

AF flows and continuous symmetries

O. Bratteli

Department of Mathematics, University of Oslo, PB 1053–Blindern, N-0316 Oslo, Norway

A. Kishimoto

Department of Mathematics, Hokkaido University, Sapporo 060, Japan

August 22, 2000

Abstract

We consider AF flows, i.e., one-parameter automorphism groups of a unital simple AF C^* -algebra which leave invariant the dense union of an increasing sequence of finite-dimensional $*$ -subalgebras, and derive two properties for these; an absence of continuous symmetry breaking and a kind of real rank zero property for the *almost fixed points*.

1 Introduction

We consider the class of AF representable one-parameter automorphism groups of a unital simple AF C^* -algebra (which will be called AF flows) and derive two properties, one of which is invariant under inner perturbations and may be used to distinguish them from other flows (i.e., one-parameter automorphism groups).

We recall that a flow α of a unital simple AF C^* -algebra A is defined to be AF locally representable or an AF flow if there is an increasing sequence (A_n) of α -invariant finite-dimensional $*$ -subalgebras of A with dense union [14], [15]. In this case there is a self-adjoint $h_n \in A_n$ such that $\alpha_t|_{A_n} = \text{Ad } e^{it h_n}|_{A_n}$ for each n . Thus the *local Hamiltonians* (h_n) mutually commute and can be considered to represent the time evolution of a classical statistical lattice model, which is a special kind of model among all the models quantum or classical. Consider the larger class of flows which are inner perturbations of AF-flows. (These are characterized by the property that the domains of the generators contains a canonical AF maximal abelian sub-algebra (masa), see [15, Proposition 3.1].) In [15, Theorem 2.1 and Remark 3.3] it was demonstrated that there are flows outside this larger class, but the proof was not easy. Our original aim was to show that all the flows which naturally arose in quantum statistical lattice models and were not obviously AF flows, were in fact beyond the class of inner perturbations of AF flows. We could not prove that there was even a single example and obtained only a weak result in this direction which

is presented in Remark 2.4. Thus we ended up presenting the two new properties of the AF flows mentioned in the abstract.

The first property we derive for AF flows can be expressed as: *there is no continuous symmetry breaking*. If δ_α denotes the generator of a general flow α , we define the *exact* symmetry group for α as $G_0 = \{\gamma \in \text{Aut}A \mid \gamma\delta_\alpha\gamma^{-1} = \delta_\alpha\}$ and the *near* symmetry group as $G_1 = \{\gamma \in \text{Aut}A \mid \gamma\delta_\alpha\gamma^{-1} = \delta_\alpha + \text{ad}ih \text{ for some } h = h^* \in A\}$. Then it is known that there is a natural homomorphism of G_0 into the affine homeomorphism group of the simplex of KMS states at each temperature. We deduce moreover in Proposition 2.1 from the perturbation theory of KMS states [1], that there is a homomorphism of G_1 into the homeomorphism group of the simplex of KMS states at each temperature, mapping the extreme points onto the extreme points. We next show in the special case of AF flows that if $\gamma \in G_0$ is connected to id in G_0 by a continuous path, then γ induces the identity map on the simplexes of KMS states. We actually show a generalization of this in Theorem 2.3: If α is an AF flow and $\gamma \in G_1$ is connected to id in G_1 by a continuous path (γ_t) such that $\gamma_t\delta_\alpha\gamma_t^{-1} = \delta_\alpha + \text{ad}ib(t)$ with $b(t)$ rectifiable in A , then γ induces a homeomorphism which fixes each extreme point. (Thus, if the homeomorphism is affine, it is the identity map. This is in particular true if $\gamma \in G_0$.)

The second property we derive for the class of inner perturbations of AF flows can be expressed as: *the almost fixed point algebra for α has real rank zero* (see Theorem 3.6). A technical lemma used to show this property is a generalization of H. Lin's result on almost commuting self-adjoint matrices [16]. The generalization says that any almost commuting pair of self-adjoint matrices, one of norm one and the other of arbitrary norm, is in fact close to an exactly commuting pair (see Theorem 3.1).

We recall here a similar kind of property in [15] saying that *the almost fixed point algebra has trivial K_1* . We will show by examples that these two properties, real rank zero and trivial K_1 for the almost fixed point algebra, are independent, as one would expect. (It is not that *the almost fixed point algebra* is actually defined as an algebra; but if α is periodic, then we can regard the almost fixed point algebra as the usual fixed point algebra, see Proposition 3.7. In general we can characterize any property of *the almost fixed point algebra* as the corresponding property of the fixed point algebra for a certain flow obtained by passing to a C^* -algebra of bounded sequences modulo c_0 , see Proposition 3.8.)

We remark that there is a flow α of a unital simple AF C^* -algebra such that $\mathcal{D}(\delta_\alpha)$ is not AF (as a Banach $*$ -algebra)(cf. [18, 19]). This was shown in [15] by constructing an example where $\mathcal{D}(\delta_\alpha)$ does not have real rank zero. Note that $\mathcal{D}(\delta_\alpha)$ has always trivial K_1 and has the same K_0 as the C^* -algebra A . Hence real rank is still the only property which has been used to distinguish α with non-AF $\mathcal{D}(\delta_\alpha)$. On the other hand even K_0 (of the almost fixed point algebra) might be used to distinguish non-AF flows (up to inner perturbations) as well as real rank and K_1 as shown above.

In the last section we will show that any quasi-free flow of the CAR algebra has the property that the almost fixed point algebra has trivial K_1 , leaving open the question of whether it is an inner perturbation of an AF flow or not and even the weaker question of

whether the almost fixed point algebra has real rank zero or not.

One of the authors (A.K.) would like to thank Professor S. Sakai for discussions and questions concerning the first property.

2 Symmetry

In the first part of this section we describe the symmetry group of a flow and how it is mapped into the homeomorphism groups of the simplex of KMS states. Then in the remaining part we discuss a theorem on a kind of absence of continuous symmetry breaking for AF flows.

In the first part A can be an arbitrary unital simple C^* -algebra. Let α be a flow of A (where we always assume strong continuity; $t \mapsto \alpha_t(x)$ is continuous for any $x \in A$), and δ_α the generator of α . Then δ_α is a closed linear operator defined on a dense $*$ -subalgebra $\mathcal{D}(\delta_\alpha)$ of A with the derivation property: $\delta_\alpha(xy) = \delta_\alpha(x)y + x\delta_\alpha(y)$, $\delta_\alpha(x)^* = \delta_\alpha(x^*)$ for $x, y \in \mathcal{D}(\delta_\alpha)$. We equip $\mathcal{D}(\delta_\alpha)$ with the norm $\|\cdot\|_{\delta_\alpha}$ obtained by embedding $\mathcal{D}(\delta_\alpha)$ into $A \otimes M_2$ by the (non $*$ -preserving) isomorphism $x \mapsto \begin{pmatrix} x & \delta_\alpha(x) \\ 0 & x \end{pmatrix}$. Note that $\mathcal{D}(\delta_\alpha)$ is a Banach $*$ -algebra. (See [6, 3, 19] for the theory of unbounded derivations.)

We call a continuous function u of \mathbf{R} into the unitary group of A an α -cocycle if $u_s \alpha_s(u_t) = u_{s+t}$, $s, t \in \mathbf{R}$. Then $t \mapsto \text{Ad } u_t \circ \alpha_t$ is a flow of A and is called a cocycle perturbation of α . If u is differentiable, then the generator of this perturbation is $\delta_\alpha + \text{ad } ih$, where $du/dt|_{t=0} = ih$ (see [14, section 1]). We define the symmetry group $G = G_\alpha$ of α as

$$\{\gamma \in \text{Aut } A \mid \gamma \alpha \gamma^{-1} \text{ is a cocycle perturbation of } \alpha\},$$

which is slightly more general than the G_1 given in the introduction, so $G_0 \subseteq G_1 \subseteq G = G_\alpha$. Then G depends on the class of cocycle perturbations of α only and is indeed a group: If $\gamma \in G$, then $\gamma \alpha_t \gamma^{-1} = \text{Ad } u_t \alpha_t$ for some α -cocycle u , which implies that

$$\gamma^{-1} \alpha_t \gamma = \text{Ad } \gamma^{-1}(u_t^*) \alpha_t.$$

We can check the α -cocycle property of $t \mapsto \gamma^{-1}(u_t^*)$ by

$$\begin{aligned} \gamma^{-1}(u_s^*) \alpha_s(\gamma^{-1}(u_t^*)) &= \gamma^{-1}(u_s^* \gamma \alpha_s \gamma^{-1}(u_t^*)) = \gamma^{-1}(u_s^* \text{Ad } u_s \alpha_s(u_t^*)) \\ &= \gamma^{-1}(\alpha_s(u_t^*) u_s^*) = \gamma^{-1}(u_{s+t}^*). \end{aligned}$$

Thus $\gamma^{-1} \in G$. If $\gamma_1, \gamma_2 \in G$, then $\gamma_i \alpha_t \gamma_i^{-1} = \text{Ad } u_{it} \alpha_t$ for some α -cocycle u_i for $i = 1, 2$. Since $\gamma_1 \gamma_2 \alpha_t (\gamma_1 \gamma_2)^{-1} = \text{Ad } \gamma_1(u_{2t}) u_{1t} \alpha_t$, we only have to check that $t \mapsto \gamma_1(u_{2t}) u_{1t}$ is an α -cocycle, which will be denoted by $\gamma_1(u_2) u_1$. We leave this simple calculation to the reader. Note that G contains the inner automorphism group $\text{Inn}(A)$ as a normal subgroup and each element of $G/\text{Inn}(A)$ has a representative $\gamma \in G$ such that γ leaves $\mathcal{D}(\delta_\alpha)$ invariant and

$$\gamma \delta_\alpha \gamma^{-1} = \delta_\alpha + \text{ad } ib$$

for some $b = b^* \in A$ (see [14, Corollary 1.2]).

We equip $G = G_\alpha$ with the topology defined by $\gamma_n \rightarrow \gamma$ in G if

- (1) $\gamma_n \rightarrow \gamma$ in $\text{Aut}(A)$ (i.e., $\|\gamma_n(x) - \gamma(x)\| \rightarrow 0$ for $x \in A$),

and

- (2) there exist α -cocycles u_n, u such that $\gamma_n \alpha_t \gamma_n^{-1} = \text{Ad } u_{nt} \alpha_t$, $\gamma \alpha_t \gamma^{-1} = \text{Ad } u_t \alpha_t$ and $\|u_{nt} - u_t\| \rightarrow 0$ uniformly in t on compact subsets of \mathbf{R} .

With this topology G is a topological group.

Let $c \in \mathbf{R} \setminus \{0\}$ and ω a state on A . We say that ω satisfies the c -KMS condition or is a c -KMS state (with respect to α) if for any $x, y \in A$ there is a bounded continuous function F on the strip $S_c = \{z \in \mathbf{C} \mid 0 \leq \Im z/c \leq 1\}$ such that F is analytic in the interior of S_c and satisfies, on the boundary of S_c ,

$$\begin{aligned} F(t) &= \omega(x\alpha_t(y)), \quad t \in \mathbf{R}, \\ F(t+ic) &= \omega(\alpha_t(y)x), \quad t \in \mathbf{R}. \end{aligned}$$

We denote by $K_c^\alpha = K_c$ the set of c -KMS states of A . Then K_c is a closed convex set of states and moreover a simplex. We denote by $\partial(K_c)$ the set of extreme points of K_c . Note that for $\omega \in K_c$, ω is extreme in K_c if and only if ω is a factorial state (see [6, 19] for details).

Proposition 2.1 *Let A be a unital simple C^* -algebra, α a flow of A , and $c \in \mathbf{R} \setminus \{0\}$. Then there is a continuous homomorphism Φ of the symmetry group G_α of α into the homeomorphism group of K_c such that $\Phi(\gamma)(\omega)$ is unitarily equivalent to $\omega\gamma^{-1}$ for each $\gamma \in G_\alpha$ and $\omega \in K_c$. Moreover $\Phi(\gamma) = \text{id}$ for any inner γ .*

Proof. Let $\gamma \in G_\alpha$ and let u be an α -cocycle such that $\gamma \alpha_t \gamma^{-1} = \text{Ad } u_t \alpha_t$. Since A is simple, u is unique up to phase factors, i.e., any other α -cocycle satisfying the same equality is given as $t \mapsto e^{ipt} u_t$ for some $p \in \mathbf{R}$.

Let $\omega \in K_c$. Then $\omega\gamma^{-1}$ is a KMS state with respect to $\gamma \alpha_t \gamma^{-1} = \text{Ad } u_t \alpha_t$. Using the fact that $\alpha_t = \text{Ad } u_t^* \gamma \alpha_t \gamma^{-1}$, there is a procedure to make a KMS positive linear functional ω' with respect to α , which depends on the choice of u ; formally it can be given as

$$\omega'(x) = \omega\gamma^{-1}(x u_{ic}^*), \quad x \in A.$$

More precisely we let $\beta_t = \text{Ad } u_t \alpha_t$ and express the β -cocycle u_t^* as

$$u_t^* = w v_t \beta_t(w^{-1})$$

such that $t \mapsto v_t$ extends to an entire function on \mathbf{C} [14, Lemma 1.1]. Then we define $\Phi(\gamma, u)\omega$ as

$$(\Phi(\gamma, u)(\omega))(a) = \omega\gamma^{-1}(w^{-1} a w v_{ic}).$$

(By a formal calculation we can see that this satisfies the c -KMS condition as follows:

$$\begin{aligned}\omega\gamma^{-1}(w^{-1}a\alpha_{ic}(b)wv_{ic}) &= \omega\gamma^{-1}(w^{-1}aw_{ic}^{-1}\beta_{ic}(b)u_{ic}wv_{ic}) \\ &= \omega\gamma^{-1}(w^{-1}awv_{ic}\beta_{ic}(w^{-1}bw)) \\ &= \omega\gamma^{-1}(w^{-1}bawv_{ic}),\end{aligned}$$

where we used that $u_{ic} = \beta_{ic}(w)v_{ic}^{-1}w^{-1}$ and that $\omega\gamma^{-1}$ is a c -KMS state for $\beta_t = \gamma\alpha_t\gamma^{-1}$. See [14].) The map

$$\Phi(\gamma) : \omega \mapsto \Phi(\gamma, u)(\omega) / \Phi(\gamma, u)(\omega)(1)$$

defines a continuous map of K_c into K_c and $\Phi(\gamma, u)(\omega)$ is quasi-equivalent (hence unitarily equivalent) to $\omega\gamma^{-1}$. (It follows from the definition of $\Phi(\gamma, v)$ that $\Phi(\gamma)(\omega)$ is quasi-contained in $\omega\gamma^{-1}$, but as w^{-1} and wv_{ic} are invertible, $\omega\gamma^{-1}$ is conversely quasi-contained in $\Phi(\gamma)(\omega)$. Since any KMS state is separating and cyclic for the weak closure, these states are unitary equivalent.) For any other choice $u'_t = e^{ipt}u_t$ for u it follows that $\Phi(\gamma, u') = e^{-cp}\Phi(\gamma, u)$. Thus $\Phi(\gamma)$ does not depend on the choice of u . For $\gamma_1, \gamma_2 \in G_\alpha$ with α -cocycles u_1, u_2 respectively, it follows that

$$\Phi(\gamma_1\gamma_2, \gamma_1(u_2)u_1) = \Phi(\gamma_1, u_1)\Phi(\gamma_2, u_2)$$

since

$$\begin{aligned}\Phi(\gamma_1, u_1)\Phi(\gamma_2, u_2)(\omega) &= \Phi(\gamma_1, u_1)(\omega\gamma_2^{-1}(\cdot u_{2,ic}^*)) \\ &= \omega\gamma_2^{-1}(\gamma_1^{-1}(\cdot u_{1,ic}^*)u_{2,ic}^*) \\ &= \omega\gamma_2^{-1}\gamma_1^{-1}(\cdot u_{1,ic}^*\gamma_1(u_{2,ic}^*)).\end{aligned}$$

This shows that Φ is a group homomorphism. If $\gamma = \text{Ad } u$, then $\Phi(\gamma, u\alpha(u^*))(\omega) = \omega$. The continuity of $\gamma \mapsto \Phi(\gamma)$ follows from the following lemma.

Lemma 2.2 *Let $(u_\infty, u_1, u_2, \dots)$ be a sequence of α -cocycles such that $\lim_{n \rightarrow \infty} u_{n,t} = u_{\infty,t}$ uniformly in t on every compact subset of \mathbf{R} . Then for any $\epsilon > 0$ there exists a sequence $(w_\infty, w_1, w_2, \dots)$ of invertible elements in A such that $\lim_{n \rightarrow \infty} w_n = w_\infty$, $\|w_n - 1\| < \epsilon$, and $v_{m,t} \equiv w_m^{-1}u_{m,t}\alpha_t(w_m)$ extends to an entire function on \mathbf{C} for $m = \infty, 1, 2, \dots$ such that $\lim_{n \rightarrow \infty} v_{n,z} = v_{\infty,z}$ for any $z \in \mathbf{C}$.*

Proof. Define a C^* -algebra B by

$$B = \{x = (x_n)_{n=1}^\infty \mid x_n \in A, \lim x_n \text{ exists}\}$$

and define a flow β on $B \otimes M_2$ by $\beta_t = \text{Ad } U \circ \alpha_t \otimes \text{id}$, where $U = (1 \oplus u_{n,t})$. We define a homomorphism φ of B onto A by $\varphi(x) = \lim x_n$ for $x = (x_n) \in B$ and note that $\varphi \circ \beta_t = \text{Ad}(1 \oplus u_{\infty,t}) \circ \alpha_t \otimes \text{id} \circ \varphi$. Let $\epsilon \in (0, 1)$. Since $(1 \oplus 0)_n$ and $(0 \oplus 1)_n$ are fixed by β , there is a $w \in B$ such that $\|w - 1\| < \epsilon$ and

$$t \mapsto \beta_t\left(\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}\right)$$

extends to an entire function on \mathbf{C} (pick an entire element y for β close to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$), and replace y by $(0 \otimes 1)_n y (1 \otimes 0)_n$. If $w = (w_n) \in B$, $v_{n,t} = w_n^{-1} u_{n,t} \alpha_t(w_n) \in A$, and $v_t = (v_{n,t}) \in B$, then we have that

$$\beta_t \left(\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ wv_t & 0 \end{pmatrix}.$$

Letting $w_\infty = \lim w_n$ and $v_{\infty,t} = \lim v_{n,t}$, the proof is complete.

Theorem 2.3 *Let A be a unital simple AF C^* -algebra and α an AF flow of A . Let $(\gamma_t)_{t \in [0,1]}$ be a continuous path in G_α such that*

$$\gamma_t \delta_\alpha \gamma_t^{-1} = \delta_\alpha + \text{ad } ib(t)$$

for some rectifiable path $(b(t))_{t \in [0,1]}$ in A_{sa} . Then it follows that $\Phi(\gamma_0)(\omega) = \Phi(\gamma_1)(\omega)$ for $\omega \in \partial(K_c)$.

Proof. Let C be a canonical AF masa in $\mathcal{D}(\delta_\alpha)$ such that $\delta_\alpha|_C = 0$. Let $\omega \in K_c$. We note that if E denotes the projection of norm one onto C , then $\omega = (\omega|_C) \circ E$, i.e., ω is determined by the restriction $\omega|_C$. (Let (A_n) be an increasing sequence of α -invariant finite dimensional subalgebras with dense union in A such that $A_n \cap C$ is masa in A_n for each n . Then $\omega|_{A_n}$ is clearly determined by $\omega|_{A_n \cap C}$, and thus ω is determined by $\omega|_C$.)

We first prove the theorem in the simpler case where $b(t) = 0$. In this case γ_t leaves the C^* -subalgebra $B = \text{Kernel}(\delta_\alpha)$ invariant, on which ω is a trace. For any projection $e \in C \subset B$, $(\gamma_t(e))$ is a continuous family of projections in B , which implies that $\gamma_0(e)$ is equivalent to $\gamma_1(e)$ in B . Hence $\omega \gamma_0(e) = \omega \gamma_1(e)$. Since C is an abelian AF algebra, this implies that $\omega \gamma_0|_C = \omega \gamma_1|_C$. Since they are KMS states, we can conclude that $\omega \gamma_0 = \omega \gamma_1$. Since this is true for any $\omega \in K_c$, it also follows that $\omega \gamma_0^{-1} = \omega \gamma_1^{-1}$.

What we will do in the following is a modification of this argument.

Let $\omega \in \partial(K_c)$. In the GNS representation associated with $\omega \in \partial(K_c)$, we define a one-parameter unitary group U by

$$U_t \pi_\omega(x) \Omega_\omega = \pi_\omega \circ \alpha_t(x) \Omega_\omega, \quad x \in A.$$

Then from the c -KMS condition on ω it follows that the modular operator Δ for Ω_ω is given by $\Delta = e^{-cH}$, where H is the generator of U ; $U_t = e^{itH}$ (See [7, Proof of Theorem 5.3.10]). We define a positive linear functional $\omega^{(h)}$ on A for $h = h^* \in A$ as the vector state given by $e^{-c(H+\pi_\omega(h))/2} \Omega_\omega$, i.e.,

$$\omega^{(h)}(x) = (\pi_\omega(x) e^{-c(H+\pi_\omega(h))/2} \Omega_\omega, e^{-c(H+\pi_\omega(h))/2} \Omega_\omega).$$

Then $\omega^{(h)}$ satisfies the c -KMS condition with respect to $\delta_\alpha + \text{ad } ih$. (See [1, 19] or [7, Theorem 5.4.4]. The relation to the previous perturbation argument in terms of cocycles

is as follows: The flow generated by $\delta_\alpha + \text{ad } ih$ is given as $\text{Ad } u_t \alpha_t$, where u is the α -cocycle with $du_t/dt|_{t=0} = ih$, and $\omega^{(h)}$ is equal to $\omega(w^{-1} \cdot wv_{ic})$, where u_t is expressed as $wv_t \alpha_t(w^{-1})$ with $t \mapsto v_t$ entire.)

For $s \in [0, 1]$ let $\omega_s = \omega^{(b(s))}$, which is a positive linear functional satisfying the c -KMS condition with respect to the generator $\delta_\alpha + \text{ad } ib(s)$. This implies that $\omega_s \gamma_s$ is a c -KMS positive linear functional with respect to $\gamma_s^{-1}(\delta_\alpha + \text{ad } ib(s))\gamma_s = \delta_\alpha$.

Let $s_1, s_2 \in [0, 1]$ and define a positive linear functional φ on $A \otimes M_2$ by

$$\varphi(a) = \omega_{s_1}(a_{11}) + \omega_{s_2}(a_{22})$$

for $a = (a_{ij}) \in A \otimes M_2$. Then φ is a c -KMS positive linear functional for the flow β of $A \otimes M_2$ defined by

$$\beta_t((a_{ij})) = \text{Ad} \begin{pmatrix} u_t^{(b(s_1))} & 0 \\ 0 & u_t^{(b(s_2))} \end{pmatrix} (\alpha_t(a_{ij})),$$

where $u_t^{(h)}$ is the α -cocycle determined by $du_t^{(h)}/dt|_{t=0} = ih$ (see [10]). The generator δ_β of β is given by

$$\delta_\beta((a_{ij})) = \begin{pmatrix} (\delta_\alpha + \text{ad } ib(s_1))(a_{11}) & \delta_\alpha(a_{12}) + ib(s_1)a_{12} - a_{12}ib(s_2) \\ \delta_\alpha(a_{21}) - a_{21}ib(s_1) + ib(s_2)a_{21} & (\delta_\alpha + \text{ad } ib(s_2))(a_{22}) \end{pmatrix}.$$

Fix $\epsilon \in (0, 1/2)$ and a C^∞ -function f on \mathbf{R} with compact support such that $f(0) = 0$ and $f(t) = t^{-1/2}$ on $[1 - \epsilon, 1]$. Let e be a projection in C . We choose $s_1, s_2 \in [0, 1]$ so that

$$\|\gamma_{s_1}(e) - \gamma_{s_2}(e)\| < \epsilon.$$

Let

$$x = \begin{pmatrix} 0 & \gamma_{s_1}(e)\gamma_{s_2}(e) \\ 0 & 0 \end{pmatrix}.$$

Then

$$x^*x = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{s_2}(e)\gamma_{s_1}(e)\gamma_{s_2}(e) \end{pmatrix}$$

and $\text{Sp}(x^*x) \subset \{0\} \cup (1 - \epsilon, 1]$. Let $v = xf(x^*x)$. Then v is a partial isometry such that

$$vv^* = \begin{pmatrix} \gamma_{s_1}(e) & 0 \\ 0 & 0 \end{pmatrix}, \quad v^*v = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{s_2}(e) \end{pmatrix}.$$

Since all the components of $\delta_\beta(x)$ are zero except for the (1,2) component and $(\delta_\alpha + \text{ad } ib(s))\gamma_s(e) = 0$, we have that

$$\begin{aligned} \|\delta_\beta(x)\| &= \|\delta_\beta(x)_{12}\| \\ &= \|\gamma_{s_1}(e)ib(s_1)\gamma_{s_2}(e) - \gamma_{s_1}ib(s_2)\gamma_{s_2}(e)\| \\ &\leq \|b(s_1) - b(s_2)\|. \end{aligned}$$

Since $\|\delta_\beta(x^*x)\| \leq 2\|b(s_1) - b(s_2)\|$, and

$$\delta_\beta(f(x^*x)) = \delta_\beta\left(\int \hat{f}(s)e^{isx^*x}ds\right) = \int \hat{f}(s) \int_0^1 e^{itsx^*x}is\delta_\beta(x^*x)e^{i(1-t)sx^*x}dtds,$$

it follows that

$$\|\delta_\beta(f(x^*x))\| \leq \int |\hat{f}(s)s|ds \cdot \|\delta_\beta(x^*x)\|.$$

Thus there is a constant $C > 0$ such that

$$\|\delta_\beta(v)\| \leq C\|b(s_1) - b(s_2)\|.$$

By the KMS condition on φ we have a continuous function f on the strip S_c between $\Im z = 0$ and $\Im z = c$, analytic in the interior, such that

$$\begin{aligned} f(t) &= \varphi(v\beta_t(v^*)), \quad t \in \mathbf{R}, \\ f(t+ic) &= \varphi(\beta_t(v^*)v), \quad t \in \mathbf{R}. \end{aligned}$$

Then f is differentiable on S_c including the boundary and satisfies that

$$\begin{aligned} f'(t) &= \varphi(v\beta_t(\delta_\beta(v^*))), \quad t \in \mathbf{R}, \\ f'(t+ic) &= \varphi(\beta_t(\delta_\beta(v^*))v), \quad t \in \mathbf{R}. \end{aligned}$$

Hence it follows that

$$\sup_{z \in S_c} |f'(z)| \leq \sup_{z \in \partial S_c} |f'(z)| \leq C \max\{\|\omega_{s_1}\|, \|\omega_{s_2}\|\} \|b(s_1) - b(s_2)\|,$$

which implies that

$$\begin{aligned} |\omega_{s_2}(\gamma_{s_2}(e)) - \omega_{s_1}(\gamma_{s_1}(e))| &= |f(ic) - f(0)| \\ &\leq |c|CM\|b(s_1) - b(s_2)\|, \end{aligned}$$

where $M = \max\{\|\omega_s\| \mid s \in [0, 1]\}$. We let $m = \min\{\|\omega_s\| \mid s \in [0, 1]\}$ and choose $t_0 = 0 < t_1 < \dots < t_k = 1$ such that

$$|c|C\frac{M}{m}\left(1 + \frac{M}{m}\right)\text{Length}(b(s), s \in [t_{i-1}, t_i]) < \frac{1}{4}.$$

Then for any projection $e \in C$, we subdivide each interval $[t_{i-1}, t_i]$ into $s_0 = t_{i-1} < s_1 < \dots < s_\ell = t_i$ such that

$$\|\gamma_{s_{j-1}}(e) - \gamma_{s_j}(e)\| < \epsilon,$$

and apply the above argument to each pair s_{j-1}, s_j to obtain that

$$|\omega_{t_{i-1}}\gamma_{t_{i-1}}(e) - \omega_{t_i}\gamma_{t_i}(e)| \leq |c|CM\text{Length}(b(s), s \in [t_{i-1}, t_i]).$$

Thus we have that for any projection $e \in C$

$$\begin{aligned} \left| \frac{\omega_{t_{i-1}} \gamma_{t_{i-1}}(e)}{\omega_{t_{i-1}}(1)} - \frac{\omega_{t_i} \gamma_{t_i}(e)}{\omega_{t_i}(1)} \right| &\leq \frac{1}{m} |\omega_{t_{i-1}} \gamma_{t_{i-1}}(e) - \omega_{t_i} \gamma_{t_i}(e)| + M \left| \frac{\omega_{t_i}(1) - \omega_{t_{i-1}}(1)}{\omega_{t_i}(1) \omega_{t_{i-1}}(1)} \right| \\ &\leq \left(\frac{1}{m} + \frac{M}{m^2} \right) |c| \text{CMLength}(b(s), s \in [t_{i-1}, t_i]) \\ &\leq 1/4. \end{aligned}$$

Let

$$\varphi_t = \frac{\omega_t \gamma_t}{\omega_t(1)}$$

and recall that φ_t is a factorial c -KMS state with respect to α . Since $\varphi_t = \varphi_t E$ with E the projection onto C and $\|(\varphi_{t_{i-1}} - \varphi_{t_i})|C\| \leq 1/2$, we have that $\|\varphi_{t_{i-1}} - \varphi_{t_i}\| \leq 1/2$. Hence $\varphi_{t_{i-1}} = \varphi_{t_i}$. Thus we conclude that $\varphi_0 = \varphi_1$ or $\Phi(\gamma_0^{-1})(\omega) = \Phi(\gamma_1^{-1})(\omega)$ for $\omega \in \partial(K_c)$. This implies that $\Phi(\gamma_0)(\omega) = \Phi(\gamma_1)(\omega)$ for $\omega \in \partial(K_c)$ as well.

Remark 2.4 Among the quantum lattice models, two or more dimensional, there are long-range interactions which exhibit continuous symmetry breaking. Let α be the flow generated by such an interaction and let γ be an action of \mathbf{T} which exactly commutes with α and acts non-trivially on the simplex of c -KMS states at some inverse temperature $c > 0$. Suppose that α is an inner perturbation of an AF flow, i.e., $\delta = \delta_\alpha + \text{ad } ib$ is the generator of an AF flow. Since $\gamma_t \delta \gamma_t^{-1} = \delta + \text{ad } i(\gamma_t(b) - b)$, we can conclude that $t \mapsto \gamma_t(b)$ is not rectifiable; thus at least b is not in the domain of the generator of γ . (Note we still cannot conclude that α is not an inner perturbation of an AF flow.)

3 Property of real rank zero

First we generalize H. Lin's result [16] and then use it to prove that *the almost fixed point algebra for an AF flow has real rank zero*.

Theorem 3.1 *For every $\epsilon > 0$ there is a $\nu > 0$ satisfying the following condition: For any $n \in \mathbf{N}$ and any pair $a, b \in (M_n)_{sa}$ with $\|b\| \leq 1$ and $\|[a, b]\| < \nu$ there exists a pair $a_1, b_1 \in (M_n)_{sa}$ such that $\|a - a_1\| < \epsilon$, $\|b - b_1\| < \epsilon$, and $[a_1, b_1] = 0$.*

If we impose the extra condition that $\|a\| \leq 1$ for a , then this result is due to H. Lin (see also [12]). Our proof is to reduce Theorem 3.1 to Lin's result.

Lemma 3.2 *Let f be a C^∞ -function on \mathbf{R} such that $f \geq 0$, $\int f(t) dt = 1$, and $\text{supp } \hat{f} \subset (-1/2, 1/2)$. For any pair a, b elements in a C^* -algebras-algebra such that $a = a^*$, define*

$$b_1 = \int f(t) e^{ita} b e^{-ita} dt.$$

Then it follows that

$$\begin{aligned}\|b - b_1\| &\leq \int f(t)|t|dt \cdot \|[a, b]\|, \\ \|[a, b_1]\| &\leq \int f(t)dt \cdot \|[a, b]\|.\end{aligned}$$

Proof. This follows from the following computations:

$$\begin{aligned}b_1 - b &= \int f(t)(e^{ita}be^{-ita} - b)dt, \\ &= \int f(t) \int_0^t e^{isa}[ia, b]e^{-isa}dsdt, \\ [a, b_1] &= \int f(t)e^{ita}[a, b]e^{-ita}dt.\end{aligned}$$

Remark 3.3 If we denote by E_a the spectral measure of a , then the b_1 defined in the above lemma satisfies that

$$E_a(-\infty, t - 1/4] b_1 E_a[t + 1/4, \infty) = 0$$

for any $t \in \mathbf{R}$, [6, Proposition 3.2.43].

Lemma 3.4 For any $\epsilon > 0$ there is a $\nu > 0$ satisfying the following condition: For any $n \in \mathbf{N}$, any pair $a, b \in (M_n)_{sa}$ with $\|b\| \leq 1$ and $\|[a, b]\| < \nu$, and any $t \in \mathbf{R}$ there exists a projection $p \in M_n$ such that

$$\begin{aligned}E_a[t + 1/4, \infty) &\leq p \leq E_a(t - 1/4, \infty), \\ \|[a, p]\| &< \epsilon, \\ \|[b, p]\| &< \epsilon,\end{aligned}$$

where E_a denotes the spectral measure associated with a .

Proof. Let f be a C^∞ -function on \mathbf{R} such that

$$f(t) = \begin{cases} 0 & t \leq -1/4 \\ 1 & t \geq 1/4 \end{cases}$$

and $f(t) \approx 2t + 1/2$, $0 < f(t) < 1$ for $t \in (-1/4, 1/4)$. Define a function g_N on \mathbf{R} for a large N by

$$g_N(t) = \min\{f(t), f(\sqrt{N} - t/\sqrt{N})\}.$$

The function g_N is C^∞ if $N - \sqrt{N}/4 > 1/4$ and satisfies that

$$g_N(t) = \begin{cases} 1 & t \in [1/4, N - \sqrt{N}/4] \\ 0 & t \leq -1/4 \text{ or } t \geq N + \sqrt{N}/4 \end{cases}.$$

If $N - \sqrt{N}/4 \geq \|a\|$, we have that

$$f(a) = g_N(a) = \int \hat{g}_N(t) e^{ita} dt,$$

where

$$\hat{g}_N(t) = \frac{1}{2\pi} \int g_N(s) e^{-its} ds.$$

Since

$$\|[b, f(a)]\| = \int \hat{g}_N(t) [b, e^{ita}] dt = \int \hat{g}_N(t) \int_0^t e^{i(t-s)a} [b, ia] e^{isa} ds dt,$$

we have that

$$\|[b, f(a)]\| \leq \int |\hat{g}_N(t) t| dt \cdot \|[b, a]\|.$$

Since

$$it\hat{g}_N(t) = -\frac{1}{2\pi} \int g_N(s) \frac{d}{ds} e^{-its} ds = \frac{1}{2\pi} \int g'_N(s) e^{-its} ds,$$

it follows for $t \neq 0$ that:

$$\begin{aligned} \lim_{N \rightarrow \infty} it\hat{g}_N(t) &= \frac{1}{2\pi} \int f'(s) e^{-its} ds - \lim_{N \rightarrow \infty} \frac{1}{2\pi\sqrt{N}} \int f'(\sqrt{N} - s/\sqrt{N}) e^{-its} ds \\ &= \frac{1}{2\pi} \int f'(s) e^{-its} ds \\ &= \hat{f}'(t). \end{aligned}$$

Since the above convergence can be estimated by

$$\frac{1}{2\pi\sqrt{N}} \int f'(\sqrt{N} - s/\sqrt{N}) e^{-its} ds = \frac{e^{-iNt}}{2\pi} \int f'(u) e^{i\sqrt{N}tu} du = e^{-iNt} \hat{f}'(-\sqrt{N}t),$$

we obtain that

$$\|[b, f(a)]\| \leq C \|[b, a]\|,$$

where

$$C = \int |\hat{f}'(t)| dt.$$

If $\|[a, b]\|$ is small enough, then $\|[b, f(a)]\|$ is so small with $\|f(a)\| \leq 1$ and $\|b\| \leq 1$ that H. Lin's result is applicable to the pair $b, c = f(a)$. Thus we obtain $b_1, c_1 \in (M_n)_{sa}$ such that

$$\|b - b_1\| \approx 0, \|c - c_1\| \approx 0, [b_1, c_1] = 0.$$

Let q be the spectral projection of c_1 corresponding to $(1/2, \infty)$. Since $\|c - c_1\| \approx 0$, and the spectral projection of c corresponding to $(0, \infty)$ (resp. $[1, \infty)$) is $E_a(-1/4, \infty)$ (resp. $E_a[1/4, \infty)$), we have that

$$\begin{aligned} E_a(-1/4, \infty)q &\approx q, \\ E_a[1/4, \infty)q &\approx E_a[1/4, \infty), \end{aligned}$$

where the approximation depends only on $\|c - c_1\|$, which in turn depends only on $\|[a, b]\|$. Hence in particular $E_a(-1/4, 1/4)$ almost commutes with q . By functional calculus we construct a projection q_0 from $E_a(-1/4, 1/4)qE_a(-1/4, 1/4)$ and set $p = q_0 + E_a[1/4, \infty)$, which is close to q , dominates $E_a[1/4, \infty)$ and is dominated by $E_a(-1/4, \infty)$. Since $[p, a] = [p, E_a(-1/4, 1/4)a] = [p - q, E_a(-1/4, 1/4)a] + [q, E_a(-1/4, 1/4)(a - f(a)/2 + 1/4)]$, we obtain that $\|[p, a]\| \leq 2\|p - q\| + 2\sup_{t \in (-1/4, 1/4)} |t - f(t)/2 + 1/4|$. Since $[p, b] = [p - q, b] + [q, b] = [p - q, b] + [q, b - b_1] + [q, b_1]$, we obtain that $\|[p, b]\| \leq 2\|b\|\|p - q\| + 2\|b - b_1\|$. Hence p is the desired projection for $t = 0$. We can apply this argument to the pair $a - t1, b$ to obtain the desired projection p for $t \in \mathbf{R}$.

Lemma 3.5 *For any $\epsilon > 0$ there exists a $\nu > 0$ satisfying the following condition: For any $n \in \mathbf{N}$, any pair $a, b \in (M_n)_{sa}$ with $\|b\| \leq 1$ and $\|[a, b]\| < \nu$ there is a family $\{p_k : k \in \mathbf{Z}\}$ of projections in M_n such that*

$$\begin{aligned} [E_a(j - 1/4, j + 1/4), p_k] &= 0, \quad j, k \in \mathbf{N}, \\ E_a[k + 1/4, k + 3/4] &\leq p_k \leq E_a(k - 1/4, k + 5/4), \\ \|[a, p_k]\| &< \epsilon, \\ \|[b, p_k]\| &< \epsilon, \\ \sum_k p_k &= 1, \end{aligned}$$

where $p_k = 0$ except for a finite number of k .

Proof. By the previous lemma we choose a $\nu > 0$ such that for a pair a, b as above, there are projections $e_k, k \in \mathbf{Z}$ such that

$$\begin{aligned} E_a[k + 1/4, \infty) &\leq e_k \leq E_a(k - 1/4, \infty), \\ \|[a, e_k]\| &< \epsilon/2, \\ \|[b, e_k]\| &< \epsilon/2. \end{aligned}$$

Then we set

$$p_k = e_k(1 - e_{k+1}) = e_k - e_{k+1}.$$

Then $\{p_k\}$ is a family of projections with $\sum_k p_k = 1$. Since

$$E_a(-\infty, k + 3/4] \leq 1 - e_{k+1} \leq E_a(-\infty, k + 5/4),$$

we see that $\{p_k\}$ satisfies the required conditions.

Proof of Theorem 3.1

By Lemma 3.2 we may assume that we are given a pair $a, b \in (M_n)_{sa}$ such that $\|b\| \leq 1$, $\|[a, b]\| < \nu$, and $E_a(-\infty, t - 1/4]bE_a[t + 1/4, \infty) = 0$ for any $t \in \mathbf{R}$, where $\nu > 0$ is given in the previous lemma. Choosing the projections $\{p_k\}$ given there, we claim that

$$\begin{aligned} \|a - \sum_k p_k a p_k\| &< 4\epsilon, \\ \|b - \sum_k p_k b p_k\| &< 4\epsilon. \end{aligned}$$

To prove this note that if $|i - j| > 1$ then $p_i a p_j = 0 = p_i b p_j$. Since

$$a - \sum_k p_k a p_k = \sum_k p_k a p_{k+1} + \sum_k p_{k+1} a p_k = \sum_k [p_k, a] p_{k+1} + \sum_k p_{k+1} [a, p_k],$$

and

$$\left\| \sum_k [p_{2k}, a] p_{2k+1} \right\|^2 = \left\| \sum_k [p_{2k}, a] p_{2k+1} [a, p_{2k}] \right\| = \sup_k \left\| [p_{2k}, a] p_{2k+1} [a, p_{2k}] \right\| < \epsilon^2,$$

and similar computations hold for three other sums and for b , we get the above assertions. We then apply H. Lin's result [16] to each pair $p_k a p_k, p_k b p_k$ which satisfies

$$\left\| [p_k a p_k, p_k b p_k] \right\| \leq \left\| p_k [a, p_k] b p_k \right\| + \left\| p_k [a, b] p_k \right\| + \left\| p_k [b, p_k] a p_k \right\| < 2\epsilon + \nu.$$

Assuming that $2\epsilon + \nu$ is sufficiently small, we obtain a pair a_k, b_k in $(p_k M_n p_k)_{sa}$ such that

$$p_k a p_k \approx a_k, \quad p_k b p_k \approx b_k, \quad [a_k, b_k] = 0.$$

We set $a' = \sum_k a_k$ and $b' = \sum_k b_k$. Then it follows that $[a', b'] = 0$ and $a \approx a', b \approx b'$ because of the inequality

$$\|a - a'\| \leq \left\| a - \sum_k p_k a p_k \right\| + \sup_k \|p_k a p_k - a_k\|$$

and a similar inequality for b, b' . This completes the proof.

For a flow α of a unital simple AF algebra we denote by δ_α the generator of α as before. We introduce the following condition on α , which we may express by saying that *the almost fixed point algebra for α has real rank zero*.

Condition F0: For any $\epsilon > 0$ there exists a $\nu > 0$ satisfying the following condition: If $h = h^* \in \mathcal{D}(\delta_\alpha)$ satisfies that $\|h\| \leq 1$ and $\|\delta_\alpha(h)\| < \nu$ there exists a pair $k = k^* \in \mathcal{D}(\delta_\alpha)$ and $b = b^* \in A$ such that $\|h - k\| < \epsilon$, $\|b\| < \epsilon$, $(\delta_\alpha + \text{ad } ib)(k) = 0$, and $\text{Sp}(k)$ is finite.

In the above condition let C be the (finite-dimensional) $*$ -subalgebra generated by k . Then h is approximated by an element of C within distance ϵ and $\|\delta_\alpha|_C\| < 2\epsilon$.

We recall from [15, Proposition 3.1] that a flow α is a cocycle perturbation of an AF flow if and only if the domain $\mathcal{D}(\delta_\alpha)$ contains a canonical AF masa. (A maximal abelian AF C^* -subalgebra C of a AF C^* -algebra A is called *canonical* if there is an increasing sequence (A_n) of finite-dimensional $*$ -subalgebras of A with dense union such that $C \cap A_n \cap A'_{n-1}$ is maximal abelian in $A_n \cap A'_{n-1}$ for each n with $A_0 = 0$.)

Theorem 3.6 *Let α be a flow of a non type I simple AF C^* -algebra. If $\mathcal{D}(\delta_\alpha)$ contains a canonical AF masa, then the above condition F0 is satisfied, i.e., the almost fixed point algebra has real rank zero.*

Proof. Let $\epsilon > 0$. We choose a $\nu > 0$ as in Theorem 3.1.

Let $h = h^* \in \mathcal{D}(\delta_\alpha)$ be such that $\|h\| \leq 1$ and $\|\delta_\alpha(h)\| < \nu$. There exists a $c = c^* \in A$ such that $\|c\| < \min\{(\nu - \|\delta_\alpha(h)\|)/2, \epsilon\}$ and $\delta_\alpha + \text{ad } ic$ generates an AF flow. Explicitly let $\{A_n\}$ be an increasing sequence of finite-dimensional *-subalgebras of A with dense union such that $A_n \subset \mathcal{D}(\delta_\alpha)$ and $(\delta_\alpha + \text{ad } ic)(A_n) \subset A_n$ for each n . There exists a sequence $\{h_n\}$ such that $h_n = h_n^* \in A_n$, $\|h_n\| \leq 1$, $\|h_n - h\| \rightarrow 0$, and $\|\delta_\alpha(h - h_n)\| \rightarrow 0$. Since $\|(\delta_\alpha + \text{ad } ic)(h)\| < \nu$, we have an n , $h_0 = h_0^* \in A_n$, and $a = a^* \in A_n$ such that $\|h_0\| \leq 1$, $\|h - h_0\| < \epsilon$, $\|(\delta_\alpha + \text{ad } ic)(h_0)\| < \nu$, and $(\delta_\alpha + \text{ad } ic)|_{A_n} = \text{ad } ia|_{A_n}$. Since A_n is a finite direct sum of matrix algebras, Theorem 3.1 is applicable to the pair a, h_0 . Thus there exists a pair $a_1, h_1 \in (A_n)_{sa}$ such that $\|a - a_1\| < \epsilon$, $\|h_0 - h_1\| < \epsilon$, and $[a_1, h_1] = 0$. Let $b = a_1 - a + c$. Then we have that $\|h - h_1\| < 2\epsilon$, $\|b\| < 2\epsilon$, $(\delta_\alpha + \text{ad } ib)(h_1) = 0$, and $\text{Sp}(h_1)$ is finite.

In the special case that α is periodic, the fact that the almost fixed point algebra has real rank zero simply means that the fixed point algebra has real rank zero:

Proposition 3.7 *Let A a non type I simple AF C^* -algebra and α a periodic flow of A . Then the following conditions are equivalent:*

1. *Condition F0 holds.*
2. *The fixed point algebra $A^\alpha = \{a \in A \mid \alpha_t(a) = a\}$ has real rank zero.*

Proof. We may suppose that $\alpha_1 = \text{id}$. Suppose (1); we have to show that $\{h \in A_{sa}^\alpha \mid \text{Sp}(h) \text{ is finite}\}$ is dense in A_{sa}^α [9]. Let $h = h^* \in A^\alpha$, $\epsilon > 0$, and $n \in \mathbb{N}$. There exist an $h_1 \in \mathcal{D}(\delta_\alpha)_{sa}$ and $b \in A_{sa}$ such that $\|h - h_1\| < \epsilon$, $\|b\| < \epsilon$, $(\delta_\alpha + \text{ad } ib)(h_1) = 0$, and $\text{Sp}(h_1)$ is finite. We approximate h_1 by an element $h_2 = \sum_{k=-n}^n (k/n)p_k$ in the *-subalgebra generated by h_1 , where (p_k) is a mutually orthogonal family of projections. We may assume that $\|h_1 - h_2\| \leq 1/n$ and hence that $\|h - h_2\| < \epsilon + 1/n$. Note that we still have that $(\delta_\alpha + \text{ad } ib)(h_2) = 0$. Since $\|\alpha_t(p_k) - p_k\| \leq |t|\|\delta(p_k)\| < 2|t|\epsilon$, we have that

$$\left\| \int_0^1 \alpha_t(p_k) - p_k \right\| < \epsilon$$

for $k = -n, -n+1, \dots, n$. If ϵ is sufficiently small, then by functional calculus we inductively define a projection $q_k \in A^\alpha$ from $(1 - \sum_{j=-n}^{k-1} q_j) \int \alpha_t p_k dt (1 - \sum_{j=-n}^{k-1} q_j)$, which belongs to A^α , such that $q_k \approx p_k$ and q_k is orthogonal to $\sum_{j=-n}^{k-1} q_j$. Then $h_3 = \sum_{k=-n}^n (k/n)q_k \approx \sum_{k=-n}^n (k/n)p_k = h_2$, where the approximation is of the order of ϵ times some function of n . Since $h_3 \in A^\alpha$, we reach the conclusion by choosing $\epsilon > 0$ sufficiently small.

The converse implication is easy to show.

If α is not periodic, we can still re-formulate Condition F0 as follows, further justifying the terminology that the almost fixed point algebra has real rank zero. We denote by ℓ^∞ the C^* -algebra of bounded sequences in A and by c_0 the closed ideal of ℓ^∞ consisting of sequences converging to zero. Then we set A^∞ to be the quotient ℓ^∞/c_0 . The flow α on

A induces a flow $\bar{\alpha}$ on ℓ^∞ by $\bar{\alpha}_t(x) = (\alpha_t(x_n))$ for $x = (x_n)$. But since $\bar{\alpha}$ is not strongly continuous (if α is not uniformly continuous), we choose the C^* -subalgebra ℓ_α^∞ consisting of $x \in \ell^\infty$ with $t \mapsto \bar{\alpha}_t(x)$ continuous. Since $\ell_\alpha^\infty \supset c_0$ and c_0 is $\bar{\alpha}$ -invariant, $\bar{\alpha}$ induces a (strongly continuous) flow on the quotient $A_\alpha^\infty = \ell_\alpha^\infty / c_0$, which will also be denoted by α . Note that A_α^∞ is inseparable even if A is separable. See [13].

Proposition 3.8 *Let A be a C^* -algebra and α a flow of A . Then the following conditions are equivalent:*

1. *Condition F0 holds.*
2. *The fixed point algebra $(A_\alpha^\infty)^\alpha$ has real rank zero.*

Proof. Suppose (1) and let $h \in (A_\alpha^\infty)_{sa}^\alpha$. We take a representative $(h_n) \in \ell_\alpha^\infty$ of h such that $h_n^* = h_n$ for all n . Taking a non-negative C^∞ function f with integral 1, we may replace each h_n by $\int f(t)\alpha_t(h_n)dt$. Thus we can assume that $h_n \in \mathcal{D}(\delta_\alpha)$ and $\|\delta_\alpha(h_n)\| \rightarrow 0$. Then for any $\epsilon > 0$ there exists a sequence of pairs $k_n \in \mathcal{D}(\delta_\alpha)_{sa}$ and $b_n \in A_{sa}$ such that $\|h_n - k_n\| < \epsilon$, $\|b_n\| \rightarrow 0$, $(\delta_\alpha + \text{ad } ib_n)(k_n) = 0$, and $\text{Sp}(k_n)$ is finite and independent of n . Hence $k = (k_n) + c_0 \in A_\alpha^\infty$ satisfies that $\|h - k\| \leq \epsilon$, $\delta_\alpha(k) = 0$, and $\text{Sp}(k)$ is finite. This shows that $(A_\alpha^\infty)^\alpha$ has real rank zero [9].

Suppose (2). If Condition F0 does not hold, we find an $\epsilon > 0$ and a sequence (h_n) in $\mathcal{D}(\delta_\alpha)_{sa}$ such that $\|h_n\| = 1$, $\|\delta_\alpha(h_n)\| \rightarrow 0$, and such that if $k \in \mathcal{D}(\delta_\alpha)_{sa}$ and $b \in A_{sa}$ satisfy that $\|h - k\| < \epsilon$, $\|b\| < \epsilon$, and $\text{Sp}(k)$ is finite, then $(\delta_\alpha + \text{ad } ib)(k) \neq 0$. Since $h = (h_n) + c_0 \in A_\alpha^\infty$ belongs to $(A_\alpha^\infty)^\alpha$, we have a $k \in (A_\alpha^\infty)_{sa}^\alpha$ such that $\|h - k\| < \epsilon$ and $\text{Sp}(k)$ is finite. By choosing an appropriate representative (consisting of projections) for each minimal spectral projection of k , we find a representative (k_n) of k such that $k_n^* = k_n$, $\text{Sp}(k_n) = \text{Sp}(k)$, and $\|\delta_\alpha(k_n)\| \rightarrow 0$. This is a contradiction.

We recall here a condition on a flow α considered in [15].

Condition F1: For any $\epsilon > 0$ there exists a $\nu > 0$ satisfying the following condition: If $u \in \mathcal{D}(\delta_\alpha)$ is a unitary with $\|\delta_\alpha(u)\| < \nu$ there is a continuous path (u_t) of unitaries in A such that $u_0 = 1$, $u_1 = u$, $u_t \in \mathcal{D}(\delta_\alpha)$, and $\|\delta_\alpha(u_t)\| < \epsilon$ for $t \in [0, 1]$.

In the above condition we can choose the path (u_t) to be continuous in the Banach $*$ -algebra $\mathcal{D}(\delta_\alpha)$. We express this condition by saying that *the almost fixed point algebra for α has trivial K_1* . What we have shown in [15] is that if α is an inner perturbation of an AF flow then the above condition holds. Actually by using the full strength of Lemma 5.1 of [2], one can show that the following stronger condition holds:

Condition F1': For any $\epsilon > 0$ there exists a $\nu > 0$ satisfying the following condition: If $u \in \mathcal{D}(\delta_\alpha)$ is a unitary with $\|\delta_\alpha(u)\| < \nu$ there is a rectifiable path (u_t) of unitaries in A such that $u_0 = 1$, $u_1 = u$, $u_t \in \mathcal{D}(\delta_\alpha)$, $\|\delta_\alpha(u_t)\| < \epsilon$ for $t \in [0, 1]$, and the length of (u_t) is bounded by C , where C is a universal constant (smaller than $3\pi + \epsilon$ for example).

Then one can show the following:

Proposition 3.9 *Let A be a unital C^* -algebra and α a flow of A . Then the following conditions are equivalent:*

1. *Condition F1' holds.*
2. *The unitary group of the fixed point algebra $(A_\alpha^\infty)^\alpha$ is path-wise connected; moreover any unitary is connected to 1 by a continuous path of unitaries whose length is bounded by a universal constant.*

We will leave the proof to the reader.

Remark 3.10 If A is a unital simple AF C^* -algebra, one can construct a periodic flow α of A , by using the general classification theory of locally representable actions [4], such that the almost fixed point algebra for α has real rank zero but does not have trivial K_1 .

Proposition 3.11 *Let A be a unital simple AF C^* -algebra. Then there exists a flow α of A such that $\mathcal{D}(\delta_\alpha)$ is AF and the almost fixed point algebra for α does not have real rank zero but has trivial K_1 (i.e., F0 holds but not F1).*

Proof. We shall use a construction used in the proof of 2.1 of [15]. Let (A_n) be an increasing sequence of finite-dimensional $*$ -subalgebras of A such that $A = \overline{\cup_n A_n}$ and let $A_n = \bigoplus_{j=1}^{k_n} A_{nj}$ be the direct sum decomposition of A_n into full matrix algebras A_{nj} . Since $K_0(A_n) \cong \mathbf{Z}^{k_n}$, we obtain a sequence of K_0 groups:

$$\mathbf{Z}^{k_1} \xrightarrow{\chi_1} \mathbf{Z}^{k_2} \xrightarrow{\chi_2} \dots,$$

where χ_n is the positive map of $K_0(A_n) = \mathbf{Z}^{k_n}$ into $K_0(A_{n+1}) = \mathbf{Z}^{k_{n+1}}$ induced by the embedding $A_n \subset A_{n+1}$. Since $K_0(A)$ is a simple dimension group different from \mathbf{Z} , we may assume that $\min_{ij} \chi_n(i, j) \rightarrow \infty$ as $n \rightarrow \infty$.

By using (A_n) we will express A as an inductive limit of C^* -algebras $A_n \otimes C[0, 1]$. First we define a homomorphism $\varphi_{n,ij}$ of $A_{nj} \otimes C[0, 1]$ into $A_{nj} \otimes M_{\chi_n(i,j)} \otimes C[0, 1]$ as follows: If $i = j = 1$ then

$$\varphi_{n,11}(x)(t) = x(t) \oplus \bigoplus_{\ell=0}^{\chi_n(1,1)-2} x\left(\frac{t+\ell}{\chi_n(1,1)-1}\right),$$

otherwise

$$\varphi_{n,ij}(x)(t) = \bigoplus_{\ell=0}^{\chi_n(1,1)-1} x\left(\frac{t+\ell}{\chi_n(1,1)}\right).$$

Especially $\varphi_{n,ij}(x)$ is of diagonal form in the matrix algebra over $A_{nj} \otimes C[0, 1]$. Then embedding

$$\bigoplus_{j=1}^{k_n} A_{nj} \otimes M_{\chi_n(i,j)} \otimes C[0, 1]$$

into $A_{n+1,i} \otimes C[0, 1]$, $(\varphi_{n,ij})$ defines an injective homomorphism $\varphi_n : A_n \otimes C[0, 1] \rightarrow A_{n+1} \otimes C[0, 1]$. Then it follows that the inductive limit C^* -algebra of $(A_n \otimes C[0, 1], \varphi_n)$ is isomorphic to the original A ; we have thus expressed A as $\overline{\cup_n B_n}$ where $B_n = A_n \otimes C[0, 1] \subset B_{n+1}$ [11].

We will define a flow or one-parameter automorphism group α of A such that $\alpha_t(B_n) = B_n$ and $\alpha_t|_{B_n}$ is inner, i.e., α is locally representable for the sequence (B_n) . First we define a sequence (H_n) with self-adjoint $H_n \in A_n \otimes 1 \subset B_n$ inductively. Let $H_1 \in A_1 \otimes 1 \subset B_1$ and let $H_n = H_{n-1} + \sum_i \sum_j h_{n,ij}$, where

$$h_{n,ij}^* = h_{n,ij} \in 1 \otimes M_{\chi_{n-1}(i,j)} \otimes 1 \subset A_{n-1,j} \otimes M_{\chi_{n-1}(i,j)} \otimes 1 \subset B_n.$$

We define $\alpha_t|_{B_n}$ by $\text{Ad } e^{itH_n}|_{B_n}$. Since $\alpha_t|_{B_n} = \text{Ad } e^{itH_{n+1}}|_{B_n}$ from the definition of H_{n+1} , $(\alpha_t|_{B_n})$ defines a flow α of A .

We fix H_1 and $h_{n,ij}$ in the following way: $\|h_{n,ij}\| \leq 1/2$ except for $h_{n,11}$ which is defined by

$$h_{n,11} = 1 \oplus 0 \oplus \cdots \oplus \in 1 \otimes M_{\chi_{n-1}(1,1)} \otimes 1 \subset A_{n,1} \otimes C[0,1].$$

We will show that the α defined this way has the desired properties.

Let x be the identity function on the interval $[0,1]$ and let $x_n = 1 \otimes x \in 1 \otimes C[0,1] \subset B_n$. To show that $\mathcal{D}(\delta_\alpha)$ is AF, it suffices to show that for each x_n , there exists a sequence $(h_m)_{m>n}$ such that $h_m = h_m^* \in B_m$, $\text{Sp}(h_m)$ is finite, and $\|x_n - h_m\|_{\delta_\alpha} \rightarrow 0$ as $m \rightarrow \infty$. For a sufficiently large $m > n$, the image $\varphi_{mn}(x_n)$ of x_n in $B_m = A_m \otimes C[0,1]$ is almost constant as a function (into the diagonal matrices in $A_m \cap A'_m$) on $[0,1]$ except for one component, which is x and appears through the first component of $\varphi_{k,11}$ for $n \leq k < m$. We will approximate this component x by a self-adjoint element with finite spectrum by using the part appearing through the components of $\varphi_{k,11}$ other than the first; they are the direct sum of $M = \prod_{k=n}^{m-1} (\chi_k(1,1) - 1)$ components $x(\frac{t+\ell}{M})$, $\ell = 0, 1, \dots, M-1$. There is a standard procedure to approximate the sum of these $M+1$ components by a self-adjoint element k with finite spectrum [2]. Since $H_m - H_n$ is $m-n$ on the support projection of x and 0 on the support projections of the other components, the $\|\cdot\|_{\delta_\alpha}$ norm of k is of the order of $\frac{m-n}{M} \approx 0$. (All the spectral projections of k are just constant at each point of $[0,1]$ perhaps except for a pair of projections, whose eigen-values are different only by the order of $1/M$, and which are of the form:

$$\begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix}, \begin{pmatrix} \sin^2 \theta & -\cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos^2 \theta \end{pmatrix}$$

in the space spanned by the support projection of x and one of the support projections of the other M components, where θ is a function in $t \in [0,1]$ which changes from $-\pi/2$ to $\pi/2$ quickly near the point in problem. This implies that $\|\delta_\alpha(k)\| \approx \frac{m-n}{M}$ and $\|x_n - k\| \approx 1/M$ for the parts of k , $x_n - k$ in question.) This concludes the proof that $\mathcal{D}(\delta_\alpha)$ is AF.

Suppose that for any $\epsilon > 0$ there exists a pair of self-adjoint elements $h, b \in A$ such that $\|h\| \leq 1$, $\|b\| < \epsilon$, $\|x_1 - h\| < \epsilon$, $(\delta_\alpha + \text{ad } ib)(h) = 0$, and $\text{Sp}(h)$ is finite, where x_1 is the element of B_1 defined above. Since $\cup_m B_m$ is dense in $\mathcal{D}(\delta_\alpha)$, we may suppose that $h \in B_m$ for some m . The image $\varphi_{m,1}(x_1)$ in $B_m \cap A'_1$ is diagonal and there is a component x , whose (one-dimensional) support projection will be denoted by Q . Let $h = \sum_i \lambda_i p_i$ be

the spectral decomposition of h and define a function θ_i by $\theta_i(t) = Qp_i(t)Q$. Then we have that

$$|t - \sum_i \lambda_i \theta_i(t)| < \epsilon, \quad t \in [0, 1].$$

Since

$$\frac{1}{2} \sum_{\lambda_i > 1/2} \theta_i(0) < \sum \lambda_i \theta_i(0) < \epsilon,$$

we obtain that

$$\sum_{\lambda_i > 1/2} \theta_i(0) < 2\epsilon.$$

Since

$$1 - \epsilon < \sum \lambda_i \theta_i(1) < \frac{1}{2} \sum_{\lambda_i \leq 1/2} \theta_i(1) + \sum_{\lambda_i > 1/2} \theta_i(1) = \frac{1}{2} + \frac{1}{2} \sum_{\lambda_i > 1/2} \theta_i(1),$$

we get

$$\sum_{\lambda_i > 1/2} \theta_i(1) > 1 - 2\epsilon.$$

Thus the projection p defined by

$$p = \sum_{\lambda_i > 1/2} p_i$$

satisfies that $\|Qp(0)Q\| < 2\epsilon$ and $\|Qp(1)Q\| > 1 - 2\epsilon$. If $\epsilon < 1/4$, there must be a point $t \in [0, 1]$ such that $\|Qp(t)Q\| = 1/2$. Then since $Qp(t)(1 - Q)p(t)Q + Qp(t)Qp(t)Q = Qp(t)Q$, we have that $\|Qp(t)(1 - Q)\| = 1/2$. Since $(H_m - H_1)Q = (m - 1)Q$ and $\|(H_m - H_1)(1 - Q)\| \leq m - 3/2$, we get that $\|\delta_\alpha(Qp(1 - Q))\| = \|Q\delta_\alpha(p)(1 - Q)\| \geq 1/4$. But since $(\delta_\alpha + \text{ad } ib)(h) = 0$, we had that $\|\delta_\alpha(p)\| \leq 2\|b\| < 2\epsilon$. For a small $\epsilon > 0$ this is a contradiction. Thus we obtain that the almost fixed point algebra does not have real rank zero.

Let u be a unitary in $\mathcal{D}(\delta_\alpha)$ such that $\delta_\alpha(u) \approx 0$. Since $\cup_m B_m$ is dense in $\mathcal{D}(\delta_\alpha)$, we may suppose that $u \in B_m = A_m \otimes C[0, 1]$. Since $H_m \in A_m \otimes 1$, the condition $\delta_\alpha(u) \approx 0$ implies that $\|[u(t), H_m]\| \approx 0$ for all $t \in [0, 1]$. Define a continuous path (u_s) of unitaries in B_m by $u_s(t) = u((1 - s)t)$. This path runs from u to the constant function $u_1 : t \mapsto u(0)$ with the estimate $\|\delta_\alpha(u_s)\| \leq \|\delta_\alpha(u)\|$. By 4.1 of [15], there is a continuous path (v_s) of unitaries in A_m from $u(0)$ to 1 such that $[v_s, H_m] \approx 0$. This concludes the proof that the almost fixed point algebra has trivial K_1 .

4 The CAR algebra

Let $A = \mathcal{A}(\mathcal{H})$ be the CAR algebra over an infinite-dimensional separable Hilbert space \mathcal{H} ; we denote by a^* the canonical linear isometric map of \mathcal{H} into the creation operators

in A , [7, Section 5.2.2.1]. Note that A , as a C^* -algebra, is isomorphic to the UHF algebra of type 2^∞ . When U is a one-parameter unitary group on \mathcal{H} , we define a flow α of A by

$$\alpha_t(a^*(\xi)) = a^*(U_t\xi), \quad \xi \in \mathcal{H},$$

which will be called the quasi-free flow induced by U . If we denote by H the generator of U , i.e., $U_t = e^{itH}$, the generator δ_α of α satisfies that

$$\delta_\alpha(a^*(\xi)) = ia^*(H\xi), \quad \xi \in \mathcal{D}(H)$$

and the $*$ -subalgebra generated by $a^*(\xi)$, $\xi \in \mathcal{D}(H)$ is dense in the Banach $*$ -algebra $\mathcal{D}(\delta_\alpha)$. If H is diagonal, i.e., has a complete orthonormal family of eigenvectors, then α is an AF flow; moreover it is of *pure product type* in the sense that (A, α) is isomorphic to (M_{2^∞}, β) , where β is given as

$$\beta_t = \otimes_{n=1}^\infty \text{Ad} \begin{pmatrix} e^{i\lambda_n t} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\{\lambda_n, n \in \mathbf{Z}\}$ are the eigenvalues of H . If H is not diagonal, α acts on a part of A in an *asymptotically abelian way*; so we can conclude that α is not an AF flow. See [7, 8, 18] for details.

Proposition 4.1 *If α is a quasi-free flow of the CAR algebra $A = \mathcal{A}(\mathcal{H})$, then the almost fixed point algebra for α has trivial K_1 .*

Proof. We use the notation given before this proposition and let E be the spectral measure of H . Let $\epsilon > 0$ and let $u \in \mathcal{D}(\delta_\alpha)$ be a unitary such that $\|\delta_\alpha(u)\| < \epsilon$. Since the $*$ -subalgebra \mathcal{P} generated by

$$a^*(\xi), \quad \xi \in \bigcup E[-n, n]\mathcal{H}$$

is dense in $\mathcal{D}(\delta_\alpha)$, we can approximate u by $x \in \mathcal{P}$. Let \mathcal{M} be the (abelian) von Neumann algebra generated by $U_t = e^{itH}$, $t \in \mathbf{R}$. We may approximate u by x in a $*$ -subalgebra \mathcal{P}_1 generated by $a^*(\xi_1), a^*(\xi_2), \dots, a^*(\xi_n)$, where all $\xi_i \in E[-N, N]\mathcal{H}$ for some N . We may further impose the following conditions on ξ_1, \dots, ξ_n :

1. $\|\xi_i\| = 1$ for all i .
2. For $i \neq j$, $\overline{\mathcal{M}\xi_i} \perp \overline{\mathcal{M}\xi_j}$.
3. Denote by S_i the smallest closed subset of \mathbf{R} such that $E(S_i)\xi = \xi$. Then either S_i is a singleton or an infinite set.

The condition 1 is trivial and the condition 3 is easy to obtain. To make sure the condition 2 holds we may argue as follows. Starting with ξ_1, \dots, ξ_n let e_1 be the projection onto $\overline{\mathcal{M}\xi_1}$. Then $\xi'_1 = \xi_1 = e_1\xi_1, \xi'_2 = e_1\xi_2, \dots, \xi'_n = e_1\xi_n$ all belong to $e_1\mathcal{H}$ on which $\mathcal{M}e_1$ is a maximal abelian von Neumann algebra. Thus there are a finite number of unit vectors

$\eta_{11}, \dots, \eta_{1m}$ in $e_1\mathcal{H}$ such that the linear span of η_{1i} 's approximately contains all ξ_j' and $\overline{\mathcal{M}\eta_{1i}} \perp \overline{\mathcal{M}\eta_{1j}}$ for $i \neq j$. We apply the same argument to the remaining (at most $n-1$) elements $(1-e_1)\xi_2, (1-e_1)\xi_3, \dots, (1-e_1)\xi_n$ in $(1-e_1)\mathcal{H}$ which is left invariant under \mathcal{M} . Next, let e_2 be the projection onto $\mathcal{M}(1-e_1)\xi_2$ (assuming this is non-zero). Note that $e_2 \leq 1-e_1$. We find a finite number of unit vectors η_{2j} in $e_2\mathcal{H}$ whose linear span approximately contains $e_2(1-e_1)\xi_2 = (1-e_1)\xi_2, e_2(1-e_1)\xi_3 = e_2\xi_3, \dots, e_2(1-e_1)\xi_n = e_2\xi_n$ such that $\overline{\mathcal{M}\eta_{2i}} \perp \overline{\mathcal{M}\eta_{2j}}$ for $i \neq j$. Note that $\overline{\mathcal{M}\eta_{1i}} \perp \overline{\mathcal{M}\eta_{2j}}$ for all i, j . Repeating this procedure we obtain a finite number of unit vectors (η_{ij}) satisfying the condition 2 whose linear span approximately contains the vectors ξ_1, \dots, ξ_n .

Since the $*$ -algebra \mathcal{P}_1 is isomorphic to M_{2^n} by [7, Theorem 5.2.5], we may further assume that x is a unitary. We express x as

$$x = \sum_{\mu\nu} a_{\mu\nu} a^*(\mu) a(\nu),$$

where $\mu = (\mu_1, \dots, \mu_\ell)$ and ν ranges over the subsequences of $(1, 2, \dots, n)$ and $a^*(\mu)$ denotes

$$a^*(\xi_{\mu_1}) a^*(\xi_{\mu_2}) \cdots a^*(\xi_{\mu_\ell})$$

with $a(\nu) = a^*(\nu)^*$. (If μ is the empty sequence, then $a^*(\mu) = 1$.) Note that the coefficients $a_{\mu\nu}$ are unique; hence the condition that x is a unitary can be read from $(a_{\mu\nu})$ only, i.e., if we replace ξ_1, \dots, ξ_n by a different orthonormal family η_1, \dots, η_n and define x by the same formula with $a^*(\mu) = a^*(\eta_{\mu_1}) \cdots a^*(\eta_{\mu_n})$, then x is still a unitary.

Let

$$\eta_i = H\xi_i - (H\xi_i|\xi_i)\xi_i.$$

If $\eta_i \neq 0$ let $\xi_{i+1/2} = \eta_i/\|\eta_i\|$ and otherwise let $\xi_{i+1/2} = 0$. Let I be the subsequence of $\{1, 3/2, 2, \dots, n+1/2\}$ with $\xi_c \neq 0$. Then the vectors $\xi_c, c \in I$ form an orthonormal family. Since $H\xi_i = (H\xi_i|\xi_i)\xi_i + \|\eta_i\|\xi_{i+1/2}$, $\delta_\alpha(x)$ is of the form

$$\delta_\alpha(x) = \sum_{\sigma\tau} b_{\sigma\tau} a^*(\sigma) a(\tau),$$

where σ, τ are subsequences of I . Again the norm $\|\delta_\alpha(x)\|$ can be read from $(b_{\sigma\tau})$ only. Note that $(b_{\sigma\tau})$ depends only on $(a_{\mu\nu}), (H\xi_i|\xi_i)$, and $\|H\xi_i - (H\xi_i|\xi_i)\xi_i\|$.

By using Lemma 4.2 below, if S_i is not a singleton, we will find a continuous path of $(\xi_{it})_{0 \leq t < 1}$ of unit vectors in $\overline{\mathcal{M}\xi_i} \subset \mathcal{H}$ such that $\xi_{i0} = \xi_i$,

$$\begin{aligned} (H\xi_{it}|\xi_{it}) &= (H\xi_i|\xi_i), \\ \|H\xi_{it} - (H\xi_{it}|\xi_{it})\xi_{it}\| &= \|H\xi_i - (H\xi_i|\xi_i)\xi_i\|, \end{aligned}$$

and $\text{supp } \xi_{it}$ shrinks to a three point set as $t \rightarrow 1$, where $\text{supp } \xi$ is the smallest closed subset S of \mathbf{R} with $E(S)\xi = \xi$. And we set $\xi_{i+1/2,t} = c_i(H\xi_{it} - (H\xi_{it}|\xi_{it})\xi_{it})$, where c_i is a positive normalizing constant. If S_i is a singleton, we set $\xi_{it} = \xi_i$. By using $\xi_{it}, 1 \leq i \leq n$ instead

of ξ_i , we define $x_t \in A$ by the same formula as x . Then we have that $(x_t)_{0 \leq t < 1}$ is a continuous family of unitaries with $x_0 = x$ satisfying

$$\delta_\alpha(x_t) = \sum_{\sigma\tau} b_{\sigma\tau} a^*(\sigma) a(\tau),$$

where σ, τ are subsequences of I and $a^*(\sigma), a(\tau)$ are now defined by using ξ_{ct} , $c \in I$ instead of ξ_c . Hence it follows that $\|\delta_\alpha(x_t)\| = \|\delta_\alpha(x)\| < \epsilon$.

We will show that for a t_0 close to 1 there is a $b = b^* \in A$ such that $\|b\| < \epsilon/2$ and $\delta_\alpha + \text{ad } ib$ leaves a finite-dimensional *-subalgebra containing x_{t_0} invariant, and such that $\|(\delta_\alpha + \text{ad } ib)x_{t_0}\|$ is sufficiently small. Then by [5] we can deform x_{t_0} to 1 in that *-subalgebra keeping the norm estimate along the path.

Suppose that t_0 is sufficiently close to 1. If S_i is a singleton, we set $\eta_{i1} = \xi_i$; otherwise we choose three unit vectors $\eta_{i1}, \eta_{i2}, \eta_{i3}$ in $\overline{\mathcal{M}\xi_i}$ such that ξ_{it_0} is a linear combination of η_{ij} 's and $\text{supp } \eta_{ij}$ is contained in a sufficiently small neighborhood of some $s_{ij} \in S_i$, where s_{i1}, s_{i2}, s_{i3} are all distinct. Let P_{ij} be the projection onto the space spanned by $\eta_{ij}, H\eta_{ij}$ and define an operator T_{ij} such that $T_{ij} = P_{ij} T_{ij} P_{ij} = T_{ij}^*$ and $T_{ij} \eta_{ij} = (s_{ij}1 - H)\eta_{ij}$. Then it follows that the projections P_{ij} are mutually orthogonal and $\|T_{ij}\| \leq 2\|(s_{ij}1 - H)\eta_{ij}\|$, which is assumed to be very small. Let $T_i = \sum_j T_{ij}$ and $T = \sum_i T_i$, where we set $T_i = 0$ if S_i is a singleton. Then $\|T\| = \sup \|T_i\|$, $\text{rank}(T) \leq 6n$, and $(H + T)\eta_{ij} = s_{ij}\eta_{ij}$. We may suppose that $\text{Tr}(|T|) < \epsilon/2$. Note that the derivation of A corresponding to T is inner and given as $\text{ad } ib$, where $b = \sum \lambda_i a^*(\zeta_i) a(\zeta_i)$, if (ζ_i) is a complete orthonormal set of eigenvectors of T with (λ_i) the corresponding eigenvalues; $T\zeta_i = \lambda_i \zeta_i$. If \mathcal{P}_2 denotes the *-algebra generated by $a^*(\eta_{ij})$, then \mathcal{P}_2 is left invariant under the derivation corresponding to $H + T$, which is $\delta_\alpha + \text{ad } ib$. Hence there is an $h = h^* \in \mathcal{P}_2$ such that $(\delta_\alpha + \text{ad } ib)|_{\mathcal{P}_2} = \text{ad } ih|_{\mathcal{P}_2}$. Since $\|b\| = \text{Tr}|T| < \epsilon/2$, we have that

$$\|\text{ad } ih(x_{t_0})\| < 2\epsilon.$$

If ϵ is sufficiently small, we have by 4.1 of [15] a continuous path (y_t) of unitaries in \mathcal{P}_2 such that

$$\text{ad } ih(y_t) \approx 0.$$

Since $\|\delta_\alpha(y_t)\| \leq \|\text{ad } ih(y_t)\| + \epsilon$, this completes the proof.

Lemma 4.2 *Let S be a compact infinite subset of \mathbf{R} and ν a probability measure on S with support S . Let H be the multiplication operator by the identity function $x \mapsto x$ on $L^2(\nu)$. If $\xi \in L^2(\nu)$ has norm one, there exists a continuous path $(\xi_t)_{0 \leq t < 1}$ of unit vectors in $L^2(\nu)$ such that $\xi_0 = \xi$, $(H\xi_t|\xi_t)$ and $\|H\xi_t - (H\xi_t|\xi_t)\xi_t\| = (\|H\xi_t\|^2 - |(H\xi_t|\xi_t)|^2)^{1/2}$ are constant in t , and $\text{supp } \xi_t$ shrinks to a three-point set as $t \rightarrow 1$.*

Proof. Since both $(H\xi|\xi) = \int_S x|\xi(x)|^2 d\nu$ and

$$\|H\xi - (H\xi|\xi)\xi\|^2 = \int_S x^2|\xi(x)|^2 d\nu - \left(\int_S x|\xi(x)| d\nu\right)^2$$

depend only on the modulus $|\xi(x)|$, we first choose a continuous path $(\xi_t)_{0 \leq t \leq 1}$ of unit vectors in $L^2(\nu)$ such that $\xi_0 = \xi$, $|\xi_t(x)| = |\xi(x)|$, and $\xi_1(x) = |\xi(x)|$. Thus we may suppose that $\xi(x) \geq 0$.

Let $a = \min S$, $b = \max S$, and

$$\begin{aligned} c &= \int_S x \xi(x)^2 d\nu(x), \\ v &= \int_S x^2 \xi(x)^2 d\nu(x) - c^2, \end{aligned}$$

where c is the *mean* of x and v is the *variance* of x with respect to the probability measure $\xi(x)^2 d\nu$. Then it follows that $a < c < b$ and $0 < v < (b-c)(c-a)$. (Note that a probability measure $d\mu$ on $[a, b]$ with $\int x d\mu = c$ can be approximated by a discrete measure

$$\sum_i \lambda_i \left(\frac{t_i - c}{t_i + s_i} \delta_{s_i} + \frac{c - s_i}{t_i + s_i} \delta_{t_i} \right),$$

where $\lambda_i > 0$, $\sum_i \lambda_i = 1$ and $a < s_i < c < t_i < b$, whose mean is c and whose variance is $\sum_i \lambda_i (t_i - c)(c - s_i)$). We find three distinct points s_1, s_2, s_3 in S such that the convex set

$$\left\{ \sum_{i=1}^3 \lambda_i \delta_{s_i} \mid \lambda_i > 0, \sum \lambda_i = 1 \right\}$$

contains a probability measure with mean c and variance v . (For example, if $c \in S$ we may take $s_1 = a, s_2 = c, s_3 = b$; otherwise set $t_1 = \max\{s \in S \mid s < c\}$ and $t_2 = \min\{s \in S \mid s > c\}$. Then there are three of the four points a, t_1, t_2, b satisfying the requirement. If $(b-c)(c-t_1) < v$, we may set $s_1 = a, s_2 = t_1, s_3 = b$; otherwise if $(t_2-c)(c-a) < v$ we may set $s_1 = a, s_2 = t_2, s_3 = b$; otherwise we may set $s_1 = a, s_2 = t_1, s_3 = t_2$.) Then for any $\epsilon > 0$ we can find a positive measurable function g on S with $\text{supp } g \subset \cup_i (s_i - \epsilon, s_i + \epsilon) \cap S$ such that

$$\begin{aligned} \int g(x) d\nu &= 1, \\ \int x g(x) d\nu &= c, \\ \int x^2 g(x) d\nu &= v + c^2. \end{aligned}$$

Define $\xi_t \in L^2(\nu)$ by

$$\xi_t(x) = ((1-t)\xi(x)^2 + tg(x))^{1/2}.$$

Then $(\xi_t)_{0 \leq t \leq 1}$ defines a continuous path of unit vectors in $L^2(\nu)$ from ξ to \sqrt{g} such that

$$\begin{aligned} (H\xi_t | \xi_t) &= (H\xi | \xi), \\ \|H\xi_t - (H\xi_t | \xi_t)\xi_t\| &= \|H\xi - (H\xi | \xi)\xi\|, \end{aligned}$$

and $\text{supp}(\xi_1) \subset \cup_i (s_i - \epsilon, s_i + \epsilon) \cap S$. Continuing this argument with $\xi = \sqrt{g}$ and a smaller ϵ , we will eventually obtain a continuous path $(\xi_t)_{0 \leq t \leq 1}$ with the required properties such that $\cap_t \cup_{s > t} \text{supp } \xi_s = \{s_1, s_2, s_3\}$. This completes the proof.

References

- [1] H. Araki, Relative Hamiltonian for faithful normal states, *Publ. RIMS, Kyoto Univ.* 9 (1973), 165–209.
- [2] B. Blackadar, O. Bratteli, G.A. Elliott, and A. Kumjian, Reduction of real rank in inductive limits of C^* -algebras, *Math. Ann.* 292 (1992), 111–126.
- [3] O. Bratteli, Derivations, dissipations and group actions on C^* -algebras, *Lecture Notes in Math.* 1229 (1986), Springer.
- [4] O. Bratteli, G.A. Elliott, D.E. Evans, and A. Kishimoto, On the classification of inductive limits of inner actions of a compact group, in *Current topics in operator algebras* edited by H. Araki et al, 13–24, 1991.
- [5] O. Bratteli, G.A. Elliott, D.E. Evans, and A. Kishimoto, Homotopy of a pair of approximately commuting unitaries in a simple C^* -algebra, *J. Funct. Anal.* 160 (1998), 466–523.
- [6] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Springer, 1987.
- [7] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics II*, Springer, 1997.
- [8] O. Bratteli, A remark on extensions of quasi-free derivations on the CAR algebra, *Letters Math. Phys.* 6 (1982), 499–504.
- [9] L. Brown and G.K. Pedersen, C^* -algebras of real rank zero, *J. Funct. Anal.* 99 (1991), 131–149.
- [10] A. Connes, Une classification des facteurs de type III, *Ann. Sci. Ecol. Norm. Sup., Paris* (4) 6 (1973), 133–252.
- [11] G.A. Elliott, On the classification of C^* -algebras of real rank zero, *J. reine angew. Math.* 443 (1993), 179–219.
- [12] P. Friis and M. Rørdam, Almost commuting self-adjoint matrices – A short proof of Huaxin Lin’s theorem, *J. Reine Angew. Math.* 479 (1996), 121–131.
- [13] A. Kishimoto, A Rohlin property for one-parameter automorphism groups, *Commun. Math. Phys.* 179 (1996), 599–622.
- [14] A. Kishimoto, Locally representable one-parameter automorphism groups of AF algebras and KMS states, *Rep. Math. Phys.* 45 (2000), 333–356.
- [15] A. Kishimoto, Examples of one-parameter automorphism groups of UHF algebras, preprint.
- [16] H. Lin, Almost commuting self-adjoint matrices and applications, in *Operator Algebras and their applications* (Waterloo, ON, 1994/1995), 193–233, *Field Inst. Commun* 13, Amer. Math. Soc. 1997.

- [17] G.K. Pedersen, *C*-algebras and their automorphism groups*, Academic Press, 1979.
- [18] S. Sakai, On one-parameter subgroups of *-automorphisms on operator algebras and the corresponding unbounded derivations, *Amer. J. Math.* 98 (1976), 427–440.
- [19] S. Sakai, *Operator Algebras in Dynamical Systems*, Cambridge Univ. Press, 1991.