The variational principle for a class of asymptotically abelian C*-algebras

Sergey Neshveyev*
B. Verkin Institute for Low Temperature Physics and Engineering
47, Lenin Ave., 310164, Kharkov, Ukraine

and

Erling Størmer†
Department of Mathematics, University of Oslo,
P.O. Box 1053, Blindern, 0316 Oslo, Norway

Abstract

Let (A, α) be a C*-dynamical system. We introduce the notion of pressure $P_\alpha(H)$ of the automorphism α at a self-adjoint operator $H \in A$. Then we consider the class of AF-systems satisfying the following condition: there exists a dense α-invariant $*$-subalgebra $\mathcal{A}$ of A such that for all pairs $a, b \in \mathcal{A}$ the C*-algebra they generate is finite dimensional, and there is $p = p(a,b) \in \mathbb{N}$ such that $[\alpha^j(a), b] = 0$ for $|j| \geq p$. For systems in this class we prove the variational principle, i.e. show that $P_\alpha(H)$ is the supremum of the quantities $h_\phi(\alpha) - \phi(H)$, where $h_\phi(\alpha)$ is the Connes-Narnhofer-Thirring dynamical entropy of α with respect to the α-invariant state ϕ. If $H \in \mathcal{A}$, and $P_\alpha(H)$ is finite, we show that any state on which the supremum is attained is a KMS-state with respect to a one-parameter automorphism group naturally associated with H. In particular, Voiculescu's topological entropy is equal to the supremum of $h_\phi(\alpha)$, and any state of finite maximal entropy is a trace.

1 Introduction

The variational principle has over the years attracted much attention both in classical ergodic theory, see e.g. [W], and in the C*-algebra setting of quantum statistical mechanics, see e.g. [BR]. In the years around 1970 there was much progress in the study of spin lattice systems. In that case one is given for each point $x \in \mathbb{Z}^\nu$ ($\nu \in \mathbb{N}$) the algebra of all linear operators $B(\mathcal{H})_x$ on a finite dimensional Hilbert space $\mathcal{H}$, and if $\Lambda \subset \mathbb{Z}^\nu$ one defines the C*-algebra $A(\Lambda) = \otimes_{x\in\Lambda} B(\mathcal{H})_x$. One then considers the UHF-algebra $A = \overline{\cup_{\Lambda\subset\mathbb{Z}^\nu} A(\Lambda)}$ with space translations α, and studies mean entropy $s(\phi)$ of invariant states and the corresponding variational principle

$$P(H) = \sup_\phi (s(\phi) - \phi(H)) \tag{1.1}$$

together with the KMS-states defined by a natural derivation associated with a self-adjoint operator H, see [BR, Chapter 6].

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With the development of dynamical entropy of automorphisms of C*-algebras [CS, CNT, V] it was natural to replace the mean entropy $s(\phi)$ by dynamical entropy $h_\phi(\alpha)$. This was done by Narnhofer [N], who considered KMS-properties of the states on which the quantity $h_\phi(\alpha) - \phi(H)$ attains its maximal value. Then Moriya [M] showed that one can replace $s(\phi)$ by the CNT-entropy $h_\phi(\alpha)$ in (1.1) and get the same result, i.e.

$$P(H) = \sup_\phi (h_\phi(\alpha) - \phi(H)). \tag{1.2}$$

If one wants to study the variational principle and equilibrium states for more general C*-dynamical systems, the mean entropy is usually not well defined, and it is necessary to consider dynamical entropy. However, in order to define time translations and extend the theory of spin lattice systems rather strong assumptions of asymptotic abelianness are needed.

In the present paper we shall study a restricted class of asymptotically abelian systems (A, α), namely we shall assume that A is a unital separable C*-algebra, and (A, α) is asymptotically abelian with locality, i.e. there exists a dense α-invariant $*$-subalgebra $\mathcal{A}$ of A such that for all pairs $a, b \in \mathcal{A}$ the C*-algebra they generate is finite dimensional, and there is $p = p(a, b) \in \mathbb{N}$ such that $[\alpha^j(a), b] = 0$ for $|j| \geq p$. In particular, A is an AF-algebra. Examples of such systems are described in [S]. They are all different shifts, on infinite tensor products of the same AF-algebra with itself, on the Temperley-Lieb algebras, on towers of relative commutants, on binary shifts algebras defined by finite subsets of the natural numbers.

In Section 2 we define the pressure $P_\alpha(H)$ of α with respect to a self-adjoint operator $H \in A$. This can be done in any unital C*-dynamical system (A, α) with A a nuclear C*-algebra, and follows closely Voiculescu's definition of topological entropy $ht(\alpha)$ in [V]. The main difference is that he considered $\operatorname{rank} B$ of a finite dimensional C*-algebra B, while we consider quantities of the form $\operatorname{Tr}_B(e^{-K})$ for K self-adjoint, where Tr_B is the canonical trace on B (so in particular we get $\operatorname{rank} B$ when $K = 0$). We can then show the analogues of several of the classical results on pressure as presented in [W].

If (A, α) is asymptotically abelian with locality we show the variational principle (1.2) in Section 3. Furthermore, if we assume $ht(\alpha) < \infty$, $H \in \mathcal{A}$ and ϕ is a β-equilibrium state at H, i.e. ϕ is α-invariant and $P_\alpha(\beta H) = h_\phi(\alpha) - \beta\phi(H)$, then we show in Section 4 via a proof modelled on the corresponding proof based on (1.1) in [BR] for spin lattice systems, that ϕ is a β-KMS state with respect to the one-parameter group defined by the derivation

$$\delta_H(x) = \sum_{j \in \mathbb{Z}} [\alpha^j(H), x], \quad x \in \mathcal{A}.$$

In particular, when $H = 0$, so $P_\alpha(0) = ht(\alpha)$, we obtain

$$ht(\alpha) = \sup_\phi h_\phi(\alpha), \tag{1.3}$$

and if $h_\phi(\alpha) = ht(\alpha)$ then ϕ is a trace.

Equation (1.3) is false in general. Indeed in [NST] there is exhibited a (non asymptotically abelian) automorphism α of the CAR-algebra for which the trace τ is the unique invariant state, $h_\tau(\alpha) = 0$, while $ht(\alpha) \geq \frac{1}{2} \log 2$. Furthermore, our assumption of locality is essential to conclude that ϕ is a trace when $h_\phi(\alpha) = ht(\alpha) < \infty$, even for asymptotically abelian systems. In Example 5.7 we show that there is a Bogoliubov automorphism α of the even CAR-algebra which is asymptotically abelian, $ht(\alpha) = 0$, while there are an infinite number of non-tracial α-invariant states.

In Section 5 we consider some other examples and special cases. First we apply our results to C*-algebras associated with certain AF-groupoids arising naturally from expansive homeomorphisms of zero-dimensional compact spaces. We show that if H lies in the diagonal then there is a one-to-one correspondence between equilibrium states on the algebra and equilibrium measures in the classical sense. We also consider unique ergodicity for non-abelian systems. If (A, α) is asymptotically abelian with locality, unique ergodicity turns out to be of marginal interest. Indeed, the unique invariant state τ is a trace, and the image of A in the GNS-representation of τ is abelian.

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2 Pressure

In order to define pressure of a C^*-dynamical system we follow the setup of Voiculescu [V] for his definition of topological entropy.

Let A be a nuclear C^*-algebra with unit and α an automorphism. Let CPA(A) denote the set of triples (ρ, ψ, B), where B is a finite dimensional C^*-algebra, and $\rho\colon A \to B$, $\psi\colon B \to A$ are unital completely positive maps. $\mathcal{P}_f(A)$ denotes the family of finite subsets of A. For $\delta > 0$, $\omega \in \mathcal{P}_f(A)$ and $H \in A_{sa}$, put

$$P(H, \omega; \delta) = \inf\{\log \mathrm{Tr}_B(e^{-\rho(H)}) \mid (\rho, \psi, B) \in \mathrm{CPA}(A),\ ||(\psi \circ \rho)(x) - x|| < \delta\ \forall x \in \omega\},$$

where Tr_B is the canonical trace on B, i.e. the trace which takes the value 1 on each minimal projection. Then set

$$\begin{aligned} P_\alpha(H, \omega; \delta) &= \limsup_{n\to\infty} \frac{1}{n} P\left(\sum_{j=0}^{n-1} \alpha^j(H), \bigcup_{j=0}^{n-1} \alpha^j(\omega); \delta\right), \\ P_\alpha(H, \omega) &= \sup_{\delta>0} P_\alpha(H, \omega; \delta). \end{aligned}$$

Definition 2.1 The *pressure of α at H* is

$$P_\alpha(H) = \sup_{\omega \in \mathcal{P}_f(A)} P_\alpha(H, \omega).$$

We have chosen the minus sign $e^{-\rho(H)}$ in the definition of $P(H, \omega; \delta)$ because of its use in physical applications, see [BR], rather than the plus sign used in ergodic theory, see [W].

It is easy to see that if $\omega_1 \subset \omega_2 \subset \dots$ is an increasing sequence of finite subsets of A such that the linear span of $\cup_n \omega_n$ is dense in A, then $P_\alpha(H) = \lim_n P_\alpha(H, \omega_n)$. If $H = 0$ the pressure coincides with the topological entropy of Voiculescu [V].

Recall that $h_\phi(\alpha)$ denotes the CNT-entropy of α with respect to an α-invariant state ϕ of A.

Proposition 2.2 *For any α-invariant state ϕ of A we have $P_\alpha(H) \geq h_\phi(\alpha) - \phi(H)$.*

Proof. The proof is a rewording of the proof of [V, Proposition 4.6]. Let N be a finite dimensional C*-algebra, and $\gamma\colon N \to A$ a unital completely positive map. Let $\omega \in \mathcal{P}_f(A)$ be such that $H \in \omega$ and $\gamma(\{x \in N \mid ||x|| \leq 1\})$ is contained in the convex hull of ω. Then if $(\rho, \psi, B) \in \mathrm{CPA}(A)$ and

$$||(\psi \circ \rho)(a) - a|| < \delta \text{ for } a \in \cup_{j=0}^{n-1} \alpha^j(\omega),$$

we obtain by [CNT, Proposition IV.3]

$$|H_\phi(\{\alpha^j \circ \gamma\}_{0\le j\le n-1}) - H_\phi(\{\psi \circ \rho \circ \alpha^j \circ \gamma\}_{0\le j\le n-1})| < n\varepsilon,$$

where $\varepsilon = \varepsilon(\delta) \to 0$ as $\delta \to 0$. If $K \in B_{sa}$ and θ is a state on B then $\log \mathrm{Tr}_B(e^{-K}) \ge S(\theta) - \theta(K)$, hence

$$\begin{aligned} H_\phi(\{\psi \circ \rho \circ \alpha^j \circ \gamma\}_{0\le j\le n-1}) &\le H_\phi(\psi) \le S(\phi \circ \psi) \\ &\le \log \mathrm{Tr}_B\left(e^{-\rho(\sum_{j=0}^{n-1}\alpha^j(H))}\right) + (\phi\circ\psi)\left(\rho\left(\sum_{j=0}^{n-1}\alpha^j(H)\right)\right) \\ &\le \log \mathrm{Tr}_B\left(e^{-\rho(\sum_{j=0}^{n-1}\alpha^j(H))}\right) + n\phi(H) + n\delta. \end{aligned}$$

Thus

$$\frac{1}{n}H_\phi(\{\alpha^j \circ \gamma\}_{0\le j\le n-1}) \le \frac{1}{n}P\left(\sum_{j=0}^{n-1}\alpha^j(H), \bigcup_{j=0}^{n-1}\alpha^j(\omega); \delta\right) + \phi(H) + \delta + \varepsilon.$$

It follows that $h_\phi(\gamma;\alpha) \le P_\alpha(H,\omega) + \phi(H)$, hence $h_\phi(\alpha) \le P_\alpha(H) + \phi(H)$.

■

Remark 2.3 If A is abelian, $A = C(X)$, and $P^{cl}_\alpha(H)$ denotes the pressure as defined in [W], then $P_\alpha(H) = P^{cl}_\alpha(-H)$. The inequality '$\le$' can be proved just the same as in [V, Proposition 4.8]. The converse inequality follows from Proposition 2.2 and the classical variational principle. In the AF-case however, i.e. when X is zero-dimensional, it is easy to give a direct proof. Indeed, if in the proof of Proposition 2.2 N was the subalgebra of A corresponding to a clopen partition P of X, then we could conclude that

$$\frac{1}{n}H_\phi(\{\alpha^j(N)\}_{0\le j\le n-1}) \le \frac{1}{n}P\left(\sum_{j=0}^{n-1}\alpha^j(H), \bigcup_{j=0}^{n-1}\alpha^j(\omega); \delta\right) + \frac{1}{n}\phi\left(\sum_{j=0}^{n-1}\alpha^j(H)\right) + \delta + \varepsilon \quad (2.1)$$

for any (not necessarily α-invariant) state ϕ of A, with ε independent of ϕ. Let T be the homeomorphism corresponding to α, so that $\alpha(f) = f \circ T$. Suppose the points $x_1, \dots, x_m$ lie in different elements of the partition $\vee_{j=0}^{n-1}T^{-j}P$. Then inequality (2.1) for the measure

$$\phi = \left(\sum_i e^{-(S_nH)(x_i)}\right)^{-1} \sum_i e^{-(S_nH)(x_i)}\delta_{x_i},$$

where $(S_nH)(x) = \sum_{j=0}^{n-1} H(T^jx)$, means that

$$\frac{1}{n}\log\sum_i e^{-(S_nH)(x_i)} \le \frac{1}{n}P\left(\sum_{j=0}^{n-1}\alpha^j(H), \bigcup_{j=0}^{n-1}\alpha^j(\omega); \delta\right) + \delta + \varepsilon.$$

Recalling the definition of pressure [W, Definition 9.7], we see that $P^{cl}_\alpha(-H) \le P_\alpha(H)$.

We list some properties of the function $H \mapsto P_\alpha(H)$ on A_{sa}.

Proposition 2.4 *The following properties are satisfied by P_α for $H, K \in A_{sa}$.*

(i) *If $H \le K$ then $P_\alpha(H) \ge P_\alpha(K)$.*

(ii) $P_\alpha(H + c1) = P_\alpha(H) - c,\ c \in \mathbb{R}$.

(iii) *$P_\alpha(H)$ is either infinite for all H or is finite valued.*

(iv) *If P_α is finite valued then $|P_\alpha(H) - P_\alpha(K)| \leq ||H - K||$.*

(v) *For $k \in \mathbb{N}$, $P_{\alpha^k}(\sum_{j=0}^{k-1} \alpha^j(H)) = kP_\alpha(H)$.*

(vi) *$P_\alpha(H + \alpha(K) - K) = P_\alpha(H)$.*

Proof. (i) Given $H \leq K$ take $\omega \in \mathcal{P}_f(A)$. If $(\rho, \psi, B) \in \mathrm{CPA}(A)$ we have

$$\log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{n-1} \alpha^j(H))}\right) \geq \log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{n-1} \alpha^j(K))}\right),$$

see e.g. [OP, Corollary 3.15]. Thus (i) follows.

(ii) As in (i) we have

$$\log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{n-1} \alpha^j(H+c1))}\right) = \log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{n-1} \alpha^j(H))}\right) - nc,$$

and (ii) follows.

(iii) By (i) and (ii) we have

$$P_\alpha(H) \geq P_\alpha(||H||) = P_\alpha(0) - ||H|| = ht(\alpha) - ||H||,$$

and similarly $P_\alpha(H) \leq ht(\alpha) + ||H||$. Thus (iii) follows.

(iv) For any $(\rho, \psi, B) \in \mathrm{CPA}(A)$ we have by the Peierls-Bogoliubov inequality [OP, Corollary 3.15]

$$\left| \frac{1}{n} \log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{n-1} \alpha^j(H))}\right) - \frac{1}{n} \log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{n-1} \alpha^j(K))}\right) \right| \leq ||H - K||.$$

Thus

$$P_\alpha(H, \omega; \delta) - ||H - K|| \leq P_\alpha(K, \omega; \delta) \leq P_\alpha(H, \omega; \delta) + ||H - K||$$

for any $\omega \in \mathcal{P}_f(A)$. Thus (iv) follows.

(v) Let $(\rho, \psi, B) \in \mathrm{CPA}(A)$ and $\omega \in \mathcal{P}_f(A)$. Given $n \in \mathbb{N}$ choose $m \in \mathbb{N}$ such that $mk \leq n < (m+1)k$. Set $H_k = \sum_{j=0}^{k-1} \alpha^j(H)$ and $\omega_k = \cup_{j=0}^{k-1} \alpha^j(\omega)$. Then

$$\begin{aligned} \log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{n-1} \alpha^j(H))}\right) &\geq \log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{mk-1} \alpha^j(H)) - k||H||}\right) \\ &= \log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{m-1} \alpha^{jk}(H_k))}\right) - k||H||. \end{aligned}$$

Similarly

$$\log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{n-1} \alpha^j(H))}\right) \leq \log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{m} \alpha^{jk}(H_k))}\right) + k||H||.$$

Since $\cup_{j=0}^{m-1} \alpha^{jk}(\omega_k) \subset \cup_{j=0}^{n-1} \alpha^j(\omega) \subset \cup_{j=0}^{m} \alpha^{jk}(\omega_k)$, it follows that

$$P_\alpha(H, \omega; \delta) = \frac{1}{k} P_{\alpha^k} \left(\sum_{j=0}^{k-1} \alpha^j(H), \bigcup_{j=0}^{k-1} \alpha^j(\omega); \delta \right),$$

and hence $P_\alpha(H) = \frac{1}{k} P_{\alpha^k}(\sum_{j=0}^{k-1} \alpha^j(H))$.

(vi) Set $H_k = \sum_{j=0}^{k-1} \alpha^j(H)$ and $H'_k = \sum_{j=0}^{k-1} \alpha^j(H + \alpha(K) - K) = H_k + \alpha^k(K) - K$. Then by (iv) and (v) we have

$$|P_\alpha(H) - P_\alpha(H + \alpha(K) - K)| = \frac{1}{k}|P_{\alpha^k}(H_k) - P_{\alpha^k}(H'_k)| \leq \frac{2||K||}{k}.$$

Thus (vi) follows.

■

The next result is the analogue of [W, Theorem 9.12], see also [R].

Proposition 2.5 *Suppose $ht(\alpha) < \infty$. Let ϕ be a self-adjoint linear functional on A. Then ϕ is an α-invariant state if and only if $-\phi(H) \leq P_\alpha(H)$ for all $H \in A_{sa}$.*

Proof. If ϕ is an α-invariant state then by Proposition 2.2

$$-\phi(H) \leq P_\alpha(H) - h_\phi(\alpha) \leq P_\alpha(H)$$

for all $H \in A_{sa}$.

Conversely if $-\phi(H) \leq P_\alpha(H)$ for all $H \in A_{sa}$ then by Proposition 2.4(vi)

$$-\phi(\alpha(H) - H) = -\frac{1}{n}\phi(\alpha(nH) - nH) \leq \frac{1}{n}P_\alpha(\alpha(nH) - nH) = \frac{1}{n}P_\alpha(0) = \frac{1}{n}ht(\alpha) \underset{n\to\infty}{\longrightarrow} 0.$$

Applying this also to $-H$ we see that ϕ is α-invariant. Furthermore, by Proposition 2.4(i),(ii)

$$-\phi(H) = -\frac{1}{n}\phi(nH) \leq \frac{1}{n}P_\alpha(nH) \leq \frac{1}{n}ht(\alpha) + ||H|| \underset{n\to\infty}{\longrightarrow} ||H||,$$

so that $||\phi|| \leq 1$. For $c \in \mathbb{R}$ we have

$$-c\phi(1) \leq P_\alpha(c1) = ht(\alpha) - c.$$

Hence $\phi(1) = 1$, and ϕ is a state.

■

Definition 2.6 We say ϕ is an *equilibrium state at H* if

$$P_\alpha(H) = h_\phi(\alpha) - \phi(H),$$

hence by Proposition 2.2

$$h_\phi(\alpha) - \phi(H) = \sup_\psi(h_\psi(\alpha) - \psi(H)),$$

where the sup is taken over all α-invariant states.

Recall that if F is a real convex continuous function on a real Banach space X, then a linear functional f on X is called a tangent functional to the graph of F at the point $x_0 \in X$ if

$$F(x_0 + x) - F(x) \geq f(x) \; \forall x \in X.$$

In the sequel we will identify self-adjoint linear functionals on A with real linear functionals on A_{sa}.

The next proposition is the analogue of [W, Theorem 9.14].

Proposition 2.7 *Suppose $ht(\alpha) < \infty$ and the pressure is a convex function on A_{sa}. Then*

(i) *if ϕ is an equilibrium state at H then $-\phi$ is a tangent functional for the pressure at H;*

(ii) *if $-\phi$ is a tangent functional for the pressure at H then ϕ is an α-invariant state.*

Proof. (i) Let $K \in A_{sa}$. Then by Proposition 2.2

$$P_\alpha(H+K) - P_\alpha(H) \geq (h_\phi(\alpha) - \phi(H+K)) - (h_\phi(\alpha) - \phi(H)) = -\phi(K),$$

so $-\phi$ is a tangent functional.

(ii) If $K \in A_{sa}$ then by Proposition 2.4(vi)

$$-\phi(\alpha(K) - K) \leq P_\alpha(H + \alpha(K) - K) - P_\alpha(H) = 0.$$

Applying this also to $-K$ we see that ϕ is α-invariant. Now note that $||\phi|| \leq 1$ by Proposition 2.4(iv). By Proposition 2.4(ii) we have also $-c \geq -c\phi(1)$ for any $c \in \mathbb{R}$. Hence $\phi(1) = 1$, and ϕ is a state.

∎

3 The variational principle

We shall prove the variational principle for the following class of C*-dynamical systems.

Definition 3.1 A unital C*-dynamical system (A, α) is called *asymptotically abelian with locality* if there is a dense α-invariant $*$-subalgebra $\mathcal{A}$ of A such that for each pair $a, b \in \mathcal{A}$ the C*-algebra generated by a and b is finite dimensional, and for some $p = p(a,b) \in \mathbb{N}$ we have $[\alpha^j(a), b] = 0$ whenever $|j| \geq p$.

We call elements of $\mathcal{A}$ for *local operators* and finite dimensional C*-subalgebras of $\mathcal{A}$ for *local algebras.*

Note that since we may add the identity operator to $\mathcal{A}$, we may assume that $1 \in \mathcal{A}$. Since each finite dimensional C*-algebra is singly generated, an easy induction argument shows that the C*-algebra generated by a finite set of local operators is finite dimensional. In particular, A is an AF-algebra. Note also that another easy induction argument shows that for each local algebra N there is $p \in \mathbb{N}$ such that $[\alpha^j(a), b] = 0$ for all $a, b \in N$ whenever $|j| \geq p$.

Theorem 3.2 *Let (A, α) be a unital separable C*-dynamical system which is asymptotically abelian with locality. Let $H \in A_{sa}$. Then*

$$P_\alpha(H) = \sup_\phi (h_\phi(\alpha) - \phi(H)),$$

where the sup is taken over all α-invariant states of A. In particular, the topological entropy

$$ht(\alpha) = \sup_\phi h_\phi(\alpha).$$

Consider first the case when there exists a finite dimensional C*-subalgebra N of A such that $H \in N$, $\alpha^j(N)$ commutes with N for $j \neq 0$, $\vee_{j \in \mathbb{Z}} \alpha^j(N) = A$.

Lemma 3.3 *Under the above assumptions there exists an α-invariant state ϕ such that*

$$P_\alpha(H) = h_\phi(\alpha) - \phi(H) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}_{\vee_{j=0}^{n-1}\alpha^j(N)} \left(e^{-\sum_{j=0}^{n-1}\alpha^j(H)}\right).$$

Proof. First note that if A_1 and A_2 are commuting finite dimensional C*-algebras, and $a_i \in A_i$, $a_i \geq 0$, $i = 1, 2$, then

$$\mathrm{Tr}_{A_1\vee A_2}(a_1a_2) \leq \mathrm{Tr}_{A_1}(a_1)\mathrm{Tr}_{A_2}(a_2),$$

since if p_i is a minimal projection in A_i, $i = 1, 2$, then p_1p_2 is either zero or minimal in $A_1 \vee A_2$. Hence the limit in the formulation of the lemma really exists. We denote it by $\tilde{P}_\alpha(H)$. It is easy to see that $\tilde{P}_\alpha(H) \geq P_\alpha(H)$.

For each $n \in \mathbb{Z}$ let N_n be a copy of N. Consider the C*-algebra $M = \otimes_{n\in\mathbb{Z}} N_n$. Let β be the shift to the right on M, and $\pi\colon M \to A$ the homomorphism which intertwines β with α, and identifies N_0 with N. Set $I = \mathrm{Ker}\,\pi$. For each $n \in \mathbb{N}$ let

$$M_n = N_0 \otimes \ldots \otimes N_{n-1}, \quad I_n = I \cap M_n, \quad \pi_n = \pi|_{M_n}.$$

Identifying M with $M_n^{\otimes\mathbb{Z}}$ consider the β^n-invariant state $\psi_n = \otimes(f_n \circ \pi_n)$ on M, where f_n is the state on $\vee_{j=0}^{n-1}\alpha^j(N)$ with density operator

$$\left(\mathrm{Tr}_{\vee_{j=0}^{n-1}\alpha^j(N)} e^{-\sum_{j=0}^{n-1}\alpha^j(H)}\right)^{-1} e^{-\sum_{j=0}^{n-1}\alpha^j(H)}.$$

Set $\phi_n = \frac{1}{n}\sum_{j=0}^{n-1} \psi_n \circ \beta^j$. Then ϕ_n is β-invariant. Using concavity of entropy we obtain

$$\begin{aligned}
h_{\phi_n}(\beta) &= \frac{1}{n}h_{\phi_n}(\beta^n) \geq \frac{1}{n^2}\sum_{j=0}^{n-1} h_{\psi_n\circ\beta^j}(\beta^n) = \frac{1}{n}h_{\psi_n}(\beta^n) = \frac{1}{n}S(f_n\circ\pi_n) = \frac{1}{n}S(f_n) \\
&= \frac{1}{n}\log \mathrm{Tr}_{\vee_{j=0}^{n-1}\alpha^j(N)}\left(e^{-\sum_{j=0}^{n-1}\alpha^j(H)}\right) + \frac{1}{n}f_n\left(\sum_{j=0}^{n-1}\alpha^j(H)\right) \\
&\geq \tilde{P}_\alpha(H) + \frac{1}{n}f_n\left(\sum_{j=0}^{n-1}\alpha^j(H)\right) \\
&= \tilde{P}_\alpha(H) + \frac{1}{n}\psi_n\left(\sum_{j=0}^{n-1}\beta^j(H)\right) = \tilde{P}_\alpha(H) + \phi_n(H).
\end{aligned}$$

Let $\tilde{\phi}$ be any weak* limit point of the sequence $\{\phi_n\}_n$. Then $\tilde{\phi}$ is β-invariant. Let B be a masa in N_0 containing H. Then B is in the centralizer of the state ϕ_n, hence

$$h_{\phi_n}(\beta) = h_{\phi_n}(B;\beta).$$

Since the mapping $\psi \mapsto h_\psi(B;\beta)$ is upper semicontinuous, we conclude that

$$h_{\tilde{\phi}}(\beta) \geq \tilde{P}_\alpha(H) + \tilde{\phi}(H).$$

Now note that $\tilde{\phi}$ is zero on I. Indeed, if $x \in I_n$ then $\beta^j(x) \in I_m$ for $j = 0, \ldots, m-n$ and $m \geq n$, whence

$$|\phi_m(x)| \leq \frac{1}{m}\sum_{j=m-n+1}^{m-1} |(\psi_m \circ \beta^j)(x)| \leq \frac{n-1}{m}||x||,$$

so $\tilde{\phi}(x) = 0$. Thus $\tilde{\phi}$ defines a state ϕ on A. We have

$$h_\phi(\alpha) = h_{\tilde{\phi}}(\beta) \geq \tilde{P}_\alpha(H) + \phi(H),$$

where the first equality follows from [CNT, Theorem VII.2]. Since by Proposition 2.2, $h_\phi(\alpha) - \phi(H) \leq P_\alpha(H) \leq \tilde{P}_\alpha(H)$, the proof of the lemma is complete.

■

We shall reduce the general case to the case considered above by replacing α by its powers. For this suppose that N is a local subalgebra of A, and $H \in N$. Choose p such that $\alpha^j(N)$ commutes with N whenever $|j| \geq p$. For $k \geq p$ set $M_k = \vee_{j=0}^{k-p} \alpha^j(N)$, $H_k = \sum_{j=0}^{k-p} \alpha^j(H)$. Then $H_k \in M_k$, and $\alpha^{jk}(M_k)$ commutes with M_k for $j \neq 0$.

Lemma 3.4 *For any finite subset ω of N we have*

$$P_\alpha(H, \omega) \leq \liminf_{k\to\infty} \frac{1}{k} P_{\alpha^k|\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)}(H_k).$$

Proof. The idea of the proof is to reduce to the situation of Lemma 3.3 by showing that the contribution of the indicies in the intervals $[jk - p + 1, jk - 1]$, $j \in \mathbb{N}$, becomes negligible for large k.

Fix $\delta > 0$. Choose $m_0 \in \mathbb{N}$ such that

$$\frac{2(p-1)||a||}{m_0} < \delta \quad \text{for} \quad a \in \omega.$$

Take any $k \geq m_0 + p$. Let $n \in \mathbb{N}$. Then $(m-1)k \leq n < mk$ for some $m \in \mathbb{N}$. Set $B_0 = \vee_{j=0}^{m} \alpha^{jk}(M_k)$ and

$$B = \underbrace{B_0 \oplus \ldots \oplus B_0}_{m_0}.$$

Choose a conditional expectation $E: A \to B_0$, and define unital completely positive mappings $\psi: B \to A$ and $\rho: A \to B$ as follows:

$$\psi(b_1, \ldots, b_{m_0}) = \frac{1}{m_0} \sum_{i=1}^{m_0} \alpha^{-i+1}(b_i),$$

$$\rho(a) = (E(a), (E \circ \alpha)(a), \ldots, (E \circ \alpha^{m_0-1})(a)).$$

For any $a \in A$ we have

$$||(\psi \circ \rho)(a) - a|| \leq \frac{2||a||}{m_0} \#\{0 \leq i \leq m_0 - 1 \,|\, \alpha^i(a) \notin B_0\},$$

where $\#S$ means the cardinality of a set S. Let $a = \alpha^l(b)$ for some $b \in \omega$ and l, $0 \leq l \leq n-1$. Then $l = jk + r$ for some j and r, $0 \leq j \leq m-1$, $0 \leq r < k$. Since $m_0 \leq k - p$, the interval $[l, l + m_0 - 1]$ is contained in $[jk, (j+1)k + k - p]$. But for $i \in [jk, (j+1)k + k - p] \backslash [jk + k - p + 1, (j+1)k - 1]$ we have $\alpha^i(N) \subset B_0$, so $\#\{0 \leq i \leq m_0 - 1 \,|\, \alpha^i(a) \notin B_0\} \leq p - 1$, and

$$||(\psi \circ \rho)(a) - a|| \leq \frac{2(p-1)||b||}{m_0} < \delta.$$

Hence

$$P\left(\sum_{j=0}^{n-1} \alpha^j(H), \bigcup_{j=0}^{n-1} \alpha^j(\omega); \delta\right) \leq \log \mathrm{Tr}_B \left(e^{-\rho(\sum_{j=0}^{n-1} \alpha^j(H))}\right).$$

Now note that for $0 \le i \le m_0 - 1$ the sets $X_i = [i, i+n-1]$ and $X = \cup_{j=0}^{m}[jk, jk+k-p]$ are contained in $Y = [0, mk+k-p]$, so

$$\begin{aligned}\#(X_i \triangle X) &\le \#(Y\backslash X_i) + \#(Y\backslash X) = (mk+k-p+1-n) + m(p-1) \\ &\le mk+k-p+1-(m-1)k + m(p-1) \le mp + 2k.\end{aligned}$$

When $j \in X_i \cap X$, $\alpha^j(H) \in B_0$. Hence

$$\left\| (E\circ\alpha^i)\left(\sum_{j=0}^{n-1}\alpha^j(H)\right) - \sum_{j=0}^{m}\alpha^{jk}(H_k)\right\| = \left\| E\left(\sum_{j\in X_i}\alpha^j(H)\right) - \sum_{j\in X}\alpha^j(H)\right\| \le (mp+2k)||H||.$$

By the Peierls-Bogoliubov inequality we obtain

$$\mathrm{Tr}_{B_0}\left(e^{-(E\circ\alpha^i)(\sum_{j=0}^{n-1}\alpha^j(H))}\right) \le e^{(mp+2k)||H||}\mathrm{Tr}_{B_0}\left(e^{-\sum_{j=0}^{m}\alpha^{jk}(H_k)}\right),$$

so

$$\mathrm{Tr}_B\left(e^{-\rho(\sum_{j=0}^{n-1}\alpha^j(H))}\right) \le m_0 e^{(mp+2k)||H||}\mathrm{Tr}_{B_0}\left(e^{-\sum_{j=0}^{m}\alpha^{jk}(H_k)}\right).$$

Taking the log, dividing by n, and letting $n \to \infty$, we obtain

$$\begin{aligned}P_\alpha(H,\omega;\delta) &\le \frac{p||H||}{k} + \frac{1}{k}\lim_{m\to\infty}\frac{1}{m}\log\mathrm{Tr}_{\vee_{j=0}^{m-1}\alpha^{jk}(M_k)}\left(e^{-\sum_{j=0}^{m-1}\alpha^{jk}(H_k)}\right) \\ &= \frac{p||H||}{k} + \frac{1}{k}P_{\alpha^k|\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)}(H_k),\end{aligned}$$

where the last equality follows from Lemma 3.3.

■

We shall need also the following

Lemma 3.5 *Let (A,α) be a C^*-dynamical system with A nuclear, B an α-invariant C^*-subalgebra of A, ϕ an α-invariant state on B. Then for any $\varepsilon > 0$ there exists an α-invariant state ψ on A such that*

$$\psi|_B = \phi \quad and \quad h_\psi(\alpha) > h_\phi(\alpha|_B) - \varepsilon.$$

Proof. Since the Sauvageot-Thouvenot entropy is not less than the CNT-entropy for general C*-systems, there exist a commutative C*-dynamical system (C,β,μ), an $(\alpha\otimes\beta)$-invariant state λ on $B\otimes C$, and a finite dimensional subalgebra P of C such that $\lambda|_B = \phi$, $\lambda|_C = \mu$ and

$$h_\phi(\alpha|_B) < H_\mu(P,P^-) - H_\lambda(P|B) + \varepsilon,$$

see [ST] for notations. Extend λ to an $(\alpha\otimes\beta)$-invariant state Λ on $A\otimes C$, and set $\psi = \Lambda|_A$. Since the conditional entropy $H_\lambda(P|B)$ is decreasing in second variable, and ST-entropy coincides with CNT-entropy for nuclear algebras, we have

$$h_\psi(\alpha) \ge H_\mu(P,P^-) - H_\Lambda(P|A) \ge H_\mu(P,P^-) - H_\lambda(P|B) > h_\phi(\alpha|_B) - \varepsilon.$$

■

Proof of Theorem 3.2. The inequality '$\ge$' has been proved in Proposition 2.2. Since the pressure is continuous by Proposition 2.4(iv), to prove the converse inequality it suffices to consider local H. Then by Lemma 3.4 we have only to show that if H is contained in a local algebra N then

$$\sup_\phi(h_\phi(\alpha) - \phi(H)) \ge \liminf_{k\to\infty}\frac{1}{k}P_{\alpha^k|\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)}(H_k).$$

By Lemma 3.3, for each $k \in \mathbb{N}$ there exists an α^k-invariant state ψ_k on $\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)$ such that

$$h_{\psi_k}(\alpha^k|_{\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)}) - \psi_k(H_k) = P_{\alpha^k|_{\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)}}(H_k).$$

By Lemma 3.5 we may extend ψ_k to an α^k-invariant state $\tilde{\phi}_k$ on A such that

$$h_{\tilde{\phi}_k}(\alpha^k) \geq h_{\psi_k}(\alpha^k|_{\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)}) - 1.$$

Set $\phi_k = \frac{1}{k}\sum_{j=0}^{k-1}\tilde{\phi}_k \circ \alpha^j$. Then as in the proof of Lemma 3.3

$$h_{\phi_k}(\alpha) \geq \frac{1}{k}h_{\tilde{\phi}_k}(\alpha^k) \geq \frac{1}{k}h_{\psi_k}(\alpha^k|_{\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)}) - \frac{1}{k}.$$

Since

$$\phi_k(H) = \frac{1}{k}\sum_{j=0}^{k-1}\tilde{\phi}_k(\alpha^j(H)) \leq \frac{1}{k}\tilde{\phi}_k(H_k) + \frac{p-1}{k}||H|| = \frac{1}{k}\psi_k(H_k) + \frac{p-1}{k}||H||,$$

we get

$$\begin{aligned} h_{\phi_k}(\alpha) - \phi_k(H) &\geq \frac{1}{k}h_{\psi_k}(\alpha^k|_{\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)}) - \frac{1}{k}\psi_k(H_k) - \frac{1+(p-1)||H||}{k} \\ &= \frac{1}{k}P_{\alpha^k|_{\vee_{j\in\mathbb{Z}}\alpha^{jk}(M_k)}}(H_k) - \frac{1+(p-1)||H||}{k}, \end{aligned}$$

and the proof is complete.

■

Corollary 3.6 *With our assumptions the pressure is a convex function of H.*

Proof. Use the affinity of the function $H \mapsto h_\phi(\alpha) - \phi(H)$.

■

Corollary 3.7 *If (A_1, α_1) and (A_2, α_2) are asymptotically abelian systems with locality then*

$$ht(\alpha_1 \otimes \alpha_2) = ht(\alpha_1) + ht(\alpha_2).$$

Proof. If ϕ_i is an α_i-invariant state, $i = 1, 2$, then by [SV, Lemma 3.4] and [V, Propositions 4.6 and 4.9],

$$h_{\phi_1}(\alpha_1) + h_{\phi_2}(\alpha_2) \leq h_{\phi_1\otimes\phi_2}(\alpha_1 \otimes \alpha_2) \leq ht(\alpha_1 \otimes \alpha_2) \leq ht(\alpha_1) + ht(\alpha_2).$$

Taking the sup over ϕ_i we get the conclusion.

■

4 KMS-states

By Corollary 3.6 and Proposition 2.7 it follows that if (A, α) is asymptotically abelian with locality and $ht(\alpha) < \infty$, then for every equilibrium state ϕ at H, $-\phi$ is a tangent functional for the pressure P_α at H. Furthermore, if ω is a tangent functional for P_α at H then $-\omega$ is an α-invariant state.

If H is local and $I \subset \mathbb{Z}$ is a subset then the derivation

$$\delta_{H,I}(x) = \sum_{j\in I}[\alpha^j(H), x], \quad x \in \mathcal{A},$$

defines a strongly continuous one-parameter automorphism group $\sigma_t^{H,I} = \exp(it\delta_{H,I})$ of A (see [BR, Theorem 6.2.6 and Example 6.2.8]). We shall mainly be concerned with the case $I = \mathbb{Z}$, and will write $\delta_H = \delta_{H,\mathbb{Z}}$, $\sigma_t^H = \sigma_t^{H,\mathbb{Z}}$. Recall that a state ϕ is a (σ_t^H, β)-KMS state if $\phi(ab) = \phi(b\sigma_{i\beta}^H(a))$ for σ_t^H-analytic elements $a, b \in A$.

We say that an α-invariant state ϕ is an *equilibrium state at H at inverse temperature β* if

$$P_\alpha(\beta H) = h_\phi(\alpha) - \beta\phi(H).$$

By Theorem 3.2, for systems which are asymptotically abelian with locality, this is equivalent to

$$h_\phi(\alpha) - \beta\phi(H) = \sup_\psi(h_\psi(\alpha) - \beta\psi(H)).$$

The main result in this section is

Theorem 4.1 *Suppose a unital separable C^*-dynamical system (A, α) is asymptotically abelian with locality, and $ht(\alpha) < \infty$. If H is a local self-adjoint operator in A and ϕ is an equilibrium state at H at inverse temperature β, then ϕ is a (σ_t^H, β)-KMS state. In particular, if $ht(\alpha) = h_\phi(\alpha)$ then ϕ is a trace.*

In order to prove the theorem we may replace H by βH and show that ϕ is a $(\sigma_t^H, 1)$-KMS state. We shall prove the following more general result.

Theorem 4.2 *If $-\phi$ is a tangent functional for P_α at H then ϕ is a $(\sigma_t^H, 1)$-KMS state.*

We shall need an explicit formula for the pressure, which is a consequence of our proof of the variational principle.

Lemma 4.3 *Let N be a local algebra. Then there exist a sequence $\{A_n\}_n$ of local algebras containing N and three sequences $\{p_n\}_n$, $\{m_n\}_n$, $\{k_n\}_n$ of positive integers such that*

(i) *$\alpha^p(A_n)$ commutes with A_n whenever $|p| \geq p_n$;*

(ii) *$\dfrac{p_n}{k_n} \to 0$ as $n \to \infty$;*

(iii) *$P_\alpha(H) = \lim_{n\to\infty} \dfrac{1}{k_n m_n} \log \mathrm{Tr}_{\vee_{j\in I_n}\alpha^j(A_n)}\left(e^{-\sum_{j\in I_n}\alpha^j(H)}\right)$ for all $H \in N_{sa}$, where* $I_n = \bigcup_{j=0}^{m_n-1}[jk_n, jk_n + k_n - p_n]$.

Proof. Let $\{A_n\}_n$ be an increasing sequence of local algebras containing N such that $\cup_n A_n$ is dense in A, ω_n a finite subset of A_n such that $\mathrm{span}(\omega_n) = A_n$. Let $\{p_n\}_n$ be a sequence satisfying condition (i). By Lemma 3.4

$$P_\alpha(H, \omega_n) \leq \liminf_{k\to\infty} \frac{1}{k} P_{\alpha^k|\vee_{j\in\mathbb{Z}}\alpha^{jk}(A_{n,k})}\left(\sum_{j=0}^{k-p_n} \alpha^j(H)\right) \quad \forall H \in N_{sa},$$

where $A_{n,k} = \vee_{j=0}^{k-p_n} \alpha^j(A_n)$. On the other hand, by the proof of Theorem 3.2

$$P_\alpha(H) \geq \limsup_{k\to\infty} \frac{1}{k} P_{\alpha^k|\vee_{j\in\mathbb{Z}}\alpha^{jk}(A_{n,k})}\left(\sum_{j=0}^{k-p_n} \alpha^j(H)\right) \quad \forall H \in N_{sa}.$$

Choose a countable dense subset X of N_{sa}. Since $P_\alpha(H, \omega_n) \nearrow P_\alpha(H)$ for any $H \in X$, we can find a sequence $\{k_n\}_n$ such that condition (ii) is satisfied and

$$P_\alpha(H) = \lim_{n\to\infty} \frac{1}{k_n} P_{\alpha^{k_n}|\vee_{j\in\mathbb{Z}}\alpha^{jk_n}(A_{n,k_n})}\left(\sum_{j=0}^{k_n-p_n} \alpha^j(H)\right) \quad \forall H \in X.$$

Since by Lemma 3.3

$$P_{\alpha^{k_n}|\vee_{j\in\mathbb{Z}}\alpha^{jk_n}(A_{n,k_n})}\left(\sum_{j=0}^{k_n-p_n} \alpha^j(H)\right) = \lim_{m\to\infty} \frac{1}{m} \log \mathrm{Tr}_{\vee_{j\in I_{n,m}}\alpha^j(A_n)}\left(e^{-\sum_{j\in I_{n,m}}\alpha^j(H)}\right),$$

where $I_{n,m} = \cup_{j=0}^{m-1}[jk_n, jk_n + k_n - p_n]$, we can choose a sequence $\{m_n\}_n$ such that condition (iii) is satisfied for all $H \in X$. But then it is satisfied for all $H \in N_{sa}$ by Proposition 2.4(iv) and the Peierls-Bogoliubov inequality.

■

Every local operator is analytic for the dynamics, and σ_t^H depends continuously on H in a fixed local algebra. More precisely, we have

Lemma 4.4

(i) *The series* $\sigma_\beta^{H,I}(a) = \sum_{n=0}^{\infty} \frac{(i\beta)^n}{n!} \delta_{H,I}^n(a)$ *converges absolutely in norm for any* $\beta \in \mathbb{C}$ *and any local operator* a.

(ii) *Given a local algebra* N, $R > 0$, $C > 0$ *and* $\varepsilon > 0$ *there exist* $q \in \mathbb{N}$ *and* $\delta > 0$ *such that*

$$||\sigma_\beta^{H_1,I_1}(a) - \sigma_\beta^{H_2,I_2}(a)|| \leq \varepsilon ||a||$$

$\forall a \in N$, $\forall H_1, H_2 \in N_{sa}$ *with* $||H_1||, ||H_2|| \leq C$ *and* $||H_1 - H_2|| < \delta$, $\forall \beta \in \mathbb{C}$ *with* $|\beta| \leq R$, $\forall I_1, I_2 \subset \mathbb{Z}$ *with* $[-q, q] \subset I_1 \cap I_2$.

Proof. We shall use the arguments of Araki [A, Theorem 4.2]. Let H and a lie in a local algebra N. Choose $p \in \mathbb{N}$ such that $\alpha^j(N)$ commutes with N for $|j| \geq p$. Then

$$\delta_{H,I}^m(a) = \sum_{j_1,\ldots,j_m} [\alpha^{j_m}(H), [\ldots, [\alpha^{j_1}(H), a]\ldots]],$$

where the sum is over all $j_1, \dots, j_m$ such that

$$j_k \in [-p, p] \bigcup \left(\bigcup_{l<k} [j_l - p, j_l + p] \right) \tag{4.1}$$

for each $k = 1, \dots, m$. But as was already noted in [GN] condition (4.1) is equivalent to

$$[j_k, j_k + p] \cap \left([0, p] \bigcup \left(\bigcup_{l<k} [j_l, j_l + p] \right) \right) \neq \emptyset.$$

Thus the lemma follows from the proof of [A, Theorem 4.2] (with $n = p$ and $r = p$).

■

The following lemma contains the main technical result needed to prove Proposition 4.2.

Lemma 4.5 *Let N be a local algebra, $H \in N_{sa}$, $-\phi \in N^*$ is a tangent functional to $(P_\alpha)|_{N_{sa}}$ at H. Let $E\colon A \to N$ be a conditional expectation. Then for any function $f \in \mathcal{D}$ (the space of C^∞-functions with compact support) and any $a, b \in N$ we have*

$$\left| \int_{\mathbb{R}} \hat{f}(t) \phi(a E(\sigma_t^H(b))) dt - \int_{\mathbb{R}} \hat{f}(t+i) \phi(E(\sigma_t^H(b)) a) dt \right|$$

$$\leq ||a|| \int_{\mathbb{R}} (|\hat{f}(t)| + |\hat{f}(t+i)|) ||\sigma_t^H(b) - E(\sigma_t^H(b))|| dt.$$

Proof. First consider the case when $(P_\alpha)|_{N_{sa}}$ is differentiable at H, in other words $-\phi$ is the unique tangent functional. With the notations of Lemma 4.3 consider the state f_n on $\vee_{j \in I_n} \alpha^j(A_n)$ with density operator

$$\left(\mathrm{Tr}_{\vee_{j \in I_n} \alpha^j(A_n)} \left(e^{-\sum_{j \in I_n} \alpha^j(H)} \right) \right)^{-1} e^{-\sum_{j \in I_n} \alpha^j(H)}.$$

Then define a positive linear functional ϕ_n on N by

$$\phi_n(x) = \frac{1}{k_n m_n} \sum_{j \in I_n} f_n(\alpha^j(x)).$$

Note that $||\phi_n|| = \phi_n(1) \leq 1$. Since $-f_n$ is a tangent functional to the convex function $x \mapsto \log \mathrm{Tr}_{\vee_{j \in I_n} \alpha^j(A_n)}(e^{-x})$ on $(\vee_{j \in I_n} \alpha^j(A_n))_{sa}$ at the point $\sum_{j \in I_n} \alpha^j(H)$, $-\phi_n$ is a tangent functional to the function $N_{sa} \ni x \mapsto \frac{1}{k_n m_n} \log \mathrm{Tr}_{\vee_{j \in I_n} \alpha^j(A_n)} \left(e^{-\sum_{j \in I_n} \alpha^j(x)} \right)$ at H. It follows that any limit point of the sequence $\{-\phi_n\}_n$ is a tangent functional to $(P_\alpha)|_{N_{sa}}$ at H. Since the latter is unique by assumption, $\phi_n \to \phi$ as $n \to \infty$.

Since f_n is a $(\sigma_t^{H, I_n}, 1)$-KMS state, by [BR, Proposition 5.3.12] we have

$$\int_{\mathbb{R}} \hat{f}(t) f_n(\alpha^j(a) \sigma_t^{H, I_n}(\alpha^j(b))) dt = \int_{\mathbb{R}} \hat{f}(t+i) f_n(\sigma_t^{H, I_n}(\alpha^j(b)) \alpha^j(a)) dt \quad \forall j \in I_n. \tag{4.2}$$

Note that $\sigma_t^{H, I_n}(\alpha^j(b)) = \alpha^j(\sigma_t^{H, I_n - j}(b))$. Fix $q \in \mathbb{N}$, and set $I_{n,q} = \cup_{j=0}^{m_n - 1} [jk_n + q, jk_n + k_n - p_n - q]$. By Lemma 4.4, if q is large enough then $\sigma_t^{H, I_n - j}(b)$ is arbitrarily close to $\sigma_t^H(b)$ for any $j \in I_{n,q}$ and any t in a fixed compact subset of $\mathbb{R}$. But then $\sigma_t^{H, I_n}(\alpha^j(b)) - \alpha^j(E(\sigma_t^H(b)))$ is arbitrarily close to $\alpha^j(\sigma_t^H(b) - E(\sigma_t^H(b)))$. In other words,

$$\left|\int_{\mathbb{R}} dt\, \hat{f}(t)\frac{1}{k_n m_n}\sum_{j\in I_{n,q}} f_n\left(\alpha^j(a)\sigma_t^{H,I_n}(\alpha^j(b)) - \alpha^j(a)\alpha^j(E(\sigma_t^H(b)))\right)\right|$$

$$\leq ||a|| \int_{\mathbb{R}} |\hat{f}(t)|\, ||\sigma_t^H(b) - E(\sigma_t^H(b))|| dt + \varepsilon(q) \quad \forall n \in \mathbb{N},$$

where $\varepsilon(q) \to 0$ as $q \to \infty$. Since $\#I_{n,q}/\#I_n \to 1$ as $n \to \infty$, letting $n \to \infty$ we may replace averaging over the set $I_{n,q}$ by averaging over I_n, and then obtain

$$\limsup_{n\to\infty}\left|\frac{1}{k_n m_n}\sum_{j\in I_n}\int_{\mathbb{R}} \hat{f}(t) f_n\left(\alpha^j(a)\sigma_t^{H,I_n}(\alpha^j(b))\right)dt - \int_{\mathbb{R}} \hat{f}(t)\phi(aE(\sigma_t^H(b)))dt\right|$$

$$\leq ||a|| \int_{\mathbb{R}} |\hat{f}(t)|\, ||\sigma_t^H(b) - E(\sigma_t^H(b))|| dt.$$

Since an analogous estimate holds for $\int_{\mathbb{R}} \hat{f}(t+i)\phi(E(\sigma_t^H(b))a)dt$, we obtain the conclusion of the lemma by virtue of (4.2).

If $(P_\alpha)|_{N_{sa}}$ is not differentiable at H then by [LR, Theorem 1], ϕ lies in the closed convex hull of those $\tilde{\phi}$, for which there exists a sequence $\{H_n\}_n \subset N_{sa}$ converging to H such that $(P_\alpha)|_{N_{sa}}$ has a unique tangent functional $-\phi_n$ at H_n and $\phi_n \to \tilde{\phi}$. Since for ϕ_n the lemma is already proved (for H_n instead of H), using Lemma 4.4 we conclude that the conclusion of the lemma is true for $\tilde{\phi}$. But then it is true for any functional in the closed convex hull of the $\tilde{\phi}$'s.

■

Proof of Theorem 4.2. If $-\phi$ is a tangent functional for P_α at H then $-\phi|_N$ is a tangent functional for $(P_\alpha)|_{N_{sa}}$ at H for any local algebra N containing H. Thus by Lemma 4.5 the equality

$$\int_{\mathbb{R}} \hat{f}(t)\phi(a\sigma_t^H(b))dt = \int_{\mathbb{R}} \hat{f}(t+i)\phi(\sigma_t^H(b)a)dt$$

holds for all $f \in \mathcal{D}$ and all local a, b, hence for all $a, b \in A$. By [BR, Proposition 5.3.12] this is equivalent to the KMS-condition.

■

Remark 4.6 Under the assumptions of Theorem 4.1, if

$$\phi \in \bigcap_{\varepsilon>0} \overline{\{\psi \,|\, P_\alpha(H) < h_\psi(\alpha) - \psi(H) + \varepsilon\}}$$

(weak* closure), then $-\phi$ is a tangent functional for the pressure at H, hence ϕ is a $(\sigma_t^H, 1)$-KMS state. In other words, any weak* limit point of a sequence on which the sup in the variational principle is attained, is a $(\sigma_t^H, 1)$-KMS state. If $ht(\alpha) = +\infty$, this is of course false in general. Moreover, for any α-invariant state ϕ there exists a sequence $\{\phi_n\}_n$ converging in norm to ϕ such that $h_{\phi_n}(\alpha) = +\infty$ for all n. Indeed, first note that by taking infinite convex combinations of states of large entropy we can find a state ψ of infinite entropy. Then $\phi_n = \frac{1}{n}\psi + \frac{n-1}{n}\phi \underset{n\to\infty}{\to} \phi$ and $h_{\phi_n}(\alpha) = +\infty$.

5 Examples

First we consider a class of systems arising naturally from systems of topological dynamics.

Let σ be an expansive homeomorphism of a zero-dimensional compact space X, G the group of uniformly finite-dimensional homeomorphisms of X in the sense of Krieger [K]. By definition, a homeomorphism T belongs to G if

$$\lim_{|n|\to\infty} \sup_{x\in X} d(\sigma^n Tx, \sigma^n x) = 0,$$

where d is a metric defining the topology of X. In other words, G consists of those homeomorphisms T of X, for which there exists a bound on the number of coordinates of any point that are changed under the action of T, when (X,σ) is represented as a subshift by means of some generator. Since the group G is locally finite, the orbit equivalence relation $\mathcal{R} \subset X \times X$ has a structure of AF-groupoid [Re]. Consider the groupoid C*-algebra $A = C^*(\mathcal{R})$ and the automorphism α of A defined by $\alpha(f) = f\circ(\sigma\times\sigma)$. The algebra $C(X)$ is a subalgebra of A, and there exists a unique conditional expectation $E\colon A \to C(X)$. Let $C_0(X)$ be the $*$-subalgebra of $C(X)$ spanned by characteristic functions of clopen sets, and $C_0(X,\mathbb{R})$ the subalgebra of $C_0(X)$ consisting of real functions. Every element $g \in G$ defines a canonical unitary $u_g \in A$ such that $u_g f u_g^* = f\circ g^{-1}$ for $f \in C(X)$. The $*$-algebra generated by $C_0(X)$ and u_g, $g\in G$, is our algebra $\mathcal{A}$ of local operators.

For $H \in C_0(X,\mathbb{R})$ consider the 1-cocycle $c_H \in Z^1(\mathcal{R},\mathbb{R})$,

$$c_H(x,y) = \sum_{j\in\mathbb{Z}} (H(\sigma^j x) - H(\sigma^j y)).$$

Recall [Re, Definition 3.15] that a measure μ on $X = \mathcal{R}^{(0)}$ satisfies the $(c_H, 1)$-KMS condition if its modular function is equal to e^{-c_H}. In other words,

$$\frac{dg_*\mu}{d\mu}(x) = e^{-c_H(g^{-1}x,x)}.$$

Proposition 5.1 *Let $H \in C_0(X,\mathbb{R})$. Then*

(i) *Any measure μ on X which is an equilibrium measure at $-H$ satisfies the $(c_H,1)$-KMS condition. In particular, any measure of maximal entropy is G-invariant.*

(ii) *The mapping $\mu \mapsto \mu\circ E$ defines a one-to-one correspondence between equilibrium measures on X at $-H$ and equilibrium states on $C^*(\mathcal{R})$ at H.*

Proof. First note that if ϕ is an α-invariant state on $C^*(\mathcal{R})$, and $\mu = \phi|_{C(X)}$ then

$$h_\mu(\sigma) = h_{\mu\circ E}(\alpha) \ge h_\phi(\alpha).$$

The equality is proved by standard arguments using [CNT, Corollary VIII.8]. The inequality follows from the fact that if ψ is a state on a finite dimensional C*-algebra M with a masa B then $S(\psi) \le S(\psi|_B)$. It follows that if μ is an equilibrium measure at $-H$ then $\mu\circ E$ is an equilibrium state at H, and if ϕ is an equilibrium state at H then $\phi|_{C(X)}$ is an equilibrium measure at $-H$. By Theorem 4.1 any equilibrium state is a $(\sigma^H_t, 1)$-KMS state. But by [Re, Proposition 5.4] any $(\sigma^H_t,1)$-KMS state has the form $\mu\circ E$ for some measure μ satisfying the $(c_H,1)$-KMS condition. From this both assertions of the proposition follow.

■

Example 5.2 As an application of Proposition 5.1 consider a topological Markov chain (X, σ) with transition matrix A_T. As is well-known, if A_T is primitive then the Perron-Frobenius theorem implies the uniqueness of the trace on $C^*(\mathcal{R})$. If A_T is only supposed to be irreducible, then the traces of $C^*(\mathcal{R})$ form a simplex with the number of vertices equal to the index of cyclicity of the matrix. The barycenter of this simplex is the unique α-invariant trace. By Proposition 5.1 we conclude that if A_T is irreducible then (X, σ) has a unique measure of maximal entropy. Thus we have recovered a well-known result of Parry (see [W, Theorem 8.10]).

While in the abelian case uniquely ergodic systems are of great interest, they are not so for asymptotically abelian systems with locality. Indeed, we have

Proposition 5.3 *Let (A, α) be a C^*-dynamical system which is asymptotically abelian with locality. If there is a unique invariant state τ, then τ is a trace, and $\pi_\tau(A)$ is an abelian algebra.*

For later use the main part of the proof will be given in a separate lemma.

Lemma 5.4 *Let (A, α) be an asymptotically abelian system with locality, τ an α-invariant ergodic trace on A, H a local self-adjoint operator. Suppose for each H' in the real linear span of $\alpha^j(H)$, $j \in \mathbb{Z}$, and for each $k \in \mathbb{N}$ there exists an α^k-invariant $(\sigma_t^{H',k\mathbb{Z}}, 1)$-KMS state ϕ such that $\tau = \frac{1}{k}\sum_{j=0}^{k-1} \phi \circ \alpha^j$. Then $\pi_\tau(H)$ is central in $\pi_\tau(A)$.*

Proof. Replacing A by $A/\mathrm{Ker}\,\pi_\tau$ we may identify A with $\pi_\tau(A) \subset B(\mathcal{H}_\tau)$.

The automorphism α being extended to $A'' \subset B(\mathcal{H}_\tau)$ is strongly asymptotically abelian. Hence for any $k \in \mathbb{N}$ the fixed point algebra $(A'')^{\alpha^k}$ is central. Since α is ergodic, this algebra is k_0-dimensional for some $k_0|k$, and we may enumerate its atoms $z_1, \dots, z_{k_0}$ in such a way that $\alpha(z_1) = z_2, \dots, \alpha(z_{k_0-1}) = z_{k_0}, \alpha(z_{k_0}) = z_1$. Now if ϕ is an α^k-invariant state such that $\tau = \frac{1}{k}\sum_{j=0}^{k-1} \phi \circ \alpha^j$ then $\phi \le k\tau$, hence $\phi(x) = \tau(xa)$ for some positive $a \in (A'')^{\alpha^k}$. In particular, ϕ is a trace. So if in addition ϕ is a $(\sigma_t^{H',k\mathbb{Z}}, 1)$-KMS state then the dynamics $\sigma_t^{H',k\mathbb{Z}}$ is trivial on $(A'')_{s(a)}$, where $s(a)$ is the support of a. Hence $\delta_{H',k\mathbb{Z}}(y)s(a) = 0$ for all local y. Then $\delta_{H',k\mathbb{Z}}(y)z_i = 0$ for some z_i $(1 \le i \le k_0)$ majorized by $s(a)$.

Fix a local $x \in A$. Choose $p \in \mathbb{N}$ such that $\alpha^j(H)$ commutes with x whenever $|j| \ge p$. Pick any $m > p$ and set $k = 2m + 1$. For $\lambda \in \mathbb{R}^k$ consider the operator

$$H(\lambda) = \sum_{j=-m}^{m} \lambda_j \alpha^j(H).$$

Applying the result of the previous paragraph to $H' = H(\lambda)$, we find i, $1 \le i \le k_0$, such that $\delta_{H(\lambda),k\mathbb{Z}}(y)z_i = 0$ for all local y. Denote by X_i the set of all $\lambda \in \mathbb{R}^k$ satisfying the latter condition. Since $\mathbb{R}^k = \cup_i X_i$, there exists i for which $\mathbb{R}^k$ coincides with the linear span of X_i. Without loss of generality we may suppose that $i = 1$. Since for any $j \in [-m+p, m-p]$, any $j' \ne 0$, and any $\lambda \in X_1$, the elements $\alpha^{j'k}(H(\lambda))$ and $\alpha^j(x)$ commute, we obtain

$$0 = \delta_{H(\lambda),k\mathbb{Z}}(\alpha^j(x))z_1 = [H(\lambda), \alpha^j(x)]z_1,$$

hence $[\alpha^{j'}(H), \alpha^j(x)]z_1 = 0$ for $j' \in [-m, m]$. In particular,

$$\alpha^j([H, x])z_1 = 0 \quad \text{for} \quad j \in [-m+p, m-p].$$

If $k_0 \neq k$ and $k - 2p \geq \frac{k}{2}(> k_0)$ then $\vee_{j=-m+p}^{m-p}\alpha^j(z_1) = 1$, so $[H,x] = 0$. If $k_0 = k$ then $[H,x]z = 0$, where

$$z = \bigvee_{j=-m+p}^{m-p} \alpha^j(z_1) = \sum_{j=-m+p}^{m-p} \alpha^j(z_1), \quad \tau(z) = \frac{k-2p}{k}.$$

Since $\frac{k-2p}{k} \to 1$ as $m \to \infty$, we conclude that $[H,x] = 0$.

■

Proof of Proposition 5.3. Since A is a unital AF-algebra, there exists a trace on A, hence there exists an α-invariant trace. It follows that the unique α-invariant state is a trace.

If H is local then for any subset I of Z there exists a $(\sigma_t^{H,I}, 1)$-KMS state. Indeed, if we take an increasing sequence of finite subsets I_n of I such that $\cup_n I_n = I$, an increasing sequence of local algebras A_n such that $\alpha^j(H) \in A_n$ for $j \in I_n$ and $\cup_n A_n$ is dense in A, and a sequence of states ϕ_n such that $\phi_n|_{A_n}$ is a $(\sigma_t^{H,I_n}, 1)$-KMS state, then any weak* limit point of the sequence $\{\phi_n\}_n$ will be a $(\sigma_t^{H,I}, 1)$-KMS state. If in addition $I + k = I$, then the state can be chosen to be α^k-invariant (since the set of $(\sigma_t^{H,I}, 1)$-KMS states is α^k-invariant). But if ϕ is an α^k-invariant state then the state $\frac{1}{k}\sum_{j=0}^{k-1} \phi \circ \alpha^j$ is α-invariant, hence it coincides with τ. Thus the conditions of Lemma 5.4 are satisfied. Hence $\pi_\tau(H)$ is central in $\pi_\tau(A)$ for any local H, so $\pi_\tau(A)$ is abelian.

■

We consider two examples illustrating Proposition 5.3.

Example 5.5 Let U be the bilateral shift on a separable Hilbert space $\mathcal{H}$, and $\alpha = \mathrm{Ad}\, U|_A$, where A is the C*-algebra $K(\mathcal{H}) + \mathbb{C}1$, $K(\mathcal{H})$ being the algebra of compact operators. Then the only α-invariant state is the trace τ, which annihilates $K(\mathcal{H})$. Then $\pi_\tau(A) = \mathbb{C}1$.

Example 5.6 More generally, consider a uniquely ergodic system (X, σ) and construct a system $(C^*(\mathcal{R}), \alpha)$ as above. Let τ be an α-invariant trace. Then $\tau = \mu \circ E$ for some measure μ, and the unique ergodicity of (X, σ) means that μ is the unique invariant measure. We check the conditions of Lemma 5.4 for any $H \in C(X_0, \mathbb{R})$.

By the same reasons as in the proof of Lemma 5.4, the fixed point algebra $(\pi_\tau(A)'')^\alpha$ is central. By [FM] the center of the algebra $\pi_\tau(A)''$ is isomorphic to $L^\infty(X, \mu)$. Since the measure μ is ergodic, we conclude that the trace τ is also ergodic.

Let $H \in C(X_0, \mathbb{R})$, $k \in \mathbb{N}$, and ϕ any α^k-invariant $(\sigma_t^{H,kZ}, 1)$-KMS state. Then $\phi = \nu \circ E$ for some σ^k-invariant measure ν. Since $\mu = \frac{1}{k}\sum_{j=0}^{k-1} \nu \circ \sigma^j$, we have $\tau = \frac{1}{k}\sum_{j=0}^{k-1} \phi \circ \alpha^j$.

Thus we can apply Lemma 5.4, and conclude that $\pi_\tau(C(X)) \cong C(\mathrm{supp}\,\mu)$ is central in $\pi_\tau(A)$. This means that G acts trivially on $\mathrm{supp}\,\mu$, and $\pi_\tau(A) = C(\mathrm{supp}\,\mu)$. By [Re, Proposition 4.5] the kernel of π_τ is the algebra corresponding to the groupoid $\mathcal{R}_{X \backslash \mathrm{supp}\,\mu}$. Since there is no non-zero finite σ-invariant measures on $X \backslash \mathrm{supp}\,\mu$, any α-invariant state is zero on $\mathrm{Ker}\,\pi_\tau$. Thus the system (A, α) is uniquely ergodic and $\pi_\tau(A) = C(\mathrm{supp}\,\mu)$.

We next give an example of an asymptotically abelian C*-dynamical system (A, α) with A an AF-algebra, for which there exist non-tracial α-invariant states with maximal finite entropy. Hence the assumption of locality in Theorem 4.1 is essential.

Example 5.7 Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, A the even CAR-algebra over $\mathcal{H}$, α the Bogoliubov automorphism corresponding to a unitary U. It easy to see that α is asymptotically abelian if and only if $(U^n f, g) \underset{n\to\infty}{\to} 0$ for any $f, g \in \mathcal{H}$. If in addition U has singular spectrum then by the proof of [SV, Theorem 5.2] we have $ht(\alpha) = 0$, while there are many non-tracial α-invariant states (for example, quasi-free states corresponding to scalars $\lambda \in (0, 1/2)$). Unitaries with such properties can be obtained using Riesz products. We shall briefly recall the construction.

Let $q > 3$ be a real number, $\{n_k\}_{k=1}^{\infty}$ a sequence of positive integers such that $\frac{n_{k+1}}{n_k} \geq q$, $\{a_k\}_{k=1}^{\infty}$ a sequence of real numbers such that $a_k \in (-1, 1)$, $a_k \to 0$ as $k \to \infty$, $\sum_k a_k^2 = \infty$. Then the sequence of measures

$$\frac{1}{2\pi}\left[\prod_{k=1}^{n}(1 + a_k \cos n_k t)\right] dt$$

on $[0, 2\pi]$ converges weakly* to a probability measure μ with Fourier coefficients

$$\hat{\mu}(n) = \mu(e^{int}) = \begin{cases} \prod_{k=1}^{\infty}\left(\frac{a_k}{2}\right)^{|\varepsilon_k|}, & \text{if } n = \sum_k \varepsilon_k n_k \text{ with } \varepsilon_k \in \{-1, 0, 1\}, \\ 0, & \text{otherwise.} \end{cases}$$

The measure μ is singular by [Z, Theorem V.7.6]. We see also that $\hat{\mu}(n) \to 0$ as $|n| \to \infty$. Thus the operator U of multiplication by e^{int} on $L^2([0, 2\pi], d\mu)$ has the desired properties.

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