Rational Curves of higher degree on a complete intersection Calabi-Yau threefold in $\mathbf{P}^3 \times \mathbf{P}^3$.

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In this note we study rational curves of degree higher than 3 on Tian-Yau's CICY (Complete Intersection Calabi-Yau) threefold in $\mathbb{P}_1^3 \times \mathbb{P}_2^3$. Tian-Yau's manifold, which is a quotient of Tian-Yau's CICY by a group acting without fixed points, gives the easiest known example of a manifold that corresponds to a three generational superstring model ([1], [6]). We prove the existence of positive dimensional families of curves of every degree greater than 3 on Tian-Yau's CICY, and, more generally, for a generic choice of defining equations for the CICY.

1. Preliminaries.

Tian-Yau's CICY is defined by the following:

$$X = Z(\sum x_i^3, \sum x_i y_i, \sum y_i^3) \subseteq \mathbb{P}_1^3 \times \mathbb{P}_2^3$$

It is defined by three polynomials of bidegrees (3,0), (1,1) and (0,3). When we speak of a *generic* Tian-Yau's CICY, we shall mean a generic choice of polynomials of these bidegrees. By a *rational curve* we shall mean a nonsingular rational curve. We introduce the following notation: F_1 (resp. F_2) is the cubic surface in $\mathbb{P}_1^3$ (resp. $\mathbb{P}_2^3$) defined by the polynomial of degree (3,0) (resp. (0,3)). The incidence variety G is defined by a polynomial of bidegree (1,1). In other words:

$$X = F_1 \times F_2 \cap G$$

On this biprojective space we have a natural notion of degree of a rational curve, by defining it to be the degree of its image in $\mathbb{P}^{15}$ via the Segre embedding. Every rational curve of degree n has a bidegree (i,j) with $i+j=n$. The Hilbert scheme Hilb_X^{nt+1} parametrising subschemes of Hilbert polynomial $p(t) = nt+1$. It has a natural partition in open-closed disjoint subschemes $\mathrm{Hilb}_X^{(i,j)t+1}$ with $i+j=n$. Let $G = Z(\sum \alpha_{ij} x_i y_j) \subset \mathbb{P}_1^3 \times \mathbb{P}_2^3$.

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Definition 1.1.
Let L be a line in $\mathbb{P}_2^3$ (resp. $\mathbb{P}_1^3$). Define $V(L)$ in $\mathbb{P}_1^3$ (resp. $\mathbb{P}_2^3$) to be the unique linear subspace of maximal dimension such that $V \times L$ (resp. $L \times V$) is contained in G.

The following lemma assures that the definition above makes sense.

Lemma 1.2.
Let $G = Z(\sum \alpha_{ij} x_i y_j) \subset \mathbb{P}_1^3 \times \mathbb{P}_2^3$. For any given line L in $\mathbb{P}_2^3$ (resp. $\mathbb{P}_1^3$), $V(L)$ has dimension at least one, and it is unique.

Proof. It is enough to prove the assertion in the case where the line L is in $\mathbb{P}_2^3$, the other case follows by symmetry. Choose coordinates s.t. L is parametrised by $(y_0, y_1, 0, 0)$. G becomes with respect to these new coordinates $Z(\sum \alpha_{ij}^1 x_i y_j)$. Let $\bar{G} = G|_{\mathbb{P}^3 \times L}$, then $\bar{G}$ is defined by the following equation:

$$(\sum \alpha_{i0}^1 x_i) y_0 + (\sum \alpha_{i1}^1 x_i) y_1 = 0$$

$Z(\sum \alpha_{i0}^1 x_i, \sum \alpha_{i1}^1 x_i)$ is obviously both maximal and unique, we get

$$V(L) = Z(\sum \alpha_{i0}^1 x_i, \sum \alpha_{i1}^1 x_i)$$

♣

Remark 1.3.
In fact we proved more: Every point $a \in \mathbb{P}_1^3$ with the property that $a \times L \subset G$, is contained in $V(L)$.

Remark 1.4.
$\dim V = 1, 2, 3$ all occur.

The general case is clearly $\dim V = 1$. The definition of V depends on L as well as on G. We are primarily interested in the case when we are in the general situation for all $L \subseteq \mathbb{P}_i^3 \quad i = 1, 2$. We make the following definition:

Definition 1.5.
G is *nice* if $\dim V(L) = 1$ for all $L \subseteq \mathbb{P}_i^3, \quad i = 1, 2$.

When G is nice, we have maps:

$$\mathrm{Grass}(\mathbb{P}_2^3) \longrightarrow \mathrm{Grass}(\mathbb{P}_1^3) \quad \text{and} \quad \mathrm{Grass}(\mathbb{P}_2^3) \longrightarrow \mathrm{Grass}(\mathbb{P}_1^3)$$

defined by sending L to $V(L)$. These maps are obviously bijective since $V(V(L)) = L$ by definition. When G is nice, we write $l(L)$ for $V(L)$ to signify that it is a line.

We have a sufficient condition for when G is nice:

Lemma 1.6.
Let $G = Z(\sum \alpha_{ij} x_i y_j)$. If the matrix $[\alpha_{ij}]$ is of maximal rank, then G is nice.

Proof.
Write $\sum \alpha_{ij}x_iy_j = xAy^t$, where $x = (x_0, \dots, x_3)$ (likewise for y) and

$$A = \begin{pmatrix} \alpha_{00} & \dots & \alpha_{03} \\ \dots & & \dots \\ \alpha_{30} & \dots & \alpha_{33} \end{pmatrix}$$

Assume that A is of maximal rank. We have to prove that for every line L in $\mathbb{P}^3_1$, $V(L)$ is of minimal dimension. Consider first the special case where $L = (y_0, y_1, 0, 0)$. This gives the following $V(L)$:

$$Z(\sum \alpha_{i1}x_i, \sum \alpha_{i0}x_i) \times L \subseteq \mathbb{P}^3_1 \times \mathbb{P}^3_2$$

Assume that $V(L)$ is not of minimal dimension, i.e. $\dim Z(\sum \alpha_{i1}x_i, \sum \alpha_{i0}x_i) \geq 2$, then there are two different possibilities: 1. $\sum \alpha_{i1}x_i \equiv 0$ or $\sum \alpha_{i0}x_i \equiv 0$. In this case α_{i0} or α_{i1} is zero for all i, which contradicts that A is of maximal rank. 2. $Z(\sum \alpha_{i1}x_i) = Z(\sum \alpha_{i0}x_i)$. This implies that $\sum \alpha_{i1}x_i = \lambda \sum \alpha_{i0}x_i$, giving $\alpha_{i1} = \lambda\alpha_{i0}$. In other words the first two rows are proportional, which contradicts that A is of maximal rank.

The final step is to reduce the general situation to the special case considered above. This is done in the following way: Choose a general L in $\mathbb{P}^3_2$. It is possible to change the coordinates on the second factor, such that L is parametrised by $(y'_0, y'_1, 0, 0)$. Call this coordinate change matrix P (i.e., $(y'_1, \cdots, y'_3)^t = P \cdot (y_1, \dots, y_3)^t$). We make the following change of coordinates on the first factor:

$$x'^t = (A^{-1})^t (P^{-1})^t A^t x^t$$

This gives $G = Z(\sum \alpha_{ij}x'_iy'_j)$ with respect to the new coordinates, since

$$\sum \alpha_{ij}x_iy_j = xAy^t = (x'APA^{-1})(AP^{-1}y'^t) = x'Ay'^t = \sum \alpha_{ij}x'_iy'_j.$$

The result now follows from the special case considered above.

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Proposition 1.7.
Let G be nice, and let $\bar{G} = G|_{\mathbb{P}^3 \times L}$.

Then $\bar{G}$ is isomorphic to the blowing-up of $\mathbb{P}^3_1$ in $l(L)$.

Proof. We can without loss of generality assume that $\bar{G}$ is defined by

$$Z(x_1y_2 - x_2y_1) \subseteq \mathbb{P}^3(x_0, \dots, x_3) \times \mathbb{P}^1(y_1, y_2)$$

(by change of coordinates). In this situation $l(L)$ is defined by $x_1 = x_2 = 0$. It is enough to check the statement locally, take for instance $x_0 = 1$. Then we have

$$Z(x_1y_2 - x_2y_1) \subseteq \mathbb{A}^3 \times \mathbb{P}^1.$$

This is in fact the blowing-up of $\mathbb{A}^3$ with center $Z(x_1, x_2)$ ([3] II.7.12.1).

♣

We have the following important corollary:

Corollary 1.8.
Let G be nice, and let $\breve{G} = G|_{H\times L}$, where H is a hyperplane and L is a line. Denote the blow-up map $\bar{G} \longrightarrow \mathbb{P}_1^3$ by π.

Then $\breve{G}$ is isomorphic to $\pi^{-1}(H)$. In the case $l(L) \not\subseteq H$ then $\breve{G}$ is isomorphic to H blown up in the point $H \cap l(L)$.

♣

2. Curves of higher degree on a generic Tian-Yau's CICY.

Let C be a rational curve in $\mathbb{P}_1^3 \times \mathbb{P}_2^3$. Let f be a parametrisation:

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "f"] & \mathbb{P}_1^3 \times \mathbb{P}_2^3 \arrow[d, "\pi_i"] \\ & \mathbb{P}_i^3 \end{tikzcd}$$

Definition 2.1.
A rational curve in $\mathbb{P}_1^3 \times \mathbb{P}_2^3$ is of *type* $(\bar{m}, \bar{n})$, if the image of the first (resp. second) projection is of degree m (resp. n).

Recall that a generic Tian-Yau CICY was defined to be a complete intersection:

$$X = F_1 \times F_2 \cap G$$

where F_1 (resp. F_2) is defined by a generic polynomial of bidegree $(3,0)$ (resp. $(0,3)$), and the incidence variety G is defined by a generic polynomial of bidegree $(1,1)$.

Proposition 2.2.
Let L be a line in $\mathbb{P}_2^3$, and let C_1 be a rational curve of degree m in $\mathbb{P}_1^3$. Furthermore let $G = Z(\sum \alpha_{ij} x_i y_j) \subseteq \mathbb{P}_1^3 \times \mathbb{P}_2^3$ be nice, and denote $G|_{\mathbb{P}_1^3 \times L}$ by $\bar{G}$. Let C be the unique component of $V = C_1 \times L \cap \bar{G}$ such that $\pi_1(C) = C_1$, where π_1 is the projection map on the first factor. Let $i = lg(C_1 \cap l(L))$. Then C is a rational curve of bidegree $(m, m-i)$ and of type $(\bar{m}, \bar{1})$.

Proof. $\bar{G}$ is isomorphic to the blow up of $\mathbb{P}_1^3$ with center $l(L)$, so C is by definition the strict transform of C_1. Moreover, $V = C \cup E_1 \cup \cdots \cup E_i$, where the E_j are the exceptional fibers corresponding to the intersection points $p_1, \ldots, p_i$ in $C_1 \cap l(L)$. C is rational ([3] V.3.7). The bidegree is easily determined [5]. (The degree on the second factor drops by one for each intersection point counted with multiplicity.)

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Our main result in this section is the following theorem:

Theorem 2.3.
Let X be a generic Tian-Yau CICY. For every n, $n \geq 4$, there exists a nonisolated rational curve of degree n.

Our aim is to prove the theorem using the criterion above. We need a result concerning certain linear systems in $\mathbb{P}^2$.

Proposition 2.4.
Fix a point p in $\mathbb{P}^2$ and let $d \geq 3$. The linear system of curves of degree d, with a point of order $(d-1)$ at p is of dimension $2d$, and a generic member is an irreducible rational curve.

Proof. Assume that it is nonempty. The dimension of the linear system of curves of degree d is $\binom{d+2}{2} - 1$. The condition that a curve has a multiple point p of order $(d-1)$, is equivalent to the vanishing of the $(d-1)$ first partial derivatives at p. This gives $1 + \cdots + (d-1)$ conditions on the coefficients, and the first statement follows. To prove prove the second statement it suffices to show that there exists an irreducible rational curve in the linear system. One can construct one in the following way: Let

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^d$$
$$f(u,v) = (u^d, u^{(d-1)}v, \cdots, v^d)$$

Let $C = f(\mathbb{P}^1) \subseteq \mathbb{P}^d$. Choose $(d-1)$ points on C. These span a linear subspace L of dimension $(d-2)$. Let $\pi : \mathbb{P}^d \longrightarrow \mathbb{P}^2$ be the linear projection from a linear subspace L' of L with the following properties: $\dim L' = (d-3)$ and $L' \cap C = \emptyset$. Let $\hat{C} = \pi(C)$, then $\hat{C}$ is a curve with the desired features.

♣

Proposition 2.5.
Let F be a nonsingular cubic surface in $\mathbb{P}^3$. For every natural number m, $m \geq 3$, there exists a a two dimensional family of nonsingular rational curves of degree m on F.

Proof. F is isomorphic to $\mathbb{P}^2$ blown up in six points $p_0, \cdots, p_5$. Consider the linear system σ^0 of curves of degree d, $d \geq 3$ and with a multiple point of order $(d-1)$ at p_0 in $\mathbb{P}^2$. We denote a generic curve of σ^0 by C_0. The strict transform of C_0 is a rational curve C_1 of degree $2d+1$. Since the dimensions of the linear systems considered down on $\mathbb{P}^2$ is at least 6 by the preceding proposition, the statement is proved for degrees $7, 9, \cdots$. For even degrees we take a sublinear system σ^1 of σ^0, by demanding the curve to pass through p_1 once. The strict transform of a generic curve is a rational curve of degree $2d$. The dimensions of these families of curves are at least 5. In the same manner we can take curves that in addition to the requirements above also pass though p_2 and so on. In each case the dimension drops by no more than one. Hence, we have inclusions $\sigma^0 \supset \sigma^1 \supset \ldots \sigma^t \supset \cdots \supset \sigma^5$ This gives the desired results for the remaining degrees $3, 4, 5$. In the case $m = 3$ (corresponding to $d = 3$ and $t = 4$) the dimension is at least equal to 2.

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Now we give a constructive proof of the theorem.

Proof of the theorem. Let L be one of the 27 lines on F_2. Consider $\bar{G} = (\mathbb{P}^3_1 \times L) \cap G$. G is nice, since we are considering a generic Tian-Yau CICY. $\bar{G}$ is the blow up of $\mathbb{P}^3_1$ in $l(L)$. Denote the intersection points of $F_1 \cap l(L)$ by a_1, a_2, a_3. (Note there are always three of them, when we consider a generic Tian-Yau CICY, because there are only 27 lines $l(L)$ in $\mathbb{P}^3_1$ with $L \subseteq F_2$. These are not tangents to the surface F_1, generically.) Fix a blowing down of a set of exceptional divisors $(E_1, \cdots, E_6)$:

$$\pi : F_1 \longrightarrow \mathbb{P}^2$$

and let $q_i = a_i$ for $i \in \{1, 2, 3\}$. Let C_1 be a curve of degree m on $F_1 \subseteq \mathbb{P}_1^3$. By Proposition 2.2, there is a uniquely determined rational curve $C \subset C_1 \times L \cap G \subset X$ of bidegree $(m, m-i)$, where i is the number of intersection points between C_1 and $l(L)$.

Consider the linear system of curves of degree m constructed in the proof of the preceding proposition, denote it by τ_m^0. A general member of τ_m^0 does not pass through any of the q_i's, giving that the strict transform of this curve up on the cubic F_1 does not pass through any of the a_i's. This gives a curve of degree $m+m = 2m$ by Proposition 2.2. Furthermore by proposition 2.4 the curves constructed in this way are members of a family of dimension at least 2. Consider the linear subsystem τ_m^1 of τ_m^0, defined by the additional requirement that the curves should pass through q_1. This of course gives a linear system of dimension one less than the one considered above. Hence this is at least one dimensional. The generic member of this linear system then gives rise to a rational curve of degree $2m-1$ by Proposition 2.2, since passing through q_i for the curve C_0 is equivalent to that the strict transform of this curve C_1 in F_1 pass through a_i. This gives at families of dimension at least 1 of curves of degrees $5, 7, \cdots$. Now only degree 4 remains. First fix a line L' on F_1. The planes in $\mathbb{P}_1^3$ containing L' is a 1-dimensional family. Denote by $H_{L'}(t)$ the pencil of planes in $\mathbb{P}_1^3$ containing L'. $F_1 \cap H_{L'}(t) = L' \cup C_{L'}(t)$ where $C_{L'}(t)$ is nonsingular conic for almost all t.

Furthermore, fix a line L in F_2. Let

$$D_{L',L} = C_{L'}(t) \times L \cap G \subseteq H_{L'}(t) \times L \cap G \simeq \hat{H_{L'}}(t)$$

where $\hat{H_{L'}}(t)$ denotes the inverse of $H_{L'}(t)$ in $\bar{G}$. Let $\pi : \hat{H_{L'}}(t) \longrightarrow H_{L'}(t)$ denote the restriction of the blowing down map. $D_{L',L}$ is of dimension 1 since it is isomorphic to $\pi^{-1}(C_{L'})$ and $C_{L'}(t) \not\subseteq l(L)$. If $C_{L'}(t) \cap l(L) = \emptyset$, then $D_{L',L}(t)$ is irreducible. Since $l(L) \cap C_1 = \emptyset$, D is isomorphic to the strict transform of C_1. In this case $D = C$. The map $\pi_2|_C : C \longrightarrow L$ is 2:1, which implies that the degree on the second factor is 2. The curve C has bidegree $(2, 2)$. It remains to prove that for almost all t, $C_{L'}(t) \cap l(L) = \emptyset$. Assume the opposite, i.e. $C_{L'}(t) \cap l(L) \neq \emptyset$ for almost all t. Then $D_{L',L}(t)$ contains a rational curve of bidegree $(2, 1)$ or $(2, 0)$ for almost all t, by Proposition 2.2. The $(2, 0)$ curves only arise when $l(L) \subseteq H_{L'}(t)$. This is possible for only finitely many t. Hence, we must have a $(2, 1)$ curve $C_{L',L}(t)$ in $D_{L',L}(t)$ for almost all t. This implies that there are infinitely many curves of degree less than 4. This is not possible on a generic Tian-Yau CICY [5]. We have established our desired contradiction, and we have a positive dimensional family of curves of degree 4.

♣

Theorem 2.6.

Let X be Tian-Yau's CICY. For every n, $n \geq 4$, there exists a nonisolated rational curve of degree n.

Proof. This is a corollary of the proof of theorem 2.3. The construction of curves relied on the use of Proposition 2.2, in other words that $G = Z(\sum_{ij} x_i y_j)$ is nice. $G = Z(\sum_i x_i y_j)$ for Tian-Yau's CICY, which is obviously nice. The last necessary ingredient in the proof is that none of the $l(L)$ where L is one of the 27 lines of

F_2 are tangent to the surface F_2. This is easily checked for Tian-Yau's CICY. The rest of the proof is identical to the proof of theorem 2.3.

♣

3. Existence of curves of various bidegrees.

The technique developed above has several applications. We want to determine the possible bidegrees for rational curves on Tian-Yau's CICY.

Proposition 3.1.
Let $G = (Z(\sum \alpha_{ij}x_iy_j)) \subseteq \mathbb{P}_1^3 \times \mathbb{P}_2^3$ be nice. Then there are no nonsingular rational curves of bidegree $(m,0)$, $m \geq 3$, on G.

Proof. Assume for contradiction that C is a nonsingular rational curve of bidegree $(m,0)$ on G, i.e., $C = C_1 \times p$, where C_1 is a nonsingular rational curve in $\mathbb{P}_1^3$ and p is a point in $\mathbb{P}_2^3$. Fix a line L in $\mathbb{P}_2^3$ passing through p. By Proposition 2.2 $l(L)$ has to be an m-secant to the curve C_1. This is impossible since a nonsingular rational curve of degree m has at most an $(m-1)$ secant.

♣

Corollary 3.2.
There are no rational curves of bidegree $(m,0)$ or $(0,m)$ on a generic Tian-Yau's CICY or Tian-Yau's CICY.

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Proposition 3.4.
Let $d \geq 3$ and let $t \in \{0,1,2,3,4,5\}$ There exist Tian-Yau CICYs with positive dimensional families of rational curves of bidegree $(2d+1-t, d+2-t)$.

Proof.
In Proposition 2.4 and Proposition 2.5 we constructed the linear systems of curves σ^t. The strict transforms of these curves have a $d-1$-secant, E_0, the exceptional divisor corresponding to p_0. Furthermore, the degree of the strict transform of a general member of σ^t is $2d+1-t$. If F_i is a nonsingular cubic surface in $\mathbb{P}_i^3$, denote by L_i^k, $i \in \{1,\ldots,27\}$, the 27 lines on F_i. Now choose a pair of cubic surfaces F_1, F_2 and a nice G with the property that there exists a pair of lines L_1^j and L_2^n such that $l(L_2^n) = L_1^j$. Using Proposition 2.2 gives the desired result.

♣

Remark 3.5.
All of these curves are of type $(\bar{m},\bar{1})$, except for the case $d=3, t=5$ which gives a bidegree $(2,0)$ curve.

Theorem 3.6.
Let X be a generic Tian-Yau CICY.
Let $m > 3$ be an integer, and $i \in \{0,1,2,3\}$. Then there exist positive dimensional families of rational curves of bidegree $(m, m-i)$.
If $m = 3$, then there exist positive dimensional families of rational curves of bidegrees $(3,3),(3,2)$.

Proof.
Let $X = F_1 \times F_2$ be a generic Tian-Yau CICY, and let L be one of the 27 lines on F_2. Let q_1, q_2, q_3 be the intersection points of $l(L) \cap F_1$. Furthermore, fix a blowing down of the exceptional divisors $\pi : F_1 \longrightarrow \mathbb{P}^2$, and let $\bar{q}_i = \pi(q_i)$ for $i = 1, 2, 3$. We will use the linear systems of curves in $\mathbb{P}^2$ considered in Prop. 2.4 and in Prop. 2.5.
Consider first $m \geq 3$ and $i = 0$. By Prop. 2.5 we have linear systems σ_t with t basepoints $p_1 \ldots p_t$. A general member of this linear system does not pass through any of the $\bar{q}_i$, i.e. it gives rise to a rational curve of bidegree (m, m) on X by Proposition 2.2. Since these linear systems are all positive dimensional, we get positive dimensional families of of bidegree (m, m) for $m > 2$ on X.
In order to prove the statement in the case $m \geq 3$ and $i = 1$, we take sublinear systems σ_t' of the σ^t considered above, by assigning the basepoint q_1. The dimension of $\sigma^{t\prime}$ is dim $\sigma^t - 1$. Proposition 2.5 then gives dim $\sigma^{t\prime} \geq 1$, and the result follows.
For $i = 2$ we take sublinear systems of $\sigma^{t\prime}$, by assigning $\bar{q}_2$ as an additional basepoint. By the same reasoning as above this gives positive dimensional families, using Prop 2.2, Prop. 2.4 and Prop. 2.5 for $m > 3$.
Finally, the case $i = 3$ is treated analogously by considering sublinear systems of σ^t by assigning $\bar{q}_1, \bar{q}_2, \bar{q}_3$ as basepoints. Using Prop. 2.2, Prop. 2.4 and Prop. 2.5 obtain positive dimensional families of bidegree $(m, m-3)$ curves for $m > 3$.
♣

Theorem 3.7.
For a generic Tian-Yau CICY there are no rational curves of bidegree $(m, m-i)$ *and of type* $(\bar{m}, \bar{1})$ *for* $m \geq i \geq 4$.

Proof.
An i secant of a curve when $i \geq 4$ has to be contained in F_1 by Bezout's theorem. In other words it has to be one of the 27 lines, but for a generic Tian-Yau CICY none of the 27 $l(L)$'s are among the 27 lines on F_1.

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