Translational Inertial Dragging

by

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Abstract

An important question in the theory of general relativity is to decide whether this theory contains the general principle of relativity for accelerated motion (including rotation). This question is also related to Mach's principle. The aim of this theisis is to throw some new light on this question.

In this connection I have deduced a solution of the full Einstein equations, describing a physical system producing perfect inertial dragging. This system is a singular shell with the metric of an accelerated black hole outside the shell and (comoving) flat spacetime inside the shell.

The effect of inertial dragging is described and previous work on translational inertial dragging is reviewed. The Israel formalism is presented in full mathematical detail and applied to a few special cases before applying it to the accelerated shell.

Acknowledgements

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Contents

Chapter 1

Introduction

The general theory of relativity is a theory not only about gravity, but also about space and time itself. Allthough the differences between this theory and Newtons theory of gravity is extremely small on experiments carried out on the Earth, the physical and philosofical consequences of the two theories couldn't be further apart. This fundamental difference, the difference between *relativism* and *absolutism*, challenges our intuition and changes how we think of space, time, and even the whole Universe.

This thesis is devoted to one of these fundamental differences, namely the effect of *inertial dragging*: That massive, accelerating bodies induce an acceleration on nearby objects. The rotational case has been widely studied, while the translational case has received far less attention. To maintain translational acceleration requires an energy source, while to maintaining rotation does not. This makes the latter case much easier to analyze.

One other rather philosofical aspect of the general theory of relativity concerns the origin of inertia. In general relativity Newton's law of inertia is replaced by the law of geodesic motion. The geodesic motion is determined from the metric tensor $g_{\mu\nu}$, which is determined from the energy-momentum tensor (energy and momentum content of space) and the choice of coordinates. In this way general relativity opens up the possibility that only relative acceleration matters.

In chapter 2 these philosofical aspects are discussed with some examples. In chapter 3 inertial dragging is explained (with examples) and some of the previous work on inertial dragging is reviewed.

In chapter 4 the Israel formalism is presented. In the sections involving this formalism a lot of the details in the beginning of the calculations are kept in order to provide a good and pedagogical introduction to this topic. The reason for this is that the formalism can only be found in a few textbooks on general relativity. Allthough these books contain a few examples, I found them a bit short on the details. (Most of the applications of the formalism are found in scientific papers involving a minimum of details.) In these sections only the calculation of the Christoffel symbols is moved to the appendix. I found that including this level of detail clarified the mathematical theory behind it as well as making it more readable.

In chapter 5 the C-metric is introduced in some of the most common representations, before transforming it to a special set of spherical coordinates. In these spherical coordinates the limits of this metric are demonstrated and the interpretation as an accelerating black hole (or two of them) become apparant.

My main contributions can be found in chapter 6 and 7: In chapter 6 the Israel formalism is applied to a spherical accelerated shell, and the shell is given a physical interpretation as a perfect fluid. In chapter 7 I discuss the properties of the shell further, and discuss the inertial dragging effect inside the shell.

Allthough the result of the examples in chapter 4 are not new, the calculations are entirely my own. The comments and discussions throughout the thesis are my own unless otherwise explicitly stated.

About notation:
In most of this thesis (the exeptions being during review of other people's work), standard relativistic units of $c = G = 1$ is used, and he signature of the metric is $(-1, 1, 1, 1)$. Greek indices run from 0 to 3, latin indices from 0 to 2. Einstein's summation convention is implied. The partial derivative of a function f with respect to x^α is denoted $f_{,\alpha}$, the covariant derivative is $f_{;\alpha}$ and total derivative is $\dot{f} = \frac{d}{d\tau}(f)$.

Chapter 2

Mach's principle and the principle of relativity

It is an observational fact that when one places a Foucault pendulum on the North pole, it's plane of rotation will rotate relative to the Earth with one revolution per 24 hours, following the rotation of the distant stars and galaxies on the night sky. This observational fact is sometimes called *"Mach Zero"* [1]. It is an experimental fact and not a principle. Mach's principle is the principle which tries to *explain* this fact from a fundamental point of view.

2.1 Mach's Principle - Historical backround

Excellent accounts for the history surrounding Mach's principle can be found in [2], and a shorter version in [3]. I will only here give a short summary of the different views of Newton, Mach and Einstein.

Newton's absolute space and time:
Newton's world was build upon the idea that space is composed of a static, unchangable, rigid invisible lattice that determines the position of everything in space. Similarly his idea of time was that it is universal, and time flowes equally fast everywhere in space, and does not depend on either position or the velocity of the observer measuring time.

An inertial frame is a frame where Newton's 1st law applies. Necessarily, an inertial frame is a frame where no inertial forces arise, and according to Newton that frame has to be non-rotating and moving with a constant velocity with respect to the rigid lattice of space.

Newton demonstrated his idea with the example of a vessel containing water ("Newton's bucket"). If one sets the water in the vessel in circulating motion without rotating the vessel, the water will experience the centrifugal "force" and be pushed towards the sides of the vessel, making the surface of the water convex. If one instead starts to rotate the vessel while the water is still at rest, there would be no deformation of the surface. This, according to Newton, was proof that in the first case the centrifugal force appeared because the water was put in rotational motion with respect to the rigid lattice of space, while in the second case, there was no centrifugal force since the water was at rest with respect to the rigid lattice of space. So the relative motion bewteen the vessel and the water (which was the same in the two cases) played no role at all.

Mach's critisism of Newton's bucket experiment:
Mach criticised Newton's notions of both absolute time and absolute space. In his book on Mechanics from 1883 he writes [2, 3]:

"Newton's experiment with the rotating vessel of water simply informs us that the relative rotation of the water with respect to the sides of the vessel produces no noticeable centrifugal forces, but that such forces are produced by its relative rotation with respect to the mass of the earth and other celestial bodies. No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick."

Mach states here that one cannot know what would happen if the vessel was very massive, and thereby implies that centrifugal forces *might* appear if the vessel was indeed very massive. On could argue that Mach here foresaw the inertial dragging effect of General Relativity.

Einstein's reading of Mach:
Einstein, who was heavily influenced by Mach, was the one who coined the term "Mach's principle" in his 1918 paper [2, p. 186]. In this paper he claims that his Theory of General Relativity rests on three principles (which were *not* independent of each other):

"a) *Relativity principle:* The laws of nature are merely statements about space-time coincidences; they therefore find their only natural expression in generally covariant equations.

b) *Equivalence principle:* Inertia and weight are identical in nature. It follows necessarily from this and from the results of the special theory of relativity that the symmetric 'fundamental tensor' $[g_{\mu\nu}]$ determines the metrical properties of space, the inertial behaviour of bodies in it, as well as gravitational effects. We shall denote the state of space described by the fundamental tensor as the 'G-field'.

c) *Mach's Principle:* The G-field is *completely* determined by the masses of bodies. Since mass and energy are identical in accordance with the results of the special theory of relativity and the energy is desribed formally by means of the symmetric energy tensor $(T_{\mu\nu})$, this means that the G-field is conditioned and determined by the energy tensor of the matter."

However, Einstein had to abandon this (and his other) formulations of Mach's principle since he could not fully implement it in his general theory of relativity.

What Mach truly meant in his discussion of Newton's bucket experiment have been debated for more than 100 years [2, p. 9]. I will in this thesis limit myself to consider only a few formulations of the so-called Mach's Principle that most physicists today will agree on.

2.2 Mach's principle - formulations

We need a modern formulation of the principle, in the "spirit" of Mach and Einstein, but which can be compatible with General Relativity.

Formulation 1: *Local inertial frames are determined by all masses in the universe.*[1]

Some authors require that the inertial mass of an object is determined this way. I will not include this in my definition of Mach's principle, as it is also a topic of controvercy (see discussion in [2, p. 96]).

Formulation 2: *Inertial forces arises due to the relative acceleration between an object and all the masses in the universe.*

I will refer to both these formulations as Mach's principle (MP) throughout this thesis, and I will argue that both are true and contained within GR. There are however a few other selected formulations worth noting:

[1]The term "all the masses in the universe" will allways refer to the *observable* universe, that is, within the current lookback distance.

Formulation 3: *A theory should not contain any absolute structures.*

This formulation is from Ehlers [2, p. 459]. It can be seen as a generalization of Mach's ideas that only relative data (positions and motions) are real and that absolute space and time are merely concepts of the imagination. For example, in special relativity, the metric is given as the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu}$, an absolute structure that decides the course of physical events, but is not itself affected by physical events. Ehlers writes: "A major goal of present-day research, the construction of a quantum theory of spacetime structure and gravitation, may be viewed as the attempt to rid quantum field theory of it's absolute elements." Thus the relevance of Mach's principle for modern research is undisputable.

2.3 The principle of relativity - relation to Mach's principle?

As Carl Hoefer points out, Mach's principle (usually together with general covariance) leads to an extended principle of relativity, encompassing arbitrary motion [2, p. 71]. This is in agreement with Einstein's wish to create a theory were all motion is relative, including rotation and translational acceleration. This is certainly in the "Machian spirit", but it is thus far not obvious that the formulations 1 and/or 2 lead to an extended principle of relativity.

Consider the two formulations of the special principle of relativity [4]:

- S1 All laws of Nature are the same in all inertial systems
- S2 Every inertial observer may consider himself/herself to be at rest

These two statements may be seen as different formulations of the same physical principle, called the special principle of relativity. Consider now the generalizations to arbitrary motion:

- G1 All laws of Nature are the same in all reference systems
- G2 Every observer in arbitrary motion may consider himself/herself to be at rest

While G1 is the generally accepted formulation of the general principle of relativity, it remains a very controversial view that G2 is true. Wether G2 is true or not was investigated in [5], and they concluded that it might be, allthough it was not proven.

Formulation 1 and 2 of Mach's principle are quite common formulations throughout the literature and is in accordance with the classical textbook formulation by Misner, Thorne and Wheeler (allthough the words are different, the content is the same). Formulation 1 only concerns how to determine how inertial frames are found, it does not equate the status of an inertial frame and a reference frame in arbitrary motion. Formulation 2 concerns how inertial forces like the centrifugal force and the Coriolis force should appear due to relative acceleration of a body with respect to the rest of the universe. In this sense formulation 2 says something more than formulation 1, and explains not only that some reference systems are not inertial, but how the inertial forces arise and that it should be calculated from the acceleration relative to the rest of the universe. So putting these two formulations together, we can both determine what frames are to be considered inertial, and what the inertial forces will be in frames that are accelerated relative to the inertial frames. If we can calculate and find the correct result using this prescription, we have a strong argument for the relativity of arbitrary motion, that is, the "strong" or "extended" principle of relativity (G2) is true.

It is extremely important to keep in mind that when one talks about scenarios which "satisfy Mach's principle" or are "Machian" (or oppositely "anti-Machian"), the whole universe has to be taken into account. Mach's principle *cannot* be used on a single subsystem of the universe. However, in some cases, subsystems can produce scenarios which mimic the effect of the entire universe. For instance, it is a fact, that to a good approximation, the "radius" of the observable part of the universe is very close to it's own Schwarzschild radius [6]. A shell of matter, representing a model of the universe, with radius equal to the Schwarzschild radius of the mass inside the lookback distance, is therefore often used as such an idealized subsystem which mimics the effect we expect from the whole universe. This is the starting point for most papers on Mach's principle and inertial dragging and shall also be considered in the following thesis.

The reason that this scenario is studied so much in context with Mach's princple can be summarized in three words: "Perfect inertial dragging".

Chapter 3

Inertial dragging

Inertial dragging (or "frame dragging"') is the effect that inertial systems are dragged along in the direction of acceleration of a nearby accelerating mass. This means, that a free test particle or observer, subject only to the "force" (effect) of a nearby accelerated gravitating body, will, as viewed from a distant observer, accelerate in the same direction as the accelerated body. The accelerating test body or observer, will not feel anything. The testbody/observer is in free fall. If the freely falling observer carries an accelerometer, it will measure zero acceleration. This is because this observer is at rest relative to the spacetime surrounding him/her locally. Like a boat that flows with the same velocity as the river, the observer falls freely with the same acceleration as that of the "river of space" [7].

3.1 Rotational inertial dragging

The case of rotational inertial dragging is the most studied case [3, 8–12]. The reason for this is that one does not need a continuous energy supply in order to keep a system rotating, in contrast to the case for accelerated translational motion. This makes the situation easier to analyze.

3.1.1 Outside a rotating sphere - the Kerr spacetime

The spacetime outside a rotating, spherically symmetric mass distribution is described by the Kerr metric:

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar}{\rho^2}\sin^2\theta dt d\phi + \frac{\rho^2}{\Delta} dr^2 \\ &+ \left(r^2 + a^2 + \frac{2Ma^2r}{\rho^2}\sin^2\theta\right)\sin^2\theta d\phi^2 + \rho^2 d\theta^2 \,, \end{aligned} \tag{3.1.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta\,,$$
$$\Delta = r^2 + a^2 - 2Mr\,, \qquad (3.1.2)$$

and M and a have dimension length and correspond to the mass of the object and angular momentum per unit mass, respectively, and (r, θ, ϕ) are spherical coordinates.

Consider a free zero angular momentum observer ("ZAMO") with Lagrangian:

$$L = -\frac{1}{2}g_{\mu\nu}d\dot{x}^\mu d\dot{x}^\nu \qquad (3.1.3)$$
$$= \frac{1}{2}\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \frac{1}{2}\frac{r^2}{\Delta}\dot{r}^2$$
$$- \frac{1}{2}\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi}^2 + \frac{2Ma}{r}\dot{t}\dot{\phi}\,, \qquad (3.1.4)$$

where we restrict the motion to the equatorial plane, $\theta = \pi/2$ and $\dot{\theta} = 0$. Since ϕ is absent in (3.1.4), ϕ is a cyclic coordinate. The corresponding conserved momentum is

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = -\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi} + \frac{2Ma}{r}\dot{t}\,. \qquad (3.1.5)$$

A freely falling ZAMO starts from infinity with $p_\phi = 0$. Since p_ϕ is conserved during the motion we have at all times during the motion:

$$\dot{\phi} = \frac{2Ma}{(r^3 + a^2 r + 2Ma^2)}\dot{t}\,. \qquad (3.1.6)$$

We also have

$$\frac{\dot{\phi}}{\dot{t}} = \frac{d\phi}{dt} \equiv \Omega\,, \qquad (3.1.7)$$

which gives

$$\Omega = \frac{2Ma}{(r^3 + a^2 r + 2Ma^2)}\,, \qquad (3.1.8)$$

which is the angular speed obtained by the ZAMO during the free fall. This is the angular velocity of the inertial frames. The limit is:

$$\lim_{r\to\infty}\Omega = 0\,. \qquad (3.1.9)$$

So infinitely far away from the source the spacetime coincides with the Minkowski spacetime. It is common to require this at infinity. This condition is called asymptotic Minkowski spacetime/metric.

Some specific values for the angular velocity around a rotating black hole are:

$$\Omega(r = R_0) = \frac{\frac{1}{2}a}{2M^2 + a^2} , \tag{3.1.10}$$

$$\Omega(r = r_+) = \frac{a}{2Mr_+} . \tag{3.1.11}$$

Where $r_+ = M + \sqrt{M^2 - a^2}$ is the horizon radius, and $R_0 = 2M$ is the radius of the stationary limit/static border for $\theta = \pi/2$.

3.1.2 Inside a rotating shell

A lot of work have been done on the dragging induced *inside* rotating mass shells. The earliest work was by Einstein and Thirring [3, 8]. Thirring derived in 1918:

$$\Omega = \frac{4M}{3R}\omega , \tag{3.1.12}$$

where Ω is the induced angular velocity of the inertial frames inside the sphere which rotates with angular velocity ω. This result is to first order in M/R (weak field) and ω.

In 1966 Brill and Cohen [9] extended this result to arbitrary order of M/R (while only in first order of ω) and found that in the collapse limit (the radius of the shell equal to its own Schwarzschild radius) the dragging coefficient approaches unity:

$$d = \frac{\Omega}{\omega} \to 1 . \tag{3.1.13}$$

This phenomenon is known as "perfect inertial dragging", that is, the inertial frames inside the shell are perfectly dragged along with the shell so that for an inertial observer inside the shell, the shell will appears to be non-rotating. An alternative deduction of this result will be shown in section 4.4.2. The authors write:

"A shell of matter of radius equal to its Schwartzschild radius has often been taken as an idealized cosmological model of our universe. Our result shows that in such a model there cannot be a rotation of the local inertial frame in the center relative to the large masses in the universe. In that sense

the result is consistent with Mach's principle."

This result was extended by Pfister and Braun to order ω^2 and ω^3 in [10, 11] respectively. They showed that to generate correct fictitious forces inside the shell, the shell will in general deviate from a spherical form, have a latitude-dependent mass density (order ω^2 corrections), and exhibit differential rotation (with faster rotation rate along the equatorial plane, an order ω^3 correction). However, these corrections did not invalidate the perfect dragging result when $R \to 2M$, in fact, in this limit the rotation becomes rigid again, and spherical shape and uniform mass density is regained.

3.1.3 A measure of absolute rotation?

In the special theory of relativity there is a class of inertial frames (which are related to each other by Lorentz transformations) which are given as "absolute elements". Rotation is not relative in this theory. This is demonstrated by experiments with the Foucault pendulum and the Sagnac effect, which can be used to argue *against* the relativity of rotation. I will demonstrate how we can re-establish the relativity of rotation in the general theory of relativity if the criteria for perfect inertial dragging is met.

Consider a circular disk of radius R which rotates with angular velocity ω. A photon emitter/receiver is fixed on the rotating disk at the circumference. It emits a photon in both directions through an optical fibre cable along the circumference. As seen from an inertial frame IF the emitter/receiver travels *with* one photon and *against* the other. Therefore the oppositely travelling photon is received first and the other one last. The difference in travel time is

$$\Delta t = t_2 - t_1 = \frac{2\pi r}{c - r\omega} - \frac{2\pi r}{c + r\omega} = \frac{4\pi r^2 \omega}{c^2 - r^2\omega^2} \, . \tag{3.1.14}$$

In the IF we have Minkowski space-time with the line element (in cylindrical coordinates):

$$ds^2 = -c^2 dT^2 + dR^2 + R^2 d\Theta^2 + dZ^2 \, . \tag{3.1.15}$$

The rotating reference frame RF rotates relative to this inertial frame, with constant angular velocity ω. The transformation is:

$$t = T \, , \quad r = R \, , \quad \theta = \Theta - \omega T \, , \quad z = Z \, . \tag{3.1.16}$$

Inserting this in the line-element gives:

$$ds^2 = -\left(1 - \frac{r^2\omega^2}{c^2}\right) c^2 dt^2 + dr^2 + r^2 d\theta^2 + 2r^2\omega d\theta dt + dz^2 \, . \tag{3.1.17}$$

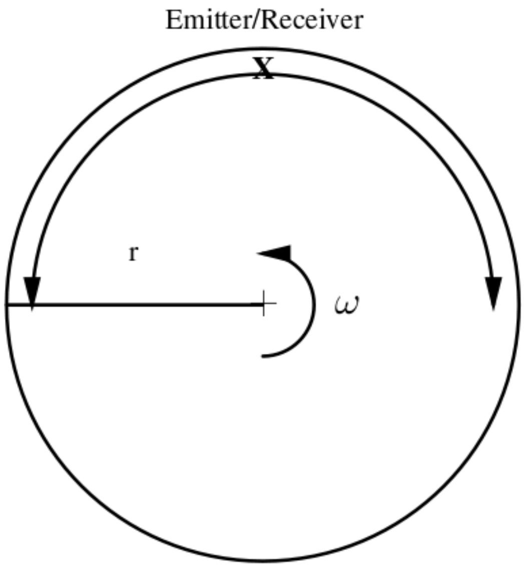

Figure 3.1: An emitter and a reciever is fixed on the rotatng disk and will travel with the disk while the signal goes around it. Picture from [4].

Light follows null-geodesic curves, so in RF this gives:

$$\begin{aligned} ds^2 &= -\left(1-\frac{r^2\omega^2}{c^2}\right)c^2dt^2 + r^2d\theta^2 + 2r^2\omega d\theta dt \\ &= 0\,. \end{aligned} \tag{3.1.18}$$

Solving this for c:

$$c = \pm\left(r\frac{d\theta}{dt} + r\omega\right)\,. \tag{3.1.19}$$

The coordinate velocity of light is $v_\pm = rd\theta/dt$ giving

$$v_\pm = -r\omega \pm c\,. \tag{3.1.20}$$

Showing that the *coordinate* velocity of light is *not* isotropic. The difference in travel time of the two photons are:

$$\Delta t = t_2 - t_1 = \frac{2\pi r}{c - r\omega} - \frac{-2\pi r}{-c - r\omega} = \frac{4\pi r^2\omega}{c^2 - r^2\omega^2}\,. \tag{3.1.21}$$

In accordance with (3.1.14). Hence the interference (be it positive or destructive) represents an invariant event.

General relativistic explanation

In special relativity this effect is seen as a measure of absolute rotation, where zero Sagnac effect indicates the non-rotating absolute frame, and all inertial frames are then defined as those reference frames which are related to this frame by a Lorentz transformation.

The observational fact "Mach 0" is then the fact that the frames of zero Sagnac effect do not rotate relative to the distant stars. If the stars would start to rotate, *imperfect* inertial dragging could be detected as the stars would only partially drag the inertial frames around (dragging factor $d < 1$), causing non-zero Sagnac effect in the reference frame where the stars are at rest. That is, the inertial frames would rotate relative to the distant stars!

However, if we have *perfect* inertial dragging, the overall rotation of the universe is unobservable. Everything within it is perfecty dragged along, and there is nothing outside that can be used as points of reference.

Only perfect inertial dragging can thus *explain* "Mach 0". So for Mach's principle to be valid, we must have that our universe fulfills the requirement for perfect inertial dragging.

Mach's principle and perfect inertial dragging can also explain the non-zero Sagnac effect: When we think about the universe as rotating (important point: Allthough overall rotation of the universe isn't physically observable, we *may* regard it as rotating and the disk as at rest if we accept the extended principle of relativity!) the inertial frames will be perfectly dragged along and a (real) force is needed to keep the disk "at rest". Since the speed of light is constant only in local inertial frames, the light signal travelling *with* the inertial frames will travel *faster* than the light signal directed *opposite* of the inertial frames, and reach the reciever first.

3.2 Translational inertial dragging

In a preliminary gravitational theory Einstein in 1912 calculated that inside a massive accelerated shell the inertial frames would be dragged along in the same direction with magnitude $a_{\text{inertial}} = \frac{3M}{2R} a_{\text{shell}}$ [3]. This is the first attempt at calculating a translational inertial dragging effect, and introduced the effective model of a spherical mass shell. Later the same effect has been considered by some authors [13–18]. The results do not all agree, as we shall see in this article, however the main point is the same: When we formulate the problem in a self-consistent way, there will indeed be an induced acceleration in the direction of the shell acceleration.

3.2.1 The weak field approximation

When space-time curvature is small, the gravitational field is "weak", and the metric is close to the Minkowski metric. We write the metric as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \; , \qquad |h_{\mu\nu}| \ll 1 \; . \tag{3.2.1}$$

where $h_{\mu\nu}$ are the deviations from the Minkowski values $\eta_{\mu\nu}$. The smallness of the deviations allow us to neglect their squares, their products, their product with a derivative and products of their derivatives. Using only this, one can rewrite Einsteins field equations to:

$$\Box h_{\mu\nu} = -2\kappa \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right) \; . \tag{3.2.2}$$

The solutions of this equation is the retarded potentials:

$$h_{\mu\nu} = \frac{\kappa}{2\pi} \int \frac{\left[T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right] (t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \; . \tag{3.2.3}$$

If we in addition require small velocities for the sources and of the particle(s) to be considered, the the geodesic equation can be written according to Davidson [13] (NB: Davidson uses the metric signature $(+,-,-,-)$.):

$$\frac{d\mathbf{v}}{dt} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \; , \tag{3.2.4}$$

where the quantities Φ and $\mathbf{A}$ are defined in terms of the *whole* $g_{\mu\nu}$:

$$(\Phi, \mathbf{A}) = \left(\frac{1}{2} g_{00}, -g_{0i} \right) \; . \tag{3.2.5}$$

Equation (3.2.4) shows some very "Machian" traits, and states that the local inertial frames are accelerated by:

(i) The gravitational attraction due to the presence of a massive body, indicated by the $-\nabla\Phi$ term.

(ii) The acceleration of massive bodies, indicated by the $-\frac{\partial \mathbf{A}}{\partial t}$ term.

(iii) The motion of the test particle relative to a rotating mass, as indicated by the term $\mathbf{v} \times (\nabla \times \mathbf{A})$.

Davidson considers the reference frame where a particle is permanently at rest, then equation (3.2.4) reduces to:

$$-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} = 0\,. \tag{3.2.6}$$

Consider now the spacetime close to a massive object with mass M which is at rest relative to the rest of the ("smoothed-out") Universe. The contribution from the rest of the Universe lies in the Minkowski values of the metric, while the contribution from the mass M represents the deviation from pure Minkowski spacetime. The metric is then:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)(dx^2 + dy^2 + dz^2)\,. \tag{3.2.7}$$

From (3.2.5) we see that in this frame the potentials are:

$$\begin{aligned} \mathbf{A} &= 0\,, \\ \Phi &= \frac{1}{2}\left(1 - \frac{2GM}{r}\right)\,. \end{aligned} \tag{3.2.8}$$

Inserted in (3.2.4), this gives the equation of motion:

$$\frac{d\mathbf{v}}{dt} = -\nabla\Phi\,, \tag{3.2.9}$$

Which is Newton's law of gravitation. In the rest frame of the falling particle (3.2.6) is valid. If we make a coordinate transformation from the previous reference frame to this one, one obtains:

$$\begin{aligned} \mathbf{A} &= (-v, 0, 0)\,, \\ \Phi &= \frac{1}{2}\left(1 - v^2 - \frac{2GM}{r}\right)\,. \end{aligned} \tag{3.2.10}$$

Inserted in (3.2.6) this gives:

$$-\frac{GM}{r^2} + \frac{dv}{dt} = 0\,, \tag{3.2.11}$$

which is also Newton's law. (The equation is scalar since it is the only non-zero component of the vector equation (3.2.6).) *Thus we see that the situation can be equally well described in both reference frames, allthough they are accelerated relative to one another.*

Since only the derivatives of Φ and $\mathbf{A}$ are included in (3.2.4), we can rewrite it in terms of the deviations $\mathbf{h} \equiv \mathbf{e}_i h_{0i}$ (sum over i). Also changing the metric signature to $(-+++)$ gives Grøn and Eriksens weak field geodesic equation [14]:

$$\mathbf{a} = \frac{1}{2}c^2 \nabla h_{00} - \frac{\partial \mathbf{h}}{\partial t} \,, \tag{3.2.12}$$

when considering translational motion. Grøn and Eriksen [14] considers a shell with *observed* acceleration (without fixing the source of acceleration) and considers the translational inertial dragging inside this shell. The shell consists of dust particles and the non-zero components of the energy-momentum tensor are given by:

$$T_{00} = \rho \,, \;\; T_{01} = -\rho v \,. \tag{3.2.13}$$

The resulting metric functions are:

$$\begin{aligned} h_{00} &= \frac{2GM}{Rc^2}\left(1 - \frac{gx}{3c^2}\right) \,, \\ h_{01} &= -\frac{4GM}{Rc^2}\left(v - \frac{Rg}{c}\right) \,. \end{aligned} \tag{3.2.14}$$

With corresponding line-element:

$$\begin{aligned} ds^2 = &- \left[1 - \frac{2GM}{Rc^2}\left(1 - \frac{gx}{3c^2}\right)\right] c^2 dt^2 - \frac{8GM}{Rc^2}\left(v - \frac{Rg}{c}\right) dt\, dx \\ &+ \left[1 + \frac{2GM}{Rc^2}\left(1 - \frac{gx}{3c^2}\right)\right] (dx^2 + dy^2 + dz^2) \,. \end{aligned} \tag{3.2.15}$$

One can calculate the observed acceleration from the geodesic equation (3.2.12) on a particle starting from rest, the x-component of the equation becomes:

$$\begin{aligned} a_x =& \frac{1}{2}c^2 \frac{\partial}{\partial x}\left[\frac{2GM}{Rc^2}\left(1 - \frac{gx}{3c^2}\right)\right] \\ &- \frac{\partial}{\partial t}\left[-\frac{4GM}{Rc^2} g \left(t - \frac{R}{c}\right)\right] \end{aligned} \tag{3.2.16}$$

$$= \frac{GM}{Rc^2} g \left(-\frac{1}{3} + 4\right) \tag{3.2.17}$$

$$= \frac{11}{3}\frac{GMg}{Rc^2} = \frac{11}{6}\frac{R_S}{R} g \,. \tag{3.2.18}$$

This is Grøn and Eriksen's final result for the dragging.

Men'shikov, Perevalova and Pinzul [15] considers the same system as Grøn and Eriksen in section 2 of their paper. Their conclusion is *opposite* to that of Grøn and Eriksen, namely *anti-dragging*, the testbody inside the shell is dragged in the opposite direction as that of the shell! However, they make a fatal error in their calculation. If we compare their result with that of Grøn and Eriksen, we see that by including only the T^{00}-component of the energy-momentum tensor they only get the first term in equation (3.2.17). A physical interpretation of this is that they have omitted alltogether the "gravitomagnetic" effect, which lies in the offdiagonal elements of $T^{\mu\nu}$, and as Grøn and Eriksens results show, these terms are the main contributers to the dragging.

It is interesting to note that they arrive at their final result, which is

$$\mathbf{a} = \frac{13}{6}\frac{R_S}{R}\mathbf{g}\,, \tag{3.2.19}$$

after considering a charged shell accelerating in a homogeneous electric field.

3.2.2 General strong gravitational fields

A few studies have been conducted to the general case which includes strong gravitational fields [16–18].

A charged, insulating spherical mass shell was studied by Lynden-Bell, Bičak and Katz in [16]. Instead of dealing with the difficult time-dependent problem of an accelerated shell, they invoked the equivalence principle and placed the shell in a uniform combined gravitational and electrical field, provided by far away mass M with charge Q, so that the gravitational "force" and the electric force cancels, reducing the problem to a static one. See figure 3.2. Translational inertial dragging is then identified by the relation between the acceleration of a free neutral test particle inside the shell, and one far from the shell (but with same distance to the source) at the point P:

$$|\mathbf{a}_P| > |\mathbf{a}_T|\,. \tag{3.2.20}$$

Their main result, in terms of the acceleration of gravity inside and outside the shell, reads:

$$\frac{g_{\text{in}}}{g_{\text{out}}} = \left(\frac{1 + M/Z}{1 + M/Z + m/b}\right)^2 . \tag{3.2.21}$$

Here, M is the gravitational length of the source (fixed), Z is the distance to the source (fixed), and m and b are the mass and radius of the shell, respectively. The authors attribute this effect to inertial dragging in the rest frame

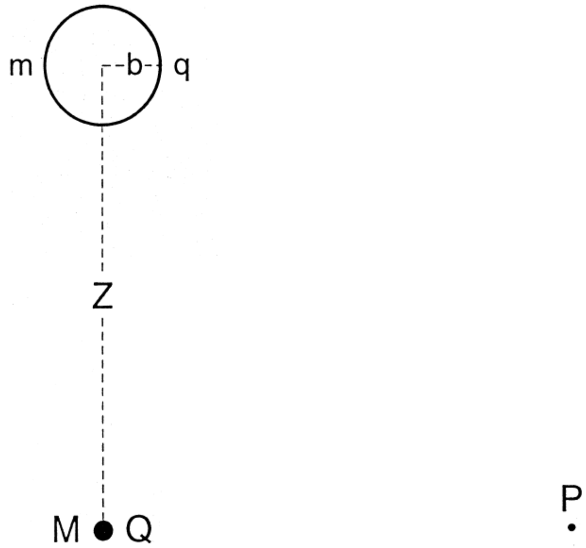

Figure 3.2: Acceleration of neutral test particles are compared inside the shell and at the point P. Translational inertial dragging can be recognized if $|\mathbf{a}_P| > |\mathbf{a}_T|$. Picture from [16].

of the freely falling test particle. In the static frame, where the (coordinate) acceleration of the shell is zero, they attribute the effect to a "diagravitational" effect in analogy with the dielectric effect in electromagnetism. This view is supported by the fact that as they show, both the electric field and gravitational field inside the shell is reduced in the static frame by the presence of the shell.

In 2012 they published a note [17] in which they corrected the fact that they only calculated for a shell with constant coordinate density and which was spherical only in the coordinate space used. When they correct for this the proper sphere of uniform proper density is neither spherical nor uniform in coordinate space. The resulting correction to (3.2.21) is:

$$\frac{g_{\text{in}}}{g_{\text{out}}} = \frac{1 + (5/3)\mu}{(1 + \mu)^3} \,, \tag{3.2.22}$$

where

$$\mu = \frac{m}{b(1 + M/Z)} \,. \tag{3.2.23}$$

The ratio in (3.2.22) is allways less than 1 for $m > 0$. Hence, the dragging effect is allways present. Note that in neither formulas is perfect inertial dragging achieved (coefficient zero), even in the limit where the shell radius is equal to it's own Schwartzschild radius ("collapse limit"). Thus we conclude that in this case the dragging is allways sympathetic but never perfect.

A charged shell accelerated by a dipolar charge distribution ("charge cloud") $\lambda\sigma(r)\sin\theta$ was studied by Pfister, Frauendiener and Hengge [18]. They studied first the weak field case (shell with small mass and charge), then an arbitrarily massive shell with a small charge, and lastly the general strong field case. In all cases the calculations were limited to first order in the dimensionless parameter λ.

In the weak field case, the result is sympathetic dragging, but never perfect dragging. The dragging factor (defined below) varies with the specific form of the charge distribution $\sigma(R)$ but is always proportional to M/R.

In the case of a massive shell with a small charge, they find an explicit formula for the dragging coefficient, defined as:

$$d_{\text{linear}} = \frac{g}{b}\,, \tag{3.2.24}$$

where g and b are the acceleration of a testparticle inside the shell and the shell itself, respectively. See equations (48)-(49) and figure 3 in their paper. In the collapse limit they find that this factor goes to unity (with a horizontal tangent), which means that perfect inertial dragging is realized in this model (for arbitrary charge distributions $\sigma(r)$).

In the general strong field case their set of differential equations was unseparable, so they turn to a numerical procedure to solve the problem. They find that the dragging factor is only exactly 1 in the special case where $R = 2M$ (collapse limit) *and* $q = 0$! Interpreting their contour plot of the dragging factor (figure 4 in their article) the dragging factor approaches unity as both $R \to 2M$ *and* $q^2/R^2 \to 0$.

The acceleration of the shell is b and must be proportional to (atleast) the product of the parameters λ and q. This means that in all cases involved in this study, only first order terms in the acceleration b is kept in the calculations. So in this sence these results are not exact, they only represent the limit where the acceleration of the shell is small.

They also analyze the weak and dominant energy conditions for the shell. Perfect inertial dragging is only realized in the regions where the dominant energy condition is violated (compare their figures 4 and 1).

3.3 Physical significance of the inertial dragging effect

Consider the kinematically equivalent (but causally different) situations:

Case 1: The stars are at rest, and a bucket containing a fluid inside a spaceship is uniformly accelerated due to the force from the engine $\mathbf{F}_e$, giving it a proper acceleration $\boldsymbol{\alpha}$. In Newtonian terms, the fluid now "feels" the inertial force $\mathbf{F}_i$.

Case 2: Consider now the situation where the stars are accelering with acceleration $-\boldsymbol{\alpha}$ (as if by magic). Due to perfect inertial dragging, the spaceship and bucket will be dragged along unless they turn on the engine. When the engine is turned on, they watch the stars accelerate and they feel the inertial force $\mathbf{F}_i$ as a cosmic dragging field. The bucket can be viewed as "at rest", but it is *not* freely falling in the gravitational field.

The point is that these two situations are *kinematically* and *dynamically* identical. The shape of the surface of the fluid is invariant. However, they are causally different, since in Case 1 the stars are actually at rest (in Newtonian terms), while in Case 2 they are actually accelerated by some engine(s) that is "magically" coordinated so that they all appear to start accelerating at the same time as viewed from the spaceship. We should say something about these hypothetical engines. They obviously cannot be ordinary fuel-engines, as then the total momentum of the stars would be conserved (considering the stars plus the exhaust) and we could not speak about the total acceleration of the stars (by which we mean the whole material content of the Universe). However, we could imagine a radiation enginge that propels the stars. The radiation carries momentum but not mass, and as long as the radiation is not absorbed at a later stage the overall material momentum of the Universe whould increase without violating conservation of energy or momentum.

Consider lastly a case similar to Case 2, but which turns out to be flawed:

Case 3: *(Flawed)* The spaceship is at rest, engine off. Suddenly, all the stars look to be accelerating in the same direction at the same time. The crew on the spaceship feels the inertial force $\mathbf{F}_i$ due to the dragging and immediately turns on the engine in order to stay at rest.

If we have perfect inertial dragging the crew would never experience the dragging because they will be dragged along while in free fall, so at no point in time could they find out that it is time to turn on the engine. The dragging field is only felt when an inertial force is present, and it is only present after they turn on the engine.

Considering this discussion I arrive at the following conclusion about the intimate relationship bewteen Mach's principle and the extended principle of relativity:

Mach's principle is the principle that states that the total rotation/acceleration of the Universe is unobservable (due to perfect inertial dragging) while the extended principle of relativity is the principle that allows us to consider the Universe as rotating/accelerating relative to something inside it, and due to the inertial "force" (created by the perfect inertial dragging) a real force is needed to keep the considered body "at rest".

3.4 Do we have perfect inertial dragging in our Universe?

The following calculation follows Grøn in [6]. The distance that light and the effect of gravity has travelled since the Big Bang is called the lookback distance:

$$R_0 = ct_0 \ .$$

The age of the Universe is measured to be:

$$t_0 = \frac{0.996}{H_0} \approx \frac{1}{H_0} \ ,$$

where H_0 is the value of the Hubble constant measured today. The Universe is also measured to be flat, so it has critical density:

$$\rho_{\rm cr} = \frac{3H_0^2}{8\pi G} \ , \tag{3.4.1}$$

which gives:

$$\frac{8\pi G\rho_{\rm cr}}{3c^2} = \left(\frac{H_0}{c}\right)^2 \approx \frac{1}{R_0^2} \ . \tag{3.4.2}$$

The cosmic mass inside the lookback distance has a Schwarzschild radius:

$$R_S = \frac{2GM}{c^2} = \left(\frac{8\pi G\rho_{\rm cr}}{3c^2}\right) R_0^3 \approx R_0 \ . \tag{3.4.3}$$

Since the cosmic mass within the lookback distance has a Schwarzschild radius approximately equal to the lookback distance, we may have perfect inertial dragging in our Universe.

Chapter 4

Israel's formalism

The framework for dealing with hypersurfaces and surface layers in general relativity in a convenient way was developed by W. Israel [19] (among others) and we call this framework the Israel formalism. I will combine the notation of Grøn and Hervik [20] with that of Poisson [21].

4.1 Hypersurfaces

Consider a spacetime $\mathcal{M}$ divided into two domains:

$$\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^- \,, \tag{4.1.1}$$

with a common boundary:

$$\Sigma = \partial\mathcal{M}^+ \cap \partial\mathcal{M}^- \,. \tag{4.1.2}$$

Such a boundary is called a hypersurface and will be $d-1$-dimensional if $\mathcal{M}$ is d-dimensional.

To specify the hypersurface one can either provide a parametrization:

$$x^\alpha = x^\alpha(y^a) \,, \tag{4.1.3}$$

where y^a are the intrinsic coordinates of the surface, or one can specify the hypersurface by a restricion on the coordinates:

$$\Phi(x^\alpha) = 0 \,. \tag{4.1.4}$$

We will need a normal vector to this surface, with normalization given by:

$$\mathbf{n} \cdot \mathbf{n} = g_{\mu\nu} n^\mu n^\nu \equiv \epsilon = \begin{cases} 1, & \text{if } \Sigma \text{ is timelike,} \\ -1, & \text{if } \Sigma \text{ is spacelike.} \end{cases} \tag{4.1.5}$$

To determine wether a surface is timelike or spacelike, consider what kind of "particles" that could constitue the surface: If the surface is made up of particles moving along timelike curves, the surface is timelike. If the particles would have to move along spacelike curves (unphysical for particles with positive non-zero mass) the surface is spacelike.

This means that there is potentially a sign difference between n_α and n^α. By convention we choose n^α in the direction of increasing Φ:

$$n^\alpha \Phi_{,\alpha} > 0 \, . \tag{4.1.6}$$

With this choice we can calculate n_α explicitly:

$$n_\alpha = \frac{\epsilon \Phi_{,\alpha}}{|g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}|^{1/2}} \, . \tag{4.1.7}$$

Tangent vectors in Σ are given by:

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a} \tag{4.1.8}$$

Displacements within Σ have line-element

$$\begin{aligned} ds^2_\Sigma &= g_{\alpha\beta} dx^\alpha dy^\beta \\ &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^a} dy^a \frac{\partial y^\beta}{\partial y^b} dy^b \\ &= h_{ab} dy^a dy^b \, , \end{aligned} \tag{4.1.9}$$

where

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b \tag{4.1.10}$$

is called the induced metric of the hypersurface.

Allthough we do not demand a continuous metric across the shell, we do demand that the induced metric is the same when looked at from both coordinate systems:

$$h^+_{ab} = h^-_{ab} \tag{4.1.11}$$

This means that if $g^+_{\alpha\beta} \neq g^-_{\alpha\beta}$ then $e^{\alpha^+}_a$ can be different from $e^{\alpha^-}_a$.

4.2 The induced energy-momentum tensor and extrinsic curvature

The energy-momentum tensor can be decomposed into three terms, one in $\mathcal{M}^+$, one in $\mathcal{M}^-$ and one on Σ:

$$T_{\alpha\beta} = S_{\alpha\beta}\delta(l) + T^+_{\alpha\beta}\theta(l) + T^-_{\alpha\beta}\theta(-l) \, , \tag{4.2.1}$$

where l is a coordinate orthogonal to Σ ($\frac{\partial}{\partial l} = \mathbf{n}$). The energy-momentum tensor on the hypersurface, $S_{\alpha\beta}$, is defined by:

$$S_{\alpha\beta} = \lim_{\tau\to 0} \int_{-\tau/2}^{\tau/2} T_{\alpha\beta} dl \ . \tag{4.2.2}$$

It can be shown that $S_{\alpha\beta}$ is tangent to the hypersurface and therefore it can be decomposed [21, p. 88]:

$$S^{\alpha\beta} = S^{ab} e^{\alpha}_{a} e^{\beta}_{b} \ , \tag{4.2.3}$$

or, equivalently:

$$S_{ab} = S_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} \ . \tag{4.2.4}$$

This is the induced energy-momentum tensor of the hypersurface.

The extrinsic curvature of a surface is related to how a surface is embedded in the surrounding spacetime. Take a cone in flat spacetime. The surface of the cone looks curved, but can be "unfolded" to a flat piece. The same is not true for a sphere. The cone has non-zero *extrinsic curvature* but zero *intrinsic curvature*. The extrinsic curvature is given by:

$$K^{\pm}_{\mu\nu} = \epsilon n_{\alpha} \Gamma^{\alpha}{}_{\mu\nu}|^{\pm} \ . \tag{4.2.5}$$

Where $\Gamma^{\alpha}{}_{\mu\nu}$ are the Christoffel symbols. We will need the intrinsic components:

$$K^{\pm}_{ab} = K^{\pm}_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} \ . \tag{4.2.6}$$

With trace:

$$K = h^{ab} K_{ab} = K^{a}{}_{a} \tag{4.2.7}$$

4.3 The Lanczos equation and equations of motion

Define the discontinuity operation:

$$[T] \equiv T^{+} - T^{-} \ . \tag{4.3.1}$$

It can be shown from Einsteins equations that S_{ab} is given by the following equation:

$$\epsilon\kappa S_{ab} = [K_{ab}] - h_{ab} [K] \ . \tag{4.3.2}$$

This equation is called Lanczos' equation and describes the mechanical properties of the shell by means of the discontinuity of the extrinsic curvature across the shell. Or equivalently, one can say that a discontinuety in the extrinsic curvature across a hypersurface implies the presence of a singular shell at this surface, with energy-momentum tensor given by the above equation.

Lanczos' equation is sometimes easier to solve for the mixed components, since the mixed components of the induced metric (or any other metric tensor) reduces to a Kronecker delta ${\delta^a}_b$:

$$\boxed{\epsilon\kappa {S^a}_b = [{K^a}_b] - {\delta^a}_b\,[K]\ .} \tag{4.3.3}$$

In addition to this, from Einstein equations, one can arrive at the following equation of motion for the shell [20]:

$${}^{(3)}\nabla_j {S^j}_i + [T_{in}] = 0\ , \tag{4.3.4}$$

where n is the index corresponding to the normal vector to the surface. Contracting this with u^i we get the equation of continuity:

$$u^{i\,(3)}\nabla_j {S^j}_i = -u^i\,[T_{in}]\ . \tag{4.3.5}$$

In the case of vanishing energy-momentum tensor outside the shell the equations of motion are:

$${}^{(3)}\nabla_j {S^j}_i = 0\ , \tag{4.3.6}$$

$$u^{i\,(3)}\nabla_j {S^j}_i = 0\ . \tag{4.3.7}$$

4.4 Application of the formalism to some known systems

In order to have something to compare our results to, it is useful to consider the formalism applied to a few special cases.

4.4.1 Schwarzschild spacetime

Let us consider a spherical shell located at $r = R$. Outside the shell there is Schwarzschild spacetime:

$$g^+_{\mu\nu} = \mathrm{diag}\left(-\left(1 - \frac{2m}{r}\right), \frac{1}{1 - \frac{2m}{r}}, r^2, r^2\sin^2\theta\right)\ . \tag{4.4.1}$$

The metric inside the spacetime is the Minkowski metric in the convenient form:

$$g^-_{\mu\nu} = \mathrm{diag}\left(-\left(1 - \frac{2m}{R}\right), 1, r^2, r^2\sin^2\theta\right)\ . \tag{4.4.2}$$

Note that by a constant adjustment of the coordinate t, the usual form of the Minkowski metric is obtained:

$$t \rightarrow t' = \frac{1}{\sqrt{1-\frac{2m}{R}}}\, t \, , \tag{4.4.3}$$

$$\Rightarrow g^{-}_{\mu\nu} = \mathrm{diag}\left(-1, 1, r^2, r^2 \sin^2\theta\right) \, . \tag{4.4.4}$$

We will work with the internal metric in the form of eq. 4.4.2. Since the surface is composed of particles moving along timelike curves, the surface is timelike and $\epsilon = 1$. This gives the normal vector:

$$n_\alpha = \frac{(0,1,0,0)}{|g^{rr}|^{1/2}} = (0, n_r, 0, 0) \, , \tag{4.4.5}$$

where

$$n_r = \begin{cases} 1, & r < R, \\ \frac{1}{\sqrt{1-\frac{2m}{r}}}, & r > R. \end{cases} \tag{4.4.6}$$

Let $(\tau, \vartheta, \varphi)$ be the intrinsic coordinates of the surface. Then the surface can be chosen to follow the parametric relations:

$$t = \frac{\tau}{\sqrt{1-\frac{2m}{R}}} \, , \;\; \theta = \vartheta \, , \; \phi = \varphi \, , \tag{4.4.7}$$

with the condition

$$r = R = \mathrm{const.} \tag{4.4.8}$$

The physical reason for the choice of τ is that we want it to represent the proper time of a standard clock at rest on the shell. The tangent vectors in Σ are:

$$e^\alpha_\tau = \left(\frac{1}{\sqrt{1-\frac{2m}{R}}}, 0, 0, 0\right) \, , \tag{4.4.9}$$

$$e^\alpha_\vartheta = (0,0,1,0) \, , \tag{4.4.10}$$

$$e^\alpha_\varphi = (0,0,0,1) \, . \tag{4.4.11}$$

The induced metric is:

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b \, . \tag{4.4.12}$$

With the following non-zero components:

$$h_{\tau\tau} = g_{tt}\left(e^t_\tau\right)^2 \ , \quad h_{\vartheta\vartheta} = g_{\theta\theta} \ , \quad h_{\varphi\varphi} = g_{\phi\phi} \ . \tag{4.4.13}$$

Demanding continuity for the induced metric tensor gives us:

$$h_{\tau\tau} = -1 \ , \quad h_{\vartheta\vartheta} = R^2 \ , \quad h_{\varphi\varphi} = R^2 \sin^2\vartheta \ . \tag{4.4.14}$$

From (4.2.5) we get following components of the extrinsic curvature tensor:

$$K^-_{tt} = 0 \ , \ K^-_{\theta\theta} = -r \ , \ K^-_{\phi\varphi} = -r\sin^2\theta \ , \tag{4.4.15}$$

$$K^+_{tt} = \frac{m}{r^2}\sqrt{1-\frac{2m}{r}} \ , \ K^+_{\theta\theta} = -r\sqrt{1-\frac{2m}{r}} \ , \ K^+_{\phi\phi} = -r\sin^2\theta\sqrt{1-\frac{2m}{r}} \ . \tag{4.4.16}$$

By(4.2.6), and the fact that K_{ab} lives on the hypersurface where $r = R =$ constant, this translates to:

$$K^-_{\tau\tau} = 0 \ , \ K^-_{\vartheta\vartheta} = -R \ , \ K^-_{\varphi\varphi} = -R\sin^2\vartheta \ , \tag{4.4.17}$$

$$K^+_{\tau\tau} = \frac{m}{R^2\sqrt{1-\frac{2m}{R}}} \ , \ K^+_{\vartheta\vartheta} = -R\sqrt{1-\frac{2m}{R}} \ , \ K^+_{\varphi\varphi} = -R\sin^2\vartheta\sqrt{1-\frac{2m}{R}} \ . \tag{4.4.18}$$

Raising an index with h^{ab} we get the mixed components:

$$K^{\tau\,-}_{\ \tau} = 0 \ , \ K^{\vartheta\,-}_{\ \vartheta} = -\frac{1}{R} = K^{\varphi\,-}_{\ \varphi} \tag{4.4.19}$$

$$K^{\tau\,+}_{\ \tau} = -\frac{m}{R^2\sqrt{1-\frac{2m}{R}}} \ , \ K^{\vartheta\,+}_{\ \vartheta} = \frac{-\sqrt{1-\frac{2m}{R}}}{R} = K^{\varphi\,+}_{\ \varphi} \tag{4.4.20}$$

Now, putting all this together the solution to the Lanczos equation (4.3.3) is:

$$S^{\tau}{}_{\tau} = \frac{-1}{4\pi R}\left(1 - \sqrt{1 - \frac{2m}{R}}\right) , \tag{4.4.21}$$

$$S^{\vartheta}{}_{\vartheta} = \frac{1 - \frac{m}{R} - \sqrt{1 - \frac{2m}{R}}}{8\pi R\sqrt{1 - \frac{2m}{R}}} \tag{4.4.22}$$

$$= S^{\varphi}{}_{\varphi} \, . \tag{4.4.23}$$

Writing the solution on perfect fluid form:

$$S^{ab} = (\sigma + p)u^a u^b + ph^{ab} \, . \tag{4.4.24}$$

The particles on the shell have intrinsic velocity $u^a = (1, 0, 0)$, with $u_a u^a = -1$, giving:

$$S^{\tau}{}_{\tau} = -\sigma \, , \tag{4.4.25}$$

$$S^{\vartheta}{}_{\vartheta} = p \tag{4.4.26}$$

$$= S^{\varphi}{}_{\varphi} \, . \tag{4.4.27}$$

This means that the mass density and pressure are given by:

$$\sigma = \frac{1}{4\pi R}\left(1 - \sqrt{1 - \frac{2m}{R}}\right) , \tag{4.4.28}$$

$$p = \frac{1 - \frac{m}{R} - \sqrt{1 - \frac{2m}{R}}}{8\pi R\sqrt{1 - \frac{2m}{R}}} \, . \tag{4.4.29}$$

The equation of continuity (4.3.7) yields:

$$\begin{aligned} 0 &= u^{a\,(3)}\nabla_b \left((\sigma + p)u^a u_b + p\delta^a{}_b\right) \\ &= u^a \left({}^{(3)}\nabla_b(\sigma + p)u^b u_a + (\sigma + p)\,{}^{(3)}\nabla_b(u^b u_a)\right) + u^{b\,(3)}\nabla_b(p) \, . \end{aligned} \tag{4.4.30}$$

Using $u^{a\,(3)}\nabla_b = \frac{d}{d\tau} \equiv \dot{}$ gives:

$$\dot{\sigma} = -(\sigma + p)\,{}^{(3)}\nabla_b(u^b) \, . \tag{4.4.31}$$

The last term is a covariant divergence which in general is given by:

$$\nabla \cdot \mathbf{A} = \nabla_\alpha A^\alpha = \frac{1}{|g|}\left(\sqrt{|g|}A^\mu\right)_{,\mu} \, . \tag{4.4.32}$$

In our case this is:

$$
\begin{aligned}
{}^{(3)}\nabla_a(u^a) &= \frac{1}{|h|}\left(\sqrt{|h|}\,u^a\right)_{,a} \\
&= \frac{1}{R^2}(R^2)_{,\tau} = 2\frac{\dot{R}}{R}\,, \qquad (4.4.33)
\end{aligned}
$$

giving the equation:

$$
\dot{\sigma} = -2(\sigma + p)\frac{\dot{R}}{R}\,. \qquad (4.4.34)
$$

Integrating yields the general result:

$$
R^2(\sigma + p) = const. \qquad (4.4.35)
$$

For a collapsing cloud of dust ($p = 0$) this equation implies conservation of the rest mass $\mu = 4\pi\sigma R^2$ during the collapse. In the Newtonian limit this rest mass is:

$$
\mu = R\left(1 - \sqrt{1 - \frac{2m}{R}}\right) \approx R\left(1 - 1 + \frac{m}{R}\right) = m\,. \qquad (4.4.36)
$$

In the static case ($\dot{R} = 0$) we can see that the rest mass density σ (and then also the rest mass) is constant from eq. (4.4.34).

The equation of motion (4.3.6) contains two more equations independent of the equation of continuity, but in this case it only shows that the pressure gradient along the shell vanishes.

The gravitational mass is the mass parameter appearing in the Schwarzschild line-element. The Tolman-Whittaker expression for the gravitation mass of a system [20, 22] is:

$$
\begin{aligned}
M &= \int_V \left(T^{\alpha}{}_{\alpha} - 2T^{0}{}_{0}\right)\sqrt{-g}\,dV \qquad (4.4.37) \\
&= \int_V \left(-T^t{}_t + T^r{}_r + T^{\theta}{}_{\theta} + T^{\phi}{}_{\phi}\right)\sqrt{-g}\,dV\,. \qquad (4.4.38)
\end{aligned}
$$

Since the volume is bounded by $r \leq R$, we use the interior metric and coordinates for the invariant volume element:

$$
\sqrt{-g}\,dV = \sqrt{1 - \frac{2m}{r}}\,r^2 \sin\theta\, dr\, d\theta\, d\phi\,. \qquad (4.4.39)
$$

The energy-momentum tensor in this basis is given by:

$$T^{\mu\nu} = S^{ab} e^{\mu}_{a} e^{\nu}_{b} \delta(r - R) \,. \tag{4.4.40}$$

Giving:

$$T^{t}_{\ t} = -\sigma\, \delta(r - R) \,, \tag{4.4.41}$$
$$T^{r}_{\ r} = 0 \,, \tag{4.4.42}$$
$$T^{\theta}_{\ \theta} = \frac{r^2}{R^2}\, p\, \delta(r - R) \,, \tag{4.4.43}$$
$$T^{\phi}_{\ \phi} = \frac{r^2}{R^2}\, p\, \delta(r - R) \,. \tag{4.4.44}$$

The integral in eq. (4.4.38) can now be evaluated:

$$\begin{aligned} M &= 4\pi \int_0^R \delta(r - R) \left(\sigma + 2\frac{r^2}{R^2} p \right) \sqrt{1 - \frac{2m}{r}}\, r^2 dr \\ &= 4\pi R^2 \left(\sigma(R) + 2p(R) \right) \sqrt{1 - \frac{2m}{R}} \\ &= m \,. \end{aligned} \tag{4.4.45}$$

This shows that the pressure contributes to the gravitational mass m appearing in the Schwarzschild line-element, and that this mass is equivalent with the Tolman-Whittaker mass M. In contrast, the rest mass of the system $\mu = 4\pi R^2 \sigma$ is not the total gravitational mass. Similarly a collapsing cloud of dust ($p = 0$) receives a contribution to the gravitational mass from the kinetic energy of the infalling dust particles. In the limit where $R \to 2M$ the pressure diverges while the gravitational mass remains constant. However it is reasonable to assume that the perfect fluid assumption breaks down if we want to keep the shell static in this limit. One cannot expect a perfect fluid to remain static in that case.

In this example we have calculated the properties of a source for the exterior Schwarzschild spacetime. This is not necessarily a point-mass source, it can equally well be a static shell with a pressure given by eq. (4.4.29).

4.4.2 Slow rotating Kerr case

Next we will consider the case of the slow rotation limit of the Kerr spacetime outside a shell, and flat spacetime inside the shell. The spacetime outside the shell has line-element:

$$ds_+^2 = -\left(1-\frac{2M}{r}\right)dt^{+2} + \frac{1}{1-\frac{2M}{R}}dr^2 + r^2 d\theta^{+2} + r^2 \sin^2\theta^+ d\phi^{+2} - \frac{4Ma}{r}\sin^2\theta^+ dt^+ d\phi^+, \tag{4.4.46}$$

where $a = \frac{J}{M} << M$ is the angular momentum per unit mass. We will work to first order in a only. Inside there is flat spacetime:

$$ds_-^2 = -dt^{-2} + dr^{-2} + r^{-2} d\theta^{-2} + r^{-2}\sin^2\theta^- d\phi^{-2} \tag{4.4.47}$$

The two spacetimes are glued together along the hypersurface Σ located at $r = R$. Displacements within Σ have line-element:

$$ds_\Sigma^{\pm 2} = -\left(1-\frac{2M}{R}\right)dt^{+2} + R^2 d\theta^{+2} + R^2\sin^2\theta^+ d\phi^{+2} - \frac{4Ma}{R}\sin^2\theta^+ dt^+ d\phi^+ \tag{4.4.48}$$

$$= -dt^{-2} + R^2 d\theta^{-2} + R^2 \sin^2\theta^- d\phi^{-2}\,. \tag{4.4.49}$$

Introducing the intrinsic coordinates $(\tau, \vartheta, \varphi)$ the line-element is:

$$ds_\Sigma^2 = -d\tau^2 + R^2 d\vartheta^2 + R^2\sin^2\vartheta d\varphi^2 \tag{4.4.50}$$

From this we have:

$$t^- = \tau\ ,\ \theta = \vartheta\ ,\ \phi = \varphi\,. \tag{4.4.51}$$

With a coordinate transformation we can remove the offdiagonal elements from ds_Σ^{+2}:

$$\psi = \phi^+ - \Omega t^+\,. \tag{4.4.52}$$

Since Ω can be assumed to be proportional to a we have:

$$d\phi^{+2} = d\psi^2 + 2\Omega dt^+ d\psi\,. \tag{4.4.53}$$

Thus the metric of eq. (4.4.48) can written in diagonal form by choosing

$$\Omega = \frac{2Ma}{R^3}\,. \tag{4.4.54}$$

This is the angular frequency that the interior coordinate system rotates with relative to the exterior system. Continuity of the induced metric, given by

(4.1.11), gives the relationship between the exterior and intrinsic coordinates. The time coordinates:

$$\left(1-\frac{2M}{R}\right)\left(e^{t^+}_\tau\right)^2=\left(e^{t^-}_\tau\right)^2=1\,, \tag{4.4.55}$$

$$e^{t^+}_\tau=\frac{1}{\sqrt{1-\frac{2M}{R}}}\,, \tag{4.4.56}$$

$$\Rightarrow t^+=\frac{1}{\sqrt{1-\frac{2M}{R}}}\tau\,. \tag{4.4.57}$$

The polar angle:

$$\theta^+=\vartheta\,. \tag{4.4.58}$$

The azimuthal angle:

$$\psi=\varphi\,. \tag{4.4.59}$$

Here ψ is an exterior coordinate. Inertial frames in $\mathcal{M}^+$ have constant values of ψ, just as inertial frames on Σ and in $\mathcal{M}^-$ have constant φ and ϕ^- respectively. Thus this equation establishes continuity of the rotation of the inertial frames across the shell.

We can now write:

$$\varphi=\phi^+-\Omega t^+\,, \tag{4.4.60}$$

or, on the form $x^{\alpha^+}=x^{\alpha^+}(y^a)$:

$$\phi^+=\varphi+\frac{\Omega}{\sqrt{1-\frac{2M}{R}}}\tau\,. \tag{4.4.61}$$

This gives the basis vectors in Σ:

$$e^{\alpha^+}_\tau=\left(\frac{1}{\sqrt{1-\frac{2m}{R}}},0,0,\frac{\Omega}{\sqrt{1-\frac{2m}{R}}}\right)\,, \tag{4.4.62}$$

$$e^{\alpha^+}_\vartheta=(0,0,1,0)\,, \tag{4.4.63}$$

$$e^{\alpha^+}_\varphi=(0,0,0,1)\,. \tag{4.4.64}$$

The components of the extrinsic curvature tensor in the exterior region is the same as in the previous section, except for the offdiagonal element:

$$K^{+}_{t\phi^{+}} = -\frac{Ma}{r^2}\sin^2\theta\sqrt{1-\frac{2M}{r}} \tag{4.4.65}$$

$$= K^{+}_{\phi t^{+}} \,. \tag{4.4.66}$$

For the interior region with no offdiagonal elements the extrinsic curvature tensor is the same as in the last section. Converting to the intrinsic coordinates and raising an index:

$$\begin{aligned} K^{\tau}{}_{\tau}^{+} &= h^{\tau\tau} K_{\tau\tau}^{+} = h^{\tau\tau} e^{\alpha^{+}}_{\tau} e^{\beta^{+}}_{\tau} K_{\alpha\beta}^{+} \\ &= h^{\tau\tau}\left(\left(e^{t^{+}}_{\tau}\right)^2 K^{+}_{t^{+}t^{+}} + 2e^{t^{+}}_{\tau} e^{\phi^{+}}_{\tau} K^{+}_{t^{+}\phi^{+}} + \left(e^{\phi^{+}}_{\tau}\right)^2 K^{+}_{\phi^{+}\phi^{+}}\right) \\ &= -\frac{1}{\sqrt{1-\frac{2M}{R}}}\frac{M}{R^2} + \mathcal{O}(a^2)\,. \end{aligned} \tag{4.4.67}$$

$$K^{\tau}{}_{\varphi}{}^{+} = h^{\tau\tau} K^{+}_{\tau\varphi} = h^{\tau\tau} e^{\alpha^{+}}_{\tau} e^{\beta^{+}}_{\varphi} K_{\alpha\beta}^{+}$$

$$= h^{\tau\tau} e^{\phi^{+}}_{\varphi}\left(e^{t^{+}}_{\tau} K^{+}_{t^{+}\phi^{+}} + e^{\phi^{+}}_{\tau} K^{+}_{\phi^{+}\phi^{+}}\right)$$

$$= \sin^2\vartheta\frac{3Ma}{R^2}\,. \tag{4.4.68}$$

$$K^{\varphi}{}_{\tau}{}^{+} = h^{\varphi\varphi} K^{+}_{\varphi\tau}$$

$$= \frac{-3Ma}{R^4}\,. \tag{4.4.69}$$

$$K^{\varphi}{}_{\varphi}{}^{+} = h^{\varphi\varphi}\left(e^{\phi^{+}}_{\varphi}\right)^2 K^{+}_{\varphi^{+}\varphi^{+}}$$

$$= -\frac{1}{R}\sqrt{1-\frac{2M}{R}} \tag{4.4.70}$$

$$= K^{\vartheta}{}_{\vartheta}{}^{+}\,. \tag{4.4.71}$$

$$K^{\vartheta}{}_{\vartheta}{}^{-} = -\frac{1}{R} \tag{4.4.72}$$

$$= K^{\varphi}{}_{\varphi}{}^{-} \tag{4.4.73}$$

From Lanczos' equation (4.3.3) we get the energy-momentum tensor of the shell:

$$S^{\tau}{}_{\tau} = -\frac{1}{4\pi R}\left(1-\sqrt{1-\frac{2M}{R}}\right) . \tag{4.4.74}$$

$$S^{\vartheta}{}_{\vartheta} = \frac{1-\frac{M}{R}-\sqrt{1-\frac{2M}{R}}}{8\pi R\sqrt{1-\frac{2M}{R}}} \tag{4.4.75}$$

$$= S^{\varphi}{}_{\varphi} . \tag{4.4.76}$$

$$S^{\tau}{}_{\varphi} = \frac{3Ma\sin^2\vartheta}{8\pi R^2} . \tag{4.4.77}$$

$$S^{\varphi}{}_{\tau} = -\frac{3Ma}{8\pi R^4} . \tag{4.4.78}$$

A particle on the shell has velocity vector u^a (normalizing to first order in a):

$$u^a = (1,0,\omega) = (\frac{d\tau}{d\tau}, 0, \frac{d\varphi}{d\tau}) , \tag{4.4.79}$$

where ω is proportional to a (ω is the angular frequency/rotational speed of the shell particles in the coordinate system (τ, ϑ, ϕ)).This gives

$$\varphi = \omega\tau = \sqrt{1-\frac{2M}{R}}\,\omega t^{+} , \tag{4.4.80}$$

for this particle. Transforming this to the exterior coordinate system:

$$\begin{aligned} u^{\alpha^+} &= e_a^{\alpha^+} u^a = e_\tau^{\alpha^+} u^\tau + e_\varphi^{\alpha^+} u^\varphi \\ &= \frac{1}{\sqrt{1-\frac{2M}{R}}}\left(1,0,0,\Omega+\sqrt{1-\frac{2M}{R}}\,\omega\right) . \end{aligned} \tag{4.4.81}$$

This means that the rotational speed of the shell as viewed from the outside is:

$$\Omega_{\text{shell}} = \Omega + \sqrt{1-\frac{2M}{R}}\,\omega . \tag{4.4.82}$$

Viewed from the outside, the interior inertial frames rotates with speed $\Omega_{\text{in}} = \Omega$. Writing the solution on perfect fluid form:

$$S^{a}{}_{b} = (\sigma + p)u^a u_b + p h^{a}{}_{b} . \tag{4.4.83}$$

We find:

$$\omega = \frac{-S^{\varphi}{}_{\tau}}{-S^{\tau}{}_{\tau} + S^{\varphi}{}_{\varphi}} \, . \tag{4.4.84}$$

The ratio

$$\frac{\Omega_{\text{in}}}{\Omega_{\text{shell}}} = \frac{\Omega}{\Omega + \sqrt{1 - \frac{2M}{R}}\,\omega} = \frac{1}{1 + \sqrt{1 - \frac{2M}{R}}\,\frac{\omega}{\Omega}} \, , \tag{4.4.85}$$

has limits:

$$\frac{\Omega_{\text{in}}}{\Omega_{\text{shell}}}(R \,=\, 2M) = 1 \, , \tag{4.4.86}$$

$$\frac{\Omega_{\text{in}}}{\Omega_{\text{shell}}}(R \gg 2M) = \frac{4M}{3R} \, . \tag{4.4.87}$$

The latter is Thirring's classic weak field result [8]. The first shows that in the limit that the radius of the shell approaches the Schwarzschild radius, the inertial frames inside the rotating shell rotates with the same angular frequency as the shell itself. Perfect inertial dragging is realized.

4.4.3 The interior and exterior Schwarzschild solution

The interior Schwarzschild solution, describing for instance the interior of a star, can be written as:

$$ds_-^2 = -\left(\frac{3}{2}\sqrt{1 - \frac{R_S}{R}} - \frac{1}{2}\sqrt{1 - \frac{R_S}{R^3}r^2}\right)^2 dt^2 \tag{4.4.88}$$

$$+ \frac{dr^2}{1 - \frac{R_S}{R^3}r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \, , \tag{4.4.89}$$

where $R_S = 2M$ and $r = r^- \leq R$, and R is the radius of the spherical object.

The exterior metric is the standard Schwarzschild metric of eq. (4.4.1). The induced metric can be chosen to be:

$$ds_\Sigma^2 = -d\tau^2 + R^2 d\vartheta^2 + R^2 \sin^2\vartheta d\varphi^2 \, . \tag{4.4.90}$$

Again demanding continuity of the induced metric as in the previous sections gives us the tangent vectors in Σ and the relation between the different sets of coordinates at the surface $r^- = r^+ = R$:

$$e_\tau^{\alpha^+} = e_\tau^{\alpha^-} = \left(\frac{1}{\sqrt{1 - \frac{R_S}{R}}}, 0, 0, 0 \right) , \tag{4.4.91}$$

$$e_\vartheta^{\alpha^+} = e_\vartheta^{\alpha^-} = (0, 0, 1, 0) , \tag{4.4.92}$$

$$e_\varphi^{\alpha^+} = e_\varphi^{\alpha^-} = (0, 0, 0, 1) . \tag{4.4.93}$$

$$\Rightarrow t^- = t^+ = \frac{\tau}{\sqrt{1 - \frac{R_S}{R}}} , \quad \phi^- = \phi^+ = \varphi , \quad \theta^- = \theta^+ = \vartheta . \tag{4.4.94}$$

The normal vector at the surface is (using $\epsilon = 1$ for a time-like surface):

$$n_r = \frac{1}{\sqrt{1 - \frac{R_S}{R}}} . \tag{4.4.95}$$

The components of the extrinsic curvature in the interior region are:

$$K^\tau{}_\tau{}^- = -\frac{M}{R^2 \sqrt{1 - \frac{2M}{R} s}} , \tag{4.4.96}$$

$$K^\vartheta{}_\vartheta{}^- = -\frac{\sqrt{1 - \frac{2M}{R}}}{R} \tag{4.4.97}$$

$$= K^\varphi{}_\varphi{}^- . \tag{4.4.98}$$

In the exterior region the components of the extrinsic curvature tensor is given by equation (4.4.20). By inspection, these are identical to the interior ones. Since there is no discontinuity in the extrinsic curvature tensor across the shell, there is no mass shell present there, $S^a{}_b = 0$. The hypersurface is simply a boundary surface between a region containing mass and a vacuum region.

This is another source of the external Schwarzschild spacetime. Allthough this result seems obvious, I have not seen it derived with this formalism before. It is therefore a nice test of consistency.

Chapter 5

An accelerated black hole

5.1 The C-metric

A generalization of the Schwarzschild metric to an accelerated mass is known as the C-metric [23–27]. The line-element is often given in the original form [23, 24]:

$$ds^2 = \frac{1}{(\tilde{x}+\tilde{y})^2}\left(-\tilde{F}d\tilde{t}^2 + \frac{d\tilde{y}^2}{\tilde{F}} + \frac{d\tilde{x}^2}{\tilde{G}} + \tilde{G}d\tilde{z}^2\right) , \tag{5.1.1}$$

where $\tilde{F}(\tilde{y})$ and $\tilde{G}(\tilde{x})$ are cubic polynomials on the form

$$\tilde{G}(\tilde{x}) = a_0 + a_1\tilde{x} + a_2\tilde{x}^2 + a_3\tilde{x}^3 , \tag{5.1.2}$$

and

$$\tilde{F}(\tilde{y}) = -\tilde{G}(-\tilde{y}) . \tag{5.1.3}$$

The choice of the constants a_i are related to the choice of coordinates. To see this consider the coordinate transformation:

$$\tilde{t} = \frac{c_0}{A}\mathrm{t} , \tag{5.1.4}$$

$$\tilde{y} = Ac_0\mathrm{y} - c_1 , \tag{5.1.5}$$

$$\tilde{x} = Ac_0\mathrm{x} + c_1 , \tag{5.1.6}$$

$$\tilde{z} = \frac{c_0}{A}\phi . \tag{5.1.7}$$

The line-element will then be adjusted to:

$$ds^2 = \frac{1}{A^2(\mathrm{x}+\mathrm{y})^2}\left(-\frac{\tilde{F}d\mathrm{t}^2}{A^2} + \frac{A^2d\mathrm{y}^2}{\tilde{F}} + \frac{A^2d\mathrm{x}^2}{\tilde{G}} + \frac{\tilde{G}d\phi^2}{A^2}\right) . \tag{5.1.8}$$

Let F $= A^{-2}\tilde{F}$ and G $= A^{-2}\tilde{G}$. This will cast the metric in another usual form[24, 25]:

$$ds^2 = \frac{1}{A^2(\mathrm{x}+\mathrm{y})^2}\left(-\mathrm{F}d\mathrm{t}^2 + \frac{d\mathrm{y}^2}{\mathrm{F}} + \frac{d\mathrm{x}^2}{\mathrm{G}} + \mathrm{G}d\phi^2\right) , \tag{5.1.9}$$

The coefficients of the functions G and F can now be adjusted by the choice of c_1 in (5.1.7) so that one of them is zero (except the cubic coefficient which does not depend on c_1). The standard choice has often been to choose $a_1 = 0$ and $a_0 = -a_2 = 1$ which gives the cubic polynomials the form

$$\mathrm{G} = 1 - \mathrm{x}^2 - 2MA\mathrm{x}^3 \ , \ \ \mathrm{F} = -1 + \mathrm{y}^2 - 2MA\mathrm{y}^3 \ . \tag{5.1.10}$$

Hong and Teo [26] used the freedom in (5.1.2) and (5.1.7) not to remove the linear terms, as has been the standard, but to make the root structure of the cubic polynomials as simple as possible. They arrive at the line-element:

$$ds^2 = \frac{1}{\alpha^2(x+y)^2}\left(-Fd\tau^2 + \frac{dy^2}{F} + \frac{dx^2}{G} + Gd\varphi^2\right) , \tag{5.1.11}$$

where

$$G = (1-x^2)(1+2\alpha m x) \ , \ \ F = -(1-y^2)(1-2\alpha m y) \ . \tag{5.1.12}$$

Notice that the coordinates (t, y, x, ϕ) have been rescaled to (τ, y, x, φ) (equations (10) and (11) in their paper), and the parameters A and M have been rescaled to α and m. In this form the coordinate x is constrained to lie between -1 and 1, and we must have $0 < 2\alpha m < 1$ in order to preserve the signature of the metric.

5.2 The C-metric in spherical coordinates

Following Griffiths, Krtouš and Podolský [27], we can now make a coordinate transformation:

$$x = \cos\theta \ , \quad y = \frac{1}{\alpha\tau} \ , \quad \tau = \alpha t \ . \tag{5.2.1}$$

The line-element then becomes:

$$ds^2 = \frac{1}{(1+\alpha r\cos\theta)^2}\left(-Qdt^2 + \frac{dr^2}{Q} + \frac{r^2 d\theta^2}{P} + Pr^2\sin^2\theta d\varphi^2\right) , \tag{5.2.2}$$

where the functions Q and P are given by:

$$Q = \left(1 - \alpha^2 r^2\right)\left(1 - \frac{2m}{r}\right) , \tag{5.2.3}$$

$$P = 1 + 2\alpha m \cos\theta . \tag{5.2.4}$$

By inspection of Q, one sees that there are two coordinate singularties which occur at

$$r = 2m , \tag{5.2.5}$$

$$r = \frac{1}{\alpha} , \tag{5.2.6}$$

where the first coordinate singularity corresponds to a black hole horizon, and the second corresponds to the horizon of a uniformly accelerating reference frame ("Rindler horizon"). In this way the metric can be viewed as a nonlinear combination of the Schwarzschild and Rindler spacetimes, thus representing the metric outside an accelerated point-particle or black hole.

In addition to the requirement $0 < 2\alpha m < 1$, we are only interested in the region of spacetime which lies inside the Rindler horizon:

$$r < \frac{1}{\alpha} . \tag{5.2.7}$$

These are our constraints. Note that while the first constraint is a general one, the second one only cuts away a part of the spacetime which we will not deal with in this thesis.

The range of the φ coordinate is $(-\pi C, \pi C)$. Griffiths *et al.* considers the circumference to radius ratio for a small circle around the two half-axes $\theta = 0$ and $\theta = \pi$. In the first case the result is $2\pi C(1 + 2\alpha m)$ and in the second case it is $2\pi C(1 - 2\alpha m)$. Since these differ from 2π we have conical singularities along these half-axes (with different conicity). We see that choosing $C = (1 \pm 2\alpha m)^{-1}$ will remove one of these singularities, but not both at the same time. Griffiths *et al.* choose $C = (1 + 2\alpha m)^{-1}$, removing the conical singularity at the $\theta = 0$ half-axis. They interpret the conical singularity at the $\theta = \pi$ half-axis as representing a *"semi-infinite cosmic string under tension"*, and that the tension in the string is the cause of the force accelerating the Schwarzschild-like particle along the $\theta = \pi$ axis.

The range of the rotational coordinate can be rescaled to 2π by:

$$\phi = C^{-1}\varphi . \tag{5.2.8}$$

With the above choice of the constant C the line-element is now:

$$ds^2 = \frac{1}{(1+\alpha r\cos\theta)^2}\left(-Qdt^2 + \frac{dr^2}{Q} + \frac{r^2 d\theta^2}{P} + \frac{Pr^2\sin^2\theta}{(1+2\alpha m)^2}d\phi^2\right) . \quad (5.2.9)$$

In order to simplify calculations later, we introduce the function D in the metric tensor:

$$D \equiv (1+\alpha r\cos\theta)^2 . \quad (5.2.10)$$

The metric tensor can then be written:

$$g_{\mu\nu} = \frac{1}{D}\,\mathrm{diag}\left(-Q, \frac{1}{Q}, \frac{r^2}{P}, \frac{Pr^2\sin^2\theta}{(1+2\alpha m)^2}\right) . \quad (5.2.11)$$

5.3 $\alpha \to 0$: The Schwarzschild limit

Taking the limit where the parameter α goes to zero, the functions Q, P and D simplify:

$$Q(\alpha = 0) = 1 - \frac{2m}{r} \;,\; P(\alpha = 0) = 1 \;,\; D(\alpha = 0) = 1 \;, \quad (5.3.1)$$

and we recover the familiar Schwarzschild solution:

$$g_{\mu\nu} = \mathrm{diag}\left(-\left(1-\frac{2m}{r}\right), \left(1-\frac{2m}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right) \quad (5.3.2)$$

In this limit the parameter m is the gravitational mass of the object. However we shall see in section 6 that this is not the case in the general scenario. This limit also serves as an important special case since our results are well known for the Schwarzschild spacetime.

5.4 $m \to 0$: The weak field limit

Taking the limit where the parameter m goes to zero, the functions Q and P simplify as follows:

$$Q(m = 0) = 1 - \alpha^2 r^2 \quad , \quad P(m = 0) = 1 \;, \quad (5.4.1)$$

and the metric reduces to:

$$g_{\mu\nu} = \frac{1}{(1+\alpha r\cos\theta)^2}\,\mathrm{diag}\left(-(1-\alpha^2 r^2), \frac{1}{1-\alpha^2 r^2}, r^2, r^2\sin^2\theta\right) . \quad (5.4.2)$$

To clarify what type of spacetime this is, let us first apply the transformation [27]:

$$\zeta = \frac{\sqrt{1-\alpha^2 r^2}}{\alpha(1+\alpha r\cos\theta}\ ,\ \rho = \frac{r\sin\theta}{1+\alpha r\cos\theta}\ ,\ \tau = \alpha t\ . \tag{5.4.3}$$

Then the line-element reduces to:

$$ds^2 = -\zeta^2 d\tau^2 + d\zeta^2 + d\rho^2 + \rho^2 d\phi^2\ . \tag{5.4.4}$$

Which is the Rindler form of Minkowski spacetime in cylindrical coordinates. Applying the transformation

$$T = \pm\zeta\sinh\tau \quad , \quad Z = \pm\zeta\cosh\tau\ , \tag{5.4.5}$$

one recovers the standard form of the Minkowski metric in cylindrical coordinates:

$$ds^2 = -dT^2 + dZ^2 + d\rho^2 + \rho^2 d\phi^2\ . \tag{5.4.6}$$

This shows that in the weak field limit the C-metric reduces to the metric of a uniformly accelerating reference frame. It follows from the knowledge of this special case that particles with a constant position in the coordinates of the C-metric (r, θ and ϕ constant) follow worldlines that in the Rindler coordinates are given by

$$Z^2 - T^2 = \frac{1-\alpha^2 r^2}{\alpha^2(1+\alpha r\cos\theta)^2}\ . \tag{5.4.7}$$

Particularly, the origin particle of the accelerated frame has the acceleration α in the positive or negative Z direction. This justifies the role of the parameter α as the acceleration of the source particle(s).

In the transformation (5.4.5) one has two choices of sign, $T = \pm\zeta\sinh\tau$ and $Z = \pm\zeta\cosh\tau$. Taking the negative transformation we get an origin particle of the accelerated frame which accelerates in the negative Z direction with acceleration α. This is because the C-metric actually describes *two* black holes, accelerating away from each other. However, they are in causally different regions of spacetime (they are outside each others Rindler horizon), therefore we are free to choose one of them and focus on that region of spacetime which corresponds to choosing only one sign in the transformation (5.4.5).

The weak field is also the field experienced far away from the source. An observer at rest in this far-away field will see the particle(s) or black hole accelerate with uniform acceleration relative to him/herself.

Chapter 6

A spherical accelerated shell

6.1 The spacetime inside and outside the shell

Outside the shell we have the C-metric. What kind of spacetime should we have inside the shell in order to produce perfect inertial dragging?

By analogy to the Schwarzschild case, it is tempting to use the C-metric with $m = 0$ inside the shell. As we have seen the metric then reduces to the metric of a uniformly accelerating reference frame. This was our initial attempt as well. However, one needs to consider how the inertial frames inside the shell behave. Reference particles in a uniformly accelerating frame feel a uniform gravitational field. Hence they are not free. We are interested in a spacetime in which *free particles accelerate along with the shell.* We must therefore choose Minkowski spacetime in spherical coordinates that follow the shell (the coordinate centers of the metrics at $r = 0$ must coincide at all times).

Therefore the metric inside the shell is:

$$g_{\mu\nu} = \text{diag}\left(-1, 1, r^2, r^2 \sin^2\theta\right) \tag{6.1.1}$$

From equation (4.2.5) and choosing a surface $r = R =$ constant it is clear that we need the ${\Gamma^r}_{\mu\nu}$ Christoffel symbols. The non-zero ones are:

$$\begin{aligned} {\Gamma^r}_{\theta\theta} &= -\,r && (6.1.2)\\ {\Gamma^r}_{\phi\phi} &= -\,r\sin^2\theta\,. && (6.1.3) \end{aligned}$$

The remaining non-zero Christoffel symbols are:

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \tag{6.1.4}$$

$$\Gamma^{\theta}{}_{r\theta} = \frac{1}{r} \tag{6.1.5}$$

$$\Gamma^{\phi}{}_{\phi r} = \frac{1}{r} \tag{6.1.6}$$

$$\Gamma^{\phi}{}_{\phi\theta} = \frac{1}{\tan\theta} = \cot\theta\ , \tag{6.1.7}$$

which are useful if one wants to solve the geodesic equation inside the shell to ensure that perfect inertial dragging is realized.

Outside the shell we have the C-metric given by (5.2.11). The $\Gamma^{r}{}_{\mu\nu}$ Christoffel symbols we need are:

$$\Gamma^{r}{}_{tt} = Q\left[\alpha^2(m-r) + \frac{m}{r^2} - \frac{\alpha\cos\theta Q}{\sqrt{D}}\right]\ , \tag{6.1.8}$$

$$\Gamma^{r}{}_{\theta\theta} = -\frac{Qr}{P\sqrt{D}}\ , \tag{6.1.9}$$

$$\Gamma^{r}{}_{\phi\phi} = -\frac{QPr\sin^2\theta}{\sqrt{D}(1+2\alpha m)^2}\ . \tag{6.1.10}$$

6.2 Solving Lanczos' equation

6.2.1 The properties of the hypersurface

We will eventually put the shell on the black hole horizon given by $r = 2m$. For now lets choose the hypersurface given by $r =$ constant. This gives the restriction:

$$\Phi(t, r, \theta, \phi) = r - R = 0\ ,\ R = \text{const.} \tag{6.2.1}$$

The hypersurface is timelike since it consists of particles moving along timelike curves: $\epsilon = 1$. The derivative of the restriction is:

$$\Phi_{,\alpha} = (0, 1, 0, 0)\ . \tag{6.2.2}$$

This gives the normal vector:

$$n_\alpha = (0, n_r, 0, 0)\ , \tag{6.2.3}$$

with

$$n_r = \frac{1}{|g^{rr}\Phi_{,r}\Phi_{,r}|^{1/2}} = \begin{cases} 1 \ , & r < R\,, \\ \frac{1}{\sqrt{DQ}}, & r > R\,, \end{cases} \tag{6.2.4}$$

and

$$n^r = g^{rr} n_r = \begin{cases} 1 \ , & r < R\,, \\ \sqrt{DQ}, & r > R\,. \end{cases} \tag{6.2.5}$$

Let $(\tau, \vartheta, \varphi)$ be the intrinsic coordinates of the hypersurface. Let $(t^+, r^+, \theta^+, \phi^+)$ denote the coordinates outside the surface and $(t^-, r^-, \theta^-, \phi^-)$ denote the coordinates inside. Then the surface is given by:

$$t^- = \tau\,,\ \theta^- = \vartheta\,,\ \phi^- = \varphi\,, \tag{6.2.6}$$

$$r^- = R = \text{ const.} \tag{6.2.7}$$

The tangent curves in Σ are:

$$e_\tau^{\alpha^-} = (1,0,0,0)\,, \tag{6.2.8}$$

$$e_\vartheta^{\alpha^-} = (0,0,1,0)\,, \tag{6.2.9}$$

$$e_\varphi^{\alpha^-} = (0,0,0,1)\,. \tag{6.2.10}$$

The induced metric is:

$$h_{ab} = g^{\pm}_{\alpha\beta} e_a^{\alpha\pm} e_b^{\beta\pm}\,. \tag{6.2.11}$$

We are free to choose either $g^+_{\mu\nu}$ or $g^-_{\mu\nu}$ on the right hand side, the easier choice being $g^-_{\mu\nu}$. With this choice (and $r^- = R$) we get the following non-zero components:

$$\begin{aligned} h_{\tau\tau} &= g^-_{tt} = -1\,, \\ h_{\vartheta\vartheta} &= g^-_{\theta\theta} = R^2\,, \\ h_{\varphi\varphi} &= g^-_{\phi\phi} = R^2\sin^2\vartheta\,. \end{aligned} \tag{6.2.12}$$

The formalism does not require a continuous metric $g_{\mu\nu}$ across the shell, however, it requires that h_{ab} (which only exists on the hypersurface $r = R$) looks the same in both coordinate systems, that is:

$$[h_{ab}] = 0\,. \tag{6.2.13}$$

This requirement can be used to find an expression for the $e_a^{\alpha^+}$ tangent vectors:

$$g^+_{\alpha\beta} e_a^{\alpha^+} e_b^{\beta^+} = g^-_{\alpha\beta} e_a^{\alpha^-} e_b^{\beta^-}\,. \tag{6.2.14}$$

Giving the tangent vectors (choosing the positive solution):

$$e_\tau^{\alpha^+} = \left(\sqrt{\frac{D}{Q}}, 0, 0, 0\right) , \tag{6.2.15}$$

$$e_\vartheta^{\alpha^+} = \left(0, 0, \frac{1}{\sqrt{PD}}, 0\right) , \tag{6.2.16}$$

$$e_\varphi^{\alpha^+} = \left(0, 0, 0, \frac{\sin\theta^+}{\sin\vartheta(1+2\alpha m)}\sqrt{\frac{P}{D}}\right) . \tag{6.2.17}$$

This gives the relationship between the exterior and intrinsic coordinates:

$$\tau = \sqrt{\frac{Q}{D}}\, t^+ \ , \ d\vartheta = \sqrt{PD}\, d\theta^+ \ , \ \varphi = \frac{\sin\vartheta(1+2\alpha m)}{\sin\theta^+}\sqrt{\frac{D}{P}}\, \phi^+ \ . \tag{6.2.18}$$

The integration over θ^+ involves an elliptical integral, the relationship between ϑ and θ^+ is therefore expressed in differential form.

6.2.2 The energy-momentum tensor of the shell

The Christoffel symbols are calculated from the metrics in eq. (5.2.11) and (6.1.1) in Appendix A. In $\mathcal{M}^-$ the relevant components of the extrinsic curvature tensor in the interior coordinate system read:

$$\begin{aligned} K^-_{tt^-} &= n^-_r \Gamma^r{}_{tt}{}^- \\ &= 0 \, , \end{aligned} \tag{6.2.19}$$

$$\begin{aligned} K^-_{\theta\theta^-} &= n^-_r \Gamma^r{}_{\phi\phi}{}^- \\ &= -r \, , \end{aligned} \tag{6.2.20}$$

$$\begin{aligned} K^-_{\phi\phi^-} &= n^-_r \Gamma^r{}_{\theta\theta}{}^- \\ &= -r\sin^2\theta \, . \end{aligned} \tag{6.2.21}$$

The extrinsic curvature tensor in $\mathcal{M}^+$, in the exterior coordinate system, has the following relevant components:

$$
\begin{aligned}
K^+_{tt^+} &= n^+_r \Gamma^r{}_{tt}{}^+ \\
&= \sqrt{\frac{Q}{D}} \left[\alpha^2 (m - r) + \frac{m}{r^2} - \frac{\alpha \cos\theta^+ Q}{\sqrt{D}} \right] , \qquad (6.2.22) \\
K^+_{\theta\theta^+} &= n^+_r \Gamma^r{}_{\phi\phi}{}^+ \\
&= -\frac{\sqrt{Q}\, r}{PD} , \qquad (6.2.23) \\
K^+_{\phi\phi^+} &= n^+_r \Gamma^r{}_{\theta\theta}{}^+ \\
&= -\frac{\sqrt{Q}\, P r^2 \sin^2\theta^+}{D(1 + 2\alpha m)^2} . \qquad (6.2.24)
\end{aligned}
$$

This transforms according to eq. (4.2.6) to the intrinsic coordinates:

$$
K^-_{\tau\tau} = 0 \, , \; K^-_{\vartheta\vartheta} = -R \, , \; K^-_{\varphi\varphi} = -R \sin^2\vartheta \, . \qquad (6.2.25)
$$

Furthermore,

$$
\begin{aligned}
K^+_{\tau\tau} &= \sqrt{\frac{D}{Q}} \left[\alpha^2 (m - R) + \frac{m}{R^2} - \frac{\alpha \cos\theta^+ Q}{\sqrt{D}} \right] , \qquad (6.2.26) \\
K^+_{\vartheta\vartheta} &= -R\sqrt{Q} \, , \qquad (6.2.27) \\
K^+_{\varphi\varphi} &= -R \sin^2\vartheta \sqrt{Q} \, . \qquad (6.2.28)
\end{aligned}
$$

Raising the indices with h^{ab} yields:

$$
K^{\tau}{}_{\tau}^{-} = 0 \, , \; K^{\vartheta}{}_{\vartheta}^{-} = -\frac{1}{R} = K^{\varphi}{}_{\varphi}^{-} \, . \qquad (6.2.29)
$$

$$
\begin{aligned}
K^{\tau}{}_{\tau}^{+} &= -\sqrt{\frac{D}{Q}} \left[\alpha^2 (m - R) + \frac{m}{R^2} - \frac{\alpha \cos\theta^+ Q}{\sqrt{D}} \right] , \qquad (6.2.30) \\
K^{\vartheta}{}_{\vartheta}^{+} &= -\frac{\sqrt{Q}}{R} \qquad (6.2.31) \\
&= K^{\varphi}{}_{\varphi}^{+} \, . \qquad (6.2.32)
\end{aligned}
$$

The discontinuities of the extrinsic curvature tensor are:

$$\begin{aligned}\left[K^{\vartheta}{}_{\vartheta}\right] &= K^{\vartheta}{}_{\vartheta}^{+} - K^{\vartheta}{}_{\vartheta}^{-} \\ &= \frac{1}{R}\left(1-\sqrt{Q}\right) && (6.2.33)\\ &= \left[K^{\varphi}{}_{\varphi}\right], && (6.2.34)\\ \left[K^{\tau}{}_{\tau}\right] &= K^{\tau}{}_{\tau}^{+} - K^{\tau}{}_{\tau}^{-} \\ &= -\sqrt{\frac{D}{Q}}\left[\alpha^2(m-R)+\frac{m}{R^2}-\frac{\alpha\cos\theta^+ Q}{\sqrt{D}}\right]. && (6.2.35)\end{aligned}$$

From the Lanczos equation (4.3.3) we have:

$$\begin{aligned}S^{\tau}{}_{\tau} &= \frac{1}{8\pi}\left(\left[K^{\tau}{}_{\tau}\right] - \delta^{\tau}{}_{\tau}\left[K\right]\right) \\ &= \frac{1}{8\pi}\left(-\left[K^{\vartheta}{}_{\vartheta}\right] - \left[K^{\varphi}{}_{\varphi}\right]\right), && (6.2.36)\\ S^{\vartheta}{}_{\vartheta} &= \frac{1}{8\pi}\left(\left[K^{\vartheta}{}_{\vartheta}\right] - \left[K\right]\right) \\ &= \left(-\left[K^{\tau}{}_{\tau}\right] - \left[K^{\varphi}{}_{\varphi}\right]\right) && (6.2.37)\\ &= \left(-\left[K^{\tau}{}_{\tau}\right] - \left[K^{\vartheta}{}_{\vartheta}\right]\right) \\ &= S^{\varphi}{}_{\varphi}, && (6.2.38)\end{aligned}$$

with the result:

$$\boxed{\begin{aligned}S^{\tau}{}_{\tau} &= -\frac{1}{4\pi R}\left(1-\sqrt{Q}\right) && (6.2.39a)\\ S^{\vartheta}{}_{\vartheta} &= \frac{1}{8\pi}\left(\frac{-1+\sqrt{Q}}{R}+\sqrt{\frac{D}{Q}}\left(\alpha^2(m-R)+\frac{m}{R^2}-\frac{\alpha\cos\theta^+ Q}{\sqrt{D}}\right)\right) && (6.2.39b)\\ &= S^{\varphi}{}_{\varphi}. && (6.2.39c)\end{aligned}}$$

6.3 Physical interpretation of the shell

The energy-momentum tensor of a perfect fluid can be written:

$$S^{ab} = (\sigma + p)u^a u^b + p h^{ab}. \qquad (6.3.1)$$

Since the shell is comoving with the coordinates, the velocity components are zero with the exception of the time component:

$$u^a = (u^\tau, 0, 0) \,. \tag{6.3.2}$$

This gives

$$S^{\tau\tau} = (\sigma + p)(u^\tau)^2 + ph^{\tau\tau} = \sigma \,, \tag{6.3.3}$$
$$S^{\vartheta\vartheta} = ph^{\vartheta\vartheta} \,, \tag{6.3.4}$$
$$S^{\varphi\varphi} = ph^{\varphi\varphi} \,. \tag{6.3.5}$$

Lowering one index we get:

$$S^\tau{}_\tau = (\sigma + p)u^\tau u_\tau + p\delta^\tau_\tau = -\sigma \,, \tag{6.3.6}$$
$$S^\vartheta{}_\vartheta = ph^\vartheta{}_\vartheta = p = S^\varphi{}_\varphi \,. \tag{6.3.7}$$

Comparing this to our final expression for $S^a{}_b$ in (6.2.39):

$$\sigma = \frac{1}{4\pi R}\left(1 - \sqrt{Q}\right) \,, \tag{6.3.8}$$
$$p = \frac{1}{8\pi R}\left(-\left(1 - \sqrt{Q}\right) + R\sqrt{\frac{D}{Q}}\left(\alpha^2(m - R) + \frac{m}{R^2} - \frac{\alpha\cos\theta^+ Q}{\sqrt{D}}\right)\right) \,. \tag{6.3.9}$$

From this we can conclude that the shell consists of a perfect fluid with proper rest mass density σ and pressure p. In the limit $R \to 2m$, the function Q goes to zero and we see that the rest mass density approaches the finite value of $(4\pi R)^{-1} = (8\pi m)^{-1}$, while the second term in the pressure diverges. This divergence is not a coordinate effect, but rather a manifestation of the fact that infinite pressure is needed in order to keep a shell at rest when it is located exactly at the Schwarzschild horizon. At this distance, space itself (or the "river of space" [7]) flows inward towards the center of the black hole with the speed of light.

Just like in the Schwarzschild case, the pressure contributes to the total gravitatonal mass. However, integrating the relativistic mass density is problematic since the pressure is a function of the exterior polar angle. Just from the fact that we know that the pressure contributes, and that the pressure diverges as $R \to 2m$ we can conclude that just as in the Schwarzschild case, we cannot expect a perfect fluid to remain static in this limit.

The equations (4.4.34) and (4.4.35) are still valid since only a perfect fluid assumption and the induced metric (which is the same as in the Schwarzschild case) was used in their derivation. However the constant on the right hand side in (4.4.35) is now only a constant with respect to time, since the pressure varies with the polar angle. This means that in the static case we still have conservation of rest mass, and in the general case we still have that $R^2(\sigma + p) = const.$ (in time).

The accelerated shell with mass density and pressure given by equations (6.3.8) and (6.3.9) is a source of the C-metric. By construction there is perfect inertial dragging inside this shell.

The limit $m \to 0$:
When the parameter m is set to zero the resulting rest mass density and pressure are:

$$\sigma = \frac{1}{4\pi R}\left(1 - \sqrt{1 - \alpha^2 R^2}\right) , \tag{6.3.10}$$

$$p = \frac{1}{8\pi R}\left(-\left(1 - \sqrt{1 - \alpha^2 R^2}\right) + \frac{-\alpha R(\alpha R + \cos\theta^+)}{\sqrt{1 - \alpha^2 R^2}}\right) . \tag{6.3.11}$$

These expressions are not vanishing. This shows that there still is a massive shell present, it is only the mass parameter m which is zero.

The Tolman-Whittaker mass appears to be non-zero:

$$M = \int_0^R \int_0^\pi \int_0^{2\pi} \delta(r - R)\left(\sigma + 2\frac{r^2}{R^2}p(\theta^+)\right) r^2 \sin\theta dr d\theta d\phi \tag{6.3.12}$$

The first term in (6.3.11) will cancel the σ term in the integral, and we are left with integration over the last term in (6.3.11). The integration over the θ^+ dependent term cannot be done analytically since θ^+ is only given as an implicit funtion of θ which cannot be inverted. A basic numerical scheme to perform the integration confirms that the result does not seem to converge to zero. See appendix B.

This means that the parameter m only represents the gravitational mass in the limit $\alpha \to 0$. In the general case a non-zero α affects the gravitational mass. The exact form of this dependence remain unknown since performing the Tolman-Whittaker integration cannot be done analytically in the general case.

The limit $\alpha \to 0$:
In this limit we recover the Schwarzschild case of section 4.4.1, and the mass parameter m is again equal to the gravitational mass.

Chapter 7

Discussion

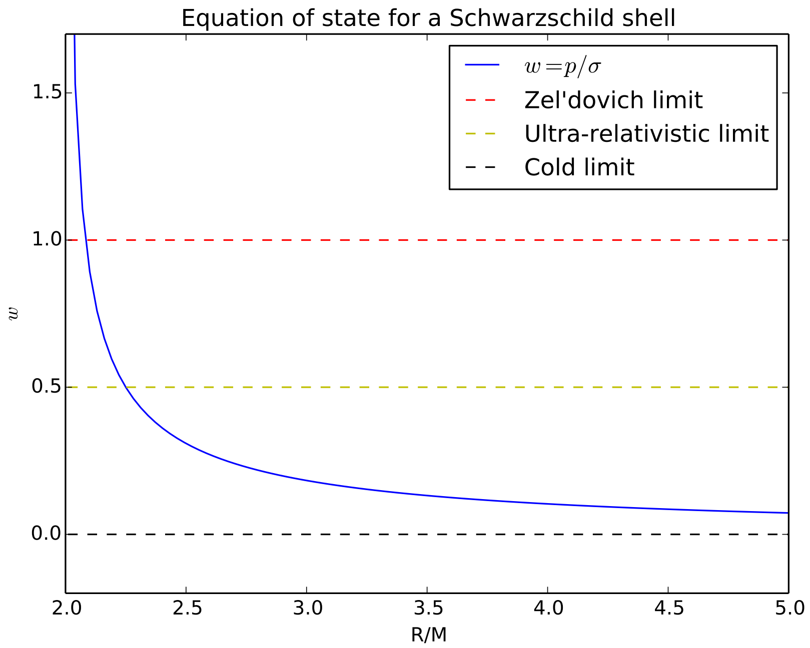

Figure 7.1: The equation of state for a perfect fluid Schwarzschild shell. This is also the equation of state for the slowly rotating Kerr case.

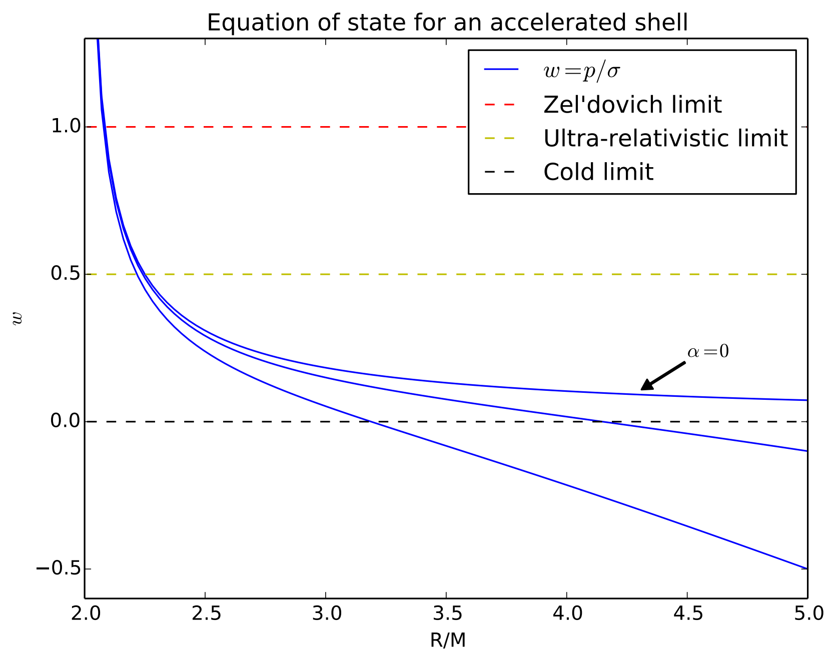

Figure 7.2: The equation of state for the accelerated shell along $\cos\theta^+ = 0$. $\alpha = 0$ corresponds to the Schwarzschild case of figure 7.1. The two lower blue curves represents increasing α, 0.05 and 0.1 respectively.

7.1 The state of the shell

In figure 7.1 we see that the static Schwarzschild shell can be desribed as a perfect fluid with a non-zero positive pressure. Radiation, or a gas consisting of ultra-relativistic particles (particles where the energy from the rest mass is negligible compared to the kinetic/total energy of the particles), have a traceless energy-momentum tensor, $T^{\alpha}{}_{\alpha} = 0$. Since $T^{r}{}_{r} = 0$ an ultra-relativistic shell has equation of state

$$p = \frac{1}{2}\sigma \,. \tag{7.1.1}$$

We see that when $R = \frac{9}{4}M$, the pressure is so large that the particles of the shell become ultra-relativistic. Past this limit the shell can still be kept static with a finite pressure. An absolute upper limit for the parameter w in general relativity is given by Zel'dovich [28]. A physical limit on the speed of sound in a material is the speed of light. In a material where these are equal, w is equal to one. According to Zel'dovich this is the upper limit alowed

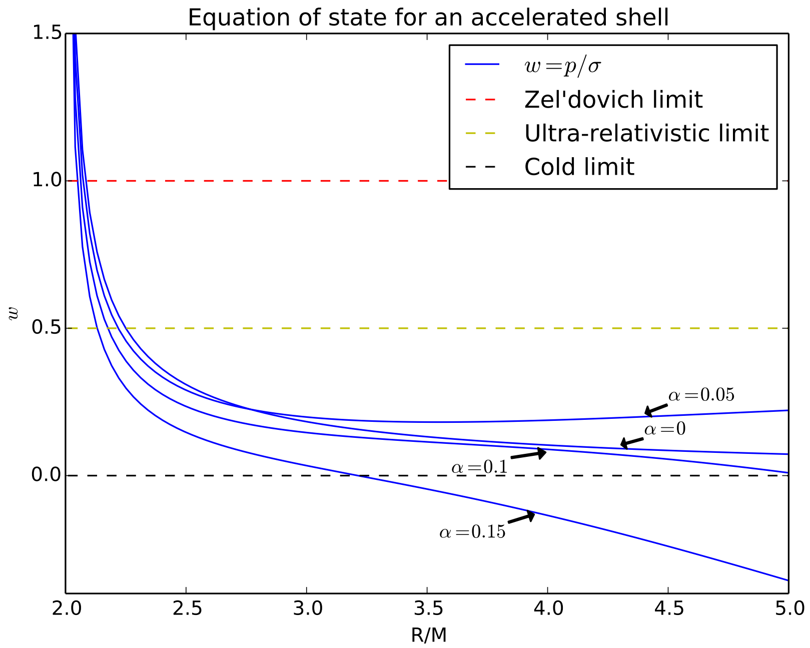

Figure 7.3: The equation of state for the accelerated shell along $\cos\theta^+ = -1$. $\alpha = 0$ corresponds to the Schwarzschild case of figure 7.1. The other blue curves represents $\alpha = 0.05$, 0.1 and 0.15.

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by general relativity. For the Schwarzschild shell this accours at $R = \frac{25}{12}M$. Past this a static schell cannot be interpreted within the general theory of relativity. In the collapse-limit the pressure diverges and one cannot expect the shell to remain static.

In figure 7.2 we see the equation of state for the accelerated shell. Since the pressure is a function of the polar angle the "equatorial" plane $\cos\theta^+ = 0$ is chosen for convenience. Increasing the acceleration parameter α lowers the pressure along this curve so much that the material obtains negative pressures for larger radia. This can ne interpreted as being due to the acceleration from the source. The source of the acceleration is a cosmic string, and like the LIVE (Lorentz Invariant Vacuum Energy) represented by the cosmological constant, this can cause repulsive gravity. At the zero-points on the graph this repulsive tendency cancels the ordinary pressure in the material, making the shell effectivey a shell of dust. However, the pressure is only zero along $\cos\theta^+ = 0$, so the whole shell will not behave as dust.

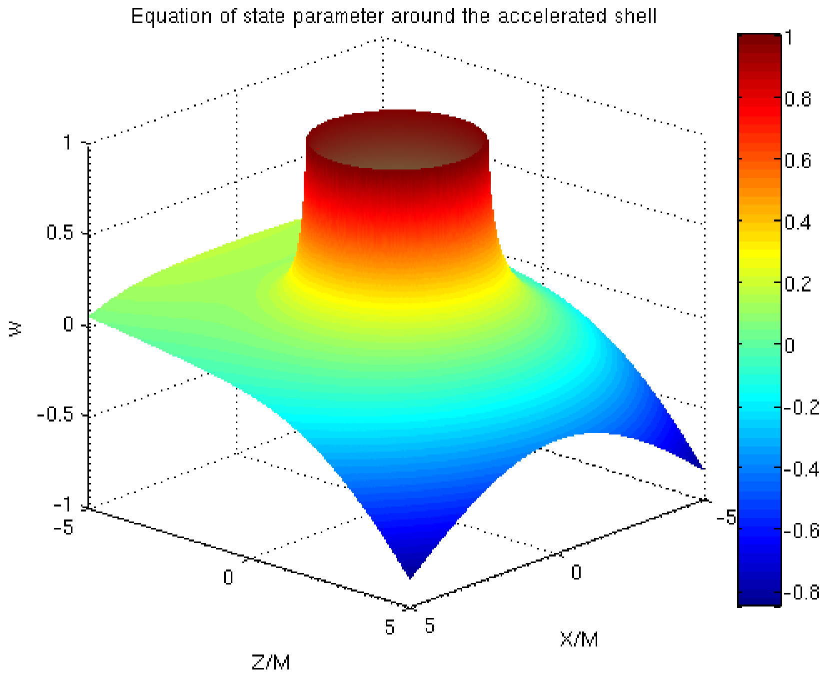

Figure 7.4: The equation of state parameter w along varying θ^+ for an accelerated shell with a radius R. The shell accelerates in the negative Z-direction ($Z = R\cos\theta^+$). $\alpha = 0.05$ is used. The pressure is constant along the ϕ^+ direction. There is a cut-off near the horizon where the pressure diverges.

We also see that the zero-point of the pressure increases as α decreases, and will go to infinity as α goes to zero. This means that a static shell of dust is only realized in the limit where both $\alpha \to 0$ and $R \to \infty$.

In figure 7.3 we see the same plot of the equation of state for the shell, now for $\cos\theta^+ = -1$, with one added value of $\alpha = 0.15$. At this angle the pressure is positive for $\alpha = (0, 0.05, 0.1)$, only $\alpha = 0.15$ becomes negative inside the domain. For larger radia the pressure increases for small α and decreases again for larger α. A more complete picture of the equation of state parameter w is shown in figure 7.4. The previous plots corresponds to vertical slices through the axes of this surface plot. $\cos\theta^+ = 0$ corresponds to a slice perpendicular to the Z-axis and $\cos\theta^+ = \pm 1$ corresponds to a slice perpendicular to the X-axis.

The fact that the coefficient w is not a constant but depends on the polar angle questions the validity of the perfect fluid assumption. It remains

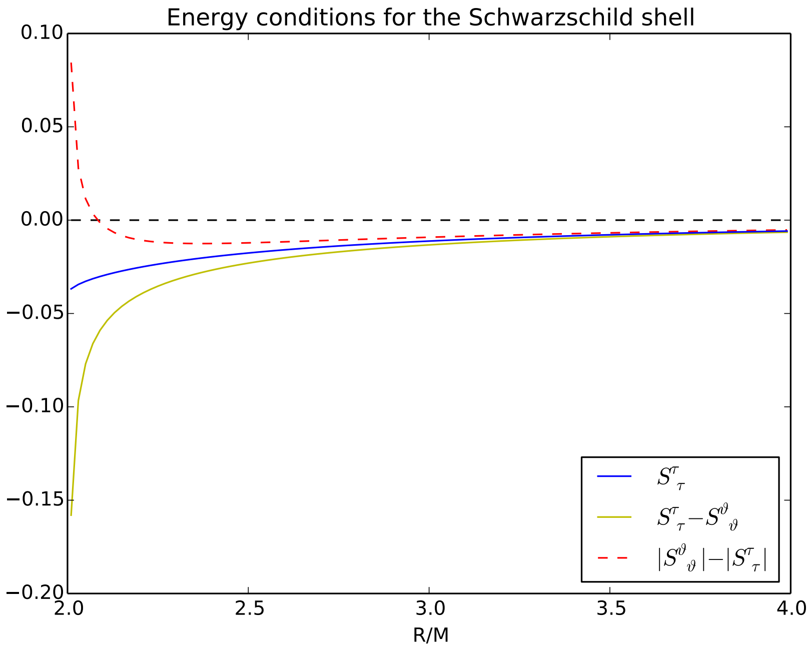

Figure 7.5: The two solid lines represent the two functions that must be below zero for the weak energy condition to hold. The red dashed line represents the function that must be below zero for the dominant energy conditions to hold.

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an open question if it is possible to choose a non-spherical shape for the shell which removes this feature.

In figure 7.5 and 7.6 we see plots of the energy condition functions - the functions that need to be below zero for the energy conditions to hold [18]. See appendix C. The two solid curves represent the function that need to be negative for the weak energy condition to hold. The dashed red curve represents the function that must be negative for the dominant energy condition to hold. We see that the weak energy condition is obeyed in the interesting regions near the collapse limit. The dominant energy condition is violated near the collapse limit. The fact that the energy conditions are violated near the acceleration horizon for the accelerated shell does not appear to be so problematic since it is in the other limit that we expect our model to give rise to perfect inertial dragging. Comparing these plots with figure 1 in [12] it seems that the violation of the dominant energy condition near the collapse limit might be overcome by giving the shell an electric charge (such a

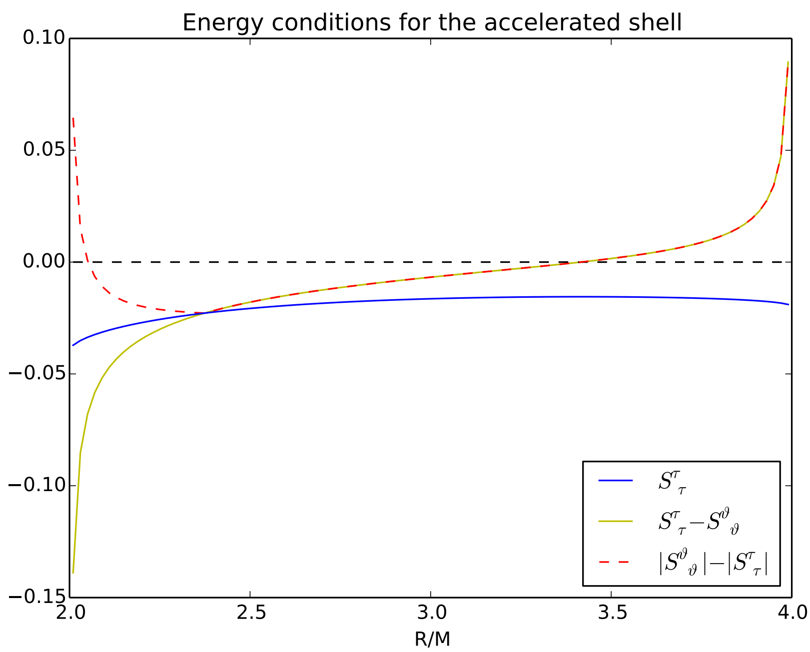

Figure 7.6: The two solid lines represent the two functions that must be below zero for the weak energy condition to hold. The red dashed line represents the function that must be below zero for the dominant energy conditions to hold. For the angle-dependent $S^{\vartheta}{}_{\vartheta}$ we again chose the "equatorial" plane $\cos\theta^{+} = 0$. Qualitatively the results are the same for all angles. $\alpha = 0.25$ is chosen so $R/M = 4$ corresponds to the acceleration horizon.

solution exists and is referred to as the charged C-metric). Wether this is true however remains an open question.

7.2 Inertial dragging

The coordinates inside the shell coincide with those on the shell. So these interior coordinates follow the shell. Free particles move in geodesics. Inside the shell the geodesics are straight lines with respect to the shell. Therefore, when the shell is viewed from an unaccelerated observer (relative to the asymptotic region) the shell and the coordinates of the shell, and hence free test particles inside it, accelerate with the same acceleration. This means that perfect inertial dragging is realised inside the shell.

The only “flaw” in this argument is that the perfect inertial dragging result does not depend on M/R at all. We cannot expect this to be true for an arbitrary light mass shell.

This means that we have described a shell where perfect inertial dragging is realized, but we have not deduced that perfect inertial dragging is the result of the presence of such an accelerated mass shell. By analogy to the rotational case, we expect that perfect inertial dragging is only realized in the limit $R \to 2M$. I suspect that it is possible to make some coordinate transformation to an unaccelerated observer somewhere outside the shell, and calculate the observed acceleration of a test particle inside the shell. Ideally the acceleration of the test particle would go to α as $R \to 2M$. Unfortunately, due to the time limitations, I have not pursued this idea.

As mentioned earlier, giving the shell a charge might salvage the violated dominant energy condition. However, from studying figure 4 in [18], we see that this invalidates their perfect inertial dragging condition.

Chapter 8

Conclusion and outlooks

In this thesis we have for the first time (as far as we know) described the physical properties of an accelerated static mass shell as a source of the (external) C-metric. By construction, perfect inertial dragging is realized inside the shell. The result is analytical and exact to all orders of the acceleration parameter α. By analogy to the well-known rotational problem, we expect that perfect inertial dragging is only realized in the limit where the radius of the shell approaches the Schwarzschild radius.

The physical properties of the shell have been analyzed and we found that the weak energy condition holds in the interesting regions (when the radius of the shell is much less than the acceleration horizon). However, near the collapse limit, the dominant energy condition is violated. This might be overcome by giving the shell a charge. To do this one needs to extend our external metric to the charged C-metric.

Near the collapse limit, the pressure diverges. This comes from the fact that we have required a static mass shell ($R = const.$). At the Schwarzschild horizon we cannot expect a mass shell to remain static. This is analogous again to the rotational case.

The main point is this: A shell of matter is only an idealized model for a Universe with lookback distance equal to the Schwarzschild radius of the mass inside this distance. We do not need this static model to behave perfectly well in this limit since it is not realized in nature. And even if a model is not realized in nature, we can still learn valuable things about nature from such models.

There are still some open questions:

- The dominant energy condition is violated near the collapse limit. For zero charge this is in agreement with [18], however in that study, for a non-zero charge within certain limits, the dominant energy condition is again obeyed even in the collapse limit. Does this mean that using the charged C-metric can repair this deficit of our model? And if so, will a charge destroy the perfect inertial dragging condition?

- The equation of state parameter w depends on θ^+. This makes a perfect fluid assumption questionable. Can this be repaired by introducin a non-spherical shape for the shell?

- The dynamics of the shell itself is lost when we restrict it to be static. Can any new insight about such shells or collapsing accelerated black holes be gained by studying a non-static shell?

- Finding the correct coordinate transformation discussed in 7.2 is not trivial. We suspect that calculations required afterwords are also not trivial. Due to time limitations I have not pursued this. Can it give us a condition for perfect inertial dragging?

Appendix A

Calculation of Christoffel symbols

The Christoffel symbols are given by:

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda} \right) \ . \tag{A.0.1}$$

Throughout this thesis we use surfaces with normal vectors $n_\alpha = (0, n_r, 0, 0)$, hence we are only interested in Christoffel symbols $\Gamma^{r}{}_{\alpha\beta}$. For a diagonal metric (in spherical coordinates), this gives

$$\Gamma^{r}{}_{\alpha\beta} = -\frac{1}{2} g^{rr} g_{\alpha\beta,r} \ , \tag{A.0.2}$$

as long as $\alpha, \beta \neq r$.

A.1 Minkowski metric in spherical coordinates

The Minkowski metric in spherical coordinates can be written:

$$g_{\mu\nu} = \mathrm{diag}\left(-1, 1, r^2, r^2 \sin^2\theta\right) \ . \tag{A.1.1}$$

In general all these components can be adjusted by a constant factor individually. As long as one is consistent with what adjustment one uses this does not matter. In section 4.4.1 g_{tt} is adjusted to $-(1-2m/R)$, in 4.4.2 and 6 it is -1. As we shall see, the Christoffel symbols we need does not depend on this choice (NB! Some of the other ones do depend on this choice.):

$$\begin{aligned} \Gamma^r{}_{tt} &= -\frac{1}{2} g^{rr} g_{tt,r} \\ &= 0 \,, \end{aligned} \tag{A.1.2}$$

$$\begin{aligned} \Gamma^r{}_{\theta\theta} &= -\frac{1}{2} g^{rr} g_{\theta\theta,r} = -\frac{1}{2} 2r \\ &= -r \,, \end{aligned} \tag{A.1.3}$$

$$\begin{aligned} \Gamma^r{}_{\phi\phi} &= -\frac{1}{2} g^{rr} g_{\phi\phi,r} = -\frac{1}{2} 2r \sin^2\theta \\ &= -r \sin^2\theta \,. \end{aligned} \tag{A.1.4}$$

A.2 Exterior Schwarzschild solution

The exterior Schwarzschild solution has metric:

$$g_{\mu\nu} = \mathrm{diag}\left(-\left(1 - \frac{2m}{r}\right), \left(1 - \frac{2m}{r}\right)^{-1}, r^2, r^2 \sin^2\theta \right) \,. \tag{A.2.1}$$

The relevant Christoffel symbols are:

$$\begin{aligned} \Gamma^r{}_{tt} &= -\frac{1}{2} g^{rr} g_{tt,r} \\ &= -\frac{1}{2} \left(1 - \frac{2m}{r}\right) \left(-\frac{2m}{r^2}\right) \\ &= \left(1 - \frac{2m}{r}\right) \frac{m}{r^2} \,. \end{aligned} \tag{A.2.2}$$

$$\begin{aligned} \Gamma^r{}_{\theta\theta} &= -\frac{1}{2} g^{rr} g_{\theta\theta,r} \\ &= -\frac{1}{2} \left(1 - \frac{2m}{r}\right) (-2r) \\ &= -\left(1 - \frac{2m}{r}\right) r \,. \end{aligned} \tag{A.2.3}$$

$$\begin{aligned} \Gamma^r{}_{\phi\phi} &= -\frac{1}{2} g^{rr} g_{\phi\phi,r} \\ &= -\frac{1}{2} \left(1 - \frac{2m}{r}\right) \left(-2r \sin^2\theta\right) \\ &= -\left(1 - \frac{2m}{r}\right) r \sin^2\theta \,. \end{aligned} \tag{A.2.4}$$

A.3 Slowly rotating Kerr solution

The line-element is:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^{+2} + \frac{1}{1-\frac{2M}{R}}dr^2 + r^2 d\theta^{+2} + r^2 \sin^2\theta^+ d\phi^{+2} - \frac{4Ma}{r}\sin^2\theta^+ dt^+ d\phi^+ . \tag{A.3.1}$$

The Christoffel symbols are:

$$\begin{aligned}
\Gamma^r{}_{tt} &= \frac{1}{2}g^{rr}(-g_{tt,r}) \\
&= \frac{1}{2}\left(1-\frac{2M}{r}\right)\frac{2M}{r^2} \\
&= \frac{M}{r^2}\left(1-\frac{2M}{r}\right) .
\end{aligned} \tag{A.3.2}$$

$$\begin{aligned}
\Gamma^r{}_{t\phi} &= \frac{1}{2}g^{rr}(-g_{t\phi,r}) \\
&= \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(-\frac{2Ma}{r^2}\sin^2\theta\right) \\
&= -\frac{Ma}{r^2}\sin^2\theta\left(1-\frac{2M}{r}\right) .
\end{aligned} \tag{A.3.3}$$

$$\begin{aligned}
\Gamma^r{}_{\theta\theta} &= \frac{1}{2}g^{rr}(-g_{\theta\theta,r}) \\
&= -\frac{1}{2}\left(1-\frac{2m}{r}\right)(-2r) \\
&= -\left(1-\frac{2m}{r}\right)r .
\end{aligned} \tag{A.3.4}$$

$$\begin{aligned}
\Gamma^r{}_{\phi\phi} &= -\frac{1}{2}g^{rr}g_{\phi\phi,r} \\
&= -\frac{1}{2}\left(1-\frac{2m}{r}\right)\left(-2r\sin^2\theta\right) \\
&= -\left(1-\frac{2m}{r}\right)r\sin^2\theta .
\end{aligned} \tag{A.3.5}$$

A.4 Interior Schwarzschild solution

The line-element is:

$$ds_-^2 = -\left(\frac{3}{2}\sqrt{1-\frac{R_S}{R}} - \frac{1}{2}\sqrt{1-\frac{R_S}{R^3}r^2}\right)^2 dt^2 \\ + \frac{dr^2}{1-\frac{R_S}{R^3}r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\ . \tag{A.4.1}$$

The Christoffel symbels are:

$$\Gamma^r{}_{tt} = \frac{1}{2}g^{rr}(-g_{tt,r})\ , \tag{A.4.2}$$

$$g_{tt,r} = \frac{\partial}{\partial r}\left[-\left(\frac{3}{2}\sqrt{1-\frac{R_S}{R}} - \frac{1}{2}\sqrt{1-\frac{R_S}{R^3}r^2}\right)^2\right]$$

$$= -\left(3\sqrt{1-\frac{R_S}{R}} - \sqrt{1-\frac{R_S}{R^3}r^2}\right)\frac{-\frac{1}{2}}{2\sqrt{1-\frac{R_S}{R^3}r^2}}\left(-2\frac{R_S}{R^3}r\right) \tag{A.4.3}$$

$$= -\left(\frac{3\sqrt{1-\frac{R_S}{R}}}{\sqrt{1-\frac{R_S}{R^3}r^2}} - 1\right)\frac{R_S r}{2R^3}\ , \tag{A.4.4}$$

$$\Gamma^r{}_{tt} = \left(1-\frac{R_S}{R^3}r^2\right)\left(\frac{3\sqrt{1-\frac{R_S}{R}}}{\sqrt{1-\frac{R_S}{R^3}r^2}} - 1\right)\frac{R_S r}{4R^3}\ . \tag{A.4.5}$$

The $\Gamma^r{}_{\theta\theta}$ and $\Gamma^r{}_{\phi\phi}$ Christoffel symbols are the same as in the previous sections.

A.5 C-metric

We use the metric in the form:

$$g_{\mu\nu} = \frac{1}{D}\ \mathrm{diag}\left(-Q, \frac{1}{Q}, \frac{r^2}{P}, \frac{Pr^2\sin^2\theta}{(1+2\alpha m)^2}\right)\ . \tag{A.5.1}$$

We will need the following partial derivatives:

$$\frac{\partial Q}{\partial r} = 2\alpha^2(m-r) + \frac{2m}{r^2}\ , \tag{A.5.2}$$

$$\frac{\partial D}{\partial r} = 2(1+\alpha r\cos\theta)\alpha\cos\theta\ , \tag{A.5.3}$$

$$\frac{\partial P}{\partial r} = 0\ . \tag{A.5.4}$$

The Christoffel symbels are:

$$\begin{aligned}
\Gamma^{r}{}_{tt} &= \frac{1}{2}g^{rr}(-g_{tt,r}) = \frac{1}{2}DQ\frac{\partial}{\partial r}\left(\frac{Q}{D}\right) \\
&= \frac{DQ}{2}\left(\frac{\frac{\partial Q}{\partial r}D - Q\frac{\partial D}{\partial r}}{D^2}\right) \\
&= Q\left(\alpha^2(m-r) + \frac{m}{r^2} - \frac{\alpha\cos\theta Q}{1+\alpha r\cos\theta}\right) \ , \qquad \text{(A.5.5)} \\
\Gamma^{r}{}_{\theta\theta} &= \frac{1}{2}g^{rr}(-g_{\theta\theta,r}) = -\frac{1}{2}DQ\frac{\partial}{\partial r}\left(\frac{r^2}{DP}\right) \\
&= -\frac{DQ}{2P}\left(\frac{2rD - r^2\frac{\partial D}{\partial r}}{D^2}\right) \\
&= -\frac{Qr}{P}\left(\frac{1}{1+\alpha r\cos\theta}\right) \ , \qquad \text{(A.5.6)} \\
\Gamma^{r}{}_{\phi\phi} &= -\frac{1}{2}g^{rr}g_{\phi\phi,r} = -\frac{1}{2}DQ\frac{\partial}{\partial r}\left(\frac{Pr^2\sin^2\theta}{D(1+2\alpha m)^2}\right) \\
&= -\frac{DQP\sin^2\theta}{2(1+2\alpha m)^2}\left(\frac{2rD - r^2\frac{\partial D}{\partial r}}{D^2}\right) \\
&= -\frac{QPr\sin^2\theta}{(1+2\alpha m)^2(1+\alpha r\cos\theta)} \ . \qquad \text{(A.5.7)}
\end{aligned}$$

Appendix B

A numerical integral

We want to evaluate the integral in eq. (6.3.12), repeated here for convenience:

$$M = \int_0^R \int_0^\pi \int_0^{2\pi} \delta(r - R) \left(\sigma + 2\frac{r^2}{R^2} p(\theta^+) \right) r^2 \sin\theta dr d\theta d\phi \,. \tag{B.0.1}$$

As mentioned in the main text, the first term in of the pressure in (6.3.11) will cancel the σ-term, and we are left with:

$$M = \frac{-4\pi\alpha R^3}{\sqrt{1 - \alpha^2 R^2}} \int_0^\pi (\alpha R + \cos\theta^+) \sin\theta d\theta \,, \tag{B.0.2}$$

where θ is the interior (or intrinsic, they are equivalent) polar coordinate. The relation between the exterior and interior polar coordinate is:

$$d\theta = \sqrt{PD}\, d\theta^+ \,. \tag{B.0.3}$$

When $m = 0$ this simplifies:

$$d\theta = (1 + \alpha R \cos\theta^+) d\theta^+ \,. \tag{B.0.4}$$

The differential equation is separated and can be integrated directly to yield:

$$\theta = \theta^+ + \alpha R \sin\theta^+ \,. \tag{B.0.5}$$

This expression cannot be inverted to give θ^+ as a function of θ, however we see that $\theta^+ = \pi \Rightarrow \theta = \pi = \theta^+$, so the limits of the integral are not affected if we do a change of variables. We can solve the integral (B.0.2) numerically in two ways after discretization:

i: By first solving (B.0.5) numerically, then summing up the contributions directly, or

Table B.1: Output from two runnings of the program.

Running with N = 1001
a = 0.1 R = 2.5
Sum1 = 0.830563163371
Sum2 = 0.83056216373
Sum3 = 0.830562752138
Running with N = 2001
a = 0.1 R = 2.5
Sum1 = 0.830563228869
Sum2 = 0.830562780581
Sum3 = 0.83056312606

ii: By changing variables to θ^+.

Note that the first term in (B.0.2) can be solved analytically, and is equal to $2\alpha R$.

The output from the program is shown in table B.1. The source code is listed in table B.2. Sum1 is the result of method i and Sum2 is the result of method ii. Sum3 is the result of method i but here the whole integral is solved numerically (in the two previous sums the first part of the integral is solved analytically). We see that the three methods agrees to the sixth significant digit, showing no sign of tending to zero if the number of gridpoints is doubled. In this way we can conclude that the gravitational mass is not zero in the limit $m \to 0$, and in general the gravitational mass is different from the mass parameter m.

Table B.2: The source code.

```
from numpy import linspace, pi, sin, cos
from random import random

# Numerical parameters:
N = 1001
theta = linspace(0, pi, N)
dtheta = pi/(N-1)

# Physical parameters:
a = 0.1 # Acceleration parameter
R = 2.5 # Radius of shell
print "Running with N =", N
```

```
print "a =", a, " R =", R

# Initializing sums:
sum = 0.
partialsum1 = 0.
partialsum2 = 0.

# Solving:
for i in xrange(N):

    # Finding thetaplus numerically:
    thetaplus = 0
    while (thetaplus + a*R*sin(thetaplus)) <= \
                                              theta[i]:
        thetaplus += dtheta
    while (thetaplus + a*R*sin(thetaplus)) >= \
                                              theta[i]:
        thetaplus -= dtheta/10.
    while (thetaplus + a*R*sin(thetaplus)) <= \
                                              theta[i]:
        thetaplus += dtheta/100.

    # To avoid a systematic error due to always
    # choosing a thetaplus-value slightly above the
    # correct value:
    if random() < 0.5:
        thetaplus -= dtheta/100.

    # Solve the whole integral in (B.2) numerically:
    sum += (a*R + cos(thetaplus)) * sin(theta[i])\
                                              *dtheta

    # Solve only the last term in (B.2) numerically:
    partialsum1 += cos(thetaplus)*sin(theta[i])\
                                              *dtheta

    # Solve the last term in (B.2) numerically with
    # a change of variables (use discretization of
    # theta for thetaplus):
    partialsum2 += cos(theta[i])*sin(theta[i] + \
                    a*R*sin(theta[i]))*\
                    (1+a*R*cos(theta[i]))*dtheta

```

```
print "Sum1 =", (partialsum1 +2*a*R)
print "Sum2 =", (partialsum2 +2*a*R)
print "Sum3 =", sum
```

Appendix C

Energy conditions

C.1 The weak energy condition

The weak energy condition is

$$T_{\mu\nu}u^{\mu}u^{\nu} \geq 0 \;\; \forall \;\text{timelike}\, u^{\mu} \,. \tag{C.1.1}$$

Since T_{rr} is vanishing we can translate this to

$$S_{ab}u^{a}u^{b} \geq 0 \;\; \forall \;\text{timelike}\, u^{a} \;\text{in}\; \Sigma \,. \tag{C.1.2}$$

A general timelike vector u^a in Σ has the form

$$u^{a} = (c, d, e) \,, \;\text{where}\; c^2 > d^2 + e^2 \,. \tag{C.1.3}$$

We have

$$\begin{aligned} 0 \leq S_{ab}u^{a}u^{b} &= h_{ac}S^{c}{}_{b}u^{a}u^{b} \\ &= S^{c}{}_{b}u_{c}u^{b} \,. \end{aligned} \tag{C.1.4}$$

For a perfect fluid with diagonal induced energy-momentum tensor this gives

$$\begin{aligned} 0 \leq S^{\tau}{}_{\tau}(-c^2) + S^{\vartheta}{}_{\vartheta}d^2 + S^{\varphi}{}_{\varphi}e^2 &= -S^{\tau}{}_{\tau}c^2 + S^{\vartheta}{}_{\vartheta}(d^2 + e^2) \\ &< -S^{\tau}{}_{\tau}c^2 + S^{\vartheta}{}_{\vartheta}c^2 \,, \end{aligned} \tag{C.1.5}$$

$$\Downarrow$$

$$0 < -S^{\tau}{}_{\tau} + S^{\vartheta}{}_{\vartheta} \,. \tag{C.1.6}$$

Or

$$S^{\tau}{}_{\tau} - S^{\vartheta}{}_{\vartheta} < 0 \,. \tag{C.1.7}$$

For the special case of the timelike vector $u^a = (1,0,0)$ (C.1.4) reduces to

$$S^{\tau}{}_{\tau} \leq 0\,. \tag{C.1.8}$$

These two last inequalities constitute the weak energy condition for perfect fluid shells with a diagonal induced energy-momentum tensor.

C.2 The dominant energy condition

The dominant energy condition is that for every timelike or null vector u^μ, the vector $-T^{\nu}{}_{\mu} u^\mu$ must be timelike or null. Again this translates to that for every timelike or null vector u^a, $-S^{b}{}_{a} u^a$ must also be timelike or null. Written out:

$$\left(-S_a{}^b u_b\right)\left(-S^a{}_b u^b\right) \leq 0\,. \tag{C.2.1}$$

This time, u^a can be either timelike or null:

$$u^a = (c,d,e)\,,\ \text{where } c^2 \geq d^2 + e^2\,. \tag{C.2.2}$$

When the induced metric tensor is diagonal, we have for the diagonal mixed components of the induced energy-momentum tensor (no sum over a):

$$S_a{}^a = h_{ab} h^{ac} S^b{}_c = S^a{}_a\,. \tag{C.2.3}$$

Using this (and $S^{\vartheta}{}_{\vartheta} = S^{\varphi}{}_{\varphi}$) in (C.2.1) yields:

$$\left(-S_a{}^b u_b\right)\left(-S^a{}_b u^b\right) = \left(S^{\tau}{}_{\tau} c - S^{\vartheta}{}_{\vartheta} d - S^{\varphi}{}_{\varphi}\right)\left(-S^{\tau}{}_{\tau} c - S^{\vartheta}{}_{\vartheta} d - S^{\varphi}{}_{\varphi}\right) \tag{C.2.4}$$

$$= -\left(S^{\tau}{}_{\tau}\right)^2 c^2 + \left(S^{\vartheta}{}_{\vartheta}\right)^2 \left(d^2 + 2de + e^2\right) \leq 0\,. \tag{C.2.5}$$

Or

$$\left(S^{\tau}{}_{\tau}\right)^2 c^2 \geq \left(S^{\vartheta}{}_{\vartheta}\right)^2 \left(d^2 + 2de + e^2\right)\,. \tag{C.2.6}$$

The right hand side is largest for the null case $c^2 = d^2 + e^2$:

$$\left(S^{\tau}{}_{\tau}\right)^2 c^2 \geq \left(S^{\vartheta}{}_{\vartheta}\right)^2 \left(c^2 + 2de\right)\,. \tag{C.2.7}$$

We can always rotate the coordinate system so that either d or e is zero. Then we get

$$\left(S^{\tau}{}_{\tau}\right)^2 \geq \left(S^{\vartheta}{}_{\vartheta}\right)^2\,, \tag{C.2.8}$$

$$\Downarrow$$

$$\left|S^{\tau}{}_{\tau}\right| \geq \left|S^{\vartheta}{}_{\vartheta}\right|\,. \tag{C.2.9}$$

This is the dominant energy condition for a perfect fluid shell with diagonal induced metric and energy-momentum tensor.

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