ISBN 82–553–1256–0
ISSN 0806–2439
Pure Mathematics

No. 23
October 2000

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PREPRINT SERIES

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF OSLO

MATEMATISK INSTITUTT/UNIVERSITETET I OSLO

Optimal consumption and portfolio in a Black-Scholes market driven by fractional Brownian motion *

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August 17, 2000

Abstract

We present a mathematical model for a Black-Scholes market driven by fractional Brownian motion $B_H(t)$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. The interpretation of the integrals with respect to $B_H(t)$ is in the sense of Itô (Skorohod-Wick), not pathwise (which are known to lead to arbitrage).

We find explicitly the optimal consumption rate and the optimal portfolio in such a market for an agent with utility functions of power type. When $H \to \frac{1}{2}+$ the results converge to the corresponding (known) results for standard Brownian motion.

1 Introduction

Let $H \in (0, 1)$ be a fixed constant. The *fractional Brownian motion with Hurst parameter H* is the Gaussian process $B_H(t) = B_H(t, \omega)$; $t \geq 0$, $\omega \in \Omega$

*Keywords: Fractional Brownian motions, fractional Itô calculus, fractional Black-Scholes market, quasi-conditional expectation, optimal consumption and portfolio.

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with mean $\mathbb{E}[B_H(t)] = 0$ for all $t \geq 0$ and covariance

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \; ; \; s, t \geq 0 \tag{1.1}$$

where $\mathbb{E} = \mathbb{E}_{\mu_H}$ denotes the expectation with respect to the law μ_H for $B_H(\cdot)$. We assume that μ_H is defined on the σ-algebra $\mathcal{F}^{(H)}$ of subsets of Ω generated by the random variables $\{B_H(t, \cdot)\}_{t \geq 0}$. We also assume that $B_H(0) = 0$.

If $H = \frac{1}{2}$ then $B_H(t)$ coincides with the standard Brownian motion $B(t)$, which has independent increments. If $H > \frac{1}{2}$ then $B_H(t)$ has a *long memory* or *strong aftereffect*, in the sense that the covariance function $\rho_H(n)$ satisfies

$$\rho_H(n) := \mathbb{E}\left[B_H(1)\left(B_H(n+1) - B_H(n)\right)\right] = \frac{1}{2}\left\{(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right\} > 0 \tag{1.2}$$

for all $n \geq 1$ and $\sum_{n=1}^{\infty} \rho_H(n) = \infty$.

On the other hand, if $0 < H < \frac{1}{2}$, then $\rho_H(n) < 0$ and $B_H(t)$ is *anti-persistent*: positive values of an increment is usually followed by negative ones and conversely.

The strong aftereffect is often observed in the logarithmic returns $\log \frac{Y_n}{Y_{n-1}}$ for financial quantities Y_n while the anti-persistence appears in turbulence and in the behavior of volatilities in finance. We refer to [19, 20, 26] for more information.

For all $H \in (0, 1)$ the process $B_H(t)$ is *self-similar*, in the sense that $B_H(\alpha t)$ has the same law as $\alpha^H B_H(t)$, for all $\alpha > 0$.

These properties make $B_H(t)$ an interesting tool for many applications. In this paper we will concentrate on applications to finance and we will assume that

$$\frac{1}{2} < H < 1. \tag{1.3}$$

We consider the classical Merton problem of finding the optimal consumption rate and the optimal portfolio in a Black-Scholes market, but now driven by fractional Brownian motion $B_H(t)$ rather than classical Brownian motion $B(t)$ (see Section 2 and problem (3.2)). We solve this problem explicitly in

Section 3 (see Theorem 3.1 and Theorem 3.2). Our solution is obtained by proving that the martingale method for classical Brownian motion (see e.g. Chapter 3 in [14]) can be adapted to work for fractional Brownian motion as well.

We now describe our approach in detail:

For $H \neq \frac{1}{2}$ the process $B_H(t)$ is not a semi-martingale, so we cannot use the well-developed theory of stochastic analysis of semimartingales to define stochastic integration with respect to $B_H(t)$. However, for $H > \frac{1}{2}$ the paths of $B_H(t)$ are smoother than the paths of classical Brownian motion $B(t)$ and a direct pathwise integration theory can be developed. To illustrate this pathwise (or ω-wise) definition we note that if

$$f(t,\omega) = \sum_{i=1}^{N} f_i(\omega)\chi_{[t_i,t_{i+})}(t) \text{ with } 0 \leq t_1 < t_2 < \cdots < t_N \tag{1.4}$$

is a step function, with $f_i(\cdot)$ bounded and measurable with respect to the σ-algebra $\mathcal{F}_{t_i}^{(H)}$ generated by $\{B_h(s)\ ;\ s \leq t_i\}$, then the pathwise integral of f with respect to $B_H(t)$ is defined by

$$\int_{\mathbb{R}} f(t,\omega)\delta B_H(t) = \sum_{i=0}^{N-1} f_i(\omega)\cdot(B_H(t_{i+1}) - B_H(t_i)). \tag{1.5}$$

(see e.g. [2], [17].)
However, it was discovered [24] that if we use this integration theory in finance, the corresponding markets may have arbitrage opportunities (see also [4] and [25]).

A different integration theory with respect to $B_H(t)$ was developed in [3] and extended to a white noise setting in [11]. When applied to the integrand $f(t,\omega)$ in (1.4) this integral, denoted by $\int_{\mathbb{R}} f(t,\omega)dB_H(t)$, is defined by

$$\int_{\mathbb{R}} f(t,\omega)dB_H(t) = \sum_{i=1}^{N}(f_i(\cdot)\diamond(B_H(t_{i+1}) - B_H(t_i)))(\omega) \tag{1.6}$$

where $\diamond$ denotes the *Wick product* (see below). This integral is then extended to the class $\mathcal{L}_\varphi^{1,2}(\mathbb{R})$ of all (t,ω)-measurable processes $f(t) = f(t,\omega)$ satisfying the condition

$$\|f\|^2_{\mathcal{L}_\varphi^{1,2}(\mathbb{R})} = \mathbb{E}\left[\int_{\mathbb{R}^2} f(s)f(t)\varphi(s,t)dsdt + \left(\int_{\mathbb{R}} D_s^\varphi f(s)ds\right)^2\right] < \infty. \tag{1.7}$$

Here

$$\varphi(s,t) = H(2H-1)|s-t|^{2H-2} \tag{1.8}$$

and

$$D_s^{\varphi} F = \int_{\mathbb{R}} \varphi(s,t) D_t F dt, \tag{1.9}$$

$D_t F$ being the (fractional) Malliavin derivative at t (see Def. 3.1 in [3], Def. 4.1 and (3.42) in [11]).

Note that if $f(t)$ is *deterministic* then

$$||f||^2_{\mathcal{L}^{1,2}_{\varphi}(\mathbb{R})} = \int_{\mathbb{R}^2} f(s)f(t)\varphi(s,t)dsdt =: |f|^2_{\varphi}.$$

The following isometry may be regarded as a fractional version of the classical Itô isometry:

$$\mathbb{E}_{\mu_H}\left[\left(\int_{\mathbb{R}} f(t,\omega)dB_H(t)\right)^2\right] = ||f||^2_{\mathcal{L}^{1,2}_{\varphi}(\mathbb{R})} \; ; \; f \in \mathcal{L}^{1,2}_{\varphi}(\mathbb{R}) \tag{1.10}$$

(see Theorem 3.7 in [3]).

Note that the only difference between (1.5) and (1.6) is that the ordinary, ω-wise product in (1.5) is replaced by the (generally non-local) Wick product $\diamond$ in (1.6). In the standard case $H = \frac{1}{2}$ these two definitions give the same result, because of the *strong independence* of $f_i(\cdot)$ and $B(t_{i+1}) - B(t_i)$ (see e.g. [8], p. 100). Thus

$$\int_{\mathbb{R}} f(t,\omega)\delta B_H(t) = \int_{\mathbb{R}} f(t,\omega)dB_H(t) \text{ for } H = \frac{1}{2} \tag{1.11}$$

and from this point of view the definition based on (1.6) is just as natural as an extension of the Itô integral to $H > \frac{1}{2}$ as (1.5). Moreover, (1.6) has some tractable Itô-integral-like features which (1.5) misses. Therefore we call $\int_{\mathbb{R}} f dB_H$ the *fractional Itô integral* and refer to $\int_{\mathbb{R}} f\delta B_H(t)$ as the *fractional pathwise integral.*

Here are some examples of properties of the fractional Itô integral:

a) Zero mean

$$\mathbb{E}_{\mu_H}\left[\int_{\mathbb{R}} f(t,\omega)dB_H(t)\right] = 0\ ;\ f \in \mathcal{L}_{\varphi}^{1,2}(\mathbb{R}). \tag{1.12}$$

b) Chaos expansion (Theorem 6.7 in [3]).

Let $F \in L^2(\mu_H)$ be $\mathcal{F}_T^{(H)}$-measurable for some $T \in (0,\infty]$. Then there exist $f_n \in \hat{L}_{\varphi}^2([0,T]^n)\ ;\ n = 0,1,2,\ldots$ such that

$$F(\omega) = \sum_{n=0}^{\infty} \int_{[0,T]^n} f_n dB_H^{\otimes n} \text{ (convergence in } L^2(\mu_H)) \tag{1.13}$$

where

$$\int_{[0,T]^n} f_n dB_H^{\otimes n} := n! \int_{0\leq s_1 < \cdots < s_n \leq T} f_n(s_1,\ldots,s_n)dB_H(s_1)\cdots dB_H(s_n) \tag{1.14}$$

is the iterated Itô fractional integral. Here $\hat{L}_{\varphi}^2([0,T]^n)$ is the set of symmetric functions $f(x_1,\ldots,x_n)$ on $[0,T]^n$ such that

$$\begin{aligned}\|f\|^2_{L^2_{\varphi}([0,T]^n)} := \int_{[0,T]^n\times[0,T]^n} & f(u_1,\ldots,u_n)f(v_1,\ldots,v_n)\varphi(u_1,v_1)\cdots \\ & \cdots\varphi(u_n,v_n)du_1\cdots du_n dv_1\cdots dv_n < \infty.\end{aligned} \tag{1.15}$$

If $f \in \hat{L}_{\varphi}^2([0,T]^n)$ and $g \in \hat{L}_{\varphi}^2([0,T]^m)$, we define the *Wick product* $\diamond$ of their iterated fractional Itô integrals as follows

$$\left(\int_{[0,T]^n} f dB_H^{\otimes n}\right) \diamond \left(\int_{[0,T]^m} f dB_H^{\otimes m}\right) = \int_{[0,T]^n\times[0,T]^m} (f\hat{\otimes}g)dB_H^{\otimes(n+m)} \tag{1.16}$$

where $f\hat{\otimes}g$ is the symmetric tensor product of f and g. This definition is then extended by linearity to sums of such integrals and then to the space $(\mathcal{S})_H^* \supset L^2(\mu_H)$ of *fractional Hida distributions* (Definition 3.7 in [11]). The Wick product $\diamond : (\mathcal{S})_H^* \times (\mathcal{S})_H^* \to (\mathcal{S})_H^*$ is a commutative and associative binary operation, distributive over addition. In particular, if $X \in (\mathcal{S})_H^*$ we can define the n-th Wick power

$$X^{\diamond n} := X \diamond X \diamond \cdots \diamond X \ (n \text{ factors})$$

and

$$\exp^\diamond(X) := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n},$$

provided the sum converges in $(\mathcal{S})_H^*$.

As an example we note that if $f \in \mathcal{L}_\varphi^{1,2}(\mathbb{R})$ is *deterministic* then

$$\exp^\diamond\left(\int_{\mathbb{R}} f(t)dB_H(t)\right) = \exp\left(\int_{\mathbb{R}} f(t)dB_H(t) - \frac{1}{2}|f|_\varphi^2\right). \tag{1.17}$$

See [11], example 3.10.

We remark that this fractional Itô integral may be regarded as a *Skorohod integral* with respect to the Gaussian process $B_H(t)$, in the sense of Skorohod [27].

c) Quasi-conditional expectation and quasi-martingales We say that a formal expansion F of the form

$$F(\omega) = \sum_{n=0}^{\infty} \int_{[0,T]^n} f_n dB_H^{\otimes n} \; ; \; f_n \in \hat{L}_\varphi^2([0,T]^n) \tag{1.18}$$

belongs to the space $\mathcal{G}^*(\mu_H)$ if there exists $q \in \mathbb{N}$ such that

$$||F||_{\mathcal{G}-q}^2 := \sum_{n=0}^{\infty} n! ||f_n||_{L_\varphi^2([0,T]^n)}^2 e^{-2qn} < \infty. \tag{1.19}$$

With this definition we have

$$L^2(\mu_H) \subset \mathcal{G}^*(\mu_H) \subset (\mathcal{S})_H^*.$$

If $F \in \mathcal{G}^*(\mu_H)$ has the expansion (1.18) we define its *quasi-conditional expectation* by

$$\tilde{E}_{\mu_H}\left[F \mid \mathcal{F}_t^{(H)}\right] = \sum_{n=0}^{\infty} \int_{[0,t]^n} f_n dB_H^{\otimes n}. \tag{1.20}$$

It can be proved that

$$\tilde{E}_{\mu_H}\left[F \mid \mathcal{F}_t^{(H)}\right] = F \text{ a.s. } \Leftrightarrow F \text{ is } \mathcal{F}_t^{(H)}\text{-measurable} \tag{1.21}$$

but in general $\tilde{E}_{\mu_H}\left[F \mid \mathcal{F}_t^{(H)}\right] \neq E_{\mu_H}\left[F \mid \mathcal{F}_t^{(H)}\right]$ (see section 4 in [11]) and the references therein.

We say that a (t,ω)-measurable $\mathcal{F}_t^{(H)}$-adapted process $M(t) = M(t,\omega)$; $t \geq 0$ is *a quasi-martingale* if $M(t) \in \mathcal{G}^*(\mu_H)$ for all t and

$$\tilde{E}_{\mu_H}\left[M(t) \mid \mathcal{F}_s^{(H)}\right] = M(s) \text{ for all } t \geq s. \tag{1.22}$$

Using the definition of the fractional Itô integral one can now prove (we omit the proof)

Lemma 1.1 *Let $f \in \mathcal{L}_\varphi^{1,2}(\mathbb{R})$. Then*

$$M(t) := \int_0^t f(s,\omega)dB_H(s) \; ; \; t \geq 0$$

is a quasi-martingale. In particular,

$$\mathbb{E}_{\mu_H}[M(t)] = \mathbb{E}_{\mu_H}[M(0)| = 0 \textit{ for all } t \geq 0.$$

This result enables us to carry over to $H > \frac{1}{2}$ many of the useful martingale methods valid for $H = \frac{1}{2}$, if we replace conditional expectation by *quasi*-conditional expectation.

Example 1.2 Let $f \in \mathcal{L}_\varphi^{1,2}(\mathbb{R})$ be deterministic. Then

$$\mathcal{E}(t) := \exp^\diamond\left(\int_0^t f(s)dB_H(s)\right) = \exp\left(\int_0^t f(s)dB_H(s) - \frac{1}{2}|f^{(t)}|_\varphi^2\right)$$

is a quasi-martingale, where $f^{(t)}(s) = f(s) \cdot \chi_{[0,t]}(s)$. In particular,

$$\mathbb{E}[\mu_H(\mathcal{E}(t)] = 1 \text{ for all } t.$$

PROOF. By Example 3.14 in [11] we have

$$d\mathcal{E}(t) = f(t)\mathcal{E}(t)dB_H(t).$$

Since $\mathcal{E}(0) = 1$ the statements follow from Lemma 1.1. □

d) A fractional Girsanov theorem. We also recall the following result, which is Theorem 3.18 in [11]:

Theorem 1.3 *[11]*

Let $T > 0$ and let $u : [0,T] \to \mathbb{R}$ be continuous. Suppose $K : [0,T] \to \mathbb{R}$ satisfies the equation

$$\int_0^T K(s)\varphi(s,t)ds = u(t) \; ; \; 0 \le t \le T \tag{1.23}$$

and extend K to $\mathbb{R}$ by putting $K(s) = 0$ outside $[0,T]$. Define the probability measure $\hat{\mu}_H$ on $\mathcal{F}_T^{(H)}$ by

$$d\hat{\mu}_H(\omega) = \exp\left\{-\int_0^T K(s)dB_H(s) - \frac{1}{2}|K|_\varphi^2\right\} d\mu_H(\omega). \tag{1.24}$$

Then

$$\hat{B}_H(t) := \int_0^t u(s)ds + B_H(t) \tag{1.25}$$

is a fractional Brownian motion with respect to $\hat{\mu}_H$.

e) A fractional Clark-Haussmann-Ocone (CHO) theorem Finally we review a fractional version of the Clark-Haussmann-Ocone (CHO) representation theorem obtained in Theorem 4.5 in [11]. See also Theorem 3.11 in [1].

Theorem 1.4 *[11]*

Let $G(\omega) \in L^2(\mu_H)$ be $\mathcal{F}_T^{(H)}$-measurable. Define

$$\psi(t,\omega) = \tilde{E}_{\mu_H}[D_t G \mid \mathcal{F}_t^{(H)}]. \tag{1.26}$$

Then

$$\psi \in \mathcal{L}_\varphi^{1,2}(\mathbb{R}) \tag{1.27}$$

and

$$G(\omega) = E_{\mu_H}[G] + \int_0^T \psi(t,\omega)dB_H(t). \tag{1.28}$$

Here $D_t G = \frac{dG}{d\omega}(t,\omega)$ is the stochastic gradient (Malliavin derivative) of G at t, which exists for a.a. $t \in [0,T]$ as an element of $\mathcal{G}^*(\mu_H)$. We refer to Section 4 in [11], for details.

2 The fractional Black and Scholes market

Suppose we have the following two investment possibilities:

1. A *bank account* or a *bond*, where the price $A(t)$ at time $t \geq 0$ is given by

$$dA(t) = rA(t)dt \ ; \ A(0) = 1 \text{ (i.e., } A(t) = e^{rt}) \tag{2.1}$$

   where $r > 0$ is a constant ; $0 \leq t \leq T$ (constant).

2. A stock, where the price $S(t)$ at time $t \geq 0$ is given by

$$dS(t) = aS(t)dt + \sigma S(t)dB_H(t) \ ; \ S(0) = s > 0 \tag{2.2}$$

   where $a > r > 0$ and $\sigma \neq 0$ are constants, $0 \leq t \leq T$.

Here the differential $dB_H(t)$ is the Itô type fractional Brownian motion differential used in [11].

Suppose an investor chooses a *portfolio* $\theta(t) = (\alpha(t), \beta(t))$ giving the number of units $\alpha(t), \beta(t)$ held at time t of bonds and stocks, respectively. We assume that $\alpha(t), \beta(t)$ are $\mathcal{F}_t^{(H)}$-adapted processes, where $\mathcal{F}_t^{(H)}$ is the σ-algebra generated by $\{B_H(s)\}_{0 \leq s \leq t}$, and that $(t, \omega) \to \alpha(t, \omega), \beta(t, \omega)$ are measurable with respect to $\mathcal{B}[0,T] \times \mathcal{F}^{(H)}$, where $\mathcal{B}[0,T]$ is the Borel σ-algebra on $[0,T]$ and $\mathcal{F}^{(H)}$ is the σ-algebra generated by $\{B_H(s)\}_{s \geq 0}$.

Suppose the investor is also free to choose a (t, ω)-measurable, adapted *consumption process* $c(t, \omega) \geq 0$. The *wealth process* $Z(t) = Z^{c,\theta}(t)$ associated to a given assumption rate c and portfolio $\theta = (\alpha, \beta)$ is defined by

$$Z(t) = \alpha(t)A(t) + \beta(t)S(t). \tag{2.3}$$

We say that θ is *self-financing* with respect to c if

$$dZ(t) = \alpha(t)dA(t) + \beta(t)dS(t) - c(t)dt. \tag{2.4}$$

From (2.3) we get

$$\alpha(t) = A^{-1}(t)[Z(t) - \beta(t)S(t)] \tag{2.5}$$

which substituted into (2.4) gives, using (2.1),

$$dZ(t) = rZ(t)dt + (a - r)\beta(t)S(t)dt + \sigma\beta(t)S(t)dB_H(t) - c(t)dt \tag{2.6}$$

or

$$d(e^{-rt}Z(t)) + e^{-rt}c(t)dt = \sigma e^{-rt}\beta(t)S(t)\left[\frac{a-r}{\sigma}dt + dB_H(t)\right]. \tag{2.7}$$

Define the measure $\hat{\mu}_H$ on $\mathcal{F}_T^{(H)}$ by

$$\frac{d\hat{\mu}_H}{d\mu_H} = \exp\left(-\int_0^T K(s)dB_H(s) - \frac{1}{2}|K|_\varphi^2\right) := \exp^\diamond\left(-\int_0^T K(s)dB_H(s)\right) =: \eta(T) \tag{2.8}$$

where ϕ is defined by (1.8),

$$K(s) = \frac{(a-r)(Ts-s^2)^{\frac{1}{2}-H}\chi_{[0,T]}(s)}{2\sigma H\cdot\Gamma(2H)\cdot\Gamma(2-2H)\cos(\pi(H-\frac{1}{2}))}, \tag{2.9}$$

and

$$|K|_\varphi^2 = \int_0^T\int_0^T K(s)K(t)\varphi(s,t)dsdt,$$

where Γ is the gamma function.

Then by the fractional Girsanov formula (Theorem 1.2), the process

$$\hat{B}_H(t) := \frac{a-r}{\sigma}t + B_H(t) \tag{2.10}$$

is a fractional Brownian motion (with Hurst parameter H) with respect to $\hat{\mu}_H$. In terms of $\hat{B}_H(t)$, we can write (2.7) as follows

$$e^{-rt}Z(t) + \int_0^t e^{-ru}c(u)du = Z(0) + \int_0^t \sigma e^{-ru}\beta(u)S(u)d\hat{B}_H(u). \tag{2.11}$$

If $Z(0) = z > 0$, we write $Z_z^{c,\theta}(t)$ for the corresponding wealth process $Z(t)$ given by (2.11).

We say that (c,θ) is *admissible* with respect to z and write $(c,\theta)\in\mathcal{A}(z)$ if $\theta = \theta(t) = (\alpha(t),\beta(t))$ with $\alpha(t)$ satisfying (2.5) and $\beta(t) = \beta(t,\omega)$ satisfying the condition

$$\beta(\cdot)S(\cdot)\in\mathcal{L}_\varphi^{1,2}(\mathbb{R}) \tag{2.12}$$

and in addition θ is self-financing with respect to c and $Z_z^{c,\theta}(T) \geq 0$ a.s.

Note that it follows from (2.12) and Lemma 1.1 that

$$M(t) := \int_0^t \sigma^{-1} e^{-ru} \beta(u) S(u) d\hat{B}_H(u) \; ; \; 0 \leq t \leq T$$

is a quasi-martingale with respect to $\hat{\mu}_H$.

In particular, $\mathbb{E}_{\hat{\mu}_H}[M(T)] = 0$. Therefore, from (2.11) we get the *budget constraint*

$$\mathbb{E}_{\hat{\mu}_H}\left[e^{-rT} Z_z^{c,\theta}(T) + \int_0^T e^{-ru} c(u) du\right] = z, \tag{2.13}$$

valid for all $(c, \theta) \in \mathcal{A}(z)$.

Conversely, suppose $c(u) \geq 0$ is a given consumption rate and $F(\omega)$ is a given $\mathcal{F}_T^{(H)}$-measurable random variable such that $\mathbb{E}_{\hat{\mu}_H}[G^2] < \infty$, where

$$G(\omega) = e^{-rT} F(\omega) + \int_0^T e^{-ru} c(u, \omega) du. \tag{2.14}$$

Then by the fractional CHO theorem (Theorem 1.3) applied to $(\hat{B}_H(\cdot), \hat{\mu}_H)$ we get

$$G(\omega) = \mathbb{E}_{\hat{\mu}_H}[G] + \int_0^T \psi(t, \omega) d\hat{B}_H(t) \tag{2.15}$$

where

$$\psi(t, \omega) := \tilde{\mathbb{E}}_{\hat{\mu}_H}[\hat{D}_t G \mid \mathcal{F}_t^{(H)}] \tag{2.16}$$

satisfies

$$\psi(\cdot) \in \mathcal{L}_\varphi^{1,2}(\mathbb{R}). \tag{2.17}$$

Therefore, if $\mathbb{E}_{\hat{\mu}_H}[G] = z$ and we define

$$\beta(t) := \sigma e^{rt} S^{-1}(t) \psi(t) \tag{2.18}$$

then $\beta(t)$ satisfies (2.12) and with $\theta = (\alpha, \beta)$ with α as in (2.5) we have by comparing (2.11) and (2.15)

$$Z_z^{c,\theta}(T) = F \text{ a.s.} \tag{2.19}$$

We have proved

Lemma 2.1 *Let $c(t) \geq 0$ be a given consumption rate and let F be a given $\mathcal{F}_T^{(H)}$-measurable random variable such that the random variable*

$$G(\omega) := e^{-rT}F(\omega) + \int_0^T e^{-ru}c(u,\omega)du \tag{2.20}$$

satisfies

$$\mathbb{E}_{\hat{\mu}_H}[G^2] < \infty. \tag{2.21}$$

Then the following, (2.22) and (2.23), are equivalent:

There exists a portfolio θ such that $(c,\theta) \in \mathcal{A}(x)$ and $Z_z^{c,\theta}(T) = F$ a.s. (2.22)

$$\mathbb{E}_{\hat{\mu}_H}[G] = z. \tag{2.23}$$

3 Optimal consumption and portfolio

Let $D_1 > 0, D_2 > 0, T > 0$ and $\gamma \in (-\infty, 1)\backslash\{0\}$ be given constants. Consider the following quantity

$$J^{(c,\theta)}(z) = \mathbb{E}_{\mu_H}\left[\int_0^T \frac{D_1}{\gamma}c^\gamma(t)dt + \frac{D_2}{\gamma}(Z_z^{c,\theta}(T))^\gamma\right], \tag{3.1}$$

where $(c,\theta) \in \mathcal{A}(z)$ and we interpret Z^γ as $-\infty$ if $Z < 0$. We may regard $J^{c,\theta}(z)$ as the total expected utility obtained from the consumption rate $c(t) \geq 0$ and the terminal wealth $Z_z^{c,\theta}(T)$. We now seek $V(z)$ and $(c^*, \theta^*) \in \mathcal{A}(z)$ such that

$$V(z) = \sup_{(c,\theta)\in\mathcal{A}(z)} J^{(c,\theta)}(z) = J^{c^*,\theta^*}(z) \; ; \; z > 0. \tag{3.2}$$

By Lemma 2.1 we see that this problem is equivalent to the *constrained* optimization problem

$$V(z) = \sup_{c,F\geq 0}\left\{\mathbb{E}_{\mu_H}\left[\int_0^T \frac{D_1}{\gamma}c^\gamma(t)dt + \frac{D_2}{\gamma}F^\gamma\right] \; ; \text{ given that } \right.$$
$$\left. \mathbb{E}_{\hat{\mu}_H}\left[\int_0^T e^{-ru}c(u)du + e^{-rT}F\right] = z\right\}, \tag{3.3}$$

where the supremum is taken over all consumption rates $c(t,\omega) \geq 0$ and $\mathcal{F}_T^{(H)}$-measurable $F(\omega) \geq 0$ such that

$$\int_0^T e^{-ru} c(u)du + e^{-rT} F \in L^2(\hat{\mu}_H). \tag{3.4}$$

Consider for each $\lambda > 0$ the following related *unconstrained* optimization problem (with $\mathbb{E} = \mathbb{E}_{\mu_H}$)

$$V_\lambda(z) = \sup_{c,F\geq 0} \left\{ \mathbb{E}\left[\int_0^T \frac{D_1}{\gamma} c^\gamma(t)dt + \frac{D_2}{\gamma} F^\gamma\right] - \lambda \mathbb{E}_{\hat{\mu}_H}\left[\int_0^T e^{-rt} c(t)dt + e^{-rT} F\right]\right\}. \tag{3.5}$$

Suppose that for each $\lambda > 0$ we can find $V_\lambda(z)$ and corresponding $c_\lambda(t,\omega) \geq 0, F_\lambda \geq 0$. Moreover, suppose that there exists $\lambda^* > 0$ such that $c_{\lambda^*}, F_{\lambda^*}$ satisfies the constraint in (3.3):

$$\mathbb{E}_{\hat{\mu}_H}\left[\int_0^T e^{-ru} c_{\lambda^*}(u)du + e^{-rT} F_{\lambda^*}\right] = z. \tag{3.6}$$

Then, $c_{\lambda^*}, F_{\lambda^*}$ actually solves the *constrained* problem (3.3), because if $c \geq 0, F \geq 0$ is another pair satisfying the constraint then

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \frac{D_1}{\gamma} c^\gamma(t)dt + \frac{D_2}{\gamma} F^\gamma\right] \\
&= \mathbb{E}\left[\int_0^T \frac{D_1}{\gamma} c^\gamma(t)dt + \frac{D_2}{\gamma} F^\gamma\right] - \lambda^* E_{\hat{\mu}_H}\left[\int_0^T e^{-ru} c(u)du + e^{-rT} F\right] + \lambda^* z \\
&\leq \mathbb{E}\left[\int_0^T \frac{D_1}{\gamma} c_{\lambda^*}^\gamma(t)dt + \frac{D_2}{\gamma} F_{\lambda^*}^\gamma\right] - \lambda^* E_{\hat{\mu}_H}\left[\int_0^T e^{-ru} c_{\lambda^*}(u)du + e^{-rT} F_{\lambda^*}\right] + \lambda^* z \\
&= \mathbb{E}\left[\int_0^T \frac{D_1}{\gamma} c_{\lambda^*}^\gamma(t)dt + \frac{D_2}{\gamma} F_{\lambda^*}^\gamma\right].
\end{aligned}$$

Finally, to solve the original problem (3.1) we use Lemma 2.1 to find θ^* such that $(c_{\lambda^*}, \theta^*) \in \mathcal{A}(z)$ and

$$Z_z^{c_{\lambda^*},\theta^*}(T) = F_{\lambda^*} \text{ a.s..}$$

Then c_{λ^*}, θ^* are optimal for (3.1) and

$$V(z) = V_{\lambda^*}(z) = E\left[\int_0^T \frac{D_1}{\gamma} c_{\lambda^*}^{\gamma}(t)dt + \frac{D_2}{\gamma}(Z_z^{c_{\lambda^*},\theta^*}(T))^{\gamma}\right]. \tag{3.7}$$

In view of the above we now proceed to solve the unconstrained optimization problem (3.5). Note that with

$$\eta(t) = \exp^{\diamond}\left(-\int_0^t K(s)dB_H(s)\right) \tag{3.8}$$

as in (2.8), we can write

$$\begin{aligned} V_\lambda(z) &= \sup_{c,F\geq 0} \mathbb{E}\left[\int_0^T \left(\frac{D_1}{\gamma}c^{\gamma}(t) - \lambda\eta(T)e^{-rt}c(t)\right)dt + \frac{D_2}{\gamma}F^{\gamma} - \lambda\eta(T)e^{-rT}F\right] \\ &= \sup_{c,F\geq 0} \mathbb{E}\left[\int_0^T \left(\frac{D_1}{\gamma}c^{\gamma}(t) - \lambda\rho(t)e^{-rt}c(t)\right)dt + \frac{D_2}{\gamma}F^{\gamma} - \lambda\eta(T)e^{-rT}F\right], \end{aligned} \tag{3.9}$$

where

$$\rho(t) = \mathbb{E}\left[\eta(T) \mid \mathcal{F}_t^{(H)}\right].$$

In the above formula we have used that

$$\begin{aligned} \mathbb{E}[\eta(T)c(t)] &= \mathbb{E}[\mathbb{E}[\eta(T)c(t) \mid \mathcal{F}_t^{(H)}]] = \mathbb{E}[c(t)\mathbb{E}[\eta(T) \mid \mathcal{F}_t^{(H)}]] \\ &= \mathbb{E}[c(t)\rho(t)]. \end{aligned}$$

The problem (3.9) can be solved by simply maximizing pointwise (for each t, ω) the two functions

$$g(c) = \frac{D_1}{\gamma}c^{\gamma} - \lambda\rho(t,\omega)e^{-rt}c\ ;\ c \geq 0 \tag{3.10}$$

$$h(F) = \frac{D_2}{\gamma}F^{\gamma} - \lambda\eta(T,\omega)e^{-rT}F\ ;\ F \geq 0 \tag{3.11}$$

for each $t \in [0, T]$ and $\omega \in \Omega$.

We have $g'(c) = 0$ for

$$c = c_\lambda(t,\omega) = \frac{1}{D_1}[\lambda e^{-rt}\rho(t,\omega)]^{\frac{1}{\gamma-1}} \tag{3.12}$$

and by concavity this is the maximum point of g.

Similarly

$$F = F_\lambda(\omega) = \frac{1}{D_2}\left[\lambda e^{-rT}\eta(T,\omega)\right]^{\frac{1}{\gamma-1}} \tag{3.13}$$

is the maximum point of h.

We now seek λ^* such that (3.6) holds, i.e.

$$\mathbb{E}\left[\int_0^T e^{-rt}\rho(t)\frac{1}{D_1}\left[\lambda e^{-rt}\rho(t)\right]^{\frac{1}{\gamma-1}}dt + e^{-rt}\eta(T)\frac{1}{D_2}[\lambda e^{-rT}\eta(T)]^{\frac{1}{\gamma-1}}\right] = z \tag{3.14}$$

or

$$\lambda^{\frac{1}{\gamma-1}}N = z,$$

where

$$N = \mathbb{E}\left[\int_0^T \frac{1}{D_1}e^{\frac{r\gamma}{1-\gamma}t}\rho(t)^{\frac{\gamma}{\gamma-1}}dt + \frac{1}{D_2}e^{\frac{r\gamma}{1-\gamma}T}\eta(T)^{\frac{\gamma}{\gamma-1}}\right] > 0. \tag{3.15}$$

Hence

$$\lambda^* = \left(\frac{z}{N}\right)^{\gamma-1}. \tag{3.16}$$

Substituted into (3.12), (3.13) this gives

$$c_{\lambda^*}(t,\omega) = \frac{z}{D_1N}e^{\frac{r}{1-\gamma}t}\rho(t,\omega)^{\frac{1}{\gamma-1}} \tag{3.17}$$

and

$$F_{\lambda^*}(\omega) = \frac{z}{D_2N}e^{\frac{r}{1-\gamma}T}\eta(T,\omega)^{\frac{1}{\gamma-1}}. \tag{3.18}$$

This is the optimal c, F for the constrained problem (3.3) and we conclude that the solution of the original problem is

$$V(z) = V_{\lambda^*}(z) = \mathbb{E}\left[\int_0^T \frac{D_1}{\gamma}c^\gamma_{\lambda^*}(t)dt + \frac{D_2}{\gamma}F^\gamma_{\lambda^*}\right]. \tag{3.19}$$

To find $V(z)$ we need to compute $\mathbb{E}\left[\rho(t)^{\frac{\gamma}{\gamma-1}}\right]$. For $t = T$, this was done in ((2.19)-(2.27) in [12]).

Define $K^{(t)} = K \cdot \chi_{[0,t]}$. From (3.6) and (3.8) of [10], we obtain

$$\begin{aligned}\rho(t) &= \mathbb{E}\left[\eta(T) \mid \mathcal{F}_t^{(H)}\right] \\ &= \exp\left\{\int_0^t \zeta(s)dB_H(s) - \frac{1}{2}|\zeta|_\phi^2\right\},\end{aligned}$$

where ζ is determined by the following equation

$$\begin{aligned}(-\Delta)^{-(H-1/2)}\zeta(s) = -(-\Delta)^{-(H-1/2)}K^{(T)}(s) \quad 0 \le s \le t \\ \zeta(s) = 0 \quad s < 0 \quad \text{or} \quad s > t\end{aligned} \tag{3.20}$$

By (6.2) of [10], we have

$$\zeta(s) = -\kappa_H s^{1/2-H}\frac{d}{ds}\int_s^t dw w^{2H-1}(w-s)^{1/2-H}\frac{d}{dw}\int_0^w dz z^{1/2-H}(w-z)^{1/2-H}g(z), \tag{3.21}$$

where $g(z) = -(-\Delta)^{-(H-1/2)}K^{(T)}(z)$ and

$$\kappa_H = \frac{2^{2H-2}\sqrt{\pi}\Gamma(-1/2)}{\Gamma(1-H)\Gamma^2(3/2-H)\cos(\pi(H-1/2))}.$$

Hence

$$\begin{aligned}\mathbb{E}\rho(t)^{\frac{\gamma}{1-\gamma}} &= \mathbb{E}e^{-\frac{\gamma}{1-\gamma}}\int_0^t \zeta(s)dB_H(s) - \frac{\gamma}{2(1-\gamma)}|\zeta|_\phi^2 \\ &= e^{\frac{2\gamma^2-\gamma}{2(1-\gamma)^2}}.\end{aligned} \tag{3.22}$$

In the special case $t = T$ we see that $\zeta = K^{(T)} = K \cdot \chi_{[0,T]}$, where

$$\int_0^T K(s)\varphi(s,t)ds = \frac{a-r}{\sigma} \text{ for } 0 \le t \le T. \tag{3.23}$$

Thus

$$
\begin{aligned}
|K^{(T)}|_\varphi^2 = |K|_\varphi^2 &= \frac{a-r}{\sigma}\int_0^T K(t)dt \\
&= \frac{(a-r)^2}{2\sigma^2 H\cdot\Gamma(2H)\cdot\Gamma(2-2H)\cos(\pi(H-\frac{1}{2}))}\int_0^T (Tt-t^2)^{\frac{1}{2}-H}dt \\
&= \frac{(a-r)^2}{\sigma^2}\Lambda_H\cdot T^{2-2H},
\end{aligned}
\tag{3.24}
$$

where

$$
\Lambda_H = \frac{\Gamma^2(\frac{3}{2}-H)}{2H\cdot(2-2H)\cdot\Gamma(2H)\cdot\Gamma(2-2H)\cos(\pi(H-\frac{1}{2}))}. \tag{3.25}
$$

Substituting (3.22) and (3.24) into (3.15), we get

$$
\begin{aligned}
N = &\frac{1}{D_1}\int_0^T \exp\left(\frac{r\gamma}{1-\gamma}t + \frac{2\gamma^2-\gamma}{2(1-\gamma)^2}|h|_\varphi^2\right)dt \\
&+\frac{1}{D_2}\exp\left(\frac{r\gamma}{1-\gamma}T + \frac{\gamma(a-r)^2\Lambda_H}{2(1-\gamma)^2\sigma^2}T^{2-2H}\right)
\end{aligned}
\tag{3.26}
$$

and (3.19) gives

$$
\begin{aligned}
V(z) = &\frac{z^\gamma}{\gamma}\left\{D_1^{1-\gamma}N^{-\gamma}\int_0^T \exp\left(\frac{r\gamma}{1-\gamma}t + \frac{2\gamma^2-\gamma}{2(1-\gamma)^2}|h|_\varphi^2\right)dt\right. \\
&\left.+D_2^{1-\gamma}N^{-\gamma}\exp\left(\frac{r\gamma}{1-\gamma}T + \frac{\gamma(a-r)^2\Lambda_H}{2(1-\gamma)^2\sigma^2}T^{2-2H}\right)\right\}.
\end{aligned}
\tag{3.27}
$$

We have proved :

Theorem 3.1 *The value function $V(z)$ of the optimal consumption and portfolio problem* (3.1) *is given by* (3.26)-(3.27). *The corresponding optimal consumption c_{λ^*} is given by* (3.17) *and the corresponding optimal terminal wealth $Z_z^{c^*_\lambda,\pi^*} = F_{\lambda^*}$ is given by* (3.18).

Remark. It is an interesting question how the value function $V(z) = V^{(H)}(z)$ of problem (3.1) depends on the Hurst parameter $H \in (\frac{1}{2}, 1)$. We will not pursue this question here, but simply note that since $\Lambda_{1/2} = 1$ we have

$$\lim_{H \to \frac{1}{2}+} V^{(H)}(z) = V^{(\frac{1}{2})}(z) \tag{3.28}$$

where $V^{(\frac{1}{2})}(z)$ is the (well known) value function in the standard Brownian motion case.

It remains to find the optimal portfolio $\theta^* = (\alpha^*, \beta^*)$ for problem (3.1). For this we use the fractional CHO theorem (Theorem 1.4) with

$$G(\omega) = e^{-rt} F_{\lambda^*}(\omega) + \int_0^T e^{-ru} c_{\lambda^*}(u, \omega) du$$

as in (2.14). Then by (2.18)

$$\beta^*(t) = \sigma e^{rt} S^{-1}(t) \widetilde{\mathbb{E}}_{\hat{\mu}_H} [\hat{D}_t G \mid \mathcal{F}_t^{(H)}]. \tag{3.29}$$

To compute this we first note that by (3.8), (1.17) and (2.10) we have

$$\begin{aligned}
\rho(t)^{\frac{1}{\gamma-1}} &= \exp\left\{ \frac{1}{1-\gamma} \int_0^t K(s) dB_H(s) + \frac{1}{2(1-\gamma)} |K^{(t)}|_\varphi^2 \right\} \\
&= \exp\left\{ \frac{1}{1-\gamma} \int_0^t K(s) d\hat{B}_H(s) - \frac{a-r}{\sigma(1-\gamma)} \int_0^t K(s) ds + \frac{1}{2(1-\gamma)} |K^{(t)}|_\varphi^2 \right\} \\
&= \exp\left\{ \frac{1}{1-\gamma} \int_0^t K(s) d\hat{B}_H(s) - \frac{1}{2(1-\gamma)^2} |K^{(t)}|_\varphi^2 + \frac{1}{2(1-\gamma)^2} |K^{(t)}|_\varphi^2 \right. \\
&\qquad \left. - \frac{a-r}{\sigma(1-\gamma)} \int_0^t K(s) ds + \frac{1}{2(1-\gamma)} |K^{(t)}|_\varphi^2 \right\} \\
&= \exp^\diamond \left\{ \frac{1}{1-\gamma} \int_0^t K(s) d\hat{B}_H(s) \right\} \cdot R(t),
\end{aligned} \tag{3.30}$$

where

$$R(t) = \exp\left\{ \frac{2-\gamma}{2(1-\gamma)^2} |K^{(t)}|_\varphi^2 - \frac{a-r}{\sigma(1-\gamma)} \int_0^t K(s) ds \right\}. \tag{3.31}$$

Hence, by (3.18)

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_{\hat{\mu}_H}\left[\widehat{D}_t\left(e^{-rT}F_{\lambda^*}\right) \mid \mathcal{F}_t^{(H)}\right] \\
&\quad= \frac{z}{D_2N}e^{-rT}e^{\frac{rT}{1-\gamma}}\widetilde{\mathbb{E}}_{\hat{\mu}_H}\left[\widehat{D}_t\left(\eta(T)^{\frac{1}{\gamma-1}}\right) \mid \mathcal{F}_t^{(H)}\right] \\
&\quad= \frac{z}{D_2N}\exp\left(\frac{r\gamma T}{1-\gamma}\right)\widetilde{\mathbb{E}}_{\hat{\mu}_H}\left[\frac{K(t)}{1-\gamma}\eta(T)^{\frac{1}{\gamma-1}} \mid \mathcal{F}_t^{(H)}\right] \\
&\quad= \frac{z}{D_2N}\exp\left(\frac{r\gamma T}{1-\gamma}\right)\frac{K(t)}{1-\gamma}R(T)\widetilde{\mathbb{E}}_{\hat{\mu}_H}\left[\exp^{\diamond}\left\{\frac{1}{1-\gamma}\int_0^T K(s)d\widehat{B}_H(s)\right\} \mid \mathcal{F}_t^{(H)}\right] \\
&\quad= \frac{z}{D_2N}\exp\left(\frac{r\gamma T}{1-\gamma}\right)\frac{K(t)}{1-\gamma}R(T)\exp^{\diamond}\left\{\frac{1}{1-\gamma}\int_0^t K(s)d\widehat{B}_H(s)\right\} \\
&\quad= \frac{z}{D_2N}\exp\left(\frac{r\gamma T}{1-\gamma}\right)R(T)\cdot\frac{K(t)}{1-\gamma}\cdot\exp\left\{\frac{1}{1-\gamma}\int_0^t K(s)d\widehat{B}_H(s) - \frac{1}{1-\gamma}|K^{(t)}|_\varphi^2\right\} \\
&\quad= \frac{zK(t)}{D_2N(1-\gamma)}\exp\left\{\frac{r\gamma T}{1-\gamma} + \frac{1}{1-\gamma}\int_0^t K(s)dB_H(s) - \frac{a-r}{\sigma(1-\gamma)}\int_t^T K(s)ds\right. \\
&\qquad\left. + \frac{2-\gamma}{2(1-\gamma)^2}|K^{(T)}|_\varphi^2 - \frac{1}{1-\gamma}|K^{(t)}|_\varphi^2\right\}.
\end{aligned}
\tag{3.32}
$$

Similarly, by (3.17) and (3.32),

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_{\hat{\mu}_H}\left[\widehat{D}_t\left(\int_0^T e^{-ru}c_{\lambda^*}(u)du\right) \mid \mathcal{F}_t^{(H)}\right] \\
&= \frac{z}{D_1 N}\widetilde{\mathbb{E}}_{\hat{\mu}_H}\left[\int_0^T \widehat{D}_t\left(e^{-ru}e^{\frac{ru}{1-\gamma}}\rho(u)^{\frac{1}{\gamma-1}}\right) du \mid \mathcal{F}_t^{(H)}\right] \\
&= \frac{z}{D_1 N}\int_0^T \exp\left(\frac{r\gamma u}{1-\gamma}\right)\widetilde{\mathbb{E}}_{\hat{\mu}_H}\left[\widehat{D}_t\left(\rho(u)^{\frac{1}{\gamma-1}}\right) \mid \mathcal{F}_t^{(H)}\right] du \\
&= \frac{zh(t)}{D_1 N(1-\gamma)}\int_0^T \exp\left(\frac{r\gamma u}{1-\gamma}\right)\exp\left\{\frac{1}{1-\gamma}\int_0^t h(s)dB_H(s) - \frac{a-r}{\sigma(1-\gamma)}\int_t^u h(s)ds\right. \\
&\quad \left. + \frac{2-\gamma}{2(1-\gamma)^2}|h|_\varphi^2 - \frac{1}{1-\gamma}|h|_\varphi^2\right\} du \\
&= \frac{zh(t)}{D_1 N(1-\gamma)}\exp\left\{\frac{1}{1-\gamma}\int_0^t K(s)dB_H(s) - \frac{1}{1-\gamma}|K^{(t)}|_\varphi^2\right\} \\
&\quad \cdot \int_0^T \exp\left\{\frac{r\gamma u}{1-\gamma} + \frac{2-\gamma}{2(1-\gamma)^2}|K^{(u)}|_\varphi^2 - \frac{a-r}{\sigma(1-\gamma)}\int_t^{u\wedge t} K(s)ds\right\} du.
\end{aligned}
\tag{3.33}
$$

Adding (3.32) and (3.33) and using (3.29) we get

Theorem 3.2 *The optimal portfolio $\theta^*(t) = (\alpha^*(t), \beta^*(t))$ for problem (3.1) is given by*

$$\beta^*(t) = \sigma e^{rt} S^{-1}(t)(Y_1 + Y_2),$$

where

$$Y_1 = \widetilde{\mathbb{E}}_{\hat{\mu}_H}\left[\widehat{D}_t\left(e^{-rT}F_{\lambda^*}\right) \mid \mathcal{F}_t^{(H)}\right]$$

is given by (3.32) and

$$Y_2 = \widetilde{\mathbb{E}}_{\hat{\mu}_H}\left[\widehat{D}_t\left(\int_0^T e^{-ru}c_{\lambda^*}(u)du\right) \mid \mathcal{F}_t^{(H)}\right]$$

is given by (3.33), and

$$\alpha^*(t) = e^{-rt}\left[Z^*(t) - \beta^*(t)S(t)\right],$$

with

$$e^{-rt}Z^*(t) + \int_0^t e^{-ru}c_{\lambda^*}(u)du = z + \int_0^t \sigma^{-1}e^{-ru}\beta^*(u)S(u)d\widehat{B}_H(u)$$

and $c_{\lambda^*}(u)$ given by (3.17).

Acknowledgments. We are grateful to Yuri Kabanov for helpful comments. This work was partially supported by the French-Norwegian cooperation project Stochastic Control and Applications, Aur 99-050 and the National Science Foundation under Grant No. EPS-9874732, matching support from the State of Kansas and General Research Fund of the University of Kansas.

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