ON OPTIMAL EXERCISE OF SWING OPTIONS IN ELECTRICITY MARKETS

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Abstract. We study the optimal exercise of a swing option in electricity markets. To this end, we set up a model in terms of a stochastic control problem. In this model, the option can be exercised in continuous time and is subject to a total volume constraint. We analyze some fundamental properties of the model and carry out a numerical analysis. Finally, the results are illustrated numerically.

1. Introduction

Swing options are contracts sold on deregulated electricity markets[1]. These contracts are designed to address some of the problems caused by the non-storability of electricity. Abrupt fluctuations in generation and demand cause spikes in prices and swing options can be used to hedge against such risk. Typically, a swing option consists of two parts, namely the forward and the swing part. The forward part is a conventional forward contract guaranteeing the delivery of electricity on a given period for a given price. The swing part is a multiple strike option with a given number of exercises (i.e, swings). When exercised, the swing part gives the holder (issuer) a right to buy (sell) electricity for a given strike price. This form of contract yields a hedge against the spikes, which is quite fitting in comparison to, say, multiple American or European options.

We study in this paper the swing part of the contract and henceforth call it the swing option. From the modeling point of view, the pricing of a swing option can be seen as a particular type of a stochastic control problem. One way of formalizing the valuation problem is to write it as a multiple strike American or Bermudan option. Here, the admissible exercise policies are of sequential type. Methodologies used to tackle these problems vary. In [3], the authors study the multiple strike American option formulation in a fairly general setting using martingale theory and propose a numerical scheme based on Malliavin calculus. The Bermudan formulation is adopted in, e.g., [8] and [11]. In [8], the authors use a multi-level lattice method to analyze the valuation of a swing option on natural gas whereas [11] relies on partial integro-differential equations (PIDE) and Fourier techniques in the analysis of an affine jump diffusion pricing model. A jump diffusion model is compared to regime switching and negative inverse Gaussian models in [7] using a Monte Carlo algorithm. In [9], the authors apply the Schwartz model to the oil price and carry out a comparison between a least-square Monte Carlo and a finite difference method. The swing option pricing can also be

[1]Swing contracts are sold also on other commodity markets. In this paper, we will focus on the electricity market.

studied via the dual formulation of the optimization problem. This approach is adopted in, e.g., [1] and [2], where the dual problem is studied using a combination of martingale and Monte Carlo simulation techniques. In [4] and [5] the authors propose and study a diffusion model, where, under some technical conditions on the diffusion parameters, the pricing problem can be solved via a set of quasi-variational inequalities.

In the study at hand, we will formalize the swing option such that it can be exercised in continuous time. Now, the total amount of the swings used follows a non-decreasing and continuous process (a monotone follower). The upper limit on swing rights is now set in terms of a total volume constraint. This formulation has appeared in the literature before. In [10], the author studies hedging of a swing option using this model under a liquid forward contract and European call options market. A viscosity solution theory for this model is developed in [15]. A similar model appears also in [12] for swing option pricing and in [14] for gas storage valuation. These studies are concerned mostly with the issue of pricing. However, it is important, especially from the holders point of view, to understand how the option should be exercised optimally. In this study, we address this question from both the analytical and numerical side. We set up a diffusion model for the electricity price. Based on this, we formulate the value process as the maximal expected present value of the cumulative exercise payoff which is attainable using an admissible exercise policy. Using Hamilton-Jacobi-Bellmann (HJB) techniques, we propose an economically reasonable characterization of the optimal exercise policy in terms of the marginal exercise payoff and lost option value. Furthermore, we study some basic quantitative properties of the marginal lost option value in order to shed more light on the optimal policy. We present also a numerical analysis of the problem for geometric Ornstein-Uhlenbeck processes. Our numerical scheme is based on the finite difference method coupled with a detailed discussion on the delicate issue of boundary conditions.

The reminder of the study is organized as follows. In Section 2 we formulate the pricing model as a monotone follower problem. In Section 3 we present and prove our main analytical results on the pricing problem. Section 4 is devoted to the numerical analysis of the model and in Section 5 we carry out numerical illustrations. The study is concluded in Section 6.

2. The pricing model

Before going into the formal description of the pricing model, we set up a stochastic modeling framework for the underlying electricity price. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, be a complete filtered probability space satisfying the usual conditions. We assume that W is a Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ and taking values in $\mathbf{R}$. We assume that the electricity price follows a strongly unique solution of the Itô stochastic differential equation

$$dP_t = \mu(t, P_t)dt + \sigma(t, P_t)dW_t \tag{2.1}$$

with $P_0 = p$, where the functions μ and σ are sufficiently well behaving Lipschitz-continuous functions – for details, see, e.g., [13], pp. 66.

Having the underlying dynamics set up, we formulate the pricing problem as a bounded variation control problem with total volume constraints. This control problem is supposed to approximate a class of swing options commonly traded in electricity markets, where the holder has the right to buy a volume of electricity at specified times over a time interval (a year, say), and where the exercise times constitute a significant portion of time interval. These options are often coined flexible load contracts. The class of admissible exercise policies consists of processes Z, which admit the representation

$$Z_t = \int_0^t u_s ds, \tag{2.2}$$

where u is progressively measurable with respect to $\mathbb{F}$ and satisfies the constraints $u_s \in [0, \bar{u}]$ for all $s \in [0, T]$ and

$$Z_T \leq M, \tag{2.3}$$

for some $M > 0$. Denote as $\mathcal{U}$ the class of processes u giving rise to admissible exercise policies Z. Let

$$\mathcal{S} := [0, T] \times [0, M] \times \mathbf{R}$$

and define the *value functional* V on $\mathcal{S}$ as the remaining discounted option value, i.e., let

$$V(t, Z_t, P_t) = \sup_{u \in \mathcal{U}} \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s ds \middle| \mathcal{F}_t\right], \tag{2.4}$$

with the final value $V(T, Z_T, P_T) = 0$. Here, $r > 0$ is the constant rate of discounting and $K > 0$ is the strike price. One objective of this paper is to study a potentially unique optimal exercise policy $Z^* = \int_0^{\cdot} u_s^* ds$ for which

$$\sup_{u \in \mathcal{U}} \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s ds \middle| \mathcal{F}_t\right] = \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s^* ds \middle| \mathcal{F}_t\right],$$

for all $t \in [0,T]$. We remark that the value functional V admits an alternative representation which is explicit in terms of Z. By applying the Itô formula to the process $t \mapsto e^{-rt}(P_t - K)Z_t$, we find that

$$\begin{aligned} e^{-rT}(P_T - K)Z_T = e^{-rt}(P_t - K)Z_t &+ \int_t^T e^{-rs}(\mu(s,P_s) - r(P_s - K))Z_s ds \\ &+ \int_t^T e^{-rs}(P_s - K)u_s ds + \int_t^T e^{-rs}\sigma(s,P_s)Z_s dW_s, \end{aligned} \tag{2.5}$$

and, consequently, the value V can be rewritten as

$$V(t,Z_t,P_t) = (K - P_t)Z_t + \sup_{u \in \mathcal{U}} \mathbf{E}\left[e^{-r(T-t)}(P_T - K)Z_T + \int_t^T e^{-r(s-t)}(r(P_s - K) - \mu(s,P_s))Z_s ds \middle| \mathcal{F}_t \right]. \tag{2.6}$$

3. On the option price and optimal exercise boundary

Having the pricing problem set up, we start the analysis of the model. As the first case, we consider Problem (2.4) in the absence of an effective final volume constraint, i.e., the case $M \geq \bar{u}T$, where the total constraint dominates the largest possible amount that can be consumed during the lifetime of the contract.

Proposition 3.1. *In the absence of an effective final volume constraint, i.e., when $M \geq \bar{u}T$, the optimal exercise policy $Z^* = \int_0^{\cdot} u_s^* ds$ is given by*

$$u_t^* = \begin{cases} \bar{u}, & P_t > K, \\ 0, & P_t \leq K, \end{cases} \tag{3.1}$$

for all $t \in [0,T]$.

Proof. Let $u \in \mathcal{U}$ and $t \in [0,T]$. Then we observe that

$$\begin{aligned} \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s ds \middle| \mathcal{F}_t \right] &= \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s \mathbf{1}_{\{P_s \leq K\}} ds \middle| \mathcal{F}_t \right] \\ &+ \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s \mathbf{1}_{\{P_s > K\}} ds \middle| \mathcal{F}_t \right] \\ &\leq \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s^* ds \middle| \mathcal{F}_t \right]. \end{aligned}$$

On the other hand, since u^* gives rise to an admissible exercise policy, the claimed result follows. □

Proposition 3.1 states an intuitively obvious result. Indeed, if there is no effective limit on the total volume, then it is optimal to use the option whenever the swing yields a positive payoff.

Lemma 3.2. *In the absence of an effective final volume constraint, i.e., when $M \geq \bar{u}T$, the marginal value $V_z \equiv 0$.*

Proof. Let $M \geq \bar{u}T$ and $t \in [0,T]$. We proved in Proposition 3.1 that the option value can be written as

$$V(t, Z_t, P_t) = \bar{u}\mathbf{E}\left[\int_t^T e^{-rt}(P_s - K)\mathbf{1}_{\{P_s - K > 0\}}ds \middle| \mathcal{F}_t\right].$$

Since the right hand side of this expression does not depend on the variable Z, the claimed result follows. □

Lemma 3.2 states another intuitively obvious result. In fact, if there is no effective limit on the total volume, then the usage of the option does not decrease the value as it depends only on the time to maturity and the price of the electricity. On the other hand, when $M < \bar{u}T$, we have the following result.

Proposition 3.3. *In the presence of an effective total volume constraint, i.e., when $M < \bar{u}T$, the marginal value $V_z \leq 0$.*

Proof. First, recall the alternative characterization of V from (2.6). Using this, we express the marginal value V_z as

(3.2)

$$\begin{aligned}
V_z(t, Z_t, P_t) &= (K - P_t) \\
&+ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Bigg\{\sup_{u^\varepsilon} \mathbf{E}\left[e^{-r(T-t)}(P_T - K)Z_T^\varepsilon + \int_t^T e^{-r(s-t)}(r(P_s - K) - \mu(s, P_s))Z_s^\varepsilon ds \middle| \mathcal{F}_t\right] \\
&- \sup_u \mathbf{E}\left[e^{-r(T-t)}(P_T - K)Z_T + \int_t^T e^{-r(s-t)}(r(P_s - K) - \mu(s, P_s))Z_s ds \middle| \mathcal{F}_t\right]\Bigg\} \\
&= (K - P_t) \\
&+ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Bigg\{\underbrace{\sup_{u^\varepsilon} \mathbf{E}\left[e^{-r(T-t)}(P_T - K)Z_T^\varepsilon + \int_t^T e^{-r(s-t)}(r(P_s - K) - \mu(s, P_s))Z_s^\varepsilon ds \middle| \mathcal{F}_t\right]}_{:=I_\varepsilon} \\
&- \underbrace{\sup_u \mathbf{E}\left[e^{-r(T-t)}(P_T - K)(Z_T + \varepsilon) + \int_t^T e^{-r(s-t)}(r(P_s - K) - \mu(s, P_s))(Z_s + \varepsilon) ds \middle| \mathcal{F}_t\right]}_{:=I_0}\Bigg\} \\
&+ \mathbf{E}\left[e^{-r(T-t)}(P_T - K) + \int_t^T e^{-r(s-t)}(r(P_s - K) - \mu(s, P_s)) ds \middle| \mathcal{F}_t\right],
\end{aligned}$$

where u^ε, $\varepsilon \geq 0$, varies over admissible exercise rates under the condition that the associated cumulative control is at time t in state $Z_t + \varepsilon$ and Z^ε denotes the resulting cumulative control on the interval $(t, T]$. To proceed, consider an arbitrary admissible $s \mapsto Z_s^\varepsilon$ on $(t, T]$ and define an associated $\check{Z}$ as

$$\check{Z}_s = Z_s^\varepsilon - \varepsilon, \tag{3.3}$$

for all $s \in (t, T]$. Using (3.3), we can map each functional I_ε for arbitrary Z^ε to the functional I_0 given by $\check{Z}$, i.e., the functionals I_ε can be embedded into the set of functionals I_0. This allows us to identify the optimization problem $\sup_{u^\varepsilon} I_\varepsilon$ as a subproblem of $\sup_u I_0$ in the sense that the optimization is done over a subset of admissible controls. Thus, we conclude that

$$\sup_{u^\varepsilon} I_\varepsilon - \sup_u I_0 \leq 0,$$

and, consequently, that

$$V_z(t, Z_t, P_t) \leq (K - P_t) + \mathbf{E}\left[e^{-r(T-t)}(P_T - K) + \int_t^T e^{-r(s-t)}(r(P_s - K) - \mu(s, P_s))ds \middle| \mathcal{F}_t\right] = 0,$$

where the final equality follows by first applying the Itô formula to $t \mapsto e^{-rt}(P_t - K)$ and then conditioning up to time t. □

Proposition 3.3 states yet another intuitively obvious result, namely that in the presence of an effective total volume constraint, the usage of the swing option will lower its value.

Proposition 3.4. *The value V is concave in z.*

Proof. Denote as Z_t^1 and Z_t^2 two different states of the control Z at time $t \geq 0$ and let u^1 and u^2 be arbitrary admissible exercise policies corresponding to these states. Define the process $\tilde{u}$ on $[0, T]$ as

$$\tilde{u}_s := \frac{u_s^1 + u_s^2}{2}.$$

First, we observe that $\tilde{u}_s \in [0, \bar{u}]$ for all s and $\int_0^T \tilde{u}_s ds \leq M$. Moreover, $\tilde{u}$ corresponds to the case where the state of the cumulative control Z at time t is $\frac{Z_t^1 + Z_t^2}{2}$. Since this construction holds for arbitrary u^i, we conclude that

$$V\left(t, \frac{Z_t^1 + Z_t^2}{2}, P_t\right) - \frac{V(t, Z_t^1, P_t) + V(t, Z_t^2, P_t)}{2} = \sup_u \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s ds \middle| \mathcal{F}_t\right]\Bigg|_{Z_t = \frac{Z_t^1 + Z_t^2}{2}} -$$
$$\frac{1}{2}\left\{\sup_{u^1} \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s^1 ds \middle| \mathcal{F}_t\right]\Bigg|_{Z_t = Z_t^1} + \sup_{u^2} \mathbf{E}\left[\int_t^T e^{-r(s-t)}(P_s - K)u_s^2 ds \middle| \mathcal{F}_t\right]\Bigg|_{Z_t = Z_t^2}\right\} \geq 0,$$

proving the claimed result. □

In economical terms, Proposition 3.4 states that exercising the option will lower the option value more for a low level of option reserve than for high level.

3.1. **Necessary conditions.** To study Problem (2.4) under an effective total volume constraint, we first derive the associated HJB-equation. Assume first that the optimal value exists. The starting point of the derivation is the Bellman principle of optimality. This principle can now be expressed as follows: for all times $0 \le t < w \le T$, the condition

$$V(t, Z_t, P_t) = \sup_u \mathbf{E}\left[\int_t^w e^{-r(s-t)}(P_s - K)u_s ds + e^{-r(w-t)}V(w, Z_w, P_w)\middle|\mathcal{F}_t\right] \tag{3.4}$$

must hold, where the supremum is taken over $u \in \mathcal{U}$ such that $\int_0^t u_s ds = Z_t \le M$. First, rewrite (3.4) as

$$\sup_u \mathbf{E}\left[\int_t^w e^{-rs}(P_s - K)u_s ds + \left(e^{-rw}V(w, Z_w, P_w) - e^{-rt}V(t, Z_t, P_t)\right)\middle|\mathcal{F}_t\right] = 0. \tag{3.5}$$

To proceed, we assume that V is smooth enough, i.e., in $C^{1,1,2}(\mathcal{S})$, and use the Itô formula to the process $t \mapsto e^{-rt}V(t, Z_t, P_t)$. Since $dZ_t = u_t dt$, we find that

$$\begin{aligned} e^{rt}d(e^{-rt}V(t, Z_t, P_t)) &= (V_t(t, Z_t, P_t) - rV(t, Z_t, P_t) + u_t V_z(t, Z_t, P_t))\, dt \\ &\quad + V_p(t, Z_t, P_t)\,(\mu(t, P_t)dt + \sigma(t, P_t)dW_t) + \frac{1}{2}\sigma^2(t, P_t)V_{pp}(t, Z_t, P_t)dt. \end{aligned}$$

Using this, we can rewrite (3.5) as

$$\sup_u \mathbf{E}\left[\frac{1}{w-t}\int_t^w e^{-rs}((P_s - K)u_s + (\mathcal{L} - r)V(s, Z_s, P_s) + u_s V_z(s, Z_s, P_s))ds + (Y_w - Y_t)\middle|\mathcal{F}_t\right] = 0, \tag{3.6}$$

where the linear operator $\mathcal{L}$ is defined on $C^{1,1,2}(\mathcal{S})$ as

$$\mathcal{L}F(t, z, p) = F_t(t, z, p) + \mu(t, p)F_p(t, z, p) + \frac{1}{2}\sigma^2(t, p)F_{pp}(t, z, p), \tag{3.7}$$

and the local martingale Y is defined as

$$Y_t = \int_0^t e^{-rs}\sigma(s, P_s)V_p(s, Z_s, P_s)dW_s.$$

We assume that the functions σ and V are regular enough for the process Y to be a true martingale. Then we know that

$$\mathbf{E}\left[\frac{1}{w-t}\int_t^w e^{-rs}((P_s - K)u_s + (\mathcal{L} - r)V(s, Z_s, P_s) + u_s V_z(s, Z_s, P_s))ds\middle|\mathcal{F}_t\right] \le 0,$$

for each admissible u. Furthermore, if there exists an optimal admissible policy u^* with which the supremum is attained, then

$$\mathbf{E}\left[\frac{1}{w-t}\int_t^w e^{-rs}((P_s-K)u_s^*+(\mathcal{L}-r)V(s,Z_s,P_s)+u_s^*V_z(s,Z_s,P_s))ds\middle|\mathcal{F}_t\right]=0.$$

Now, under appropriate regularity conditions on V, see e.g., [6], we can pass to the limit $w\to t$ in (3.6) and, consequently, end up into the HJB-equation

$$V_t(t,z,p)+\frac{1}{2}\sigma^2(t,p)V_{pp}(t,z,p)+\mu(t,p)V_p(t,z,p)-rV(t,z,p)+\sup_u\{u(t)(p-K+V_z(t,z,p))\}=0, \tag{3.8}$$

where u varies over the set of functions defined on $[0,T]$ satisfying the conditions $0\le u(t)\le\bar{u}$, $\int_0^t u(s)ds=z$, and $\int_0^T u(t)dt\le M$. We remark that if the value V does not satisfy the required smoothness conditions, it can still be identified as a viscosity solution of (3.8) – for details, see [15].

Intuitively, we can interpret the quantity V_z in equation (3.8) as the (marginal) lost option value. From (3.8) it seems clear that the boundary $p-K=-V_z(t,z,p)$ plays a key role when determining the optimal exercise rule. Indeed, if $p-K>-V_z(t,z,p)$ for $t\in[0,T]$, that is, payoff dominates the lost option value, then

$$\sup_u\{u(t)(p-K+V_z(t,z,p))\}=\bar{u}(p-K+V_z(t,z,p)).$$

On the other hand, if $p-K\le-V_z(t,z,p)$, then $\sup_u\{u(p-K+V_z(t,z,p))\}=0$. Define now the admissible exercise policy $\hat{Z}=\int_0^{\cdot}\hat{u}_t dt$ via

$$\hat{u}_t=\begin{cases}\bar{u}, & P_t-K>-V_z(t,Z_t,P_t),\\ 0, & P_t-K\le-V_z(t,Z_t,P_t),\end{cases} \tag{3.9}$$

for all $t\in[0,T]$. We remark that when $M\ge\bar{u}T$, the marginal value $V_z\equiv 0$ (see Lemma 3.2) and, consequently, the exercise policy $\hat{Z}$ coincides with the optimal policy Z^* described in Proposition 3.1. In the presence of an effective total volume constraint, we proved in Proposition 3.3 that the marginal value V_z is non-positive. Thus, the introduction of the total volume constraint to the model postpones the optimal exercise of the swing option. Furthermore, we note that if V satisfies (3.8), then

$$\hat{u}(t)\geqq 0\Longleftrightarrow(\mathcal{L}-r)V(t,z,p)\leqq 0,$$

for all $(t,z,p)\in\mathcal{S}$.

3.2. **Sufficient conditions.** In the previous section, we derived necessary conditions for the option value V defined in (2.4). As a result, we proposed the candidate $\hat{u}$ for the optimal exercise policy. In this section, we study the conditions under which this candidate is indeed optimal. A set of such conditions are stated in the following verification theorem. This verification theorem also ensures that the HJB-equation (3.8) is the correct one for the original optimization problem (2.4).

Theorem 3.5. *Assume that a function $F : \mathcal{S} \to \mathbf{R}$ satisfies the following conditions*

(i) $F(T, \cdot, \cdot) \equiv 0$ *and* $F \in C^{1,1,2}(\mathcal{S})$,

(ii) $(\mathcal{L} - r)F(t, z, p) + u(t)\left(F_z(t, z, p) + (p - K)\right) \leq 0$ *for all* $(t, z, p) \in \mathcal{S}$ *and* $u \in \mathcal{U}$, *where the operator* $\mathcal{L}$ *is defined in* (3.7),

(iii) *the process* $Y : t \mapsto \int_0^t e^{-rs}\sigma(s, P_s)F_p(s, Z_s, P_s)dW_s$ *is a martingale.*

Then $F(t, Z_t, P_t) \geq V(t, Z_t, P_t)$ *for all* t *and* ω. *In addition, if there exist an admissible* u^o *such that*

$$\begin{aligned} &(\mathcal{L} - r)F(t, z, p) + \sup_u \left(u(t)\left(F_z(t, z, p) + (p - K)\right)\right) = \\ &\qquad (\mathcal{L} - r)F(t, z, p) + u^o(t)\left(F_z(t, z, p) + (p - K)\right) = 0, \end{aligned} \tag{3.10}$$

for all $(t, z, p) \in \mathcal{S}$, *then* $u^o = u^*$ *and the function* F *coincides with the value function* V.

Proof. Let $u \in \mathcal{U}$ and $t \in [0, T]$. First, we find using the Itô formula to the process $t \mapsto e^{-rt}F(t, Z_t, P_t)$ that

$$d(e^{-rt}F(t, Z_t, P_t)) = e^{-rt}\left[(\mathcal{L} - r)F(t, Z_t, P_t) + u_t F_z(t, Z_t, P_t) + \sigma(t, P_t)F_p(t, Z_t, P_t)dW_t\right].$$

Using assumption (i) and conditioning up to time t, we find that

$$\begin{aligned} 0 = e^{-rt}F(t, Z_t, P_t) &+ \mathbf{E}\left[\int_t^T e^{-rs}(\mathcal{L} - r)F(s, Z_s, P_s)ds \middle| \mathcal{F}_t\right] \\ &+ \mathbf{E}\left[\int_t^T e^{-rs}F_z(s, Z_s, P_s)u_s ds \middle| \mathcal{F}_t\right] + \mathbf{E}\left[(Y_T - Y_t)\middle|\mathcal{F}_t\right]. \end{aligned}$$

Now, assumptions (ii) and (iii) imply that

$$0 \leq e^{-rt}F(t, Z_t, P_t) - \mathbf{E}\left[\int_t^T e^{-rs}(P_s - K)u_s ds \middle| \mathcal{F}_t\right], \tag{3.11}$$

for all ω, which is equivalent to the first claim.

Assume now that there exist an admissible u^o such that the condition (3.10) holds. Then with exactly the same calculation as above we find that

$$0 = e^{-rt}F(t, Z_t, P_t) - \mathbf{E}\left[\int_t^T e^{-rs}(P_s - K)u_s^o ds \middle| \mathcal{F}_t\right], \tag{3.12}$$

for all ω. □

4. A numerical solution of the HJB-equation

In this section we present a numerical scheme for solving the HJB-equation (3.8). More precisely, we present a finite difference approximation of $V(t, z, p)$ which solves the equation

$$V_t(t,z,p) + \frac{\sigma^2}{2}V_{pp}(t,z,p) + \mu V_p(t,z,p) - rV(t,z,p) + \sup_u \{u(p - K + V_z(t,z,p)))\} = 0, \quad (t,z,p) \in \mathcal{S}\,, \tag{4.1}$$

with given boundary and terminal conditions. These conditions will be stated later.

In the numerical examples, we assume that the underlying price process P_t follows a geometric Ornstein-Uhlenbeck process. However, the numerical approach applied here will be valid for a much more general class of diffusion process. Only the boundary conditions on the truncated boundary must be changed, since, as we shall see, they will depend crucially on the properties of the price process chosen. In this section we will first give a description of the boundary conditions and then describe how to compute a finite difference approximation of (4.1).

4.1. **The boundary conditions.** In order for the HJB–equation to be well posed we need sufficient boundary conditions. The terminal condition at $t = T$ and the condition for no optionality left $z = M$ are both zero since the option contract in those cases are worthless. We have,

$$V(T, z, p) = 0, \quad (z, p) \in [0, M] \times \mathbf{R}, \qquad \text{and} \qquad V(t, M, p) = 0, \quad (t, p) \in [0, T] \times \mathbf{R}. \tag{4.2}$$

The last boundary conditions which need to be imposed are at the endpoints in the $p-$direction. Dependent on the upper and lower bounds on the underlying process these endpoints could be unbounded. The Ornstein–Uhlenbeck process is unbounded from below and above, but the exponential Ornstein–Uhlenbeck process is unbounded from above and bounded by zero from below.

We will in the following assume that we have chosen p_{min} and p_{max} so small and large, respectively, that the process P_t is unlikely to be outside the interval (p_{min}, p_{max}). For the exponential Ornstein-Uhlenbeck process, one could naturally choose $p_{min} = 0$, but we keep the general view here. We will assume that p_{min}

is so small that the dominating behavior of the log-price process $\ln(P_t)$ until the time of maturity T is to increase due to mean-reversion, while p_{max} is so large that the process is dominated by a decreasing behavior.

The idea now is to use these assumptions to determine how the holder will optimally use her optionality, i.e. determine the optimal control u_s. Inserting this into (2.4), we can compute

$$V(t, Z_t, P_t) = \mathbf{E}\left[\int_t^T e^{-r(s-t)} P_s u_s ds \middle| \mathcal{F}_t\right], \tag{4.3}$$

and thereby obtain boundary conditions for our HJB-equation also in the price direction.

The optimal operational behavior when $p = p_{max}$ is to use as much as possible of the option until $z = M$. This is because p_{max} is much larger than the long time expectation and the price is then expected to decrease until maturity. Thus

$$u_s = \begin{cases} \bar{u}, & s \in (t, t + \frac{M-z}{\bar{u}}) \\ 0, & s \in (t + \frac{M-z}{\bar{u}}, T) \end{cases}. \tag{4.4}$$

Then the price can be calculated

$$V(t, Z_t, p_{max}) = \bar{u}\mathbf{E}\left[\int_t^{t+\frac{M-z}{\bar{u}}} e^{-r(s-t)} P_s ds \middle| \mathcal{F}_t\right]. \tag{4.5}$$

Similarly for the case of $p = p_{min}$, the optimal behavior of the operator is to wait as long as possible before exercising. This is because p_{min} is much smaller than the long time expectation and the price is then expected to increase until maturity. Thus

$$u_s = \begin{cases} 0, & s \in (t, T - \frac{M-z}{\bar{u}}) \\ \bar{u}, & s \in (T - \frac{M-z}{\bar{u}}, T) \end{cases}, \tag{4.6}$$

and the price can be calculated by

$$V(t, Z_t, p_{min}) = \bar{u}\mathbf{E}\left[\int_{T-\frac{M-z}{\bar{u}}}^{T} e^{-r(s-t)} P_s ds \middle| \mathcal{F}_t\right]. \tag{4.7}$$

In Appendix A we compute the expectations in (4.5) and (4.7) explicitly for the case of P_t being an exponential Ornstein–Uhlenbeck process. The conclusion of these derivations is Dirichlet boundary conditions on the form

$$V(t, z, p_{min}) = g_1(t, z), \quad V(t, z, p_{max}) = g_2(t, z),$$

where g_1 and g_2 are defined in Appendix A. It may be desirable to have Neumann conditions instead, and they will take the form

$$V_p(t,z,p_{min}) = \hat{g}_1(t,z), \quad V_p(t,z,p_{max}) = \hat{g}_2(t,z),$$

for function $\hat{g}_1$ and $\hat{g}_2$ derived in Appendix A.

The computations in Appendix A are valid only if the underlying price follows an exponential Ornstein–Uhlenbeck process. Similar calculations might be harder to carry out for other price processes. In such cases Monte Carlo simulations might be used to calculate (4.5) and (4.7).

In other works studying swing options, a second order boundary condition is used on the truncated boundary. In e.g. [11] the condition

$$V_{pp}(t,z,p_{min}) = 0, \quad V_{pp}(t,z,p_{max}) = 0. \tag{4.8}$$

We have two objections to such boundary conditions. The first is that it seems a little unmotivated, and in general it should not be approximately correct for large and small values of p_{max} and p_{min}. This could be fixed by doing calculations similar to those in Appendix A, or a corresponding Monte Carlo simulation. The other problem with such a second order boundary condition is that even though a specific numerical scheme might be stable, the problem is not well posed from the PDE point of view. The scheme presented below, with a discrete version of (4.8) is stable and is implemented for comparison.

We point out that no boundary conditions are needed on the boundaries $z = 0$ and $z = \bar{u}t$. The reason for this is that the transport of (4.1) is towards these boundaries. Thus, in some sense, these boundary conditions are taken care of by the HJB–equation itself.

4.2. **The numerical scheme.** For the finite difference scheme we use a uniform grid in both $t-$ and $z-$directions defined as follows. The $t-$dimension is divided into N uniform intervals, i.e. $0 = t_0 < t_1 < \cdots < t_N = T$, where the interval length is $\Delta t = t_n - t_{n-1}$. The $z-$dimension is similarly divided into N_z uniform intervals, i.e. $0 = z_0 < z_1 < \cdots < z_{N_z} = M$, where the interval length is $\Delta z = z_i - z_{i-1}$. In our numerical approximation of the value function, we truncate the p-direction to the interval $p \in (p_{min}, p_{max})$. To be able to choose p_{min} small and p_{max} large without increasing the number of grid points too much we use an adaptive grid in the $p-$direction, i.e. $p_{min} = p_0 < p_1 < \cdots < p_{N_p} = p_{max}$, where the interval length $\Delta p_j = p_j - p_{j-1}$ may vary.

The numerical scheme will be defined by a first order backward time-stepping scheme where we use a nearly second order approximation of the p-derivatives and a first order approximation of the z-derivative.

The terms including p-derivatives are solved implicitly in time, the other terms are solved explicitly. More precisely, let $V_{i,j}^n$ denote the approximation of $V(t_n, z_i, p_j)$. We then use the approximations

$$V_t(t_n, z_i, p_j) \approx \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta t},$$

$$V_z(t_n, z_i, p_j) \approx \frac{V_{i+1,j}^{n+1} - V_{i,j}^{n+1}}{\Delta z},$$

$$V_p(t_n, z_i, p_j) \approx \frac{1}{\Delta p_{j+1} + \Delta p_j} \left(-\frac{\Delta p_{j+1}}{\Delta p_j} V_{i,j-1}^n + \left(\frac{\Delta p_{j+1}}{\Delta p_j} - \frac{\Delta p_j}{\Delta p_{j+1}} \right) V_{i,j}^n + \frac{\Delta p_j}{\Delta p_{j+1}} V_{i,j+1}^n \right), \tag{4.9}$$

$$V_{pp}(t_n, z_i, p_j) \approx \frac{2}{\Delta p_{j+1} + \Delta p_j} \left(\frac{1}{\Delta p_j} V_{i,j-1}^n + \left(-\frac{1}{\Delta p_{j+1}} - \frac{1}{\Delta p_j} \right) V_{i,j}^n + \frac{1}{\Delta p_{j+1}} V_{i,j+1}^n \right); \tag{4.10}$$

see Appendix B for a further analysis of these approximations. Inserting these into (4.1) and approximating $V(t_n, z_i, p_j) \approx V_{i,j}^n$ and $p \approx p_j$ we get

$$\begin{aligned} \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta t} &+ \frac{(\sigma_j^n)^2}{\Delta p_{j+1} + \Delta p_j} \left(\frac{1}{\Delta p_j} V_{i,j-1}^n + \left(-\frac{1}{\Delta p_{j+1}} - \frac{1}{\Delta p_j} \right) V_{i,j}^n + \frac{1}{\Delta p_{j+1}} V_{i,j+1}^n \right) \\ &+ \frac{\mu_j^n}{\Delta p_{j+1} + \Delta p_j} \left(-\frac{\Delta p_{j+1}}{\Delta p_j} V_{i,j-1}^n + \left(\frac{\Delta p_{j+1}}{\Delta p_j} - \frac{\Delta p_j}{\Delta p_{j+1}} \right) V_{i,j}^n + \frac{\Delta p_j}{\Delta p_{j+1}} V_{i,j+1}^n \right) \\ &- r V_{i,j}^n + u_{i,j}^{n+1} \left(p_j - K + \frac{V_{i+1,j}^{n+1} - V_{i,j}^{n+1}}{\Delta z} \right) = 0, \quad \forall i, j, n \end{aligned} \tag{4.11}$$

where $u_{i,j}^{n+1} = H(p_j - K + \frac{V_{i+1,j}^{n+1} - V_{i,j}^{n+1}}{\Delta z})\bar{u}$ and $H(\cdot)$ denotes the Heaviside function.

This scheme is numerically stable for any choice of the Δp_j's. However, the CFL condition requires the following relation on Δz and Δt for the numerical scheme to be stable

$$\Delta t \bar{u} \leq \Delta z\,.$$

In Appendix B we study the truncation error of the numerical scheme, and it is seen that truncation error converges to zero as the discretization parameters goes to zero. Thus the scheme is consistent. Now, according to the Lax–Richtmyer Equivalence Theorem, the finite difference approximation will converge to the original partial differential equation since the scheme is both consistent and stable.

The numerical scheme defined in (4.11) should be solved recursively and backward in time. When solving (4.11) for a given time level n, all values with super-index $n+1$ are known. Since the discretization of V_z is done at time level $n+1$, there is no implicit coupling in the $z-$direction in the numerical scheme. This means that we only have to solve a tridiagonal linear system for each value of n and i.

As described in the terminal condition below, the last time level is known in advance, i.e. $V_{i,j}^N$, $\forall i, j$ are known initially. Further, the boundary conditions give the solution at the edges of the grid, $V_{N_z,j}^n$, $\forall j, n$,

$V^n_{i,0}$, $\forall i, n$, and V^n_{i,N_p}, $\forall i, n$. Together with (4.11), this is sufficient to find the solution in all points given by $i = 0, 1, \ldots N_z$, $j = 0, 1, \ldots N_p$ and $n = 0, 1, \ldots N$.

An important observation is that this scheme corresponds to a finite difference approximation of a Bermudan swing option, that is, a swing option with finite discrete exercise times (c.f. [11]) and where the number of exercise times is N. As mentioned earlier in the introduction, a continuous-time formulation of the control can be considered as an approximation of a contract with a very high number of exercise rights. Thus, we solve in fact numerically the options we are in practice interested in. On the other hand, the continuous-time framework puts us in a situation where nice stochastic control theory may be applied.

The terminal and boundary conditions described in (4.2) are incorporated into the scheme by

$$V^N_{i,j} = 0, \quad \forall i, j, \qquad \text{and} \qquad V^n_{N_z,j} = 0, \quad \forall j, n\,.$$

On the truncated boundary the Dirichlet conditions are,

$$V^n_{i,0} = g_1(t_n, z_i), \quad \forall i, n, \qquad \text{and} \qquad V^n_{i,N_p} = g_2(t_n, z_i), \quad \forall i, n\,.$$

Choosing Neumann conditions instead, we will have

$$V_p(t_n, z_i, p_0) = \hat{g}_1(t_n, z_i), \quad \forall i, n, \qquad \text{and} \qquad V_p(t_n, z_i, p_{N_p}) = \hat{g}_2(t_n, z_i), \quad \forall i, n\,,$$

where $V_p(t_n, z_i, p_j)$ is approximated by (4.9). In Appendix C we present the complete numerical scheme with these boundary conditions incorporated.

5. Numerical experiments

In this section we present a numerical example analyzing a so-called flexible load contract (FLC). This type of contract is much used in the electricity market, and gives the holder the right to buy up to a certain volume of electricity at fixed prices over a large amount of exercise times in a period. In fact, in many circumstances these exercise dates can fill up more than 80-90% of the total amount of days in the period. Typical for many of these contracts in the market place is that $K = 0$, so the holder faces the problem of picking the most favorable prices at the provided exercise times. We run an experiment with $M = \frac{1}{2}$ and $\bar{u} = 1$, that is, a contract where the holder can buy 1MWh of electricity every year, but only 50% of the total time interval being set to one year $T = 1$. For simplicity, we choose the risk-free interest rate equal to zero, $r = 0$.

For the dynamics for P we use the exponential Ornstein–Uhlenbeck process

$$P_t = \exp(X_t)\,, \qquad dX_t = \kappa(\mu - X_t)dt + \sigma dW_t. \tag{5.1}$$

The parameters are shown in Table 1 The parameters κ and σ are approximately the same as the estimated

Table 1. The parameters in the exponential Ornstein–Uhlenbeck process

Parameters	κ	σ	μ
	0.4	0.55	3.5

parameters in the paper [16], and $\mu = \ln 40 - \frac{\sigma^2}{4\kappa} \approx 3.5$ corresponds to a long time expectation of 40 EUR/MWh.

In order to find a sufficiently fine grid for the numerical solution we have first solved the problem on a very fine mesh. This solution is then used as a reference for comparison. To define a criterion for determining what solution is accurate enough, we study the optimal exercise curves, c.f. Figure 2. We consider these curves as a function of z. Denote the numerical solution $P_h(z)$ and the "exact" solution $P_e(z)$. Then we say that the solution is accurate enough if the average error, in the L^1-norm sense, is less than 0.01 EUR/MWh, i.e.

$$\frac{1}{M}\int_0^M |P_h(z) - P_e(z)|dz < 0.01.$$

At Nordpool the prices are given with 2 digits.

In the presented experiments we have used a mesh defined by $N_p = 180$, $N_z = 225$ and $N = 450$. We create the grid for p to be sparse close to the endpoints and dense near the optimal exercise curves. In more details the grid is created the following way. Let $\{x_i\}_{i=1}^{N_p}$ be a uniform grid with $N_p = 180$ grid points. The grid $\{p_i\}_{i=1}^{N_p}$ is defined by $p_i = \exp(\hat{f}(x_i))$, where $\hat{f}(\cdot)$ is the normal inverse cumulative distribution function with expectation $= \mu$ and variance ≈ 0.15, such that $p_{min} = 21.6$ and $p_{max} = 73.9$. The grid in the three dimensions was determined by trial and error.

In Figure 1 we depict the value function $V(t, z, p)$ for $t = 0.5$, as a function of z and p, using the above grid. We observe that it is decreasing in z as Prop. 3.3 predicts. The numerical solution is also concave, however, very close to be linear in z.

Figure 2 shows the optimal exercise curves at different times as functions of z and p. The curves are moving downwards with decreasing times, and it is optimal to not use the control (or the exercise right) when you are to the left of such a curve. Once the price P_t is hitting the curve, it is optimal to start using the

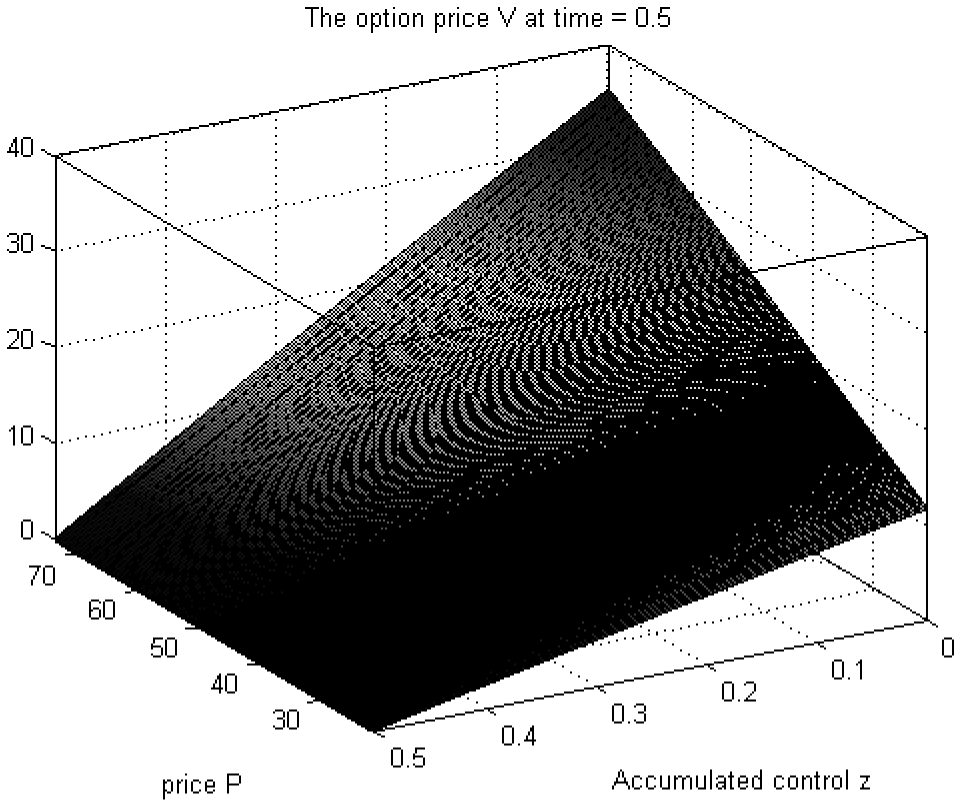

Figure 1. The option price

control. As functions of p, the curves are increasing. At a fixed time, the more we have used of our control (the higher value of z, that is), the higher price we must have before we are willing to exercise. For a given level of z, on the other hand, the price level to exercise is decreasing with time, which is intuitive since we have less time left for using our rights, and thus are willing to interfere at lower prices. Note that the optimal exercise curves are stopping when $z > \bar{u}t$, for which the option does not exist. We point out that the exercise

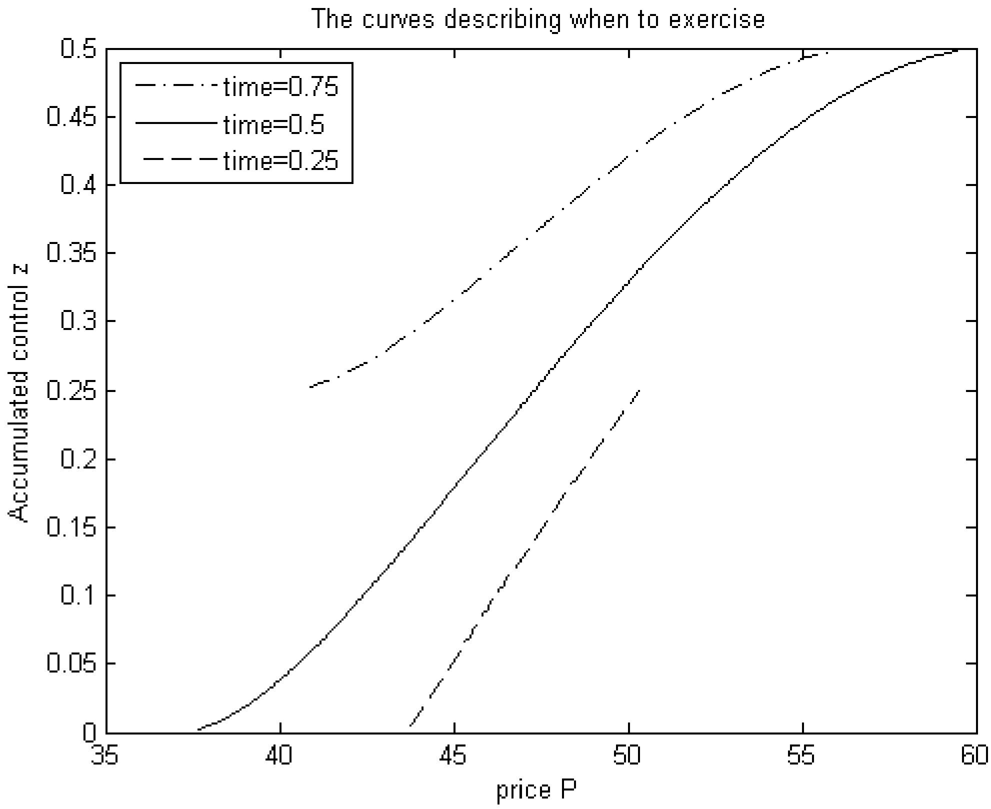

FIGURE 2. The exercise curves. For prices to the right of the curve it is optimal to exercise the option.

curves for $t = 0.5$ and 0.75 cut off at both ends. This is due to the chosen resolution of the numerical grid. Indeed, if we use higher resolution, then the curves would continue further to both left and right.

In Table 2 we show the accuracy of the three schemes using each type of boundary condition. The tests are done in *Matlab*$^{\text{TM}}$ implementations of the schemes on a laptop from 2010 with Intel Core i7 processor

TABLE 2. Numerical tests for the three types of boundary conditions

Boundary condition	Neumann	Dirichlet	$V_{pp} = 0$
Average error in P	0.0099 EUR/MWh	0.0110 EUR/MWh	0.0115 EUR/MWh
Computation time	$\approx$ 15 sec	$\approx$ 15 sec	$\approx$ 1 sec

with 2.66 GHz core speed (with dual core, but *Matlab*™ uses only one core). The extra time needed for the Neumann and Dirichlet conditions is because of the integrals to calculate the boundary conditions.

It is also interesting to know the accuracy in option price V. For the Neumann conditions the average error in V for $P \in (30, 55)$ and $z \in (0, \frac{1}{2})$ is ≈ 0.045 EUR/MWh.

Comparing the numerical solutions for V for all the three boundary conditions, it can be verified that the calculations for the Dirichlet and Neumann conditions are right, i.e.

$$V(t, z, p_{min}) \approx g_1(t_n, z_i), \quad \text{and} \quad V(t, z, p_{max}) \approx g_2(t_n, z_i),$$

$$V_p(t, z, p_{min}) \approx \hat{g}_1(t_n, z_i), \quad \text{and} \quad V_p(t, z, p_{max}) \approx \hat{g}_2(t_n, z_i).$$

The fact that $g_k(t_n, z_i)$ and $\hat{g}_k(t_n, z_i)$ for $k = 1, 2$ are calculated in different way from the PDE solver, gives us a verification that the code is correct. Verifications like this are often omitted when PDEs are solved numerically.

As mentioned before the presented discretization scheme of the discussed contracts, namely swing options with continuous time exercise, can be viewed as a finite difference approximation of a Bermudian swing option with N exercise opportunities, where N is the number of discretization points in time. In this paper we have shown that for an exponential Ornstein–Uhlenbeck model with realistic parameters it is enough to have $N = 450$. Notice that the original FLC–contracts might have hourly exercises during one year, which means that $N = 8760$. Thus we conclude that the continuous exercise approximation of the FLC–contracts is a very accurate approximation.

We end this section by including two plots to describe the sensitivity in the optimal exercise curves w.r.t. the parameters σ and κ.

6. Concluding comments

We studied in this paper the optimal exercise of a swing option. To this end, we formulated the pricing model as a bounded variation control problem under a total volume constraint. We proved some basic qualitative properties of the model which are intuitively appealing. First we studied the model in the absence of an effective volume constraint as a limiting case. In this case, we found the optimal exercise policy and

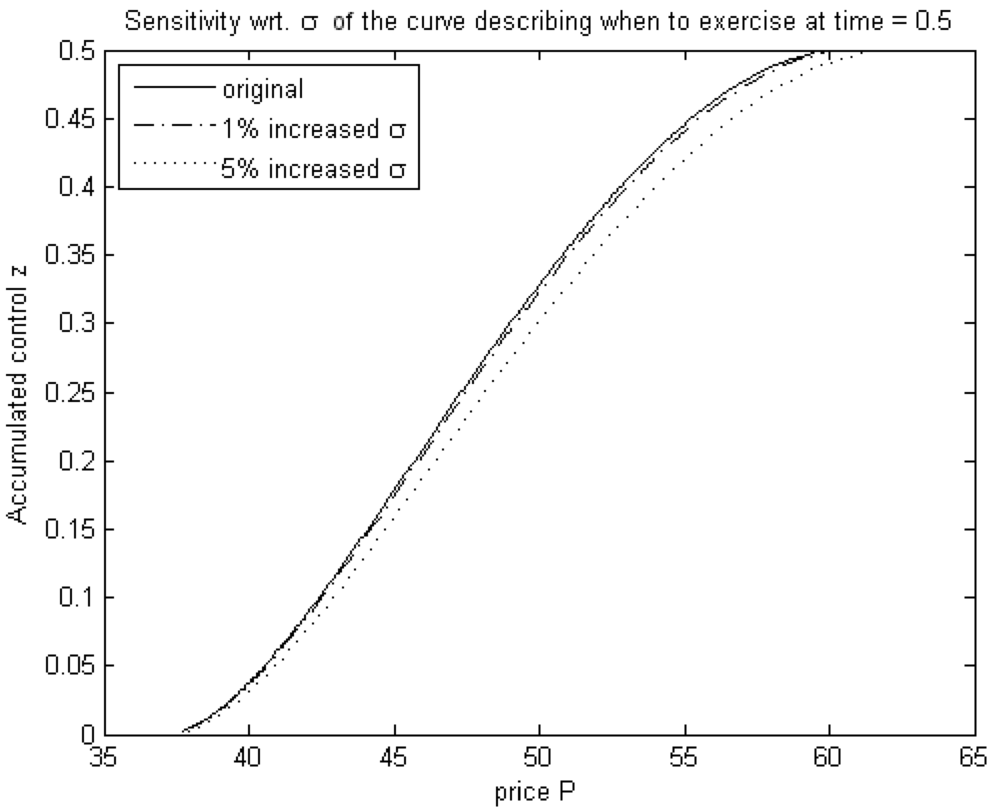

Figure 3. When the volatility is increased, the holder should require higher prices before exercising. The average change in $P(z)$ is 0.17 EUR/MWh for the 1% change and 0.86 EUR/MWh for the 5% change.

proved that the lost marginal option value is zero. In the presence of an effective total volume constraint we found that the value function is decreasing and concave in the direction of the cumulative control (z, that is). This means that the option loses more value when the volume reserve is low in comparison to high reserve. We derived a candidate for the optimal exercise policy and value for the option using an HJB-equation. In

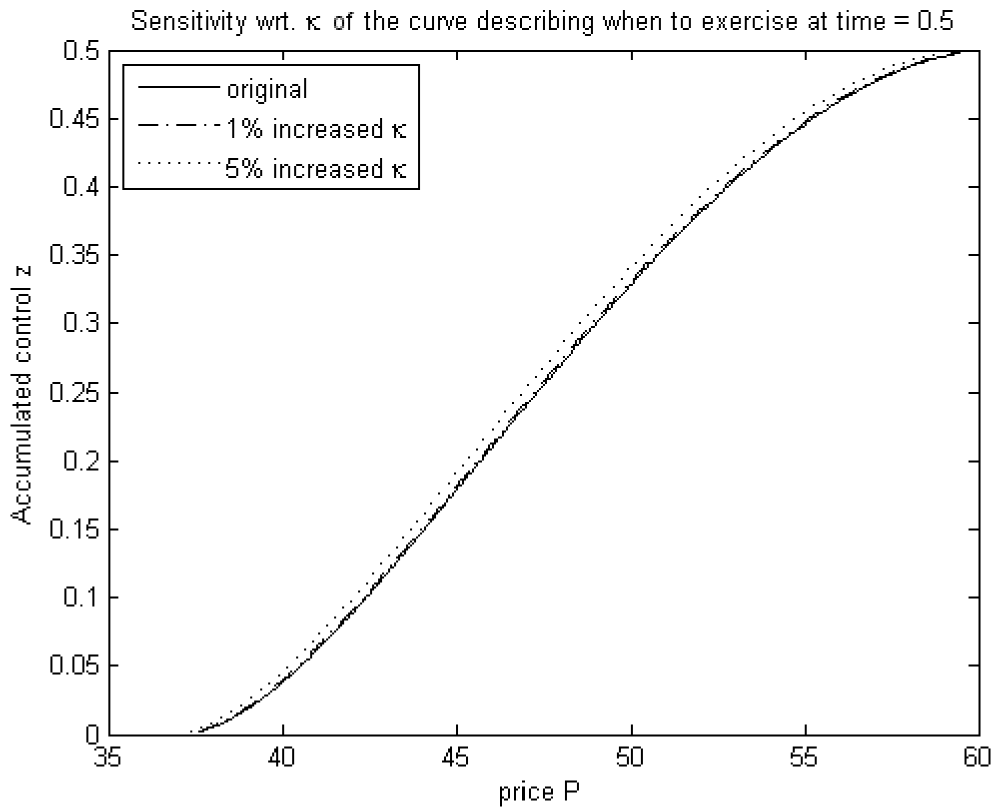

FIGURE 4. When the speed of mean reversion is increased, the holder should exercise for lower prices. The change measured in L^1 is is 0.088 EUR/MWh for the 1% change and 0.42 EUR/MWh for the 5% change.

particular, we characterized the associated trigger rule as as simple comparison principle between the instant exercise payoff and the lost option value. We proved also a verification theorem, which gives a set of sufficient conditions when a given function coincides with the value function.

We carried out also a numerical analysis of the model and, in particular, discussed the issue of boundary conditions. We set up a numerical scheme for solving the HJB-equation and illustrated it with an exponential Ornstein-Uhlenbeck process. Finally, we concluded that, given some assumption on the underlying price process, the continuous time exercise swing option is an accurate approximation of flexible load contracts, which is Bermudan swing options with high number of exercise opportunities.

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Appendix A. Boundary conditions at the truncated boundary

We recall that the solution of the Langevin-type SDE

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t$$

at time s, given the state at time $t < s$, is

$$X_s = (X_t - \mu)e^{-\kappa(s-t)} + \mu + \sigma \int_t^s e^{\kappa(u-s)} dW_u.$$

We next calculate the stochastic integrals defined by the boundary conditions on the truncated boundary, i.e. we compute

$$\begin{aligned} V(t, z, p_{max}) &= \bar{u} E\left[\int_t^{\tau_1} e^{-r(s-t)} e^{X_s} ds \middle| \mathcal{F}_t \right], \\ V(t, z, p_{min}) &= \bar{u} E\left[\int_{\tau_2}^{T} e^{-r(s-t)} e^{X_s} ds \middle| \mathcal{F}_t \right], \end{aligned}$$

for the case $P_s = \exp(X_s)$, where $\tau_1 = t + \frac{M-z}{\bar{u}}$ and $\tau_2 = T - \frac{M-z}{\bar{u}}$. We find

$$\begin{aligned} g_2(t, z) = V(t, z, p_{max}) &= \bar{u} E\left[\int_t^{\tau_1} e^{-r(s-t)} e^{X_s} ds \middle| \mathcal{F}_t \right] \\ &= \bar{u} e^{rt + \mu + \frac{\sigma^2}{4\kappa}} \int_t^{\tau_1} \exp\left[-rs + (\ln(p_{max}) - \mu)e^{-\kappa(s-t)} - \frac{\sigma^2}{4\kappa} e^{-2\kappa(s-t)} \right] ds, \\ g_1(t, z) = V(t, z, p_{min}) &= \bar{u} E\left[\int_{\tau_2}^{T} e^{-r(s-t)} e^{X_s} ds \middle| \mathcal{F}_t \right] \\ &= \bar{u} e^{rt + \mu + \frac{\sigma^2}{4\kappa}} \int_{\tau_2}^{T} \exp\left[-rs + (\ln(p_{min}) - \mu)e^{-\kappa(s-t)} - \frac{\sigma^2}{4\kappa} e^{-2\kappa(s-t)} \right] ds, \end{aligned}$$

and the corresponding Neumann conditions are

$$\begin{aligned} \hat{g}_1(t, z) = V_p(t, z, p_{max}) &= p_{max}^{-1} \bar{u} e^{(r+\kappa)t + \mu + \frac{\sigma^2}{4\kappa}} \int_t^{\tau_1} \exp\left[-(\kappa + r)s + (\ln(p_{max}) - \mu)e^{-\kappa(s-t)} - \frac{\sigma^2}{4\kappa} e^{-2\kappa(s-t)} \right] ds, \\ \hat{g}_2(t, z) = V_p(t, z, p_{min}) &= p_{min}^{-1} \bar{u} e^{(r+\kappa)t + \mu + \frac{\sigma^2}{4\kappa}} \int_{\tau_2}^{T} \exp\left[-(\kappa + r)s + (\ln(p_{min}) - \mu)e^{-\kappa(s-t)} - \frac{\sigma^2}{4\kappa} e^{-2\kappa(s-t)} \right] ds. \end{aligned}$$

These integrals are calculated numerically.

Appendix B. The local truncation error for the numerical scheme

In this appendix we investigate the local truncation error for the numerical scheme presented in Section 4. We will observe that the approximations for V_p and V_{pp} are nearly second order accurate. Using Taylor expansion we see that (we skip the t and q dependency for notational ease):

$$\begin{aligned} V(p_{j+1}) &= V(p_j) + \Delta p_{j+1} V_p(p_j) + \frac{\Delta p_{j+1}^2}{2} V_{pp}(p_j) + \frac{\Delta p_{j+1}^3}{6} V_{ppp}(p_j) + \mathcal{O}(\Delta p_{j+1}^4), \\ V(p_{j-1}) &= V(p_j) - \Delta p_j V_p(p_j) + \frac{\Delta p_j^2}{2} V_{pp}(p_j) - \frac{\Delta p_j^3}{6} V_{ppp}(p_j) + \mathcal{O}(\Delta p_j^4). \end{aligned}$$

Putting this expansion into the approximation for V_p we get

$$\begin{aligned} V_p(t_n, z_i, p_j) &\approx \frac{1}{\Delta p_{j+1} + \Delta p_j} \left(\frac{\Delta p_j}{\Delta p_{j+1}} V_{i,j+1}^n + \left(\frac{\Delta p_{j+1}}{\Delta p_j} - \frac{\Delta p_j}{\Delta p_{j+1}} \right) V_{i,j}^n - \frac{\Delta p_{j+1}}{\Delta p_j} V_{i,j-1}^n \right) \\ &= V_p(p_j) + \mathcal{O}((\Delta p_{j+1} + \Delta p_j)^2). \end{aligned}$$

This is a second order accurate approximation. The local truncation error for the approximation of V_{pp} is

$$\begin{aligned} V_{pp}(t_n, z_i, p_j) &\approx \frac{2}{\Delta p_{j+1} + \Delta p_j} \left(\frac{1}{\Delta p_{j+1}} V_{i,j+1}^n + \left(-\frac{1}{\Delta p_{j+1}} - \frac{1}{\Delta p_j} \right) V_{i,j}^n + \frac{1}{\Delta p_j} V_{i,j-1}^n \right) \\ &= V_{pp}(p_j) + \mathcal{O}((\Delta p_{j+1} + \Delta p_j)^2, \Delta p_{j+1} - \Delta p_j). \end{aligned}$$

We see that this is second order accurate if $\Delta p_{j+1} - \Delta p_j = \mathcal{O}((\Delta p_{j+1} + \Delta p_j)^2)$. This means that if two neighboring discretization intervals in the $p-$direction are always almost at the same size, then we have second order accuracy in the approximation. We may call this property quasi-uniform grid.

Appendix C. Boundary conditions in the numerical scheme

Now over to the implementation of the boundary conditions in the numerical scheme. We denote the Dirichlet conditions with $g_1(t)$ and $g_2(t)$ or discretely in time

$$g_1^n = V(t^n, z, p_{min}), \qquad g_2^n = V(t^n, z, p_{max}).$$

Inserting this into (4.11) for $j = 0$ and $j = N_p$ we obtain an equation to be solved for each time step on the form

$$(I - \Delta t A) V^n = V^{n+1} + \Delta t G^{n+1} + BC^n, \tag{C.1}$$

where $A = \frac{\sigma^2}{2} D_2 + \mu D_1 - rI$, I is the identity matrix, e_k is the $k-$th unit vector and

$$BC^n = \frac{\Delta t}{\Delta p_2 + \Delta p_1}\left(\frac{(\sigma_1^n)^2}{\Delta p_1} - \frac{\mu_1^n \Delta p_2}{\Delta p_1}\right) g_1^n\, e_1 + \frac{\Delta t}{\Delta p_N + \Delta p_{N-1}}\left(\frac{(\sigma_{N-1}^n)^2}{\Delta p_N} + \frac{\mu_{N-1}^n \Delta p_{N-1}}{\Delta p_N}\right) g_2^n\, e_{N-1},$$

$$G_j^{n+1} = u_{i,j}^{n+1}\left(p_j - K + \frac{V_{i+1,j}^{n+1} - V_{i,j}^{n+1}}{\Delta z}\right),$$

$$D_2 = \begin{pmatrix} \frac{2}{\Delta p_2+\Delta p_1}\left(-\frac{1}{\Delta p_2} - \frac{1}{\Delta p_1}\right) & \frac{2}{\Delta p_2+\Delta p_1}\frac{1}{\Delta p_2} & 0 \\ \frac{2}{\Delta p_{j+1}+\Delta p_j}\frac{1}{\Delta p_j} & \frac{2}{\Delta p_{j+1}+\Delta p_j}\left(-\frac{1}{\Delta p_{j+1}} - \frac{1}{\Delta p_j}\right) & \frac{2}{\Delta p_{j+1}+\Delta p_j}\frac{1}{\Delta p_{j+1}} \\ 0 & \frac{2}{\Delta p_{N_p}+\Delta p_{N_p-1}}\frac{1}{\Delta p_{N_p-1}} & \frac{2}{\Delta p_{N_p}+\Delta p_{N_p-1}}\left(-\frac{1}{\Delta p_{N_p}} - \frac{1}{\Delta p_{N_p-1}}\right) \end{pmatrix} \in \mathbf{R}^{N-1,N-1},$$

$$D_1 = \begin{pmatrix} \frac{1}{\Delta p_2+\Delta p_1}\left(\frac{\Delta p_2}{\Delta p_1} - \frac{\Delta p_1}{\Delta p_2}\right) & \frac{1}{\Delta p_2+\Delta p_1}\frac{\Delta p_1}{\Delta p_2} & 0 \\ -\frac{1}{\Delta p_{j+1}+\Delta p_j}\frac{\Delta p_{j+1}}{\Delta p_j} & \frac{1}{\Delta p_{j+1}+\Delta p_j}\left(\frac{\Delta p_{j+1}}{\Delta p_j} - \frac{\Delta p_j}{\Delta p_{j+1}}\right) & \frac{\Delta p_j}{(\Delta p_{j+1}+\Delta p_j)\Delta p_{j+1}} \\ 0 & -\frac{\Delta p_{N_p}}{(\Delta p_{N_p}+\Delta p_{N_p-1})\Delta p_{N_p-1}} & \frac{1}{\Delta p_{N_p}+\Delta p_{N_p-1}}\left(\frac{\Delta p_{N_p}}{\Delta p_{N_p-1}} - \frac{\Delta p_{N_p-1}}{\Delta p_{N_p}}\right) \end{pmatrix} \in \mathbf{R}^{N-1,N-1}.$$

D_1 and D_2 are matrices approximating the first and second order differential operators.

We use the same notation for the Neumann conditions $\hat{g}_1(t)$ and $\hat{g}_2(t)$ or discretely in time

$$\hat{g}_1^n = V_p(t^n, z, p_{min}), \qquad \hat{g}_2^n = V_p(t^n, z, p_{max}).$$

By introducing the shadow points $V_{i,-1}^n$ and V_{i,N_p+1}^n and inserting this into (4.11) for $j = 0$ and $j = N_p$ we obtain an equation to be solved for each time step on the form

$$(I - \Delta t A)u^n = u^{n+1} + \Delta t G^n + \Delta t BC^n, \tag{C.2}$$

where $A = \frac{\sigma^2}{2} D_2 + \mu D_1 - rI$ and

$$BC^n = \left(-\frac{(\sigma_0^n)^2}{h_0} + \mu_0^n\right)\hat{g}_1^n e_1 + \left(\frac{(\sigma_{N_p}^n)^2}{h_{N_p}} + \mu_{N_p}^n\right)\hat{g}_2^n e_{N+1},$$

$$D_2 = \begin{pmatrix} -\frac{2}{\Delta p_1^2} & \frac{2}{\Delta p_1^2} & 0 \\ \frac{2}{\Delta p_{j+1}+\Delta p_j}\frac{1}{\Delta p_j} & \frac{2}{\Delta p_{j+1}+\Delta p_j}\left(-\frac{1}{\Delta p_{j+1}} - \frac{1}{\Delta p_j}\right) & \frac{2}{\Delta p_{j+1}+\Delta p_j}\frac{1}{\Delta p_{j+1}} \\ 0 & \frac{2}{\Delta p_{N_p}^2} & -\frac{2}{\Delta p_{N_p}^2} \end{pmatrix} \in \mathbf{R}^{N+1,N+1},$$

$$D_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\Delta p_j}{(\Delta p_{j+1}+\Delta p_j)\Delta p_{j+1}} & \frac{1}{\Delta p_{j+1}+\Delta p_j}\left(\frac{\Delta p_{j+1}}{\Delta p_j} - \frac{\Delta p_j}{\Delta p_{j+1}}\right) & \frac{1}{\Delta p_{j+1}+\Delta p_j}\frac{\Delta p_{j+1}}{\Delta p_j} \\ 0 & 0 & 0 \end{pmatrix} \in \mathbf{R}^{N+1,N+1}.$$

D_1 and D_2 are still matrices approximating the first and second order differential operators, but now with Neumann conditions incorporated. Above it is implicitly assumed that the shadow points are located such that $\Delta p_{-1} = \Delta p_0$ and $\Delta p_{N_p+1} = \Delta p_{N_p}$.

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