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A. Øksendal

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A solvable irreversible investment problem with transaction costs

Anders Øksendal
Department of Mathematics
University of Oslo
P.O.Box 1053 Blindern
N-0316 Oslo
Norway
E-mail: anderso@math.uio.no

ABSTRACT: This paper mathematically treats the following economic problem: A company wants to expand its capacity in investments that are irreversible. The problem is to find the best investment strategy taking the fluctuating market into account. We assume that to each investment there is associated a fixed transaction cost, in addition to the cost of the actual investment. We solve an example explicitly and show that

$$\lim_{C \to 0^+} J_C = J_0$$

where J_C denotes the value function with transaction cost C. We also show that the value function is not robust with respect to the transaction costs (at C=0) in the sense that

$$\lim_{C \to 0^+} \frac{J_C}{dC} = -\infty$$

KEY WORDS Impulse control, Irreversible investments, Fixed transaction costs, Quasi-variational inequalities, Nonrobustness feature.

1 Introduction

This paper focuses on the problem of investing in an uncertain market, when the investments are considered to be irreversible. This means that once an investment has been made and the market later drops to a less favorable state, we cannot undo the investment. The risk of overinvesting means that we should wait longer to invest, than if the investments were totally or partially reversible. On the other hand we do not want to wait too long and miss out on any profits due to our lack of capacity. In addition we have a fixed transaction cost associated with each investment. The problem then is to find a proper investment strategy taking the fluctuating marked into account. Numerous examples of irreversible investments exist, for example purchase of highly specified production machinery, educating staff members or spending money on advertising. See Dixit and Pindyck [2] for further economic discussions of the problem. See also Kobila [3] and Øksendal [5] for a treatment of the same problem without transaction costs.

We will assume that our investments have two effects on our economy. The first is that the income increases. In general the income will depend on the current state of the market and the current investment level, which will be denoted by θ and k, respectively. This is reasonable since a favourable market could for instance mean greater sales of a product. On the other hand a high capacity could result in higher maintenance costs. The income function will be denoted by

$$\Pi(\theta, k) : E \times [0, \infty) \to [0, \infty)$$

The second effect of an investment is obviously that it costs money. We will assume that an investment has different costs depending on what our capacity is and naturally also depending on how much we want to increase our capacity. The function $\Gamma(k)$ will be such that an increase in capacity from k to q will cost

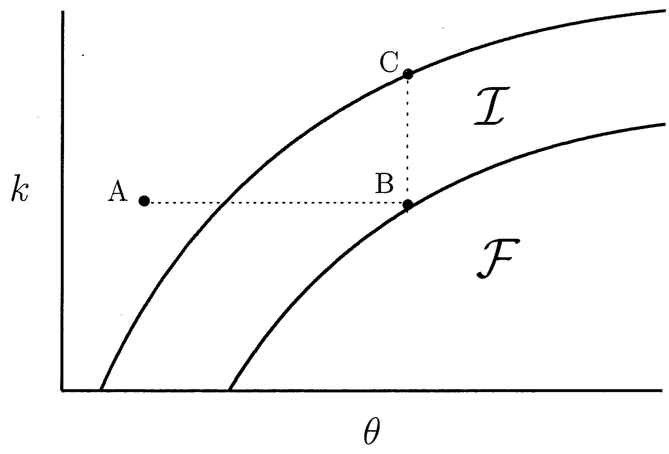
$$\Gamma(q) - \Gamma(k)$$

Also we will have a transaction cost associated with each investment. This will be constant and denoted by C.

In addition we have a discount factor λ built into the model. This factor is considered to be strictly positive and constant.

Another assumption is that the market process is not affected by the investments made. This is a valid assumption if we are considered to be small investors in a large market.

The market process will be denoted by Θ_t. It takes values in the interval $E \subseteq R$. We will show that the solution to the problem is to find a forbidden region $\mathcal{F} \subseteq E \times [0, \infty)$ and an investment region $\mathcal{I} \subseteq E \times [0, \infty)$ such that the optimal solution is to invest whenever (Θ_t, k) hits $\mathcal{F}$ and then invest until we are outside $\mathcal{I}$. Suppose we start in the point A in the figure below. Then we should wait until (Θ_t, k) hits $\mathcal{F}$ (point B) before we invest. The optimal strategy is to invest our way out of $\mathcal{I}$ (point C). Then we should wait until we hit $\mathcal{F}$ again before investing further and so on.



This paper is organized as follows: Section 2 gives some preliminary results, section 3 gives sufficient conditions in the general case. In section 4 we give an example that is explicitly solvable by using the methods developed in section 3. In section 5 we show that if J_C denotes the optimal value function with the transaction cost C then

$$\lim_{C \to 0+} J_C = J_0$$

and

$$\lim_{C \to 0+} \frac{dJ_C}{dC} = -\infty$$

2 Preliminary results

2.1 The Market Process

The marked process, Θ_t, is assumed to be an Ito diffusion on the interval $E \subseteq R$

$$d\Theta_t = \mu(\Theta_t)dt + \sigma(\Theta_t)dB_t$$

2.2 The Controls

We will let functions of the form

$$K_t(\omega) : [0, \infty) \times \Omega \to [0, \infty)$$

represent the investment strategies. They are required to be:

1. Measurable with respect to the σ-algebra $\mathcal{B}_{[0,\infty)} \times \mathcal{M}$
2. Non-decreasing as a function of t, for a.e. ω
3. Right-continuous as a function of t, for a.e. ω
4. Adapted to the filtration $\mathcal{M}_t = \{\sigma(\Theta_s); s \leq t\}$

Because of our transaction cost a large family of these controls will not be optimal. If we consider our total income to be finite, we can only have a finite amount of investments within each finite period, if not our transaction costs will become infinitely large. Therefore we can assume that our controls for each $\omega \in \Omega$ only have a countable amount of jumps. Then we only need to consider controls of the form:

$$K_t = \sum_{i=0}^{\infty} k_i(\omega)\chi_{\tau_i \leq t < \tau_{i+1}} \tag{1}$$

where $\{\tau_i\}_{i=1}^{\infty}$ is a family of Θ_t-stopping times such that $\tau_i \to \infty$ a.s. as $i \to \infty$.

2.3 Conditions

We assume that Π and Γ are continuous and continuously differentiable wrt. k. Furthermore if we define

$$N(\theta) \triangleq \sup_{k\in[0,\infty)} \{(\Pi - \lambda\Gamma)(\theta, k)\}$$

then we assume that $E^\theta \left[\int_0^\infty e^{-\lambda t}|N(\Theta_t)|dt\right] < \infty$ for all θ. Then we can easily prove the following result

Lemma 2.1. *For all controls K_t of the form in (1):*

$$\lim_{n\to\infty} E^\theta \left[\int_0^{\tau_n} e^{-\lambda t}(\Pi - \lambda\Gamma)(\Theta_t, K_t)dt\right] = E^\theta \left[\int_0^{\infty} e^{-\lambda t}(\Pi - \lambda\Gamma)(\Theta_t, K_t)dt\right]$$

Proof. Since $N(\theta) \geq (\Pi - \lambda\Gamma)(\theta, k)$ we have by monotone convergence:

$$\lim_{n\to\infty} E^\theta \left[\int_0^{\tau_n} e^{-\lambda t}(N(\Theta_t) - (\Pi - \lambda\Gamma)(\Theta_t, K_t))dt\right] = E^\theta \left[\int_0^{\infty} e^{-\lambda t}(N(\Theta_t) - (\Pi - \lambda\Gamma)(\Theta_t, K_t))dt\right]$$

The result easily follows. □

2.4 The mathematical problem

Now we are ready to present the precise mathematical problem. We want to maximize the expected profit after we have deducted the expected cost of our investments and the transaction costs. The total expected discounted income is given by

$$E^{\theta}\left[\int_0^{\infty} e^{-\lambda t}\Pi(\Theta_t, K_t)\, dt\right]$$

where $\Pi(\theta, k)$ denotes the income rate when the market is in the state θ and our investment level is k. The expected reduction of funds due to the investments is a bit different. The total expected cost of the investments (not including the transaction cost) is given by

$$\sum_{i=1}^{\infty} E^{\theta}\left[e^{-\lambda\tau_i}\Gamma(k_i(\omega)) - e^{-\lambda\tau_i}\Gamma(k_{i-1}(\omega))\right]$$

Changing the indexes then gives

$$= \sum_{i=0}^{\infty} E^{\theta}\left[e^{-\lambda\tau_i}\Gamma(k_i(\omega)) - e^{-\lambda\tau_{i+1}}\Gamma(k_i(\omega))\right] - \Gamma(k_0)$$

$$= \sum_{i=0}^{\infty} E^{\theta}\left[\int_{\tau_i}^{\infty} e^{-\lambda t}\lambda\Gamma(k_i(\omega)) - \int_{\tau_{i+1}}^{\infty} e^{-\lambda t}\lambda\Gamma(k_i(\omega))dt\right] - \Gamma(k_0)$$

$$= E^{\theta}\left[\int_0^{\infty} e^{-\lambda t}\lambda\Gamma(K_t)dt\right] - \Gamma(k_0)$$

Therefore the total expected profit for the control K_t is

$$E^{\theta}\left[\int_0^{\infty} e^{-\lambda t}(\Pi - \lambda\Gamma)(\Theta_t, K_t)\, dt\right] - \sum_{i=0}^{\infty} E^{\theta}\left[Ce^{-\lambda\tau_i}\right] + \Gamma(k_0)$$

But $\Gamma(k_0)$ does not depend on the control K_t. Then if we define

$$J^{(\theta,k)}(K_t) \triangleq E^{\theta}\left[\int_0^{\infty} e^{-\lambda t}(\Pi - \lambda\Gamma)(\Theta_t, K_t)\, dt\right] - \sum_{i=0}^{\infty} E^{\theta}\left[Ce^{-\lambda\tau_i}\right]$$

the problem is to find the control that maximizes this expression. Or in other words finding a K_t^* such that

$$J^{(\theta,k)}(K_t^*) = \sup_{K_t}\{J^{(\theta,k)}(K_t)\}$$

We also want to know what the maximum value $J^{(\theta,k)}(K_t^*)$ is.

3 Constructing a solution

3.1 Sufficient conditions

With the conditions on Π and Γ given earlier we now present a verification theorem using quasi variational inequalities (See [1] for an elaborate treatment of QVIs)

Proposition 3.1. *Define*

$$L\nu \triangleq -\lambda\nu + \mu(\theta)\frac{\partial\nu}{\partial\theta} + \frac{1}{2}\sigma^2(\theta)\frac{\partial^2\nu}{\partial\theta^2}$$

Suppose we are given $\nu : E \times [0,\infty) \to R$ and two strictly increasing locally Lipschitz continuous functions $\phi, \psi : [0,\infty) \to E$ such that the following holds:

1. $\nu \in C^1(E \times [0,\infty))$
2. $\nu \in C^{2\times 0}(E \times [0,\infty))$
3. $$\nu \in C^2(E \times [0,\infty)\backslash\partial\psi)$$
   and the second order derivatives are locally bounded near $\partial\psi$, where $\partial\psi$ denotes the set $\{(\theta,k) : \psi(k) = \theta\}$.
4. $$L\nu \begin{cases} \le -(\Pi - \lambda\Gamma)(\theta,k) & \text{when} \quad \psi(k) < \theta \\ = -(\Pi - \lambda\Gamma)(\theta,k) & \text{when} \quad \psi(k) \ge \theta \end{cases}$$
5. $$\nu(\theta,k) - \nu(\theta,q) \le C$$
   for all θ and all $q \le k$.
6. $$\nu(\theta,k) - \nu(\theta,q) = C$$
   for $k = \phi^{-1}(\theta)$ and $q \le \psi^{-1}(\theta)$.
7. *There exists a function $h(\theta) \ge \sup_k \nu(\theta,k)$ such that $\{e^{-\lambda\tau}h(\Theta_\tau)\}_\tau$ is uniformly integrable and*
   $$\lim_{t\to\infty} e^{-\lambda t}h(\Theta_t) = 0 \qquad a.s.$$

Then

$$\nu(\theta,k) \ge \sup_{K_t} E^\theta\left[\int_0^\infty e^{-\lambda t}(\Pi - \lambda\Gamma)(\Theta_t, K_t)dt - \sum_{i=1}^\infty Ce^{-\lambda\tau_i}\right]$$

Let $k_0 = k$, $\tau_0 = 0$. Define $\{k_i\}_{i=1}^\infty$ inductively by

$$k_{i+1} = \phi^{-1}(\psi(k_i))$$

Let τ_{i+1} denote the first hitting time of the process Θ_t to the set $[\psi(k_i), \infty)$ and K_t^* denote the control:

$$K_t^* = \sum_{i=0}^{\infty} k_i \chi_{\tau_i \leq t < \tau_{i+1}}$$

Suppose further that

8. $\tau_i < \infty$ *a.s.*

9. $\{e^{-\lambda\tau}\nu(\Theta_\tau, K_\tau^*)\}_{\tau \leq \tau_i}$ *is uniformly integrable for all i.*

10. $$\lim_{n\to\infty} E^\theta \left[e^{-\lambda\tau_n}\nu(\Theta_{\tau_n}, K_{\tau_n}^*) \right] = 0$$

Then K_t^* is optimal.

Proof. From Øksendal [6] (Theorem D.1) we know that there exists a sequence of C^2-functions $\{\nu_j\}_{j=1}^{\infty}$ such that

$$\nu_j \to \nu \qquad \text{uniformly on compacts}$$

$$A\nu_j - \lambda\nu_j \to A\nu - \lambda\nu \text{ uniformly on compact subsets of } E \times [0,\infty) \backslash \partial\psi$$

$$\{L\nu_j\}_{j=1}^{\infty} \qquad \text{is locally bounded}$$

Let K_t be a given control. The Ito formula for semimartingales (see Protter [4]) gives

$$e^{-\lambda t}\nu_j(\Theta_t, K_t) - \nu_j(\theta, k) = \int_0^t e^{-\lambda s} L\nu_j(\Theta_s, K_s)ds + \sum_{0<s\leq t} e^{-\lambda s}\left(\nu_j(\Theta_s, K_s) - \nu_j(\Theta_{s-}, K_{s-})\right)$$

$$+ \int_0^t e^{-\lambda s}\nu_j(\Theta_s, K_s)dB_s$$

Define $\tau_i^R \triangleq \min\{\tau_i, R, \inf\{t > 0 : |\Theta_t| + |K_t| \geq R\}\}$. Then

$$\nu_j(\theta, k) = -E^\theta \left[\int_0^{\tau_n^R} e^{-\lambda s} L\nu_j(\Theta_s, K_s)ds - \sum_{i=1}^{n} e^{-\lambda\tau_i^R} \left(\nu_j(\Theta_{\tau_i^R}, k_{i+1}) - \nu_j(\Theta_{\tau_i^R}, k_i) \right) \chi_{\tau_i^R = \tau_i} \right]$$

$$+ E^\theta \left[e^{-\lambda\tau_n^R}\nu_j(\Theta_{\tau_n^R}, K_{\tau_n^R}) \right]$$

Letting $j \to \infty$ we get

$$\nu(\theta, k) = -E^\theta \left[\int_0^{\tau_n^R} e^{-\lambda s} L\nu(\Theta_s, K_s)ds - \sum_{i=1}^{n} e^{-\lambda\tau_i^R} \left(\nu(\Theta_{\tau_i^R}, k_{i+1}) - \nu(\Theta_{\tau_i^R}, k_i) \right) \chi_{\tau_i^R = \tau_i} \right] \quad (2)$$

$$+ E^\theta \left[e^{-\lambda\tau_n^R}\nu(\Theta_{\tau_n^R}, K_{\tau_n^R}) \right] \quad (3)$$

using assumptions 4) and 5) and taking $\limsup_{R\to\infty}$ and $\limsup_{n\to\infty}$ on both sides we get

$$\nu(\theta,k) \geq \limsup_{n\to\infty}\limsup_{R\to\infty} E^\theta\left[\int_0^{\tau_n^R} e^{-\lambda t}(\Pi-\lambda\Gamma)(\Theta_t,K_t)dt\right] - \limsup_{n\to\infty}\limsup_{R\to\infty} E^\theta\left[\sum_{i=1}^n e^{-\lambda\tau_i}C\chi_{\tau_i^R=\tau_i}\right]$$

$$+\limsup_{n\to\infty}\limsup_{R\to\infty} E^\theta\left[e^{-\lambda\tau_n^R}\nu(\Theta_{\tau_n}^R, K_{\tau_n^R})\right]$$

using lemma 2.1 and assumption 7)

$$\nu(\theta,k) \geq E^\theta\left[\int_0^\infty e^{-\lambda t}(\Pi-\lambda\Gamma)(\Theta_t,K_t)dt - \sum_{i=1}^\infty Ce^{-\lambda\tau_i}\right] \tag{4}$$

To see that K_t^* gives equality in (4) note that (Θ_t, K_t^*) never enters $\mathcal{F}$ and always invests from ψ to ϕ. Thus using assumption 3) and 5) in equation (2)-(3) we get

$$\nu(\theta,k) = E^\theta\left[\int_0^{\tau_n^R} e^{-\lambda t}(\Pi-\lambda\Gamma)(\Theta_t,K_t^*)dt - \sum_{i=1}^n Ce^{-\lambda\tau_i}\chi_{\tau_i^R=\tau_i} + e^{-\lambda\tau_n^R}\nu(\Theta_{\tau_n^R}, K^*_{\tau_n^R})\right]$$

Letting $R\to\infty$ and using lemma 2.1 and assumption 9) this equals

$$= E^\theta\left[\int_0^{\tau_n} e^{-\lambda t}(\Pi-\lambda\Gamma)(\Theta_t,K_t^*)dt - \sum_{i=1}^n Ce^{-\lambda\tau_i} + e^{-\lambda\tau_n}\nu(\Theta_{\tau_n}, K^*_{\tau_n})\right]$$

Letting $n\to\infty$ and using lemma 2.1 and assumption 10) we get

$$= E^\theta\left[\int_0^\infty e^{-\lambda t}(\Pi-\lambda\Gamma)(\Theta_t,K_t^*)dt - \sum_{i=1}^\infty Ce^{-\lambda\tau_i}\right]$$

$\square$

3.2 Nescessary conditions

We now present two conditions that heuristically have to be satisfied in order for the problem to have a solution of the form given in proposition 3.1. Suppose τ_ψ and τ_ϕ denote the first hitting times for Θ_t to the sets $[\psi(k),\infty)$ and $[\phi(k),\infty)$, respectively. Then we must have:

$$E^\theta\left[\int_{\tau_\phi}^{\tau_\psi} e^{-\lambda t}(\Pi_k-\lambda\Gamma_k)(\Theta_t,k)dt\right] = 0 \tag{5}$$

To see why this is reasonable let K_t^* denote the optimal control given in 3.1. Suppose that the expression above is strictly positive for $k=k_1$. Then there exists a $\delta\in(0,k_2-k_1)$ such that

$$E^\theta\left[\int_{\tau_\phi}^{\tau_\psi} e^{-\lambda t}(\Pi-\lambda\Gamma)(\Theta_t,k_1+\delta)\right] > E^\theta\left[\int_{\tau_\phi}^{\tau_\psi} e^{-\lambda t}(\Pi-\lambda\Gamma)(\Theta_t,k_1)\right]$$

Then compare the strategies

$$K_t^* = \sum_{i=0}^{\infty} k_i \chi_{\tau_i \le t \le \tau_{i+1}}$$

and

$$\tilde{K}_t = \sum_{i=0}^{\infty} \tilde{k}_i \chi_{\tau_i \le t \le \tau_{i+1}}$$

where $\tilde{k}_i = k_i$, whenever $i \neq 1$ and $\tilde{k}_1 = k_1 + \delta$. $\tilde{K}_t$ is an increasing control from the choice of δ. Comparing the value functions for the controls K_t^* and $\tilde{K}_t$ we get

$$J^{(\theta,k)}(K_t^*) - J^{(\theta,k)}(\tilde{K}_t) = E^\theta \left[\int_0^\infty e^{-\lambda t} (\Pi - \lambda\Gamma)(\Theta_t, K_t) dt \right] - \sum_{i=1}^{\infty} E^\theta \left[Ce^{-\lambda \tau_i} \right]$$

$$-E^\theta \left[\int_0^\infty e^{-\lambda t} (\Pi - \lambda\Gamma)(\Theta_t, \tilde{K}_t) dt \right] + \sum_{i=1}^{\infty} E^\theta \left[Ce^{-\lambda \tau_i} \right]$$

$$= E^\theta \left[\int_{\tau_\phi}^{\tau_\psi} e^{-\lambda t} (\Pi - \lambda\Gamma)(\Theta_t, k_1) \right] - E^\theta \left[\int_{\tau_\phi}^{\tau_\psi} e^{-\lambda t} (\Pi - \lambda\Gamma)(\Theta_t, k_1 + \delta) \right] < 0$$

In other words the control K_t^* is not optimal. A similar argument applies if we assume the expression in (5) to be strictly negative.

The other nescessary condition is a consequence of the fact that it should decrease the value function if we invest before or after we hit the function ψ. Suppose we keep $k \in [0, \infty)$ fixed. Let τ_ψ be the first hitting time of the process Θ_t to the set $[\psi(k), \infty)$. Then it must be suboptimal to invest slightly before or slightly after Θ_t hits $[\psi(k), \infty)$. Suppose we invest slightly after $\psi(k)$, say at the time when Θ_t hits $\hat{\psi}$, where $\hat{\psi} > \psi(k)$. Let $\kappa_{\hat{\psi}}$ denote the first hitting time for the process Θ_t to the set $[\hat{\psi}, \infty)$. Consider the strategy $\hat{K}_t$ given by

$$\hat{K}_t = k_i \chi_{\kappa_i \le t \le \kappa_{i+1}}$$

where $\kappa_i = \tau_i$ for $i \neq 1$ and $\kappa_1 = \kappa_{\hat{\psi}}$. Then comparing the value functions of K_t^* and $\hat{K}_t$ gives:

$$J(K_t^*) - J(\hat{K}_t) = E^\theta \left[\int_{\tau_\psi}^{\kappa_{\hat{\psi}}} e^{-\lambda t} (\Pi - \lambda\Gamma)(\Theta_t, \phi^{-1}(\psi(k))) dt \right]$$

$$-E^\theta \left[\int_{\tau_\psi}^{\kappa_{\hat{\psi}}} e^{-\lambda t} (\Pi - \lambda\Gamma)(\Theta_t, k) dt \right] - E^\theta \left[Ce^{-\lambda \kappa_\psi} \right] + E^\theta \left[Ce^{-\lambda \tau_{\hat{\psi}}} \right]$$

Viewing this expression as a function of $\hat{\psi}$ it must be decreasing in $\hat{\psi} = \psi(k)$, otherwise K_t^* is not an optimal control. Similar argumentation for $\hat{\psi} < \psi(k)$ gives that the expression above must be decreasing for $\hat{\psi} = \psi(k)$. Then supposing that the expression above is differentiable wrt. $\hat{\psi}$ we must have

$$\frac{d}{d\hat{\psi}} \left[E^\theta \left[\int_{\tau_\psi}^{\kappa_{\hat{\psi}}} e^{-\lambda t} \left[(\Pi - \lambda\Gamma)(\Theta_t, \phi^{-1}(\psi(k))) - (\Pi - \lambda\Gamma)(\Theta_t, k) - \lambda C \right] dt \right] \right]^{\hat{\psi} = \psi} = 0 \quad (6)$$

4 A simple example

It turns out that in some cases the conditions we presented in the previous section are enough to determine the functions ϕ and ψ. A similar example for a controlled Brownian motion can be found in [6].

$$d\Theta_t = \sqrt{2}\, dB_t$$

$$\Pi(\theta, k) = \theta k$$

$$\Gamma(k) = \eta k^2$$

$$\lambda = 1$$

In this case we have the following equality:

$$E^\theta \left[e^{-\tau_\psi}\right] = e^{\theta-\psi}$$

Then (5) becomes

$$E^\theta \left[\int_{\tau_\phi}^{\tau_\psi} e^{-\lambda t}(\Pi_k - \lambda\Gamma_k)(\Theta_t, k)dt\right] = E^\theta \left[\int_{\tau_\phi}^{\tau_\psi} e^{-t}(\Theta_t - 2\eta k)dt\right]$$

Using Dynkin's theorem this equals

$$= E^\theta \left[e^{-\tau_\phi} E^\phi \left[\int_0^\infty e^{-t}(\Theta_t - 2\eta k)dt\right]\right] - E^\theta \left[e^{-\tau_\psi} E^\psi \left[\int_0^\infty e^{-t}(\Theta_t - 2\eta k)dt\right]\right]$$

$$= e^{\theta-\phi}(\phi - 2\eta k) - e^{\theta-\psi}(\psi - 2\eta k)$$

This expression has to be zero, hence we obtain the equality:

$$e^{-\phi}(\phi - 2\eta k) = e^{-\psi}(\psi - 2\eta k)$$

Now we see that if we choose

$$\psi(k) = 2\eta k + V$$

and

$$\phi(k) = 2\eta k + W$$

where V and W are constants, then the equality above is independent of k and can be written

$$We^{-W} = Ve^{-V} \tag{7}$$

The question now is whether the condition given in (6) agrees with our assumptions on the form of $\psi(k)$ and $\phi(k)$. For simplicity of notation we will set $d \triangleq \frac{V-W}{2\eta}$. From (6) we have the expression

$$E^\theta \left[\int_{\tau_\psi}^{\tau_{\hat{\psi}}} e^{-\lambda t} \left[(\Pi - \lambda\Gamma)(\Theta_t, \phi^{-1}(\psi(k)))dt - (\Pi - \lambda\Gamma)(\Theta_t, k) - C\right] dt\right]$$

$$= E^{\theta}\left[\int_{\tau_{\psi}}^{\tau_{\hat{\psi}}} e^{-t}\left[\Theta_t(k+d) - \eta(k+d)^2 - (\Theta_t k - \eta k^2) - C\right] dt\right]$$

$$= E^{\theta}\left[\int_{\tau_{\psi}}^{\tau_{\hat{\psi}}} e^{-t}(\Theta_t d - 2\eta k d - \eta d^2 - C)dt\right]$$

Using Dynkin's theorem this equals

$$= E^{\theta}\left[e^{-\tau_{\psi}} E^{\psi}\left[\int_0^{\infty} e^{-t}(\Theta_t d - 2\eta k d - \eta d^2 - C)dt\right]\right]$$

$$-E^{\theta}\left[e^{-\tau_{\hat{\psi}}} E^{\hat{\psi}}\left[\int_0^{\infty} e^{-t}(\Theta_t d - 2\eta k d - \eta d^2 - C)dt\right]\right]$$

Computing this expression explicitly then gives

$$= e^{\theta-\psi}(\psi d - 2\eta k d - \eta d^2 - C) - e^{\theta-\hat{\psi}}(\hat{\psi} d - 2\eta k d - \eta d^2 - C)$$

This function is differentiable wrt.$\hat{\psi}$, and the derivative is

$$e^{\theta-\hat{\psi}}(\hat{\psi} d - 2\eta k d - \eta d^2 - C - d)$$

which must be zero for $\hat{\psi} = \psi$ for every k. Thus

$$\psi d - 2\eta k d - \eta d^2 - C - d = 0 \tag{8}$$

$\psi(k)$ is assumed to be of the form $\psi(k) = 2\eta k + V$. Inserting this then gives

$$V d - \eta d^2 - d = C \tag{9}$$

But $W = V - 2\eta d$ and inserting this in (7) we get

$$V = \frac{2\eta d}{1 - e^{-2\eta d}} \tag{10}$$

inserting this in (9) we get

$$\frac{2\eta d^2}{1 - e^{-2\eta d}} - \eta d^2 - d = C \tag{11}$$

The function $C(d) = \frac{2\eta d^2}{1-e^{-2\eta d}} - \eta d^2 - d$ is strictly increasing for $d \geq 0$, hence to each C there is only one solution $d \geq 0$. Furthermore from (10) there is only one pair V and W for each d. Then we have uniquely determined two functions $\psi(k)$ and $\phi(k)$ that agree with the nescessary conditions. We now need to show that the value function associated with these functions satisfies proposition 3.1.

4.1 The value function

With the candidate for the optimal investment strategy given in the previous section it is possible to compute the value function. In this section we will compute this value function and show that it satisfies the conditions given in proposition 3.1. The control in question always has a constant investment increase of $d = \frac{V-W}{2\eta}$. τ_i denotes the first hitting time of the process Θ_t to the value $\psi_{i-1} = \psi(k_{i-1}) = 2\eta k + V + (V-W)(i-1)$ for $i \geq 1$ and $\tau_0 = 0$. It is easily seen that $k_i(\omega) = k + id$. The value function for this control is

$$J^{(\theta,k)}(K_t^*) = \sum_{i=0}^{\infty} E^\theta \left[\int_{\tau_i}^{\tau_{i+1}} e^{-t}(\Theta_t k_i - \eta k_i^2) dt \right] - \sum_{i=1}^{\infty} E^\theta \left[Ce^{-\tau_i} \right]$$

Using Dynkin's theorem we get

$$= \sum_{i=0}^{\infty} E^\theta \left[e^{-\tau_i} E^{\Theta_{\tau_i}} \left[\int_0^\infty e^{-t}(\Theta_t k_i - \eta k_i^2) dt \right] \right] - \sum_{i=0}^{\infty} E^\theta \left[e^{-\tau_{i+1}} E^{\Theta_{\tau_{i+1}}} \left[\int_0^\infty e^{-t}(\Theta_t k_i - \eta k_i^2) dt \right] \right]$$

$$- \sum_{i=1}^{\infty} E^\theta \left[Ce^{-\tau_i} \right]$$

Changing the summation index on the first sum and using that $\Theta_{\tau_{i+1}}$ is constant this equals

$$= E^\theta \left[\int_0^\infty e^{-t}(\Theta_t k - \eta k^2) dt \right] + \sum_{i=0}^{\infty} E^\theta \left[e^{-\tau_{i+1}} \right] \left(E^{\Theta_{\tau_{i+1}}} \left[\int_0^\infty e^{-t}(\Theta_t k_{i+1} - \eta k_{i+1}^2) dt \right] \right.$$

$$\left. - E^{\Theta_{\tau_{i+1}}} \left[\int_0^\infty e^{-t}(\Theta_t k_i - \eta k_i^2) dt \right] - C \right)$$

Inserting $k_{i+1} = k_i + d$

$$= E^\theta \left[\int_0^\infty e^{-t}(\Theta_t k - \eta k^2) dt \right] + \sum_{i=0}^{\infty} E^\theta \left[e^{-\tau_{i+1}} \right] E^{\Theta_{\tau_{i+1}}} \left[\int_0^\infty e^{-t}(\Theta_t d - 2\eta k_i d - \eta d^2 - C) dt \right]$$

This expression can be computed explicitly using that $E^\theta[\Theta_t] = \theta$ and $E^\theta[e^{-\tau_{i+1}}] = e^{\theta - \psi_i}$, since τ_{i+1} denotes the first hitting time for the process to the value ψ_i. This gives

$$= \theta k - \eta k^2 + \sum_{i=0}^{\infty} e^{\theta - \psi_i} \left(\psi_i d - 2\eta k_i d - \eta d^2 - C \right)$$

Then by using (8)

$$= \theta k - \eta k^2 + \sum_{i=0}^{\infty} e^{\theta - \psi_i} d$$

Inserting $\psi_i = 2\eta k + V + (V-W)i$

$$= \theta k - \eta k^2 + de^{\theta - 2\eta k - V} \sum_{i=0}^{\infty} e^{-(V-W)i}$$

$$= \theta k - \eta k^2 + d\frac{e^{\theta-2\eta k-V}}{1-e^{W-V}}$$

Recall from (7) that $e^{W-V} = \frac{W}{V}$

$$= \theta k - \eta k^2 + \frac{V}{2\eta}e^{\theta-2\eta k-V}$$

This is the value function if we start above the function ψ. If we start below, then we should immedeately invest until we hit $\phi^{-1}(\theta)$ and then follow the same strategy as if we started in the state $(\theta, \phi^{-1}(\theta))$. Then the value function would be $J^{(\theta,k)}(K_t^*) = J^{(\theta,\phi^{-1}(\theta))}(K_t^*) - C$. This equals $J^{(\theta,\psi^{-1}(\theta))}(K_t^*)$. Thus our candidate for the optimal value function is

$$\nu(\theta,k) = \begin{cases} \theta k - \eta k^2 + \frac{V}{2\eta}e^{\theta-2\eta k-V} & \text{whenever } \psi(k) \geq \theta \\ \frac{\theta^2}{4\eta} - \frac{V^2}{4\eta} + \frac{V}{2\eta} & \text{whenever } \psi(k) < \theta \end{cases} \tag{12}$$

We now need to check that $\nu(\theta,k)$ satisfies the conditions in proposition 3.1.

Lemma 4.1. *The function $\nu(\theta,k)$ in 12 satisfies proposition 3.1.1-10*

Proof. **Proof of 1) - 4)** is left to the reader.
Proof of 5) and 6) For $k > \psi^{-1}(\theta)$ we have

$$\nu_k(\theta,k) = \theta - 2\eta k - Ve^{\theta-2\eta k-V}$$

Then it is easily seen that

$$\nu_k(\theta,k) = \begin{cases} \leq 0 & \text{whenever } \phi^{-1}(\theta) < k \\ \geq 0 & \text{whenever } \psi^{-1}(\theta) \leq k \leq \phi^{-1}(\theta) \\ = 0 & \text{whenever } \quad k < \psi^{-1}(\theta) \end{cases}$$

Thus the maximum value for $\nu(\theta,q) - \nu(\theta,k)$ is obtained for $q = \phi^{-1}(\theta)$ and $k = \psi^{-1}(\theta)$. The construction of $\nu(\theta,k)$ shows that this maximum is C.
Proof of 7) It is easily seen that $\nu(\theta,k) \leq K(1+e^\theta)$, for some constant K. The result easily holds for this function.
Proof of 8) Since τ_i is the first hitting time to an interval for the one dimentional brownian motion the result easily follows.
Proof of 9) $\tau \leq \tau_i$ implies that K_τ is bounded. Then $\nu(\Theta_\tau, K_\tau)$ is bounded from below by the function $-|\Theta_\tau|K_{\tau_i} - K_{\tau_i}^2$. The result follows.
Proof of 10) It is easily seen that

$$e^{-\lambda\tau_n}\nu(\Theta_{\tau_n}, K_{\tau_n}^*) \geq 0 \qquad \text{for } n \geq 1$$

Combined with 6) this gives the result. □

Remark 4.2. *The contruction of $\nu(\theta,k)$ obviously shows that K_t^* is optimal. Hence it was not nessescary to show 3.1.7-9.*

5 Discussion

We now discuss the nature of the solution further. First notice that the set where it is locally profitable to invest, or in other words the set

$$U = \{(\theta, k) : (\Pi_k - \lambda\Gamma_k)(\theta, k) \geq 0\}$$

in this case is the set where $\theta \geq 2k$. It is never optimal to invest outside this region, since W then would be negative, and then there exists no $V \neq W$ such that $We^{-W} = Ve^{-V}$.

Recall from (11) that

$$\frac{2\eta d^2}{1 - e^{-2\eta d}} - \eta d^2 - d = C \tag{13}$$

Thus as the transactions cost C increases to infinity, then d increases to infinity also. Recall from (10) that

$$V = \frac{2\eta d}{1 - e^{-2\eta d}} \tag{14}$$

Then $V \to \infty$ and $W \to 0$ as $C \to 0$.

A perhaps more interesting question is what happends when $C \to 0$, and how does this relate to the case where $C = 0$? From (13) it is seen that $d \to 0$ as $C \to 0$. Then from (14) we see that as $d \to 0$ then $V, W \to 1$. The case where $C = 0$ can be solved using the methods developed in Øksendal [5]. This gives that the solution in the case $C = 0$ is to invest infinitesimaly to stay above the function $\Upsilon(k) = 2k+1$. This is the same strategy we get from letting $C \to 0$. It is also easily verified that the value function in (12) converges to

$$J_0^{(\theta,k)} = \begin{cases} \theta k - \eta k^2 + \frac{1}{2\eta} e^{\theta - 2\eta k - 1} & \text{for } \theta \leq 2\eta k + 1 \\ \frac{\theta^2}{4\eta} + \frac{1}{4\eta} & \text{for } \theta > 2\eta k + 1 \end{cases}$$

as $C \to 0$. Again the results in Øksendal [5] can be used to show that this is the optimal value function in the case $C = 0$.

We now want to show the nonrobustness of value function. Let $J_C^{(\theta,k)}$ denote the optimal value function with transaction cost C. Then we have the following result:

Proposition 5.1.

$$\lim_{C \to 0^+} \frac{dJ_C^{(\theta,k)}}{dC} = -\infty$$

Proof. To show this we note by (12) that the only term in the value function that is dependent of C is V. Then it is easily seen that

$$\lim_{C \to 0^+} \frac{J_C^{(\theta,k)}}{dC} = \begin{cases} \lim_{C \to 0^+} \frac{1}{2\eta}(1 - V) e^{\theta - 2\eta k - 1} \frac{dV}{dC} & \text{whenever } 2\eta k + 1 \geq \theta \\ \lim_{C \to 0^+} \frac{1}{2\eta}(1 - V) \frac{dV}{dC} & \text{whenever } 2\eta k + 1 < \theta \end{cases}$$

But

$$\lim_{C \to 0^+} (1 - V) \frac{dV}{dC} = \lim_{d \to 0^+} \frac{(1 - V)\frac{dV}{dd}}{\frac{dC}{dd}}$$

Inserting from (13) and (14) we get

$$= \lim_{d\to 0^+} \frac{\frac{1-e^{-2\eta d}-2\eta d}{1-e^{-2\eta d}} \frac{2\eta(1-e^{-2\eta d})-4\eta^2 de^{-2\eta d}}{(1-e^{-2\eta d})^2}}{\frac{4\eta d(1-e^{-2\eta d})-4\eta^2 d^2 e^{-2\eta d}}{(1-e^{-2\eta d})^2} - 2\eta d - 1}$$

But note that $\lim_{d\to 0^+} \frac{1-e^{-2\eta d}-2\eta d}{(1-e^{-2\eta d})^2} = -\frac{1}{2}$. Then the expression above equals

$$= -\frac{1}{2} \lim_{d\to 0^+} \frac{\frac{2\eta(1-e^{-2\eta d})-4\eta^2 de^{-2\eta d}}{(1-e^{-2\eta d})}}{\frac{4\eta d(1-e^{-2\eta d})-4\eta^2 d^2 e^{-2\eta d}}{(1-e^{-2\eta d})^2} - 2\eta d - 1}$$

But this equals

$$= -\frac{1}{2} \lim_{d\to 0^+} \frac{1}{\frac{d}{1-e^{-2\eta d}} + \frac{2\eta d-(2\eta d+1)(1-e^{-2\eta d})}{2\eta(1-e^{-2\eta d})-4\eta^2 de^{-2\eta d}}}$$

The denominator is easily seen to tend to zero as $d \to 0^+$ by l'Hopitals rule. The result follows. $\square$

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