(g,h)$ converges. Now

$$\otimes_i U_i(gh)$$

and

$$(\otimes_i U_i(g))(\otimes_i U_i(h)) = \otimes_i U_i(g)U_i(h) = \otimes_i u_i(g,h)\, U_i(gh)$$

are both unitaries. Putting $a_i = (\, U_i(gh))\phi_i\, ,\, \phi_i\,)$, we deduce from Proposition 2.3 that

$$\sum_i |\, 1 - a_i\,| < \infty \quad \text{and} \quad \sum_i |\, 1 - u_i(g,h) a_i\,| < \infty.$$

This implies that $\sum_i |\, 1 - u_i(g,h)\,| < \infty$, and therefore that $\prod_i u_i(g,h)$ converges, as desired. (We use here implicitely that whenever $z \in \mathbb{T}$ and $a \in \mathbb{C}$, then $|1 - z| = |1 - \bar{z}| \leq |1 - a| + |a - \bar{z}| = |1 - a| + |za - 1|$).

ii) Using i) we get

$$U^\phi(x)\, U^\phi(y) = \otimes_i u_i(x,y)\, U_i(xy) = (\prod_i u_i(x,y)) \,\otimes_i\, U_i(xy) = u(x,y)\, U^\phi(xy)$$

for all $x, y \in G$, as asserted. □

The following corollary is now easily deduced from Proposition 2.2 :

Corollary 3.3. *Under the same assumptions and with the same notation as in Theorem 3.2, we have:*

i) If U_k is equal to λ_{u_k} for one k, then U^ϕ is unitarily equivalent to $\lambda_u \otimes I_{\mathcal{H}'}$, where $\mathcal{H}'$ denotes the infinite tensor product $\otimes^{\phi'}_{i \neq k} \mathcal{H}_i$ along the sequence $\phi' = (\phi_i)_{i \neq k}$.

ii) $\lambda \otimes U^\phi$ is unitarily equivalent to $\lambda_u \otimes I_{\mathcal{H}^\phi}$.

The case where infinitely many of the U_i's in Theorem 3.2 are projective regular representations of G can not occur when G is uncountable or non-amenable. This follows easily from our next theorem:

Theorem 3.4. *Let (u_i) be a sequence in $Z^2(G, \mathbb{T})$ and set $U_i = \lambda_{u_i}$ for every i. Let $\phi = (\phi_i)$ be a sequence of unit vectors in $\ell^2(G)$. Assume that $\otimes_i U_i(x)$ exists on $\mathcal{H}^\phi = \otimes^\phi_i \ell^2(G)$ for each $x \in G$. Then G is σ-amenable.*

Proof. Let us first mention that if $u_i \sim 1$ for all but finitely many i's, then the result is quite easily obtained by reorganizing part of the proof of Theorem 5 in [1]. Furthermore, the proof of the general case is similar and requires only that one now takes advantage of the existence of an "absolute value" in $\ell^2(G)$, cf. [6] for this terminology. A sketch is as follows.

By hypothesis, $\sum_i |1 - (U_i(x)\phi_i, \phi_i)| < \infty$ for every $x \in G$. Notice that

$$|(U_i(x)\phi_i, \phi_i)| \leq (\lambda(x)|\phi_i|, |\phi_i|) \leq 1.$$

Therefore, setting $f_i(x) := |(U_i(x)\phi_i, \phi_i)| \geq 0$ we have $0 \leq f_i \leq 1$, $f_i \in C_0(G)$ and $f_i \to 1$ pointwise. Then $f_i^{-1}([1/2, 1]) =: H_i$ is finite, and $G = \cup_i H_i$ is therefore countable. Moreover, we get

$$(\lambda(x)|\phi_i|\,,\,|\phi_i|) \to 1 \qquad (x \in G),$$

so $1 \prec \lambda$ and the amenability of G follows. $\square$

We now turn our attention to the problem of showing that it is possible to form the infinite tensor product of a sequence of projective unitary representations of G, at least in some specific situations. Some concrete examples illustrating our results will be given in the next section.

In view of Theorem 3.4, it is quite natural to wonder whether some converse holds. So we assume that G is σ-amenable, let (u_i) be a product sequence in $Z^2(G, \mathbb{T})$ and set $U_i = \lambda_{u_i}$ for every i. The question is then whether it does *always* exist a sequence $\phi = (\phi_i)$ of unit vectors in $\ell^2(G)$ such that $\otimes_i U_i$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \ell^2(G)$, i. e. such that

$$\sum_i |1 - (U_i(x)\phi_i, \phi_i)| < \infty \quad \text{for all } x \in G.$$

It is conceivable that the answer to this question is positive and we shall provide a partial answer in this direction. Our approach is based on the following inequality :

$$(*)\ \sum_i |1 - (U_i(x)\phi_i, \phi_i)| \leq \sum_i |1 - (\lambda(x)\phi_i, \phi_i)| + \sum_i |(\,(\lambda(x) - U_i(x))\phi_i, \phi_i\,)|$$

which is valid for every $x \in G$ and every sequence $(\phi_i) \subset \ell^2(G)$, as follows from the triangle inequality.

Now, since G is assumed to be σ-amenable, we can surely find some sequence (ϕ_i) of unit vectors in $\ell^2(G)$ making the first sum of the right hand side of the above inequality convergent for every $x \in G$, and the problem is then whether (ϕ_i) can also be chosen so that the second sum is convergent for every $x \in G$. There is some flexibility of choice here and it is not difficult to see that this might be achieved if one is willing to eventually replace (U_i) by one suitably chosen subsequence if necessary. We illustrate this by showing that all ϕ_i's may then even be chosen as normalized characteristic functions associated with some σF-sequence for G.

We first record an easy calculation. Let χ_F denote the characteristic function of some finite (non-empty) subset $F \subset G$ and set $\phi_F := \chi_F/(\#F)^{1/2}$. Let $u \in Z^2(G, \mathbb{T})$. Then we have

$$(\lambda_u(x)\phi_F, \phi_F) = \frac{1}{\#F} \sum_{y \in F \cap xF} u(y^{-1}, x)$$

and therefore

$$((\lambda(x) - \lambda_u(x))\phi_F, \phi_F) = \frac{1}{\#F} \sum_{y \in F \cap xF} (1 - u(y^{-1}, x))$$

for all $x, y \in G$.

Theorem 3.5. *Let (u_i) be a sequence in $Z^2(G, \mathbb{T})$ and assume that G (is σ-amenable and) has a σF-sequence (F_i) which satisfies*

$$(1) \quad \sum_i \frac{1}{\#F_i} \sum_{y \in F_i} |1 - u_i(y^{-1}, x)| < \infty \text{ for all } x \in G,$$

or

$$(2) \quad \sum_i \frac{1}{\#F_i} \mid \sum_{y \in F_i \cap xF_i} (1 - u_i(y^{-1}, x)) \mid < \infty \text{ for all } x \in G.$$

Set $U_i = \lambda_{u_i}$ and $\phi_i := \chi_{F_i}/(\#F_i)^{1/2}$ for every i.
Then $\phi = (\phi_i)$ is a sequence of unit vectors in $\ell^2(G)$ such that $\otimes_i U_i$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \ell^2(G)$.

Proof. Since (1) clearly implies (2), we assume that (2) holds. Using the inequality $(*)$ above, we get

$$\sum_i |1 - (U_i(x)\phi_i, \phi_i)| \leq$$

$$\sum_i |1 - \frac{\#(F_i \cap xF_i)}{\#F_i}| + \sum_i \frac{1}{\#F_i} \mid \sum_{y \in F_i \cap xF_i} (1 - u_i(y^{-1}, x)) \mid$$

for all $x \in G$. By assumption, both sums on the right hand side of this inequality are convergent for all $x \in G$, and we can therefore conclude that the left hand side converges for all $x \in G$, as desired. $\square$

Clearly, if $u_i = 1$ for all but finitely many i's, any σF-sequence (F_i) for G trivially satisfies (1) (and (2)). In this case, the above theorem could also have been deduced from [9] or [1].

This theorem enables us to obtain the following general existence results.

Corollary 3.6. *Let G be σ-amenable and let (v_j) be a product sequence in $Z^2(G, \mathbb{T})$. Then there exist a subsequence (u_i) of (v_j) and a sequence $\phi = (\phi_i)$ of unit vectors in $\ell^2(G)$ such that $\otimes_i \lambda_{u_i}$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \ell^2(G)$.*

Proof. First we pick a σF-sequence (F_i) for G and a growing sequence (H_i) of non-empty finite subsets of G satisfying $\cup_i H_i = G$. Since the (pointwise) product $\prod_j v_j$ exists, we can choose a subsequence (u_i) of (v_j) satisfying

$$|1 - u_i(y^{-1}, x)| \leq 1/i^2 \text{ for all } x \in H_i, y \in F_i, i \in \mathbb{N}.$$

Let $x \in G$ and choose $N \in \mathbb{N}$ such that $x \in H_N$. Then we get

$$\sum_i \frac{1}{\#F_i} \sum_{y \in F_i} |1 - u_i(y^{-1}, x)|$$

$$\leq \sum_{i<N} 2 + \sum_{i \geq N} \frac{1}{\#F_i} \sum_{y \in F_i} 1/i^2$$

$$= 2(N-1) + \sum_{i \geq N} 1/i^2 < \infty.$$

This shows that (F_i) satisfies (1) in Theorem 3.5, from which the result then clearly follows. □

Corollary 3.7. *Let G be σ-amenable. Then there always exist some 1-free product sequence (u_i) in $Z^2(G, \mathbb{T})$ and some sequence $\phi = (\phi_i)$ of unit vectors in $\ell^2(G)$ such that $\otimes_i \lambda_{u_i}$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \ell^2(G)$. If $H^2(G, \mathbb{T})$ is non-trivial and $1 \neq [u] \in H^2(G, \mathbb{T})$, then the sequence (u_i) above may chosen so that $u = \prod_i u_i$.*

Proof. Since 1-free product sequences do exist in $B^2(G, \mathbb{T})$ and 1-freeness is clearly preserved when passing to subsequences, the first assertion follows from the previous corollary. The 1-free product sequence (u_i) is then in $B^2(G, \mathbb{T})$. Therefore (by closedness) $\prod_i u_i \in B^2(G, \mathbb{T})$, so we may write it as $\mathrm{d}\rho$ for some normalized $\rho : G \to \mathbb{T}$. Assume now $H^2(G, \mathbb{T})$ is non-trivial and $1 \neq [u] \in H^2(G, \mathbb{T})$. Set $v_1 = \overline{d\rho}\, u$ and $v_i = u_{i-1}, i > 1$. Then (v_i) is a 1-free product sequence satisfying $u = \prod_i v_i$. Further we can define a new sequence $\psi = (\psi_i)$ of unit vectors in $\ell^2(G)$, by setting $\psi_1 = \delta_e$ and $\psi_i = \psi_{i-1}, i > 1$. It is then obvious that $\otimes_i \lambda_{v_i}$ exists on $\mathcal{H}^\psi$, which proves the second assertion. □

Remarks.

1) It follows from Corollary 3.3 that representations obtained as the infinite tensor product of projective regular representations are never irreducible.

2) It is unknown to us whether the second assertion of the Corollary 3.7 may be strengthened into that *all* u_i may be chosen to be non-trivial in cohomology if one assumes that $H^2(G, \mathbb{T})$ is infinite (the above proof ensures only that one of the u_i's satisfies this requirement). We will see that this may be done in an example considered in Section 4.

3) Let G be σ-amenable and let (u_i) and (v_i) be two sequences in $Z^2(G, \mathbb{T})$ satisfying $v_i \sim_{\rho_i} u_i$ for every i. Assume that $\otimes_i \lambda_{u_i}$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \ell^2(G)$ for some sequence $\phi = (\phi_i)$ of unit vectors in $\ell^2(G)$. As $\prod_i v_i$ does not necessarily exist, it may happen that $\otimes_i \lambda_{v_i}$ can not be formed at all (cf. Theorem 3.2). However, it is quite clear that $\rho_1 \lambda_{v_1} \otimes \rho_2 \lambda_{v_2} \otimes \cdots$ exists on $\otimes^{\psi_i} \ell^2(G)$, where ψ_i is defined by $\psi_i(x) = \overline{\rho_i(x^{-1})} \phi_i(x)$, and this may be considered as a problem of gauge fixing. On the other hand, let us also assume that $\otimes_i \lambda_{v_i}$ exists on

$\mathcal{H}^\psi = \otimes_i^\psi \ell^2(G)$ for some sequence $\psi = (\psi_i)$ of unit vectors in $\ell^2(G)$. Then we may conclude that $\otimes_i\lambda_{v_i}$ is, up to unitary equivalence, just a perturbation of $\otimes_i\lambda_{u_i}$.

(To prove this, we first appeal to Theorem 3.2 and obtain that both $u = \prod_i u_i$ and $v = \prod_i v_i$ exist. Using Lemma 2.1 we may then write $v = \mathrm{d}\rho\, u$ for some normalized $\rho : G \to \mathbb{T}$. Now, using that $\lambda_v \simeq \rho\lambda_u$ and Corollary 3.3, we get

$$\otimes_i\lambda_{v_i} \simeq \lambda_v \otimes I \simeq \rho(\lambda_u \otimes I) \simeq \rho \otimes_i \lambda_{u_i},$$

where I denotes the identity representation of G on any infinite separable Hilbert space)

4) To produce examples of infinite tensor product of projective unitary representations of not necessarily amenable groups, one can proceed as follows. Let G be any countable group possessing a non-trivial amenable factor group K (one can here for instance let G be any non-perfect, non-amenable group, e. g. any non-abelian countable free group, since $G/[G,G]$ is then non-trivial and abelian) and let (v_i) be a sequence in $Z^2(K,\mathbb{T})$ such that $\otimes_i\lambda_{v_i}$ exists on $\otimes_i^\phi \ell^2(K)$. Using the canonical homomorphism $\pi : G \to K$, we may lift each v_i to a $u_i \in Z^2(G,\mathbb{T})$ in an obvious way. Now, set $U_i(x) := \lambda_{v_i}(\pi(x)), (x \in G)$, for each i. It is then a simple matter to check that each U_i is a projective unitary representation of G on $\ell^2(K)$ associated to u_i, and that $\otimes_i U_i$ exists on $\otimes_i^\phi \ell^2(K)$.

We now turn to another class of examples which is in spirit related to the setting of the Stone-Mackey-von Neumann theorem, i. e. with so called CCR-representations of a locally compact group and its dual (cf. [24]). Our approach is different from the one in [17], which deals with CCR-representations of the direct sum of countably many copies of $\mathbb{R}$ on infinite tensor product spaces.

We consider two discrete groups A and B satisfying the following weak form of duality: we assume that there exists a non-trivial bilinear map $\sigma : A \times B \to \mathbb{T}$. This amounts to assume that the abelianized groups A_{ab} and B_{ab} satisfy the same weak form of duality, but it seems worthwile not restricting at once to the case of abelian groups, even if this may cause some degeneracy. We just mention here one simple example where this type of weak duality is present: Let $A = \mathbb{Z}$ and B be any group satisfying $B_{ab} \neq \{0\}$. Then pick $1 \neq \gamma \in \mathrm{Hom}(B,\mathbb{T})$ and define $\sigma(n,b) = \gamma(b)^n$ $(n \in \mathbb{Z}, b \in B)$.

Going back to our general setting, we set $G = A \times B$ and define $u_\sigma : G \times G \to \mathbb{T}$ by

$$u_\sigma(\,(a_1,b_1),(a_2,b_2)\,) = \overline{\sigma(a_2,b_1)}.$$

There is some obvious arbitrariness in this definition, and our choice is governed by what follows. First, it is an easy exercise to check that u_σ is a non-trivial 2-cocycle on G (in fact a bicharacter). When both A and B are abelian, then $[u_\sigma] \neq 1$ in $H^2(G,\mathbb{T})$, as follows from [18] since u_σ is clearly non-symmetric. Secondly, it is well known that there is a canonical way to produce a projective unitary representation U_σ of G on $\ell^2(B)$ associated with u_σ. We recall this construction (and remark that a similar representation can be constructed on $\ell^2(A)$ in an analogous way):

For each $a \in A, b \in B$ we set $\sigma_a(b) = \sigma(a,b)$, so the map $(a \mapsto \sigma_a)$ belongs to $\mathrm{Hom}(A, \hat{B})$ where $\hat{B} := \mathrm{Hom}(B, \mathbb{T})$. Let then $V_\sigma(a)$ denote the multiplication operator by the function σ_a on $\ell^2(B)$ and λ_B be the left regular representation of B on $\ell^2(B)$. By computation we have

$$V_\sigma(a)\lambda_B(b) = \sigma(a,b)\ \lambda_B(b)V_\sigma(a)$$

for all $a \in A, b \in B$. If we now put $U_\sigma(a,b) := V_\sigma(a)\lambda_B(b)$ for all $(a,b) \in G$, then U_σ is as desired. The triple $\{V_\sigma, \lambda_B, \ell^2(B)\}$ is a CCR-representation of σ if we agree to call a triple $\{V, W, \mathcal{H}\}$ for a CCR-representation of σ whenever V and W are unitary representations of respectively A and B on $\mathcal{H}$ which satisfy the CCR-relation

$$V(a)W(b) = \sigma(a,b)\ W(b)V(a)$$

for all $a \in A, b \in B$. There is an obvious 1-1 correspondence between CCR-representations of σ, projective unitary representations of G associated with u_σ and nondegenerate representations of $C^*(G, u_\sigma)$. For the sake of completeness, we mention that the C^*-algebra $C^*(G, u_\sigma)$ may be decomposed as the ordinary crossed product $C^*(B) \times_\alpha A$ where α is the action of A on $C^*(B)$ naturally induced by the homomorphism $(a \mapsto \sigma_a)$ from A into $\hat{B}$ (and anagously as the crossed product of $C^*(A)$ by the induced action of B).

Assume now that (σ_i) is a sequence of bilinear maps from $A \times B$ into $\mathbb{T}$. (In the example mentioned earlier, this is achieved by first picking a sequence in $\mathrm{Hom}(B, \mathbb{T})$). Then we set $U_i := U_{\sigma_i}$ and consider the question: when is it possible to form $\otimes_i U_i$ on $\otimes_i^\phi \ell^2(B)$ for some sequence $\phi = (\phi_i)$ of unit vectors in $\ell^2(B)$? Or, equivalently, when is it possible to form the infinite tensor product of the CCR-representations associated with the σ_i's ? In the case of a positive answer $\prod_i u_{\sigma_i}$ will exist (as a consequence of Theorem 3.2), so $\prod_i \sigma_i$ will then exist too and the infinite tensor product of the CCR-representations associated with the σ_i's will be a CCR-representation of this product map.

Since $U_i(e,b) = \lambda_B(b)$, we must at least require that B is σ-amenable and ϕ is chosen so that $\otimes_i \lambda_B$ exists on $\otimes_i^\phi \ell^2(B)$, in accordance with Theorem 3.4. The question reduces then clearly to whether ϕ can also be chosen so that $\otimes_i V_{\sigma_i}$ exist on $\otimes_i^\phi \ell^2(B)$, i. e. whether

$$\sum_i |1 - (V_{\sigma_i}(a)\phi_i, \phi_i)| = \sum_i |1 - ((\sigma_i)_a\, \phi_i, \phi_i)| < \infty$$

holds for every $a \in A$. Remark that our first requirement on ϕ prevents us from choosing $\phi_i = \delta_e$ for every i, which would have made all these sums convergent in a trivial way. Choosing ϕ to be associated with a σF-sequence for B leads to the following:

Theorem 3.8. *Let notation be as introduced above.*
Assume that B is σ-amenable and let (F_i) be a σF-sequence for B.
Set $\phi = (\phi_i)$ where $\phi_i := \chi_{F_i}/\#(F_i)^{1/2}$.
Then $\otimes_i U_i$ exists (as a representation of $G = A \times B$) on $\otimes_i^{\phi} \ell^2(B)$ whenever

$$\sum_i \frac{1}{\#(F_i)} \mid \sum_{b \in F_i} (1 - \sigma_i(a, b)) \mid < \infty$$

for every $a \in A$.

Proof. As we have

$$|1 - ((\sigma_i)_a \, \phi_i, \phi_i)| = \frac{1}{\#(F_i)} \mid \sum_{b \in F_i} (1 - \sigma_i(a, b))|$$

for every $a \in A$, this is just a consequence of the above discussion. □

4 The case of free abelian groups

The purpose of this section is to exhibit concrete situations where our results in the previous section apply. We consider only the simple but instructive case where G is a finitely generated free abelian group, although it could be interesting to consider other groups for which the second cohomology group has been computed, e. g. the 3-dimensional integer Heisenberg goup (see [20]).

We let $N \in \mathbb{N}$ and set $G = \mathbb{Z}^N$. When $x = (x_1, \dots, x_N) \in G$, we set

$$|x|_1 = \sum_{j=1}^{N} |x_j| \quad \text{and} \quad |x|_\infty = \max\{|x_i|, i = 1, \dots, N\}.$$

When $m \in \mathbb{N}$, we define $K_m \subset G$ by

$$K_m = \{x \in G \mid |x|_\infty \leq m\} \ (= \{0, 1, \dots, m\}^N).$$

To each $N \times N$ real matrix A, one may associate $u_A \in Z^2(G, \mathbb{T})$ by

$$u_A(x, y) = e^{ix \cdot (Ay)}.$$

Obviously, one may assume without loss of generality that $A \in M_N((-\pi, \pi])$, i.e. all of A's coefficients belong to $(-\pi, \pi]$. It is quite easy to see that $u_A \in B^2(G, \mathbb{T})$ if and only if A is symmetric. In fact we have $[u_A] = [u_B]$ where B denotes the skew-symmetric part of A, i.e. $B = (A - A^t)/2$. Thus every element in $H^2(G, \mathbb{T})$ may be written as $[u_A]$ for some skew-symmetric $A \in M_N((-\pi, \pi])$. We refer to [2, 3] for proofs of these facts. Twisted group C^*-algebras of the form $C^*(G, u_A)$ are often called noncommutative N-tori (since $C^*(G, u_A)$ is isomorphic to the continuous functions on the N-torus in the case where A is symmetric).

It will be convenient for us to use the following norm on $M_N(\mathbb{R})$: when $A = [a_{ij}] \in M_N(\mathbb{R})$, we set $|A|_\infty = \max\{|a_{ij}|, 1 \le i, j \le N\}$.

We first record a technical lemma.

Lemma 4.1. *Let $A \in M_N((-\pi, \pi])$, $x, y \in G$ and $m \in \mathbb{N}$. Then*

(1) $|1 - u_A(x, y)| \le |A|_\infty |x|_1 |y|_1$

(2) $\sum_{x \in K_m} |x|_1 = \frac{Nm(m+1)^N}{2}$

(3) $1 - \frac{\#((x+K_m) \cap K_m)}{\#K_m} \le \frac{|x|_1}{m+1}$.

Proof. 1) follows from $|1 - e^{ix \cdot (Ay)}| \le |x \cdot (Ay)| \le |A|_\infty |x|_1 |y|_1$.

2) $\sum_{x \in K_m} |x|_1 = \sum_{j=1}^N \sum_{x \in K_m} |x_j| = N(m+1)^{N-1} (\sum_{k=0}^m k) = \frac{Nm(m+1)^N}{2}$.

3) $1 - \frac{\#((x+K_m) \cap K_m)}{\#K_m} = \frac{\#(K_m \setminus (x+K_m))}{\#K_m} \le \frac{(m+1)^{N-1}}{(m+1)^N} |x|_1 = \frac{|x|_1}{m+1}$. □

Proposition 4.2. *Let (A_i) be a sequence in $M_N((-\pi, \pi])$ and (m_i) be a sequence in $\mathbb{N}$. For each $i \in \mathbb{N}$, we set*

$$F_i = K_{m_i} \subset G,$$

$$\phi_i = \frac{1}{(\#F_i)^{1/2}} \chi_{F_i} \in \ell^2(G),$$

$$u_i = u_{A_i} \in Z^2(G, \mathbb{T}).$$

Then we have:

(1) (F_i) is a F-sequence for G if and only if $m_i \to +\infty$.

(2) (F_i) is a σF-sequence for G if and only if $\sum_{i=1}^\infty \frac{1}{m_i} < \infty$.

(3) $\prod_i u_i$ exists $\Leftrightarrow$ $\sum_i |A_i|_\infty < \infty$.

(4) The projective unitary representation $\otimes_i \lambda_{u_i}$ of G exists on $\otimes_i^{\phi_i} \ell^2(G)$ whenever

$$\sum_{i=1}^\infty \frac{1}{m_i} < \infty \text{ and } \sum_{i=1}^\infty m_i |A_i|_\infty < \infty$$

(and then $\prod_i u_i$ is the associated 2-cocycle).

Proof. The nontrivial parts of (1) and (2) are consequences of Lemma 4.1, part (3).

Assertion (3) relies on the inequality $2|\theta|/\pi \leq |1 - e^{i\theta}| \leq |\theta|$ which holds when $|\theta| \leq \pi$.

Concerning (4) let $x, y \in G$. Then we have

$$\begin{aligned}
\frac{1}{\#F_i} \sum_{y \in F_i} |1 - u_i(-y,x)| &\leq \frac{1}{(m_i+1)^N} \Big(\sum_{y \in F_i} |A_i|_\infty |x|_1 |y|_1 \Big) \quad \text{(by Lemma 4.1, (1))} \\
&= \frac{|x|_1 |A_i|_\infty}{(m_i+1)^N} \sum_{y \in F_i} |y|_1 \\
&= \frac{|x|_1 |A_i|_\infty}{(m_i+1)^N} \frac{N m_i (m_i+1)^N}{2} \quad \text{(by Lemma 4.1, (2))} \\
&= \frac{N|x|_1}{2} m_i |A_i|_\infty
\end{aligned}$$

for every $i \in \mathbb{N}$. Hence we have

$$\sum_i \frac{1}{\#F_i} \sum_{y \in F_i} |1 - u_i(-y,x)| \leq \frac{N|x|_1}{2} \sum_i m_i |A_i|_\infty .$$

Now if we assume that $\sum_{i=1}^\infty \frac{1}{m_i} < \infty$ and $\sum_{i=1}^\infty m_i |A_i|_\infty < \infty$, then $\{F_i\}$ is a σF-sequence for G (by (2)) and $\sum_i \frac{1}{\#F_i} \sum_{y \in F_i} |1 - u_i(-y,x)| < \infty$ for all $x \in G$, and the conclusion follows from Theorem 3.5 (since condition (1) in this theorem is satisfied). □

Example. Let $A \in M_N((-\pi,\pi])$ and set $A_i = 2^{-i}A$ and $u_i = u_{A_i}$ $(i \in \mathbb{N})$. Then clearly $u_A = \prod_i u_{A_i}$. Further, if we let $m_i = i^2$, then $\sum_i 1/m_i < \infty$ and $\sum_i m_i |A_i|_\infty = |A|_\infty \sum_i i^2/2^i < \infty$ so (4) in the above Proposition applies. Corollary 3.3 then gives

$$\lambda_{u_A} \otimes I \cong \otimes_i \lambda_{u_i},$$

thus producing an infinite tensor product decomposition of the amplification of λ_{u_A}. Using this we may clearly obtain a faithful representation of the noncommutative N-torus $C^*(G, u_A) \simeq C^*_r(G, u_A)$ onto the C^*-algebra generated by $\otimes_i \lambda_{u_i}$. Notice also that $[u_A]$ is non-trivial if and only if all $[u_i]$ are non-trivial (cf. our earlier remarks in Sections 2 and 3).

Remark. We don't know whether statement (4) in Proposition 4.2 may be strenghtened to the optimal statement: $\otimes_i \lambda_{u_i}$ exists on $\otimes_i^{\phi_i} \ell^2(G)$ whenever $\sum 1/m_i < \infty$ and $\sum_i |A_i|_\infty < \infty$. In the proof of (4), we appeal to condition (1) in Theorem 3.5. One may wonder whether this better result could be obtained by appealing to condition (2) in Theorem 3.5 .

We shall now exhibit projective unitary representations arising from CCR-representations of bilinear maps on some direct product decomposition of G.

We assume from now on that $N \geq 2$ and write $G = \mathbb{Z}^N \simeq \mathbb{Z}^P \times \mathbb{Z}^Q$ where $1 < P, Q < N$ and $P + Q = N$.

To each $P \times Q$ matrix D with coefficients in $(-\pi, \pi]$, one may associate a bilinear map $\sigma_D : \mathbb{Z}^P \times \mathbb{Z}^Q \to \mathbb{T}$ by

$$\sigma_D(a, b) = e^{ia\cdot(Db)}.$$

Following the construction described at the end of the previous section, we obtain a CCR-representation of σ_D on $\ell^2(\mathbb{Z}^Q)$, or, equivalently, a projective unitary representation U_D of $G = \mathbb{Z}^N$ with associated 2-cocycle u^D. This cocycle is easy to describe: a simple computation gives

$$u^D(x, y) = e^{ix\cdot(\tilde{D}y)} \quad (x, y \in G)$$

where $\tilde{D}$ is the $N \times N$ matrix given by

$$\tilde{D} = \begin{pmatrix} 0 & 0 \\ -D^t & 0 \end{pmatrix}.$$

Notice that $u^D = u_{\tilde{D}}$ and $[u^D]$ is non-trivial whenever $D \neq 0$.

Proposition 4.3. *Let (D_i) be a sequence of $P \times Q$ matrices with coefficients in $(-\pi, \pi]$, and let $(U_i) = (U_{D_i})$ be the associated sequence of projective unitary representations of G on $\ell^2(\mathbb{Z}^Q)$. Let (n_i) be a sequence in $\mathbb{N}$.*
Set $H_i = \{b \in \mathbb{Z}^Q \mid |b|_\infty \leq n_i\}$ and $\psi_i = 1/(\#H_i)^{1/2}\chi_{H_i}$ $(i \in \mathbb{N})$.
Then $\otimes_i U_i$ exists on $\otimes_i^{\psi_i} \ell^2(\mathbb{Z}^Q)$ whenever $\sum_i 1/n_i < \infty$ and $\sum_i n_i |D_i|_\infty < \infty$.

Proof. This follows from Theorem 3.8. As the details are quite similar to the proof of the previous proposition, we leave these to the reader. □

Example. We take $P = Q = 1$ so that $G = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$, and let $(D_j) = (\theta_j)$ be a sequence in $(-\pi, \pi]$. This gives rise to the sequence (U_j) of representations of $\mathbb{Z}^2$ on $\ell^2(\mathbb{Z})$ with associated 2-cocycles

$$u_j(x, y) = e^{-i\theta_j x_1 y_2} \; (x, y \in \mathbb{Z}^2).$$

By Proposition 4.3 we can then form the infinite tensor representation $\otimes_j U_j$ whenever we can choose a sequence (n_j) in $\mathbb{N}$ such that $\sum_j 1/n_j < \infty$ and $\sum_j n_j |\theta_j| < \infty$ (e.g. $n_j = j^2$ will do if $(j^4 |\theta_j|)$ is bounded).

A more careful analysis of this situation (still based on Theorem 3.8) involving the familiar Dirichlet sums gives that $\otimes_j U_j$ will exist whenever we can choose (n_j) such that

$$\sum_j \frac{1}{n_j} < \infty \text{ and } \sum_j |1 - \frac{1}{2n_j + 1} \frac{\sin((2n_j + 1)\theta_j/2)}{\sin(\theta_j/2)}| < \infty.$$

Assuming that $\sum_j |\theta_j| < \infty$ (so $\prod_j u_j$ exists), it would be interesting to know whether such a choice of (n_j) can always be made.

It is well known that U_j is an irreducible projective representation of $\mathbb{Z}^2$ on $\ell^2(\mathbb{Z})$ if and only if θ_j/π is irrational. A problem which ought to be investigated in the future is to find conditions (if any) ensuring that $\otimes_j U_j$ (exists and) is irreducible.

5 Infinite products of actions

For each $i \in \mathbb{N}$ let $\mathcal{H}_i$ be a Hilbert space, $\phi_i \in \mathcal{H}_i$ a unit vector, $\mathcal{M}_i \subset \mathcal{B}(\mathcal{H}_i)$ a von Neumann algebra and $\alpha_i : G \to \mathrm{Aut}(\mathcal{M}_i)$ an action of G on $\mathcal{M}_i$. We denote by I_i the identity operator on $\mathcal{H}_i$. We can form the *-algebra $\overset{a}{\otimes}_i \mathcal{M}_i$ (resp. von Neumann algebra $\otimes_i(\mathcal{M}_i, \phi_i)$) acting on $\otimes_i^{(\phi_i)}\mathcal{H}_i$ generated by operators of the form $\otimes_i T_i$ where $T_i \in \mathcal{M}_i$ and $T_i = I_i$ for all but finitely many i's. At the *-algebraic level we can easily define an action $\overset{a}{\otimes}_i \alpha_i$ of G on $\overset{a}{\otimes}_i \mathcal{M}_i$ such that for every finite $J \subset \mathbb{N}$ we have

$$\overset{a}{\otimes}_i \alpha_i((\otimes_{i\in J}T_i) \otimes (\otimes_{i\notin J}I_i)) = (\otimes_{i\in J}\alpha_i(T_i)) \otimes (\otimes_{i\notin J}I_i).$$

One natural question is whether $\overset{a}{\otimes}_i \alpha_i$ may be extended to an action of G on the von Neumann algebra $\otimes_i(\mathcal{M}_i, \phi_i)$. As we shall see, the answer may be negative in some situations, regardless of the choice of unit vectors ϕ_i.

We consider only the case where each $\alpha_i(g)$ is unitarily implemented, i. e. for every i and g we can write $\alpha_i(g) = \mathrm{Ad}\,(U_i(g))$, where $U_i(g)$ is a unitary on $\mathcal{H}_i$. This assumption is automatically satisfied for many classes of von Neumann algebras (see [25], §8).

Let us first remark that the following condition:

$$(*) \quad \sum_i (1 - |(U_i(g)\phi_i, \phi_i)|) < \infty \qquad (g \in G)$$

clearly holds whenever $\otimes_i U_i(g)$ exists on $\otimes_i^{\phi_i}\mathcal{H}_i$ $(g \in G)$. In fact, using [14, §1.2], $(*)$ is equivalent to the following condition:

$$(**)\ \exists\ \rho_i : G \to \mathbb{T}, \rho_i(e) = 1 \text{ such that } \otimes \rho_i(g)U_i(g) \text{ exists on } \otimes_i^{\phi_i} \mathcal{H}_i \ (g \in G).$$

Theorem 5.1. *With the assumptions and notation introduced above we have:*

1) If condition $(*)$ holds, then $\overset{a}{\otimes}_i \alpha_i$ extends to a unitarily implemented automorphic action α on $\otimes_i(\mathcal{M}_i, \phi_i)$. Moreover if $U_i(g) \in \mathcal{M}_i$ for every i and $g \in G$, then every α_g is an inner automorphism of $\otimes_i(\mathcal{M}_i, \phi_i)$.

2) Conversely suppose that an extension α of $\overset{a}{\otimes}_i \alpha_i$ exists on $\otimes_i(\mathcal{M}_i, \phi_i)$ for some choice of unit vectors ϕ_i in such a way that $\alpha_g = \mathrm{Ad}\,(U(g))$ with $U(g) \in \mathcal{U}(\otimes_i^{\phi_i}\mathcal{H}_i)$ for every $g \in G$. Then condition $(*)$ holds in the following two cases:

i) the $\mathcal{M}_i$ have the property that any automorphism is unitarily implemented (on $\mathcal{H}_i$), and the same is true for every tensor product $\otimes_{s\in S}(\mathcal{M}_s, \phi_s)$ with $S \subset \mathbb{N}$ (S possibly infinite);

ii) the $\mathcal{M}_i$ are factors, $U_i(g) \in \mathcal{M}_i$ and $U(g) \in \otimes_i(\mathcal{M}_i, \phi_i)$.

Proof. 1) If $(*)$ holds then $(**)$ holds and one can take $\alpha_g = \mathrm{Ad}\,(U(g))$ where $U(g) = \otimes_i\rho_i(g)U_i(g)$ is well defined on $\otimes_i^{\phi_i}\mathcal{H}_i$. If $U_i(g) \in \mathcal{M}_i$ then we have $U(g) \in \otimes_i(\mathcal{M}_i, \phi_i)$ and α_g is inner for every $g \in G$.

2) Assume that an extension α of $\overset{a}{\otimes}_i\ \alpha_i$ exists on $\mathcal{M}^\phi := \otimes_i(\mathcal{M}_i, \phi_i)$ for some choice of unit vectors ϕ_i in such a way that $\alpha_g = \mathrm{Ad}\,(U(g))$ with $U(g) \in \mathcal{U}(\otimes_i^{\phi_i}\mathcal{H}_i)$ for every $g \in G$.

We assume first that we are in the situation described in case i).
Let J be a non-empty finite subset of $\mathbb{N}$. Then we may identify $\mathcal{M}^\phi$ with $(\otimes_{i\in J}\mathcal{M}_i) \otimes {}_J\mathcal{M}$ where ${}_J\mathcal{M} := \otimes_{i\notin J}(\mathcal{M}_i, \phi_i)$. We may then consider ${}_J\mathcal{M}$ as a von Neumann subalgebra of $\mathcal{M}^\phi$ in the obvious way. It is easy to see that α restricts to an action ${}_J\alpha$ of G on ${}_J\mathcal{M}$ and that we then have $\alpha = (\otimes_{i\in J}\alpha_i) \otimes {}_J\alpha$.

For each $g \in G$, using our hypothesis, we may write ${}_J\alpha_g = \mathrm{Ad}\, {}_JU(g)$ for some ${}_JU(g) \in \mathcal{U}(\otimes_{i\notin J}^{\phi_i}\mathcal{H}_i)$. Set now $U_J(g) = \otimes_{i\in J}U_i(g)$ for each $g \in G$. Then $\alpha_g = \mathrm{Ad}\,(U_J(g) \otimes {}_JU(g))$. Therefore, for each $g \in G$, there exists some $z(g) \in \mathbb{T}$ such that $U(g) = z(g)U_J(g) \otimes {}_JU(g)$. Since $U(g) \neq 0$ we can pick two elementary decomposable vectors $\otimes\psi_i$ and $\otimes\xi_i$ in $\otimes_i^{\phi_i}\mathcal{H}_i$ which do not depend on J satisfying

$$0 \neq c(g) := |(U(g) \otimes \psi_i, \otimes\xi_i)| = \prod_{i\in J} |(U_i(g)\psi_i, \xi_i)|\ |({}_JU(g) \otimes_{i\notin J} \psi_i, \otimes_{i\notin J}\xi_i)|$$

for each $g \in G$. Since $|({}_JU(g) \otimes_{i\notin J} \psi_i, \otimes_{i\notin J}\xi_i)| \leq 1$ we get

$$0 < c(g) \leq \prod_{i\in J} |(U_i(g)\psi_i, \xi_i)|.$$

As this holds fore every J, one easily deduces that $\prod_{i\in\mathbb{N}} |(U_i(g)\psi_i, \xi_i)|$ converges to a non-zero number. Since $\psi_i = \xi_i = \phi_i$ for all but finitely many i's, this implies that $(*)$ holds, as claimed.

Assume now that we are in the situation described in case ii).
We define $U_J(g)$ as in i) and set $V_J(g) = (\,U_J(g) \otimes (\otimes_{i\notin J}I_i)\,)^*U(g)$. Then, using that we may write $\alpha = (\otimes_{i\in J}\alpha_i) \otimes {}_J\alpha$, we get

$$V_J(g) \in (\otimes_i(\mathcal{M}_i, \phi_i)) \cap ((\otimes_{i\in J}\mathcal{M}_i) \otimes (\otimes_{i\notin J}\mathbb{C}I_i))'.$$

Using now that all $\mathcal{M}_i$ are factors, it is a simple exercise to deduce that $V_J(g) \in (\otimes_{i\in J}\mathbb{C}I_i) \otimes (\otimes_{i\notin J}(\mathcal{M}_i, \phi_i))$. We may therefore write $V_J(g) = (\otimes_{i\in J}I_i) \otimes {}_JV(g)$ for some unitary ${}_JV(g) \in \otimes_{i\notin J}(\mathcal{M}_i, \phi_i)$. This gives $U(g) = U_J(g) \otimes {}_JV(g)$ and we can clearly proceed further in the same way as above to show that $(*)$ holds. $\square$

The proof of the above result is nearly connected to the proof of a lemma in [26] (see also [13]).

The case of interest for us in this paper is the one where we set $\mathcal{M}_i = \mathcal{B}(\mathcal{H}_i)$ for every i. As is well-known, every automorphism of a type I factor is inner.

Therefore we can then write $\alpha_i(g) = \operatorname{Ad} U_i(g)$ for every i $(g \in G)$, where $g \mapsto U_i(g)$ is easily seen to be a projective unitary representation of G on $\mathcal{H}_i$.

Then we have

Corollary 5.2. *Let $\phi_i \in \mathcal{H}_i$ be a sequence of unit vectors. Then $\overset{a}{\otimes}_i \alpha_i$ extends (uniquely) to an action $\alpha := \otimes\alpha_i$ on $\otimes_i(\mathcal{B}(\mathcal{H}_i), \phi_i)$ ($\simeq \mathcal{B}(\otimes_i^{\phi_i}\mathcal{H}_i)$ [14, Proposition 1.6]) if and only if $\otimes_i\rho_i U_i$ exists on $\otimes_i^{\phi_i}\mathcal{H}_i$.*

Proof. This follows from theorem 5.1, using again the fact that every automorphism a type I factor is inner. □

In view of our positive existence results concerning infinite tensor product of projective unitary representations, it is clear that one may use the above corollary to produce examples where $\overset{a}{\otimes}\ \alpha_i$ extends to an action on $\otimes_i(\mathcal{B}(\mathcal{H}_i), \phi_i)$ for some suitable choice of the sequence (ϕ_i). We can also use it to present two different types of obstruction for the existence of such an extension, regardless of the choice of the vectors ϕ_i.

Corollary 5.3. *Let u_i be a sequence in $Z^2(G, \mathbb{T})$ and $\alpha_i = \operatorname{Ad} \lambda_{u_i}$ be the associated sequence of actions of G on $\mathcal{B}(\ell^2(G))$. If G is non σ-amenable group, then $\overset{a}{\otimes}_i\ \alpha_i$ does not extend to an infinite tensor product action of G on $\otimes_i(\mathcal{B}(\ell^2(G)), \phi_i)$, regardless of the choice of the vectors ϕ_i.*

Proof. According to Corollary 5.2, the existence of such an extension would imply the existence of $\otimes_i\rho_i\lambda_{u_i}$ on $\otimes_i^{\phi_i}\ell^2(G)$ for some choice of functions $\rho_i : G \to \mathbb{T}$ with $\rho_i(e) = 1$. It is straightforward to see that this amounts to the existence of $\otimes_i\lambda_{v_i}$ on some $\otimes_i^{\psi_i}\ell^2(G)$ for some $v_i \in Z^2(G, \mathbb{T})$ with $v_i \sim u_i$. This is impossible if G is not σ-amenable, as follows from Theorem 3.4. □

Corollary 5.4. *Let α_i be a sequence of actions of G on $\mathcal{B}(\mathcal{H}_i)$, so that $\alpha_i = \operatorname{Ad} U_i(g)$ where U_i are projective representations of G with associated 2-cocycles u_i. Assume that $[u_i] = [u]$ for every i and $[u] \neq [1]$ in $H^2(G, \mathbb{T})$. Then $\overset{a}{\otimes}_i\ \alpha_i$ does not extend to $\otimes_i(\mathcal{B}(\mathcal{H}_i), \phi_i)$ for any choice of the ϕ_i.*

Proof. If such an extension exists for some choice of ϕ_i, (*) holds and therefore $\otimes\rho_i U_i$ exists on $\otimes_i^{\phi_i}\mathcal{H}_i$. It follows then from Theorem 3.2 that $\prod_i(\mathrm{d}\rho_i)u_i$ exists. Hence $\mathrm{d}\rho_i u_i \to 1$ (in the pointwise topology). As each $u_i = (\mathrm{d}\rho_i')u$ for some ρ_i', we get that u is a limit of 2-coboundaries. Since $B^2(G, \mathbb{T})$ is closed, this means that u is itself a coboundary, i. e. $[u] = 1$, which gives a contradiction. □

The simplest example of the above situation occurs when choosing $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. In fact, let

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$VW = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -WV.$$

A projective (irreducible) unitary representation of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{C}^2$ is defined by setting $U((a,b)) = V^a W^b$ $(a,b \in \mathbb{Z}_2)$. Since $V^a W^b = \sigma(a,b) W^b U^a$ where $\sigma(a,b) = -1$ if $a = b = 1$ and 1 otherwise, the associated cocycle is easily computed to be $u((a_1,b_1),(a_2,b_2)) = (-1)^{a_2 b_1}$ and its class is not trivial in cohomology. Remark that U is nothing but the projective representation associated to the CCR representation of σ on $\mathbb{C}^2 = \ell^2(\mathbb{Z}_2)$ determined by V and W. Consider the action α of G on $M_2(\mathbb{C})$ given by $\alpha_{(a,b)} = \mathrm{Ad}\,(U((a,b)))$. Then, according to the above result, the infinite tensor product of α does not make sense as an action on the type I factor $\otimes_i(M_2(\mathbb{C}), \phi_i)$. Of course, the normalized trace of $\mathcal{A} = M_2(\mathbb{C})$ is α-invariant. Therefore $\otimes_i \alpha$, considered as a product action on the UHF algebra of type 2^∞, still extends to the weak closure in the GNS representation given by the (unique) tracial state (= the infinite tensor product of the trace states), which is the well known unique hyperfinite factor of type II_1.

If G is a non amenable group and we consider the action $\alpha = \mathrm{Ad}\,\lambda$ on $\mathcal{B}(\ell^2(G))$, then there are no α-invariant states (see e. g. [5]). Therefore the invariance argument sketched above to extend the algebraic tensor power of α is not available. It is conceivable that it is impossible to extend this algebraic tensor power action to $\otimes_i(\mathcal{B}(\ell^2(G)), \rho_i)$ regardless of the choice of normal states ρ_i on $\mathcal{B}(\ell^2(G))$ (here we are using the same notation as in [26]).

6 Further comments

For the sake of completeness we include in this section some comments on projective unitary representations of restricted direct product of groups and their associated C^*-algebras. Our discussion is based on [14], where Guichardet deals with the non-projective case.

Let $(G_i)_{i \in I}$ be a family of discrete groups, $e_i \in G_i$ their neutral elements, $U_i : G_i \to \mathcal{U}(H_i)$ a family of projective representations with associated 2-cocycles u_i, $\phi_i \in H_i$ *any* family of unit vectors, and consider the restricted product $\oplus_i G_i (\subset \times_i G_i)$. Then

$$\oplus_i G_i \ni g = (g_i) \mapsto U(g) = \otimes_i U_i(g_i) \in \mathcal{U}(\otimes^\phi H_i)$$

(notation: $U \equiv \otimes^{\phi_i} U_i$) is a projective representation of $\oplus_i G_i$ on $\otimes^{\phi_i} H_i$ with 2-cocycle

$$u(g,h) := \prod_i u_i(g_i, h_i), \qquad g, h \in \oplus_i G_i, \quad g = (g_i), h = (h_i).$$

Moreover $U(\oplus_i G_i)'' = \otimes^{\phi_i} U_i(G_i)''$.

A natural example of this situation is as follows. If $\mathcal{H}_i = \ell^2(G_i)$, $\phi_i = \delta_{e_i}$, $U_i = (\lambda_{G_i})_{u_i}$ with $u_i \in Z^2(G_i, \mathbb{T})$ then we have $\otimes^\phi \mathcal{H}_i \cong \ell^2(\oplus_i G_i)$ via the unitary transformation T defined by

$$\otimes_i f_i \mapsto (\oplus_i G_i \ni g = (g_i) \to \prod_i f_i(g_i))$$

(where $f_i = \phi_i$ for all but finitely many indices) [14, Corollary 1.2]. In fact, using T as intertwiner, one easily checks that

$$\otimes^{\phi_i} U_i \cong (\lambda_{\oplus_i G_i})_u$$

with u as above (cf. [14, Proposition 1.7] for the non-projective case). Thus we get the natural identification $\otimes_i^{\delta_{e_i}} VN(G_i, u_i) \cong VN(\oplus_i G_i, u)$, and also $\otimes_i^{\min} C_r^*(G_i, u_i) \cong C_r^*(\oplus_i G_i, u)$ by appealing to [15, part II, Proposition 14].

Recall now that if all the C^*-algebras $\mathcal{A}_i$ are nuclear then the natural surjection $\otimes_i^{\max} \mathcal{A}_i \to \otimes_i^{\min} \mathcal{A}_i$ is in fact an isomorphism. This is proved in [14, Proposition 2.3] in the special case where all the A_i are GCR (= type I), but the proof goes through in the general case. Moreover there is a natural identification $\otimes_i^{\max} C^*(G_i) \simeq C^*(\oplus_i G_i)$, [14, Corollary 2.3]. In a similar vein, it is possible to show the existence of an isomorphism

$$\otimes_i^{\max} C^*(G_i, u_i) \to C^*(\oplus_i G_i, u).$$

In fact this follows by setting $A_i = \mathbb{C}$ in the more general statement

$$\otimes_i^{\max} (A_i \times_{\alpha_i, u_i} G_i) \simeq (\otimes_i^{\max} A_i) \times_{\alpha, u} (\oplus_i G_i)$$

(cf. [14, Proposition 2.13]) obtained by exploiting the relevant universal properties. Here $\times$ refers to the twisted crossed product construction [21] and α and u are naturally defined in terms of the α_i's and the u_i's.

Assume that all G_i's are amenable. Then $\oplus_i G_i$ is amenable as well, and $\otimes_i^{\max} C^*(G_i) \simeq \otimes_i^{\min} C^*(G_i)$ [14, Corollary 2.4]. The obvious twisted version of this statement, namely

$$\otimes_i^{\max} C^*(G_i, u_i) \simeq \otimes_i^{\min} C^*(G_i, u_i),$$

follows then immediately from the nuclearity of $C^*(G_i, u_i)$, cf. [21, Corollary 3.9].

Going back to our situation in the previous sections, we may consider the case where all G_i's coincide with a given group G. Our results can be obviously strenghtened to get (projective) unitary representations of the subgroup generated by $\oplus_i G_i$ and the diagonal copy of G embedded in the unrestricted direct product $\times_i G_i$, acting on $\otimes^{\phi_i} H_i$ for suitably chosen vectors ϕ_i (which can definitely not be chosen as some Dirac delta functions). We will not address the apparently complicated problem of representing the whole unrestricted direct product or any other subgroup of it containing the diagonal.

Acknowledgements. Most of this work was made while R. C. was visiting the Department of Mathematics at the University of Oslo during the academic year 1998-1999 supported by the EU TMR network "Non Commutative Geometry". He is very grateful to the members of the Operator Algebra group in Oslo for their friendly hospitality and for providing perfect working conditions. Both authors would like to thank Ola Bratteli and Erling Størmer for some helpful discussions.

References

[1] M. Aita, W. Bergmann, R. Conti: Amenable groups and generalized Cuntz algebras, *J. Funct. Anal.* **150**, 48–64 (1997).

[2] N. B. Backhouse : Projective representations of space groups, II: Factor systems, *Quart. J. Math. Oxford* **21** (1970), 223-242.

[3] N. B. Backhouse, C. J. Bradley : Projective representations of space groups, I: Translation groups, *Quart. J. Math. Oxford* **21** (1970), 203-222.

[4] E. Bédos: On the C^*-algebra generated by the left regular representation of a locally compact group, *Proc. Amer. Math. Soc.* **120**, 603–608 (1994).

[5] M. E. B. Bekka: Amenable unitary representation of locally compact groups, *Invent . Math.* **100**, 383–401 (1990).

[6] W. Bergmann, R. Conti: Group representations and generalized Cuntz algebras, in preparation.

[7] K. S. Brown: Cohomology of groups, GTM in Mathematics 87, Springer-Verlag, New-York (1982).

[8] R. Conti: Amenability and infinite tensor products, in Operator Theory, Operator Algebras and Related Topics, Proceedings of the 16th International Conference on Operator Theory, Timisoara 1996, The Theta Foundation, Bucharest 1997.

[9] R. Conti: On groups with Kazhdan's property T and Cuntz algebras, in preparation.

[10] J. Dixmier: Les C*-algebres et leurs representations, Gauthiers-Villars, Paris (1969).

[11] D. E. Evans, Y. Kawahigashi: Quantum symmetries on operator algebras, Oxford : Oxford University Press (1998).

[12] P. Eymard: Moyennes invariantes et representations unitaires, *Lect. Notes in Math.* **300**, Berlin-Heidelberg-New York, Springer Verlag (1972).

[13] G. A. Elliott: Finite projections in tensor products of von Neumann algebras, *Trans. Amer. Math. Soc.* **212**, 47–60 (1975).

[14] A. Guichardet: Produits tensoriels infinis et représentations des relations d'anticommutation, *Ann. scient. Éc. Norm. Sup.* **3**, 1–52 (1966).

[15] A. Guichardet: Tensor product of C*-algebras, part I and II, *Lect. Notes Series No* **12,13**, Aarhus University (1969).

[16] T. Hirai, H. Shimomura: Relations between unitary representations of diffeomorphism groups and those of the infinite symmetric group or of related permutation groups, *J. Math. Kyoto Univ.* **37**, 261–316 (1997).

[17] J. R. Klauder, J. McKenna, E. J. Woods: Direct product representations of the canonical commutation relations, *J. Math. Phys.* **7**, 822–828 (1966).

[18] A. Kleppner: Multipliers on Abelian groups, *Math. Ann.* **158**, 11–34 (1965).

[19] J. von Neumann: On infinite direct products, *Compos. Math.* **6**, 1–77 (1938).

[20] J. A. Packer: C*-algebras generated by projective representations of the discrete Heisenberg group, *J. Operator Th.* **18**, 41–66 (1987).

[21] J. A. Packer, I. Raeburn: Twisted crossed product of C^*-algebras, *Math. Proc. Camb. Phil. Soc.* **106**, 293–311 (1989).

[22] A. Paterson: Amenability, Math. Surveys and Monographs 29, Amer. Math. Soc. (1988).

[23] J. P. Pier: Amenable locally compact groups, John Wiley & Sons Inc. (1984).

[24] J. Slawny: On factor representations and the C^*-algebra of canonical commutation relations, *Comm. Math. Phys.* **24**, 151–170 (1972).

[25] S. Strătilă, L. Zsidó: Lectures on von Neumann algebras, Abacus Press, Tunbridge Wells, Kent (1979).

[26] E. Størmer: On infinite tensor products of von Neumann algebras, *Amer. J. Math.* **93**, 810–818 (1971).

[27] G. Zeller-Meier: Produits croisés d'une C*-algèbre par un groupe d'automorphismes, *J. Math. Pures Appl.* **47**, 101-239 (1968).

ISBN 82–553–1209–9
ISSN 0806–2439
Pure Mathematics

No. 26
December 1999

On infinite tensor products of projective unitary representations

by

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PREPRINT SERIES

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSLO

MATEMATISK INSTITUTT/UNIVERSITETET I OSLO

On infinite tensor products of projective unitary representations

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Abstract

We initiate a study of infinite tensor products of projective unitary representations of a discrete group G. Special attention is given to regular representations twisted by 2-cocycles and to projective representations associated with CCR-representations of bilinear maps. Detailed computations are presented in the case where G is a finitely generated free abelian group. We also apply our results to discuss an extension problem about product type actions of G, where the projective representation theory of G plays a central role.

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* partially supported by the Norwegian Research Council
** supported by the EU TMR network "Non Commutative Geometry"

and infinite tensor products. Section 3 contains our main results concerning necessary conditions and sufficient conditions for the existence of infinite tensor products of projective unitary representations. We especially display some sufficient conditions for countable amenable groups in the case of projective regular representations and in the case of projective representations associated with CCR-representations of bilinear maps. In order to illustrate our work with some concrete examples we present in section 4 some explicit computations concerning finitely generated free abelian groups. The next section deals with some applications to the existence problem for infinite tensor products of actions of a group G on von Neumann algebras. One of our result exhibits an obstruction for extending some algebraic tensor power action of G to the weak closure that lies in the second cohomology group $H^2(G,\mathbb{T})$. In another result, the obstruction lies in the non-amenability of G. In the final section we collect some related remarks about projective unitary representations of restricted direct products of groups and associated operator algebras.

2 Preliminaries

Throughout this note G denotes a non-trivial discrete group, with neutral element e, while λ denotes the (left) regular representation of G acting on $\ell^2(G)$.

A *2-cocycle* (or multiplier) on G with values in the circle group $\mathbb{T}$ is a map $u : G \times G \to \mathbb{T}$ such that

$$u(x,y)u(xy,z) = u(y,z)u(x,yz) \qquad (x,y,z \in G),$$

see e.g. [7, Chapter IV]. We will consider only *normalized* 2-cocycles, satisfying

$$u(x,e) = u(e,x) = 1 \qquad (x \in G).$$

The set of all such 2-cocycles, which is denoted by $Z^2(G,\mathbb{T})$, becomes an abelian group under pointwise product. We equip $Z^2(G,\mathbb{T})$ with the topology of pointwise convergence.

A 2-cocycle v on G is called a *coboundary* whenever $v(x,y) = \rho(x)\rho(y)\overline{\rho(xy)}$ $(x,y \in G)$ for some $\rho : G \to \mathbb{T}$, $\rho(e) = 1$, in which case we write $v = \mathrm{d}\rho$ (such a ρ is uniquely determined up to multiplication by a character). The set of all coboundaries, which is denoted by $B^2(G,\mathbb{T})$, is a subgroup of $Z^2(G,\mathbb{T})$, which is easily seen to be closed (using Tychonov's theorem). The quotient group $H^2(G,\mathbb{T}) := Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$ is called the *second cohomology group* of G with values in $\mathbb{T}$. We denote elements in $H^2(G,\mathbb{T})$ by $[u]$ and write $v \sim u$ when $[v] = [u]$ $\quad (u, v \in Z^2(G,\mathbb{T}))$. We also write $v \sim_\rho u$ when we have $v = (\mathrm{d}\rho)u$ for some coboundary $\mathrm{d}\rho$.

We shall be interested in *product* sequences in $Z^2(G,\mathbb{T})$: we call a sequence (u_i) in $Z^2(G,\mathbb{T})$ for a *product* sequence whenever the (pointwise) infinite product $u = \prod_i u_i$ exists on $G \times G$ (u being then obviously a 2-cocycle itself). Such a product sequence will occasionally be called *1-free* if $u_i \neq 1$ for every i. Notice that 1-free product sequences are easily seen to exist since we allow

For $i = 1, 2$, let U_i be a projective unitary representation of G on a Hilbert space $\mathcal{H}_i$ associated with $u_i \in Z^2(G, \mathbb{T})$. Then the naturally defined tensor product representation $U_1 \otimes U_2$ is easily seen to be a projective unitary representation of G on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ associated with the product cocycle $u_1 u_2$. In the case of ordinary unitary representations of a group, it is a classical result of Fell (cf. [10], 13.11.3) that the (left) regular representation acts in an absorbing way with respect to tensoring (up to multiplicity and equivalence). In the projective case we have the following analogue.

Proposition 2.2. *Let u, v be elements in $Z^2(G, \mathbb{T})$ and let V be any projective unitary representation of G on a Hilbert space $\mathcal{H}$ associated with v. Then the tensor product representation $\lambda_u \otimes V$ is unitarily equivalent to $\lambda_{uv} \otimes I_{\mathcal{H}}$, i.e. to (dim V) $\cdot \lambda_{uv}$.*

Proof. We leave to the reader to check that the same unitary operator W as in the non-projective case (which is determined on $\ell^2(G) \otimes \mathcal{H}$ ($\cong \ell^2(G, \mathcal{H})$) by $(W(f \otimes \psi))(x) = f(x)V(x^{-1})\psi$) implements the asserted equivalence. □

We conclude this section by recalling a few facts and some notation concerning infinite tensor products.

Let $\mathcal{H} = \{\mathcal{H}_i\}$ denote a sequence of Hilbert spaces and $\phi = \{\phi_i\}$ be a sequence of unit vectors where $\phi_i \in \mathcal{H}_i$ for each $i \geq 1$. We denote by $\mathcal{H}^\phi$ or by $\bigotimes_i^\phi \mathcal{H}_i$ the associated infinite tensor product Hilbert space of the $\mathcal{H}_i$'s along the sequence ϕ (sometimes called the incomplete direct product space detemined by ϕ), whose construction goes back to von Neumann [19]. We will follow the slightly different approach given by Guichardet in [14, 15]. We give here only a short account, and the reader should consult these papers for full details on this matter.

For any sequence $\psi_i \in \mathcal{H}_i$ such that

$$\sum_i |\, 1 - \|\psi_i\| \,| < \infty \text{ and } \sum_i |\, 1 - (\psi_i, \phi_i) \,| < \infty$$

there corresponds a so-called decomposable vector

$$\otimes_i \psi_i = \lim_n \psi_1 \otimes \ldots \otimes \psi_n \otimes \phi_{n+1} \otimes \phi_{n+2} \otimes \ldots \in \mathcal{H}^\phi$$

depending linearly on each ψ_i (in fact one gets here convergence over the net consisting of nonempty finite subsets of $\mathbb{N}$ ordered by inclusion, cf. [15, Part II, Proposition 5]). If $\otimes_i \xi_i$ is another decomposable vector in $\mathcal{H}^\phi$, then

$$(\otimes_i \psi_i, \otimes_i \xi_i) = \prod_i (\psi_i, \xi_i)$$

(where the infinite product above is convergent in the unordered sense, cf. [19, §2]). Each finite tensor product $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_k$ is embedded in $\mathcal{H}^\phi$ by extending the map identifying a simple tensor of the form $\psi_1 \otimes \ldots \otimes \psi_k$ with the elementary

3 Infinite tensor products of projective unitary representations

In this section we shall discuss the following (loosely formulated) problem: If U_i is a sequence of projective unitary representations of a group G, when is it possible to form the infinite tensor product $\otimes_i U_i$?

The most elementary case to consider consists obviously of picking just one projective unitary representation U of G on a Hilbert space $\mathcal{H}$ with associated 2-cocycle u and trying to form the infinite tensor product of U with itself infinitely many times, i.e. its infinite tensor power. For each $i \in \mathbb{N}$, put then $U_i = U$, $\mathcal{H}_i = \mathcal{H}$ and let $\phi = \{\phi_i\}$ be a sequence of unit vectors in $\mathcal{H}$.

Now, if we assume that $U^{\otimes\infty}(x) := \otimes_i U_i(x)$ exists on $\bigotimes_i^{\phi} \mathcal{H}_i$ for all $x \in G$, then Proposition 2.3 gives

$$\sum_i |\, 1 - (U(x)\phi_i, \phi_i) \,| < \infty,$$

and especially

$$\lim_i \, (U(x)\phi_i, \phi_i) = 1$$

for all $x \in G$. Letting then ω be any weak* limit point of the sequence (ω_{ϕ_i}) of vector states of $\mathcal{B}(\mathcal{H})$, we have $\omega(U(x)) = 1$ for all $x \in G$, from which it follows (using the Cauchy-Schwarz inequality for states, as in [4]) that ω is multiplicative at each $U(x)$. This implies that

$$1 = \omega(\, U(x) \,)\, \omega(\, U(y) \,) = \omega(\, U(x)U(y) \,)$$

$$= \omega(\, u(x,y)U(xy) \,) = u(x,y)\omega(\, U(xy) \,) = u(x,y)$$

for all $x, y \in G$, i.e. u is the trivial 2-cocycle on G and consequently, U is an ordinary unitary representation of G.

Concerning infinite tensor powers of unitary representations, we refer to [1] in the case of the regular representation and [6] in the case of the adjoint representation. More generally, we have (cf. [9]):

Proposition 3.1. *Let U be a unitary representation of G on $\mathcal{H}$.*

1) If $U^{\otimes\infty}$ exists (i.e. $U^{\otimes\infty}(x)$ exists for all $x \in G$) on $\mathcal{H}^{\phi} = \bigotimes_i^{\phi} \mathcal{H}_i$ for some sequence $\phi = \{\phi_i\}$ of unit vectors in $\mathcal{H}$, then 1 (= the trivial one-dimensional representation of G) is weakly contained in U in the sense of Fell ([10]).(As usual, we denote this by $1 \prec U$).

2) If G is countable and $1 \prec U$, then there exists a sequence $\phi = (\phi_i)$ of unit vectors in $\mathcal{H}$ such that $U^{\otimes\infty}$ exists on $\mathcal{H}^{\phi}$ (and $U^{\otimes\infty}$ is then a unitary representation of G).

Proof. For completeness, we sketch the proof.

1) When $U^{\otimes\infty}$ exists on $\mathcal{H}^{\phi}$, we have $\lim_i (U(x)\phi_i, \phi_i) = 1$ for all $x \in G$, so $1 \prec U$ by [10], 18.1.4.

σF-sequence. Finally, when (F_i) is a σF-sequence for G, χ_{F_i} denotes the characteristic function of F_i and we set $\phi_i := \chi_{F_i}/\#(F_i)^{1/2}$, then $\lambda^{\otimes\infty}$ exists along the sequence (ϕ_i) of unit vectors in $\ell^2(G)$: this readily follows from the equation

$$(\lambda(x)\chi_F, \chi_F) = \#(F \cap xF)$$

which holds for every $x \in G$ and every non-empty, finite subset F of G.

Concerning amenability of representations, we recall that a unitary representation π of G on a Hilbert space $\mathcal{H}$ is called *amenable* (in the sense of Bekka, cf. [5]) whenever there exists a state ω on $\mathcal{B}(\mathcal{H})$ satisfying $\omega(\pi(x)T\pi(x)^*) = \omega(T)$ for all $x \in G$ and all $T \in \mathcal{B}(\mathcal{H})$. It is easy to see that π is amenable whenever $1 \prec \pi$, while the converse implication is not necessarily true. Bekka has shown that π is amenable if and only if $1 \prec \pi \otimes \overline{\pi}$, and that amenability of a group is characterized by the fact that all its unitary representations are amenable (and it suffices to check this for its regular representation). However, many non-amenable groups, such as non-abelian free groups, do have amenable representations. Hence, using the above proposition, it is clear that one may produce examples of unitary representations of non-amenable groups for which the associated infinite tensor power representation exists.

The next natural step now is to try to form the infinite tensor product of a sequence of (possibly) different unitary representations of G. In the simple case where $G = \mathbb{Z}$, this boils down to the question of existence of the infinite tensor product of a sequence of unitary operators, and we have no more conceptual answer to this question than the one provided by Proposition 2.3. In the case where G is a group acting on some standard Borel space S with a quasi-invariant measure, one may construct sequences of unitary representations of G associated with suitably chosen Borel 1-cocycles on $S \times G$ and study the existence of the resulting infinite tensor products, essentially along the same lines as in [16]. Since this would lead us too far away from our main task, we don't elaborate on this matter here. Therefore, we arrive to the final step of generality, which is to consider a sequence of *projective* unitary representations. Before attacking the (main) problem whether it is possible to form an infinite tensor product of such a sequence in some cases, we first show that this construction, when possible, produces a new projective unitary representation of G, and also make some general observations.

Theorem 3.2. *Let U_i be a sequence of projective unitary representations of G acting respectively on a Hilbert space $\mathcal{H}_i$ and with associated $u_i \in Z^2(G, \mathbb{T})$. Let $\phi = (\phi_i)$ be a sequence of unit vectors where each $\phi_i \in \mathcal{H}_i$. Assume that $\otimes_i U_i(x)$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \mathcal{H}_i$ for each $x \in G$. Then we have*

i) (u_i) is a product sequence in $Z^2(G, \mathbb{T})$.

ii) The map $x \to U^\phi(x) := \otimes_i U_i(x)$ is a projective unitary representation of G on $\mathcal{H}^\phi$ with $u = \prod_i u_i$ as its associated 2-cocycle.

Proof. We notice first that Proposition 2.3 implies that each $U^\phi(x) := \otimes_i U_i(x)$ is a unitary.

Therefore, setting $f_i(x) := |(U_i(x)\phi_i, \phi_i)| \geq 0$ we have $0 \leq f_i \leq 1$, $f_i \in C_0(G)$ and $f_i \to 1$ pointwise. Then $f_i^{-1}([1/2,1]) =: H_i$ is finite, and $G = \cup_i H_i$ is therefore countable. Moreover, we get

$$(\lambda(x)|\phi_i|\,,\,|\phi_i|) \to 1 \qquad (x \in G),$$

so $1 \prec \lambda$ and the amenability of G follows. □

We now turn our attention to the problem of showing that it is possible to form the infinite tensor product of a sequence of projective unitary representations of G, at least in some specific situations. Some concrete examples illustrating our results will be given in the next section.

In view of Theorem 3.4, it is quite natural to wonder whether some converse holds. So we assume that G is σ-amenable, let (u_i) be a product sequence in $Z^2(G,\mathbb{T})$ and set $U_i = \lambda_{u_i}$ for every i. The question is then whether it does *always* exist a sequence $\phi = (\phi_i)$ of unit vectors in $\ell^2(G)$ such that $\otimes_i U_i$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \ell^2(G)$, i. e. such that

$$\sum_i |1 - (U_i(x)\phi_i, \phi_i)| < \infty \quad \text{for all } x \in G.$$

It is conceivable that the answer to this question is positive and we shall provide a partial answer in this direction. Our approach is based on the following inequality :

$$(*) \ \sum_i |1 - (U_i(x)\phi_i, \phi_i)| \leq \sum_i |1 - (\lambda(x)\phi_i, \phi_i)| + \sum_i |(\,(\lambda(x) - U_i(x))\phi_i, \phi_i\,)|$$

which is valid for every $x \in G$ and every sequence $(\phi_i) \subset \ell^2(G)$, as follows from the triangle inequality.

Now, since G is assumed to be σ-amenable, we can surely find some sequence (ϕ_i) of unit vectors in $\ell^2(G)$ making the first sum of the right hand side of the above inequality convergent for every $x \in G$, and the problem is then whether (ϕ_i) can also be chosen so that the second sum is convergent for every $x \in G$. There is some flexibility of choice here and it is not difficult to see that this might be achieved if one is willing to eventually replace (U_i) by one suitably chosen subsequence if necessary. We illustrate this by showing that all ϕ_i's may then even be chosen as normalized characteristic functions associated with some σF-sequence for G.

We first record an easy calculation. Let χ_F denote the characteristic function of some finite (non-empty) subset $F \subset G$ and set $\phi_F := \chi_F/(\#F)^{1/2}$. Let $u \in Z^2(G,\mathbb{T})$. Then we have

$$(\lambda_u(x)\phi_F, \phi_F) = \frac{1}{\#F} \sum_{y \in F \cap xF} u(y^{-1}, x)$$

Let $x \in G$ and choose $N \in \mathbb{N}$ such that $x \in H_N$. Then we get

$$\sum_i \frac{1}{\#F_i} \sum_{y \in F_i} |1 - u_i(y^{-1}, x)|$$

$$\leq \sum_{i<N} 2 + \sum_{i \geq N} \frac{1}{\#F_i} \sum_{y \in F_i} 1/i^2$$

$$= 2(N-1) + \sum_{i \geq N} 1/i^2 < \infty.$$

This shows that (F_i) satisfies (1) in Theorem 3.5, from which the result then clearly follows. □

Corollary 3.7. *Let G be σ-amenable. Then there always exist some 1-free product sequence (u_i) in $Z^2(G, \mathbb{T})$ and some sequence $\phi = (\phi_i)$ of unit vectors in $\ell^2(G)$ such that $\otimes_i \lambda_{u_i}$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \ell^2(G)$. If $H^2(G, \mathbb{T})$ is non-trivial and $1 \neq [u] \in H^2(G, \mathbb{T})$, then the sequence (u_i) above may chosen so that $u = \prod_i u_i$.*

Proof. Since 1-free product sequences do exist in $B^2(G, \mathbb{T})$ and 1-freeness is clearly preserved when passing to subsequences, the first assertion follows from the previous corollary. The 1-free product sequence (u_i) is then in $B^2(G, \mathbb{T})$. Therefore (by closedness) $\prod_i u_i \in B^2(G, \mathbb{T})$, so we may write it as $\mathrm{d}\rho$ for some normalized $\rho : G \to \mathbb{T}$. Assume now $H^2(G, \mathbb{T})$ is non-trivial and $1 \neq [u] \in H^2(G, \mathbb{T})$. Set $v_1 = \overline{d\rho}\, u$ and $v_i = u_{i-1}, i > 1$. Then (v_i) is a 1-free product sequence satisfying $u = \prod_i v_i$. Further we can define a new sequence $\psi = (\psi_i)$ of unit vectors in $\ell^2(G)$, by setting $\psi_1 = \delta_e$ and $\psi_i = \psi_{i-1}, i > 1$. It is then obvious that $\otimes_i \lambda_{v_i}$ exists on $\mathcal{H}^\psi$, which proves the second assertion. □

Remarks.

1) It follows from Corollary 3.3 that representations obtained as the infinite tensor product of projective regular representations are never irreducible.

2) It is unknown to us whether the second assertion of the Corollary 3.7 may be strengthened into that *all* u_i may be chosen to be non-trivial in cohomology if one assumes that $H^2(G, \mathbb{T})$ is infinite (the above proof ensures only that one of the u_i's satisfies this requirement). We will see that this may be done in an example considered in Section 4.

3) Let G be σ-amenable and let (u_i) and (v_i) be two sequences in $Z^2(G, \mathbb{T})$ satisfying $v_i \sim_{\rho_i} u_i$ for every i. Assume that $\otimes_i \lambda_{u_i}$ exists on $\mathcal{H}^\phi = \otimes_i^\phi \ell^2(G)$ for some sequence $\phi = (\phi_i)$ of unit vectors in $\ell^2(G)$. As $\prod_i v_i$ does not necessarily exist, it may happen that $\otimes_i \lambda_{v_i}$ can not be formed at all (cf. Theorem 3.2). However, it is quite clear that $\rho_1 \lambda_{v_1} \otimes \rho_2 \lambda_{v_2} \otimes \cdots$ exists on $\otimes^{\psi_i} \ell^2(G)$, where ψ_i is defined by $\psi_i(x) = \overline{\rho_i(x^{-1})} \phi_i(x)$, and this may be considered as a problem of gauge fixing. On the other hand, let us also assume that $\otimes_i \lambda_{v_i}$ exists on

construction (and remark that a similar representation can be constructed on $\ell^2(A)$ in an analogous way):

For each $a \in A, b \in B$ we set $\sigma_a(b) = \sigma(a,b)$, so the map $(a \mapsto \sigma_a)$ belongs to $\mathrm{Hom}(A, \hat{B})$ where $\hat{B} := \mathrm{Hom}(B, \mathbb{T})$. Let then $V_\sigma(a)$ denote the multiplication operator by the function σ_a on $\ell^2(B)$ and λ_B be the left regular representation of B on $\ell^2(B)$. By computation we have

$$V_\sigma(a)\lambda_B(b) = \sigma(a,b)\ \lambda_B(b)V_\sigma(a)$$

for all $a \in A, b \in B$. If we now put $U_\sigma(a,b) := V_\sigma(a)\lambda_B(b)$ for all $(a,b) \in G$, then U_σ is as desired. The triple $\{V_\sigma, \lambda_B, \ell^2(B)\}$ is a CCR-representation of σ if we agree to call a triple $\{V, W, \mathcal{H}\}$ for a CCR-representation of σ whenever V and W are unitary representations of respectively A and B on $\mathcal{H}$ which satisfy the CCR-relation

$$V(a)W(b) = \sigma(a,b)\ W(b)V(a)$$

for all $a \in A, b \in B$. There is an obvious 1-1 correspondence between CCR-representations of σ, projective unitary representations of G associated with u_σ and nondegenerate representations of $C^*(G, u_\sigma)$. For the sake of completeness, we mention that the C^*-algebra $C^*(G, u_\sigma)$ may be decomposed as the ordinary crossed product $C^*(B) \times_\alpha A$ where α is the action of A on $C^*(B)$ naturally induced by the homomorphism $(a \mapsto \sigma_a)$ from A into $\hat{B}$ (and anagously as the crossed product of $C^*(A)$ by the induced action of B).

Assume now that (σ_i) is a sequence of bilinear maps from $A \times B$ into $\mathbb{T}$. (In the example mentioned earlier, this is achieved by first picking a sequence in $\mathrm{Hom}(B, \mathbb{T})$). Then we set $U_i := U_{\sigma_i}$ and consider the question: when is it possible to form $\otimes_i U_i$ on $\otimes_i^\phi \ell^2(B)$ for some sequence $\phi = (\phi_i)$ of unit vectors in $\ell^2(B)$? Or, equivalently, when is it possible to form the infinite tensor product of the CCR-representations associated with the σ_i's ? In the case of a positive answer $\prod_i u_{\sigma_i}$ will exist (as a consequence of Theorem 3.2), so $\prod_i \sigma_i$ will then exist too and the infinite tensor product of the CCR-representations associated with the σ_i's will be a CCR-representation of this product map.

Since $U_i(e,b) = \lambda_B(b)$, we must at least require that B is σ-amenable and ϕ is chosen so that $\otimes_i \lambda_B$ exists on $\otimes_i^\phi \ell^2(B)$, in accordance with Theorem 3.4. The question reduces then clearly to whether ϕ can also be chosen so that $\otimes_i V_{\sigma_i}$ exist on $\otimes_i^\phi \ell^2(B)$, i. e. whether

$$\sum_i |1 - (V_{\sigma_i}(a)\phi_i, \phi_i)| = \sum_i |1 - ((\sigma_i)_a\, \phi_i, \phi_i)| < \infty$$

holds for every $a \in A$. Remark that our first requirement on ϕ prevents us from choosing $\phi_i = \delta_e$ for every i, which would have made all these sums convergent in a trivial way. Choosing ϕ to be associated with a σF-sequence for B leads to the following:

It will be convenient for us to use the following norm on $M_N(\mathbb{R})$: when $A = [a_{ij}] \in M_N(\mathbb{R})$, we set $|A|_\infty = \max\{|a_{ij}|, 1 \le i, j \le N\}$.

We first record a technical lemma.

Lemma 4.1. *Let $A \in M_N((-\pi, \pi])$, $x, y \in G$ and $m \in \mathbb{N}$. Then*

(1) $|1 - u_A(x, y)| \le |A|_\infty |x|_1 |y|_1$

(2) $\sum_{x \in K_m} |x|_1 = \frac{Nm(m+1)^N}{2}$

(3) $1 - \frac{\#((x+K_m) \cap K_m)}{\#K_m} \le \frac{|x|_1}{m+1}$.

Proof. 1) follows from $|1 - e^{ix\cdot(Ay)}| \le |x \cdot (Ay)| \le |A|_\infty |x|_1 |y|_1$.

2) $\sum_{x \in K_m} |x|_1 = \sum_{j=1}^N \sum_{x \in K_m} |x_j| = N(m+1)^{N-1} (\sum_{k=0}^m k) = \frac{Nm(m+1)^N}{2}$.

3) $1 - \frac{\#((x+K_m)\cap K_m)}{\#K_m} = \frac{\#(K_m \setminus (x+K_m))}{\#K_m} \le \frac{(m+1)^{N-1}}{(m+1)^N} |x|_1 = \frac{|x|_1}{m+1}$. □

Proposition 4.2. *Let (A_i) be a sequence in $M_N((-\pi, \pi])$ and (m_i) be a sequence in $\mathbb{N}$. For each $i \in \mathbb{N}$, we set*

$$F_i = K_{m_i} \subset G,$$

$$\phi_i = \frac{1}{(\#F_i)^{1/2}} \chi_{F_i} \in \ell^2(G),$$

$$u_i = u_{A_i} \in Z^2(G, \mathbb{T}).$$

Then we have:

(1) (F_i) is a F-sequence for G if and only if $m_i \to +\infty$.

(2) (F_i) is a σF-sequence for G if and only if $\sum_{i=1}^\infty \frac{1}{m_i} < \infty$.

(3) $\prod_i u_i$ exists $\Leftrightarrow \sum_i |A_i|_\infty < \infty$.

(4) The projective unitary representation $\otimes_i \lambda_{u_i}$ of G exists on $\otimes_i^{\phi_i} \ell^2(G)$ whenever

$$\sum_{i=1}^\infty \frac{1}{m_i} < \infty \text{ and } \sum_{i=1}^\infty m_i |A_i|_\infty < \infty$$

(and then $\prod_i u_i$ is the associated 2-cocycle).

We assume from now on that $N \geq 2$ and write $G = \mathbb{Z}^N \simeq \mathbb{Z}^P \times \mathbb{Z}^Q$ where $1 < P, Q < N$ and $P + Q = N$.

To each $P \times Q$ matrix D with coefficients in $(-\pi, \pi]$, one may associate a bilinear map $\sigma_D : \mathbb{Z}^P \times \mathbb{Z}^Q \to \mathbb{T}$ by

$$\sigma_D(a, b) = e^{ia\cdot(Db)}.$$

Following the construction described at the end of the previous section, we obtain a CCR-representation of σ_D on $\ell^2(\mathbb{Z}^Q)$, or, equivalently, a projective unitary representation U_D of $G = \mathbb{Z}^N$ with associated 2-cocycle u^D. This cocycle is easy to describe: a simple computation gives

$$u^D(x, y) = e^{ix\cdot(\tilde{D}y)} \quad (x, y \in G)$$

where $\tilde{D}$ is the $N \times N$ matrix given by

$$\tilde{D} = \begin{pmatrix} 0 & 0 \\ -D^t & 0 \end{pmatrix}.$$

Notice that $u^D = u_{\tilde{D}}$ and $[u^D]$ is non-trivial whenever $D \neq 0$.

Proposition 4.3. *Let (D_i) be a sequence of $P \times Q$ matrices with coefficients in $(-\pi, \pi]$, and let $(U_i) = (U_{D_i})$ be the associated sequence of projective unitary representations of G on $\ell^2(\mathbb{Z}^Q)$. Let (n_i) be a sequence in $\mathbb{N}$.*
Set $H_i = \{b \in \mathbb{Z}^Q \mid |b|_\infty \leq n_i\}$ and $\psi_i = 1/(\#H_i)^{1/2}\chi_{H_i}$ $(i \in \mathbb{N})$.
Then $\otimes_i U_i$ exists on $\otimes_i^{\psi_i}\ell^2(\mathbb{Z}^Q)$ whenever $\sum_i 1/n_i < \infty$ and $\sum_i n_i|D_i|_\infty < \infty$.

Proof. This follows from Theorem 3.8. As the details are quite similar to the proof of the previous proposition, we leave these to the reader. □

Example. We take $P = Q = 1$ so that $G = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$, and let $(D_j) = (\theta_j)$ be a sequence in $(-\pi, \pi]$. This gives rise to the sequence (U_j) of representations of $\mathbb{Z}^2$ on $\ell^2(\mathbb{Z})$ with associated 2-cocycles

$$u_j(x, y) = e^{-i\theta_j x_1 y_2} \ (x, y \in \mathbb{Z}^2).$$

By Proposition 4.3 we can then form the infinite tensor representation $\otimes_j U_j$ whenever we can choose a sequence (n_j) in $\mathbb{N}$ such that $\sum_j 1/n_j < \infty$ and $\sum_j n_j|\theta_j| < \infty$ (e.g. $n_j = j^2$ will do if $(j^4|\theta_j|)$ is bounded).

A more careful analysis of this situation (still based on Theorem 3.8) involving the familiar Dirichlet sums gives that $\otimes_j U_j$ will exist whenever we can choose (n_j) such that

$$\sum_j \frac{1}{n_j} < \infty \textit{ and } \sum_j |1 - \frac{1}{2n_j + 1}\frac{\sin((2n_j + 1)\theta_j/2)}{\sin(\theta_j/2)}| < \infty.$$

Assuming that $\sum_j |\theta_j| < \infty$ (so $\prod_j u_j$ exists), it would be interesting to know whether such a choice of (n_j) can always be made.

It is well known that U_j is an irreducible projective representation of $\mathbb{Z}^2$ on $\ell^2(\mathbb{Z})$ if and only if θ_j/π is irrational. A problem which ought to be investigated in the future is to find conditions (if any) ensuring that $\otimes_j U_j$ (exists and) is irreducible.

2) Assume that an extension α of $\overset{a}{\otimes}_i \alpha_i$ exists on $\mathcal{M}^\phi := \otimes_i(\mathcal{M}_i, \phi_i)$ for some choice of unit vectors ϕ_i in such a way that $\alpha_g = \mathrm{Ad}\,(U(g))$ with $U(g) \in \mathcal{U}(\otimes_i^{\phi_i}\mathcal{H}_i)$ for every $g \in G$.

We assume first that we are in the situation described in case i).
Let J be a non-empty finite subset of $\mathbb{N}$. Then we may identify $\mathcal{M}^\phi$ with $(\otimes_{i\in J}\mathcal{M}_i) \otimes {}_J\mathcal{M}$ where ${}_J\mathcal{M} := \otimes_{i\notin J}(\mathcal{M}_i, \phi_i)$. We may then consider ${}_J\mathcal{M}$ as a von Neumann subalgebra of $\mathcal{M}^\phi$ in the obvious way. It is easy to see that α restricts to an action ${}_J\alpha$ of G on ${}_J\mathcal{M}$ and that we then have $\alpha = (\otimes_{i\in J}\alpha_i) \otimes {}_J\alpha$.

For each $g \in G$, using our hypothesis, we may write ${}_J\alpha_g = \mathrm{Ad}\,{}_JU(g)$ for some ${}_JU(g) \in \mathcal{U}(\otimes_{i\notin J}^{\phi_i}\mathcal{H}_i)$. Set now $U_J(g) = \otimes_{i\in J}U_i(g)$ for each $g \in G$. Then $\alpha_g = \mathrm{Ad}\,(U_J(g) \otimes {}_JU(g))$. Therefore, for each $g \in G$, there exists some $z(g) \in \mathbb{T}$ such that $U(g) = z(g)U_J(g) \otimes {}_JU(g)$. Since $U(g) \neq 0$ we can pick two elementary decomposable vectors $\otimes\psi_i$ and $\otimes\xi_i$ in $\otimes_i^{\phi_i}\mathcal{H}_i$ which do not depend on J satisfying

$$0 \neq c(g) := |(U(g) \otimes \psi_i, \otimes\xi_i)| = \prod_{i\in J} |(U_i(g)\psi_i, \xi_i)|\; |({}_JU(g) \otimes_{i\notin J} \psi_i, \otimes_{i\notin J}\xi_i)|$$

for each $g \in G$. Since $|({}_JU(g) \otimes_{i\notin J} \psi_i, \otimes_{i\notin J}\xi_i)| \leq 1$ we get

$$0 < c(g) \leq \prod_{i\in J} |(U_i(g)\psi_i, \xi_i)|.$$

As this holds fore every J, one easily deduces that $\prod_{i\in\mathbb{N}} |(U_i(g)\psi_i, \xi_i)|$ converges to a non-zero number. Since $\psi_i = \xi_i = \phi_i$ for all but finitely many i's, this implies that $(*)$ holds, as claimed.

Assume now that we are in the situation described in case ii).
We define $U_J(g)$ as in i) and set $V_J(g) = (\,U_J(g) \otimes (\otimes_{i\notin J}I_i)\,)^*U(g)$. Then, using that we may write $\alpha = (\otimes_{i\in J}\alpha_i) \otimes {}_J\alpha$, we get

$$V_J(g) \in (\otimes_i(\mathcal{M}_i, \phi_i)) \cap ((\otimes_{i\in J}\mathcal{M}_i) \otimes (\otimes_{i\notin J}\mathbb{C}I_i))'.$$

Using now that all $\mathcal{M}_i$ are factors, it is a simple exercise to deduce that $V_J(g) \in (\otimes_{i\in J}\mathbb{C}I_i) \otimes (\otimes_{i\notin J}(\mathcal{M}_i, \phi_i))$. We may therefore write $V_J(g) = (\otimes_{i\in J}I_i) \otimes {}_JV(g)$ for some unitary ${}_JV(g) \in \otimes_{i\notin J}(\mathcal{M}_i, \phi_i)$. This gives $U(g) = U_J(g) \otimes {}_JV(g)$ and we can clearly proceed further in the same way as above to show that $(*)$ holds. $\square$

The proof of the above result is nearly connected to the proof of a lemma in [26] (see also [13]).

The case of interest for us in this paper is the one where we set $\mathcal{M}_i = \mathcal{B}(\mathcal{H}_i)$ for every i. As is well-known, every automorphism of a type I factor is inner.

We have

$$VW = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -WV.$$

A projective (irreducible) unitary representation of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{C}^2$ is defined by setting $U((a,b)) = V^a W^b$ $(a, b \in \mathbb{Z}_2)$. Since $V^a W^b = \sigma(a,b) W^b U^a$ where $\sigma(a,b) = -1$ if $a = b = 1$ and 1 otherwise, the associated cocycle is easily computed to be $u((a_1, b_1), (a_2, b_2)) = (-1)^{a_2 b_1}$ and its class is not trivial in cohomology. Remark that U is nothing but the projective representation associated to the CCR representation of σ on $\mathbb{C}^2 = \ell^2(\mathbb{Z}_2)$ determined by V and W. Consider the action α of G on $M_2(\mathbb{C})$ given by $\alpha_{(a,b)} = \mathrm{Ad}\,(U((a,b)))$. Then, according to the above result, the infinite tensor product of α does not make sense as an action on the type I factor $\otimes_i (M_2(\mathbb{C}), \phi_i)$. Of course, the normalized trace of $\mathcal{A} = M_2(\mathbb{C})$ is α-invariant. Therefore $\otimes_i \alpha$, considered as a product action on the UHF algebra of type 2^∞, still extends to the weak closure in the GNS representation given by the (unique) tracial state (= the infinite tensor product of the trace states), which is the well known unique hyperfinite factor of type II_1.

If G is a non amenable group and we consider the action $\alpha = \mathrm{Ad}\,\lambda$ on $\mathcal{B}(\ell^2(G))$, then there are no α-invariant states (see e. g. [5]). Therefore the invariance argument sketched above to extend the algebraic tensor power of α is not available. It is conceivable that it is impossible to extend this algebraic tensor power action to $\otimes_i (\mathcal{B}(\ell^2(G)), \rho_i)$ regardless of the choice of normal states ρ_i on $\mathcal{B}(\ell^2(G))$ (here we are using the same notation as in [26]).

6 Further comments

For the sake of completeness we include in this section some comments on projective unitary representations of restricted direct product of groups and their associated C^*-algebras. Our discussion is based on [14], where Guichardet deals with the non-projective case.

Let $(G_i)_{i \in I}$ be a family of discrete groups, $e_i \in G_i$ their neutral elements, $U_i : G_i \to \mathcal{U}(H_i)$ a family of projective representations with associated 2-cocycles u_i, $\phi_i \in H_i$ any family of unit vectors, and consider the restricted product $\oplus_i G_i (\subset \times_i G_i)$. Then

$$\oplus_i G_i \ni g = (g_i) \mapsto U(g) = \otimes_i U_i(g_i) \in \mathcal{U}(\otimes^\phi H_i)$$

(notation: $U \equiv \otimes^{\phi_i} U_i$) is a projective representation of $\oplus_i G_i$ on $\otimes^{\phi_i} H_i$ with 2-cocycle

$$u(g,h) := \prod_i u_i(g_i, h_i), \qquad g, h \in \oplus_i G_i, \quad g = (g_i), h = (h_i).$$

Moreover $U(\oplus_i G_i)'' = \otimes^{\phi_i} U_i(G_i)''$.

Acknowledgements. Most of this work was made while R. C. was visiting the Department of Mathematics at the University of Oslo during the academic year 1998-1999 supported by the EU TMR network "Non Commutative Geometry". He is very grateful to the members of the Operator Algebra group in Oslo for their friendly hospitality and for providing perfect working conditions. Both authors would like to thank Ola Bratteli and Erling Størmer for some helpful discussions.

References

[1] M. Aita, W. Bergmann, R. Conti: Amenable groups and generalized Cuntz algebras, *J. Funct. Anal.* **150**, 48–64 (1997).

[2] N. B. Backhouse : Projective representations of space groups, II: Factor systems, *Quart. J. Math. Oxford* **21** (1970), 223-242.

[3] N. B. Backhouse, C. J. Bradley : Projective representations of space groups, I: Translation groups, *Quart. J. Math. Oxford* **21** (1970), 203-222.

[4] E. Bédos: On the C^*-algebra generated by the left regular representation of a locally compact group, *Proc. Amer. Math. Soc.* **120**, 603–608 (1994).

[5] M. E. B. Bekka: Amenable unitary representation of locally compact groups, *Invent . Math.* **100**, 383–401 (1990).

[6] W. Bergmann, R. Conti: Group representations and generalized Cuntz algebras, in preparation.

[7] K. S. Brown: Cohomology of groups, GTM in Mathematics 87, Springer-Verlag, New-York (1982).

[8] R. Conti: Amenability and infinite tensor products, in Operator Theory, Operator Algebras and Related Topics, Proceedings of the 16th International Conference on Operator Theory, Timisoara 1996, The Theta Foundation, Bucharest 1997.

[9] R. Conti: On groups with Kazhdan's property T and Cuntz algebras, in preparation.

[10] J. Dixmier: Les C*-algebres et leurs representations, Gauthiers-Villars, Paris (1969).

[11] D. E. Evans, Y. Kawahigashi: Quantum symmetries on operator algebras, Oxford : Oxford University Press (1998).