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THE PRESSURE EQUATION FOR FLUID FLOW IN A STOCHASTIC MEDIUM

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Abstract

An equation modelling the pressure $p(x) = p(x, \omega)$ at $x \in D \subset \mathbf{R}^d$ of an incompressible fluid in a heterogeneous, isotropic medium with a stochastic permeability $k(x, \omega) \geq 0$ is the stochastic partial differential equation

$$\begin{cases} \operatorname{div}(k(x,\omega) \diamond \nabla p(x,\omega)) = -f(x) & ; x \in D \\ p(x,\omega) = 0 & ; x \in \partial D \end{cases}$$

where f is the given source rate of the fluid, $\diamond$ denotes Wick product.

We represent k as the positive noise given by the Wick exponential of white noise, and we find an explicit formula for the (unique) solution $p(x, \omega)$, which is proved to belong to the space $(\mathcal{S})^{-1}$ of generalized white noise distributions.

§1. INTRODUCTION

If fluid is injected into a region $D \subset \mathbf{R}^d$ at the density rate $f(x)$ at the point $x \in \mathbf{R}^d$, then the pressure $p(x)$ of the fluid at x will satisfy the following partial differential equation:

$$\operatorname{div}(k(x) \cdot \nabla p(x)) = -f(x) \quad ; \quad x \in D \tag{1.1}$$

where $k(x) \geq 0$ is the *permeability* of the medium at x. (We assume that the fluid is incompressible and that the medium is *isotropic*. In the anisotropic case $k(x)$ must be replaced by a symmetric, non-negative definite $d \times d$ matrix $K(x)$). (See e.g. [LØU 3], [MØ] for more details). In addition, let us for simplicity assume that the pressure is kept equal to 0 at the boundary ∂D of D:

$$p(x) = 0 \quad ; \quad x \in \partial D \tag{1.2}$$

In many important applications, e.g. oil flow in porous rocks, the permeability is a rapidly fluctuating, irregular function. Therefore it is natural to represent $k(x)$ as a generalized stochastic process $k(x, \omega); \omega \in \Omega$, where $(\Omega, \mathcal{F}, P)$ is a suitable probability space.

It is usually assumed that such a stochastic permeability process should have, at least approximately, the following properties:

(1.3) (Independence) If $x_1 \neq x_2$ then $k(x_1, \cdot)$ and $k(x_2, \cdot)$ are independent

(1.4) (Lognormality) For each x the random variable $k(x, \cdot)$ is lognormal

(1.5) (Stationarity) For all $x_1, \cdots, x_n \in \mathbf{R}^d$ and $h \in \mathbf{R}^d$ the random variable

$$Y = (k(x_1 + h, \cdot), \cdots, k(x_n + h, \cdot))$$

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has a distribution which is independent of h.

A natural (almost canonical) generalized process $k(x,\omega)$ satisfying (1.3) - (1.5) is

$$k(x,\omega) = a\,\mathrm{Exp}[bW_x(\omega)] \tag{1.6}$$

where $a > 0, b > 0$ are constants, $W_x(\omega)$ denotes *d-parameter white noise* and Exp denotes the *Wick exponential* (see §2). The process $k(x,\omega)$ given by (1.6) belongs to the space of Hida distributions, $(\mathcal{S})^*$, which again is a subset of a certain space $(\mathcal{S})^{-1}$ of *generalized white noise functionals* (see §2). In this space $(\mathcal{S})^{-1}$ there is a natural product, called *the Wick product* and denoted by $\diamond$. If we use this product, our interpretation of (1.1) becomes

$$\mathrm{div}(a\,\mathrm{Exp}[bW_x] \diamond \nabla p(x,\cdot)) = -f(x); \quad x \in D \tag{1.7}$$

regarded as an equation in $(\mathcal{S})^{-1}$.

Using the Wick product in (1.7) corresponds to adopting the *Ito* approach to the stochastic partial differential equation. (See §2).

If the permeability process $k(x,\omega)$ is represented by a more regular stochastic process than in (1.6), then one can also consider the *Stratonovich* interpretation of the equation, obtained by using the ordinary pointwise product instead of the Wick product in (1.7):

$$\mathrm{div}(k(x,\omega)\cdot\nabla p(x,\omega)) = -f(x); \quad x \in D \tag{1.8}$$

In the case when $k(x,\omega)$ is bounded and bounded away from 0 this Stratonovich equation has been studied by Dikow and Hornung [DH], who prove the existence and uniqueness of a weak solution $p(x,\omega)$ as an element of a suitable Sobolov space of functions with values in $L^2(\Omega,\mu)$. Note that k given by (1.6) is neither bounded nor bounded away from 0.

The model (1.6) for the stochastic permeability was suggested in [LØU 3] and there the explicit solution $p(x,\cdot)$ of the corresponding (Ito) stochastic boundary value problem

$$\begin{cases} \mathrm{div}(\mathrm{Exp}W_x \diamond \nabla p(x,\cdot)) = -f(x) & x \in D \\ p(x,\cdot) = 0 & ; x \in \partial D \end{cases} \tag{1.9}$$

was given (formula (7.3) p.170 in [LØU 3]), but without proof. The result has subsequently been announced by one of us (B. Øksendal) in several conferences, including the Seminar on Stochastic Analysis, Random Fields and Applications in Monte Veritá, Ascona, June 1993. There the proof was also presented. More precisely, the lecture gave a proof that the formula solves the equation in a weak (generalized Hermite transform) sense.

The purpose of this paper is to give a complete proof that formula (7.3) in [LØU 3] solves equation (1.9) in the space of *generalized white noise distributions*, $(\mathcal{S})^{-1}$, which was recently constructed by [AKS]. See Theorem 3.1. Moreover, with applications in mind we give interpretations of this solution concept.

§2. SOME PRELIMINARIES IN WHITE NOISE CALCULUS

Here we briefly recall some of the basic definitions and results from white noise calculus. For more information the reader is referred to [HKPS].

In the following we fix the parameter dimension d and let $\mathcal{S} = \mathcal{S}(\mathbf{R}^d)$ denote the Schwartz space of rapidly decreasing smooth (C^∞) functions on $\mathbf{R}^d$. The dual $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^d)$ is the space of tempered distributions. By the Bochner-Minlos theorem [GV] there exists a probability measure μ on the Borel subsets $\mathcal{B}$ of $\mathcal{S}'$ with the property that

$$\int_{\mathcal{S}'} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2}; \forall\phi \in \mathcal{S} \tag{2.1}$$

where $\langle\omega,\phi\rangle$ denotes the action of $\omega\in\mathcal{S}'$ on $\phi\in\mathcal{S}$ and $\|\phi\|^2=\int_{\mathbf{R}^d}|\phi(x)|^2dx$. The triple $(\mathcal{S}',\mathcal{B},\mu)$ is called *the white noise probability space.*

The white noise process is the map $W:\mathcal{S}\times\mathcal{S}'\to\mathbf{R}$ defined by

$$W(\phi,\omega)=W_\phi(\omega)=\langle\omega,\phi\rangle;\quad \omega\in\mathcal{S}',\phi\in\mathcal{S} \tag{2.2}$$

Expressed in terms of Ito integrals with respect to d-parameter Brownian motion B we have

$$W_\phi(\omega)=\int_{\mathbf{R}^d}\phi(x)dB_x(\omega)\quad;\ \phi\in\mathcal{S}. \tag{2.3}$$

The *Hermite polynomials* are defined by

$$h_n(x)=(-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}(e^{-\frac{x^2}{2}});\quad n=0,1,2,\cdots \tag{2.4}$$

and the *Hermite functions* are defined by

$$\xi_n(x)=\pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}}e^{-\frac{x^2}{2}}h_{n-1}(\sqrt{2}x)\quad;\quad n\geq 1 \tag{2.5}$$

In the following we let $\{e_1,e_2,\cdots\}\subset\mathcal{S}$ denote a fixed orthonormal basis for $L^2(\mathbf{R}^d)$. For most purposes the basis can be arbitrary, but it is sometimes convenient to assume that the e_n's are obtained by taking tensor products of $\xi_k(x)$. Define

$$\theta_j(\omega):=W_{e_j}(\omega)=\int_{\mathbf{R}^d}e_j(x)dB_x(\omega)\quad;\quad j=1,2,\cdots \tag{2.6}$$

If $\alpha=(\alpha_1,\cdots,\alpha_m)$ is a multi-index of non-negative integers we put

$$H_\alpha(\omega)=\prod_{j=1}^m h_{\alpha_j}(\theta_j) \tag{2.7}$$

The *Wiener-Ito chaos expansion theorem* says that any $X\in L^2(\mu)$ can be (uniquely) written

$$X(\omega)=\sum_\alpha c_\alpha H_\alpha(\omega) \tag{2.8}$$

Moreover,

$$\|X\|^2_{L^2(\mu)}=\sum_\alpha \alpha!c_\alpha^2\quad\text{where}\quad \alpha!=\alpha_1!\alpha_2!\cdots\alpha_m! \tag{2.9}$$

The Hida test function space $(\mathcal{S})$ and the Hida distribution space $(\mathcal{S})^*$ can be given the following characterization, due to T.-S. Zhang [Z]:

THEOREM 2.1 ([Z])

Part a): A function $f=\sum_\alpha c_\alpha H_\alpha\in L^2(\mu)$ belongs to $(\mathcal{S})$ if and only if

$$\sup_\alpha c_\alpha^2\alpha!(2\mathrm{N})^{\alpha k}<\infty\quad\forall k<\infty \tag{2.10}$$

where

(2.11)
$$(2\mathrm{N})^\alpha := \prod_{j=1}^{m} (2^d \beta_1^{(j)} \cdots \beta_d^{(j)})^{\alpha_j} \quad \text{if } \alpha = (\alpha_1, \cdots, \alpha_m)$$

Here $\beta^{(j)} = (\beta_1^{(j)}, \cdots, \beta_d^{(j)})$ is multi-index nr. j in the fixed ordering of all d-dimensional multi-indices $\beta = (\beta_1, \cdots, \beta_d)$, related to the basis $\{e_j\}$ by

(2.12)
$$e_j = \xi_{\beta_1^{(j)}} \otimes \cdots \otimes \xi_{\beta_d^{(j)}}.$$

Part b): A formal series $F = \sum_\alpha b_\alpha H_\alpha$ belongs to $(\mathcal{S})^*$ if and only if

(2.13)
$$\sup_\alpha b_\alpha^2 \alpha! (2\mathrm{N})^{-\alpha q} < \infty \quad \text{for some} \quad q < \infty$$

The action of $F = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^*$ on $f = \sum_\alpha c_\alpha H_\alpha \in (\mathcal{S})$ is given by

(2.14)
$$\langle F, f \rangle = \sum_\alpha \alpha! b_\alpha c_\alpha$$

EXAMPLE The pointwise (or singular) white noise W_x is defined by

(2.15)
$$W_x(\omega) = \sum_{k=1}^{\infty} e_k(x) H_{\epsilon_k}(\omega) = \sum_{k=1}^{\infty} e_k(x) h_1(\theta_k)$$

where $\epsilon_k = (0, 0, \cdots, 0, 1)$ with 1 on k'th place.

In this case
$b_\alpha = b_{\epsilon_k} = e_k(x)$ if $\alpha = \epsilon_k$ for some k
$b_\alpha = 0$ if $\alpha \neq \epsilon_k$ for all k

Moreover, if $\alpha = \epsilon_k$ we have
$$(2\mathrm{N})^\alpha = 2^d \beta_1^{(k)} \cdots \beta_d^{(k)}$$

So in this case we get
$$\sup_\alpha b_\alpha^2 \alpha! (2\mathrm{N})^{-\alpha q} = \sup_k e_k^2(x) (2^d \beta_1^{(k)} \cdots \beta_d^{(k)})^{-q} < \infty$$

for all $q > 0$, since
$$\sup_{t \in \mathbf{R}} |\xi_k(t)| = 0(k^{-\frac{1}{12}}) \quad ([\mathrm{HP}])$$

We conclude that $W_x(\omega) \in (\mathcal{S})^*$.
Note that if $1 < p < \infty$ we have

(2.16)
$$(\mathcal{S}) \subset L^p(\mu) \subset (\mathcal{S})^*$$

However,

(2.17)
$$L^1(\mu) \not\subset (\mathcal{S})^* \quad \text{(see e.g. [HLØUZ 1])}$$

For our purposes it turns out to be convenient to work with spaces $(\mathcal{S})^1$ and $(\mathcal{S})^{-1}$ which are related to $(\mathcal{S})$ and $(\mathcal{S})^*$ as follows:

(2.18)
$$(\mathcal{S})^1 \subset (\mathcal{S}) \subset (\mathcal{S})^* \subset (\mathcal{S})^{-1}$$

The spaces $(\mathcal{S})$ and $(\mathcal{S})^{-1}$ have recently been constructed by Albeverio, Kondratiev and Streit [AKS]. (Related spaces, denoted by $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}^*$, are constructed in [CY].) We recall here their basic properties, stated in forms which are convenient for our purposes. For details and proofs we refer to [AKS].

DEFINITION 2.2 [AKS]

Part a): For $0 \leq \rho \leq 1$ let $(\mathcal{S})^{\rho}$ consist of all

$$f = \sum_{\alpha} c_\alpha H_\alpha \in L^2(\mu) \quad \text{such that}$$

(2.19)
$$\|f\|^2_{\rho,k} := \sum_{\alpha} c^2_\alpha (\alpha!)^{1+\rho} (2\mathrm{N})^{\alpha k} < \infty \quad \text{for all } k < \infty$$

Part b): The space $(\mathcal{S})^{-\rho}$ consists of all formal expansions

$$F = \sum_{\alpha} b_\alpha H_\alpha$$

such that

(2.20)
$$\sum_{\alpha} b^2_\alpha (\alpha!)^{1-\rho} (2\mathrm{N})^{-\alpha q} < \infty \quad \text{for some } q < \infty$$

The family of seminorms $\|f\|^2_{\rho,k}$; $k = 1, 2, \cdots$ gives rise to a topology on $(\mathcal{S})^{\rho}$ and we can then regard $(\mathcal{S})^{-\rho}$ as the dual of $(\mathcal{S})^{\rho}$ by the action

(2.21)
$$\langle F, f \rangle = \sum_{\alpha} b_\alpha c_\alpha \alpha!$$

if $F = \sum b_\alpha H_\alpha \in (\mathcal{S})^{-\rho}$ and $f = \sum c_\alpha H_\alpha \in (\mathcal{S})^{\rho}$.

REMARKS.

1) Regarding (2.21), note that

$$\begin{aligned}\sum_{\alpha} |b_\alpha c_\alpha| \alpha! &= \sum_{\alpha} |b_\alpha c_\alpha| (\alpha!)^{\frac{1-\rho}{2}} (\alpha!)^{\frac{1+\rho}{2}} \cdot (2\mathrm{N})^{\frac{\alpha k}{2}} (2\mathrm{N})^{-\frac{\alpha k}{2}} \\ &\leq [\sum_{\alpha} b^2_\alpha (\alpha!)^{1-\rho} (2\mathrm{N})^{-\alpha k}]^{\frac{1}{2}} \cdot [\sum_{\alpha} c^2_\alpha (\alpha!)^{1+\rho} (2\mathrm{N})^{\alpha k}]^{\frac{1}{2}} \\ &< \infty \quad \text{for } k \text{ large enough.}\end{aligned}$$

2) Putting $\rho = 0$ we see by comparing (2.19), (2.20) with (2.10), (2.13) that $(\mathcal{S}) = (\mathcal{S})^0$ and $(\mathcal{S})^* = (\mathcal{S})^{-0}$. So for general $\rho \in [0, 1]$ we have

(2.22)
$$(\mathcal{S})^1 \subset (\mathcal{S})^{\rho} \subset (\mathcal{S})^0 = (\mathcal{S}) \subset (\mathcal{S})^* = (\mathcal{S})^{-0} \subset (\mathcal{S})^{-\rho} \subset (\mathcal{S})^{-1}$$

(Observe that with this notation $(\mathcal{S})^0$ and $(\mathcal{S})^{-0}$ are different spaces).

DEFINITION 2.3

The Wick product $F \diamond G$ of two elements

(2.23)
$$F = \sum_{\alpha} b_\alpha H_\alpha, G = \sum_{\beta} a_\beta H_\beta \text{ in } (\mathcal{S})^{-1} \text{ is defined by}$$
$$F \diamond G = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta}$$

From Def. 2.2 we get

LEMMA 2.4

(i) $F, G \in (\mathcal{S})^{-1} \Rightarrow F \diamond G \in (\mathcal{S})^{-1}$
(ii) $f, g \in (\mathcal{S})^{1} \Rightarrow f \diamond g \in (\mathcal{S})^{1}$

The *Hermite transform* [LØU 1-3] has a natural extension to $(\mathcal{S})^{-1}$:

DEFINITION 2.5 If $F = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^{-1}$ then the Hermite transform of $F, \mathcal{H}F = \tilde{F}$, is defined by

$$\tilde{F}(z) = \mathcal{H}F(z) = \sum_\alpha b_\alpha z^\alpha \quad \text{(whenever convergent)} \tag{2.24}$$

where $z = (z_1, z_2, \cdots)$ and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m} \text{ if } \alpha = (\alpha_1, \cdots, \alpha_m).$$

Note that if $F \in (\mathcal{S})^{-\rho}$ for $\rho < 1$ then $(\mathcal{H}F)(z_1, z_2, \cdots)$ converges for all finite sequences $(z_1, \cdots, z_m)$ of complex numbers. To see this we write

$$\begin{aligned} \sum_\alpha |b_\alpha||z^\alpha| &= \sum_\alpha |b_\alpha|(\alpha!)^{\frac{1-\rho}{2}}(\alpha!)^{\frac{\rho-1}{2}}|z^\alpha| \cdot (2\mathrm{N})^{-\frac{\alpha q}{2}} \cdot (2\mathrm{N})^{\frac{\alpha q}{2}} \\ &\leq [\sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho}(2\mathrm{N})^{-\alpha q}]^{\frac{1}{2}} \cdot [\sum_\alpha |z^\alpha|^2 (\alpha!)^{\rho-1}(2\mathrm{N})^{\alpha q}]^{\frac{1}{2}} \end{aligned} \tag{2.25}$$

Now if $z = (z_1, \cdots, z_m)$ with $|z_j| \leq M$ then

$$\sum_\alpha |z^\alpha|^2 (\alpha!)^{\rho-1}(2\mathrm{N})^{\alpha q} \leq \sum_\alpha M^{|\alpha|}(\alpha!)^{\rho-1} 2^{dq|\alpha|} N^{|\alpha| q} < \infty$$

where $N = \sup\{\beta_k^{(j)}; 1 \leq k \leq d, 1 \leq j \leq m\}$. So if q is large enough, then by (2.20) the expression (2.25) is finite.

If $F \in (\mathcal{S})^{-1}$, however, we can only obtain convergence of $\mathcal{H}F(z_1, z_2, \cdots)$ in a neighbourhood of the origin: We have

$$\sum_\alpha |b_\alpha||z^\alpha| \leq |\sum_\alpha b_\alpha^2 (2\mathrm{N})^{-\alpha q}]^{\frac{1}{2}} \cdot [\sum_\alpha |z^\alpha|^2 (2\mathrm{N})^{\alpha q}]^{\frac{1}{2}},$$

where the first factor on the right hand side converges for q large enough. For such a value of q we have convergence of the second factor if $z = (z_1, \cdots, z_m)$ with

$$|z_j| < (2^d \mathrm{N})^{-q} \quad \text{for all } j.$$

The next result is an immediate consequence of Def. 2.3 and Def. 2.5:

LEMMA 2.6 If $F, G \in (\mathcal{S})^{-1}$ then

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)$$

for all z such that $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exist.

The topology on $(\mathcal{S})^{-1}$ can conveniently be expressed in terms of Hermite transforms as follows:

LEMMA 2.7 [AKS]

The following are equivalent
(i) $X_n \to X$ in $(\mathcal{S})^{-1}$

(ii) $\exists \delta > 0, q < \infty, M < \infty$ such that

$$\mathcal{H}X_n(z) \to \mathcal{H}X(z) \quad \text{as } n \to \infty \quad \text{for } z \in \mathbf{B}_q(0,\delta)$$

and

$$|\mathcal{H}X_n(z)| \leq M \quad \text{for all } n = 1, 2, \cdots; z \in \mathbf{B}_q(0,\delta)$$

where

$$\mathbf{B}_q(\delta) = \{\zeta = (\zeta_1, \zeta_2, \cdots) \in \mathbf{C}_0^{\mathbf{N}}; \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2\mathbf{N})^{\alpha q} < \delta^2\} \tag{2.26}$$

LEMMA 2.8 [AKS]

Suppose $g(z_1, z_2, \cdots)$ is a bounded analytic function on $\mathbf{B}_q(\delta)$ for some $\delta > 0, q < \infty$. Then there exists $X \in (\mathcal{S})^{-1}$ such that

$$\mathcal{H}X = g$$

From this we deduce the following useful result:

LEMMA 2.9

Suppose $g = \mathcal{H}X$ for some $X \in (\mathcal{S})^{-1}$. Let f be an analytic function in a neighbourhood of $\zeta_0 = E[X] = g(0)$ in $\mathbf{C}$. Then there exists $Y \in (\mathcal{S})^{-1}$ such that

$$\mathcal{H}Y = f \circ g$$

Proof. Let $r > 0$ be such that f is bounded analytic on $\{\zeta \in \mathbf{C}; |\zeta - \zeta_0| < r\}$. Then choose $\delta > 0$ and $q < \infty$ such that the function $z \to g(z)$ is bounded analytic on $\mathbf{B}_q(\delta)$ and such that $|g(z) - \zeta_0| < r$ for $z \in \mathbf{B}_q(\delta)$. Then $f \circ g$ is bounded analytic in $\mathbf{B}_q(\delta)$, so the result follows from Lemma 2.8.

EXAMPLE 2.10

a) Let $X \in (\mathcal{S})^{-1}$. Then $X \diamond X = X^{\diamond 2} \in (\mathcal{S})^{-1}$ and more generally $X^{\diamond n} \in (\mathcal{S})^{-1}$ for all natural numbers n. Define *the Wick exponential of* X, Exp X, by

$$\text{Exp } X = \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}$$

Then by Lemma 2.9 Exp $X \in (\mathcal{S})^{-1}$ also.

b) In particular, if we choose $X = W_x$ (the singular white noise) then $K_0 := \text{Exp}W_x$ is in fact in $(\mathcal{S})^*$. As suggested in [LØU 3] the process $K_0(x,\omega)$ is a natural model for the stochastic permeability $k(x,\cdot)$ discussed in §1. The reason for this is the following:

Choose a test function $\phi \in \mathcal{S}$. Define the x-shifts $\phi_x(\cdot)$ of ϕ by

$$\phi_x(y) = \phi(y - x) \tag{2.27}$$

and consider the smoothed version of K_0:

$$K(x,\omega) := \text{Exp } W_{\phi_x}(\omega) \tag{2.28}$$

Since for general $\psi \in L^2$ we have

$$\text{Exp } W_\psi = \exp(\int \psi dB - \frac{1}{2}\|\psi\|^2), \tag{2.29}$$

we see that $K(x, \omega)$ does indeed satisfy the 3 requirements (1.3)-(1.5) to a stochastic permeability, except that the independence requirement (1.3) is weakened to

(2.30) (Independence in a weak sense). If supp $\phi_{x_1}(\cdot) \cap$ supp $\phi_{x_2}(\cdot) = \emptyset$, then $K(x_1, \cdot)$ and $K(x_2, \cdot)$ are independent.

The test function ϕ is more than just a technical convenience to avoid too singular mathematical objects, it also has an important physical interpretation: ϕ_x represents the macroscopic average (or *window*) that is used when the value of the permeability at the point x is measured. Typically ϕ will be chosen such that supp ϕ corresponds to what is called a *representative elementary volume*, which is large compared to the pores of the medium but small compared to the macroscopic properties of the flow.

c) If we choose $\lambda \in \mathbf{R}$ and put $X = W_{\lambda e_1} = \lambda H_{\epsilon_1}$ (where e_1 is the first basis element for $L^2(\mathbf{R}^d)$), then $Y = \text{Exp} W_{\lambda e_1} \in (\mathcal{S})^1$ for $|\lambda|$ small enough. To see this we note that the expansion of Y is

$$Y = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n H_{n\epsilon_1} = \sum_{\alpha} c_\alpha H_\alpha,$$

so that

$$\sum_{\alpha} c_\alpha^2 (\alpha!)^2 (2\mathrm{N})^{\alpha q} = \sum_{n=0}^{\infty} (\frac{1}{n!})^2 |\lambda|^{2n} \cdot (n!)^2 \cdot (2^d \beta_1^{(1)} \cdots \beta_d^{(1)})^{nq}$$
$$\leq \sum_{n=0}^{\infty} |\lambda|^{2n} (2^d N)^{nq} < \infty$$

if $|\lambda|^2 < (2^d N)^q$, with $N = \max\{|\beta_j^{(1)}|^d; 1 \leq j \leq d\}$.

Since we are free to choose $e_1 \in \mathcal{S}$ with $\|e_1\| = 1$ we conclude that

$$\text{Exp } W_{\lambda\phi} = \text{Exp}\langle\cdot, \lambda\phi\rangle \in (\mathcal{S})^1$$

for all $\phi \in \mathcal{S}$ if $|\lambda|$ is small enough (depending on ϕ).

d) Other useful applications of Lemma 2.9 include the *Wick logarithm* $Y =$ Log X, which is defined (in $(\mathcal{S})^{-1}$) for all $X \in (\mathcal{S})^{-1}$ with $E[X] \neq 0$. For such X we have

$$\text{Exp}(\text{Log } X) = X$$

and for all $Z \in (\mathcal{S})^{-1}$ we have

$$\text{Log}(\text{Exp } Z) = Z$$

e) Similarly we note that the *Wick-inverse* $X^{\diamond(-1)}$ exists in $(\mathcal{S})^{-1}$ for all $X \in (\mathcal{S})^{-1}$ with $E[X] \neq 0$. This is useful in the discussion of the 1-dimensional pressure equation (see §3).

DEFINITION 2.11. The last observation enables us to extend the concept of $\mathcal{S}$-transform from $(\mathcal{S})^*$ to $(\mathcal{S})^{-1}$ as follows:

For $F \in (\mathcal{S})^{-1}$ define the $\mathcal{S}$-transform of F, $\mathcal{S}F$, by

(2.31) $$(\mathcal{S}F)(\lambda\phi) = \langle F, \text{Exp}\langle\cdot, \lambda\phi\rangle\rangle$$

for $\lambda \in U$, a small neighbourhood of 0 in $\mathbf{R}$.

The argument above shows that the function

$$\lambda \to (\mathcal{S}F)(\lambda\phi) \quad ; \quad \lambda \in U$$

extends to an analytic function

$$z \to (\mathcal{S}F)(z\phi)$$

defined in a neighbourhood of 0 in $\mathbf{C}$.

If $F = \sum c_\alpha H_\alpha$ and $|z_k|$ is small enough, we have

$$(\mathcal{S}F)(z_k e_k) = \langle F, \mathrm{Exp}\langle\cdot, z_k e_k\rangle\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} z_k^n \langle F, \langle\cdot, e_k\rangle^{\diamond n}\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z_k^n \langle F, H_{n\epsilon_k}\rangle n! = \sum_{n=0}^{\infty} c_{n\epsilon_k} z^{n\epsilon_k} = \mathcal{H}F(0, 0, \cdots, z_k)$$

Hence we see that the connection between the $\mathcal{S}$- and $\mathcal{H}$- transform is

(2.32) $$\mathcal{H}F(z_1, z_2, \cdots, z_m) = (\mathcal{S}F)(z_1 e_1 + \cdots + z_m e_m)$$

REMARK. It is important to note that (2.31) actually allows us to define what we could call *generalized expectation* of an arbitrary $F \in (\mathcal{S})^{-1}$, in spite of the fact that such an F need not even be in $L^1(\mu)$: If $F_0 \in L^p(\mu)$ for $p > 1$ then the action of F_0 on an element $\psi \in (\mathcal{S})^1$ is given by

(2.33) $$\langle F_0, \psi\rangle = E[F_0\psi] = \int_{\mathcal{S}'} F_0(\omega)\psi(\omega)d\mu(\omega),$$

so if $\psi \equiv 1$ then $\langle F_0, \psi\rangle = \langle F_0, 1\rangle$ gives us the expectation of F_0. Similarly, choosing $\lambda = 0$ in (2.31) we get

$$(\mathcal{S}F)(0) = \langle F, 1\rangle$$

as a *generalized expectation* of an arbitrary $F \in (\mathcal{S})^{-1}$.

More generally, expanding (2.31) in a power series in z we see that the $\mathcal{S}$-transform (and the Hermite transform) gives us all the *actions*

$$\langle F, \langle\cdot, \phi\rangle^{\diamond n}\rangle$$

of $F \in (\mathcal{S})^{-1}$ on $\langle\cdot, \phi\rangle^{\diamond n} \in (\mathcal{S})^1$. Therefore, although F need not exist as a random variable, it exists as a stochastic distribution: Given a stochastic test function it computes its associated average. See the concluding remarks in the end of §3.

§3. SOLUTION OF THE STOCHASTIC PRESSURE EQUATION

We now proceed to solve the stochastic pressure equation (1.9). As explained in §2 there are both physical and mathematical reasons for interpreting the equation in the smoothed out sense, i.e. we choose a test function (window) ϕ and represent the stochastic permeability by $K(x,\omega) = \mathrm{Exp}\, W_{\phi_x}(\omega)$. The resulting solution $p(x,\omega)$ will be a function of ϕ also, $p = p(\phi, x, \omega)$. Such processes are called *functional processes* in [HLØUZ 1].

THEOREM 3.1. Let D be a bounded domain in $\mathbf{R}^d$ and let f be a Hölder continuous (deterministic) function on $\overline{D}$ (i.e. $|f(x) - f(y)| \le c|x-y|^\delta$ for all $x, y \in \overline{D}; \delta > 0, c < \infty$ constants). Then for all $\phi \in \mathcal{S}$ there is a unique solution $p(x,\cdot) \in (\mathcal{S})^{-1}$ of the stochastic pressure equation

(3.1) $$\mathrm{div}(\mathrm{Exp}W_{\phi_x}(\cdot) \diamond \nabla p(x,\cdot)) = -f(x) \quad ; x \in D$$

(3.2) $$p(x,\cdot) = 0 \quad ; x \in \partial D$$

The solution is given by

(3.3)
$$p(x,\omega)=\frac{1}{2}\operatorname{Exp}(-\frac{1}{2}W_{\phi_x}(\omega))\diamond \hat{E}^x[\int_0^{\tau} f(b_t)\cdot \operatorname{Exp}\{-\frac{1}{2}W_{\phi_{b_t}}(\omega) -\frac{1}{4}\int_0^t [\frac{1}{2}(\nabla W_{\phi_x})^{\diamond 2}(\omega)+\Delta W_{\phi_x}(\omega)]_{x=b_s}ds\}dt],$$

where $(b_t(\hat{\omega}),\hat{P}^x)$ is a (1-parameter) standard Brownian motion in $\mathbf{R}^d$ (independent of B_x), $\hat{E}^x$ denotes expectation with respect to $\hat{P}^x$ and

(3.4)
$$\tau=\tau(\hat{\omega})=\inf\{t>0; b_t\notin D\}$$

(As before all differentiations are taken w.r.t. x). We have used the "vector-Wick product" notation

$$(\vec{X})^{\diamond 2}=\vec{X}\diamond\vec{X}=\sum_{i=1}^d X_i\diamond X_i \quad \text{if} \quad \vec{X}=(X_1,\cdots,X_d)\in(\mathcal{S})^{-1}\times(\mathcal{S})^{-1}\times\cdots\times(\mathcal{S})^{-1}.$$

REMARK Since we have no smoothness conditions on D, the requirement (3.2) must be interpreted in the weak sense

(3.2)′
$$\lim_{\substack{x\to y\\ x\in D}} p(x,\cdot)=0 \quad \text{for all} \quad y\in\partial_R D,$$

where $\partial_R D$ is the set of points $y\in\partial D$ which are *regular* for the classical Dirichlet problem in D.

Proof of Theorem 3.1

Note that in (3.1) the derivatives are taken w.r.t. x in $(\mathcal{S})^{-1}$. For example, by saying that $\frac{\partial p}{\partial x_k}$ exists and is equal to $G(x,\cdot)\in(\mathcal{S})^{-1}$ we mean that

$$\lim_{\epsilon\to 0}\frac{p(x+\epsilon u_k,\cdot)-p(x,\cdot)}{\epsilon}=G(x,\cdot) \quad (\text{limit in } (\mathcal{S})^{-1})$$

where u_k is the k'th unit vector in $\mathbf{R}^n$. From Lemma 2.7 this is equivalent to saying that $\exists\delta>0, q<\infty$ such that

$$\lim_{\epsilon\to 0}\frac{\tilde{p}(x+\epsilon u_k,z)-\tilde{p}(x,z)}{\epsilon}=\tilde{G}(x,z)$$

pointwise boundedly for $z\in\mathbf{B}_q(\delta)$, where as before $\sim$ denotes the Hermite transform. In other words, the statement

(3.5)
$$\frac{\partial p}{\partial x_k}(x,\cdot)=G(x,\cdot)\in(\mathcal{S})^{-1}$$

is equivalent to $\exists\delta>0, q<\infty$ such that

(3.6)
$$\frac{\partial\tilde{p}}{\partial x_k}(x,z)=\tilde{G}(x,z)$$

as elements of the space of functions $\mathbf{R}^d\to A_b(\mathbf{B}_q(\delta))$, where $A_b(\mathbf{B}_q(\delta))$ is the space of all bounded analytic functions on $\mathbf{B}_q(\delta)$ with the topology of pointwise, bounded convergence on $\mathbf{B}_q(\delta)$.

Therefore, to solve (3.1), (3.2) it suffices to find $\delta>0, q<\infty$ and $u(x,\cdot)(=\tilde{p}(x,\cdot))\in A_b(\mathbf{B}_q(\delta))$ such that

(3.7)
$$\operatorname{div}(\exp\tilde{W}_{\phi_x}(z)\cdot\nabla u(x,z))=-f(x); x\in D, z\in\mathbf{B}_q(\delta)$$

and

$$\lim_{\substack{x \to y \\ x \in D}} u(x,z) = 0 \quad \text{for all} \quad y \in \partial_R D, z \in \mathbf{B}_q(\delta) \tag{3.8}$$

Put

$$\gamma(x,z) = \tilde{W}_{\phi_x}(z) = \sum_k (\phi_x, e_k) z_k.$$

Then we can rewrite (3.7) as

$$L^{(z)} u(x,z) := \frac{1}{2}\Delta u(x,z) + \frac{1}{2}\nabla\gamma(x,z) \cdot \nabla u(x,z) = -F(x,z); \quad x \in D \tag{3.9}$$

where

$$F(x,z) = f(x)\exp(-\frac{1}{2}\gamma(x,z)). \tag{3.10}$$

Now assume that $z_k = \xi_k \in \mathbf{R}$ for all k. Since the operator $L^{(\xi)}$ defined by (3.9) is uniformly elliptic in D, we know that equations (3.9), (3.8) have a unique solution $u(\cdot,\xi) \in C^2(D)$ (the twice continuously differentiable functions on D) for each $\xi \in \mathbf{R}_0^{\mathbf{N}}$.

Let $(x_t = x_t^{(\xi)}(\tilde{\omega}), \tilde{P}^x)$ be the solution of the (ordinary) Ito stochastic differential equation

$$dx_t = \frac{1}{2}\nabla\gamma(x_t,\xi)dt + db_t \; ; \; x_0 = x \tag{3.11}$$

where $(b_t(\hat{\omega}), \hat{P}^x)$ is the d-dimensional Brownian motion described below (3.3). Then the generator of $x_t^{(\xi)}$ is $L^{(\xi)}$, so by Dynkin's formula we have, for $x \in U \subset\subset D$,

$$\tilde{E}^x[u(x_{\tilde{\tau}_U},\xi)] = u(x,\xi) + \tilde{E}^x[\int_0^{\tilde{\tau}_U} L^{(\xi)} u(x_t,\xi)dt] \tag{3.12}$$

where $\tilde{E}^x$ denotes expectation w.r.t. $\tilde{P}^x$ and

$$\tilde{\tau}_U = \tilde{\tau}_U(\tilde{\omega}) = \inf\{t > 0; x_t(\tilde{\omega}) \notin U\}$$

is the first exit time from U for x_t. By the Cameron-Martin-Girsanov formula this can be expressed in terms of the probability law $\hat{P}^x$ of b_t as follows:

$$\hat{E}^x[u(b_\tau,\xi)\mathcal{E}(\tau,\xi)] = u(x,\xi) + \hat{E}^x[\int_0^\tau L^{(\xi)} u(b_t,\xi)\mathcal{E}(t,\xi)dt], \tag{3.13}$$

where

$$\mathcal{E}(t,z) = \exp\{\frac{1}{2}\int_0^t \nabla\gamma(b_s,z)db_s - \frac{1}{8}\int_0^t (\nabla\gamma)^2(b_s,z)ds\}, \tag{3.14}$$

$\hat{E}^x$ denotes expectation w.r.t. $\hat{P}^x$ and

$$\tau_\tau = \tau_U(\hat{\omega}) = \inf\{t > 0; b_t(\hat{\omega}) \notin U\}$$

Letting $U \uparrow D$ we get from (3.13), (3.9) and (3.8)

$$u(x,\xi) = \hat{E}^x[\int_0^\tau F(b_t,\xi)\mathcal{E}(t,\xi)dt] \tag{3.15}$$

By Ito's formula we have

$$\gamma(b_t,\xi) = \gamma(b_0,\xi) + \int_0^t \nabla\gamma(b_s,\xi)db_s + \frac{1}{2}\int_0^t \Delta\gamma(b_s,\xi)ds \tag{3.16}$$

or

$$\frac{1}{2}\int_0^t \nabla\gamma(b_s,\xi)db_s = \frac{1}{2}\gamma(b_t,\xi) - \frac{1}{2}\gamma(b_0,\xi) - \frac{1}{4}\int_0^t [\Delta\gamma(b_s,\xi)]ds \tag{3.17}$$

Substituting (3.17) and (3.10) in (3.15) we get

$$u(x,\xi) = \frac{1}{2}\exp(-\frac{1}{2}\gamma(x,\xi))\cdot\hat{E}^x[\int_0^\tau f(b_t)\cdot\exp\{-\frac{1}{2}\gamma(b_t,\xi) - \frac{1}{4}\int_0^t [\frac{1}{2}(\nabla\gamma)^2(b_s,\xi) + \Delta\gamma(b_s,\xi)]ds\}dt \tag{3.18}$$

for all $\xi \in \mathbf{R}_0^{\mathbf{N}}$.

Since $\gamma(x,\xi) = \sum_k(\phi_x, e_k)\xi_k; \xi_k \in \mathbf{R}$ has an obvious analytic extension to $z_k \in \mathbf{C}$ given by $\gamma(x,z) = \sum_k(\phi_x, e_k)z_k$ and similarly with

$$\nabla\gamma(x,z) = \sum_k \nabla_x(\phi_x(\cdot), e_k)z_k,\ \Delta\gamma(x,z) = \sum_k \Delta_x(\phi_x(\cdot), e_k)z_k,$$

we see that $\xi \to u(x,\xi); \xi \in \mathbf{R}_0^{\mathbf{N}}$ given by (3.18) has an analytic extension given by

$$\begin{aligned} w(x,z) = \frac{1}{2}\exp(-\frac{1}{2}\gamma(x,z))\cdot\hat{E}^x[\int_0^\tau f(b_t)\cdot\exp\{-\frac{1}{2}\gamma(b_t,z) - \frac{1}{4}\int_0^t [\frac{1}{2}(\nabla\gamma)^2(b_s,z) \\ + \Delta\gamma(b_s,z)]ds\}dt] \end{aligned} \tag{3.19}$$

provided the expression converges. If $z \in \mathbf{B}_q(\delta)$ then

$$\begin{aligned} |\gamma(x,z)|^2 &= |\sum_k(\phi_x(\cdot), e_k)z_k|^2 \le [\sum_k(\phi_x(\cdot), e_k)^2]\cdot[\sum_k |z_k|^2] \\ &\le \|\phi\|^2\cdot\sum_\alpha |z^\alpha|^2(2\mathbf{N})^{\alpha q} \le \delta^2\|\phi\|^2 \quad \text{for all } q > 0, \end{aligned}$$

and similarly with $|\nabla\gamma(x,z)|$ and $|\Delta\gamma(x,z)|$.

This gives

$$|w(x,z)| \le C_1\exp(\frac{\delta}{2}\|\phi\|)\cdot\hat{E}^x[\int_0^\tau \exp\{\frac{\delta}{2}\|\phi\| + \frac{1}{4}(\frac{\delta^2}{2}\|\nabla\phi_x(\cdot)\|^2 + \delta\|\Delta\phi_x(\cdot)\|)t\}dt]$$

where C_1 is a constant. Since D is bounded there exists $\rho > 0$ such that

$$\hat{E}^x[\exp(\rho\tau)] < \infty.$$

Therefore, if we choose $\delta > 0$ such that

$$\frac{1}{4}(\frac{\delta^2}{2}\|\nabla\phi_x(\cdot)\|^2 + \delta\|\Delta\phi_x(\cdot)\|) < \rho$$

we obtain that $w(x,z)$ is bounded for $z \in \mathbf{B}_q(\delta)$.

Therefore $w(x,\cdot) \in A_b(\mathbf{B}_q(\delta))$.

It remains to verify that $w(\cdot,z)$ satisfies the equations (3.7), (3.8). From (3.18) we know that this is the case when $z = \xi \in \mathbf{R}^{\mathbf{N}}$. Moreover, the solution $u(x,\xi)$ is real analytic in a neighbourhood of $\xi = 0$, so we can write

$$u(x,\xi) = \sum_\alpha c_\alpha(x)\xi^\alpha$$

Similarly we may write $F(x,z) = \sum a_\alpha(x)z^\alpha$ and we have

$$\nabla\gamma(x,z) = \sum_k \nabla(\phi_x, e_k)z_k.$$

Substituted in (3.9) this gives

$$\sum_\alpha \frac{1}{2}\Delta c_\alpha(x)\xi^\alpha + \sum_{\beta,k}\nabla(\phi_x,e_k)\cdot\nabla c_\beta(x)\cdot\xi^{\beta+\epsilon_k} = \sum_\alpha a_\alpha(x)\xi^\alpha. \tag{3.20}$$

i.e.

$$\sum_\alpha[\frac{1}{2}\Delta c_\alpha(x) + \sum_{\substack{\beta,k\\ \beta+\epsilon_k=\alpha}} \nabla(\phi_x,e_k)\cdot\nabla c_\beta(x)]\xi^\alpha = \sum_\alpha a_\alpha(x)\cdot\xi^\alpha \tag{3.21}$$

Since this holds for all ξ small enough, we conclude that

$$\frac{1}{2}\Delta c_\alpha(x) + \sum_{\substack{\beta,k\\ \beta+\epsilon_k=\alpha}} \nabla(\phi_x,e_k)\cdot\nabla c_\beta(x) = a_\alpha(x) \tag{3.22}$$

for all multi-indices α. But then (3.21), and hence (3.20), also hold when ξ is replaced by small enough $z \in \mathbf{C}_0^{\mathbf{N}}$. In other words, the analytic extension $w(x,z)$ of $u(x,\xi)$ does indeed solve (3.9).

The proof that $w(x,z)$ satisfies (3.8) follows standard arguments from stochastic potential theory and is omitted. (See e.g. [Ø, Th. 9.16]).

Finally, to complete the proof of Theorem 3.1 we note that formula (3.3) follows directly from formula (3.19) by means of Lemma 2.6 and the fact that

$$\mathcal{H}(\text{Exp}X) = \exp(\mathcal{H}X) \quad \text{for all } X \in (\mathcal{S})^{-1}.$$

THE 1-DIMENSIONAL CASE

When $d = 1$ it is possible to solve equations (3.1), (3.2) directly, using Wick calculus:

THEOREM 3.2 La $a, b \in \mathbf{R}, a < b$ and assume that $f \in L^1[a,b]$ is a deterministic function. Then for all $\phi \in \mathcal{S}$ the unique solution $p(x,\cdot) \in (\mathcal{S})^{-1}$ of the 1-dimensional pressure equation

$$(\text{Exp } W_{\phi_x}(\cdot) \diamond p'(x,\cdot))' = -f(x) \quad ; \ x \in (a,b) \tag{3.23}$$

$$p(a,\cdot) = p(b,\cdot) = 0 \tag{3.24}$$

is given by

$$p(x,\cdot) = A \diamond \int_a^x \operatorname{Exp}(-W_{\phi_t}(\cdot))dt - \int_a^x (\int_a^t f(s)ds)\operatorname{Exp}(-W_{\phi_t}(\cdot))dt, \tag{3.25}$$

where

$$A = A(\omega) = [\int_a^b \operatorname{Exp}(-W_{\phi_t}(\omega))dt]^{\diamond(-1)} \diamond \int_a^b (\int_a^t f(s)ds)\operatorname{Exp}(-W_{\phi_t}(\omega))dt \in (\mathcal{S})^{-1} \tag{3.26}$$

Proof. Integrating (3.23) we get

$$\operatorname{Exp} W_{\phi_x}(\cdot) \diamond p'(x,\cdot) = A - \int_a^x f(t)dt \quad ; x \in (a,b),$$

where $A = A(\omega)$ does not depend on x. Since $\operatorname{Exp}(-X) \diamond \operatorname{Exp}(X) = 1$ for all $X \in (\mathcal{S})^{-1}$ we can write this as

$$p'(x,\cdot) = A \diamond \operatorname{Exp}(-W_{\phi_x}(\cdot)) - \int_a^x f(s)ds \cdot \operatorname{Exp}(-W_{\phi_x}(\cdot)). \tag{3.27}$$

Using the condition $p(a,\cdot) = 0$ we deduce from (3.27) that

$$p(x,\cdot) = A \diamond \int_a^x \operatorname{Exp}(-W_{\phi_t}(\cdot))dt - \int_a^x (\int_a^t f(s)ds)\operatorname{Exp}(-W_{\phi_t}(\cdot))dt. \tag{3.28}$$

It remains to determine A. The condition $p(b,\cdot) = 0$ leads to

$$A \diamond \int_a^b \operatorname{Exp}(-W_{\phi_t}(\cdot))dt = \int_a^b (\int_a^t f(s)ds)\operatorname{Exp}(-W_{\phi_t}(\cdot))dt \tag{3.29}$$

Put

$$Y = \int_a^b \operatorname{Exp}(-W_{\phi_t}(\cdot))dt. \tag{3.30}$$

We have $Y \in (\mathcal{S})^{-1}$ and $E[Y] = b - a \neq 0$. Therefore $Y^{\diamond(-1)} \in (\mathcal{S})^{-1}$ exists by Example 2.9 e). So

$$A := Y^{\diamond(-1)} \diamond \int_a^b (\int_a^t f(s)ds)\operatorname{Exp}(-W_{\phi_t}(\cdot))dt \in (\mathcal{S})^{-1}$$

and with this choice of A in (3.28) we see that $p(x,\cdot)$ given by (3.28) solves (3.23), (3.24).

CONCLUDING REMARKS.

We emphasize that although the solution $p(x,\cdot)$ lies in the abstract space $(\mathcal{S})^{-1}$ of generalized white noise distributions, it does have a physical interpretation. For example, as explained in the end of §2 we can associate to $p(x,\cdot)$ a *generalized expected value* $E[p(x,\cdot)]$ defined by

$$E[p(x,\cdot)] = < p(x,\cdot), 1 > = \tilde{p}(x,0)$$

Putting $z = 0$ in (3.7) we see that the (generalized) expected value $\overline{p}(x) := E[p(x,\cdot)]$ satisfies the equation

$$\Delta\overline{p}(x) = -f(x) \quad ; x \in D,$$

i.e. the equation obtained by replacing the stochastic permeability $K(x,\omega) = \text{ Exp } W_{\phi_x}(\omega)$ by its average

$$\overline{K}(x) := E[K(x,\cdot)] = 1,$$

which corresponds to a completely homogeneous medium.

We may regard $\overline{p}(x) = E[p(x,\cdot)]$ as the best ω-*constant* approximation to $p(x,\omega)$. This ω-constant coincides with the 0-order term $c_0(x)$ of the generalized Wiener-Ito expansion for $p(x,\omega)$,

$$p(x,\omega) = \sum_\alpha c_\alpha(x) H_\alpha(\omega) \tag{3.31}$$

Having found $\overline{p}(x) = c_0(x)$, we may proceed to find the best *Gaussian* approximation $p_1(x,\omega)$ to $p(x,\omega)$. This coincides with the sum of the first order terms:

$$p_1(x,\omega) = \sum_{|\alpha|\leq|} c_\alpha(x) H_\alpha(\omega) = c_0(x) + \sum_{j=1}^{\infty} c_{\epsilon_j}(x) < \omega, e_j > \tag{3.32}$$

From (3.22) we can find $c_{\epsilon_j}(x)$ when $c_0(x)$ is known:

$$\frac{1}{2}\Delta c_{\epsilon_j}(x) + \nabla(\phi_x, e_j)\cdot\nabla c_0(x) = -f(x)\cdot(\phi_x, e_j) \tag{3.34}$$

which gives

$$c_{\epsilon_j}(x) = \hat{E}^x[\int_0^T \{f(x)\cdot(\phi_x, e_j) + \nabla(\phi_x, e_j)\cdot\nabla c_0(x)\}_{x=b_s} ds] \tag{3.35}$$

Similarly one can proceed by induction to find higher order approximations of $p(x,\omega)$.

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