

A survey of noncommutative dynamical entropy

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1 Introduction

With the success of entropy in classical ergodic theory it became a natural problem to extend the entropy concept to operator algebras. In the classical case we are given a probability space (X, \mathcal{B}, μ) together with a measure preserving nonsingular transformation T of X . If $P = \{P_1, \dots, P_k\}$ is a partition of X by sets in \mathcal{B} then the entropy of P is

$$(1.1) \quad H(P) = \sum_{i=1}^k \eta(\mu(P_i)),$$

where η is the real function on $[0, \infty)$ defined by $\eta(0) = 0$, $\eta(t) = -t \log t$. One shows that the limit

$$(1.2) \quad H(P, T) = \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_0^{n-1} T^{-i} P\right)$$

exists and define the entropy of T by

$$(1.3) \quad H(T) = \sup_P H(P, T),$$

where the sup is taken over all finite partitions P as above. Since T defines an automorphism α_T of $L^\infty(X, \mathcal{B}, \mu)$ by $\alpha_T(f)(x) = f(T^{-1}(x))$, $x \in X$, the entropy definition immediately extends to an entropy $H(\alpha_T)$, where in the definition we replace P by the finite dimensional subalgebra of $L^\infty(X, \mathcal{B}, \mu)$ spanned by the characteristic functions χ_{P_i} .

If we want to extend this definition to the noncommutative setting the obvious first try is first to define the entropy $H_\varphi(N)$ of a finite dimensional algebra with respect to a state φ . Then by analogy with (1.2) and (1.3) if α is a φ -invariant state, to consider $\frac{1}{n} H_\varphi\left(\bigvee_0^{n-1} \alpha^i(N)\right)$, where $\bigvee_{i=1}^r A_i$ stands for the von Neumann algebra generated by the algebras A_1, \dots, A_r . This approach does not work, because the C^* -algebra generated by two finite dimensional C^* -algebras need not be finite dimensional. There have been several approaches to circumvent this difficulty. We shall consider the two we consider most successful. The first was initiated by Connes and Størmer [C-S] and consisted of defining a function $H(N_1, \dots, N_k)$ on finite families N_1, \dots, N_k of finite dimensional

subalgebras of a von Neumann algebra with a normal tracial state, which satisfies many of the same properties as the entropy function $H(N_1 \vee \dots \vee N_k)$ in the abelian case. This was possible due to the beautiful properties of relative entropy of states and the function $\eta(t)$. Later on Connes, Narnhofer and Thirring [CNT] extended this definition to entropy with respect to invariant states on C^* -algebras.

The second approach due to Voiculescu [V] is a refinement of the mean entropy described above. Instead of starting with a finite dimensional subalgebra N and looking at $\bigvee_0^{n-1} \alpha^i(N)$, he considered finite subsets ω of the von Neumann algebra and then looked

for finite dimensional subalgebras which approximately contained $\bigcup_0^{n-1} \alpha^i(\omega)$. Then the entropy, or the rank, of this algebra was used in the definition. There are several variations of this definition depending on how the approximation is taken. They all majorize the C-S or CNT-entropies indicated in the previous paragraph.

The two definitions behave quite differently with respect to tensor products. The CNT-entropy is superadditive, i.e. $h(\alpha \otimes \beta) \geq h(\alpha) + h(\beta)$, while those of Voiculescu are subadditive. In many cases the two entropies coincide, so that the tensor product formula $h(\alpha \otimes \beta) = h(\alpha) + h(\beta)$ holds.

Having the different definitions of entropy the natural question is: what do they tell us about the automorphism? In the classical cases in addition to being a good conjugacy invariant, entropy roughly measures the amount of ergodicity of the transformation and how fast and how far finite dimensional subalgebras are moved with increasing powers of the transformation. In the noncommutative situation much the same is true, except for one major difference. The entropy also measures the amount of commutativity between finite dimensional subalgebras and their images under the action. Thus in highly noncommutative cases like infinite free products of an algebra with itself and the shift, the entropy is zero even though the shift is extremely ergodic. On the other hand, shifts on infinite tensor products behave like classical shifts.

The aim of these notes is to describe all the above in more detail together with the most studied examples. They will usually be introduced in places where they illustrate and show applications of the theory. We shall rarely give complete proofs, but will indicate the main ideas in many cases in order to exhibit the mathematical techniques and ideas involved. The bibliography is not meant to be complete; we have as a rule tried to include references to papers directly related to the text. For other approaches and references see [A-F], [Hu], [T].

The notes are organized as follows

In Section 2 we treat the entropy of Connes and Størmer on finite von Neumann algebras. We start with the background on the operator concave function $\eta(t)$ and relative entropy. Then we define the entropy function $H(N_1, \dots, N_k)$ and discuss its properties. After defining entropy of a trace invariant automorphism and stating its basic properties we illustrate the results by looking at noncommutative Bernoulli shifts.

Section 3 is devoted to the extension of Connes, Narnhofer and Thirring of the results in Section 2 to automorphisms and invariant states of C^* -algebras. They replace the entropy function $H(N_1, \dots, N_k)$ by a similar function $H(\gamma_1, \dots, \gamma_k)$ defined on completely positive maps $\gamma_1, \dots, \gamma_k$ from finite dimensional C^* -algebras into the C^* -algebra.

In Section 4 we consider the example which has attracted most attention in the theory, namely quasifree states of the CAR-algebra and invariant Bogoliubov automorphisms. The formula for the CNT-entropy we shall discuss, has been gradually extended to more general situations, starting with [SV] and now being completed in [N].

Section 5 is devoted to the entropy of Sauvageot and Thouvenot [S-T]. This entropy is a variation of the CNT-entropy, and they coincide for nuclear C*-algebras and injective von Neumann algebras. As an illustration type I algebras are considered in some detail.

In Section 6 we define and study Voiculescu's approximation entropies [V] together with Brown's extension [Br1] of topological entropy to exact C*-algebras. These entropies majorize the CNT-entropy and comparison of them can yield much information on the C*-dynamical system under consideration.

Section 7 is devoted to crossed products. If (A, φ, α) is a C*-dynamical system, i.e. A is a C*-algebra, φ a state and α a φ -invariant automorphism, then α extends to an inner automorphism $\hat{\alpha}$ of $A \times_{\alpha} \mathbb{Z}$. Using the theory from Section 6 we obtain results, even in more general situations, to the effect that the entropies of α and $\hat{\alpha}$ are the same. In particular the shift on \mathcal{O}_{∞} has topological entropy zero.

In Section 8 we study the most noncommutative setting, namely shifts on infinite free products $(*A_i, *\varphi_i)$ where the A_i 's and the φ_i 's are equal. These automorphisms are "extremely" ergodic, but still their entropies vanish.

Section 9 is devoted to binary shifts on the CAR-algebra, arising from sequences of 0's and 1's. Different bitstreams give rise to C*-dynamical systems of quite different nature. The entropies with respect to the trace are in most computed cases equal to $\frac{1}{2} \log 2$, but there are examples with entropy zero.

In Section 10 on generators we consider W*-dynamical systems (M, τ, α) with M a von Neumann algebra with a faithful normal tracial state τ , where the entropy is a mean entropy. In such cases the C-S entropy tends to coincide with one of Voiculescu's approximation entropies. The concept of generator can be made quite general, and we get in some cases the analogue of the classical formula when the entropy of a transformation T is the relative (or conditional) entropy $H(\bigvee_0^{\infty} T^{-i} P \mid \bigvee_1^{\infty} T^{-i} P)$. Applications are given to subfactors and the canonical endomorphism Γ on the hyperfinite II_1 -factor defined by an inclusion of subfactors of finite index.

Finally, Section 11 is devoted to the variational principle. Several of the well-known results from the classical case and from spin lattice systems in the C*-algebra formalism of quantum statistical mechanics are extended to a class of asymptotically abelian C*-algebras.

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2 Entropy in finite von Neumann algebras

In this section we shall define and sketch the proofs of the main properties of the entropy function $H(N_1, \dots, N_k)$ and the corresponding entropy of a trace invariant automorphism. For this we need to study the function $\eta(t) = -t \log t$, $t > 0$, $\eta(0) = 0$, and relative entropy in some detail. The first goes back to early work on entropy of states, see [N-U]. Recall

that $B(H)$ denotes the bounded linear operators on a Hilbert space H , and $B(H)^+$ the positive operators in $B(H)$.

Lemma 2.1 (i) *The function $\log t$ is operator increasing on $B(H)^+$, i.e. if $0 \leq x \leq y$ in $B(H)^+$ then $\log x \leq \log y$.*

(ii) *The function $\eta(t)$ is strictly operator concave on $B(H)^+$, i.e.*

$$\eta\left(\frac{1}{2}(x+y)\right) \geq \frac{1}{2}\eta(x) + \frac{1}{2}\eta(y), \quad x, y \in B(H)^+$$

with equality only if $x = y$.

Proof. For $t \geq 0$

$$\log t = \int_0^{\infty} \left(\frac{1}{1+x} - \frac{1}{t+x} \right) dx,$$

providing (i). Multiplying by t we get

$$\eta(t) = \int_0^{\infty} \left(1 - \frac{t}{1+x} - \frac{x}{t+x} \right) dx.$$

When we take convex combinations, the first two summands cancel out, so the lemma follows from the inequality

$$\left(\frac{1}{2}(z+w)\right)^{-1} \leq \frac{1}{2}(z^{-1} + w^{-1})$$

for positive invertible operators z and w . □

Instead of using partitions of unity consisting of orthogonal projections as in the classical case it will be necessary to look at more general partitions of unity.

Notation 2.2 Let M be a von Neumann algebra, and let $k \in \mathbb{N}$. Then

$$S_k = S_k(M) = \{(x_{i_1, \dots, i_k}) : x_{i_1, \dots, i_k} \in M^+ \text{ and equal to } 0 \\ \text{except for a finite number of indices, } \sum_{i_1 \dots i_k} x_{i_1 \dots i_k} = 1\}$$

Let for $j \in \{1, \dots, k\}$

$$x_{i_j}^j = \sum_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k} x_{i_1 \dots i_k}.$$

One can then show the following inequality [H-S].

Lemma 2.3 *Let M be a von Neumann algebra with a normal tracial state τ . Let $\|x\|_2 = \tau(x^*x)^{1/2}$ for $x \in M$. Let $(x_{ij}) \in S_2$, $i = 1, \dots, m$, $j = 1, \dots, n$. Then*

$$\sum_i \tau \eta(x_i^1) + \sum_j \tau \eta(x_j^2) - \sum_{i,j} \tau \eta(x_{ij}) \geq \frac{1}{2} \sum_{ij} \|[(x_i^1)^{1/2}, (x_j^2)^{1/2}]\|_2$$

where $[a, b] = ab - ba$.

In particular the left side of the inequality is nonnegative. This follows also from the joint convexity of *relative entropy*, defined as follows: If $x, y \in M^+$ with $x \leq \lambda y$ for some $\lambda > 0$,

$$(2.1) \quad S(x, y) = \tau(x(\log x - \log y)) .$$

More generally if M is a von Neumann algebra and φ, ψ normal states we can define their relative entropy as follows, [A], [O-P]. We may assume φ and ψ are vector states ω_{ξ_φ} and ω_{ξ_ψ} respectively and for simplicity that ξ_φ is separating and cyclic for M . We define

$$S_{\psi, \varphi}(x\xi_\varphi) = x^*\xi_\psi$$

If $\bar{S}_{\psi\varphi}$ is the closure, the relative modular operator is

$$\Delta_{\psi, \varphi} = S_{\psi\varphi}^* \bar{S}_{\psi\varphi} .$$

Then the relative entropy is

$$S(\varphi, \psi) = -(\log \Delta_{\psi, \varphi} \xi_\varphi, \xi_\varphi) .$$

There are also integral formulas due to Pusz, Woronowicz and Kosaki which yield generalizations to C^* -algebras, see [O-P]. One can show that S is jointly convex in φ and ψ and $S(\lambda\varphi, \lambda\psi) = \lambda S(\varphi, \psi)$. Furthermore, if $x \in M^+$ and S is given by (2.1) then

$$S(x, \tau(x)) = \tau(x(\log x - \log \tau(x))) = \eta(\tau(x)) - \tau\eta(x) .$$

This together with joint convexity of S yields the inequality

$$(2.2) \quad \eta\tau(x + y) - \tau\eta(x + y) \leq (\eta\tau(x) - \tau\eta(x)) + (\eta\tau(y) - \tau\eta(y)) .$$

If φ is a normal state on M then there exists a positive self-adjoint operator $h_\varphi \in L^1(M, \tau)$ such that $\varphi(x) = \tau(h_\varphi x)$. Then the relative entropy of φ and ω is given by

$$S(\varphi, \omega) = S(h_\varphi, h_\omega) = \varphi(\log h_\varphi - \log h_\omega)$$

whenever it is defined.

If $N \subset M$ is a von Neumann subalgebra we denote by E_N the trace invariant conditional expectation of M onto N defined by the identity

$$\tau(E_N(x)y) = \tau(xy) \quad \text{for } x \in M, y \in N .$$

If φ and ω are normal states of N and M respectively we have, see [O-P, Thm. 5.15],

$$(2.3) \quad S(\omega, \varphi \circ E_N) = S(\omega|_N, \varphi) + S(\omega, \omega \circ E_N) .$$

If $\varphi, \psi, \omega \in M_*^+$ and $\omega \leq \psi$ then by [O-P, Cor. 5.12]

$$(2.4) \quad S(\varphi, \psi) \leq S(\varphi, \omega) .$$

After these preliminaries we now define the entropy function $H(N_1, \dots, N_k)$.

Definition 2.4 [C-S] *Let N_1, \dots, N_k be finite dimensional von Neumann subalgebras of M . Then*

$$H(N_1, \dots, N_k) = \sup_{x_{i_1 \dots i_k} \in S_k} \left\{ \sum_{i_1 \dots i_k} \eta \tau(x_{i_1 \dots i_k}) - \sum_{j=1}^k \sum_{i_j} \tau \eta(E_{N_j} x_{i_j}^j) \right\}.$$

The definition can be rewritten in terms of relative entropy as follows. Write $x_{(i)}$ for $x_{i_1 \dots i_k}$. Let $\tau_{(i)}$ and $\tau_{i_j}^j$ denote the positive linear functionals

$$\tau_{(i)}(a) = \tau(x_{(i)} a), \quad \tau_{i_j}^j(a) = \tau(x_{i_j}^j a).$$

Since

$$\begin{aligned} S(\tau_{i_j}^j |_{N_j}, \tau |_{N_j}) &= \tau(E_{N_j}(x_{i_j}^j)) (\log E_{N_j}(x_{i_j}^j) - \log E_{N_j}(1)) \\ &= -\tau \eta(E_{N_j} x_{i_j}^j), \end{aligned}$$

the definition of H becomes

$$(2.5) \quad H(N_1, \dots, N_k) = \sup_{(\tau_{(i)})} \left\{ \sum_{(i)} \eta \tau_{(i)}(1) + \sum_{j=1}^k \sum_{i_j} S(\tau_{i_j}^j |_{N_j}, \tau |_{N_j}) \right\}.$$

The main properties of H are summarized in

Theorem. 2.5 [C-S] *For finite dimensional von Neumann subalgebras N, N_i, P_j of M we have*

- (A) $H(N_1, \dots, N_k) \leq H(P_1, \dots, P_k)$ when $N_i \subset P_i$, $i = 1, \dots, k$.
- (B) $H(N_1, \dots, N_k, N_{k+1}, \dots, N_p) \leq H(N_1, \dots, N_k) + H(N_{k+1}, \dots, N_p)$
- (C) $N_1, \dots, N_k \subset N \Rightarrow H(N_1, \dots, N_k, N_{k+1}, \dots, N_p) \leq H(N, N_{k+1}, \dots, N_p)$
- (D) *For any family of minimal projections of N , $(e_j)_{j \in I}$, such that $\sum_{j \in I} e_j = 1$ we have*

$$H(N) = \sum_{j \in I} \eta \tau(e_j).$$

- (E) *If P_i pairwise commute, $P_i \subset N_i$, and $\bigvee_{i=1}^k P_i = \bigvee_{i=1}^k N_i$ then*

$$H(N_1, \dots, N_k) = H\left(\bigvee_{i=1}^k N_i\right).$$

Indication of proof

(A) A variant of Jensen's inequality states that $\eta(E_N(x)) \geq E_N(\eta(x))$ for $x \in M^+$. If $N_i \subset P_i$ then $\eta(E_{N_i}(x)) = \eta(E_{N_i}E_{P_i}(x)) \geq E_{N_i}\eta(E_{P_i}(x))$, hence $\tau\eta(E_{N_i}(x)) \geq \tau\eta(E_{P_i}(x))$, proving (A).

(B) This is a reduction to subadditivity of $H(P)$ in the classical case.

(C) This is a consequence of the positivity of the left side of the inequality in Lemma 2.3.

(D) The proof of this property is helpful in understanding the need for the second sum in Definition 2.4. Let $(e_j)_{j \in I}$ be as in (D). Let $(x_i) \in S_1$. Since $\tau(x_i) = \tau(E_N x_i)$, we may assume $x_i \in N$. Thus we have to show

$$\sum_i \eta\tau(x_i) - \sum_i \tau\eta(x_i) \leq \sum_{j \in I} \tau\eta(e_j).$$

By inequality (2.2) we can reduce to the case when each x_i is of rank 1, i.e. $x_i = \lambda_i p_i$ with $\lambda_i > 0$, p_i a minimal projection in N . Computing and noting that $\eta(p_i) = 0$ we have

$$\begin{aligned} \sum (\eta\tau(x_i) - \tau\eta(x_i)) &= \sum (\eta\tau(\lambda_i p_i) - \tau\eta(\lambda_i p_i)) \\ &= \sum \lambda_i \eta(\tau(p_i)) + \eta(\lambda_i)\tau(p_i) - \lambda_i \tau\eta(p_i) - \eta(\lambda_i)\tau(p_i) \\ &= \sum \lambda_i \eta(\tau(p_i)). \end{aligned}$$

If we write N as a direct sum of factors, we reduce to the case when $N \cong M_n(\mathbb{C})$, $n \in \mathbb{N}$, so that

$$1 = \sum \tau(\lambda_i p_i) = \left(\sum \lambda_i \right) \frac{1}{n},$$

hence $\sum \lambda_i = n$, and so

$$\sum \lambda_i \eta(\tau(p_i)) = n \cdot \eta\left(\frac{1}{n}\right) = \log n = \sum \tau\eta(e_j).$$

(E) This property shows that the definition of H generalizes the abelian case. In the proof we can replace each N_i by P_i . If A_i is a masa, i.e. a maximal abelian subalgebra of P_i , and $A = \bigvee_{i=1}^k A_i$ is the masa they generate in $\bigvee_{i=1}^k P_i$, then by (D), $H(A) = H\left(\bigvee_{i=1}^k P_i\right)$. Thus (E) follows from the abelian case. \square

The function $(N_1, \dots, N_k) \rightarrow H(N_1, \dots, N_k)$ is from the above a function of the sizes of the N_i 's together with their relative positions. It seems to be very difficult to formulate a general theorem in the converse direction. One simple result follows from (D), namely if $P \subset N$ and $H(P) = H(N)$ then each masa in P is a masa in N , i.e. $\text{rank } P = \text{rank } N$, where the rank of $N - \text{rank } N = \dim A$, where A is a masa in N . Note that $\dim N \leq (\text{rank } N)^2$. So far the only theorem in the literature along the line discussed above is

Theorem. 2.6 [H-S] *Let M and τ be as before. Let N_1, \dots, N_k be finite dimensional von Neumann subalgebras of M , and let $N = \bigvee_{i=1}^k N_i$. Then the following two conditions are equivalent.*

(i) $H(N_1, \dots, N_k) = H(N)$

(ii) *There exists a masa $A \subset N$ such that $A = \bigvee_{i=1}^k (A \cap N_i)$.*

In particular, if the above conditions hold then N is finite dimensional, and $\text{rank } N \leq \prod_{i=1}^k \text{rank } N_i$.

Note that the implication (ii) \Rightarrow (i) is an easy consequence of Theorem 2.5. Indeed

$$\begin{aligned} H(N) &\geq H(N_1, \dots, N_k) && \text{by (C)} \\ &\geq H(A \cap N_1, \dots, A \cap N_k) && \text{by (A)} \\ &= H\left(\bigvee_{i=1}^k (A \cap N_i)\right) && \text{by (E)} \\ &= H(A) \\ &= H(N) && \text{by (D)} \end{aligned}$$

For the converse we must attack the definition of H , Definition 2.4, directly. Choose $(x_{(i)}) \in S_k$ for which the right side of Definition 2.4 almost takes the value $H(N_1, \dots, N_k)$. By using the k -dimensional version of the inequality in Lemma 2.3 it follows that the operators $x_{i_j}^j$ almost commute for different j 's, and taking limits of such families $(x_{(i)}) \in S_k$ we can conclude that the $x_{i_j}^j$ belong to pairwise commuting algebras P_j . Taking masas A_j in these P_j we get the desired A as $A = \bigvee_{j=1}^k A_j$. \square

In the classical case two finite dimensional algebras A and B (identified with the partition of unities of their atoms) are said to be independent if $\mu(fg) = \mu(f)\mu(g)$, $f \in A$, $g \in B$, or equivalently $H(A \vee B) = H(A) + H(B)$. This equivalence is false in the noncommutative case. However, we have

Corollary 2.7 [H-S] *Let $N_1, \dots, N_k \subset M$ as before and put $N = \bigvee_{i=1}^k N_i$. Then the following two conditions are equivalent.*

(i) $H(N) = H(N_1, \dots, N_k) = \sum_{i=1}^k H(N_i)$.

(ii) *There exists a masa $A \subset N$ such that $A_i = A \cap N_i$ is a masa in N_i for each i , and A_1, \dots, A_k are independent.*

We shall next consider continuity of H .

Definition 2.8 If $N, P \subset M$ are finite dimensional subalgebras their relative entropy is

$$H(N|P) = \sup_{x \in S_1} \sum_i (\tau \eta(E_P x_i) - \tau \eta(E_N x_i))$$

(If we compare with the classical situation we should perhaps rather have used the name “conditional entropy”). The following properties are immediate consequences of Definition 2.8

- (F) $H(N_1, \dots, N_k) \leq H(P_1, \dots, P_k) + \sum_{j=1}^k H(N_j|P_j)$.
- (G) $H(N|Q) \leq H(N|P) + H(P|Q)$.
- (H) $H(N|P)$ is increasing in N and decreasing in P .

If $N \supset P$ the definition makes sense even when N and P are infinite dimensional, as noted by Pimsner and Popa [P-P]. They computed $H(N|P)$ in several cases relating it in particular to the Jones index, see Section 10.

In the classical case the crucial result which makes entropy useful, is the Kolmogoroff-Sinai theorem, see [Sh] for a natural proof using continuity of relative entropy. Continuity in our case takes the form of the following lemma. For $N, P \subset M$ and $\delta > 0$ we write $N \subset^\delta P$ if for each $x \in N$, $\|x\| \leq 1$, there exists $y \in P$, $\|y\| \leq 1$, such that $\|x - y\|_2 < \delta$.

Lemma 2.9 [C-S] *Let M and τ be as before and $n \in \mathbb{N}$, $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all pairs of von Neumann subalgebras $N, P \subset M$ we have:*

$$\dim N = n, \quad N \subset^\delta P \Rightarrow H(N|P) < \varepsilon.$$

Definition 2.10 [C-S] *Let α be an automorphism of M which is τ -invariant, i.e. $\tau \circ \alpha = \tau$. If N is a finite dimensional von Neumann subalgebra of M , put*

$$H(N, \alpha) = \lim_{k \rightarrow \infty} \frac{1}{k} H(N, \alpha(N), \dots, \alpha^{k-1}(N)).$$

This limit exists by property (B), see [W, Thm. 4.9]. The entropy $H_\tau(\alpha)$, or $H(\alpha)$, of α is

$$H(\alpha) = \sup_N H(N, \alpha),$$

where the sup is taken over all N as above. The Kolmogoroff-Sinai theorem takes the form, see [W, Thm. 4.22] for the classical analogue.

Theorem. 2.11 [C-S] *Let M be hyperfinite, and τ and α as above. Let $P_j)_{j \in \mathbb{N}}$ be an increasing sequence of finite dimensional subalgebras of M with $\left(\bigcup_{j=1}^{\infty} P_j\right)'' = M$. Then*

$$H(\alpha) = \lim_{j \rightarrow \infty} H(P_j, \alpha).$$

Proof. Let $N \subset M$ be finite dimensional and $\varepsilon > 0$. By hypothesis and Lemma 2.9 there exists $j \in \mathbb{N}$ such that $H(N|P_j) < \varepsilon$. Thus by property (F)

$$\begin{aligned} H(N, \alpha) &= \lim_k \frac{1}{k} H(N, \alpha(N), \dots, \alpha^{k-1}(N)) \\ &\leq \lim_k \frac{1}{k} H(P_j, \alpha(P_j), \dots, \alpha^{k-1}(P_j)) + \lim_k \frac{1}{k} \sum_{i=0}^{k-1} H(\alpha^i(N), \alpha^i(P_j)) \\ &\leq H(P_j, \alpha) + \varepsilon. \end{aligned}$$

□

It is clear that $H(\alpha)$ is a conjugacy invariant, i.e. if γ is an automorphism of M then $H(\gamma\alpha\gamma^{-1}) = H(\alpha)$.

In the classical case we have $H(\alpha^p) = |p|H(\alpha)$ for $p \in \mathbb{Z}$. In our case we have,

Proposition 2.12 [C-S] (i) $H(\alpha^p) \leq |p|H(\alpha)$.

(ii) If M is hyperfinite, $H(\alpha^p) = |p|H(\alpha)$.

Note also that by property (C) $H(\alpha)$ is monotone, i.e. if $N \subset R$ is a von Neumann subalgebra such that $\alpha(N) = N$, then $H(\alpha|_N) \leq H(\alpha)$.

A problem which has attracted much attention in noncommutative entropy is that of additivity under tensor products. If (M_i, α_i, τ_i) are W^* -dynamic systems like (M, τ, α) above, $i = 1, 2$, then the problem is whether

$$H_{\tau_1 \otimes \tau_2}(\alpha_1 \otimes \alpha_2) = H_{\tau_1}(\alpha_1) + H_{\tau_2}(\alpha_2) ?$$

This is well-known in the classical case. In our case we can only conclude that

$$(2.6) \quad H_{\tau_1 \otimes \tau_2}(\alpha_1 \otimes \alpha_2) \geq H_{\tau_1}(\alpha_1) + H_{\tau_2}(\alpha_2).$$

Indeed, if $N_i \subset M_1$, $P_i \subset M_2$, $i = 1, \dots, k$ are finite dimensional then there are more families $(x_{(i)}) = (x_{i_1}, \dots, x_{i_k}) \in S_k(M_1 \otimes M_2)$ than there are families $(y_{(i)} \otimes z_{(i)}) = (y_{i_1 \dots i_k} \otimes z_{i_1 \dots i_k})$ in $S_k(M_1) \otimes S_k(M_2)$, hence

$$H_{\tau_1 \otimes \tau_2}(N_1 \otimes P_1, \dots, N_k \otimes P_k) \geq H_{\tau_1}(N_1, \dots, N_k) + H_{\tau_2}(P_1, \dots, P_k).$$

Remark 2.13 The n -shift The first nontrivial example that was computed was the entropy of the n -shift. Let $n \in \mathbb{N}$, $M_i = M_n(\mathbb{C})$, $i \in \mathbb{Z}$, and τ_i be the tracial state on M_i . Let $B = \bigotimes_{i \in \mathbb{Z}} M_i$, $\tau = \bigotimes_{i \in \mathbb{Z}} \tau_i$, be the C^* -tensor product, and consider B as a subalgebra of the II_1 -factor R obtained from the GNS-representation of τ . Let α be the shift on B identified with its extension to R . Let

$$P_j = \cdot \otimes 1 \otimes \bigotimes_{-j}^j M_i \otimes 1 \otimes \dots, \quad j \in \mathbb{N}$$

be the finite tensor product of the M_i from $-j$ to j considered as a subalgebra of R . Let D_i be the diagonal in M_i and

$$D_{pq} = \dots \otimes 1 \otimes \bigotimes_{-p}^q D_i \otimes \dots, \quad j \in \mathbb{N}.$$

As an illustration of the techniques developed we give two computations of $H(\alpha)$. The first which is the original from [C-S], is quite helpful in understanding Definition 2.4. Let e_j be the minimal projection in D_i which is 1 in the j 'th row. Let

$$x_{i_1 \dots i_k} = \dots \otimes 1 \otimes e_{i_1} \otimes \dots \otimes e_{i_k} \otimes 1 \in D_{1k} .$$

Then $(x_{i_1, \dots, i_k}) \in S_k$, and

$$x_{i_j}^i = \dots \otimes 1 \otimes e_{i_j} \otimes 1 \otimes \dots \in D_j .$$

Thus

$$\begin{aligned} H(M_1, \alpha(M_1), \dots, \alpha^{k-1}(M_1)) &= H(M_1, \dots, M_k) \\ &\geq \sum_{i_1 \dots i_k} \eta \tau(x_{i_1 \dots i_k}) - \sum_{j=1}^k \sum_{i_j=1}^n \tau \eta(E_{M_j} x_{i_j}^j) \\ &= n^k \eta(n^{-k}) - \sum_j \sum_{i_j} \tau \eta(e_{i_j}) \\ &= k \log n - 0 , \end{aligned}$$

so that $H(M_1, \alpha) \geq \log n$.

To prove the opposite inequality we use that $(M_1 \cup M_2 \cup \dots \cup M_k)''$ is a factor of type I_{n^k} , hence has entropy $k \log n$. The rest of the proof consists of an application of the Kolmogoroff-Sinai theorem to the sequence (P_j) together with an application of property (E).

The other proof is quicker. Fix $q \in \mathbb{N}$. Then $A_q = D_{-q, q}$ is a masa in P_q . If $k \in \mathbb{N}$ let $A = \bigvee_{j=0}^{k-1} \alpha^j(A_q)$. Then $A = D_{-q, q+k-1}$ is a masa in $\bigvee_0^{k-1} \alpha^j(P_q)$ such that $A \cap \alpha^j(P_q) = \alpha^j(A_q)$ is a masa in $\alpha^j(P_q)$. Thus by the easy part of Theorem 2.6,

$$H(P_q, \alpha) = \lim_{k \rightarrow \infty} \frac{2q + k - 1}{k} \log n = \log n ,$$

so that by the Kolmogoroff-Sinai theorem, $H(\alpha) = \log n$.

2.14 Bernoulli shifts The above arguments can be extended to noncommutative Bernoulli shifts of the hyperfinite II_1 -factor R . Let $h \in M_0^+$ with $\text{Tr}(h) = 1$, Tr denoting the usual trace on $M_n(\mathbb{C})$, with eigenvalues h_1, \dots, h_n . Let φ_0 be the state $\varphi_0(x) = \text{Tr}(hx)$ for $x \in M_0$.

Let $\varphi_i = \varphi_0$ on M_i and $\varphi = \bigotimes_{i \in \mathbb{Z}} \varphi_i$ denote the corresponding product state on $B = \bigotimes_{i \in \mathbb{Z}} M_i$. In the GNS-representation of B due to φ the centralizer R of the weak closure is the hyperfinite II_1 -factor and contains the algebras A_q above. Since α is φ -invariant, the extension of α to the GNS-representation restricts to an automorphism of R , which we call a Bernoulli shift. A slight extension of the argument from 2.13 shows that

$$H(\alpha) = \sum_{i=1}^n \eta(h_i) = S(\varphi_0) ,$$

where $S(\varphi_0)$ is the entropy of the state φ_0 on $M_n(\mathbb{C})$. □

There is another natural definition of Bernoulli shift on R . Let T be a (classical) Bernoulli shift on a probability space (X, \mathcal{B}, μ) . Then $R = L^\infty(X, \mathcal{B}, \mu) \times_T \mathbb{Z}$, and T extends to an inner automorphism $\text{Ad } u_T$ on R . Both the von Neumann algebra generated by the A_q 's above and $L^\infty(X, \mathcal{B}, \mu)$ are Cartan subalgebras of R , i.e. they are masas whose normalizers generate R , hence they are conjugate by [CFW]. Thus we have one outer and one inner automorphism on R which act as the same Bernoulli shift on a Cartan subalgebra and have the same entropy (see Theorem 7.1 below).

Other extensions of classical shift automorphisms have been studied by Besson [B] for Markov shifts and Quasthoff [Q]. In all examples there is a masa like A in 2.19 and the entropy is the same as the classical counterpart.

3 Entropy in C*-algebras

After the appearance of [C-S] an obvious problem was to extend the definition from the tracial case to that of general states. It took 10 years before Connes [Co] saw what had to be done. If one looks at the rewritten form of $H(N_1, \dots, N_k)$ in equation 2.5 and notes that states are of the form $\varphi(x) = \tau(hx)$, $h \in L^1(M, \tau)^+$, it is obvious what to do. Let φ be a normal state of a von Neumann algebra M . Modify Notation (2.2) as follows: For $k \in \mathbb{N}$ put

$$S_{k, \varphi} = \{ \varphi_{i_1 \dots i_k} \in M_*^+, \quad i_j \in \mathbb{N}, \quad \varphi_{i_1 \dots i_k} = 0 \text{ except for} \\ \text{a finite number of indices,} \quad \sum_{i_1 \dots i_k} \varphi_{i_1 \dots i_k} = \varphi \}.$$

Let

$$\varphi_{i_j}^j = \sum_{i_1 \dots i_{j-1} i_{j+1} \dots i_k} \varphi_{i_1 \dots i_k}.$$

If $N_1, \dots, N_k \subset M$ are finite dimensional von Neumann subalgebras we let [Co]

$$(3.1) \quad H_\varphi(N_1, \dots, N_k) = \sup_{(\varphi_{i_j \dots i_k}) \in S_{k, \varphi}} \left\{ \sum_{i_1 \dots i_k} \eta(\varphi_{i_1 \dots i_k}(1)) + \sum_{j=1}^k \sum_{i_j} S(\varphi_{i_j}^j |_{N_j}, \varphi|_{N_j}) \right\}.$$

This definition even makes sense for C*-algebras, because, as we pointed out earlier, Pusz, Woronowicz and Kosaki extended the definition of relative entropy to C*-algebras. Since C*-algebras may have no finite dimensional C*-subalgebras except the scalars, the definition above would only be useful for AF-algebras and their like. Connes, together with Narnhofer and Thirring [CNT] circumvented the problem by replacing the algebras N_j by completely positive maps γ_j from finite dimensional algebras into the C*-algebra. The definition is as follows.

Let A be a unital C*-algebra with a state φ . Let N_1, \dots, N_k be finite dimensional C*-algebras and $\gamma_j : N_j \rightarrow A$ a unital completely positive map, $j = 1, \dots, k$. Let B be

a finite dimensional abelian C*-algebra and $P : A \rightarrow B$ a unital positive linear map such that there is a state μ on B with $\mu \circ P = \varphi$. Let p_1, \dots, p_r be the minimal projections in B . Then there are states $\hat{\varphi}_1, \dots, \hat{\varphi}_r$ on A such that

$$(3.2) \quad P(x) = \sum_{i=1}^r \hat{\varphi}_i(x) p_i,$$

and

$$(3.3) \quad \varphi = \sum_{i=1}^r \mu(p_i) \hat{\varphi}_i,$$

is φ written as a convex sum of states. Put

$$\varepsilon_\mu(P) = \sum \mu(p_i) S(\hat{\varphi}_i, \varphi).$$

where $S(\hat{\varphi}_i, \varphi)$ is the relative entropy. Let the *entropy defect* be

$$s_\mu(P) = S(\mu) - \varepsilon_\mu(P),$$

where $S(\mu) = \sum_{i=1}^r \eta(\mu(p_i))$ is the entropy of μ .

Suppose B_1, \dots, B_k are C*-subalgebras of B and $E_j : B \rightarrow B_j$ the μ -invariant conditional expectation. Then the quadruple (B, E_j, P, μ) is called an *abelian model* for $(A, \varphi, \gamma_1, \dots, \gamma_k)$, and its entropy is defined to be

$$(3.4) \quad S\left(\mu \Big|_{\bigvee_{j=1}^k B_j}\right) - \sum_{j=1}^k s_\mu(P_j),$$

where $P_j = E_j \circ P \circ \gamma_j : N_j \rightarrow B_j$, and the definition of $s_\mu(P_j)$ is the same as for P above, where we replace μ by $\mu|_{B_j}$, φ by $\varphi \circ \gamma_j$.

Definition 3.1 [CNT] $H_\varphi(\gamma_1, \dots, \gamma_k) = \sup$ of (3.4) over all abelian models.

In the special case when $N_1, \dots, N_k \subset A$ and $\gamma_j : N_j \rightarrow A$ is the inclusion map let (B, E_j, P, μ) be an abelian model. We may assume $B = \bigvee_{j=1}^k B_j$. Let $\{p_{i_j}\}$ be the set of minimal projections in B_j . Then $\{p^{(i)} = p_{i_1 \dots i_k} = p_{i_1} \dots p_{i_k}\}$ is the set of minimal projections in B . If we use the abbreviation following Definition 2.4, equations (3.2) and (3.3) can be written

$$P(x) = \sum_{(i)} \hat{\varphi}^{(i)}(x) p^{(i)},$$

$$\varphi = \sum_{(i)} \mu(p^{(i)}) \hat{\varphi}^{(i)}.$$

Let $\varphi_{(i)}(x) = \mu(p_{(i)})\hat{\varphi}_i(x)$, so $\varphi = \sum \varphi_{(i)}$. We have

$$E_j(p_{(i)}) = \mu(p_{(i)})\mu(p_{i_j}^j)^{-1}p_{i_j}^j ,$$

where $p_{i_j} = p_{i_j}^j$ in the notation of (2.2). Hence if $x \in N_j$ we have

$$P_j(x) = E_j \circ P(x) = \sum_{(i)} \frac{\varphi_{(i)}^j(x)}{\mu(p_{i_j})} \mu(p_{(i)})p_{i_j} = \sum_{i_j} \frac{\varphi_{i_j}^j(x)}{\mu(p_{i_j})} p_{i_j} .$$

Thus

$$\varepsilon_\mu(P_j) = \sum_{i_j} \mu(p_{i_j}) S\left(\frac{\varphi_{i_j}^j}{\mu(p_{i_j})} \Big|_{N_j}, \varphi|_{N_j}\right) .$$

From the identity

$$\lambda S(\rho, \psi) = \eta(\lambda) + S(\lambda\rho, \psi) ,$$

as is easily shown in the finite dimensional case, we have

$$\begin{aligned} \varepsilon_\mu(P_j) &= \sum_{i_j} \{\eta(\mu(p_{i_j})) + S(\varphi_{i_j}^j|_{N_j}, \varphi|_{N_j})\} \\ &= S(\mu|_{B_j}) + \sum_{i_j} S(\varphi_{i_j}^j|_{N_j}, \varphi|_{N_j}) . \end{aligned}$$

Note that $\varphi_{(i)}(1) = \mu(p_{(i)})$. Thus we find that the entropy of the abelian model (B, E, P, μ) is

$$S(\mu|_B) - \sum_{j=1}^k \{S(\mu|_{B_j}) - \varepsilon_\mu(P_j)\} = \sum_{(i)} \eta(\varphi_{(i)}(1)) + \sum_{j=1}^k \sum_{i_j} S(\varphi_{i_j}^j|_{N_j}, \varphi|_{N_j}) ,$$

which is the same as the expression in (3.1).

We note that if we are given $(\varphi_{(i)}) \in S_{k,\varphi}$ and $N_1, \dots, N_k \subset M$ it is not hard to construct an abelian model like (B, E, P, μ) above, so that (3.1) defines $H_\varphi(\gamma_1, \dots, \gamma_k)$ when $\gamma_j : N_j \rightarrow M$ is the inclusion map. Thus $H_\varphi(\gamma_1, \dots, \gamma_k)$ is a direct generalization of $H_\tau(N_1, \dots, N_k)$ defined in Definition 2.4. One can show similar properties to (A)–(E) in Section 2, see [CNT], hence we can define the entropy of an automorphism

Definition 3.2 [CNT] *Let A be a unital C^* -algebra, φ a state and α a φ -invariant automorphism of A . Let C be a finite dimensional C^* -algebra and $\gamma : C \rightarrow A$ a unital completely positive map. Then*

$$h_{\varphi,\alpha}(\gamma) = \lim_{k \rightarrow \infty} \frac{1}{k} H_\varphi(\gamma, \alpha \circ \gamma, \dots, \alpha^{k-1} \circ \gamma)$$

exists. We define the entropy of α with respect to φ to be

$$h_\varphi(\alpha) = \sup_{(C,\gamma)} h_{\varphi,\alpha}(\gamma) ,$$

where the sup is taken over all pairs (C, γ) .

The Kolmogoroff-Sinai Theorem, cf. Theorem 2.11, now takes the following form.

Theorem. 3.3 [CNT] *Let A, φ, α be as above. Suppose (τ_n) is a sequence of unital completely positive maps $\tau_n : A_n \rightarrow A$ from finite dimensional C^* -algebras A_n into A such that there exist unital completely positive maps $\sigma_n : A \rightarrow A_n$ for which $\tau_n \circ \sigma_n \rightarrow \text{id}_A$ in the pointwise norm topology. Then*

$$\lim_{n \rightarrow \infty} h_{\varphi, \alpha}(\tau_n) = h_{\varphi}(\alpha).$$

In particular if $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ is an AF-algebra, and we identify A_n with its inclusion map $A_n \rightarrow A$, we have,

$$h_{\varphi}(\alpha) = \lim_{n \rightarrow \infty} h_{\varphi, \alpha}(A_n).$$

Theorem 3.3 is applicable if A is nuclear. In that case we have [CNT]

- (i) $h_{\varphi}(\alpha) = h_{\varphi}(\gamma\alpha\gamma^{-1})$ if $\gamma \in \text{Aut } A$.
- (ii) $h_{\varphi}(\alpha^n) = |n|(h_{\varphi}(\alpha))$, $n \in \mathbb{Z}$.
- (iii) If φ_1 and φ_2 are α -invariant and $\lambda \in [0, 1]$, $h_{\lambda\varphi_1+(1-\lambda)\varphi_2}(\alpha) \geq \lambda h_{\varphi_1}(\alpha) + (1-\lambda)h_{\varphi_2}(\alpha)$.

Definition 3.2 also makes sense for normal states of von Neumann algebras. If (A, φ, α) is as above, and $(\pi_{\varphi}, H_{\varphi}, \xi_{\varphi})$ is the GNS-representation of φ , and $\bar{\alpha}$ the extension of α to $M = \pi_{\varphi}(A)''$, then [CNT],

$$(3.5) \quad h_{\varphi}(\alpha) = h_{\omega_{\xi_{\varphi}}}(\bar{\alpha}).$$

Thus we can freely move back and forth between A and M in our computations of entropy.

If $(A_i, \varphi_i, \alpha_i)$, $i = 1, 2$ are C^* -dynamical systems we have as in the tracial case (2.6),

$$(3.6) \quad h_{\varphi_1 \otimes \varphi_2}(\alpha_1 \otimes \alpha_2) \geq h_{\varphi_1}(\alpha_1) + h_{\varphi_2}(\alpha_2),$$

because there are many more choices of abelian models to compute the left side of (3.6) than the right.

In nicer cases a useful criterion for computing entropy on von Neumann algebras is to consider the restriction of the automorphism to the centralizer of the state.

Proposition 3.4 [CNT] *Let M be a von Neumann algebra and φ a normal state. Let N_1, \dots, N_k be finite dimensional von Neumann subalgebras of M . Suppose they contain abelian subalgebras $A_j \subset N_j \cap M_{\varphi}$, where M_{φ} is the centralizer in M , such that the A_j pairwise commute and $A = \bigvee_{j=1}^k A_j$ is a masa in $N = \bigvee_{j=1}^k N_j$. Then*

$$H_{\varphi}(N_1, \dots, N_k) = S(\varphi|_N).$$

A good illustration of Proposition 3.4 is the case of Bernoulli shifts as described in Theorem 2.14. In the notation of 2.14 the state $\varphi = \bigotimes_{i \in \mathbb{Z}} \varphi_i$ on $B = \bigotimes_{i \in \mathbb{Z}} M_i$ satisfies the conditions of the proposition, so with α the shift, $h_{\varphi}(\alpha) = H_{\varphi|_{M_{\varphi}}}(\alpha|_{M_{\varphi}}) = S(\varphi_0)$.

4 Bogoliubov automorphisms

The main examples for which the C*-algebra entropy have been computed, are those of quasifree states of the CAR- and CCR-algebras and invariant Bogoliubov (or quasifree) automorphisms. The computations and results are quite similar, so for simplicity we restrict attention to the CAR-algebra. Let us recall the definitions.

Let H be a complex Hilbert space. The CAR-algebra $\mathcal{A}(H)$ over H is a C*-algebra with the property that there is a linear map $f \rightarrow a(f)$ of H into $\mathcal{A}(H)$ whose range generates $\mathcal{A}(H)$ as a C*-algebra and satisfies the canonical anticommutation relations

$$\begin{aligned} a(f)a(g)^* + a(g)^*a(f) &= (f, g)1, & f, g \in H, \\ a(f)a(g) + a(g)a(f) &= 0, \end{aligned}$$

where (\cdot, \cdot) is the inner product on H and 1 the unit of $\mathcal{A}(H)$. If $0 \leq A \leq 1$ is an operator on H , then the quasifree state ω_A on $\mathcal{A}(H)$ is defined by its values on products of the form $a(f_n)^* \dots a(f_1)^* a(g_1) \dots a(g_m)$ given by

$$\omega_A(a(f_n)^* \dots a(f_1)^* a(g_1) \dots a(g_m)) = \delta_{nm} \det((Ag_i, f_j)).$$

If U is a unitary operator on H then U defines an automorphism α_U on $\mathcal{A}(H)$, called a *Bogoliubov automorphism*, determined by

$$\alpha_U(a(f)) = a(Uf).$$

If U and A commute it is an easy consequence of the above definition of ω_A that α_U is ω_A -invariant. We shall in this section state a formula for the entropy $h_{\omega_A}(\alpha_U)$ and indicate the ideas in the computation when A is a scalar operator.

Connes suggested to the author that if ω_A is the trace τ the answer should be

$$(4.1) \quad h_\tau(\alpha_U) = \frac{\log 2}{2\pi} \int_0^{2\pi} m(U)(\theta) d\theta,$$

where $m(U)$ is the multiplicity function of the absolutely continuous part U_a of U . Then Voiculescu and Størmer [SV] showed this and more by solving the problem when A has pure point spectrum. Later on Narnhofer and Thirring [N-T1] and Park and Shin [P-S] independently extended the result to more general A . Finally Neshveyev [N] settled the problem completely in the general case. He and Golodets [G-N1], and before them Bezuglyi and Golodets [B-G] considered more general group actions than \mathbb{Z} .

If A has pure point spectrum there is an orthonormal basis (f_n) of H such that $Af_n = \lambda_n f_n$, $n \in \mathbb{N}$, $0 \leq \lambda_n \leq 1$. Define recursively operators

$$\begin{aligned} V_0 &= 1, & V_n &= \prod_{i=1}^n (1 - 2a(f_i)^* a(f_i)), & e_{11}^{(n)} &= a(f_n) a(f_n)^* \\ e_{12}^{(n)} &= a(f_n) V_{n-1}, & e_{21}^{(n)} &= V_{n-1} a(f_n)^*, & e_{22}^{(n)} &= a(f_n)^* a(f_n). \end{aligned}$$

Then the $e_{ij}^{(n)}$, $i, j = 1, 2$ form a complete set of 2×2 matrix units generating a I_2 -factor $M_2(\mathbb{C})_n$, and for distinct n and m $e_{ij}^{(n)}$ and $e_{kl}^{(m)}$ commute. Thus $\mathcal{A}(H) \simeq \bigotimes_1^\infty M_2(\mathbb{C})_n$, and

ω_A is a product state $\omega_A = \bigotimes_1^\infty \omega_{\lambda_n}^0$ with respect to this factorization, where ω_λ^0 is the state on $M_2(\mathbb{C})$ given by

$$\omega_\lambda^0 \left(\begin{pmatrix} a & b \\ c & a \end{pmatrix} \right) = (1 - \lambda)a + \lambda d.$$

In case $A = \lambda 1$ we write ω_λ for ω_A . Then α_U is ω_λ -invariant for all U . We consider the entropy $h_{\omega_\lambda}(\alpha_U)$.

Each unitary U is a direct sum $U = U_a \oplus U_s$, where U_a has spectral measure absolutely continuous with respect to Lebesgue measure $d\theta$ on the circle, while U_s has spectral measure singular with respect to $d\theta$. We shall as above denote by $m(U)$ the multiplicity function of U_a . The idea is now to approximate the case when

$$U = U_s \oplus U_1 \oplus \cdots \oplus U_n,$$

where each U_i acts on a Hilbert space H_i , $i = 1, \dots, n$, and U_i is unitarily equivalent to V^{p_i} , where V is a bilateral shift. Let us for simplicity ignore the complications due to the grading of $\mathcal{A}(H)$ as a direct sum of its even and odd parts. Then

$$\alpha_U = \alpha_{U_s} \otimes \alpha_{U_1} \otimes \cdots \otimes \alpha_{U_n}$$

and

$$\omega_\lambda = \omega_\lambda|_{\mathcal{A}(H_s)} \otimes \omega_\lambda|_{\mathcal{A}(H_1)} \otimes \cdots \otimes \omega_\lambda|_{\mathcal{A}(H_n)}.$$

Thus we could hope that

$$(4.2) \quad h_{\omega_\lambda}(\alpha_U) = h_{\omega_\lambda|_{\mathcal{A}(H_s)}}(\alpha_{U_s}) + \sum_{i=1}^n h_{\omega_\lambda|_{\mathcal{A}(H_i)}}(\alpha_{U_i}),$$

and thus restrict attention to the case when U is singular or a power of a bilateral shift. We do have problems because of (3.6), but life turns out nicely because we can as with the shift in 2.13 restrict attention to the diagonal, and the diagonal is contained in the even CAR-algebra, where the tensor product formulas above hold. First we take care of the singular part U_s .

Lemma 4.1 *If U has spectral measure singular with respect to the Lebesgue measure, and α_U is φ -invariant for a state φ , then $h_\varphi(\alpha_U) = 0$.*

Thus in (4.2) we can forget about U_s . If $U = V^p$ with V a bilateral shift and $p \in \mathbb{Z}$, then

$$h_{\omega_\lambda}(\alpha_U) = h_{\omega_\lambda}((\alpha_V)^p) = |p| h_{\omega_\lambda}(\alpha_V).$$

If we write $\mathcal{A}(H) = \bigotimes_{n=-\infty}^\infty M_2(\mathbb{C})_n$, then on the diagonal α_V is the shift, so like in 2.14, see also Proposition 3.4, we get

$$h_{\omega_\lambda}(\alpha_V) = \eta(\lambda) + \eta(1 - \lambda).$$

Now $|p|$ is the multiplicity $m(U)$ of U , and since $\frac{1}{2\pi}d\theta$ is the normalized Haar measure on the circle, it is not surprising that we have

Theorem. 4.2 Let U be a unitary operator on H and $\lambda \in [0, 1]$. Then $h_{\omega_\lambda}(\alpha_U) = \frac{1}{2\pi} \left(\eta(\lambda) + \eta(1 - \lambda) \right) \int_0^{2\pi} m(U)(\theta) d\theta$.

Note that if $\lambda = 1/2$, $\omega_\lambda = \tau$, so we get formula (4.1). For more general A we use direct integral theory with respect to the von Neumann algebra generated by U_a . If A commutes with U , $A = A_a \oplus A_s$, where $A_a = \int_0^{2\pi \oplus} A(\theta) d\theta$, $H = \int_0^{2\pi \oplus} H_\theta d\theta$, and $H_\theta = 0$ if $m(U)(\theta) = 0$, and $A(\theta) \subset B(H_\theta)$.

We can now state the main theorem in this section, see [SV], [P-S], [N-T1] and [N].

Theorem. 4.3 Let $0 \leq A \leq 1$ and U be a unitary operator commuting with A . Then

$$h_{\omega_A}(\alpha_U) = \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}(\eta(A(\theta)) + \eta(1 - A(\theta))) d\theta.$$

Furthermore, [N], $h_{\omega_A}(\alpha_U) < \infty$ if and only if $A(\theta)$ has pure point spectrum for almost all $\theta \in [0, 2\pi)$.

Entropy is far from a complete conjugacy invariant for Bogoliubov automorphisms. It has been shown by Golodets and Neshveyev [G-N3] in the case of the CCR-algebra, that there exists a quasifree state ω and a one-parameter family α_θ , $\theta \in [0, 2\pi)$, of ω -invariant Bogoliubov automorphisms with the same positive entropy $h_\omega(\alpha_\theta)$ and such that if M is the weak closure of the CCR-algebra in the GNS-representation of ω , then the W^* -dynamical systems $(M, \omega, \alpha_\theta)$ are pairwise nonconjugate. In this case M is a factor of type III₁ and the centralizer M_ω of ω in M is the scalars. Thus the situation is quite different from that encountered in Proposition 3.4. Such an example had previously been found by Connes [Co].

5 The entropy of Sauvageot and Thouvenot

Sauvageot and Thouvenot [S-T] have given an alternative definition of entropy, which is close to that of [CNT], but which has technical advantages in some cases. Again we look at all possible ways a state can be written as convex combinations of other states.

Let (A, φ, α) be a unital C^* -dynamical system, and let (C, μ, β) be an abelian C^* -dynamical system. A *stationary coupling* of these two systems is an $\alpha \otimes \beta$ -invariant state λ on $A \otimes C$ such that $\lambda|_A = \varphi$, $\lambda|_C = \mu$. If P is a finite dimensional C^* -subalgebra of C with atoms p_1, \dots, p_r let

$$\varphi_i(x) = \mu(p_i)^{-1} \lambda(x \otimes p_i),$$

whenever $\mu(p_i) \neq 0$, as we may assume. Then

$$\varphi = \sum_{i=1}^r \mu(p_i) \varphi_i$$

is φ written as a convex sum of states. Let

$$P^- = \bigvee_{i=1}^{\infty} \beta^i(P) ;$$

then from the classical theory

$$H_{\mu}(P, \beta) = \lim_{n \rightarrow \infty} \frac{1}{n} H_{\mu} \left(\bigvee_{i=0}^{n-1} \beta^i(P) \right) = H_{\mu}(P|P^-) .$$

Definition 5.1 *The Sauvageot-Thouvenot entropy $h'_{\varphi}(\alpha)$ of the system (A, φ, α) is the supremum over all abelian systems (C, μ, β) as above of the quantities*

$$H_{\mu}(P|P^-) - H_{\mu}(P) + \sum_{i=1}^r \mu(p_i) S(\varphi_i, \varphi) .$$

Note that if $\lambda = \varphi \otimes \mu$ then $\varphi_i = \varphi$, hence $S(\varphi_i, \varphi) = 0$, and $H(P|P^-) - H_{\mu}(P) \leq 0$. Thus if $\lambda = \varphi \otimes \mu$ is the only stationary coupling then $h'_{\varphi}(\alpha) = 0$, a fact we shall need later.

Theorem. 5.2 [S-T] *If (A, φ, α) is a C^* -dynamical system with A nuclear or a W^* -dynamical system with A injective then $h'_{\varphi}(\alpha)$ equals the CNT-entropy $h_{\varphi}(\alpha)$.*

In order to understand the relation between h_{φ} and h'_{φ} better, let us prove the inequality $h'_{\varphi}(\alpha) \leq h_{\varphi}(\alpha)$ in some detail. The opposite inequality holds in general.

Let (C, μ, β) be an abelian system and λ a stationary coupling. Let P be a finite dimensional subalgebra of C with atoms p_1, \dots, p_r . Let $m \in \mathbb{N}$. We shall show

$$(5.1) \quad h_{\varphi}(\alpha) \geq H_{\mu}(P|P^-) - H_{\mu}(P) + \sum_{i=1}^r \mu(p_i) S(\varphi_i, \varphi) ,$$

where $\varphi_i(x) = \mu(p_i)^{-1} \lambda(x \otimes p_i)$.

φ has the decomposition

$$\varphi = \sum_{i_1, \dots, i_m} \varphi_{i_1 \dots i_m} ,$$

where

$$\varphi_{i_1 \dots i_m}(x) = \lambda(x \otimes p_{i_1} \beta(p_{i_2}) \dots \beta^{m-1}(p_{i_m})) .$$

We thus have a completely positive map $\rho : A \rightarrow C$ defined by

$$\rho(x) = \sum \frac{\varphi_{i_1 \dots i_m}(x)}{\mu(p_{i_1 \dots i_m})} p_{i_1 \dots i_m} ,$$

where $p_{i_1 \dots i_m} = p_{i_1} \beta(p_{i_2}) \dots \beta^{m-1}(p_{i_m})$ are the atoms in $B = \bigvee_0^{m-1} \beta^j(P)$. The μ -invariant conditional expectation $E_j : B \rightarrow \beta^j(P)$ is determined by

$$E_j(p_{i_1 \dots i_m}) = \frac{\mu(p_{i_1 \dots i_m})}{\mu(\beta^j(p_{i_j}))} \beta^j(p_{i_j}).$$

Let γ be a unital completely positive map from a finite dimensional C*-algebra into A . We found above an abelian model for $(A, \varphi, \gamma, \alpha \circ \gamma, \dots, \alpha^{m-1} \circ \gamma)$, namely (B, E_j, ρ, μ) . We find

$$\begin{aligned} E_j \circ \rho(x) &= \sum_{i_1 \dots i_m} \frac{\varphi_{i_1 \dots i_m}(x)}{\mu(\beta^j(p_{i_j}))} \beta^j(p_{i_j}) \\ &= \sum_{i_j} \frac{\lambda(x \otimes \beta^j(p_{i_j}))}{\mu(\beta^j(p_{i_j}))} \beta^j(p_{i_j}) \\ &= \sum_i \frac{\lambda(\alpha^{-j}(x) \otimes p_i)}{\mu(p_i)} \beta^j(p_i) \\ &= \sum_i \varphi_i(\alpha^{-j}(x)) \beta^j(p_i) \end{aligned}$$

by invariance of μ and λ with respect to β and $\alpha \otimes \beta$ respectively. From Definition 3.1 we obtain

$$\begin{aligned} H_\varphi(\gamma, \alpha \circ \gamma, \dots, \alpha^{m-1} \circ \gamma) &\geq S(\mu|B) - \sum_{j=0}^{m-1} S(\mu|\beta^j(P)) \\ &\quad + \sum_{j=0}^{m-1} \sum_{i=1}^r \mu(\beta^j(p_i)) S(\varphi_i \circ \alpha^{-j} \circ \alpha^j \circ \gamma, \varphi \circ \alpha^j \circ \gamma) \\ &= S(\mu|B) - mS(\mu|P) + m \sum_{i=1}^r \mu(p_i) S(\varphi_i \circ \gamma, \varphi \circ \gamma). \end{aligned}$$

Thus

$$\frac{1}{m} H_\varphi(\gamma, \alpha \circ \gamma, \dots, \alpha^{m-1} \circ \gamma) \geq H_\mu(P|P^-) - H_\mu(P) + \sum_i \mu(p_i) S(\varphi_i \circ \gamma, \varphi \circ \gamma).$$

If A is an injective von Neumann algebra or a nuclear C*-algebra we can find a net of maps $(\gamma_\omega)_\omega$ such that $S(\varphi_i \circ \gamma_\omega, \varphi \circ \gamma_\omega) \rightarrow S(\varphi_i, \varphi)$, see [O-P, 5.29 and 5.30]. Thus (5.1) holds, and $h_\varphi(\alpha) \geq h'_\varphi(\alpha)$. \square

If α is an automorphism of a C*-algebra A , and B is a C*-subalgebra of A such that $\alpha(B) = B$, then every α -invariant state on B has an α -invariant extension to A . If we apply this to a stationary coupling λ on $N \otimes C$ we can extend λ to $A \otimes C$ and prove the following result.

Proposition 5.3 [N-S1] *Let A be a unital C^* -algebra and α an automorphism of A . Let B be an α -invariant C^* -subalgebra of A and ψ an α -invariant state of B . Then for each $\varepsilon > 0$ there exists an α -invariant state φ of A such that $\varphi|_B = \psi$ and $h'_\varphi(\alpha) > h'_\psi(\alpha|_B) - \varepsilon$.*

It is well-known from subfactor theory that properties of a subfactor of finite index often are kept by the larger factor. The following is an analogous result for entropy.

Proposition 5.4 [N-S1] *Let (A, φ, α) be a unital C^* -dynamical system. Let $B \subset A$ be an α -invariant C^* -subalgebra with $1 \in B$. Suppose there is a conditional expectation $E : A \rightarrow B$ such that $E \circ \alpha = \alpha \circ E$, $\varphi \circ E = \varphi$, and $E(x) \geq cx$ for all $x \in A^+$ for some real number $c > 0$. Then*

$$h'_\varphi(\alpha) = h'_\varphi(\alpha|_B).$$

Proof. The inequality $h'_\alpha(\alpha) \geq h'_\varphi(\alpha|_B)$ follows by monotonicity. To prove the opposite inequality we consider A in its GNS-representation with respect to φ , so we may assume A and B are von Neumann algebras and φ and E normal. Let (C, μ, β) be as before. Then $\varphi_i \circ E \geq c\varphi_i$, hence by (2.4) $S(\varphi_i, \varphi_i \circ E) \leq S(\varphi_i, c\varphi_i) = -\log c$. Thus by (2.3)

$$\begin{aligned} \sum \mu(p_i) S(\varphi_i, \varphi) &= \sum \mu(p_i) (S(\varphi_i|_B, \varphi|_B) + S(\varphi_i, \varphi_i \circ E)) \\ &\leq \sum \mu(p_i) S(\varphi_i|_B, \varphi|_B) - \log c \end{aligned}$$

It follows from Definition 5.1 that $h'_\varphi(\alpha) \leq h'_\varphi(\alpha|_B) - \log c$. But this inequality must hold for α^m , $m \in \mathbb{N}$, as well. Hence

$$h'_\varphi(\alpha) = \frac{1}{m} h'_\varphi(\alpha^m) \leq \frac{1}{m} h'_\varphi(\alpha^m|_B) - \frac{1}{m} \log c = h'_\varphi(\alpha|_B) - \frac{1}{m} \log c,$$

and $h'_\varphi(\alpha) \leq h'_\varphi(\alpha|_B)$. □

From Theorem 5.2 it follows that if A and B are nuclear C^* -algebras or injective von Neumann algebras then

$$h_\varphi(\alpha) = h_\varphi(\alpha|_B).$$

Theorem. 5.5 [N-S1] *If (M, φ, α) is a W^* -dynamical system with M of type I and Z is the center of M , then $h_\varphi(\alpha) = h_\varphi(\alpha|_Z)$.*

Indeed, if M is homogeneous of type I and φ is a trace the result is immediate from Proposition 5.4, since the center valued trace satisfies the conditions of E . The proof in the general case is a technical adjustment to this idea.

By a similar argument, see [G-N2], one can show that if N is an injective von Neumann algebra with a normal state ω , and (M, φ, α) is a W^* -dynamical system then

$$(5.2) \quad h_{\omega \otimes \varphi}(\text{id} \otimes \alpha) = h_\varphi(\alpha).$$

When we apply Theorem 5.5 to inner automorphisms we obtain

Corollary 5.6 [N-S1] *If (A, φ, α) (resp. (M, φ, α)) is a C^* - (resp. W^* -) dynamical system with A (resp. M) of type I, and $\alpha = \text{Ad } u$ is an inner automorphism, then $h_\varphi(\alpha) = 0$.*

Corollary 5.7 [S2], [N-S1] *Let R be the hyperfinite II_1 -factor, A a Cartan subalgebra of R and u a unitary operator in A . If φ is a normal state such that u belongs to its centralizer R_φ , then $h_\varphi(\text{Ad } u) = 0$.*

Proof. As pointed out in 2.14 A is conjugate to the infinite tensor product of the diagonals D_i considered in 2.13. Thus there exists an increasing sequence $N_1 \subset N_2 \subset \dots$ of full matrix algebras with union dense in R such that $A \cong A_n \otimes B_n$, where $A_n = N_n \cap A$, $B_n = N'_n \cap A$. Then $M_n = N_n \otimes B_n$ is of type I and contains A . By Corollary 5.6 $h_\varphi(\text{Ad } u|_{M_n}) = 0$. Since $\bigcup_{n=1}^{\infty} M_n$ is weakly dense in R , $h_\varphi(\text{Ad } u) = 0$. \square

In particular it follows that if α_T is the automorphism of $L^\infty(X, B, \mu)$ induced by an ergodic measure preserving transformation T and u_T is the unitary in $R = L^\infty(X, B, \mu) \rtimes_{\alpha_T} \mathbb{Z}$ which implements α_T , and A is the masa in R generated by u_T , then A is a singular masa, i.e. not Cartan, whenever $H(T) > 0$, because by monotonicity $H(\text{Ad } u_T) \geq H(\alpha_T) = H(T)$.

A related consequence is due to Brown [Br2]. We say a C^* -algebra A is AT if it is an inductive limit of circle algebras, i.e. $A = \overline{\bigcup_{i=1}^r A_i}$, norm closure, where $A_1 \subset A_2 \subset \dots$ are C^* -algebras of the form

$$B = \bigoplus_{j=1}^k M_{n_j}(C(X_j)),$$

where X_j is homeomorphic to either the circle \mathbb{T} , $[0, 1]$, or a point.

Corollary 5.8 [Br2] *If A is an AT algebra and $B \subset A$ is a circle algebra we cannot always find a sequence (A_i) as above with $A_1 = B$.*

Proof. Let $C = \bigotimes_{i \in \mathbb{Z}} M_i$, where $M_i = M_2(\mathbb{C})$, and let α be the 2-shift on C , and τ the unique tracial state, see Remark 2.13. Then $H_\tau(\alpha) = \log 2$. Let $A = C \rtimes_{\alpha} \mathbb{Z}$ (see section 7 for the detailed definition), and let u be the unitary operator in A which implements α . By [BKRS] A is an AT algebra, and by monotonicity $h_\tau(\text{Ad } u) \geq h_\tau(\alpha) = \log 2 > 0$. However, if u belongs to a circle algebra A_i in a sequence as above, then by Corollary 5.6 $h_\tau(\text{Ad } u|_{A_i}) = 0$, hence $h_\tau(\text{Ad } u) = 0$, which proves that the circle algebra $C^*(u)$ cannot be A_1 in a sequence (A_i) as above. \square

For another entropy result on AT algebras see [De].

6 Voiculescu's approximation entropies

As mentioned in the introduction Voiculescu [V] has introduced entropies which are refinements of mean entropy and which provide a very nice technique to study entropy.

Let M be a hyperfinite von Neumann algebra with a faithful normal tracial state τ . Let $Pf(M)$ denote the family of finite subsets of M . Modifying the notation introduced before Lemma 2.9 we write $\omega \subset^\delta \mathcal{X}$ if $\omega \in Pf(M)$, $\mathcal{X} \subset M$, if for each $x \in \omega$ there is a $y \in \mathcal{X}$ such that $\|x - y\|_2 < \delta$. Let further $\mathcal{F}(M)$ denote the family of finite dimensional C^* -subalgebras of M . As noted in Section 2, if $A \in \mathcal{F}(M)$ then $\text{rank } A$ is the dimension of a masa in A .

Definition 6.1 [V] *If $\omega \in Pf(M)$, $\delta > 0$ put*

$$r_\tau(\omega, \delta) = \inf\{\text{rank } A : A \in \mathcal{F}(M), \omega \subset^\delta A\},$$

called the δ -rank of ω .

Note that a slightly different choice for $r_\tau(\omega, \delta)$ would be to replace $\text{rank } A$ by $\exp(H_\tau(A))$, see [C5] and [G-S2].

Definition 6.2 [V] *If α is a τ -invariant automorphism of M and $\delta > 0$, $\omega \in Pf(M)$ we put:*

$$\begin{aligned} ha_\tau(\alpha, \omega, \delta) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_\tau\left(\bigcup_{j=0}^{n-1} \alpha^j(\omega), \delta\right) \\ ha_\tau(\alpha, \omega) &= \sup_{\delta > 0} ha_\tau(\alpha, \omega, \delta) \\ ha_\tau(\alpha) &= \sup\{ha_\tau(\alpha, \omega) : \omega \in Pf(M)\}. \end{aligned}$$

$ha_\tau(\alpha)$ is the approximation entropy of α .

An alternative is to take \liminf in the definition of $ha_\tau(\alpha, \omega, \delta)$. Then we get the *lower approximation* entropy $\ell ha_\tau(\alpha)$.

As for the previous entropies we have $ha_\tau(\alpha^k) = |k|ha_\tau(\alpha)$, $k \in \mathbb{Z}$. The proof that $ha_\tau(\alpha^{-1}) = ha_\tau(\alpha)$ is very easy; indeed

$$r_\tau\left(\bigcup_0^{n-1} \alpha^j(\omega), \delta\right) = r_\tau\left(\alpha^{-n+1}\left(\bigcup_0^{n-1} \alpha^j(\omega)\right), \delta\right) = r_\tau\left(\bigcup_0^{n-1} \alpha^{-j}(\omega), \delta\right).$$

The analogue of the Kolmogoroff-Sinai Theorem takes the following form.

Proposition 6.3 [V] *Let $\omega_j \in Pf(M)$, $j \in \mathbb{N}$, $\omega_1 \subset \omega_2 \subset \dots$ be a sequence such that $\bigcup_{j \in \mathbb{N}} \bigcup_{n \in \mathbb{Z}} \alpha^n(\omega_j)$ generates M as a von Neumann algebra. Then*

$$ha_\tau(\alpha) = \sup_{j \in \mathbb{N}} ha_\tau(\alpha, \omega_j).$$

Proposition 6.4 [V] (i) If $A \in \mathcal{F}(M)$ and $\omega \in Pf(M)$ generates A as a C^* -algebra then $H_\tau(A, \alpha) \leq lha_\tau(\alpha, \omega)$.
(ii) $H(\alpha) \leq lha_\tau(\alpha) \leq ha_\tau(\alpha)$.

Proof. It suffices to show (i). Let $\varepsilon > 0$. By Lemma 2.9 there exists $\delta > 0$ such that if $B \in \mathcal{F}(M)$ satisfies $A \subset^\delta B$ then $H(A|B) < \varepsilon$. By hypothesis on ω there exists therefore $\delta_1 > 0$ such that if $\omega \subset^{\delta_1} B$ then $H(A|B) < \varepsilon$. This also implies that if $\alpha^j(\omega) \subset^{\delta_1} B$ then $H(\alpha^j(A)|B) < \varepsilon$. Put $r(n) = r_\tau(\bigcup_0^{n-1} \alpha^j(\omega), \delta_1)$. Then there exists $B \in \mathcal{F}(M)$ with $\text{rank } B = r(n)$ and $\alpha^j(A) \subset^{\delta_1} B$ for $0 \leq j \leq n-1$. Hence by Property (F) in Section 2,

$$\begin{aligned} H(A, \alpha(A), \dots, \alpha^{n-1}(A)) &\leq H(B) + \sum_{j=0}^{n-1} H(\alpha^j(A)|B) \\ &\leq \log r(n) + n\varepsilon, \end{aligned}$$

so that $H(A, \alpha) \leq ha_\tau(\alpha, \omega, \delta_1) + \varepsilon$, proving the proposition. \square

In general it can be quite difficult to know when an algebra B as in the above proof satisfies $\text{rank } B = r(n)$, hence to compute $r(n)$. A case when it is easy is that of the n -shift. In the notation of Remark 2.13 let $R = \bigotimes_{i \in \mathbb{Z}} (M_i, \tau_i)$ with $M_i = M_n(\mathbb{C})$, and α be the shift. Let $A = M_0 \in \mathcal{F}(R)$. Let, as is often done, ω be a complete set of matrix units for A . By Proposition 6.3 $ha_\tau(\alpha, \omega) = ha_\tau(\alpha)$. On the other hand $\bigcup_{j=0}^{k-1} \alpha^j(\omega) \subset \bigvee_{j=0}^{k-1} \alpha^j(A)$, which is a I_{n^k} -factor, hence

$$r_\tau\left(\bigcup_{j=0}^{n-1} \alpha^j(\omega), \delta\right) \leq n^k \quad \text{for all } \delta > 0.$$

Thus, by the above and Proposition 6.4 $ha_\tau(\alpha) = ha_\tau(\alpha, \omega) \leq \log n = H(\alpha) \leq ha_\tau(\alpha)$, so $ha_\tau(\alpha) = \log n$.

One test for any definition of entropy is that it should coincide with the classical entropy in the abelian case. Via an application of the Shannon, Breiman, McMillan Theorem the approximation entropy does this [V].

We remarked in (2.6) that the entropy $H(\alpha)$ is superadditive on tensor products. For the approximation entropy the inequality goes the other way, i.e.

$$ha_{\tau_1 \otimes \tau_2}(\alpha_1 \otimes \alpha_2) \leq ha_{\tau_1}(\alpha_1) + ha_{\tau_2}(\alpha_2).$$

Hence to show equality it suffices to show that the two entropies coincide. In Section 10 we shall look at such cases.

In the above treatment of the approximation entropy the trace played a minor role. If A is an AF-algebra we can do essentially the same, where we now replace the distance $\|\cdot\|_2$ with the operator norm. Then we get the entropy Voiculescu denotes by $\hat{h}(\alpha)$ – the *topological approximation entropy* of α .

The most flexible and therefore probably the most useful of Voiculescu's approximation entropies are the completely positive ones. We consider the von Neumann algebra definition first. Let (M, φ, α) be a W^* -dynamical system with M injective and φ faithful. Let $\|x\|_\varphi = \varphi(x^*x)^{1/2}$ be the φ -norm on M . Let

$$CPA(M, \varphi) = \{(\rho, \psi, B) : B \text{ a finite dimensional } C^*\text{-algebra, } \rho : M \rightarrow B, \psi : B \rightarrow M \\ \text{are unital completely positive maps such that } \varphi \circ \psi \circ \rho = \varphi\}.$$

Definition 6.5 [V] *If $\omega \in Pf(M)$ and $\delta > 0$ the completely positive δ -rank is*

$$rcp_\varphi(\omega, \delta) \\ = \inf\{\text{rank } B : (\rho, \psi, B) \in CPA(M, \varphi), \|\psi \circ \rho(x) - x\|_\varphi < \delta \text{ for all } x \in \omega\}.$$

Then we continue as in Definition 6.2 to define the *completely positive approximation entropy* $h_{cpa_\varphi}(\alpha)$. Again we can prove much the same results as for the approximation entropy $h_{a_\tau}(\alpha)$.

The C^* -algebra version of the above definition is like the corresponding entropy hat $\hat{h}(\alpha)$ independent of invariant states. Voiculescu defined this entropy for nuclear C^* -algebras, but later on Brown [Br1] saw that one can develop the theory for exact C^* -algebras.

Definition 6.6 [Br1] *Let A be a C^* -algebra and $\pi : A \rightarrow B(H)$ a faithful $*$ -representation. Then*

$$CPA(\pi, A) = \{(\rho, \psi, B) : \rho : A \rightarrow B, \psi : B \rightarrow B(H) \text{ are contractive} \\ \text{completely positive maps, } B \text{ is finite dimensional } C^*\text{-algebra}\}$$

Let $\omega \in Pf(A)$, $\delta > 0$. Then

$$rcp(\pi, \omega, \delta) = \inf\{\text{rank } B : (\rho, \psi, B) \in CPA(\pi, A), \text{ and} \\ \|\psi \circ \rho(x) - \pi(x)\| < \delta \text{ for all } x \in \omega\}.$$

It follows from [K] that the C^* -algebras for which this definition makes sense are the exact C^* -algebras. We shall therefore assume A is exact and define the *topological entropy* of $\alpha \in \text{Aut}(A)$, denoted by $ht(\pi, \alpha)$ as in Definition 6.2.

The first result to be proved is that the definition is independent of π , hence we can define

$$ht(\alpha) = ht(\pi, \alpha),$$

or if $A \subset B(H)$ as $ht(\text{id}_A, \alpha)$. The proof is a good illustration of the techniques involved. We may assume A is unital. Let $\pi_i : A \rightarrow B(H_i)$, $i = 1, 2$, be faithful $*$ -representations. Let $\omega \in Pf(A)$, $\delta > 0$. It suffices by symmetry to show

$$(6.1) \quad rcp(\pi_1, \omega, \delta) \geq rcp(\pi_2, \omega, \delta).$$

Choose $(\rho, \psi, B) \in CPA(\pi_1, A)$ such that $\text{rank } B = rcp(\pi_1, \omega, \delta)$, and

$$\|\psi \circ \rho(x) - \pi_1(x)\| < \delta, \quad x \in \omega.$$

Consider the map $\pi_2 \circ \pi_1^{-1} : \pi_1(A) \rightarrow B(H_2)$. From Arveson's extension theorem for completely positive maps [Ar] there exists a unital completely positive map $\Phi : B(H_1) \rightarrow B(H_2)$ extending $\pi_2 \circ \pi_1^{-1}$. Thus we have $(\rho, \Phi \circ \psi, B) \in CPA(\pi_2, A)$ and $\|\Phi \circ \psi \circ \rho(x) - \pi_2(x)\| < \delta$ for $x \in \omega$, since $\pi_2(x) = \Phi \circ \pi_1(x)$. Thus (6.1) follows. \square

Again we can prove the basic properties of entropy. Note that monotonicity is an easy consequence of the fact that a C^* -subalgebra of an exact C^* -algebra is itself exact. The analogous result is not true for nuclear C^* -algebra. We conclude this section with a theorem which compares the entropies defined so far.

Theorem. 6.7 [V] (i) If (M, τ, α) is a W^* -dynamical system with τ a trace then $H_\tau(\alpha) \leq hcpa_\tau(\alpha) \leq ha_\tau(\alpha)$
(ii) [V] If (A, φ, α) is a C^* -dynamical system with A an AF -algebra then $ht(\alpha) \leq hat(\alpha)$.
(iii) [V], [D2] If in (ii) A is exact then $h_\varphi(\alpha) \leq ht(\alpha)$.

7 Crossed products

If (A, ϕ, α) is a C^* -dynamical system a natural problem is to compute the entropy of the extension of α to the crossed product $A \times_\alpha \mathbb{Z}$. More generally, if G is a discrete subgroup of $\text{Aut } A$ and $\beta \in \text{Aut } A$ commutes with G , compute the entropy of the extension of β to $A \times G$. The first positive result is due to Voiculescu [V], who showed that for an ergodic measure preserving Bernoulli transformation T on a Lebesgue probability space (X, B, μ) , $H(T) = H(\text{Ad } u_T)$, where u_T is the unitary operator in $L^\infty(X, B, \mu) \times_T \mathbb{Z}$ which implements T . Later on several extensions have appeared, see [Br1], [B-C], [D-S], [G-N2]. We first recall the definition of crossed products.

Let A be a unital C^* -algebra, G a discrete group, and $\alpha : G \rightarrow \text{Aut } A$ a group homomorphism. Let $\sigma : A \rightarrow B(H)$ be a faithful nondegenerate representation. Let

$$\pi : A \rightarrow B(\ell^2(G, H)) \cong B(\ell^2(G)) \otimes B(H)$$

be the representation given by

$$(\pi(x)\xi)(h) = \sigma(\alpha_{h^{-1}}(x))(\xi(h)), \quad x \in A, \quad \xi \in \ell^2(G, H), \quad h \in G,$$

and let λ be the unitary representation of G on $\ell^2(G, H)$ given by

$$(\lambda_g \xi)(h) = \xi(g^{-1}h), \quad \xi \in \ell^2(G, H), \quad g, h \in G.$$

Then we have

$$\lambda_{g^{-1}} \pi(x) \lambda_g = \pi(\alpha_g(x)), \quad x \in A, \quad g \in G.$$

The *reduced crossed product C^* -algebra* $A \times_\alpha G$ is the norm closure of the linear span of the set $\{\pi(x)\lambda_g : x \in A, g \in G\}$. Up to isomorphism $A \times_\alpha G$ is independent of the choice of σ , so for simplicity we assume henceforth that σ is the identity map. Let $\{\xi_h\}_{h \in G}$ be

the standard orthonormal basis in $\ell^2(G)$, so $\xi_h(g) = \delta_{g,h}$, $g, h \in G$. Then if $\xi = \xi_h \otimes \psi$ with $\psi \in H$ we have

$$(\lambda_g \xi)(h) = \xi_{g^{-1}h} \otimes \psi = ((\ell_g \otimes 1)\xi)(h),$$

where ℓ_g is the left regular representation of G . Furthermore

$$\pi(x)\xi = \pi(x)(\xi_h \otimes \psi) = \xi_h \otimes \alpha_{g^{-1}}(x)\psi.$$

By the above, since we may consider A as a subalgebra of $B(\ell^2(G, H))$, we may also assume from the outset that α is implemented by a unitary representation $g \rightarrow U_g$, $g \in G$, of G . Thus we have

$$(1 \otimes U_g)^* \pi(x) \lambda_h (1 \otimes U_g) = \pi(\alpha_g(x)) \lambda_h.$$

For simplicity let us assume G is abelian – the argument works for G amenable. We follow the approach of [S-S] and [Br1]. Let $e_{p,q} \in B(\ell^2(G))$ denote the standard matrix units, i.e.

$$e_{p,q}(\xi_t) = \delta_{q,t} \xi_p,$$

where $\delta_{q,t}$ is the Kronecker δ . Then we have

$$\pi(x) \lambda_g = \sum_{t \in G} e_{t,t-g} \otimes \alpha_{-t}(x), \quad x \in A, \quad g \in G.$$

In particular, if $\beta \in \text{Aut } A$, and we assume as before that $\beta = \text{Ad } v$ for a unitary $v \in B(H)$, then if β commutes with all α_g then

$$\begin{aligned} \text{Ad}(1 \otimes v)(\pi(x) \lambda_g) &= \sum e_{t,t-g} \otimes v \alpha_{-t}(x) v^* = \\ &= \sum e_{t,t-g} \otimes \alpha_{-t}(\beta(x)) \\ &= \pi(\beta(x)) \lambda_g. \end{aligned}$$

Thus β extends to an automorphism $\hat{\beta} = \text{Ad}(1 \otimes v)$ of $A \times_\alpha G$.

If $F \subset G$ is a finite set let P_F denote the orthogonal projection of $\ell^2(G)$ onto span $\{\xi_t : t \in F\}$. Then we find

$$(P_F \otimes 1)(\pi(x) \lambda_g)(P_F \otimes 1) = \sum_{t \in F \cap (F+g)} e_{t,t-g} \otimes \alpha_{-t}(x) \in M_F \otimes A,$$

where $M_F = P_F B(\ell^2(G)) P_F$.

In order to compute the entropy of $\hat{\beta}$ on $A \times_\alpha G$ the idea is now to start with a triple $(B, \rho, \psi) \in CPA(\text{id}_A, A)$ and extend it to a triple $(M_F \otimes B, \Phi, \Psi) \in CPA(\text{id}_{A \times_\alpha G}, A \times_\alpha G)$ such that we can control the estimates. If $f \in L^\infty(G)$ has support contained in F let m_f denote the corresponding multiplication operator on $\ell^2(G)$, and define

$$T_f(x) = \sum_{t \in G} \ell_g^* \otimes U_g(m_f \otimes 1)x((m_f^* \otimes 1)\ell_g \otimes U_g^*), \quad x \in B(\ell^2(G, H)).$$

Note that by amenability we can assume $\|f\|_2 = 1$ and $f * \tilde{f}(g_i)$ is close to 1 on a given set g_1, \dots, g_k determining F where $\tilde{f}(g) = \overline{f(-g)}$. With ρ and ψ as above we put

$$\Phi_F(x) = (P_F \otimes 1)x(P_F \otimes 1), \quad x \in B(\ell^2(G, H)).$$

Then $\Phi_F(A \times_\alpha G) \subset M_F \otimes A$, so that

$$(M_F \otimes B, (1 \otimes \rho) \circ \Phi_F, T_f \circ (1 \otimes \psi)) \in CPA(\text{id}_{A \times_\alpha G}, A \times_\alpha G)$$

is the desired triple extending (B, ρ, ψ) . Since $\text{rank}(M_F \otimes B) = \text{card } F \cdot \text{rank } B$, all that remains is to choose F with some care depending on a given set $\omega \in Pf(A \times_\alpha G)$, which we may suppose is of the form $\omega = \{\pi(x_i)\lambda_{g_i}; i = 1, 2, \dots, n\}$.

The above construction essentially works for all the different entropies considered, and even for $\beta \in \text{Aut } A$ commuting with all α_g when G is amenable.

Theorem. 7.1 *Let A be a unital C^* -algebra, G a discrete amenable group and $\alpha : G \rightarrow \text{Aut } A$ a representation. Let $\beta \in \text{Aut } A$ commute with all $\alpha_g, g \in G$. Let $\hat{\beta}$ be the natural extension of β to $\text{Aut}(A \times_\alpha G)$. Then we have*

(i) [D-S], [C6]. *If A is exact then $ht(\hat{\beta}) = ht(\beta)$.*

(ii) [G-N2]. *If A is an injective von Neumann algebra and φ a normal state which is both G - and β -invariant then, if φ is identified with its canonical extension to $A \times_\alpha G$,*

$$h_{cpa_\varphi}(\hat{\beta}) = h_{cpa_\varphi}(\beta), \quad \text{and} \quad h_\varphi(\hat{\beta}) = h_\varphi(\beta).$$

Note that when $A = L^\infty(X, B, \mu)$ and $\beta = \alpha_1 = \alpha_T, G = \mathbb{Z}$, the theorem implies the result of Voiculescu alluded to in the first paragraph of the section. When G is abelian and $\beta = \alpha_g$ some $g \in G$, part (i) was proved by Brown [Br1]. A variation of (ii) can also be found in [B-C].

Sometimes one can prove results on operator algebras by representing them as crossed products, see e.g. the proof of Corollary 5.8. Another example is \mathcal{O}_∞ – the universal C^* -algebra generated by isometries $\{S_i\}_{i \in \mathbb{Z}}$ which satisfy the relation

$$\sum_{i=-r}^r S_i S_i^* \leq 1 \quad \text{for all } r \in \mathbb{N}.$$

Every bijection $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}$ defines an automorphism, also denoted by α of \mathcal{O}_∞ by $\alpha(S_i) = S_{\alpha(i)}$. By [Cu] there exist an AF-algebra $B, \Phi \in \text{Aut } B$, an imbedding $\pi : \mathcal{O}_\infty \rightarrow B \times_\Phi \mathbb{Z}$, and a projection $p \in B$, such that $\pi(\mathcal{O}_\infty) = p(B \times_\Phi \mathbb{Z})p$. By using techniques similar to those used to prove Theorem 7.1 we have

Theorem. 7.2 [B-C] *If $\alpha \in \text{Aut } \mathcal{O}_\infty$ is induced by a bijective function $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}$ then $ht(\alpha) = 0$. In particular, if φ is an α -invariant state on \mathcal{O}_∞ then $h_\varphi(\alpha) = 0$.*

Note that the last statement follows from the first and Theorem 6.7 since $h_\varphi(\alpha) \leq ht(\alpha)$, since \mathcal{O}_∞ is nuclear. For a closely related result see [C-N]. This theorem is the first we shall encounter, which shows that if a C*-dynamical system (A, φ, α) is highly nonabelian then the entropy of α tends to be small.

A problem related to the above is the computation of the entropy of the canonical endomorphism Φ of the C*-algebra \mathcal{O}_n of Cuntz [Cu], which is the C*-algebra generated by n isometries S_1, \dots, S_n such that $\sum_{i=1}^n S_i S_i^* = 1$. Analogously to \mathcal{O}_∞ , \mathcal{O}_n can be written as a crossed product $B \times_\sigma \mathbb{N}$, where $B = \bigotimes_{i \in \mathbb{N}} M_i$ with $M_i = M_n(\mathbb{C})$, σ is the shift to the right, and φ the canonical state extending the trace on B . The *canonical endomorphism* Φ is defined by

$$\Phi(x) = \sum_{i=1}^n S_i x S_i^*, \quad x \in \mathcal{O}_n.$$

It is a simple task to extend the entropies h_φ and ht to endomorphisms. We have

Theorem. 7.3 [C4] *The canonical endomorphism Φ on \mathcal{O}_n satisfies*

$$ht(\Phi) = h_\varphi(\Phi) = \log n.$$

The result has a natural extension to the Cuntz-Krieger algebra \mathcal{O}_A defined by an irreducible $n \times n$ matrix which is not a permutation matrix. Then we have [Bo-Go]

$$ht(\alpha) = \log r(A),$$

where $r(A)$ is the spectral radius of A . For further extensions see [PWY].

8 Free products

In Theorem 7.2 we saw that the shift on \mathcal{O}_∞ has entropy zero. The first example of a highly nonabelian dynamical system where the entropy is zero, was the shift on the II_1 -factor $L(\mathbb{F}_\infty)$ obtained from the left regular representation of the free group in infinite number of generators [S1]. This phenomenon was rather surprising because the shift is so ergodic that there is no globally invariant injective von Neumann subalgebra except for the scalars. We shall in the present section study extensions of the above shift.

We first recall the definitions. Let I be an index set, and for each $\iota \in I$ let A_ι be a unital C*-algebra and φ_ι a state on A_ι . Let $(\pi_\iota, H_\iota, \xi_\iota)$ be the GNS-representation of φ_ι , $\iota \in I$. Let $H_\iota^0 = H_\iota \ominus \mathbb{C}\xi_\iota$ and $(H, \xi) = \ast_{\iota \in I} (H_\iota, \xi_\iota)$ be the free product. Put

$$H(\iota) = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \left(\bigoplus_{\substack{\iota_1 \neq \iota_2 \neq \dots \neq \iota_n \\ \iota_1 \neq \iota}} H_{\iota_1}^0 \otimes \dots \otimes H_{\iota_n}^0 \right)$$

We have unitary operators $V_\iota : H_\iota \otimes H(\iota) \rightarrow H$ defined by

$$\begin{aligned} \xi_\iota \otimes \xi &\rightarrow \xi \\ H_\iota^0 \otimes \xi &\rightarrow H_\iota^0 \quad \text{by } \eta \otimes \xi \rightarrow \eta \\ \xi_\iota \otimes (H_{\iota_1}^0 \otimes \cdots \otimes H_{\iota_n}^0) &\rightarrow H_{\iota_1}^0 \otimes \cdots \otimes H_{\iota_n}^0 \quad \text{by } \xi_\iota \otimes \eta \rightarrow \eta, \iota_1 \neq \iota \\ H_\iota^0 \otimes (H_{\iota_1}^0 \otimes \cdots \otimes H_{\iota_n}^0) &\rightarrow H_\iota^0 \otimes H_{\iota_1}^0 \otimes \cdots \otimes H_{\iota_n}^0 \quad \text{by } \psi \otimes \eta \rightarrow \psi \otimes \eta, \iota_1 \neq \iota. \end{aligned}$$

The representation $\lambda_\iota : A_\iota \rightarrow B(H)$ is defined by

$$\lambda_\iota(x) = V_\iota(\pi_\iota(x) \otimes 1_{H(\iota)})V_\iota^*, \quad x \in A_\iota.$$

The free product representation $\pi = \ast_{\iota \in I} \pi_\iota : \ast A_\iota \rightarrow B(H)$ is the \ast -homomorphism of the free product C^\ast -algebra $(\ast A_\iota, \ast \lambda_\iota) \rightarrow B(H)$, using the universal property of the free product. When we write $(A, \varphi) = (\ast A_\iota, \ast \varphi_\iota)_{\iota \in I}$ we shall mean $\ast A_\iota$ in the representation π , i.e. we shall mean $\pi(\ast A_\iota) \subset B(H)$.

There are now two approaches; the first is to show that $ht(\alpha)$ for α the shift, or rather α defined via a bijection $\mathbb{Z} \rightarrow \mathbb{Z}$ like for \mathcal{O}_∞ in Theorem 7.2, and use the inequalities in Theorem 6.7 to conclude that the other entropies are zero. This works if the A_ι are all exact, because then A is exact [D1].

Theorem. 8.1 [D2] *In the above notation assume each A_ι is finite dimensional and let σ be a permutation of I such that for all $\iota \in I$ there is an isomorphism $\alpha_\iota : A_\iota \rightarrow A_{\sigma(\iota)}$ such that $\varphi_{\sigma(\iota)} \circ \alpha_\iota = \varphi_\iota$. Then there exists a unique $\alpha \in \text{Aut } A$ such that $\alpha \circ \lambda_\iota = \lambda_{\sigma(\iota)} \circ \alpha_\iota$, $\iota \in I$, and $ht(\alpha) = 0$.*

Dykema proves a more general result than the above. The theorem is not as restricted as it looks, because if we let $B_\iota = A_{2\iota} \ast A_{2\iota+1}$ we can write A as a free product $\ast B_\iota$ of a large class of C^\ast -algebras. For example in this way we can imbed \mathcal{O}_∞ in a free product to recover Theorem 7.2. Similarly we can obtain the announced result on the shift of $C_r^\ast(\mathbb{F}_\infty)$, and hence by (3.5) of $L^\infty(\mathbb{F}_\infty)$, from the inequality $h_\varphi(\alpha) \leq ht(\alpha)$.

For the other approach we note that we have no Kolmogoroff-Sinai Theorem in the nonnuclear case, so we must go directly at the definition of $h_\varphi(\gamma_1, \dots, \gamma_n)$. Recall that we then considered a positive map $P : A \rightarrow B$, where B is a finite dimensional abelian C^\ast -algebra with a state μ such that $\mu \circ P = \varphi$. Denote by $\|x\|_\mu = \mu(x^\ast x)^{1/2}$.

Lemma 8.2 [S3] *Let $(A, \varphi) = (\ast A_\iota, \ast \varphi_\iota)_{\iota \in I}$ be a free product of unital C^\ast -algebras. Let B be an abelian C^\ast -algebra with a state μ . Suppose $P : A \rightarrow B$ is a unital positive linear map such that $\mu \circ P = \varphi$. Then given $\varepsilon > 0$ there is $J \subset I$ with $\text{card } J \leq \left\lceil \frac{1000}{\varepsilon} \right\rceil + 1$ such that*

$$\|P(x) - \varphi(x)1\|_\mu < \varepsilon \|x\|, \quad x \in A_\iota, \quad \iota \notin J.$$

Thus P is essentially almost constant outside the subalgebra $\ast_{\iota \in J} A_\iota$. If we now assume $A_\iota = A_0$, $\iota \in I = \mathbb{Z}$ and letting for example α be the free shift on A arising from the shift $\iota \rightarrow \iota + 1$ on \mathbb{Z} , then it is not hard to go through the different steps in the definition of the CNT-entropy $h_\varphi(\alpha)$ to conclude:

Theorem. 8.3 [S3] *If A_0 is a unital C^* -algebra and φ_0 a state on A_0 , and $A_i = A_0$, $\varphi_i = \varphi_0$, $i \in \mathbb{Z}$, then the free shift α on $(A, \varphi) = (*A_i, *\varphi_i)_{i \in \mathbb{Z}}$ has entropy $h_\varphi(\alpha) = 0$.*

Remark 8.4 It should be noted that by a result of Avitzour [Av] it follows that every stationary coupling extending φ is of the form $\lambda = \varphi \otimes \mu$, hence the entropy $h'_\varphi(\alpha)$ of Sauvageot and Thouvenot is zero. One can further show [C3] that if (C, ρ, γ) is a C^* -dynamical system then

$$h'_{\varphi*\rho}(\alpha * \gamma) = h'_\rho(\gamma).$$

Similar results hold for other entropies, see [C6].

9 Binary shifts

A rich class of C^* -dynamical systems is obtained from bitstreams, i.e. sequences $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \{0, 1\}$. Denote by X the subset of \mathbb{N} , $X = \{n \in \mathbb{N} : x_n = 1\}$. We can construct a sequence $(s_n)_{n \in \mathbb{N}}$ of symmetries, i.e. self-adjoint unitary operators on a Hilbert space which satisfy the commutation relations

$$s_i s_j = \begin{cases} s_j s_i & \text{if } |i - j| \notin X, \text{ i.e. } x_{|i-j|} = 0 \\ -s_j s_i & \text{if } |i - j| \in X, \text{ i.e. } x_{|i-j|} = 1, \end{cases}$$

see e.g. [Vi].

Let $A(X)$ denote the C^* -algebra generated by the s_n , $n \in \mathbb{Z}$. The canonical trace τ on $A(X)$ is the one which takes the value zero on all products $s_{i_1} \dots s_{i_n}$, where $i_1 < i_2 < \dots < i_n$, and $\tau(1) = 1$. We denote by α the shift automorphism of $A(X)$ defined by $\alpha(s_i) = s_{i+1}$. Then $(A(X), \tau, \alpha)$ is a C^* -dynamical system. Well-known situations from both C^* -algebras and the classical case are represented as special cases, e.g. asymptotically abelian, proximally asymptotically abelian, K -systems, and completely positive entropy, see [G-S1]. We shall assume we are in the nontrivial case when then set $-X \cup \{0\} \cup X$ is nonperiodic. Let $A_n = C^*(s_0, \dots, s_{n-1})$ be the C^* -algebra generated by s_0, \dots, s_{n-1} , so that

$$A_n = \bigvee_0^{n-1} \alpha^i(C^*(s_0)).$$

We list some properties of A_n and $A(X)$ which will be used in the sequel, see [E], [Po-Pr] or [Vi]. Denote by Z_n the center of A_n . Then we have:

$$(9.1) \quad \text{There are } c_n, d_n \in \mathbb{N} \cup \{0\} \text{ such that } n = 2d_n + c_n, \quad A_n \cong M_{2d_n}(\mathbb{C}) \otimes Z_n$$

where $Z_n \cong C(\{0, 1\}^{c_n})$.

$$(9.2) \quad \text{If } e \text{ is a minimal projection in } Z_n \text{ then } \tau(e) = 2^{-c_n}.$$

$$(9.3) \quad \text{There is a sequence } (m_i) \text{ in } \mathbb{N} \text{ such that } (c_n) \text{ consists of the concatenation of the strings } (1, \dots, m_i - 1, m_i, m_i - 1, \dots, 1, 0).$$

In particular $c_n = 0$ for an infinite number of n 's, hence $A_n = M_{2n/2}(\mathbb{C})$ for these n 's, and so $A(X)$ is the CAR-algebra.

$$(9.4) \quad H(A_n) = \log \text{rank } A_n = (c_n + d_n) \log 2 .$$

By (9.1) $2d_n \leq n \leq 2d_n + 2c_n$, hence

$$\liminf_n \frac{1}{n} H(A_n) = \liminf_n \frac{1}{n} (c_n + d_n) \log 2 \leq \frac{1}{2} \log 2 .$$

Since also $\frac{1}{n}(c_n + d_n) \geq \frac{1}{2}$ we find

$$(9.5) \quad \liminf_n \frac{1}{n} H(A_n) = \frac{1}{2} \log 2$$

$$(9.6) \quad \lim_{n \rightarrow \infty} \frac{c_n}{n} = 0 \quad \text{if and only if} \quad \lim_n \frac{1}{n} H(A_n) = \frac{1}{2} \log 2 .$$

Indeed, if $\frac{c_n}{n} \rightarrow 0$ then $\frac{d_n}{n} \rightarrow \frac{1}{2}$, hence by (9.4) $\frac{1}{n} H(A_n) \rightarrow \frac{1}{2} \log 2$. Conversely, if $\lim_n \frac{1}{n} H(A_n) = \frac{1}{2} \log 2$ then by (9.4) $\frac{1}{n}(c_n + d_n) \rightarrow \frac{1}{2}$, hence by (9.1) $\frac{c_n}{n} \rightarrow 0$.

Since $A(X)$ is the CAR-algebra the weak closure of its image in the GNS-representation of τ is the hyperfinite II_1 -factor R . When we in the sequel consider the approximation entropies of α defined in Definition 6.2 it is really the extension of α to R that we consider. While $h_{\alpha, \tau}$ is subadditive on tensor products the lower approximation entropy $\ell h_{\alpha, \tau}$ only satisfies

$$\ell h_{\alpha, \tau \otimes \tau}(\alpha \otimes \alpha) \leq 2 \ell h_{\alpha, \tau}(\alpha) .$$

Lemma 9.1 [NST], [G-S2] *With α and X as before we have:*

$$(i) \quad h_{\tau \otimes \tau}(\alpha \otimes \alpha) = \log 2$$

$$(ii) \quad \ell h_{\alpha, \tau}(\alpha) = \frac{1}{2} \log 2$$

Proof. Let A_0 denote the C^* -subalgebra of $A(X) \otimes A(X)$ generated by the symmetries $s_i \otimes s_i$, $i \in \mathbb{Z}$. Then A_0 is abelian, and $\tau \otimes \tau$ vanishes on each $s_i \otimes s_i$. Thus the C^* -dynamical system $(A_0, \tau \otimes \tau, \alpha \otimes \alpha)$ is isomorphic to the 2-shift, hence has entropy $\log 2$, hence by monotonicity $h_{\tau \otimes \tau}(\alpha \otimes \alpha) \geq \log 2$. Thus by Proposition 6.4, and the inequality preceding the lemma

$$2 \ell h_{\alpha, \tau}(\alpha) \geq \ell h_{\alpha, \tau \otimes \tau}(\alpha \otimes \alpha) \geq h_{\tau \otimes \tau}(\alpha \otimes \alpha) \geq \log 2 .$$

The converse inequality follows from easy estimates using (9.5). □

From the above lemma $h_{\tau}(\alpha) \in [0, \frac{1}{2} \log 2]$. In many cases $h_{\tau}(\alpha) = \frac{1}{2} \log 2$.

Theorem. 9.2 [C1], [G-S1] *Suppose X satisfies one of the following: X is finite, $\mathbb{N} \setminus X$ is finite, X is contained in the even or odd numbers, X contains the odd numbers. Then $h_{\tau}(\alpha) = \frac{1}{2} \log 2$.*

Consider the case when $X \supset \{1, 3, 5, \dots\}$. Let $t_j = s_{2j-1}s_{2j}$, $j \in \mathbb{Z}$. Then the t_j all commute, and as in the proof of Lemma 9.1 α^2 acts as a 2-shift on the C^* -algebra they generate. Thus

$$h_\tau(\alpha) = \frac{1}{2}h_\tau(\alpha^2) \geq \frac{1}{2}\log 2,$$

and the conclusion follows. \square

For other examples when $h_\tau(\alpha) = \frac{1}{2}\log 2$ see [Pr]. If $J \subset \mathbb{N}$ is finite, $J = \{i_1, \dots, i_n : i_1 < i_2 < \dots < i_n\}$ then the operator $s_{i_1} \dots s_{i_n}$ or $is_{i_1} \dots s_{i_n}$ is self-adjoint. Denote the self-adjoint operator by s_J . Then $\{\alpha^j(s_J)\}_{j \in \mathbb{Z}}$ is a sequence of symmetries which either commute or anticommute. Let

$$X(J) = \{j \in \mathbb{N} : \alpha^j(s_J)s_J + s_J\alpha^j(s_J) = 0\}$$

be the corresponding subset of \mathbb{N} . If $I \subset \mathbb{N}$ we denote by $I - I = \{n - m : m, n \in I\}$.

Theorem 9.3 [NST] (i) *Assume for each finite subset $J \subset \mathbb{N}$ there exists an infinite subset $I \subset \mathbb{N}$ such that $(I - I) \cap \mathbb{N} \subset X(J)$. Then $h_\tau(\alpha) = 0$.*

(ii) *There exists $X \subset \mathbb{N}$ such that (i) holds.*

The proof of (i) consists of showing that the Sauvageot-Thouvenot entropy $h'_\tau(\alpha) = 0$. This is done by showing that a stationary coupling λ corresponding to an abelian system (B, μ, β) necessarily is of the form $\lambda = \tau \otimes \mu$.

If we combine Theorem 9.3 with Lemma 9.1 we have

Corollary 9.4 [NST] *There exists $X \subset \mathbb{N}$ such that $h_{\tau \otimes \tau}(\alpha \otimes \alpha) = \log 2$ while $h_\tau(\alpha) = 0$.*

We leave it as open problems whether there exists $X \subset \mathbb{N}$ such that $0 < h_\tau(\alpha) < \frac{1}{2}\log 2$, and whether there exists $X \subset \mathbb{N}$ such that, cf. Lemma 9.1 (ii), $h_{\tau \otimes \tau}(\alpha \otimes \alpha) > \frac{1}{2}\log 2$.

Another example of a C^* -dynamical system (A, τ, α) for which $h_{\tau \otimes \tau}(\alpha \otimes \alpha) > h_\tau(\alpha) + h_\tau(\alpha) = 0$ has been exhibited by Sauvageot [Sa], see also [N-T2].

Let $\theta \in \mathbb{R}$ and A_θ be the C^* -algebra generated by two unitaries U and V such that

$$VU = e^{2\pi i\theta}UV,$$

so A_θ is the irrational rotation algebra when θ is irrational. If $\mu = (m, n) \in \mathbb{Z}^2$ let

$$W_\theta(\mu) = e^{i\pi\theta mn}U^mV^n.$$

Then linear combinations of the $W_\theta(\mu)$, $\mu \in \mathbb{Z}^2$, are dense in A_θ . In analogy with Bogoliubov automorphisms each matrix $A \in SL(2, \mathbb{Z})$ defines an automorphism σ_A of A_θ by

$$\sigma_A(W_\theta(\mu)) = W_\theta(A\mu).$$

A_θ has a canonical trace τ_θ such that $\tau_\theta(W_\theta(\mu)) = 0$ whenever $\mu \neq (0, 0)$. We make the assumption that A has two real eigenvalues λ and λ^{-1} with $|\lambda| > 1$. Then we have

Theorem. 9.5 [Sa], [N-T2] *With the above assumption there is a subset $\Omega \subset \mathbb{R}$ for which the complement Ω^c has Lebesgue measure zero and $\mathbb{Q} + \lambda\mathbb{Q} \subset \Omega^c$, with the following properties:*

(i) *If $\theta \in \mathbb{Q} + \lambda\mathbb{Q}$ then $h_{\tau_\theta}(\sigma_A) > 0$.*

(ii) *If $\theta \in \Omega$ then τ_θ is the unique invariant state and $h_{\tau_\theta}(\sigma_A) = 0 = h_{\tau_{-\theta}}(\sigma_A)$.*

Furthermore, in case (ii) $h_{\tau_\theta \otimes \tau_{-\theta}}(\sigma_A \otimes \sigma_A) > 0$.

10 Generators

Corollary 9.4 shows that the tensor product formula $h_{\varphi \otimes \varphi}(\alpha \otimes \beta) = h_\varphi(\alpha) + h_\varphi(\beta)$ can only hold in special cases. By the superadditivity of h_φ , see (3.6), and the subadditivity of the approximation entropy ha_τ , the tensor product formula holds whenever the two entropies coincide. Since the latter is a refined version of mean entropy we may therefore expect the tensor product formula to hold whenever the CNT-entropy h_τ is a mean entropy.

In the classical case of a probability space (X, \mathcal{B}, μ) with a measure preserving nonsingular transformation T , a partition P is called a generator if the σ -algebra $\bigvee_{i \in \mathbb{Z}} T^{-i}P = \mathcal{B}$, or equivalently, if A is the C^* -algebra generated by the atoms in P , then $\bigvee_{i \in \mathbb{Z}} \alpha^i(A) = L^\infty(X, \mathcal{B}, \mu)$. In [G-S2] different candidates for nonabelian generators were considered.

Definition 10.1 *Let M be a hyperfinite von Neumann algebra with a faithful normal tracial state τ . Let α be a τ -invariant automorphism. A finite dimensional von Neumann subalgebra N of M is a generator for α if*

$$(i) \quad \bigvee_{i \in \mathbb{Z}} \alpha^i(N) = M.$$

$$(ii) \quad \bigvee_{i=m}^n \alpha^i(N) \text{ is finite dimensional whenever } m < n, m, n \in \mathbb{Z}.$$

$$(iii) \quad H(N, \alpha) = \limsup_n \frac{1}{n} H \left(\bigvee_0^{n-1} \alpha^i(N) \right).$$

If (iii) is replaced by

$$(iv) \quad H(\alpha) = \limsup_n \frac{1}{n} H \left(\bigvee_0^{n-1} \alpha^i(N) \right).$$

then N is called a mean generator.

If N is a generator then N is a mean generator, and

$$H(\alpha) = H(N, \alpha) = \lim_n \frac{1}{n} H \left(\bigvee_0^{n-1} \alpha^i(N) \right).$$

We say N is a *lower generator* (resp. *lower mean generator*) if we replace \limsup by \liminf in (iii) (resp. (iv)). Recall from Definition 6.2 that if we replace $\text{rank } A$ in the definition of $ha_\tau(\alpha)$ and $\ell ha_\tau(\alpha)$ by $\exp(H(A))$ we obtained two entropies we denote by $Ha_\tau(\alpha)$ and $\ell Ha_\tau(\alpha)$. An easy consequence of the definitions is then

Proposition 10.2 *Let $N \subset M$ be finite dimensional. If N is a mean generator (resp. lower mean generator) then $H(\alpha) = Ha_\tau(\alpha)$ (resp. $H(\alpha) = \ell Ha_\tau(\alpha)$).*

Corollary 10.3 *Let (M_i, τ_i, α_i) be W^* -dynamical systems as above, $i = 1, 2$. If α_1 and α_2 have mean generators then*

$$H_{\tau_1 \otimes \tau_2}(\alpha_1 \otimes \alpha_2) = H_{\tau_1}(\alpha_1) + H_{\tau_2}(\alpha_2).$$

Since the CNT-entropy remains the same when we imbed C^* -dynamical system into the W^* -dynamical system obtained via the GNS-representation due to the invariant state (3.5) our definitions clearly make sense for AF-algebras.

Several well-known examples of C^* -dynamical systems have generators. We list a few.

10.4 Temperley-Lieb algebras

Let $(e_i)_{i \in \mathbb{Z}}$ be a sequence of projections with the properties.

- (a) $e_i e_{i \pm 1} e_i = \lambda e_i$ for some $\lambda \in (0, \frac{1}{4}] \cup \{\frac{1}{4} \sec^2(\frac{\pi}{m}) : m \geq 3\}$
- (b) $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.
- (c) $\tau(\omega e_i) = \lambda \tau(\omega)$ if ω is a word in 1 and e_j , $j < i$.

Then the von Neumann algebra R generated by the e_i 's is the hyperfinite II_1 -factor. The shift automorphism α_λ determined by $\alpha_\lambda(e_i) = e_{i+1}$ has $C^*(e_0)$ as a mean generator, see [G-S2].

10.5 Noncommutative Bernoulli shifts

In the notation of (2.14) assume $d = 2$. Let N denote the centralizer of φ in $M_0 \otimes M_1$. Then N is a generator for α [G-S2].

10.6 Binary shifts

Assume as before $-X \cup \{0\} \cup X$ is nonperiodic. By Theorem 9.3 and Corollary 10.3 we do not in general have generators for binary shifts. In the notation of (9.1)–(9.6) the following hold.

- (i) If $H(\alpha) = \frac{1}{2} \log 2$ then A_1 is a lower mean generator. If moreover $\lim_n \frac{c_n}{n} = 0$ then A_1 is a mean generator.
- (ii) If X is contained in the even numbers then A_2 is a lower generator. If moreover $\lim_n \frac{c_n}{n} = 0$ then A_2 is a generator.
- (iii) If X is contained in the odd numbers then A_1 is a lower generator. If moreover $\lim_n \frac{c_n}{n} = 0$ then A_1 is a generator.

There are many ways to generalize the concept of generators. Instead of looking at $\bigvee_0^{n-1} \alpha^i(N)$ we can consider an increasing sequence of finite dimensional algebras and endomorphisms instead of automorphisms.

Definition 10.7 [S4] *Let M be a hyperfinite von Neumann algebra with a faithful normal tracial state. Let α be a τ -invariant endomorphism. An increasing sequence $(A_n)_{n \in \mathbb{N}}$ of finite dimensional von Neumann subalgebras of M is a generating sequence for α if*

- (i) $\bigvee_{n=1}^{\infty} A_n = M$,
- (ii) $\alpha(A_n) \subset A_{n+1}$, $n \in \mathbb{N}$,
- (iii) $H(\alpha) = \lim_n \frac{1}{n} H(A_n)$.

By (iii) it is clear that the tensor product formula holds in the presence of generating sequences.

We say a sequence $(A_n)_{n \in \mathbb{N}}$ such that (i) and (ii) hold, satisfies the *commuting square condition* if the diagram

$$\begin{array}{ccc} A_{n+1} & \subset & M \\ \cup & & \cup \\ \alpha(A_n) & \subset & \alpha(A_{n+1}) \end{array}$$

is a commuting square for all $n \in \mathbb{N}$, i.e. $E_{\alpha(A_n)} = E_{\alpha(A_{n+1})} \circ E_{A_{n+1}}$.

With the above assumptions we can generalize the classical result that if P is a generator for a measure preserving nonsingular transformation T on probability space (X, \mathcal{B}, μ) then $H(T)$ is given by relative entropy,

$$H(T) = \lim_{n \rightarrow \infty} H \left(\bigvee_0^n T^{-i}(P) \middle| \bigvee_1^n T^{-i}(P) \right).$$

In our case relative entropy is defined in Definition 2.8. If N is a von Neumann algebra we denote by $Z(N)$ the center of N .

Theorem. 10.8 [S4] *Let M, τ, α be as above and suppose $H(\alpha) < \omega$. Suppose $(A_n)_{n \in \mathbb{N} \cup \{0\}}$ is a generating sequence for α satisfying the commuting square condition. Then we have:*

- (i) $\lim_{n \rightarrow \infty} \frac{1}{n} H(Z(A_n))$ exists.
- (ii) $H(\alpha) = \frac{1}{2} H(M|\alpha(M)) + \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{n} H(Z(A_n))$.

Furthermore, if M is of type I then $H(\alpha) = H(M|\alpha(M))$.

Pimsner and Popa [P-P] found an explicit formula for the relative entropy $H(P|Q)$ when $P \supset Q$ are finite dimensional C^* -algebras. For the sequence (A_n) the formula becomes

$$(10.1) \quad \begin{aligned} H(A_n)|\alpha(A_{n-1}) \\ = 2(H(A_n) - H(\alpha(A_{n-1}))) - (H(Z(A_n)) - H(Z(\alpha(A_{n-1})))) + c_n, \end{aligned}$$

where c_n is a real number depending on the multiplicity of the imbedding $\alpha(A_{n-1}) \subset A_n$. Since $H(\alpha(A_{n-1})) = H(A_{n-1})$ and similarly for the centers, if we add the equations in (10.1) we find,

$$\begin{aligned} & \frac{1}{k} \sum_{n=1}^k H(A_n | \alpha(A_{n-1})) \\ &= \frac{2}{k} H(A_k) - \frac{2}{k} H(A_0) - \frac{1}{k} H(Z(A_k)) + \frac{1}{k} H(Z(A_0)) + \frac{1}{k} \sum_{n=1}^k c_n . \end{aligned}$$

Since (A_n) satisfies the commuting square condition the left side converges to $H(M | \alpha(M))$, [P-P]. Some analysis shows $\frac{1}{k} \sum_{i=1}^k c_n \rightarrow 0$. Since the terms involving A_0 go to zero and $\frac{2}{k} H(A_k) \rightarrow 2H(\alpha)$, the proof is completed. \square

If R is the hyperfinite II_1 -factor then by a formula of Pimsner and Popa [P-P] we get

Corollary 10.9 [S4] *If in Theorem 10.8 $M = R$ is the hyperfinite II_1 -factor we have*

- (i) $\lim_n \frac{1}{n} H(Z(A_n))$ exists.
- (ii) $R \cap \alpha(R)'$ is atomic with minimal projections f_k , $\sum_k f_k = 1$,
- (iii) $H(\alpha) = H(R \cap \alpha(R)') + \frac{1}{2} \sum_k \tau(f_k) \log[R_{f_k} : \alpha(R)_{f_k}] + \frac{1}{2} \lim_n \frac{1}{n} H(Z(A_n))$.

Here R_f means the R cut down by f , and $[P : Q]$ is the Jones index [J]. In particular, if $R \cap \alpha(R)' = \mathbb{C}$ then

$$(10.2) \quad H(\alpha) = \frac{1}{2} \log[R : \alpha(R)] + \frac{1}{2} \lim_n \frac{1}{n} H(Z(A_n)) .$$

The natural sequences considered in Examples (10.4)–(10.6) all satisfy the commuting square conditions, and in (10.4) and (10.6) (10.2) reduces to

$$(10.3) \quad H(\alpha) = \frac{1}{2} \log[R : \alpha(R)] ,$$

a formula which is well-known in those cases, see [J], [P-P], [P].

If $N \subset M$ is an inclusion of type III factors with a common separating and cyclic vector ξ the conjugations J_M and J_N defined by ξ define an endomorphism γ of M into N by

$$\gamma(x) = J_N J_M x J_M J_N , \quad x \in M ,$$

see [L]. If the index is finite Choda [C2] showed a formula like (10.3) for the entropy of γ with respect to a natural invariant state.

The II_1 -analogue Γ of γ is defined for an irreducible inclusion of II_1 -factors $N \subset M$ of finite index. We then get a tower

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset \dots$$

The trace τ of M is extended naturally to a trace, also denoted by τ , on each M_k . The canonical conjugation J_k on the Hilbert space $L^2(M_k, \tau)$ is defined by $J_k x = x^*$, $x \in M_k$. We denote by M_∞ the closure of $\bigcup M_k$ in the GNS-representation of τ and consider M as a subfactor of M_∞ . Let $R = M' \cap M_\infty$. Then R is the hyperfinite II_1 -factor, and the canonical shift Γ on R is defined by

$$\Gamma(x) = J_{n+1} J_n x J_n J_{n+1}$$

for $x \in M' \cap M_{2k}$, $n \geq k$. This definition does not depend on n . Let $A_n = M' \cap M_{2n}$. Then the sequence (A_n) is a generating sequence which satisfies the commuting square condition. It follows from Corollary 10.9 that $H(\Gamma)$ is given by (10.2). This formula was first shown by Hiai [H]. With certain extra assumptions he and Choda [C2], [C-H] showed that $H(\Gamma)$ is given by (10.3).

11 The variational principle

The variational principle appears as an important ingredient both in classical ergodic theory and in spin lattice systems in the C^* -algebra formalism of quantum statistical mechanics. The principle can be well described in the finite dimensional case. Indeed, let B be a finite dimensional C^* -algebra and Tr_B the canonical trace on B , i.e. $\text{Tr}_B(e) = 1$ for all minimal projections $e \in B$. If φ is a state on B let $K_\varphi \in B^+$ be its density operator, so $\varphi(x) = \text{Tr}_B(K_\varphi x)$. Then the mean entropy of φ is

$$S(\varphi) = \text{Tr}_B(\eta(K_\varphi)) .$$

The variational principle takes the form: If $H \in B_{\text{sa}}$ then

$$(11.1) \quad S(\varphi) - \varphi(H) \leq \log \text{Tr}_B(e^{-H}) ,$$

with equality if and only if

$$K_\varphi = \frac{e^{-H}}{\text{Tr}_B(e^{-H})} .$$

In this case φ is called the Gibbs state. Note that φ is then a KMS-state at $\beta = 1$ for the one-parameter group

$$\sigma_t^H(x) = e^{-itH} x e^{itH} ,$$

where we recall that a state φ on a C^* -algebra A is a KMS-state for a one-parameter group σ_t at temperature β if

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all analytic elements $a, b \in A$, see [B-R, Ch. 5].

The above variational principle has been extended to automorphisms of $C(X)$ when X is a compact metric space, see [W, Ch. 9], and to spin lattice systems, see [B-R,

Ch. 6]. In the latter case the C^* -algebra is an infinite tensor product indexed by \mathbb{Z}^ν , $\nu \in \mathbb{N}$, and the automorphism group consists of the shifts. In the years around 1970 the variational principle was intensely studied by mathematical physicists, see the comments to Section 6.2.4 in [B-R], and solved in a way naturally extending the results in the finite dimensional case. The entropy used, was mean entropy. However, Moriya [M] has shown that one gets the same results with CNT-entropy replacing mean entropy.

The C^* -dynamical systems (A, α) considered above are all asymptotically abelian, i.e. $\|[\alpha^n(x), y]\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x, y \in A$. Following work of Neshveyev and the author [N-S2] we shall in the present section see that the variational principle has a natural extension to a class of asymptotically abelian C^* -algebras. For another approach in the case of Cuntz-Krieger algebras see [PWY]. First we shall need to extend the right side of (11.1) to C^* -algebras. We modify the definition of topological entropy in Definition 6.6.

Let A be a nuclear C^* -algebra with unit and α an automorphism. Let

$$\text{CPA}(A) = \{(\rho, \psi, B) : B \text{ is finite dimensional } C^*\text{-algebra,} \\ \rho : A \rightarrow B, \psi : B \rightarrow A \text{ are unital completely positive maps}\}$$

For $\delta > 0$, $\omega \in P_f(A)$, and $H \in A_{\text{sa}}$ put

$$P(H, \omega, \delta) = \inf\{\log \text{Tr}_B(e^{-\rho(H)}) : (\rho, \psi, B) \in \text{CPA}(A), \|(\psi \circ \rho)(x) - x\| < \delta, \forall x \in \omega\},$$

where the inf is taken over all $(\rho, \psi, B) \in \text{CPA}(A)$. Let

$$P_\alpha(H, \omega, \delta) = \limsup_{n \rightarrow \infty} \frac{1}{n} P\left(\sum_{j=0}^{n-1} \alpha^j(H), \bigcup_{j=0}^{n-1} \alpha^j(\omega), \delta\right), \\ P_\alpha(H, \omega) = \sup_{\delta > 0} P_\alpha(H, \omega, \delta).$$

Definition 11.1 *The pressure of α at H is*

$$P_\alpha(H) = \sup_{\omega \in P_f(A)} P_\alpha(H, \omega).$$

Note that $\text{Tr}_B(1) = \text{rank } B$, so when $H = 0$ the above definition restricts to Voiculescu's definition of topological entropy $ht(\alpha)$. The basic properties of pressure are contained in

Proposition 11.2 *The following properties are satisfied by P_α for $H, K \in A_{\text{sa}}$.*

(i) *If φ is an α -invariant state of A then*

$$P_\alpha(H) \geq h_\varphi(\alpha) - \varphi(H),$$

where $h_\varphi(\alpha)$ is the CNT-entropy of α .

(ii) *If $H \leq K$ then $P_\alpha(H) \geq P_\alpha(K)$.*

(iii) *$P_\alpha(H + c1) = P_\alpha(H) - c$, $c \in \mathbb{R}$.*

(iv) *$P_\alpha(H)$ is either infinite for all H or is finite valued.*

(v) *If P_α is finite valued then $|P_\alpha(H) - P_\alpha(K)| \leq \|H - K\|$.*

(vi) *For $k \in \mathbb{N}$ $P_{\alpha^k}\left(\sum_{j=0}^{k-1} \alpha^j(H)\right) = kP_\alpha(H)$.*

(vii) *$P_\alpha(H + \alpha(K) - K) = P_\alpha(H)$.*

The proof of (i) is a modification of [V, Prop. 4.6] using (11.1). The others are modifications of the corresponding results in the classical case, [W, Thm. 9.7]. In the proof of (ii), (v), (vi) we make use of the important Peierls-Bogoliubov inequality [O-P, Cor. 3.15],

$$(11.2) \quad \log \operatorname{Tr}_B(e^H) \leq \log \operatorname{Tr}_B(e^K) \quad \text{if } H \leq K .$$

The pressure function has very strong properties, as our next result shows.

Proposition 11.3 *Suppose $ht(\alpha) < \infty$, and let φ be a self-adjoint linear functional on A . Then φ is an α -invariant state if and only if $-\varphi(H) \leq P_\alpha(H) \quad \forall H \in A_{sa}$.*

The proof is a good illustration of the basic properties of P_α . If φ is an α -invariant state then by (i) in Proposition 11.2, $-\varphi(H) \leq P_\alpha(H) - h_\varphi(\alpha) \leq P_\alpha(H)$.

Conversely, if $-\varphi(H) \leq P_\alpha(H)$ for all H then by (vii) applied to $H = 0$,

$$\begin{aligned} -\varphi(\alpha(K) - K) &= -\frac{1}{n}\varphi(\alpha(nK) - nK) \leq \frac{1}{n}P_\alpha(\alpha(nK) - nK) \\ &= \frac{1}{n}P_\alpha(0) \rightarrow 0 \quad \text{as } n \rightarrow \infty . \end{aligned}$$

Application of this to $-K$ yields α -invariance of φ .

By properties (ii) and (iii)

$$\begin{aligned} -\varphi(H) &= -\frac{1}{n}\varphi(nH) \leq \frac{1}{n}P_\alpha(nH) \\ &\leq \frac{1}{n}P_\alpha(0 - n\|H\|) = \frac{1}{n}P_\alpha(0) + \|H\| \rightarrow \|H\| . \end{aligned}$$

Thus $\|\varphi\| \leq 1$. Since for $c \in \mathbb{R}$,

$$-c\varphi(1) \leq P_\alpha(c1) = ht(\alpha) - c$$

we see that $\varphi(1) = 1$, so φ is a state. □

Definition 11.4 *We say an α -invariant state is an equilibrium state at H if*

$$P_\alpha(H) = h_\varphi(\alpha) - \varphi(H) ,$$

or equivalently by property (i)

$$h_\varphi(\alpha) - \varphi(H) = \sup_{\psi} (h_\psi(\alpha) - \psi(H)) ,$$

where the sup is taken over all α -invariant states.

In the finite dimensional case an equilibrium state corresponds to the Gibbs state.

Recall that if F is a real convex function on a real Banach space X then a linear functional f on X is called a *tangent functional to F at $x_0 \in X$* if

$$F(x_0 + x) - F(x_0) \geq f(x) \quad (= f(x_0 + x) - f(x_0))$$

for all $x \in X$.

Proposition 11.5 *Suppose $ht(\alpha) < \infty$ and the pressure is a convex function on A_{sa} .*

(i) *If φ is an equilibrium state at H then $-\varphi$ is a tangent functional to P_α at H .*

(ii) *If $-\varphi$ is a tangent functional for P_α at H then φ is an α -invariant state.*

The proof of (i) is immediate from property (i) in Proposition 11.2. Since $P_\alpha(H) = h_\varphi(\alpha) - \varphi(H)$,

$$P_\alpha(H + K) - P_\alpha(H) \geq (h_\varphi(\alpha) - \varphi(H + K)) - (h_\varphi(\alpha) - \varphi(H)) = -\varphi(K).$$

The proof of (ii) is similar to that of Proposition 11.3.

With the definition and the main properties of pressure well taken care of, we now embark on the variational principle. The setting will be a restricted class of asymptotically abelian C^* -algebras.

Definition 11.6 *A unital C^* -dynamical system (A, α) is called asymptotically abelian with locality if there is a dense α -invariant $*$ -subalgebra \mathcal{A} of A such that for each pair $a, b \in \mathcal{A}$ the C^* -algebra they generate is finite dimensional, and for some $p = p(a, b) \in \mathbb{N}$ we have $[\alpha^j(a), b] = 0$ whenever $|j| \geq p$.*

We call elements of \mathcal{A} *local operators* and finite dimensional C^* -subalgebras of \mathcal{A} *local algebras*. We may add the identity 1 to \mathcal{A} , so we assume $1 \in \mathcal{A}$. Since each finite dimensional C^* -algebra is singly generated an easy induction argument shows $C^*(a_1, \dots, a_r)$ is a local algebra whenever $a_1, \dots, a_r \in \mathcal{A}$. In particular if A is separable then A is an AF-algebra. A similar argument shows that for each local algebra N there is $p \in \mathbb{N}$ such that $\alpha^j(N)$ commutes with N whenever $|j| \geq p$.

Theorem. 11.7 *Let (A, α) be a unital separable C^* -dynamical system which is asymptotically abelian with locality. Let $H \in A_{sa}$. Then*

$$P_\alpha(H) = \sup_{\varphi} (h_\varphi(\alpha) - \varphi(H)),$$

where the sup is taken over all α -invariant states. In particular, the topological entropy satisfies

$$ht(\alpha) = \sup_{\varphi} h_\varphi(\alpha).$$

For the proof two lemmas are needed. We first consider the simplest case, which is close to that of shifts on infinite tensor products. To simplify notation we put for each $k \in \mathbb{N}$, and local algebra N

$$H_k = \sum_{j=0}^k \alpha^j(H), \quad N_k = C^*(\alpha^j(N) : 0 \leq j \leq k).$$

Lemma 11.8 *Let A, α, H be as in Theorem 11.7. Suppose there exists a local algebra N such that $H \in N$, all $\alpha^j(N)$, $j \neq 0$, commute with N , and $C^*(\alpha^j(N), j \in \mathbb{Z}) = A$. Then there exists an α -invariant state φ such that*

$$P_\alpha(H) = h_\varphi(\alpha) - \varphi(H) = \lim_{k \rightarrow \infty} \frac{1}{k+1} \text{Tr}_{N_k}(e^{-H_k}).$$

The lemma is easiest to understand if we note that

$$\begin{aligned} \frac{1}{k+1} (h_\varphi(\alpha^{k+1}) - \varphi(H_k)) &= h_\varphi(\alpha) - \varphi(H) \\ \frac{1}{k+1} P_{\alpha^{k+1}}(H_k) &= P_\alpha(H), \end{aligned}$$

and that the expression inside the limit is $\frac{1}{k+1}$ times the pressure of H_k in N_k as in equation (11.1).

Lemma 11.9 *Let $\omega \subset \mathcal{A}$ be a finite set containing H . Let N be a local algebra with $\omega \subset N$. Choose $p \in \mathbb{N}$ such that $\alpha^j(N)$ commutes with N for $|j| \geq p$. Let $k \geq p$, and $M_k = C^*(\alpha^{jk}(N_{k-p}); j \in \mathbb{Z})$. Then*

$$P_\alpha(H, \omega) \leq \liminf_{k \rightarrow \infty} \frac{1}{k} P_{\alpha^k|_{M_k}}(H_{k-p}).$$

Note that $\alpha^{jk}(N_{k-p})$ commutes with N_{k-p} for all $j \neq 0$. Furthermore

$$M_k = C^*(\alpha^{jk+i}(N); 1 \leq i \leq k-p, j \in \mathbb{Z}),$$

hence the $\alpha^\ell(N)$'s which appear in M_k are those for which

$$\ell \notin \bigcup_{j \in \mathbb{Z}} \{jk - p + 1, jk - p + 2, \dots, jk - 1\}.$$

The proof consists of showing that the contribution of these ℓ 's is negligible for large k .

In order to complete the proof of Theorem 11.7 apply Lemma 11.8 to M_k and α^k . We get an α^k -invariant state ψ_k on M_k such that

$$h_{\psi_k}(\alpha^k|_{M_k}) - \psi_k(H_{k-p}) = P_{\alpha^k|_{M_k}}(H_{k-p}).$$

By Theorem 5.2 and Proposition 5.3 we can extend ψ_k to an α^k -invariant state $\tilde{\varphi}_k$ on A such that

$$h_{\tilde{\varphi}_k}(\alpha^k) \geq h_{\psi_k}(\alpha^k|_{M_k}) - 1.$$

Put

$$\varphi_k = \frac{1}{k} \sum_{j=0}^{k-1} \tilde{\varphi}_k \circ \alpha^j.$$

Then φ_k is α -invariant, and by concavity of $\varphi \rightarrow h_\varphi(\alpha)$,

$$h_{\varphi_k}(\alpha) \geq \frac{1}{k} h_{\bar{\varphi}_k}(\alpha^k) \geq \frac{1}{k} h_{\psi_k}(\alpha^k|_{M_k}) - \frac{1}{k},$$

so that

$$h_{\varphi_k}(\alpha) - \varphi_k(H) \geq \frac{1}{k} P_{\alpha^k|_{M_k}}(H_{k-p}) - \varepsilon,$$

where ε is small. We then find that

$$\sup(h_{\varphi_k}(\alpha) - \varphi_k(H)) \geq \liminf_{k \rightarrow \infty} \frac{1}{k} P_{\alpha^k|_{M_k}}(H_{k-p}),$$

and the theorem follows from Lemma 11.9.

Corollary 11.10 *With our assumptions on (A, α) the pressure is a convex function of H .*

Proof. Use the affinity of the function $H \rightarrow h_\varphi(\alpha) - \varphi(H)$.

Let (A, α) be asymptotically abelian with locality as before and H a self-adjoint local operator. Put

$$\delta_H(x) = \sum_{j \in \mathbb{Z}} [\alpha^j(H), x].$$

Then δ_H is a derivation on \mathcal{A} and defines a one-parameter group $\sigma_t^H = \exp(it \delta_H)$ on A . Let $\beta \geq 0$. We say an α -invariant state φ is an *equilibrium state at H at inverse temperature β* if

$$P_\alpha(\beta H) = h_\varphi(\alpha) - \beta \varphi(H) \quad (= \sup_{\psi} (h_\psi(\alpha) - \beta \psi(H))).$$

Theorem. 11.11 *Suppose a unital separable C^* -dynamical system (A, α) is asymptotically abelian with locality, and $ht(\alpha) < \infty$. If H is a self-adjoint local operator in A and φ is an equilibrium state at H at inverse temperature β , then φ is a KMS-state for σ^H at β . In particular, if $ht(\alpha) = h_\varphi(\alpha)$, then φ is a trace.*

In order to prove the theorem we may replace H by $\beta(H)$ and show φ is a KMS-state for σ^H at 1. By Proposition 11.5 Theorem 11.11 is a consequence of

Theorem. 11.12 *Let A, α, H be as in Theorem 11.11. If $-\varphi$ is a tangent functional for P_α at H then φ is a KMS-state for σ^H at 1.*

The proof of this theorem is modelled on the corresponding proof for spin lattice systems, see [B-R, Ch. 6].

Several examples encountered in the previous sections are asymptotically abelian with locality. They are:

- (11.3) Shifts on $\bigotimes_{i \in \mathbb{Z}} B_i$, where $B_i = B_0$ is an AF-algebra.
- (11.4) Binary shifts $(A(X), \alpha)$ as defined in Section 9 with $X \subset \mathbb{N}$ a finite set, see [G-S1].
- (11.5) The shift on the Jones projections in the Temperley-Lieb algebra, see 10.4.
- (11.6) The canonical shift defined by a subfactor $N \subset M$ of finite index, see Section 10.
- (11.7) Let $A = K(H) + \mathbb{C}1$, where $K(H)$ is the compact operators on a separable Hilbert space. Let $\alpha = \text{Ad } u|_A$, where U is the bilateral shift on H .

11.13. Counter examples Theorem 11.7 is false without the assumption of asymptotic abelianness. Indeed Theorem 9.3 provides a counter example. In that case τ is the unique α -invariant state, and $ht(\alpha) \geq \frac{1}{2} \log 2$.

The assumption $ht(\alpha) < \infty$ is necessary in Theorem 11.11, as is immediate from example (11.3) above. If B_0 is infinite dimensional there exist many α -invariant states with infinite entropy.

Locality is necessary in Theorem 11.11. For let U be a unitary operator on a separable Hilbert space H with singular spectrum such that $(U^n f, g) \rightarrow 0$ as $n \rightarrow \infty$, for all $f, g \in H$. Let A be the even CAR-algebra, i.e. the fixed points in the CAR-algebra of the Bogoliubov automorphism α_{-1} , see Section 4. From the proof of Lemma 4.1 we have $ht(\alpha_U) = 0$. However, there exist many α_U -invariant states, e.g. all quasi-free states ω_λ , $0 \leq \lambda \leq 1$.

References

- [A-F] **Alicki, R.** and **Fannes, M.** Defining quantum dynamical entropy, *Lett. Math. Phys.* 32 (1994), 75–82.
- [A] **Araki, H.** Relative entropy for states of von Neumann algebras II, *Publ. RIMS Kyoto Univ.* 13 (1977), 173–192.
- [Ar] **Arveson, W.** Subalgebras of C^* -algebras, *Acta Math.* 123 (1969), 141–224.
- [Av] **Avitzour, D.** Free products of C^* -algebras and von Neumann algebras, *Trans. Amer. Math. Soc.* 217 (1982), 423–435.
- [B] **Besson, O.** On the entropy of quantum Markov states, *Lecture Notes in Math.* 1136 (1985), 81–89, Springer-Verlag.
- [B-G] **Bezuglyi, S.I.** and **Golodets, V.Ya.** Dynamical entropy for Bogoliubov actions of free abelian groups on the CAR-algebra, *Ergod. Th. & Dynam. Sys.* 17 (1997), 757–782.

- [Bo-Go] **Boca, F.** and **Goldstein, P.** Topological entropy for the canonical endomorphism of Cuntz-Krieger algebras, preprint 1999.
- [BKRS] **Bratteli, O., Kishimoto, A., Rørdam, M.** and **Størmer, E.** The crossed product of a UHF-algebra by a shift, *Ergod. Th. & Dynam. Sys.* 13 (1993), 615–626.
- [B-R] **Bratteli, O.** and **Robinson, D.W.** Operator algebras and quantum statistical mechanics II. Springer-Verlag 1981.
- [Br1] **Brown, N.** Topological entropy in exact C^* -algebras, *Math. Ann.* 314 (1999), 347–367.
- [Br2] **Brown, N.** A note on topological entropy, embeddings and unitaries in nuclear quasidiagonal C^* -algebras, preprint 1998.
- [B-C] **Brown, N.** and **Choda, M.** Approximation entropies in crossed products with an application to free shifts, preprint 1999.
- [C1] **Choda, M.** Entropy for $*$ -endomorphisms and relative entropy for subalgebras, *J. Operator Th.* 25 (1991), 125–140.
- [C2] **Choda, M.** Entropy for canonical shifts, *Trans. Amer. Math. Soc.* 334 (1992), 827–849.
- [C3] **Choda, M.** Reduced free products of completely positive maps and entropy for free products of automorphisms, *Publ. RIMS Kyoto Univ.* 32 (1996), 371–382.
- [C4] **Choda, M.** Entropy of Cuntz's canonical endomorphism, *Pacific J. Math.* (1999), 235–245.
- [C5] **Choda, M.** A dynamical entropy and applications to canonical endomorphisms, preprint 1998.
- [C6] **Choda, M.** Entropy on crossed products and entropy on free products, preprint 1999.
- [C-H] **Choda, M.** and **Hiai, F.** Entropy for canonical shifts II. *Publ. RIMS Kyoto Univ.* 27 (1991), 461–489.
- [C-N] **Choda, M.** and **Natsume, T.** Reduced C^* -crossed products by free shifts, *Ergod. Th. & Dynam. Sys.* 18 (1998), 1075–1096.
- [Co] **Connes, A.** Entropie de Kolmogoroff Sinai et mécanique statistique quantique, *C. R. Acad. Sci. Paris* 301 (1985), 1–6.
- [CFW] **Connes, A., Feldmann, J.** and **Weiss, B.** An amenable equivalence relation is generated by a single transformation, *Ergod. Th. & Dynam. Sys.* 1 (1981), 431–450.
- [CNT] **Connes, A., Narnhofer, H.** and **Thirring, W.** Dynamical entropy of C^* -algebras and von Neumann algebras, *Commun. Math. Phys.* 112 (1987), 691–719.

- [C-S] **Connes, A.** and **Størmer, E.** Entropy of automorphisms of II_1 -von Neumann algebras, *Acta Math.* 134 (1975), 289–306.
- [Cu] **Cuntz, J.** Simple C^* -algebras generated by isometries, *Commun. Math. Phys.* 57 (1977), 173–185.
- [De] **Deaconu, V.** Entropy estimates for some C^* -endomorphisms, *Proc. Amer. Math. Soc.* 127 (1999), 3653–3658.
- [D1] **Dykema, K.** Exactness of reduced amalgamated free products of C^* -algebras, preprint 1999.
- [D2] **Dykema, K.** Topological entropy of some automorphisms of reduced amalgamated free product C^* -algebras, preprint 1999.
- [D-S] **Dykema, K.** and **Shlyakhtenko, D.** Exactness of Cuntz-Pimsner C^* -algebras, preprint 1999.
- [E] **Enomoto, M., Nagisa M., Watatani, Y.** and **Yoshida, H.** Relative commutant algebras of Powers' binary shifts on the hyperfinite II_1 -factor, *Math. Scand.* 68 (1991), 115–130.
- [G-N1] **Golodets, V. Ya.** and **Neshveyev, S.** Dynamical entropy for Bogoliubov actions of torsion-free abelian groups on the CAR-algebra, preprint 1998.
- [G-N2] **Golodets, V. Ya.** and **Neshveyev, S.** Entropy of automorphisms of II_1 -factors arising from the dynamical systems theory, preprint 1999.
- [G-N3] **Golodets, V. Ya.** and **Neshveyev, S.** Non-Bernoullian quantum K -systems. *Commun. Math. Phys.* 195 (1998), 213–232.
- [G-S1] **Golodets, V. Ya.** and **Størmer, E.** Entropy of C^* -dynamical systems defined by bitstreams, *Ergod. Th. & Dynam. Sys.* 18 (1998), 859–874.
- [G-S2] **Golodets, V. Ya.** and **Størmer, E.** Generators and comparison of entropies of automorphisms of finite von Neumann algebras, *J. Funct. Anal.* 164 (1999), 110–133.
- [H-S] **Haagerup, U.** and **Størmer, E.** Maximality of entropy in finite von Neumann algebras, *Invent. Math.* 132 (1998), 433–455.
- [H] **Hiai, F.** Entropy for canonical shifts and strong amenability, *Int. J. Math.* 6 (1995), 381–396.
- [Hu] **Hudetz, T.** Quantum dynamical entropy revisited, *Quantum probability* (1997), 241–251, Banach Center Publ. 43.
- [J] **Jones, V.F.R.** Index for subfactors, *Invent. Math.* 72 (1983), 1–25.
- [K] **Kirchberg, E.** On subalgebras of the CAR-algebra, *J. Funct. Anal.* 129 (1995), 35–63.

- [L] **Longo, R.** Index of subfactors and statistics of quantum fields, *Commun. Math. Phys.* 126 (1989), 145–155.
- [M] **Moriya, H.** Variational principle and the dynamical entropy of space translations, preprint 1999.
- [N-U] **Nakamura, M.** and **Umegaki, H.** A note on the entropy for operator algebras, *Proc. Japan Acad.* 37 (1961), 149–154.
- [NST] **Narnhofer, H., Størmer, E.** and **Thirring, W.** C^* -dynamical systems for which the tensor product formula for entropy fails, *Ergod. Th. & Dynam. Sys.* 15 (1995), 96/9/68.
- [N-T1] **Narnhofer, H.** and **Thirring, W.** Dynamical entropy of quantum systems and their abelian counterpart. *On Klauder Path: A field trip*, Emch, G.G., Hegerfeldt, G.C., Streit, L. World Scientific; Singapore, 1994.
- [N-T2] **Narnhofer, H.** and **Thirring, W.** C^* -dynamical systems that are highly anti-commutative,
- [N] **Neshveyev, S.** Entropy of Bogoliubov automorphisms of CAR and CCR algebras with respect to quasi-free states, *Rev. Math. Phys.*
- [N-S1] **Neshveyev, S.** and **Størmer, E.** Entropy in type I algebras, preprint 2000.
- [N-S2] **Neshveyev, S.** and **Størmer, E.** The variational principle for a class of asymptotically abelian C^* -algebras, preprint 2000.
- [O-P] **Ohya, M.** and **Petz, D.** Quantum entropy and its use. *Texts and Monographs in Physics*, Springer-Verlag, 1993.
- [O] **Ornstein, D.S.** Bernoulli shifts with the same entropy are isomorphic, *Adv. Math.* 4 (1970), 337–352.
- [P-S] **Park, Y.M.** and **Shin, H.H.** Dynamical entropy of space translations of CAR and CCR algebras with respect to quasi-free states, *Commun. Math. Phys.* 152 (1993), 497–537.
- [P-P] **Pimsner, M.** and **Popa, S.** Entropy and index for subfactors, *Ann. Sci. Ecole Norm. Sup.* 19 (1986), 57–106.
- [PWY] **Pinzari, C., Watatani, Y.** and **Yonetani, K.** KMS states, entropy and the variational principle in full C^* -dynamical systems, preprint.
- [P] **Powers, R.T.** An index theory for semigroups of $*$ -endomorphisms of $B(H)$ and type II_1 -factors, *Canad. J. Math.* 40 (1988), 86–114.
- [Po-Pr] **Powers, R.T.** and **Price, G.** Binary shifts on the hyperfinite II_1 -factor, *Contemp. Math.* 145 (1993), 453–464.

- [Pr] **Price, G.** The entropy of rational Powers shifts, Proc. Amer. Math. Soc.
- [Q] **Quasthoff, V.** Shift automorphisms of the hyperfinite factor, Math. Nachrichten 131 (1987), 101–106.
- [Sa] **Sauvageot, J.** Ergodic properties of the action of a matrix in $SL(2, \mathbb{Z})$ on a non commutative torus, preprint.
- [S-T] **Sauvageot, J.-L.** and **Thouvenot, J.-P.** Une nouvelle definition de l'entropie dynamique des systems non commutatifs, Commun. Math. Phys. 145 (1998), 521–542.
- [Sh] **Shields, P.** The theory of Bernoulli shifts, Univ. Chicago Press, 1973.
- [S-S] **Sinclair, A.** and **Smith, R.R.** The completely bounded approximation property for discrete crossed products, Indiana Univ. Math. J. 46 (1997), 1311–1322.
- [S1] **Størmer, E.** Entropy of some automorphisms of the II_1 -factor of the free group in infinite number of generators, Invent. Math. 110 (1992), 63–73.
- [S2] **Størmer, E.** Entropy of some inner automorphisms of the hyperfinite II_1 -factor, Int. J. Math. 4 (1993), 319–322.
- [S3] **Størmer, E.** States and shifts on infinite free products of C^* -algebras, Fields Inst. Commun. 12 (1997), 281–291.
- [S4] **Størmer, E.** Entropy of endomorphisms and relative entropy in finite von Neumann algebras, J. Funct. Anal. 171 (2000), 34–52.
- [SV] **Størmer, E.** and **Voiculescu, D.** Entropy of Bogoliubov automorphisms of the canonical anticommutation relations, Commun. Math. Phys. 133 (1990), 521–542.
- [T] **Thomsen, K.** Topological entropy for endomorphisms of local C^* -algebras, Commun. Math. Phys. 164 (1994), 181–193.
- [Vi] **Vik, S.** Fock representation of the binary shift algebra, Math. Scand.
- [V] **Voiculescu, D.** Dynamical approximation entropies and topological entropy in operator algebras, Commun. Math. Phys. 170 (1995), 249–281.
- [W] **Walters, P.** An introduction to ergodic theory, Graduate texts in Math. 79, Springer-Verlag, 1982.