Optimal portfolio problems under model ambiguity

by

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THESIS

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Abstract

The topic of this thesis is portfolio optimization under model ambiguity, i.e. a situation when the probability distribution of the events in the sample space is not known. The financial market studied is driven by a Brownian motion: a continuous driving element, and a doubly stochastic Poisson random process: a discontinuous driving element. What separates the doubly stochastic Poisson random process from the standard Poisson case, is that the jump intensity is a stochastic process. From a modeling point of view, this adds more flexibility in capturing hidden random effects. Models of this type appear in the literature of credit risk and in financial price modeling in the class of stochastic volatility models. See e.g. [9] and [4], respectively.

In this thesis we assume that the investing agent is ambiguity averse, i.e. the agent does not take any risk with respect to the uncertainty of the probability distribution, and thus relates to the worst case probability distribution. A dynamic risk measure that respects the agents ambiguity aversion is applied to quantify the risk of a hedging strategy, and the agent wishes to make the risk vanish at all times.

The optimization problem takes the form of a stochastic differential game, in which the agent minimizes the risk of the hedging strategy, while the opponent drives in the opposite way proposing a probability distribution yielding the worst case scenario. This thesis entails two methods of solving this stochastic differential game. First, through backward stochastic differential equations (BSDEs), and then using the maximum principle. In this thesis, the approach with BSDEs will give a solution of a price process at all times in the give horizon, while the approach using the maximum principle will give a solution at the initial time.

This approach of pricing is of great interest to insurers, as it gives a supplement to Value-at-Risk and Estimated Shortfall calculations, and gives a benchmark for capital needed to withstand extreme scenarios.

Contents

Preface

This thesis is for the degree of Master in Science, written under the Mathematics program, with specialization mathematical finance, at the Department of Mathematics, University of Oslo. The work of this thesis has been done at the Mathematics Department, Blindern, lasting from fall 2013 to spring 2014, and corresponds to 60 credits. The work has been done independently, but with the guidance of my supervisor, Professor Giulia Di Nunno (Center of Mathematics for Applications, University of Oslo).

Acknowledgments: Jeg vil først og fremst takke min veileder, Giulia, som alltid har tatt seg tid til å svare på spørsmål, og som har vært til stor hjelp når jeg har stått fast. Jeg har lært utrolig mye av deg underveis i dette prosjektet. Videre vil jeg takke Kjersti for støtte og omsorg i perioder da stresset stormet som størst. Jeg vil også takke min familie, spesielt mine foreldre, for at de alltid har støttet meg og for at jeg alltid er hjertelig velkommen hjem.

1 Introduction

1.1 Project description

The material for my thesis originates mainly from two sources. Firstly, I used the research paper *BSDEs driven by time-changed Lévy noises and optimal control* by Giulia Di Nunno and Steffen Sjursen [8] to study *doubly stochastic Poisson random fields*, and the stochastic analysis and BSDEs connected to such measures. Secondly, I used the book *Backward stochastic differential equations with jumps and their actuarial and financial applications* by Łukasz Delong [5] as introductory reading on backward stochastic differential equations and their link to optimization problems.

The progress of the project (as intended) was as follows:

1. Study the background on BSDEs driven by Gaussian and centered Poisson noises.

2. Acquire basic knowledge about dynamic risk measures and their connection with BSDEs. Link the use of risk measure to model ambiguity.

3. Understand the solution to the optimization problem

$$Y(t) = \operatorname*{ess\,inf}_{\pi} \operatorname*{ess\,sup}_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}\left[-\left(e^{\int_t^T r(s)ds} X^{\pi}(T) - X^{\pi}(t) - e^{\int_t^T r(s)ds} F\right) \middle| \mathcal{M}_t \right],$$

   for $0 \leq t \leq T$, in the case of mixture of Gaussian and centered Poisson noises, when $(\mathcal{M}_t)_{t\in[0,T]} = \mathbb{G}$ and $(\mathcal{M}_t)_{t\in[0,T]} = \mathbb{F}$. (See equation (3.15) and the corresponding section for details.)

4. Extend 3. to include the doubly stochastic Poisson noise.

1.-3. were used as an introduction to the subject, and 4. is the main contribution of the thesis. In addition to the intended progress, I looked at another approach to solve the problem, namely by the maximum principle for stochastic differential games. Background theory for this approach is found in *BSDEs driven by time-changed Lévy noises and optimal control* by Giulia Di Nunno and Steffen Sjursen [8], and *Maximum principle for stochastic differential games with partial information* by An Thi Kieu Ta and Bernt Øksendal [1].

Some challenges I experienced during the project was in the work beyond centered Poisson noises, the study of the statistical properties, and the stochastic calculus of such. Much of my work in this project has been to acquire knowledge and extend results from Lévy noise to doubly stochastic Poisson noise in the stochastic integration scheme, change of measure and optimization.

1.2 My contributions

A main contribution of this thesis is the derivation of the dynamics of the price processes and the solution of the optimization problem (3.18) for the setup with information flow $\mathbb{G}$ and information flow $\mathbb{F}$. See equation (3.35) (equation (3.54)) for the dynamics and page 58 (page 68) for a summary of the optimal solution with information flow $\mathbb{G}$ (information flow $\mathbb{F}$). In addition to the aforementioned results, the following summarizes my contributions:

- Chapter 2:
  - Section 2.4 is a fusion of the theory of the maximum principle in [8] in a setup with Lévy process, and the theory of the maximum principle in [1] in a setup with doubly stochastic Poisson random field.
  - Theorem 2.17 (Maximum principle III, for the solution via the maximum principle) and its proof.
- Chapter 3:
  - Theorem 3.1 and its proof are modifications of Theorem 1.35 in [14]. The modification is to extend the theorem from a Poisson measure to a doubly stochastic Poisson random field. We prove that the Girsanov change of measure preserves the property of martingale random fields and the independence between the Brownian motion W^θ and the random jump measure $\tilde{H}^\theta$ for some equivalent probability measures. (See Section 3.1, Theorem 3.1 for details.)
  - Corollary 3.2 (preservation of Poisson structure under Girsanov change of measures) and its proof.
  - The derivation of the optimization problem (Section 3.3).
  - Theorem 3.8 (The optimization theorem II, for the solution via BSDEs) and its proof.
  - The derivation of the dynamics of the price processes for the setup with information flow $\mathbb{G}$ (3.35), along with the optimal solution (p. 58.)
  - The proof of Lemma 3.12.
  - The derivation of the dynamics of the price processes for the setup with information flow $\mathbb{F}$ (3.54), along with the optimal solution (p. 68.)
  - The analysis and the toy example in Section 3.6.

My contributions are marked with ♣ in the text.

1.3 Historical introduction to model ambiguity

In 1954, Leonard J. Savage postulated a method in his book *Foundations of Statistics* called the *subjective expected utility*, where he proposed that people's choices can be explained through a mathematical function. The theory of subjective expected utility explains the relation between subjective assumptions and choices, given that people act rationally. Rationality is here thought of as an ability to rank in order all possible situations from worst to best and act upon this. This is a seemingly fair assumption to make.

In order to give a swift mathematical explanation of this theory, we need a utility function, u, and a probability distribution, P, for an uncertain event. It must be emphasized that the utility function and the probability distribution are subjectively chosen by an agent. The probability distribution is based on beliefs of the acting agent. The agent is faced with a choice leading to different sets of outcomes $\{x_i\}$ and $\{y_i\}$, with subjective utilities $\{u(x_i)\}$ and $\{u(y_i)\}$ and probabilities $\{P(x_i)\}$ and $\{P(y_i)\}$. Which choice the agent prefers, according to the theory of subjective expected utility, depends on which subjective expected utility is higher. As an example, say $\sum_i u(x_i)P(x_i) > \sum_i u(y_i)P(y_i)$, then the agent will choose the option leading to the outcome $\{x_i\}$. However, experiments have been carried out that shows evidence that people do not act according to the theory of subjective expected utility.

There is extreme complexity related to the optimization of the composition of a portfolio. There is a long list of decisions that will affect the end result, such as selection of statistical model, the amount of data available, how we choose to estimate the parameters, the choice of optimization criteria, and we may even have to approximate the solution numerically.

In other words, we most likely do not have the right model to explain the financial market we are interested in modeling. It is even daring to suggest that there actually exist such a model, meaning that the world is actually made up of laws, and that the laws of nature are not just man made simplifications of the world made for us to better understand it. This is a philosophic question this paper will no further discuss, and we rather just accept that the model we choose is most likely not the right one.

An investing agent in a financial market is exposed to model ambiguity from the arguments above, or at least assumed to be in this thesis. The agent may respond differently to this exposure, but the attitude towards this risk affects the investments of the agent, and puts conditions on the portfolio optimization.

We will show evidence of this alleged aversion of ambiguity by a paradox by Ellsberg in his 1961 paper *Risk, Ambiguity, and the Savage Axioms.* The paradox contradicts the postulate of Leonard J. Savage, about the rational agent, thus there is evidence of deviation between what agents do and Savage says they should do.

Ellsberg's paradox

There exists more than one version of this paradox, and here we choose to introduce the one-urn problem with three different colored balls. The Ellsberg's Paradox is a paradox in decision theory, and is taken as evidence for existence of ambiguity aversion. This contradicts Savage's axioms, that a rational agent faced with model ambiguity will maximize his utility with respect to *one* subjectively chosen prior.

The experiment goes like this: Suppose we have an urn containing 90 balls of three different colors; red, blue and yellow. We know for certain that there are 30 yellow balls, but we do not know the distribution of blue and red balls of the remaining 60 balls. This is a problem in ambiguity as we defined it, as there is no way to measure the risk exactly due to the uncertainty in the model.

What Ellsberg did, was to introduce two games of chance to the subjects. In the first game, he asked whether they would bet on a yellow or a red ball being pulled up from the urn. In the second game, he asked whether they would bet on a yellow or blue ball, or if they would bet on a red or blue ball being pulled up from the urn. Most subjects in his experiment, which later experiments have confirmed, prefer to bet on the yellow ball in the first gamble and red or blue in the second. Figure 1.1 provides an overview of the games of chance and the preferences of the majority of the subjects.

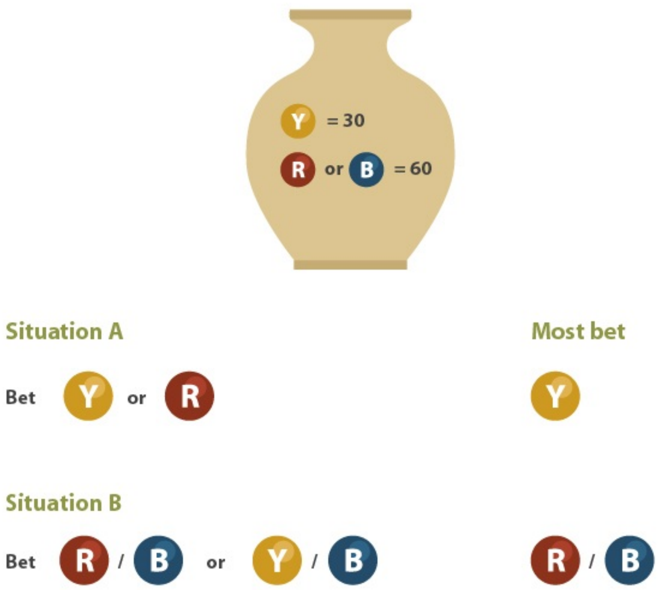

Figure 1.1: Ellsberg's experiment and the results. In the first game of chance, Situation A, most subjects preferred the yellow ball over the red ball. This implies that the subjects think there are more yellow balls than red balls. In the second game, Situation B, most subjects preferred the red or blue alternative, meaning the subjects think there are more red and blue balls combined than yellow and blue. This contradicts the first assumption, and leads to a paradox when assuming Savage's axioms.

According to the figure, most subjects preferred the yellow option in Situation A, and the red or blue option in Situation B. So how does this contradict Savage's axioms? Well, the choice of most subjects in Situation A reflects that they think there are more yellow balls than red balls in the urn. In order to make it easier to keep up with the train of thought, say the subject believes there are 20 red balls, which implies there are 40 blue balls. Then, making another bet on the same urn, the logical choice of the alternatives red or blue and yellow or blue under the same beliefs, is to bet on the yellow or blue option to be in line with Savage's axioms. This would give a 70/90 chance of winning, while the red or blue choice will only give a 60/90 chance of winning. The result from the experiment shows that most subjects prefer the red or blue bet to the yellow or blue, although this contradicts their first choice according the theory of subjective expected utility.

Ambiguity aversion is a possible, and perhaps probable, explanation to this phenomenon. Maximizing the expected payoff of the games with certainty is impossible, so therefore one might go for robustness. The subjects seek for known probabilities and known payoff, instead of going for one specific *assumption* of the distribution of the colored balls. The specific strategy (Y,R/B) in the experiment is robust in that no matter what the distribution of the colored balls is, the expected payoff of the strategy is the same. This is a suboptimal solution with respect to assumed probability distribution implied by the choice A in all cases except for when the distribution is $(30, 30, 30)$. But for players who treasure certainty more than possible gain, this is an optimal solution.

This suboptimal solution is a robust solution in that the variation is small under neighboring probability models. In contrast, non-robust decisions may drop a lot in performance under neighboring probability models. As an example of this, as mentioned above, the variation of the expected payoff of the strategy (Y,R/B) is zero. The variation of the optimal subjective expected utility strategy (Y,Y/B) (or (R,R/B) if one assumes R>Y) is $\pm 1/3$ of the payoff for picking the right colored ball.

1.4 A measure for the risk of ambiguity

In this thesis, we take the perspective of an ambiguity averse agent who wishes to hedge a contingent claim in a market driven by a Brownian motion and a doubly stochastic Poisson random field. The degree of aversion may vary, but in this thesis we assume the agent to be strongly ambiguity averse, in that the agent wishes to evaluate the portfolio in the worst-case scenario and hence obtain a robust suboptimal solution. With realistic applications in mind, such a robust strategy is of great interest e.g. for insurers that want to put stress-test on model parameters to ensure the company has enough capital to withstand extreme scenarios.

The uncertainty in the market is captured by the distribution law of the random space. The possible probability distributions of the random space are the equivalent probability measures of the measure under which the financial market is defined.

The measure for the ambiguity aversion of the agent is a dynamic risk measure ρ_t that measures the risk of a financial position ξ to be

$$\rho_t(\xi) := \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-\xi|\mathcal{M}_t], \quad 0 \leq t \leq T.$$

Negative ξ means loss, while positive ξ means profit. The set $\mathcal{Q}$ represents the set of the possible distribution laws, and the minus sign inside the expectation indicates that the risk measure measures losses positively. The essential supremum over all losses evaluates ξ under the least favorable conditions. The conditional expectations are stochastic variables, and must be evaluated almost everywhere, thus we take the *essential* supremum. $\mathcal{M}_t$ denotes the information available in the market at time $t \in [0, T]$, and in this thesis two information filtrations are considered. ρ_t is a specific example of a dynamic risk measure. For more information about dynamic risk measures see Appendix A.

Financially speaking, the agent has a liability of which he wants a "best possible" replication, i.e. the best replication with respect to the ambiguity aversion through the dynamic risk measure. The agent invests in a portfolio π in the financial market, and choose a portfolio that hedges the liability under the specified risk measure. Mathematically presented, the problem is to find π such that

$$\operatorname*{ess\,inf}_{\pi} \; \rho_t\Big(e^{-\int_t^T r(s)ds}[X^\pi(T) - F]\Big) = 0, \quad 0 \leq t \leq T.$$

Here $X^\pi(t)$ is the value process of the portfolio, and F represents a financial claim.

As mentioned above, two information filtrations are considered. The first is the filtration generated by the natural filtrations of the Brownian motion, the doubly stochastic Poisson process, and the stochastic intensity process of the doubly stochastic Poisson random field. The second, is generated by the filtrations of the Brownian motion and the doubly stochastic Poisson process, and at all times contains the complete filtration of the stochastic intensity process. Respectively, the filtrations is denoted $\mathbb{F}$ and $\mathbb{G}$. The $\mathbb{G}-$filtration is interesting for its statistical properties, while the $\mathbb{F}-$filtration is interesting from the modeling point of view.

In Chapter 2, we introduce preliminary theory for the calculations that will come in Chapter 3. In Appendix B there are more theoretical results to support and elaborate the theory in Chapter 2. This theory is not necessary for the reading, but included in the appendices for completion.

In Section 2.1, we define the jump process in the financial market, or more precisely the random field of the jump process, and the random intensity process.

In Section 2.2, we define the stochastic integration with respect to this random field in a classical Itô integration scheme. In Section 2.3, a short introduction to backward stochastic differential equations and a theorem for comparison of solutions of such BSDEs are included. In Section 2.4 theory on the maximum principle is introduced, and an optimization theorem for both filtrations $\mathbb{F}$ and $\mathbb{G}$ is given.

In Section 3.1, we define the equivalent probability measures to the measure under which the financial market is modeled. In Section 3.2, the financial market is defined by the processes introduced in Chapter 2. In Section 3.3, we make clear the optimization problem in order to do calculations via BSDEs, and we also define the sets of admissible portfolios and equivalent probability measures. In Section 3.4, we solve the optimization problem for both filtrations $\mathbb{F}$ and $\mathbb{G}$ via BSDEs. In Section 3.5, we solve the optimization problem for both filtrations $\mathbb{F}$ and $\mathbb{G}$ via the maximum principle at the initial time $t = 0$.

If the reader is in need of more introductory theory on stochastic processes and statistical properties of such, the books [13] and [14] are suggested.

2 Preliminary theory

2.1 The mathematical framework

In this section the statistical properties of the stochastic processes are introduced, and in the upcoming sections, these statistical properties are used to build up an integration scheme. The framework is a modification of the one in the paper [8], in that there is not assumed any time-distortion on the continuous process in this thesis, only in the jump process.

In this thesis we consider the complete probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$. We shall define a doubly stochastic Poisson random field, and in order to do that we need to define a distortion process. Let λ be a stochastic process such that the following properties are satisfied:

(i) $\lambda(t) \geq 0 \ \ \mathbb{P}-a.s. \quad$ for all $\quad t \in [0, T]$,

(ii) $\lim_{h \to 0} \mathbb{P}(|\lambda(t+h) - \lambda(t)| \geq \epsilon) = 0$ for all $\epsilon > 0$ and almost all $t \in [0, T]$,

(iii) $\mathbb{E}[\int_0^T \lambda(t)dt] < \infty$.

Denote the space of processes satisfying $(i) - (iii)$ by $\mathcal{L}$.

We define a random measure on $[0, T] \times \mathbb{R}_0$ by

$$\Lambda(\Delta) := \int_0^T \int_{\mathbb{R}_0} \mathbb{1}_\Delta(t, z)\nu(dz)\lambda(t)dt,$$

for all $\Delta \in \mathcal{B}_{[0,T]\times\mathbb{R}_0}$. This measure is assumed to be non-atomic, meaning that $\mathbb{P}(\Lambda(\{(t, z)\}) = 0,$ for all $(t, z) \in \mathcal{B}_{[0,T]\times\mathbb{R}_0}) = 1$, to ensure certain needed properties of its natural filtration, which will be explained in more detail below. Here ν is a deterministic, σ-finite measure on $\mathcal{B}_{\mathbb{R}_0}$ satisfying

$$\int_{\mathbb{R}_0} (1 \wedge z^2)\nu(dz) < \infty.$$

We denote the $\sigma-$algebra generated by the values of Λ by $\mathcal{F}_T^\Lambda$, in particular the $\sigma-$algebra generated by the values of $\Lambda(\Delta),\ \Delta \in [0, t] \times \mathbb{R}_0$, by $\mathcal{F}_t^\Lambda$.

The driving noises in our market are a Brownian motion $W := (W(t),\ 0 \leq t \leq T)$, and a doubly stochastic Poisson random field.

Definition 2.1 *W is a signed random measure on the Borel sets of $[0, T]$ satisfying*

(i) $\mathbb{P}(W(\Delta) \leq x) = \Phi\left(\frac{x}{\sqrt{\int_0^T \mathbb{1}_\Delta(t)dt}}\right), \quad x \in \mathbb{R},\ \Delta \in [0, T]$,

(ii) $W(\Delta_1)$ and $W(\Delta_2)$ are independent whenever Δ_1 and Δ_2 are disjoint sets.

Here Φ *stands for the cumulative probability distribution function of a standard normal random variable.*

Definition 2.2 *H is a random measure on the Borel sets of* $[0,T] \times \mathbb{R}_0$ *satisfying*

(iii) $\mathbb{P}(H(\Delta) = k|\mathcal{F}_T^\Lambda) = \mathbb{P}(H(\Delta) = k|\Lambda(\Delta)) = \frac{\Lambda(\Delta)^k}{k!} e^{-\Lambda(\Delta)}$,

(iv) $H(\Delta_1)$ *and* $H(\Delta_2)$ *are conditionally independent given* $\mathcal{F}_T^\Lambda$ *whenever* $\Delta_1 \cap \Delta_2 = \emptyset$.

In addition we assume that

(v) W and H are independent.[1]

Let the signed random measure $\tilde{H} := H - \Lambda$ be defined on the Borel sets of $[0,T] \times \mathbb{R}_0$ by

$$\tilde{H}(\Delta) = H(\Delta) - \Lambda(\Delta), \quad \Delta \in \mathcal{B}_{[0,T]\times\mathbb{R}_0}.$$

This is a *centered doubly stochastic Poisson random field.* From Definition 2.2 we can conclude that the conditional first and second moment of $\tilde{H}$ are given by

$$\mathbb{E}[\tilde{H}(\Delta)|\mathcal{F}_T^\Lambda] = 0$$
$$\mathbb{E}[\tilde{H}(\Delta)^2|\mathcal{F}_T^\Lambda] = \Lambda(\Delta)$$

We call $M(\Delta) := \mathbb{E}[\tilde{H}(\Delta)^2|\mathcal{F}_T^\Lambda] = \Lambda(\Delta)$ the conditional variance measure, and $m(\Delta) := \mathbb{E}[\Lambda(\Delta)]$ the variance measure.

Let $\mathcal{F}^W$ and $\mathcal{F}^H$ be the natural filtrations of the Brownian motion and the doubly stochastic Poisson random field, respectively. By Theorem 2.8 in [8] we have that $\mathcal{F}_t^W \vee \mathcal{F}_t^{\tilde{H}} = \mathcal{F}_t^W \vee \mathcal{F}_t^H \vee \mathcal{F}_t^\Lambda$, $t \in [0,T]$. Here our assumption on Λ being non-atomic is crucial for this to be true. The result is proved in a more general framework in [8], where Λ (α in [8]) is defined on $\mathbb{X}$, a locally compact, second countable Hausdorff topological space. We simply let $\mathbb{X} := [0,T] \times \mathbb{R}$, and apply the result in our more specific framework. We define two filtrations that will be important in our work:

$$\mathbb{F} = \{\mathcal{F}_t,\ t \in [0,T]\}, \quad \mathcal{F}_t = \bigcap_{r>t} \left(\mathcal{F}_r^W \vee \mathcal{F}_r^{\tilde{H}}\right)$$
$$\mathbb{G} = \{\mathcal{G}_t,\ t \in [0,T]\}, \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_T^\Lambda$$

[1]The work of this thesis could be considered with a conditional Brownian motion W with respect to some time-distortion process Λ_W. Then (i) and (ii) would be replaced to hold conditionally to the natural filtration $\mathcal{F}^{\Lambda_W}$, and (v) would be replaced such that W and H are conditionally orthogonal with respect to the filtration generated by $\Lambda = \Lambda_W + \Lambda_H$. This would imply to extend the concept of $[,]$ and $\langle , \rangle$ (see Appendix D) to conditional information.

$\mathbb{F}$ is right-continuous by definition, and $\mathbb{G}$ is right-continuous from the relation with $\mathbb{F}$. By definition of $\mathbb{G}$ we have that $\mathcal{G}_T = \mathcal{F}_T$. Note that $\mathcal{G}_0 = \mathcal{F}_T^\Lambda$, while $\mathcal{F}_0$ is trivial. From now on we set $\mathcal{F} = \mathcal{F}_T$ $(= \mathcal{G}_T.)$

W is a $(\mathbb{G}, \mathbb{P})-$martingale by being a Brownian motion, and $\tilde{H}$ is a $(\mathbb{G}, \mathbb{P})-$*martingale random field* with conditionally orthogonal values in the sense of [[6], Definition 2.1], i.e.

Definition 2.3 (Martingale Random Field) *$\tilde{H}$ is a* $(\mathbb{G}, \mathbb{P})-$martingale random field *with conditionally orthogonal values if*

(i) *$\tilde{H}$ is $\mathbb{G}$-adapted;*

(ii) *$\tilde{H}$ has the martingale property, i.e. for any $t \in [0,T]$ and any $\Delta \in (t,T] \times \mathbb{R}_0$ such that $m(\Delta) < \infty$, we have*

$$\mathbb{E}[\tilde{H}(\Delta)|\mathcal{G}_t] = \mathbb{E}[\tilde{H}(\Delta)|\mathcal{F}_t \vee \mathcal{F}_T^\Lambda] = \mathbb{E}[\tilde{H}(\Delta)|\mathcal{F}_T^\Lambda] = 0;$$

(iii) *$\tilde{H}$ has a* tight *σ-finite variance measure $m : \mathbb{R}_0 \times [0,T] \to \mathbb{R}_+$ such that*

$$m(\Delta) := \mathbb{E}[\tilde{H}(\Delta)^2], \quad \Delta \in \mathcal{B}_{[0,T]\times\mathbb{R}_0},$$

and $m(\mathbb{R}_0 \times \{0\}) = 0$;

(iv) *$\tilde{H}$ is additive, and $\sigma-$additive in $L_2(\mathbb{P})$, i.e. for pairwise disjoint sets $\Delta_1, \Delta_2, ... : \Lambda(\Delta_k) < \infty$ $\mathbb{P} - a.s.$, $K < \infty$*

$$\tilde{H}(\bigcup_{k=1}^{K} \Delta_k) = \sum_{k=1}^{K} \tilde{H}(\Delta_k),$$

and

$$\tilde{H}(\bigcup_{k=1}^{\infty} \Delta_k) = \sum_{k=1}^{\infty} \tilde{H}(\Delta_k),$$

with convergence in $L_2(\Omega, \mathcal{F}, \mathbb{P})$;

(v) *$\tilde{H}$ has conditionally orthogonal values, i.e. for $\Delta_1, \Delta_2 \in (t,T] \times \mathbb{R}_0$, $\Delta_1 \cap \Delta_2 = \emptyset$, then*

$$\mathbb{E}[\tilde{H}(\Delta_1)\tilde{H}(\Delta_2)|\mathcal{G}_t] = \mathbb{E}[\tilde{H}(\Delta_1)|\mathcal{F}_T^\Lambda]\mathbb{E}[\tilde{H}(\Delta_2)|\mathcal{F}_T^\Lambda] = 0.$$

Remark. A measure is *tight* if for every $\delta > 0$ there exists a compact X_δ such that $m(X \setminus X_\delta) < \delta$. $\diamond$

Remark. Note that since both W and $\tilde{H}$ are $\mathbb{F}-$adapted, they are also an $(\mathbb{F}, \mathbb{P})-$martingale and an $(\mathbb{F}, \mathbb{P})-$martingale random field. (iii) and (iv) are the same, and (ii) and (v) can be deduced by applying the double expectation rule. $\diamond$

2.2 Stochastic integration and representation theorems

Given the martingale structure of the centered doubly stochastic Poisson random field $\tilde{H}$ with respect to the filtrations $\mathbb{F}$ and $\mathbb{G}$, we can construct a non-anticipating stochastic integration of Itô type according to the classical scheme.

Since $\tilde{H}$ is a martingale random field for both the filtration $\mathbb{F}$ and $\mathbb{G}$, one may use one of the two as reference information flow in the integration. From the modeling point of view, it would be a natural choice to build an integration scheme around $\mathbb{F}$, since this filtration represents no future insight on the time-distortion. Still we want to build up an integration scheme around $\mathbb{G}$, because this integration scheme possesses an integration representation where all the processes are known. This will be of great importance in solving our optimization problem for the case of $\mathbb{G}-$information.

Non-anticipating stochastic integration with respect to general martingale random fields is treated in [6], and this theory can be adapted in the setup of this thesis by specifying the spaces and processes involved. We present the main features of the integration scheme, leading to an integral and a martingale representation theorem as treated in [8]. The theory is presented in a shortened version for quick reading. The missing links can be found in Appendix B.

To begin with, we introduce the integrands of the martingale random field as integrator. The theory is from Section 3 in [6].

Let $\mathcal{P}$ be the $\mathbb{G}-$*predictable* $\sigma-$algebra generated by sets of the form

$$F \times (s,u] \times B, \quad F \in \mathcal{F}_s^H \vee \mathcal{F}_T^\Lambda,\ 0 \le s \le u \le T,\ B \in \mathcal{B}_{\mathbb{R}_0},$$

and let $\mathcal{P}_0$ be the $\mathbb{G}-$predictable $\sigma-$algebra generated by sets of the form

$$F \times (s,u], \quad F \in \mathcal{F}_s^W \vee \mathcal{F}_T^\Lambda,\ 0 \le s \le u \le T. \tag{2.1}$$

Definition 2.4 *A measurable function*

$$\gamma : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}$$

is a simple integrand *if it admits the following representation*

$$\gamma(t,z) = \sum_{k=1}^{K} \gamma_k \chi_{\Delta_k}(t,z)$$

where $\Delta_1, ..., \Delta_K$ are pairwise disjoint sets of the form $\Delta_k = (s_k, u_k] \times B_k$ with $m(\Delta_k) < \infty$, and the values γ_k are $\mathcal{G}_{s_k}-$measurable random variables satisfying

$$\mathbb{E}[\int_{\Delta_k} \gamma_k^2(t,z)\Lambda(dt,dz)] < \infty$$

Thus, the simple integrands are elements of

$$L_2(\mathbb{P}\times\Lambda) := L_2(\Omega\times[0,T]\times\mathbb{R}_0, (\mathcal{F}_T^H\vee\mathcal{F}_T^\Lambda)\times\mathcal{B}_{[0,T]}\times\mathcal{B}_{\mathbb{R}_0}, \mathbb{P}\times\Lambda),$$

with the finite norm

$$\|\gamma\|_{L_2(\mathbb{P}\times\Lambda)} := \Big(\mathbb{E}[\int_0^T\int_{\mathbb{R}_0}\gamma^2(t,z)\Lambda(dt,dz)]\Big)^{1/2}.$$

Note that a simple integrand is a predictable function, i.e.

$$\gamma\in L_2(\mathcal{P}) := L_2(\Omega\times[0,T]\times\mathbb{R}_0, \mathcal{P}, \mathbb{P}\times\Lambda).$$

Definition 2.5 *A measurable function*

$$\gamma : \Omega\times[0,T]\times\mathbb{R}_0\to\mathbb{R}$$

is a (general) integrand *if it can be represented as a limit $\gamma = \lim_{n\to\infty}\gamma_n$ with convergence in $L_2(\mathbb{P}\times\Lambda)$ of a sequence $(\gamma_n)_{n\geq 1}$ of simple integrands.*

Remark 3.1 in [6]. The set of general integrands corresponds to $L_2(\mathcal{P})$, that is the subspace of elements in $L_2(\mathbb{P}\times\Lambda)$ admitting a predictable representative. $\diamond$

According to classical Itô integration scheme, for any integrand $\gamma = \lim_{n\to\infty}\gamma_n$, we can define the non-anticipating integral $\mathcal{J}$ as the limit

$$\mathcal{J}(\gamma) = \int_0^T\int_{\mathbb{R}_0}\gamma(t,z)\tilde{H}(dt,dz) := \lim_{n\to\infty}\int_0^T\int_{\mathbb{R}_0}\gamma_n(t,z)\tilde{H}(dt,dz),$$

with convergence in $L_2(\mathbb{P})$. For this the Itô isometry is crucial:

$$\mathbb{E}\Big[\Big(\int_0^T\int_{\mathbb{R}_0}\gamma(t,z)\tilde{H}(dt,dz)\Big)^2\Big] = \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0}\gamma^2(t,z)\Lambda(dt,dz)\Big].$$

Moreover, the basic rules of calculus hold:

$$\mathbb{E}\left[\int_\Delta\gamma(s,z)\tilde{H}(ds,dz)\Big|\mathcal{G}_t\right] = 0, \tag{2.2}$$

$$\begin{aligned}\mathbb{E}\Big[\int_\Delta\gamma(s,z)\tilde{H}(ds,dz)\int_\Delta\theta(s,z)\tilde{H}(ds,dz)\Big|\mathcal{G}_t\Big]\\ = \int_\Delta\mathbb{E}[\gamma(s,z)\theta(s,z)|\mathcal{G}_t]\Lambda(ds,dz),\end{aligned}$$

$$\mathbb{E}\left[\int_{\Delta_1}\gamma(s,z)\tilde{H}(ds,dz)\int_{\Delta_2}\theta(s,z)\tilde{H}(ds,dz)\Big|\mathcal{G}_t\right] = 0, \tag{2.3}$$

for $\Delta,\Delta_1,\Delta_2\in\mathcal{B}_{(t,T]\times\mathbb{R}_0}$ and $\Delta_1\cap\Delta_2=\emptyset$.

Remark 3.2 in [6]. Let $\gamma \in L_2(\mathcal{P})$. For any $\Delta \in \mathcal{B}_{[0,T]} \times \mathcal{B}_{\mathcal{R}_0}$ define

$$\mathcal{J}(\gamma, \Delta) := \int_{\Delta} \gamma(t,z)\tilde{H}(dt,dz).$$

From (2.2) and (2.3) we see that the stochastic measure $\mathcal{J}(\gamma, \Delta)$, $\Delta \in \mathcal{B}_{[0,T]} \times \mathcal{B}_{\mathbb{R}_0}$ is a martingale random field with corresponding conditional variance and variance measures given by

$$\Lambda(\gamma, \Delta) = \int_{\Delta} \gamma^2(t,z)\Lambda(dt,dz)$$

and

$$m(\gamma, \Delta) = \mathbb{E}\left[\int_{\Delta} \gamma^2(t,z)\Lambda(dt,dz)\right].$$

Thus martingale random fields appear naturally after non-anticipating integration with respect to another martingale random field as integrator. $\diamond$

Next we give a proposition that holds for the integration scheme of information flow $\mathbb{G}$ only. This is a critical result that separates the representation properties for the integration scheme around $\mathbb{G}$ and $\mathbb{F}$. This is Proposition 4.3 in [7].

Proposition 2.6 *Consider the $\mathcal{F}_T^{\Lambda}-$measurable $\beta \in L_2(\Omega, \mathcal{G}_T, \mathbb{P})$ and $\gamma \in L_2(\mathcal{P})$. Then*

$$\beta\mathcal{J}(\gamma) = \mathcal{J}(\beta\gamma),$$

if either side exists as an element of $L_2(\Omega, \mathcal{G}_T, \mathbb{P})$.

Proof: As in [7]. Assume γ is a simple integrand (Definition 2.4) and β is bounded. Then, for every k, $\beta\gamma_k$ is $\mathcal{G}_{s_k}-$measurable and

$$\beta\mathcal{J}(\gamma) = \sum_{k=1}^{K} \beta\gamma_k \tilde{H}((s_k, u_k] \times B_k) = \mathcal{J}(\beta\gamma).$$

The general case follows by taking limits. ■

Until now, nothing has been said about the stochastic integration with respect to the Brownian motion, but this follows from the same Itô calculus structure. In particular the integrands of the integration with respect to the Brownian motion are elements of $L_2(\mathcal{P}_0)$, where we recall that $\mathcal{P}_0$ is a set of $\mathbb{G}-$predictable integrands. (See definition in (2.1).) Here we also use $\mathbb{G}$ as the information flow and the corresponding result of Proposition 2.6 holds.

Definition 2.7 *Define the two spaces of integrands*

$$\mathcal{I}_{\mathbb{G}} := \{\theta = (\theta_0, \theta_1) \mid (\theta_0, \theta_1) \in L_2(\mathcal{P}_0) \times L_2(\mathcal{P})\},$$

and

$$\mathcal{I}_{\mathbb{F}} := \{\theta = (\theta_0, \theta_1) \mid (\theta_0, \theta_1) \in \mathcal{I}_{\mathbb{G}},\ \mathbb{F}\text{−}predictable\}.$$

Clearly we have $\mathcal{I}_{\mathbb{F}} \subset \mathcal{I}_{\mathbb{G}}$.

Next we give an integral and martingale representation theorem for the integration scheme with respect to the information flow $\mathbb{G}$. These representations are nice in the way that they give an explicit representation without unknown processes. See Appendix B for details on these results, or in the paper [8], where the theory is taken from.

Theorem 2.8 (Integral Representation) *Assume* $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$. *Then there exists a unique* $\theta \in \mathcal{I}_{\mathbb{G}}$ *such that*

$$\xi = \mathbb{E}[\xi|\mathcal{F}_T^\Lambda] + \int_0^T \theta_0(t)dW(t) + \int_0^T \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{H}(dt,dz), \tag{2.4}$$

where $\mathbb{E}[\xi|\mathcal{F}_T^\Lambda] \in L_2(\Omega, \mathcal{F}_T^\Lambda, \mathbb{P})$. *Moreover, the* $\mathbb{E}[\xi|\mathcal{F}_T^\Lambda]$ *and* $\int_0^T \theta_0(t)dW(t) + \int_0^T \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{H}(dt,dz)$ *are orthogonal in* $L_2(\mathbb{P})$.

Proof: See Appendix B, or proof of Theorem 3.3 in [8].

Theorem 2.9 (Martingale Representation) *Let* $M(t)$, $t \in [0,T]$, *be a* $(\mathbb{G}, \mathbb{P})$*-martingale. Then there exists a unique* $\theta \in \mathcal{I}_{\mathbb{G}}$ *such that*

$$M(t) = \mathbb{E}[M(T)|\mathcal{F}_T^\Lambda] + \int_0^t \theta_0(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \theta_1(s,z)\tilde{H}(ds,dz), \quad t \in [0,T], \tag{2.5}$$

where $\mathbb{E}[M(T)|\mathcal{F}_T^\Lambda] \in L_2(\Omega, \mathcal{F}_T^\Lambda, \mathbb{P})$. *Moreover,* $\mathbb{E}[\xi|\mathcal{F}_T^\Lambda]$ *and* $\int_0^t \theta_0(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \theta_1(s,z)\tilde{H}(ds,dz)$ *are for all* $t \in [0,T]$ *orthogonal in* $L_2(\mathbb{P})$.

Proof: See Appendix B.

2.3 Backward stochastic differential equations

When considering information flow $\mathbb{G}$, we work with BSDEs of the form

$$\begin{aligned}Y(t) = \xi + \int_t^T g(s, \lambda(s), Y(s), Z(s), U(s,z))ds - \int_t^T Z(s)dW(s) \\ - \int_t^T \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz), \end{aligned} \tag{2.6}$$

Here the random variable $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ is the terminal condition, and the function g is the generator of the BSDE. The stochastic process $Z : \Omega \times [0,T] \to \mathbb{R}$ and the stochastic field $U : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}$ are predictable processes in $\mathcal{I}_{\mathbb{G}}$. Below we determine sufficient conditions on the couple (ξ, g) in order to have a unique solution (Y, Z, U) to the BSDE (2.6). First we define the domain of the solution Y.

Definition 2.10 *The space* $(\mathbb{S}_2(\mathbb{P}), \|Y\|^2_{\mathbb{S}_2})$ *defines a Banach space, where the*

$$\mathbb{S}_2(\mathbb{P}) := \{Y : \Omega \times [0,T] \to \mathbb{R} \mid \mathbb{G}-\textit{adapted, càdlàg, and } \mathbb{E}\Big[\sup_{t\in[0,T]} |Y(t)|^2\Big] < \infty\},$$

and

$$\|Y\|^2_{\mathbb{S}_2} := \mathbb{E}\Big[\sup_{t\in[0,T]} e^{\rho t}|Y(t)|^2\Big],$$

for some $\rho > 0$.

In order to have a unique solution $(Y, Z, U) \in \mathbb{S}_2(\mathbb{P}) \times \mathcal{I}_{\mathbb{G}}$ to the BSDE (2.6), the couple (ξ, g) must consist of *standard parameters*. Before we can define what standard parameters are, we need to introduce the function space $\mathcal{R}$:

Definition 2.11 *Let* $\mathcal{R}$ *be space of functions* $u : \mathbb{R} \to \mathbb{R}$ *that satisfy*

$$\int_{\mathbb{R}_0} u^2(x)\nu(dx) < \infty,$$

where ν *is the jump measure of the compensator of the (doubly stochastic Poisson) random measure.*

Definition 2.12 *The couple* (ξ, g) *are standard parameters, with respect to* $\mathbb{G}$*, for a BSDE on* $(\Omega, \mathcal{F}_T, \mathbb{P})$ *if* $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ *and* $g : \Omega \times [0,T] \times [0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies, for some* $K_g > 0$*, the following:*

(i) $g(\cdot,\cdot,\lambda, Y, Z, U(\cdot))$ *is* $\mathbb{G}$*-adapted for all* $\lambda \in \mathcal{L}$*,* $Y \in \mathbb{S}_2(\mathbb{P})$*,* $(Z,U) \in \mathcal{I}_{\mathbb{G}}$*;*

(ii) For all $\lambda \in \mathcal{L}$ *we have* $g(\cdot,\cdot,\lambda(\cdot),0,0,0)$ $\mathbb{G}$*-predictable and*

$$\mathbb{E}\left[\int_0^T g^2(t,\lambda(t),0,0,0)dt\right] < \infty;$$

(iii) For all $\lambda \in [0,\infty), y_1, y_2, z_1, z_2 \in \mathbb{R}$ *and* $u_1, u_2 \in \mathcal{R}$ *we have*

$$\begin{aligned}|g(t,&\lambda, y_1, z_1, u_1) - g(t,\lambda,y_2,z_2,u_2)| \leq \\ &K_g\left(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\int_{\mathbb{R}_0} |u_1(x) - u_2(x)|^2 \nu(dx)\lambda}\right).\end{aligned}$$

With (ξ, g) being standard parameters, we know that there exists a unique solution to the equation (2.6) $\mathbb{P} \times dt-$a.e. by the next theorem.

Theorem 2.13 *Let* (ξ, g) *be standard parameters. Then there exists a unique triple* $(Y,Z,U) \in \mathbb{S}_2(\mathbb{P}) \times \mathcal{I}_{\mathbb{G}}$*,* $\mathbb{P} \times dt-a.e.$ *such that*

$$\begin{aligned}Y(t) = \xi + &\int_t^T g(s,\lambda(s),Y(s),Z(s),U(s,z))ds - \int_t^T Z(s)dW(s) \\ &- \int_t^T \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz).\end{aligned}$$

Proof: See Appendix B or Theorem 4.5 in [8].

Next, we state a comparison theorem from [8] that compares two solutions of the BSDE (2.6) with two different sets of standard parameters. The theorem states that for certain assumptions on the standard parameters, we can compare the solutions $\mathbb{P} \times dt-$a.e:

Theorem 2.14 *Let* (ξ_1, g_1) *and* (ξ_2, g_2) *be two sets of standard parameters for the BSDEs with solutions* $(Y_1, Z_1, U_1), (Y_2, Z_2, U_2) \in \mathbb{S}_2(\mathbb{P}) \times \mathcal{I}_{\mathbb{G}}$*. Assume that*

$$g_2(t,\lambda,y,z,u(\cdot)) = f(t,y,z\alpha(t), \int_{\mathbb{R}_0} \beta(t,x)u(x)\nu(dx)\sqrt{\lambda}),$$

where $(\alpha, \beta) \in \mathcal{I}_{\mathbb{G}}$ *and satisfies* $0 \leq \beta(t,x) < Kx$ *for* $x \in \mathbb{R}_0$*, and* $|\alpha(t)| < K$ $\mathbb{P} \times dt-a.e.$ *for some* $K > 0$*. Assume the function* $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* *(i)* $|f(t,y,b,h) - f(t,y',b',h')| \leq K_h(|y-y'|+|b-b'|+|h-h'|)$

(ii) $\mathbb{E}\left[\int_0^T |f(t,0,0,0)|^2 dt\right] < \infty$

If

$$\xi_1 \leq \xi_2 \quad \mathbb{P} - a.s.$$

and

$$g_1(t,\lambda(t),Y^1(t),Z^1(t),U^1(t,\cdot)) \leq g_2(t,\lambda(t),Y^1(t),Z^1(t),U^1(t,\cdot)) \quad \mathbb{P} \times dt - a.e.,$$

then

$$Y^1(t) \leq Y^2(t) \quad \mathbb{P} \times dt - a.e.$$

Proof: See [8], Theorem 5.2.

Remark. It can be shown that f is indeed a standard parameter and satisfies $(i) - (iii)$ in Definition 2.12. $\diamond$

2.4 Theory on the maximum principle

This section contains theory on the maximum principle, which later will be used to find a solution to the optimization problem at the initial time $t = 0$. We look at two different performance criteria: The performance of the optimization problem at the initial time for the information flow $\mathbb{G}$, and the performance of the optimization problem at the initial time for the information flow $\mathbb{F}$.

The theory is inspired by the paper [1]. The modifications from the original paper is the following: Firstly, in the case of information flow $\mathbb{G}$, the performance criterion is in this thesis conditioned on $\mathcal{G}_0 = \mathcal{F}_T^\Lambda$. Secondly, the assumptions on the processes are different. This is because the driving processes in [1] are Brownian noises and Poisson noises, while in this thesis we have extended the jump noise.

The two situations need their own theorem, but the processes involved have the same form. The processes are different in the two situations in that the parameters are adapted to different filtrations. We introduce the processes independently of the situation of the information flow.

First, we introduce a general state process. π and $\theta = (\theta_0, \theta_1)$ are the control parameters we wish to optimize:

$$\begin{aligned} dX(t) &= b(t, X(t), \lambda(t), \pi(t), \theta(t))dt + \sigma(t, X(t), \lambda(t), \pi(t), \theta(t))dW(t) \\ &\quad + \int_{\mathbb{R}_0} \gamma(t, X(t^-), \lambda(t), \pi(t), \theta(t), z)\tilde{H}(ds, dz) \qquad (2.7) \\ X(0) &= x \in \mathbb{R} \end{aligned}$$

Here, $b : [0,T] \times \mathbb{R} \times \mathbb{R}_+ \times K^3 \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times \mathbb{R}_+ \times K^3 \to \mathbb{R}$ and $\gamma : [0,T] \times \mathbb{R} \times \mathbb{R}_+ \times K^3 \times \mathbb{R}_0 \to \mathbb{R}$ are given continuous functions, differentiable in x, and K is a given closed, convex subset of $\mathbb{R}$.

The performance criteria are given by:

Performance criterion for the information flow $\mathbb{G}$

$$J_1(\pi, \theta) = \mathbb{E}\left[\int_0^T f(s, X(s^-), \lambda(s), \pi(s), \theta(s))ds + l(X(T)) \middle| \mathcal{F}_T^\Lambda \right].$$

Performance criterion for the information flow $\mathbb{F}$

$$J_2(\pi, \theta) = \mathbb{E}\left[\int_0^T f(s, X(s^-), \lambda(s), \pi(s), \theta(s))ds + l(X(T)) \right].$$

Here $x \mapsto f(t, x, \lambda, \pi, \theta)$ is a real, differentiable function, and $x \mapsto l(x)$ is a real concave differentiable function.

Define the optimization problem by

$$J(\hat{\pi},\hat{\theta}) = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}^1}\left(\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^2} J(\pi,\theta)\right), \tag{2.8}$$

where $\mathcal{A}^1$ and $\mathcal{A}^2$ are the admissible sets of controls. The admissible controls $(\pi,\theta)\in\mathcal{A}^1\times\mathcal{A}^2$ are either $\mathbb{G}-$predictable or $\mathbb{F}-$predictable and we denote these sets as $\mathcal{A}^i_{\mathbb{G}}$ and $\mathcal{A}^i_{\mathbb{F}}$ for $i=1,2$, respectively.

Define the Hamiltonian, $\mathcal{H}:[0,T]\times\mathbb{R}\times[0,\infty)\times K^3\times\mathbb{R}^2\times\mathcal{R}$ by

$$\begin{aligned}\mathcal{H}(t,x,\lambda,\pi,\theta,p,q,r) &= f(t,x,\lambda,\pi,\theta)+b(t,x,\lambda,\pi,\theta)p+\sigma(t,x,\lambda,\pi,\theta)q\\ &\quad+\int_{\mathbb{R}_0}\gamma(t,x,\lambda,\pi,\theta,z)r(z)\nu(dz)\lambda \end{aligned}\tag{2.9}$$

Corresponding to (X,π,θ) is the solution (p,q,r) to the adjoint equation

$$\begin{aligned}p(t) &= \partial_x l(X(T))+\int_t^T \partial_x\mathcal{H}(s,X(s),\lambda(s),\pi(t),\theta(t),p(s),q(s),r(s,\cdot))ds\\ &\quad-\int_t^T q(s)dW(s)-\int_t^T\int_{\mathbb{R}_0} r(s^-,z)\tilde{H}(ds,dz)\end{aligned}\tag{2.10}$$

For standard parameters $(\partial_x\mathcal{H},\partial_x l(X(T)))$, there exists a unique solution to the BSDE (2.10) by Theorem 2.13.

Now we state two theorems with slight differences. The first solves the control problem of (2.8) for J_1, i.e. solves

$$J_1(\hat{\pi},\hat{\theta}) = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}^1_{\mathbb{G}}}\left(\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^2_{\mathbb{G}}} J_1(\pi,\theta)\right), \tag{2.11}$$

and the second solves the control problem of (2.8) for J_2, i.e. solves

$$J_2(\hat{\pi},\hat{\theta}) = \inf_{\pi\in\mathcal{A}^1_{\mathbb{F}}}\left(\sup_{\theta\in\mathcal{A}^2_{\mathbb{F}}} J_2(\pi,\theta)\right). \tag{2.12}$$

The following theorem is a slight modification of Theorem 2.1 in [1] in order to fit the the problem in (2.11).

Notation: We use the notation $X^{\pi}(t):=X^{\pi,\hat{\theta}}(t)$ and $X^{\theta}(t):=X^{\hat{\pi},\theta}(t)$. Moreover, denote $\hat{X}(t):=X^{\hat{\pi},\hat{\theta}}(t)$, and $X(t):=X^{\pi,\theta}(t)$. We also abbreviate the parameters, e.g. denote $\sigma(t,X(t),\lambda(t),\pi(t),\theta(t))=\sigma(X(t),\pi(t),\theta(t))$, etc. This makes the calculations more readable.

Theorem 2.15 (Maximum principle I) *Let* $(\hat{\pi}, \hat{\theta}) \in \mathcal{A}^1_{\mathbb{G}} \times \mathcal{A}^2_{\mathbb{G}}$. *Suppose there exists a solution* $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ *of the adjoint equation (2.10) such that, for all* $\pi \in \mathcal{A}^1_{\mathbb{G}}$ *and* $\theta \in \mathcal{A}^2_{\mathbb{G}}$, *we have*

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T |\hat{p}(s^-)\sigma(\hat{X}(s^-), \hat{\pi}(s), \theta(s))|^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}_0} |\hat{p}(s^-)\gamma(X^\theta(s^-), \hat{\pi}(s), \theta(s), z)|^2 \Lambda(ds, dz) \Big| \mathcal{F}_T^\Lambda\Big] < \infty, \\
&\mathbb{E}\Big[\int_0^T |\hat{p}(s^-)\sigma(\hat{X}(s^-), \pi(s), \hat{\theta}(s))|^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}_0} |\hat{p}(s^-)\gamma(X^\pi(s^-), \pi(s), \hat{\theta}(s), z)|^2 \Lambda(ds, dz) \Big| \mathcal{F}_T^\Lambda\Big] < \infty, \\
&\mathbb{E}\Big[\int_0^T |(\hat{X}(s^-) - X^\pi(s^-))\hat{q}(s)|^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}_0} |(\hat{X}(s^-) - X^\pi(s^-))\hat{r}(s,z)|^2 \Lambda(ds, dz) \Big| \mathcal{F}_T^\Lambda\Big] < \infty, \\
&\mathbb{E}\Big[\int_0^T |(\hat{X}(s^-) - X^\theta(s^-))\hat{q}(s)|^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}_0} |(\hat{X}(s^-) - X^\theta(s^-))\hat{r}(s,z)|^2 \Lambda(ds, dz) \Big| \mathcal{F}_T^\Lambda\Big] < \infty,
\end{aligned}$$

$\mathbb{P}-$*a.s. Denote* $\hat{\mathcal{H}}$ *by*

$$\hat{\mathcal{H}}(X(t^-), \pi(t), \theta(t)) = \mathcal{H}(t, X(t^-), \lambda(t), \pi(t), \theta(t), \hat{p}(t^-), \hat{q}(t), \hat{r}(t, \cdot)).$$

Suppose for all $t \in [0,T]$ *the following holds* $\mathbb{P}-$*a.s.:*

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} \hat{\mathcal{H}}(\hat{X}(t^-), \pi, \hat{\theta}(t)) \\
&\qquad = \hat{\mathcal{H}}(\hat{X}(t^-), \hat{\pi}(t), \hat{\theta}(t)) \\
&\qquad\qquad = \operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} \hat{\mathcal{H}}(\hat{X}(t^-), \hat{\pi}(t), \theta).
\end{aligned} \tag{2.13}$$

(i) Suppose that, for all $t \in [0,T]$, $l(x)$ *is convex and*

$$(x, \pi) \mapsto \hat{\mathcal{H}}(x, \pi, \hat{\theta}(t)),$$

is convex. Then,

$$J_1(\hat{\pi}, \hat{\theta}) \le J_1(\pi, \hat{\theta}), \quad \textit{for all } \pi \in \mathcal{A}^1_{\mathbb{G}},\ \mathbb{P} - a.s.,$$

and

$$J_1(\hat{\pi}, \hat{\theta}) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} J_1(\pi, \hat{\theta}).$$

(ii) Suppose that, for all $t \in [0,T]$, $l(x)$ is concave and

$$(x, \theta) \mapsto \hat{\mathcal{H}}(x, \hat{\pi}(t), \theta),$$

is concave. Then,

$$J_1(\hat{\pi}, \hat{\theta}) \geq J_1(\hat{\pi}, \theta), \quad \textit{for all } \theta \in \mathcal{A}^2_{\mathbb{G}},\ \mathbb{P} - a.s.,$$

and

$$J_1(\hat{\pi}, \hat{\theta}) = \operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} J_1(\hat{\pi}, \theta).$$

(iii) If both (i) and (ii) hold, i.e. l is linear, then $(\hat{\pi}, \hat{\theta})$ are optimal controls and we have

$$\operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} \left(\operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} J(\pi, \theta) \right) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} \left(\operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} J(\pi, \theta) \right).$$

Proof: We divide the problem in two. First assume that (i) holds. Choose $(\pi, \theta) \in \mathcal{A}_{\mathbb{G}}^1 \times \mathcal{A}_{\mathbb{G}}^2$ and consider

$$J_1(\hat{\pi}, \hat{\theta}) - J_1(\pi, \hat{\theta}) = I_1 + I_2,$$

where

$$I_1 = \mathbb{E}\bigg[\int_0^T f(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - f(X^\pi(s^-), \pi(s), \hat{\theta}(s))ds\bigg|\mathcal{F}_T^\Lambda\bigg],$$

and

$$I_2 = \mathbb{E}\bigg[l(\hat{X}(T)) - l(X^\pi(T))\bigg|\mathcal{F}_T^\Lambda\bigg].$$

Since l is convex, we have by convex property

$$\begin{aligned} I_2 &= \mathbb{E}\bigg[l(\hat{X}(T)) - l(X^\pi(T))\bigg|\mathcal{F}_T^\Lambda\bigg] \\ &\leq \mathbb{E}\bigg[(\hat{X}(T) - X^\pi(T))\partial_x l(\hat{X}(T))\bigg|\mathcal{F}_T^\Lambda\bigg] \\ &= \mathbb{E}\bigg[(\hat{X}(T) - X^\pi(T))\hat{p}(T)\bigg|\mathcal{F}_T^\Lambda\bigg]. \end{aligned}$$

The dynamics of $(\hat{X}(t) - X^\pi(t))\hat{p}(t)$ are given by

$$\begin{aligned} &d[(\hat{X}(t) - X^\pi(t))\hat{p}(t)] \\ &\quad= (\hat{X}(t^-) - X^\pi(t^-))d\hat{p}(t) + \hat{p}(t^-)(d\hat{X}(t) - dX^\pi(t)) \\ &\qquad+ [\sigma(\hat{X}(t^-), \hat{\pi}(t), \hat{\theta}(t)) - \sigma(X^\pi(t^-), \pi(t), \hat{\theta}(t))]\hat{q}(t)dt \\ &\qquad+ \int_{\mathbb{R}_0}[\gamma(\hat{X}(t^-), \hat{\pi}(t), \hat{\theta}(t), z) - \gamma(X^\pi(t^-), \pi(t), \hat{\theta}(t), z)]\hat{r}(t, z)\Lambda(dt, dz). \end{aligned}$$

Thus,

$$\begin{aligned} I_2 \leq \mathbb{E}\bigg[&(\hat{X}(0) - X^\pi(0))\hat{p}(0) + \int_0^T (\hat{X}(s^-) - X^\pi(s^-))d\hat{p}(s) \\ &+ \int_0^T \hat{p}(s^-)(d\hat{X}(s) - dX^\pi(s)) + \int_0^T [\sigma(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) \\ &- \sigma(X^\pi(s^-), \pi(s), \hat{\theta}(s))]\hat{q}(s)ds \\ &+ \int_0^T \int_{\mathbb{R}_0} [\gamma(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s), z) \\ &- \gamma(X^\pi(s^-), \pi(s), \hat{\theta}(s), z)]\hat{r}(s, z)\Lambda(ds, dz)\bigg|\mathcal{F}_T^\Lambda\bigg] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\bigg[\int_0^T (\hat{X}(s^-) - X^\pi(s^-)) \cdot (-\partial_x \hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)))ds \bigg| \mathcal{F}_T^\Lambda \bigg] \\
&\quad + \mathbb{E}\bigg[\int_0^T \hat{p}(s^-)(b(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - b(X^\pi(s^-), \pi(s), \hat{\theta}(s)))ds \\
&\quad + \int_0^T [\sigma(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - \sigma(X^\pi(s^-), \pi(s), \hat{\theta}(s))]\hat{q}(s)ds \\
&\quad + \int_0^T \int_{\mathbb{R}_0} [\gamma(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s), z) \\
&\quad - \gamma(X^\pi(s^-), \pi(s), \hat{\theta}(s), z)]\hat{r}(s,z)\Lambda(ds,dz) \bigg| \mathcal{F}_T^\Lambda \bigg].
\end{aligned} \tag{2.14}
$$

Over to I_1, we have by the definition of $\mathcal{H}$ in (2.9) the following relations

$$
\begin{aligned}
I_1 &= \mathbb{E}\bigg[\int_0^T f(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - f(X^\pi(s^-), \pi(s), \hat{\theta}(s))ds \bigg| \mathcal{F}_T^\Lambda \bigg] \\
&= \mathbb{E}\bigg[\int_0^T [\hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - \hat{\mathcal{H}}(X^\pi(s^-), \pi(s), \hat{\theta}(s))]ds \bigg| \mathcal{F}_T^\Lambda \bigg] \\
&\quad - \mathbb{E}\bigg[\int_0^T \hat{p}(s^-)(b(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - b(X^\pi(s^-), \pi(s), \hat{\theta}(s)))ds \\
&\quad + \int_0^T [\sigma(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - \sigma(X^\pi(s^-), \pi(s), \hat{\theta}(s))]\hat{q}(s)ds \qquad (2.15) \\
&\quad + \int_0^T \int_{\mathbb{R}_0} [\gamma(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s), z) \\
&\quad - \gamma(X^\pi(s^-), \pi(s), \hat{\theta}(s), z)]\hat{r}(s,z)\Lambda(ds,dz) \bigg| \mathcal{F}_T^\Lambda \bigg].
\end{aligned}
$$

Thus, combining (2.14) and (2.15) we have $\mathbb{P}-$a.s.

$$
\begin{aligned}
I_1 + I_2 \le \mathbb{E}\bigg[\int_0^T &\Big\{ \hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - \hat{\mathcal{H}}(X^\pi(s^-), \pi(s), \hat{\theta}(s)) \Big\} \\
&- \Big\{ (\hat{X}(s^-) - X^\pi(s^-)) \cdot \partial_x \hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) \Big\} ds \bigg| \mathcal{F}_T^\Lambda \bigg]
\end{aligned} \tag{2.16}
$$

By the convexity of $\mathcal{H}$ in x and π, we have $\mathbb{P} \times ds-$a.e.

$$
\begin{aligned}
&\hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - \hat{\mathcal{H}}(X^\pi(s^-), \pi(s), \hat{\theta}(s)) \\
&\quad \le \partial_x \hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s))(\hat{X}(s^-) - X^\pi(s^-)) \\
&\qquad + \partial_\pi \hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s))(\hat{\pi}(s) - \pi(s)).
\end{aligned} \tag{2.17}
$$

Since minimum of $\pi \mapsto \hat{\mathcal{H}}(X^\pi(s^-), \pi(s), \hat{\theta}(s))$ is attained for $\pi = \hat{\pi}(t)$ by (2.13), we get that

$$
\partial_\pi \hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s))(\hat{\pi}(s) - \pi(s)) \le 0, \quad \mathbb{P} \times ds - a.e. \tag{2.18}
$$

Consequently, from (2.16), (2.17) and (2.18), we have

$$J_1(\hat{\pi}, \hat{\theta}) - J_1(\pi, \hat{\theta}) = I_1 + I_2 \leq 0, \quad \mathbb{P} - a.s. \tag{2.19}$$

We therefore conclude that $J_1(\hat{\pi}, \hat{\theta}) \leq J_1(\pi, \hat{\theta})$, $\mathbb{P} - a.s.$, for all $\pi \in \mathcal{A}^1_{\mathbb{G}}$.

Now assume (ii) holds. Following the same steps, but now with concavity of $\mathcal{H}$ in x and θ, we obtain the inequality

$$J_1(\hat{\pi}, \hat{\theta}) \geq J_1(\hat{\pi}, \theta), \quad \mathbb{P} - a.s. \text{ for all } \theta \in \mathcal{A}^2_{\mathbb{G}}. \tag{2.20}$$

If both (i) and (ii) hold, then it follows from (2.19) and (2.20) that

$$J_1(\hat{\pi}, \theta) \leq J_1(\hat{\pi}, \hat{\theta}) \leq J_1(\pi, \hat{\theta}), \quad \mathbb{P} - a.s., \tag{2.21}$$

for any $(\pi, \theta) \in \mathcal{A}^1_{\mathbb{G}} \times \mathcal{A}^2_{\mathbb{G}}$. So by the second inequality in (2.21), we have

$$J_1(\hat{\pi}, \hat{\theta}) \leq \operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} J_1(\pi, \hat{\theta}) \leq \operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} \left(\operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} J_1(\pi, \theta) \right)$$

On the other hand, by the first inequality in (2.21), we have

$$J_1(\hat{\pi}, \hat{\theta}) \geq \operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} J_1(\hat{\pi}, \theta) \geq \operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} \left(\operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} J_1(\pi, \theta) \right).$$

Now, due to the inequality

$$\operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} \left(\operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} J_1(\pi, \theta) \right) \geq \operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} \left(\operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} J_1(\pi, \theta) \right),$$

we have

$$\operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} \left(\operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} J_1(\pi, \theta) \right) = \operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} \left(\operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} J_1(\pi, \theta) \right). \quad \blacksquare$$

The following theorem is a slight modification of Theorem 2.1 in [1] in order to fit the the problem in (2.12).

Theorem 2.16 (Maximum principle II) *Let* $(\hat{\pi}, \hat{\theta}) \in \mathcal{A}^1_{\mathbb{F}} \times \mathcal{A}^2_{\mathbb{F}}$. *Suppose there exists a solution* $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ *of the adjoint equation (2.10) such that, for all* $\pi \in \mathcal{A}^1_{\mathbb{F}}$ *and* $\theta \in \mathcal{A}^2_{\mathbb{F}}$, *we have*

$$\mathbb{E}\Big[\int_0^T |\hat{p}(s^-)\sigma(\hat{X}(s^-), \hat{\pi}(s), \theta(s))|^2 ds + \int_0^T \int_{\mathbb{R}_0} |\hat{p}(s^-)\gamma(X^\theta(s^-), \hat{\pi}(s), \theta(s), z)|^2 \Lambda(ds, dz)\Big] < \infty,$$

$$\mathbb{E}\Big[\int_0^T |\hat{p}(s^-)\sigma(\hat{X}(s^-), \pi(s), \hat{\theta}(s))|^2 ds + \int_0^T \int_{\mathbb{R}_0} |\hat{p}(s^-)\gamma(X^\pi(s^-), \pi(s), \hat{\theta}(s), z)|^2 \Lambda(ds, dz)\Big] < \infty,$$

$$\mathbb{E}\Big[\int_0^T |(\hat{X}(s^-) - X^\pi(s^-))\hat{q}(s)|^2 ds + \int_0^T \int_{\mathbb{R}_0} |(\hat{X}(s^-) - X^\pi(s^-))\hat{r}(s,z)|^2 \Lambda(ds, dz)\Big] < \infty,$$

$$\mathbb{E}\Big[\int_0^T |(\hat{X}(s^-) - X^\theta(s^-))\hat{q}(s)|^2 ds + \int_0^T \int_{\mathbb{R}_0} |(\hat{X}(s^-) - X^\theta(s^-))\hat{r}(s,z)|^2 \Lambda(ds, dz)\Big] < \infty.$$

Define

$$\hat{\mathcal{H}}^{\mathbb{F}}(X(t^-), \pi(t), \theta(t)) = \mathbb{E}[\hat{\mathcal{H}}(X(t^-), \pi(t), \theta(t)) | \mathcal{F}_t].$$

We cannot decide in general whether $\mathcal{H}$ *is* $\mathbb{F}-$*adapted, since it depends on the solution* (p, q, r) *of the adjoint equation. Suppose, for all* $t \in [0,T]$, *the following holds* $\mathbb{P}-a.s.$:

$$\operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{F}}} \hat{\mathcal{H}}^{\mathbb{F}}(\hat{X}(t^-), \pi, \hat{\theta}(t)) = \hat{\mathcal{H}}^{\mathbb{F}}(\hat{X}(t^-), \hat{\pi}(t), \hat{\theta}(t)) = \operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{F}}} \hat{\mathcal{H}}^{\mathbb{F}}(\hat{X}(t^-), \hat{\pi}(t), \theta). \tag{2.22}$$

(i) Suppose that, for all $t \in [0,T]$, $l(x)$ *is convex and*

$$(x, \pi) \mapsto \hat{\mathcal{H}}(x, \pi, \hat{\theta}(t)),$$

is convex. Then,

$$J(\hat{\pi}, \hat{\theta}) \leq J(\pi, \hat{\theta}), \quad \textit{for all } \pi \in \mathcal{A}^1_{\mathbb{F}},$$

and

$$J(\hat{\pi},\hat{\theta}) = \inf_{\pi\in\mathcal{A}^1_{\mathbb{F}}} J(\pi,\hat{\theta}).$$

(ii) Suppose that, for all $t\in[0,T]$, $l(x)$ is concave and

$$(x,\theta)\mapsto \hat{\mathcal{H}}(x,\hat{\pi}(t),\theta),$$

is concave. Then,

$$J(\hat{\pi},\hat{\theta}) \geq J(\hat{\pi},\theta), \quad \textit{for all } \theta\in\mathcal{A}^2_{\mathbb{F}},$$

and

$$J(\hat{\pi},\hat{\theta}) = \sup_{\theta\in\mathcal{A}^2_{\mathbb{F}}} J(\hat{\pi},\theta).$$

(iii) If both (i) and (ii) hold, i.e. l is linear, then $(\hat{\pi},\hat{\theta})$ are optimal controls and we have

$$\sup_{\theta\in\mathcal{A}^2_{\mathbb{F}}}\Big(\inf_{\pi\in\mathcal{A}^1_{\mathbb{F}}} J(\pi,\theta)\Big) = \inf_{\pi\in\mathcal{A}^1_{\mathbb{F}}}\Big(\sup_{\theta\in\mathcal{A}^2_{\mathbb{F}}} J(\pi,\theta)\Big).$$

Proof: The argumentation is as in the proof of the previous theorem. Assume that (i) holds, then the arguments of Theorem 2.15 are leading to the corresponding of (2.16)

$$\begin{aligned}
&J_2(\hat{\pi},\hat{\theta}) - J_2(\pi,\hat{\theta}) \\
&\leq \mathbb{E}\Big[\int_0^T \Big\{\hat{\mathcal{H}}(\hat{X}(s^-),\hat{\pi}(s),\hat{\theta}(s)) - \hat{\mathcal{H}}(X^\pi(s^-),\pi(s),\hat{\theta}(s))\Big\} \\
&\quad - \Big\{(\hat{X}(s^-) - X^\pi(s^-))\cdot\partial_x\hat{\mathcal{H}}(\hat{X}(s^-),\hat{\pi}(s),\hat{\theta}(s))\Big\}ds\Big]
\end{aligned} \tag{2.23}$$

By convexity of $\mathcal{H}$ in x and π, we have (2.17). Since the minimum of $\pi\mapsto\hat{\mathcal{H}}^{\mathbb{F}}(X^\pi(t^-),\pi(t),\hat{\theta}(t))$ is attained for $\pi=\hat{\pi}(t)$ by (2.22), and since $\pi(t),\hat{\pi}(t)$ are $\mathcal{F}_t-$measurable, we get

$$\begin{aligned}
&\mathbb{E}\Big[\partial_\pi\hat{\mathcal{H}}(\hat{X}(s^-),\hat{\pi}(s),\hat{\theta}(t))(\hat{\pi}(t)-\pi(t))\Big|\mathcal{F}_s\Big] \\
&\quad = \partial_\pi\hat{\mathcal{H}}^{\mathbb{F}}(\hat{X}(s^-),\hat{\pi}(s),\hat{\theta}(t))(\hat{\pi}(t)-\pi(t)) \\
&\quad \leq 0.
\end{aligned}$$

By the inequality above and (2.17), we see that (2.23) becomes

$$
\begin{aligned}
&J(\hat{\pi}, \hat{\theta}) - J(\pi, \hat{\theta}) \\
&\le \mathbb{E}\Big[\int_0^T \mathbb{E}\Big[\hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - \hat{\mathcal{H}}(X^\pi(s^-), \pi(s), \hat{\theta}(s))] \\
&\qquad + \partial_x \hat{\mathcal{H}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s))(\hat{X}(s^-) - X^\pi(s^-))\Big|\mathcal{F}_s\Big] ds\Big] \\
&= \mathbb{E}\Big[\int_0^T \hat{\mathcal{H}}^{\mathbb{F}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s)) - \hat{\mathcal{H}}^{\mathbb{F}}(X^\pi(s^-), \pi(s), \hat{\theta}(s))] \\
&\qquad + \partial_x \hat{\mathcal{H}}^{\mathbb{F}}(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s))(\hat{X}(s^-) - X^\pi(s^-)) ds\Big] \\
&\le 0
\end{aligned}
$$

(ii) is proved in a similar way, now with concavity. (iii) is proved as in the proof of Theorem 2.15, now with infimum and supremum instead of essential infimum and supremum. ■

The next result is a theorem that was formed in the study of the maximum principle and by the desire to have an optimization theorem that did not require a saddle point optimum. We do not use the theorem is this thesis, but the theorem is included as a curiosity.

The theorem is for the filtration $\mathbb{G}$, but a similar result can perhaps be found for $\mathbb{F}$, e.g. with a similar modifications as the one from Theorem 2.15 to Theorem 2.16.

Notation: We use the following notation in the next theorem:

$$\begin{aligned} p^\pi(t) = \partial_x l(X^\pi(T)) + \int_t^T \partial_x \mathcal{H}(s, X^\pi(s^-), \lambda(s), \pi(t), \hat{\theta}(t), p^\pi(s^-), q^\pi(s), r^\pi(s,\cdot))ds \\ - \int_t^T q^\pi(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} r^\pi(s,z)\tilde{H}(ds,dz), \end{aligned}$$

Theorem 2.17 (Maximum principle III ♣) *Let $(\hat{\pi}, \hat{\theta}) \in \mathcal{A}^1_{\mathbb{G}} \times \mathcal{A}^2_{\mathbb{G}}$. Suppose there exists a solution $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ of the adjoint equation (2.10) such that, for all $\pi \in \mathcal{A}^1_{\mathbb{G}}$ and $\theta \in \mathcal{A}^2_{\mathbb{G}}$, we have*

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T |\hat{p}(s^-)\sigma(X^\pi(s^-), \pi(s), \hat{\theta}(s))|^2 ds \\ &\quad + \int_0^T \int_{\mathbb{R}_0} |\hat{p}(s^-)\gamma(X^\pi(s^-), \pi(s), \hat{\theta}(s), z)|^2 \Lambda(ds,dz) \Big| \mathcal{F}_T^\Lambda \Big] < \infty, \\ &\mathbb{E}\Big[\int_0^T |\hat{p}(s^-)\sigma(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s))|^2 ds \\ &\quad + \int_0^T \int_{\mathbb{R}_0} |\hat{p}(s^-)\gamma(\hat{X}(s^-), \hat{\pi}(s), \hat{\theta}(s), z)|^2 \Lambda(ds,dz) \Big| \mathcal{F}_T^\Lambda \Big] < \infty, \\ &\mathbb{E}\Big[\int_0^T |(\hat{X}(s^-) - X^\pi(s^-))\hat{q}(s)|^2 ds \\ &\quad + \int_0^T \int_{\mathbb{R}_0} |(\hat{X}(s^-) - X^\pi(s^-))\hat{r}(s,z)|^2 \Lambda(ds,dz) \Big| \mathcal{F}_T^\Lambda \Big] < \infty, \end{aligned}$$

$\mathbb{P}-a.s.$ *In addition,there exists a solution $(p^\pi(t), q^\pi(t), r^\pi(t,z))$ of the adjoint* *equation (2.10) such that, for all $\pi \in \mathcal{A}^1_{\mathbb{G}}$ and $\theta \in \mathcal{A}^2_{\mathbb{G}}$, we have*

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T |p^\pi(s^-)\sigma(X^{\pi,\theta}(s^-),\pi(s),\theta(s))|^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}_0} |p^\pi(s^-)\gamma(X^{\pi,\theta}(s^-),\pi(s),\theta(s),z)|^2 \Lambda(ds,dz) \Big| \mathcal{F}^\Lambda_T \Big] < \infty, \\
&\mathbb{E}\Big[\int_0^T |p^\pi(s^-)\sigma(X^{\pi}(s^-),\pi(s),\hat{\theta}(s))|^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}_0} |p^\pi(s^-)\gamma(X^{\pi}(s^-),\pi(s),\hat{\theta}(s),z)|^2 \Lambda(ds,dz) \Big| \mathcal{F}^\Lambda_T \Big] < \infty, \\
&\mathbb{E}\Big[\int_0^T |(X^{\pi,\theta}(s^-) - X^\pi(s^-))q^\pi(s)|^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}_0} |(X^{\pi,\theta}(s^-) - X^\pi(s^-))r^\pi(s,z)|^2 \Lambda(ds,dz) \Big| \mathcal{F}^\Lambda_T \Big] < \infty,
\end{aligned}$$

$\mathbb{P}-$*a.s. Denote* $\hat{\mathcal{H}}$ *by*

$$\hat{\mathcal{H}}(X(t^-),\pi(t),\theta(t)) = \mathcal{H}(t, X(t^-), \lambda(t), \pi(t), \theta(t), \hat{p}(t^-), \hat{q}(t), \hat{r}(t,\cdot)),$$

and $\mathcal{H}^\pi$ *by*

$$\mathcal{H}^\pi(X(t^-),\pi(t),\theta(t)) = \mathcal{H}(t, X(t^-), \lambda(t), \pi(t), \theta(t), p^\pi(t^-), q^\pi(t), r^\pi(t,\cdot)),$$

Suppose for all $t \in [0,T]$ the following holds $\mathbb{P}-a.s.$:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} \mathcal{H}^\pi(X^\pi(t^-),\pi(t),\theta) = \mathcal{H}^\pi(X^\pi(t^-),\pi(t),\hat{\theta}(t)), \quad \textit{for all } \pi \in \mathcal{A}^1_{\mathbb{G}} \qquad (2.24) \\
&\operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} \hat{\mathcal{H}}(\hat{X}(t^-),\pi,\hat{\theta}(t)) = \hat{\mathcal{H}}(\hat{X}(t^-),\hat{\pi}(t),\hat{\theta}(t)).
\end{aligned}$$

(i) Suppose that, for all $t \in [0,T]$, $l(x)$ is concave and

$$(x,\theta) \mapsto \mathcal{H}^\pi(x,\pi(t),\theta), \quad \textit{for all } \pi \in \mathcal{A}^1_{\mathbb{G}},$$

is concave. Then,

$$J_1(\pi,\theta) \leq J_1(\pi,\hat{\theta}), \quad \textit{for all } \pi \in \mathcal{A}^1_{\mathbb{G}},\ \mathbb{P} - a.s.,$$

and

$$J_1(\pi,\hat{\theta}) = \operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} J_1(\pi,\theta) \quad \textit{for all } \pi \in \mathcal{A}^1_{\mathbb{G}},\ \mathbb{P} - a.s.$$

(ii) Suppose that, for all $t \in [0,T]$, $l(x)$ is convex and

$$(x,\pi) \mapsto \hat{\mathcal{H}}(x,\pi,\hat{\theta}(t)),$$

is convex. Then,

$$J_1(\hat{\pi}, \hat{\theta}) \leq J_1(\pi, \hat{\theta}), \quad \mathbb{P} - a.s.,$$

and

$$J_1(\hat{\pi}, \hat{\theta}) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} J_1(\pi, \hat{\theta}).$$

(iii) If both (i) and (ii) hold, i.e. l is linear, then $(\hat{\pi}, \hat{\theta})$ are optimal controls and we have

$$J(\hat{\pi}, \hat{\theta}) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}^1_{\mathbb{G}}} \Big(\operatorname*{ess\,sup}_{\theta \in \mathcal{A}^2_{\mathbb{G}}} J(\pi, \theta) \Big).$$

Proof: We divide the problem in two. First assume that (i) holds. Choose $(\pi, \theta) \in \mathcal{A}^1_{\mathbb{G}} \times \mathcal{A}^2_{\mathbb{G}}$ and consider

$$J(\pi, \theta) - J(\pi, \hat{\theta}) = I_1 + I_2,$$

where

$$I_1 = \mathbb{E}\Big[\int_0^T f(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) - f(X^{\pi}(s^-), \pi(s), \hat{\theta}(s)) ds \Big| \mathcal{F}^{\Lambda}_T \Big],$$

and

$$I_2 = \mathbb{E}\Big[l(X^{\pi,\theta}(T)) - l(X^{\pi}(T)) \Big| \mathcal{F}^{\Lambda}_T \Big].$$

Since l is concave, we have by concave property the following

$$\begin{aligned} I_2 &= \mathbb{E}\Big[l(X^{\pi,\theta}(T)) - l(X^{\pi}(T)) \Big| \mathcal{F}^{\Lambda}_T \Big] \\ &\leq \mathbb{E}\Big[(X^{\pi,\theta}(T) - X^{\pi}(T)) \partial_x l(X^{\pi}(T)) \Big| \mathcal{F}^{\Lambda}_T \Big] \\ &= \mathbb{E}\Big[(X^{\pi,\theta}(T) - X^{\pi}(T)) p^{\pi}(T) \Big| \mathcal{F}^{\Lambda}_T \Big]. \end{aligned} \tag{2.25}$$

The dynamics of $(X^{\pi,\theta}(t) - X^{\pi}(t)) p^{\pi}(t)$ are given by

$$\begin{aligned} & d[(X^{\pi,\theta}(t) - X^{\pi}(t)) p^{\pi}(t)] \\ &\quad = (X^{\pi,\theta}(t^-) - X^{\pi}(t^-)) dp^{\pi}(t) + p^{\pi}(t^-)(dX^{\pi,\theta}(t) - dX^{\pi}(t)) \\ &\qquad + [\sigma(X^{\pi,\theta}(t^-), \pi(t), \theta(t)) - \sigma(X^{\pi}(t^-), \pi(t), \hat{\theta}(t))] q^{\pi}(t) dt \\ &\qquad + \int_{\mathbb{R}_0} [\gamma(X^{\pi,\theta}(t^-), \pi(t), \theta(t), z) \\ &\qquad\qquad - \gamma(X^{\pi}(t^-), \pi(t), \hat{\theta}(t), z)] r^{\pi}(t, z) \Lambda(dt, dz). \end{aligned} \tag{2.26}$$

From (2.25) and (2.26), we get

$$
\begin{aligned}
I_2 \leq \mathbb{E}\bigg[&(X^{\pi,\theta}(0) - X^{\pi}(0))p^{\pi}(0) + \int_0^T (X^{\pi,\theta}(s^-) - X^{\pi}(s^-))dp^{\pi}(s) \\
&+ \int_0^T p^{\pi}(s^-)(dX^{\pi,\theta}(s) - dX^{\pi}(s)) + \int_0^T [\sigma(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) \\
&- \sigma(X^{\pi}(s^-), \pi(s), \hat{\theta}(s))]q^{\pi}(s)ds \\
&+ \int_0^T \int_{\mathbb{R}_0} [\gamma(X^{\pi,\theta}(s^-), \pi(s), \theta(s), z) \\
&- \gamma(X^{\pi}(s^-), \pi(s), \hat{\theta}(s), z)]r^{\pi}(s,z)\Lambda(ds,dz)\bigg|\mathcal{F}_T^{\Lambda}\bigg] \\
= \mathbb{E}\bigg[&\int_0^T (X^{\pi,\theta}(s^-) - X^{\pi}(s^-)) \cdot (-\partial_x \mathcal{H}^{\pi}(X^{\pi}(s^-), \pi(s), \hat{\theta}(s)))ds\bigg|\mathcal{F}_T^{\Lambda}\bigg] \\
&+ \mathbb{E}\bigg[\int_0^T p^{\pi}(s^-)(b(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) - b(X^{\pi}(s^-), \pi(s), \hat{\theta}(s)))ds \\
&\quad + \int_0^T [\sigma(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) - \sigma(X^{\pi}(s^-), \pi(s), \hat{\theta}(s))]q^{\pi}(s)ds \\
&\quad + \int_0^T \int_{\mathbb{R}_0} [\gamma(X^{\pi,\theta}(s^-), \pi(s), \theta(s), z) \\
&\quad - \gamma(X^{\pi}(s^-), \pi(s), \hat{\theta}(s), z)]r^{\pi}(s,z)\Lambda(ds,dz)\bigg|\mathcal{F}_T^{\Lambda}\bigg].
\end{aligned} \tag{2.27}
$$

Over to I_1, we have by the definition of $\mathcal{H}$ in (2.9) the following relations

$$
\begin{aligned}
I_1 &= \mathbb{E}\bigg[\int_0^T f(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) - f(X^{\pi}(s^-), \pi(s), \hat{\theta}(s))ds\bigg|\mathcal{F}_T^{\Lambda}\bigg] \\
&= \mathbb{E}\bigg[\int_0^T [\mathcal{H}^{\pi}(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) - \mathcal{H}^{\pi}(X^{\pi}(s^-), \pi(s), \hat{\theta}(s))]ds\bigg|\mathcal{F}_T^{\Lambda}\bigg] \\
&\quad - \mathbb{E}\bigg[\int_0^T p^{\pi}(s^-)(b(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) - b(X^{\pi}(s^-), \pi(s), \hat{\theta}(s)))ds \\
&\qquad + \int_0^T [\sigma(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) - \sigma(X^{\pi}(s^-), \pi(s), \hat{\theta}(s))]q^{\pi}(s)ds \\
&\qquad + \int_0^T \int_{\mathbb{R}_0} [\gamma(X^{\pi,\theta}(s^-), \pi(s), \theta(s), z) \\
&\qquad - \gamma(X^{\pi}(s^-), \pi(s), \hat{\theta}(s), z)]r^{\pi}(s,z)\Lambda(ds,dz)\bigg|\mathcal{F}_T^{\Lambda}\bigg].
\end{aligned} \tag{2.28}
$$

From (2.27) and (2.28), we have $\mathbb{P}-$a.s.

$$I_1 + I_2 \le \mathbb{E}\Big[\int_0^T \Big\{\mathcal{H}^\pi(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) - \mathcal{H}^\pi(X^\pi(s^-), \pi(s), \hat{\theta}(s))\Big\} \tag{2.29}$$
$$- \Big\{(X^{\pi,\theta}(s^-) - X^\pi(s^-)) \cdot \partial_x \mathcal{H}^\pi(X^\pi(s^-), \pi(s), \hat{\theta}(s))\Big\} ds \Big| \mathcal{F}_T^\Lambda\Big].$$

By the concavity of $\mathcal{H}^\pi$ in x and θ for all $\pi \in \mathcal{A}_{\mathbb{G}}^1$, we have $\mathbb{P} \times ds-$a.e.

$$\begin{aligned}\mathcal{H}^\pi(X^{\pi,\theta}(s^-), \pi(s), \theta(s)) &- \mathcal{H}^\pi(X^\pi(s^-), \pi(s), \hat{\theta}(s))\\ &\le \partial_x \mathcal{H}^\pi(X^\pi(s^-), \pi(s), \hat{\theta}(s))(X^{\pi,\theta}(s^-) - X^\pi(s^-))\\ &\quad + \partial_\theta \mathcal{H}^\pi(X^\pi(s^-), \pi(s), \hat{\theta}(s))(\theta(s) - \hat{\theta}(s)).\end{aligned} \tag{2.30}$$

Since maximum of $\theta \mapsto \mathcal{H}^\pi(X^\pi(s^-), \pi(s), \theta(s))$ is attained for $\theta = \hat{\theta}(s)$ for all $\pi \in \mathcal{A}_{\mathbb{G}}^1$ by (2.24), we get that

$$\partial_\theta \mathcal{H}^\pi(X^\pi(s^-), \pi(s), \hat{\theta}(s))(\theta(s) - \hat{\theta}(s)) \le 0, \quad \mathbb{P} \times ds - a.e.$$

Consequently, we get from (2.29) and (2.30) that

$$J_1(\pi, \theta) - J_1(\pi, \hat{\theta}) = I_1 + I_2 \le 0, \quad \mathbb{P} - a.s.$$

We therefore conclude that $J_1(\pi, \theta) \le J_1(\pi, \hat{\theta}),\ \mathbb{P} - a.s.$, for all $\pi \in \mathcal{A}_{\mathbb{G}}^1,\ \theta \in \mathcal{A}_{\mathbb{G}}^2$.

A direct consequence, is that we have

$$\operatorname*{ess\,sup}_{\theta \in \mathcal{A}_{\mathbb{G}}^2} J_1(\pi, \theta) \le J_1(\pi, \hat{\theta}), \quad \mathbb{P} - a.s. \text{ for all } \pi \in \mathcal{A}_{\mathbb{G}}^1$$

Moreover, by the definition of essential supremum, we have

$$\operatorname*{ess\,sup}_{\theta \in \mathcal{A}_{\mathbb{G}}^2} J_1(\pi, \theta) \ge J_1(\pi, \hat{\theta}), \quad \mathbb{P} - a.s. \text{ for all } \pi \in \mathcal{A}_{\mathbb{G}}^1$$

for some $\hat{\theta} \in \mathcal{A}_{\mathbb{G}}^2$. Hence,

$$\operatorname*{ess\,sup}_{\theta \in \mathcal{A}_{\mathbb{G}}^2} J_1(\pi, \theta) = J_1(\pi, \hat{\theta}), \quad \mathbb{P} - a.s. \text{ for all } \pi \in \mathcal{A}_{\mathbb{G}}^1 \tag{2.31}$$

Second, assume (ii) holds. Following the same steps, but now with convexity, we obtain the inequality

$$J_1(\hat{\pi}, \hat{\theta}) \le J_1(\pi, \hat{\theta}), \quad \mathbb{P} - a.s. \text{ for all } \pi \in \mathcal{A}_{\mathbb{G}}^1,$$

and in a similar manner as above, we show that

$$J_1(\hat{\pi}, \hat{\theta}) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_{\mathbb{G}}^1} J_1(\pi, \hat{\theta}), \quad \mathbb{P} - a.s., \tag{2.32}$$

If both (i) and (ii) hold, then it follows from (2.31) and (2.32) that

$$J_1(\hat{\pi}, \hat{\theta}) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_{\mathbb{G}}^1} J_1(\pi, \hat{\theta}) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_{\mathbb{G}}^1} \Big\{ \operatorname*{ess\,sup}_{\theta \in \mathcal{A}_{\mathbb{G}}^2} J_1(\pi, \theta) \Big\}, \quad \mathbb{P} - a.s. \quad ♣$$

3 Financial model ambiguity and optimization

In this chapter we give a presentation of the optimization problem together with the details of the processes involved. The problem can be written as a dynamic stochastic differential game between the agent and the opponent, which here will be called "the market". By "the market" we mean to personify the source of the model ambiguity as an opponent to the agent because of his ambiguity aversion. We simulate the situation by pretending that "the market" finds the worst possible admissible probability distribution for the portfolio of the agent. The agent, on the other hand, finds the portfolio that minimize risk and hedge the contingent claim.

First, we solve the problem directly through BSDEs by a comparison theorem. This gives a dynamic solution of a price process where the problem is evaluated at all times in the given horizon. Thereafter, we solve the problem using the maximum principle. By the maximum principle we get a solution of the problem evaluated at the initial time.

For both methods the problem is laid out in two ways: First, by find the optimal solution when the controls are $\mathbb{G}-$predictable, and then by assuming the controls are $\mathbb{F}-$predictable. Recall that $\mathbb{F} \subset \mathbb{G}$, such that the problem assuming information flow $\mathbb{F}$ is in a way a problem of partial information, even though this is a filtration of perfect information of the driving stochastic processes.

Before we formally define the optimization problem in Section 3.3 (equation (3.18)), we introduce necessary concepts involved. In Section 3.1 we define the equivalent probability measures of $\mathbb{P}$, which serves as a base for the admissible controls of the "the market". In Section 3.2 we define the financial market (Definition 3.3), and its self-financing strategies, which serves as a base for the admissible controls of the agent. The common admissible controls of the agent (Definition 3.6), and the common admissible controls of "the market" (Definition 3.5) are defined in the end of Section 3.3 together with the optimization problem (3.18).

Further, the optimization problem is solved dynamically via BSDEs for both the information flow $\mathbb{G}$ (pp. 58) and $\mathbb{F}$ (pp. 68). Then the optimization problem is solved via the maximum principle evaluated at the initial time for both the information flow $\mathbb{G}$ (pp. 75) and $\mathbb{F}$ (pp. 80). An analysis and comparison of the solutions follows.

3.1 Equivalent probability measures

We define a set of equivalent probability measures to $\mathbb{P}$, and stochastic processes which have the martingale property under these new probability measures with respect to the filtration $\mathbb{G}$. (Thus, these stochastic processes have also the martingale property with respect to the filtration $\mathbb{F}$ by double expectation. Recall the second remark to Definition 2.3)

We introduce the Radon-Nikodym density process. Define the process $(Z(t),\ 0 \le t \le T)$ by

$$\begin{aligned} &\frac{dZ(t)}{Z(t^-)} = \theta_0(t)dW(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{H}(dt,dz), \\ &Z(0) = 1,\ \theta \in \mathcal{I}:\ \theta_1(t,z) > -1\ \ \mathbb{P}\times\Lambda - a.e. \end{aligned} \tag{3.1}$$

Note that we do not specify in which set of predictable integrands θ is. In fact, we intend that it has to be predictable with respect to the given filtration of reference, either $\mathbb{G}$ or $\mathbb{F}$ depending on the specific study.

The solution of the diffusion process (3.1) is obtained by application of the Itô formula.

$$\begin{aligned} &d(\ln Z(t)) \\ &= \frac{Z(t^-)}{Z(t^-)}\theta_0(t)dW(t) + \frac{1}{2}(Z(t^-)\theta_0(t))^2(-\frac{1}{Z(t^-)})dt \\ &\quad + \int_{\mathbb{R}_0}\Big[\ln(Z(t^-) + Z(t^-)\theta_1(t,z)) - \ln(Z(t^-)) - \frac{Z(t^-)\theta_1(t,z)}{Z(t^-)}\Big]\nu(dz)\lambda(t)dt \\ &\quad + \int_{\mathbb{R}_0}\Big[\ln(Z(t^-) + Z(t^-)\theta_1(t,z)) - \ln(Z(t^-))\Big]\tilde{H}(dt,dz) \\ &= \theta_0(t)dW(t) - \frac{1}{2}{\theta_0}^2(t)dt \\ &\quad + \int_{\mathbb{R}_0}\Big[\ln(Z(t^-)) + \ln(1+\theta_1(t,z)) - \ln(Z(t^-)) - \theta_1(t,z)\Big]\nu(dz)\lambda(t)dt \\ &\quad + \int_{\mathbb{R}_0}\Big[\ln(Z(t^-)) + \ln(1+\theta_1(t,z)) - \ln(Z(t^-))\Big]\tilde{H}(dt,dz) \\ &= \theta_0(t)dW(t) - \frac{1}{2}{\theta_0}^2(t)dt + \int_{\mathbb{R}_0}\Big[\ln(1+\theta_1(t,z)) - \theta_1(t,z)\Big]\nu(dz)\lambda(t)dt \\ &\quad + \int_{\mathbb{R}_0}\Big[\ln(1+\theta_1(t,z))\Big]\tilde{H}(dt,dz). \end{aligned}$$

From $Z(0) = 1$ we obtain

$$\begin{aligned} \ln Z(t) - \ln Z(0) = &\int_0^t \theta_0(s)dW(s) - \int_0^t \frac{1}{2}{\theta_0}^2(s)ds \\ &+ \int_0^t\int_{\mathbb{R}_0}\Big[\ln(1+\theta_1(s,z)) - \theta_1(s,z)\Big]\nu(dz)\lambda(s)ds \\ &+ \int_0^t\int_{\mathbb{R}_0}\Big[\ln(1+\theta_1(s,z))\Big]\tilde{H}(ds,dz). \end{aligned}$$

Hence,

$$
\begin{aligned}
Z(t) = \exp\Bigg(& \int_0^t \theta_0(s)dW(s) - \int_0^t \frac{1}{2}{\theta_0}^2(s)ds \\
& + \int_0^t \int_{\mathbb{R}_0} [\ln(1+\theta_1(s,z)) - \theta_1(s,z)]\nu(dz)\lambda(s)ds \\
& + \int_0^t \int_{\mathbb{R}_0} [\ln(1+\theta_1(s,z))]\tilde{H}(ds,dz)\Bigg). \qquad (3.2)
\end{aligned}
$$

Since we have assumed $\theta_1(t,z) > -1 \quad \mathbb{P}\times\Lambda-$a.e. and the conditions on the parameters, we know that $\ln(1+\theta_1(s,z))$ and the stochastic integration are well-defined. A generalized version of the Novikov condition found in [10] ensure uniformly integrability of Z.

Theorem 3.1 (Girsanov Theorem) ♣ *Let W and $\tilde{H}$ be a $\mathbb{P}-$Brownian motion and a $(\mathbb{G},\mathbb{P})-$centered Poisson random field, respectively. Assume that $(Z(t),\ 0 \le t \le T)$, from (3.2) with $\theta \in \mathcal{I}_{\mathbb{G}}$, is a positive uniformly integrable $(\mathbb{G},\mathbb{P})-$martingale with $\mathbb{E}[Z^2(T)] < \infty$, and define the probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T).$$

Define the processes W^θ and $\tilde{H}^\theta$ by the dynamics

$$
\begin{aligned}
dW^\theta(t) &:= dW(t) - \theta_0(t)dt, \\
\tilde{H}^\theta(dt,dz) &:= \tilde{H}(dt,dz) - \theta_1(t,z)\Lambda(dt,dz),
\end{aligned}
$$

Moreover, for any bounded predictable ψ such that $\int_0^T \int_{\mathbb{R}_0} \psi(t,z)\Lambda(dt,dz) < \infty$, $\mathbb{P}-$a.s., define the process $M(t,\psi)$ by

$$M(t,\psi) := \int_0^t \int_{\mathbb{R}_0} \psi(s,z)\tilde{H}^\theta(ds,dz), \quad 0 \le t \le T.$$

Then W^θ is a $\mathbb{Q}-$Brownian motion, and $M(\psi)$ is a $(\mathbb{G},\mathbb{Q})-$martingale, where $\tilde{H}^\theta$ is a compensated jump random field.

Moreover, if

$$\mathbb{E}_{\mathbb{Q}}\Bigg[\sup_{t\in[0,T]} |[W^\theta, M(\psi)](t)|\Bigg] < \infty, \qquad (3.3)$$

for $\psi(t,z) = \mathbb{1}_\Delta(t,z),\ \Delta \in \mathcal{B}_{[0,T]\times\mathbb{R}_0}$, then W^θ and $M(\psi)$ are strongly orthogonal under $\mathbb{Q}$, and $\tilde{H}^\theta$ is a $(\mathbb{G},\mathbb{Q})-$martingale random field.

Proof: ♣ With $\epsilon \in [0,1]$, define

$$X_\epsilon(t) := \epsilon W^\theta(t) + M(\psi, t).$$

We have

$$\begin{aligned} dX_\epsilon(t) &= -\epsilon\theta_0(t)dt + \epsilon dW(t) - \int_{\mathbb{R}_0} \psi(t,z)\theta_1(t,z)\nu(dz)\lambda(t)dt + \int_{\mathbb{R}_0} \psi(t,z)\tilde{H}(dt,dz) \\ &= -\alpha_\epsilon(t)dt + \epsilon dW(t) + \int_{\mathbb{R}_0} \psi(t,z)\tilde{H}(dt,dz), \end{aligned}$$

where

$$\alpha_\epsilon(t) = \epsilon\theta_0(t) + \int_{\mathbb{R}_0} \psi(t,z)\theta_1(t,z)\nu(dz)\lambda(t).$$

From Lemma 1.27 in [14], we know that if $Z(t)X_\epsilon(t)$ is a local $(\mathbb{G}, \mathbb{P})-$martingale, then $X_\epsilon(t)$ is a local $(\mathbb{G}, \mathbb{Q})-$martingale. From Definition 1.28 and Example 1.29 in [14], and recalling that

$$dZ(t) = Z(t^-)\Big(\theta_0(t)dW(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{H}(dt,dz)\Big),$$

we get the following:

$$\begin{aligned} d(Z(t)X_\epsilon(t)) &= Z(t^-)dX_\epsilon(t) + X_\epsilon(t^-)dZ(t) + dZ(t)dX_\epsilon(t) \\ &= Z(t^-)\Big(- \alpha_\epsilon(t)dt + \epsilon dW(t) + \int_{\mathbb{R}_0} \psi(t,z)\tilde{H}(dt,dz)\Big) \\ &\quad + X_\epsilon(t^-)Z(t^-)\Big(\theta_0(t)dW(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{H}(dt,dz)\Big) \\ &\quad + Z(t^-)\epsilon\theta_0(t)dt + Z(t^-)\int_{\mathbb{R}_0} \psi(t,z)\theta_1(t,z)H(dt,dz) \end{aligned}$$

$$\begin{aligned} &= Z(t^-)\Big(- \alpha_\epsilon(t) + \Big(\epsilon\theta_0(t) + \int_{\mathbb{R}_0} \psi(t,z)\theta_1(t,z)\nu(dz)\lambda(t)\Big)\Big)dt \\ &\quad + Z(t^-)\Big(\epsilon dW(t) + \int_{\mathbb{R}_0} \psi(t,z)\tilde{H}(dt,dz)\Big) \\ &\quad + X_\epsilon(t^-)Z(t^-)\Big(\theta_0(t)dW(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{H}(dt,dz)\Big) \\ &\quad + Z(t^-)\int_{\mathbb{R}_0} \psi(t,z)\theta_1(t,z)\tilde{H}(dt,dz) \\ &= Z(t^-)\Big(\epsilon + X_\epsilon(t^-)\theta_0(t)\Big)dW(t) \\ &\quad + Z(t^-)\int_{\mathbb{R}_0} \Big(X_\epsilon(t^-)\theta_1(t,z) + \psi(t,z) + \psi(t,z)\theta_1(t,z)\Big)\tilde{H}(dt,dz) \end{aligned}$$

Thus, $X_\epsilon(t)$ is a local $(\mathbb{G},\mathbb{Q})-$martingale for all $\epsilon\in[0,1]$. In particular, $X_0(t) = M(t,\psi)$ is a local $(\mathbb{G},\mathbb{Q})-$martingale. Moreover, $W^\theta(t) = X_1(t) - M(t,\psi)$ is also a local $(\mathbb{G},\mathbb{Q})-$martingale.

Since W^θ is a continuous local martingale, with quadratic variation $[W^\theta, W^\theta](t) = [W,W](t) = t$ (the quadratic variation is invariant under equivalent measure change), then W^θ is a $\mathbb{Q}-$Brownian motion by the Lévy characterization of Brownian motion, see Theorem 39.II.[11].

As for $M(\psi)$, we can see that its quadratic variation is

$$[M(\psi), M(\psi)](t) = \int_0^t \int_{\mathbb{R}_0} \psi^2(s,z) H(ds,dz).$$

Now, let $\psi(t,z) = \mathbb{1}_{(0,t]\times B}(t,z)$ for $t\in[0,T]$ and $B\in\mathcal{B}_{\mathbb{R}_0}$. Then

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[[M(\psi), M(\psi)](T)] &= \mathbb{E}_{\mathbb{Q}}\Big[\int_0^T \int_{\mathbb{R}_0} \mathbb{1}_{(0,t]\times B}(t,z) H(dt,dz)\Big] \\ &= \mathbb{E}\Big[Z(T)\int_0^T \int_{\mathbb{R}_0} \mathbb{1}_{(0,t]\times B}(t,z) H(dt,dz)\Big].\end{aligned}$$

By Hölder's inequality, we have that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[[M(\psi), M(\psi)](T)] &\le \mathbb{E}[Z^2(T)]^{1/2}\cdot \mathbb{E}\Big[\Big(\int_0^T \int_{\mathbb{R}_0} \mathbb{1}_{(0,t]\times B}(t,z) H(dt,dz)\Big)^2\Big]^{1/2} \\ &< \infty. \end{aligned}\tag{3.4}$$

This is finite by the assumption on Z and the definition of H. H is a conditional Poisson random variable, and its variance is finite. By Corollary D.5 in Appendix D, $M(\psi)$ is a $(\mathbb{G},\mathbb{Q})-$martingale.

Denote B a semi-ring generating $\mathcal{B}_{\mathbb{R}_0}$. We can regard the $\sigma-$algebra $\mathcal{B}_{(0,T]}$ as generated by the semi-ring of intervals of the form $(s,t]$, where $0\le s<t\le T$. The $\sigma-$algebra $\mathcal{B}_{(0,T]\times\mathbb{R}_0}$ is generated by the semi-ring of sets $(s,t]\times B$, where $B\in\mathcal{B}$.

For an element $(s,t]\times B$ in the semi-ring, let $\psi(t,z) = \mathbb{1}_{(0,t]\times B}(t,z)$. Then we have

$$\begin{aligned}\tilde{H}^\theta((s,t]\times B) &= \int_0^t \int_{\mathbb{R}_0} \mathbb{1}_B(z)\tilde{H}^\theta(du,dz) - \int_0^s \int_{\mathbb{R}_0} \mathbb{1}_B(z)\tilde{H}^\theta(du,dz) \\ &= M(t,\psi) - M(s,\psi).\end{aligned}\tag{3.5}$$

By (3.4) and (3.5) $\tilde{H}^\theta$ is $\sigma-$finite on the semi-ring $\mathbb{P}-$a.s. (equivalently $\mathbb{Q}-$a.s.), hence we can uniquely extend (3.5) to the $\sigma-$algebra $\mathcal{B}_{[0,T]\times\mathbb{R}_0}$. See Theorem 11.3 and Theorem 10.3 in [2]. Hence, the compensated jump random field $\tilde{H}^\theta$

has the $(\mathbb{G},\mathbb{Q})-$martingale property, conditionally orthogonal values with respect to $(\mathbb{G},\mathbb{Q})$, and the variance measure is $\sigma-$finite. $\tilde{H}^\theta$ is clearly $\mathbb{G}-$adapted by its definition, and $\tilde{H}^\theta$ is additive and $\sigma-$additive in $L_2(\mathbb{Q})$ by its integral form and condition on θ_1. In conclusion, $\tilde{H}^\theta$ is a $(\mathbb{G},\mathbb{Q})-$martingale random field with conditionally orthogonal values.

Finally, we show that W^θ and $M(\psi)$ are strongly orthogonal under $\mathbb{Q}$. Recall property (v) on page 10, i.e.

$$\Big\langle W, \int_0^{\cdot}\int_{\mathbb{R}_0} \tilde{H}(dt,dz)\Big\rangle(t) = 0.$$

From the Girsanov change of measure, the processes W^θ and $\tilde{H}^\theta$ are defined on sharp brackets form (see e.g. proof of Lemma 2.2 in [3]) by

$$\begin{aligned} W^\theta(t) &:= W(t) - \Big\langle W, \int_0^{\cdot} \theta_0(s)dW(s)\Big\rangle(t),\\ \int_0^t\int_{\mathbb{R}_0} \tilde{H}^\theta(dt,dz) &:= \int_0^t\int_{\mathbb{R}_0}\tilde{H}(dt,dz)\\ &\quad - \Big\langle \int_0^{\cdot}\int_{\mathbb{R}_0}\tilde{H}(ds,dz), \int_0^{\cdot}\int_{\mathbb{R}_0}\theta_1(s,z)\tilde{H}(ds,dz)\Big\rangle(t). \end{aligned}$$

Now, set $\psi(s,z) = \mathbb{1}_{(0,t]\times\mathbb{R}_0}(s,z)$. For this type of ψ, (3.5) becomes

$$\tilde{H}^\theta((0,t]\times\mathbb{R}_0) = M(t,\psi).$$

The sharp bracket process of a martingale and a predictable process is zero, hence by the bilinearity of the sharp bracket operator, we have the following

$$\Big\langle W^\theta, M(\psi)\Big\rangle(t) = \Big\langle W, \int_0^{\cdot}\int_{\mathbb{R}_0}\tilde{H}(dt,dz)\Big\rangle(t) = 0.$$

From this we know that $[W^\theta, M(\psi)](t)$ is a local martingale, and by (3.3) we get from Theorem D.4 in Appendix D that $[W^\theta, M(\psi)]$ is a uniformly integrable martingale. By Definition D.8 $W^\theta(t)$ and $M(t,\psi)$ are strongly orthogonal square integrable martingales for $\psi(t,z) = \mathbb{1}_{(0,t]\times\mathbb{R}_0}(t,z)$. ♣

The following corollary to the Girsanov Theorem is a curiosity, and not necessary for the upcoming calculations. It states sufficient conditions for $\tilde{H}^\theta$ to be a doubly stochastic Poisson random field under $\mathbb{Q}$.

Corollary 3.2 ♣ *Let $\tilde{H}^\theta$ and Z be defined as in Theorem 3.1. If the stochastic field θ_1 is deterministic, then $\tilde{H}^\theta$ is a $(\mathbb{Q},\mathbb{G})-$centered doubly stochastic Poisson random field. Moreover, if $\theta_1(t,z) = \theta_1(t)$ the new jump measure and the new time distortion process are given by*

$$\nu^\theta(dz) = \nu(dz), \qquad \lambda^\theta(\omega,t) = \{1+\theta_1(t)\}\lambda(\omega,t).$$

If $\theta_1(t,z)=\theta_1(z)$ the new jump measure and the new time distortion process are given by

$$\nu^\theta(dz)=\{1+\theta_1(z)\}\nu(dz), \qquad \lambda^\theta(\omega,t)=\lambda(\omega,t).$$

Proof: ♣ Let $\Delta\in\mathcal{B}_{[0,T]\times\mathbb{R}_0}$. We compare the conditional expectation of the exponential of $\tilde H(\Delta)$ with respect to $\mathcal{F}_T^\Lambda$ under the probability measure $\mathbb{P}$ to the same of $\tilde H^\theta(\Delta)$ under the probability measure $\mathbb{Q}$. By the Lévy-Khintchine theorem for random variable $\tilde H(\Delta)$ under $\mathbb{P}$ conditioned on $\mathcal{F}_T^\Lambda$, we have the following:

$$\begin{aligned} E[e^{iu\tilde H(\Delta)}|\mathcal{F}_T^\Lambda] &= e^{\int_\Delta [e^{iu}-1-iu]\nu(dz)\lambda(s)ds} \\ &= e^{(e^{iu}-1-iu)\int_\Delta \nu(dz)\lambda(s)ds}. \end{aligned}$$

The characteristic function for $\tilde H^\theta(\Delta)$ under $\mathbb{Q}$ given $\mathcal{F}_T^\Lambda$ is given by

$$\begin{aligned} \varphi(u) &= \mathbb{E}_\mathbb{Q}[e^{iu\tilde H^\theta(\Delta)}|\mathcal{F}_T^\Lambda] = \frac{\mathbb{E}[Z(T)e^{iu\tilde H^\theta(\Delta)}|\mathcal{F}_T^\Lambda]}{Z(0)} \\ &= \mathbb{E}[e^{\int_0^T\int_\mathbb{R}\ln(1+\theta_1(s,z))\tilde H(ds,dz)+\int_0^T\int_\mathbb{R}[\ln(1+\theta_1(s,z))-\theta_1(s,z)]\nu(dz)\lambda(s)ds}e^{iu\tilde H^\theta(\Delta)}|\mathcal{F}_T^\Lambda] \\ &= \mathbb{E}[e^{\int_\Delta \ln(1+\theta_1(s,z))\tilde H(ds,dz)+\int_\Delta[\ln(1+\theta_1(s,z))-\theta_1(s,z)]\nu(dz)\lambda(s)ds}e^{iu\tilde H^\theta(\Delta)}|\mathcal{F}_T^\Lambda] \\ &= \mathbb{E}[e^{\int_\Delta \ln(1+\theta_1(s,z))\tilde H(ds,dz)+\int_\Delta[\ln(1+\theta_1(s,z))-\theta_1(s,z)]\nu(dz)\lambda(s)ds} \\ &\quad \cdot e^{\int_\Delta iu(\tilde H(ds,dz)-\theta_1(s,z)\nu(dz)\lambda(s)ds)}|\mathcal{F}_T^\Lambda] \\ &= \mathbb{E}[e^{\int_\Delta[\ln(1+\theta_1(s,z))+iu]\tilde H(ds,dz)+\int_\Delta[\ln(1+\theta_1(s,z))-\theta_1(s,z)(1+iu)]\nu(dz)\lambda(s)ds}|\mathcal{F}_T^\Lambda] \end{aligned}$$

If θ_1 is deterministic, then by Lévy-Khintchine

$$\begin{aligned} &\mathbb{E}[e^{\int_\Delta[\ln(1+\theta_1(s,z))+iu]\tilde H(ds,dz)+\int_\Delta[\ln(1+\theta_1(s,z))-\theta_1(s,z)(1+iu)]\nu(dz)\lambda(s)ds}|\mathcal{F}_T^\Lambda] \\ &= e^{\int_\Delta e^{\ln(1+\theta_1(s,z))+iu}-1-(\ln(1+\theta_1(s,z))+iu)\nu(dz)\lambda(s)ds+\int_\Delta[\ln(1+\theta_1(s,z))-\theta_1(s,z)(1+iu)]\nu(dz)\lambda(s)ds} \\ &= e^{\int_\Delta(1+\theta_1(s,z))e^{iu}-1-(\ln(1+\theta_1(s,z))+iu)+\ln(1+\theta_1(s,z))-\theta_1(s,z)(1+iu)\nu(dz)\lambda(s)ds} \\ &= e^{\int_\Delta[(1+\theta_1(s,z))e^{iu}-1-iu(1+\theta_1(s,z))-\theta_1(s,z)]\nu(dz)\lambda(s)ds} \\ &= e^{\int_\Delta[(1+\theta_1(s,z))e^{iu}-(1+\theta_1(s,z))-iu(1+\theta_1(s,z))]\nu(dz)\lambda(s)ds} \\ &= e^{\int_\Delta(e^{iu}-1-iu)(1+\theta_1(s,z))\nu(dz)\lambda(s)ds} \\ &= e^{(e^{iu}-1-iu)\int_\Delta(1+\theta_1(s,z))\nu(dz)\lambda(s)ds}. \end{aligned}$$

So the characteristic functions are on the same form, and thus they will have the same kind of distribution. The compensator of H under $\mathbb{Q}$ is $\Lambda^\theta(dt,dz)=(1+\theta_1(t,z))\nu(dz)\lambda(t)dt$. Since $\tilde H$ is a doubly stochastic Poisson random field, so will also $\tilde H^\theta$ be. ♣

3.2 The financial market and its self-financing portfolios

In this section we define the financial market (Definition 3.3), and solve the dynamics of a value process of a self-financing portfolio in this market (3.11). As anticipated in the introduction to this chapter, the self-financing portfolios constitutes the base of the admissible portfolios of the agent, and a unique strong solution to (3.11) will be required.

We start by defining the financial market. The specifications of the processes and their stochastic properties were defined in Chapter 2.

Definition 3.3 (The Financial Market) *We consider a market model with two investment possibilities in a finite time horizon $T > 0$. The first investment option is a risk-free bond, S_0:*

(i)

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad S_0(0) = 1,\ t \in [0, T], \tag{3.6}$$

The second investment option is a risky asset, S_1, driven by the Brownian motion W and the centered doubly stochastic Poisson random field $\tilde{H}$:

(ii)

$$\frac{dS_1(t)}{S_1(t^-)} = \mu(t)dt + \sigma(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t, z)\tilde{H}(dt, dz), \quad S_1(0) > 0,\ t \in [0, T]. \tag{3.7}$$

The parameters $r(t), \mu(t), \sigma(t)$ and $\gamma(t, z)$, $t \in [0, T], z \in \mathbb{R}$, are càglàd stochastic processes and $\mathbb{F}-$adapted. Moreover, $|r(t)| < C$, $C > 0$, $\gamma(t, z) > -1$, $\mathbb{P}\times\Lambda-$a.e., and

$$\mathbb{E}\left[\int_0^T \{|\mu(t)|+|\sigma(t)|^2+\int_{\mathbb{R}_0} |\ln(1 + \gamma(t, z)) - \gamma(t, z)|\nu(dz)\lambda(t)\}dt\right] < \infty.$$

Let the value process of the investor's portfolio be denoted by $(X^\pi(t),\ 0 \le t \le T)$, where $(\pi(t),\ 0 \le t \le T)$ denotes the portfolio that generates the value process. Here we choose to let $\pi(t)$ be *the amount of wealth* invested in the risky asset at time t, thus $(X^\pi(t) - \pi(t))$ will be the amount of wealth invested in the risk-free bond at time t. We say that π is a *self-financing* strategy if

$$\mathbb{E}\left[\int_0^T [|\mu(t) - r(t)||\pi(t)|+|\pi(t)\sigma(t)|^2+\int_{\mathbb{R}_0} |\pi(t)\gamma(t, z)|^2\nu(dz)\lambda(t)]dt\right] < \infty, \tag{3.8}$$

and if the dynamics of the value process of the portfolio are of the form

$$dX^{\pi}(t) = \frac{\pi(t)}{S_1(t^-)} dS_1(t) + \frac{X^{\pi}(t) - \pi(t)}{S_0(t)} dS_0(t). \tag{3.9}$$

By (3.6) and (3.7), the dynamics (3.9) have following form

$$\begin{aligned} dX^{\pi}(t) &= \frac{\pi(t)}{S_1(t^-)} dS_1(t) + \frac{X^{\pi}(t) - \pi(t)}{S_0(t)} dS_0(t) \\ &= \pi(t) \frac{dS_1(t)}{S_1(t^-)} + (X^{\pi}(t) - \pi(t)) \frac{dS_0(t)}{S_0(t)} \\ &= \pi(t) \left(\mu(t)dt + \sigma(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{H}(dt,dz) \right) \\ &\quad + (X^{\pi}(t) - \pi(t))\, r(t)dt \qquad (3.10) \\ X^{\pi}(0) &= x > 0 \end{aligned}$$

Here x is the initial wealth.

If there exists a unique strong solution to the dynamics in (3.10), then the solution of the value process X^{π} is given by

$$\begin{aligned} X^{\pi}(t) &= x \cdot e^{\int_0^t r(u)du} + \int_0^t e^{\int_s^t r(u)du} \pi(s)(\mu(s) - r(s))ds \\ &\quad + \int_0^t e^{\int_s^t r(u)du} \pi(s)\sigma(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} e^{\int_s^t r(u)du} \pi(s)\gamma(s,z)\tilde{H}(ds,dz) \end{aligned} \tag{3.11}$$

3.3 The optimization problem and the admissible controls ♣

In this section we formally define the optimization problem and the admissible set of controls of the agent and "the market", in our imagined stochastic differential game between the two opponents.

The optimization problem is derived from the desire of the agent to hedge himself from the risk of the contingent claim F, with respect to his ambiguity aversion to the uncertainty in the market model.

The stochastic differential game is a zero-sum stochastic game, i.e. there is balance between the gain and loss of the participants of the game. In other words, the loss of the agent is the gain of "the market", and vice versa.

The dynamic risk measure of the agent's preferences is defined as follows:

Definition 3.4 *If $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ is a contingent claim, then we can measure its risk at time t by the dynamic risk measure over all probability measures in the set of admissible measures $\mathcal{Q}$*

$$\rho_t(\xi) := \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-\xi | \mathcal{M}_t], \tag{3.12}$$

where the filtration $\mathbb{M} := (\mathcal{M}_t,\ 0 \leq t \leq T)$ is the information available to the manager. This risk measure is called the least favorable measure*. (Also, worst case scenario.)*

For supplemental reading on dynamic risk measures via $g-$expectations, see Appendix A.

Recall the definition of the value process $X^\pi(t)$ in equation (3.10). This is the value of the portfolio $\pi(t)$ of the agent at time t. The agent wishes to hedge the contingent claim $F \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ at time $t = T$ minimizing the risk evaluated by the dynamic risk measure ρ_t at every time $t \in [0, T]$. By discounting the difference between the value process at maturity time and the claim F, the random variable

$$\rho_t(e^{-\int_t^T r(s)ds}(X^\pi(T) - F)), \quad t \in [0, T],$$

is the risk of the portfolio $\pi(t)$ at time $t = t$ with time value of money considered. The agent wants to make the expected terminal shortfall vanish at all times. Formally, we can state the problem to be:

The problem of the agent

The agent aims to find a portfolio $\hat{\pi}$ in the set of admissible portfolios Π such that

$$\rho_t(e^{-\int_t^T r(s)ds}(X^{\hat{\pi}}(T) - F)) = \operatorname*{ess\,inf}_{\pi\in\Pi} \rho_t(e^{-\int_t^T r(s)ds}(X^{\pi}(T) - F)) = 0, \quad \forall t. \tag{3.13}$$

If the value process $X^{\hat{\pi}}$ is $\mathbb{M}-$adapted, the price process of the optimal strategy in the worst case scenario $Y(t) := X^{\hat{\pi}}(t)$ is given by

$$\begin{aligned} Y(t) &= X^{\hat{\pi}}(t) \\ &= \rho_t\left(e^{-\int_t^T r(s)ds}(X^{\hat{\pi}}(T) - F)\right) + X^{\hat{\pi}}(t) \\ &= \rho_t\left(e^{-\int_t^T r(s)ds}(X^{\hat{\pi}}(T) - F) - X^{\hat{\pi}}(t)\right) \\ &= \operatorname*{ess\,inf}_{\pi\in\Pi} \rho_t\left(e^{-\int_t^T r(s)ds}(X^{\pi}(T) - F) - X^{\pi}(t)\right), \quad 0 \le t \le T. \end{aligned} \tag{3.14}$$

Here we used the translation invariance of the dynamic risk measure and the optimality condition in (3.13). By definition of ρ_t, the equivalent of (3.14) is

$$Y(t) = \operatorname*{ess\,inf}_{\pi\in\Pi} \left\{ \operatorname*{ess\,sup}_{\theta\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-(e^{-\int_t^T r(s)ds} X^{\pi}(T) - X^{\pi}(t) - e^{-\int_t^T r(s)ds} F)|\mathcal{M}_t] \right\}. \tag{3.15}$$

The reason we say "filtration $\mathbb{M}$", is because we want to solve this problem both under the $\mathbb{F}-$filtration and under the $\mathbb{G}-$filtration.

From the expression in (3.11) we expand the expression inside the expectation of (3.15):

$$\begin{aligned} &X^{\pi}(t) - e^{-\int_t^T r(s)ds} X^{\pi}(T) = \\ &\quad \Big(x_\pi e^{\int_0^t r(u)du} + \int_0^t e^{\int_s^t r(u)du} \pi(s)(\mu(s) - r(s))ds + \int_0^t e^{\int_s^t r(u)du} \pi(s)\sigma(s)dW(s) \\ &\qquad + \int_0^t \int_{\mathbb{R}_0} e^{\int_s^t r(u)du} \pi(s)\gamma(s,z)\tilde{H}(ds,dz) \Big) \\ &\quad - e^{-\int_t^T r(u)du} \Big(x_\pi e^{\int_0^T r(u)du} + \int_0^T e^{\int_s^T r(u)du} \pi(s)(\mu(s) - r(s))ds \\ &\qquad + \int_0^T e^{\int_s^T r(u)du} \pi(s)\sigma(s)dW(s) + \int_0^T \int_{\mathbb{R}_0} e^{\int_s^T r(u)du} \pi(s)\gamma(s,z)\tilde{H}(ds,dz) \Big). \end{aligned} \tag{3.16}$$

Notice that

$$e^{-\int_t^T r(u)du} e^{\int_s^T r(u)du} = e^{\int_s^t r(u)du} = e^{-\int_t^s r(u)du},$$

so (3.16) becomes

$$\begin{aligned}
&X^\pi(t) - e^{-\int_t^T r(s)ds} X^\pi(T) = \\
&\quad - \int_t^T e^{-\int_t^s r(u)du} \pi(s)(\mu(s) - r(s))ds - \int_t^T e^{-\int_t^s r(u)du} \pi(s)\sigma(s)dW(s) \\
&\quad - \int_t^T \int_{\mathbb{R}_0} e^{-\int_t^s r(u)du} \pi(s)\gamma(s,z)\tilde{H}(ds,dz).
\end{aligned} \tag{3.17}$$

Therefore, by inserting the expression in (3.17) into (3.15) we obtain the equivalent of (3.15) and the expression of the optimization problem we are going to use in the future calculations via BSDEs. See (3.18) below.

Next, we define the sets of admissible equivalent measures and admissible portfolios, followed by the expression of the optimization problem we use for calculations in this thesis.

Summarizing the optimization problem

We define the common assumptions on the parameters for the two methods of solving the optimization problem. In addition to these assumptions, both methods will need further conditions for technical reasons. These extra conditions are defined in the start of the sections where they will be needed. Define the sets of *admissible equivalent probability measures* to $\mathbb{P}$ by:

Definition 3.5 (Admissible equivalent probability measures) *Let the process $(Z(t),\ 0 \leq t \leq T)$ be a $(\mathbb{G}, \mathbb{P})-$martingale defined by*

$$\frac{dZ(t)}{Z(t^-)} = \theta_0(t)dW(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{H}(dt,dz),$$
$$\theta \in \mathcal{I}_{\mathbb{M}},\ Z(0) = 1,\ \textit{and}\ \theta_1(t,z) > -1\ \ \mathbb{P} \times \Lambda - a.e.,$$

such that $Z \in L_2(\mathbb{P})$ and (3.3) is satisfied. Define the sets of admissible equivalent probability measures to $\mathbb{P}$ by

$$\mathcal{Q}_{\mathbb{G}} := \{\theta \in \mathcal{I}_{\mathbb{G}} \mid \frac{d\mathbb{Q}}{d\mathbb{P}} = Z^{\theta}(T)\},$$

and

$$\mathcal{Q}_{\mathbb{F}} := \{\theta \in \mathcal{I}_{\mathbb{F}} \mid \frac{d\mathbb{Q}}{d\mathbb{P}} = Z^{\theta}(T)\}.$$

A choice of $\theta \in \mathcal{Q}_{\mathbb{M}}$ induces an equivalent probability measure $\mathbb{Q}$ of $\mathbb{P}$, for $\mathbb{M} = \mathbb{F}, \mathbb{G}$.

Define the sets of *admissible portfolios* by

Definition 3.6 (Admissible portfolios) *A strategy $\pi : [0,T] \times \Omega \to \mathbb{R}$ is called admissible if:*

(i) $\pi(t)\gamma(t,z) > -1, \quad$ *a.s.*

(ii) π *is self-financing by (3.8) and (3.9), and in addition satisfies for all* $\mathbb{Q} \in \mathcal{Q}_{\mathbb{M}}$

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T [|\mu(t) - r(t)||\pi(t)| + |\pi(t)\sigma(t)|^2 + \int_{\mathbb{R}_0} |\pi(t)\gamma(t,z)|^2 \nu(dz)\lambda(t)]dt\right] < \infty,$$

and satisfies either

(iii) π *is $\mathbb{G}-$predictable such that there exist a unique strong càdlàg $\mathbb{G}-$adapted solution X^{π} to the dynamics in (3.10) on $[0,T]$,*

or

(iv) π *is* $\mathbb{F}-$*predictable such that there exist a unique strong càdlàg* $\mathbb{F}-$*adapted solution* X^{π} *to the dynamics in (3.10) on* $[0,T]$.

Notation: We say that strategies that satisfies (i), (ii) with $\mathbb{Q} \in \mathcal{Q}_{\mathbb{G}}$, and (iii) belongs to $\Pi_{\mathbb{G}}$. Strategies that satisfies (i), (ii) with $\mathbb{Q} \in \mathcal{Q}_{\mathbb{F}}$, and (iv) belong to $\Pi_{\mathbb{F}}$. It is clear that $\Pi_{\mathbb{F}} \subset \Pi_{\mathbb{G}}$.

The **optimal portfolio,** $\hat{\pi}$, of the agent, the parameter of the **optimal probability measure**, $\hat{\theta}$, of "the market", and the **optimal price process**, $\hat{Y}(t),\ 0 \leq t \leq T$, induced by $(\hat{\pi}, \hat{\theta})$, are the solutions of the following problem:

$$
\begin{aligned}
Y(t) = \operatorname*{ess\,inf}_{\pi \in \Pi_{\mathbb{M}}} \Bigg\{ \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}_{\mathbb{M}}} \mathbb{E}_{\mathbb{Q}} \Bigg[& e^{-\int_t^T r(s)ds} F - \int_t^T e^{-\int_t^s r(u)du} \pi(s)(\mu(s) - r(s))ds \\
& - \int_t^T e^{-\int_t^s r(u)du} \pi(s)\sigma(s)dW(s) \\
& - \int_t^T \int_{\mathbb{R}_0} e^{-\int_t^s r(u)du} \pi(s)\gamma(s,z)\tilde{H}(ds,dz) \Bigg| \mathcal{M}_t \Bigg] \Bigg\}.
\end{aligned} \tag{3.18}
$$

3.4 Solution via BSDEs ♣

In this section we use the theory presented in Section 2.3. Recall the main definitions and theorems: The definition of standard parameters in Definition 2.12; the existence and uniqueness of solutions for BSDEs with standard parameters in Theorem 2.13; and the comparison of solutions of BSDEs, Theorem 2.14.

We look at two cases of the problem in equation (3.18): First, where $\mathbb{M} = \mathbb{G}$, and the controls are in the $\mathbb{G}-$predictable sets $(\pi, \theta) \in \Pi_{\mathbb{G}} \times \mathcal{Q}_{\mathbb{G}}$, and secondly where $\mathbb{M} = \mathbb{F}$, and the controls are in the $\mathbb{F}-$predictable sets $(\pi, \theta) \in \Pi_{\mathbb{F}} \times \mathcal{Q}_{\mathbb{F}}$. In both cases we find the controls that solves (3.18).

From the comparison theorem we form an optimization theorem, which provides sufficient conditions for obtaining an optimal solution to the problem. The theorem and its proof are obtained from the paper [15].

In the method of the BSDEs, the admissible controls $(\pi, \theta) \in \Pi \times \mathcal{Q}$ must in addition satisfy:

For $K > 0$:

$$|\theta_0(t)| < K, \quad 0 \leq \theta_1(t, z)\sqrt{\lambda(t)} < K \cdot z, \; z \in \mathbb{R}_0, \qquad \mathbb{P} \times dt - a.e. \tag{3.19}$$

We still call the sets by the same names $\Pi_{\mathbb{M}}$ and $\mathcal{Q}_{\mathbb{M}}$, for $\mathbb{M} = \mathbb{F}, \mathbb{G}$.

Notation: First of all, denote $g(\pi(t), \theta(t)) = g(t, \lambda, y, z, u(\cdot), \pi(t), \theta(t))$. The solution of the BSDE (see Theorem 2.13) with the generator $g(\pi(t), \hat{\theta}(t))$, where the $\hat{\theta}$ is an optimal process, is denoted by $(Y^\pi(t), Z^\pi(t), U^\pi(t, \cdot))$. Similar notation is used for the solution of $g(\pi(t), \theta(t))$ and $g(\hat{\pi}(t), \theta(t))$. The solution of $g(\hat{\pi}(t), \hat{\theta}(t))$ is denoted $(\hat{Y}(t), \hat{Z}(t), \hat{U}(t, \cdot))$.

Theorem 3.7 (Optimization Theorem I) *Let (ξ, g) be standard parameters. Suppose that for all $(\omega, t, \lambda, y, z, u(\cdot)) \in \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathcal{R}$ there exist $\hat{\pi}(t) = \hat{\pi}(\omega, t, \lambda, y, z, u(\cdot))$ and $\hat{\theta}(t) = \hat{\theta}(\omega, t, \lambda, y, z, u(\cdot))$ such that for all admissible portfolios π and all admissible probability measures given by θ, we have:*

$$g(\hat{\pi}(t), \theta(t)) \leq g(\pi(t), \theta(t)) \leq g(\pi(t), \hat{\theta}(t)), \tag{3.20}$$

for a.a. (ω, t). Assume g satisfies the criteria of g_2 in Theorem 2.14. Suppose $\hat{\pi}$ and $\hat{\theta}$ are admissible. Suppose that for all admissible (π, θ) there exists a unique solution to the BSDE with $(\xi, g(\pi(t), \theta(t)))$ as terminal condition and generator, respectively. Then

$$\hat{Y}(t) = \operatorname*{ess\,inf}_{\pi \in \Pi} Y^\pi(t) = Y(t) = \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} \left\{ \operatorname*{ess\,inf}_{\pi \in \Pi} Y^{\pi,\theta}(t) \right\} = \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} Y^\theta(t)$$

Proof: By applying the comparison theorem to the solutions of the BSDEs of the couples $(F, g(\hat{\pi}(t), \theta(t))), (F, g(\pi(t), \theta(t)))$, $(F, g(\pi(t), \hat{\theta}(t)))$, by (3.20) we get that $Y^{\theta}(t) \leq Y^{\pi,\theta}(t) \leq Y^{\pi}(t)$, for all admissible (π, θ), thus

$$\text{For all } \theta\text{:} \quad Y^{\theta}(t) \leq \operatorname*{ess\,inf}_{\pi \in \Pi} Y^{\pi,\theta}(t) \quad \mathbb{P} \times dt - a.e. \tag{3.21}$$

$$\text{For all } \pi\text{:} \quad \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} Y^{\pi,\theta}(t) \leq Y^{\pi}(t) \quad \mathbb{P} \times dt - a.e. \tag{3.22}$$

From definition of essential supremum and (3.21), we get

$$\hat{Y}(t) \leq \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} Y^{\theta}(t) \leq \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} \left(\operatorname*{ess\,inf}_{\pi \in \Pi} Y^{\pi,\theta}(t) \right).$$

From (3.22) and definition of essential infimum, we get

$$Y(t) = \operatorname*{ess\,inf}_{\pi \in \Pi} \left\{ \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} Y^{\pi,\theta}(t) \right\} \leq \operatorname*{ess\,inf}_{\pi \in \Pi} Y^{\pi}(t) \leq \hat{Y}.$$

Hence, we obtain the following chain of inequalities:

$$\begin{aligned} Y(t) = \operatorname*{ess\,inf}_{\pi \in \Pi} \left\{ \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} Y^{\pi,\theta}(t) \right\} &\leq \operatorname*{ess\,inf}_{\pi \in \Pi} Y^{\pi}(t) \leq \hat{Y} \\ &\leq \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} Y^{\theta}(t) \leq \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}} \left(\operatorname*{ess\,inf}_{\pi \in \Pi} Y^{\pi,\theta}(t) \right). \end{aligned}$$

Since sup(inf) $\leq$ inf(sup) we get equality between all terms. This completes the proof. ■

As in Section 2.4, we state a theorem that evolved during the study of the optimization by BSDEs. Again, we wanted to form a theorem that do not require a saddle point optimum. We do not use the theorem is this thesis, but the theorem is included as a curiosity.

Theorem 3.8 (Optimization Theorem II ♣) *Let (ξ, g) be standard parameters. Suppose that for all $(\omega, t, \lambda, y, z, u(\cdot)) \in \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathcal{R}$ there exist $\hat{\pi}(t) = \hat{\pi}(\omega, t, \lambda, y, z, u(\cdot))$ and $\hat{\theta}(t) = \hat{\theta}(\omega, t, \lambda, y, z, u(\cdot))$ such that for all admissible portfolios and all admissible probability measures, we have:*

$$\begin{aligned} &g(\hat{\pi}(t), \hat{\theta}(t)) \leq g(\pi(t), \hat{\theta}(t)), \quad \textit{and} \\ &g(\pi(t), \theta(t)) \leq g(\pi(t), \hat{\theta}(t)), \end{aligned}$$

for a.a. (ω, t). Assume g satisfies the criteria of g_2 in Theorem 2.14. Suppose $\hat{\pi}$ and $\hat{\theta}$ are admissible. Suppose that for all admissible (π, θ) there exists a unique *solution to the BSDE with $(\xi, g(\pi(t), \theta(t)))$ as terminal condition and generator, respectively. Then*

$$\hat{Y}(t) = \operatorname*{ess\,inf}_{\pi\in\Pi} Y^{\pi}(t) = \operatorname*{ess\,inf}_{\pi\in\Pi} \left\{ \operatorname*{ess\,sup}_{\theta\in\mathcal{Q}} Y^{\pi,\theta}(t) \right\} = Y(t) \tag{3.23}$$

Proof: ♣ By applying the comparison theorem to the solutions of the BSDEs of the couples $(F, g(\hat{\pi}(t), \hat{\theta}(t))), (F, g(\pi(t), \hat{\theta}(t))),\ (F, g(\pi(t), \theta(t)))$, we get that $\hat{Y}(t) \leq Y^{\pi}(t)$ and $Y^{\pi,\theta}(t) \leq Y^{\pi}(t)$, for all admissible (π, θ), thus

$$\hat{Y}(t) \leq \operatorname*{ess\,inf}_{\pi\in\Pi} Y^{\pi}(t) \quad \mathbb{P}\times dt - a.e. \tag{3.24}$$

$$\text{For all } \pi\text{:} \quad \operatorname*{ess\,sup}_{\theta\in\mathcal{Q}} Y^{\pi,\theta}(t) \leq Y^{\pi}(t) \quad \mathbb{P}\times dt - a.e. \tag{3.25}$$

From the definition of essential infimum we have

$$\hat{Y}(t) \geq \operatorname*{ess\,inf}_{\pi\in\Pi} Y^{\pi}(t). \tag{3.26}$$

Together with (3.24), (3.26) gives the following equality

$$\hat{Y}(t) = \operatorname*{ess\,inf}_{\pi\in\Pi} Y^{\pi}(t). \tag{3.27}$$

From the definition of essential supremum we have

$$\operatorname*{ess\,sup}_{\theta\in\mathcal{Q}} Y^{\pi,\theta}(t) \geq Y^{\pi}(t), \quad \text{for all admissible } \pi. \tag{3.28}$$

Together with (3.25), (3.28) gives the following equality

$$\operatorname*{ess\,sup}_{\theta\in\mathcal{Q}} Y^{\pi,\theta}(t) = Y^{\pi}(t), \quad \text{for all admissible } \pi. \tag{3.29}$$

Combining (3.27) and (3.29) gives (3.23), and completes the proof. ♣

3.4.1 Case I: Knowledge of the time-distortion

First we consider the problem where $\mathcal{M}_t = \mathcal{G}_t$, i.e. full information on the driver processes and complete future insight in the time-distortion process.

We find the BSDE for

$$Y^{\pi,\theta}(t) = \mathbb{E}_{\mathbb{Q}}\bigg[e^{-\int_t^T r(s)ds}F - \int_t^T e^{-\int_t^s r(u)du}\pi(s)(\mu(s)-r(s))ds$$
$$- \int_t^T e^{-\int_t^s r(u)du}\pi(s)\sigma(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} e^{-\int_t^s r(u)du}\pi(s)\gamma(s,z)\tilde{H}(ds,dz)\bigg|\mathcal{G}_t\bigg], \tag{3.30}$$

where $\pi \in \Pi_{\mathbb{G}}$ and $\theta \in \mathcal{Q}_{\mathbb{G}}$. By the Girsanov Theorem (Theorem 3.1), we can define a $\mathbb{Q}-$Brownian motion and a $\mathbb{Q}-$martingale random field by

$$dW^\theta(t) := dW(t) - \theta_0(t)dt,$$

and

$$\tilde{H}^\theta(dt,dz) := \tilde{H}(dt,dz) - \theta_1(t,z)\nu(dz)\lambda(t)dt.$$

We use this representation and the martingale property of $\int_0^t dW^\theta(s)$ and $\int_0^t \int_{\mathbb{R}_0} \tilde{H}^\theta(ds,dz)$ with respect to $(\mathbb{G},\mathbb{Q})$, to eliminate the stochastic integrators inside the expectation of (3.30):

$$\begin{aligned}
Y^{\pi,\theta}(t) &= \mathbb{E}_{\mathbb{Q}}\bigg[e^{-\int_t^T r(s)ds}F - \int_t^T e^{-\int_t^s r(u)du}\pi(s)(\mu(s)-r(s))ds \\
&\quad - \int_t^T e^{-\int_t^s r(u)du}\pi(s)\sigma(s)dW(s) \pm \int_t^T e^{-\int_t^s r(u)du}\pi(s)\sigma(s)\theta_0(s)ds \\
&\quad - \int_t^T \int_{\mathbb{R}_0} e^{-\int_t^s r(u)du}\pi(s)\gamma(s,z)\tilde{H}(ds,dz) \\
&\quad \pm \int_t^T \int_{\mathbb{R}_0} e^{-\int_t^s r(u)du}\pi(s)\gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s)ds\bigg|\mathcal{G}_t\bigg] \\
&= \mathbb{E}_{\mathbb{Q}}\bigg[e^{-\int_t^T r(s)ds}F - \int_t^T e^{-\int_t^s r(u)du}\pi(s)\bigg((\mu(s)-r(s)) + \sigma(s)\theta_0(s) \\
&\quad + \int_{\mathbb{R}_0}\gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s)\bigg)ds - \int_t^T e^{-\int_t^s r(u)du}\pi(s)\sigma(s)dW^\theta(s) \\
&\quad - \int_t^T \int_{\mathbb{R}_0} e^{-\int_t^s r(u)du}\pi(s)\gamma(s,z)\tilde{H}^\theta(ds,dz)\bigg|\mathcal{G}_t\bigg] \\
&= \mathbb{E}_{\mathbb{Q}}\bigg[e^{-\int_t^T r(s)ds}F - \int_t^T e^{-\int_t^s r(u)du}\pi(s)\bigg((\mu(s)-r(s)) + \sigma(s)\theta_0(s) \\
&\quad + \int_{\mathbb{R}_0}\gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s)\bigg)ds\bigg|\mathcal{G}_t\bigg].
\end{aligned} \tag{3.31}$$

To find the BSDE corresponding to $Y^{\pi,\theta}$, we use the martingale representation (2.5) from Theorem 2.9. The right-hand side of (3.31) is not a $(\mathbb{G}, \mathbb{Q})-$martingale, but by manipulating the expression inside the expectation to become an element of $L_2(\Omega, \mathcal{F}_T, \mathbb{Q})$, we obtain a $(\mathbb{G}, \mathbb{Q})-$martingale term plus a residual term.

First multiply both sides of (3.31) by the $\mathcal{G}_t-$adapted process

$$e^{-\int_0^t r(u)du}.$$

Since this process is $\mathcal{G}_t-$adapted, we can take it inside the expectation and get

$$\begin{aligned}
&e^{-\int_0^t r(u)du} Y^{\pi,\theta}(t) \\
&= e^{-\int_0^t r(u)du} \mathbb{E}_{\mathbb{Q}}\Bigg[e^{-\int_t^T r(s)ds} F - \int_t^T e^{-\int_t^s r(u)du} \pi(s) \\
&\quad \cdot \Big((\mu(s) - r(s)) + \sigma(s)\theta_0(s) \quad + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s) \Big) ds \Big| \mathcal{G}_t \Bigg] \\
&= \mathbb{E}_{\mathbb{Q}}\Bigg[e^{-\int_0^T r(s)ds} F - \int_t^T e^{-\int_0^s r(u)du} \pi(s) \Big((\mu(s) - r(s)) + \sigma(s)\theta_0(s) \\
&\quad + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s) \Big) ds \Big| \mathcal{G}_t \Bigg].
\end{aligned} \tag{3.32}$$

Now, by adding and subtracting the missing piece of the integrals in the expectation, which is $\mathcal{G}_t-$adapted too, we obtain the martingale term we have been looking for:

$$\begin{aligned}
&e^{-\int_0^t r(u)du} Y^{\pi,\theta}(t) \\
&= \mathbb{E}_{\mathbb{Q}}\Bigg[e^{-\int_0^T r(s)ds} F - \int_t^T e^{-\int_0^s r(u)du} \pi(s) \\
&\quad \cdot \Big((\mu(s) - r(s)) + \sigma(s)\theta_0(s) + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s) \Big) ds \Big| \mathcal{G}_t \Bigg] \\
&\pm \int_0^t e^{-\int_0^s r(u)du} \pi(s) \Big((\mu(s) - r(s)) + \sigma(s)\theta_0(s) + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s) \Big) ds \\
&= \mathbb{E}_{\mathbb{Q}}\Bigg[e^{-\int_0^T r(s)ds} F - \int_0^T e^{-\int_0^s r(u)du} \pi(s) \\
&\quad \cdot \Big((\mu(s) - r(s)) + \sigma(s)\theta_0(s) + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s) \Big) ds \Big| \mathcal{G}_t \Bigg] \\
&+ \int_0^t e^{-\int_0^s r(u)du} \pi(s) \Big((\mu(s) - r(s)) + \sigma(s)\theta_0(s) + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s) \Big) ds
\end{aligned} \tag{3.33}$$

To make the calculations more readable, rename the expression inside the expectation of (3.33), and call it

$$\xi = e^{-\int_0^T r(s)ds} F - \int_0^T e^{-\int_0^s r(u)du} \pi(s) \cdot \Big((\mu(s) - r(s)) + \sigma(s)\theta_0(s) + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s) \Big) ds. \tag{3.34}$$

We write (3.33) over again, now with the abbreviation (3.34) inserted:

$$\begin{aligned} &e^{-\int_0^t r(u)du} Y^{\pi,\theta}(t) \\ &\qquad = \mathbb{E}_{\mathbb{Q}}[\xi|\mathcal{G}_t] + \int_0^t e^{-\int_0^s r(u)du} \pi(s) \\ &\qquad\quad \cdot \Big((\mu(s) - r(s)) + \sigma(s)\theta_0(s) + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s) \Big) ds. \end{aligned}$$

We apply the Martingale theorem (Theorem 2.9) on this conditional expectation, which yields

$$\begin{aligned} &e^{-\int_0^t r(u)du} Y^{\pi,\theta}(t) \\ &\qquad = \mathbb{E}_{\mathbb{Q}}[\xi|\mathcal{F}_T^{\Lambda}] + \int_0^t Z^{\pi,\theta}(s) dW^{\theta}(s) + \int_0^t \int_{\mathbb{R}_0} U^{\pi,\theta}(s,z) \tilde{H}^{\theta}(ds,dz) \\ &\qquad + \int_0^t e^{-\int_0^s r(u)du} \pi(s) \Big((\mu(s) - r(s)) + \sigma(s)\theta_0(s) \\ &\qquad\quad + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s) \Big) ds, \end{aligned}$$

for $(Z, U) \in \mathcal{I}_{\mathbb{G}}$. The differentiated form of this is obtained by using Itô's lemma on the left-hand side:

$$\begin{aligned} &-r(t)e^{-\int_0^t r(u)du} Y^{\pi,\theta}(t)dt + e^{-\int_0^t r(u)du} dY^{\pi,\theta}(t) \\ &\quad = e^{-\int_0^t r(u)du} \pi(t) \Big((\mu(t) - r(t)) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\theta_1(t,z)\nu(dz)\lambda(t) \Big) dt \\ &\qquad + Z^{\pi,\theta}(t) dW^{\theta}(t) + \int_{\mathbb{R}_0} U^{\pi,\theta}(t,z) \tilde{H}^{\theta}(dt,dz). \end{aligned}$$

By rearranging this expression, we end up with $Y^{\pi,\theta}$ in form of a BSDE:

$$\begin{aligned}
dY^{\pi,\theta}(t) &= r(t)Y^{\pi,\theta}(t)dt + \pi(t)\Big((\mu(t) - r(t)) + \sigma(t)\theta_0(t) \\
&\quad + \int_{\mathbb{R}_0} \gamma(t,z)\theta_1(t,z)\nu(dz)\lambda(t)\Big)dt \\
&\quad + e^{\int_0^t r(u)du} Z^{\pi,\theta}(t)dW^\theta(t) + \int_{\mathbb{R}_0} e^{\int_0^t r(u)du} U^{\pi,\theta}(t,z)\tilde{H}^\theta(dt,dz) \\
&= \Big(r(t)Y^{\pi,\theta}(t) + \pi(t)\{\mu(t) - r(t)\} + \{\pi(t)\sigma(t) - e^{\int_0^t r(u)du} Z^{\pi,\theta}(t)\}\theta_0(t) \\
&\quad + \int_{\mathbb{R}_0} \{\pi(t)\gamma(t,z) - e^{\int_0^t r(u)du} U^{\pi,\theta}(t,z)\}\theta_1(t,z)\nu(dz)\lambda(t)\Big)dt \\
&\quad + e^{\int_0^t r(u)du} Z^{\pi,\theta}(t)dW(t) + \int_{\mathbb{R}_0} e^{\int_0^t r(u)du} U^{\pi,\theta}(t,z)\tilde{H}(dt,dz), \qquad (3.35) \\
Y^{\pi,\theta}(T) &= F,
\end{aligned}$$

$Z^{\pi,\theta}$ and $U^{\pi,\theta}$ can be found by the Integral representation (Theorem 2.8) of the martingale $\mathbb{E}_{\mathbb{Q}}[\xi|\mathcal{G}_t]$ for $t = T$.

$$\xi = \mathbb{E}_{\mathbb{Q}}[\xi|\mathcal{F}_T^\Lambda] + \int_0^T Z^{\pi,\theta}(s)dW^\theta(s) + \int_0^T \int_{\mathbb{R}_0} U^{\pi,\theta}(s,z)\tilde{H}^\theta(ds,dz).$$

From (2.6) we recall how to find the generator of a BSDE, and we see that the generator of the BSDE (3.35) is

$$\begin{aligned}
g(\cdot,\lambda,y,z,u(\cdot),\pi,\theta) =& -yr - (\mu - r)\pi - (\pi\sigma - e^{\int_0^\cdot r(s)ds} z)\theta_0 \\
& - \int_{\mathbb{R}_0} (\pi\gamma(\cdot,x) - e^{\int_0^\cdot r(s)ds} u(\cdot,x))\theta_1(\cdot,x)\nu(dx)\lambda. \qquad (3.36)
\end{aligned}$$

The unique optimal solution

We have to verify that the generator (3.36) of the BSDE (3.35) satisfy Theorem 3.7, and we need to find $(\hat{\pi}, \hat{\theta}) \in \Pi_{\mathbb{G}} \times \mathcal{Q}_{\mathbb{G}}$ that satisfy the following:

(1)
$$\frac{\partial g}{\partial \pi}(\hat{\pi}, \hat{\theta}) = 0,$$

(2)
$$\frac{\partial g}{\partial \theta_0}(\hat{\pi}, \hat{\theta}) = 0,$$

(3)
$$\frac{\partial g}{\partial \theta_1}(\hat{\pi}, \hat{\theta}) = 0.$$

First we check if the generator (3.36) satisfies Theorem 2.14, i.e. if there exists $(\alpha, \beta) \in \mathcal{I}_{\mathbb{G}}$ and $K > 0$, with $0 \leq \beta(t, z) < Kx,\ x \in \mathbb{R}_0$, and $|\alpha(t)| < K\ d\mathbb{P} \times dt-$a.e. such that

$$\begin{aligned} g(t, \lambda, &y, z, u(\cdot), \pi(t), \theta(t)) \\ &= f(t, y, z\alpha(t), \int_{\mathbb{R}_0} \beta(t, z) u(z) \nu(dz) \sqrt{\lambda}, \pi(t), \theta(t)), \end{aligned}$$

for a function f that satisfies

(i) $|f(t, y, b, h, \pi(t), \theta(t)) - f(t, y', b', h', \pi(t), \theta(t))| \leq K_h(|y-y'| + |b-b'| + |h-h'|)$,

(ii) $\mathbb{E}\Big[\int_0^T |f(t, 0, 0, 0, \pi(t), \theta(t))|^2 dt \Big] < \infty.$

From (3.36) we have

$$\begin{aligned} g(t, \lambda, y, z, u(\cdot), \pi(t), \theta(t)) \\ = &- yr(t) - (\mu(t) - r(t))\pi(t) - (\pi(t)\sigma(t) - e^{\int_0^t r(s)ds} z)\theta_0(t) \\ &- \int_{\mathbb{R}_0} (\pi(t)\gamma(t, z) - e^{\int_0^t r(s)ds} u(z))\theta_1(t, z)\nu(dz)\lambda \\ = &- yr(t) - \pi(t)\Big[\mu(t) - r(t) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}_0} \gamma(t, z)\theta_1(t, z)\nu(dz)\lambda\Big] \\ &+ e^{\int_0^t r(s)ds} z\theta_0(t) + \int_{\mathbb{R}_0} e^{\int_0^t r(s)ds} u(z)\sqrt{\lambda}\theta_1(t, z)\nu(dz)\sqrt{\lambda}. \end{aligned}$$

We see that we must have

$$\alpha(t) = e^{\int_0^t r(s)ds}\theta_0(t), \quad \beta(t,z) = e^{\int_0^t r(s)ds}\sqrt{\lambda(t)}\theta_1(t,z), \tag{3.37}$$

By the definition of admissible equivalent probability measures and the assumption that $r(t) < C$, (3.37) holds. We get the following representation of the new generator f:

$$\begin{aligned} f(t, y(t), b(t), h(t), \pi(t), \theta(t)) = & -yr(t) - \pi(t)\Big[\mu(t) - r(t) + \sigma(t)\theta_0(t) \\ & + \int_{\mathbb{R}_0} \gamma(t,z)\theta_1(t,z)\nu(dz)\lambda(t)\Big] + b(t) + h(t). \end{aligned}$$

(i) follows from the boundedness of $r(t)$, and (ii) holds by the definition of an admissible portfolio.

So we know the generator satisfies the conditions in Theorem 3.7. Now we find a solution that satisfies (1), (2) and (3) above:

1.
$$\frac{\partial g}{\partial \pi}(\hat{\pi}, \hat{\theta}) = -(\mu - r) - \sigma\hat{\theta}_0 - \int_{\mathbb{R}_0} \gamma(x)\hat{\theta}_1(x)\nu(dx)\lambda.$$

Hence, the optimal parameters for change of measure must satisfy:

$$(\mu(t) - r(t)) + \sigma(t)\hat{\theta}_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\hat{\theta}_1(t,z)\nu(dz)\lambda(t) = 0.$$

2.
$$\frac{\partial g}{\partial \theta_0}(\hat{\pi}, \hat{\theta}) = e^{\int_0^{\cdot} r(s)ds}z - \hat{\pi}\sigma.$$

Hence, the optimal portfolio must satisfy:

$$e^{\int_0^t r(s)ds}\hat{Z}(t) - \hat{\pi}(t)\sigma(t) = 0.$$

3.
$$\frac{\partial g}{\partial \theta_1}(\hat{\pi}, \hat{\theta}) = \int_{\mathbb{R}_0} (e^{\int_0^{\cdot} r(s)ds}u(x) - \hat{\pi}\gamma(x))\nu(dx)\lambda.$$

Hence, the optimal portfolio must also satisfy:

$$\int_{\mathbb{R}_0} (e^{\int_0^t r(s)ds}\hat{U}(t,z) - \hat{\pi}(t)\gamma(t,z))\nu(dz)\lambda(t) = 0.$$

Summary

The unique optimal solution of the problem (3.18), when $\mathcal{M}_t = \mathcal{G}_t$, is to find $(\hat{\pi}, \hat{\theta}) \in \Pi_{\mathbb{G}} \times \mathcal{Q}_{\mathbb{G}}$ so that the following conditions hold

$$e^{\int_0^t r(s)ds} \hat{Z}(t) - \hat{\pi}(t)\sigma(t) = 0, \tag{3.38}$$

$$\int_{\mathbb{R}_0} (e^{\int_0^t r(s)ds} \hat{U}(t,z) - \hat{\pi}(t)\gamma(t,z))\nu(dz)\lambda(t) = 0, \tag{3.39}$$

$$(\mu(t) - r(t)) + \sigma(t)\hat{\theta}_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\hat{\theta}_1(t,z)\nu(dz)\lambda(t) = 0. \tag{3.40}$$

The dynamics of the price process are then

$$\begin{aligned}
&d\hat{Y}(t) = r(t)\hat{Y}(t)dt + e^{\int_0^t r(s)ds} \hat{Z}(t) dW^{\hat{\theta}}(t) + \int_{\mathbb{R}_0} e^{\int_0^t r(s)ds} \hat{U}(t,z) \tilde{H}^{\hat{\theta}}(dt,dz), \\
&\hat{Y}(T) = F,
\end{aligned}$$

with solution

$$\begin{aligned}
Y(t) = e^{-\int_t^T r(s)ds} F &- \int_t^T e^{\int_0^t r(u)du} \hat{Z}(s) dW^{\hat{\theta}}(s) \\
&- \int_t^T \int_{\mathbb{R}_0} e^{\int_0^t r(u)du} \hat{U}(s,z) \tilde{H}^{\hat{\theta}}(ds,dz).
\end{aligned}$$

$(\hat{Z}, \hat{U}) \in \mathcal{I}_{\mathbb{G}}$ are the solutions of

$$e^{-\int_0^T r(t)dt} F = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(t)dt} F | \mathcal{F}_T^{\Lambda}] + \int_0^T \hat{Z}(s) dW^{\hat{\theta}}(s) + \int_0^T \int_{\mathbb{R}_0} \hat{U}(s,z) \tilde{H}^{\hat{\theta}}(ds,dz). \tag{3.41}$$

Moreover, the optimal $\hat{\theta}$ is a martingale measure to the price process, since we have from (3.32) that

$$e^{-\int_0^t r(s)ds} \hat{Y}(t) = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(s)ds} \hat{Y}(T) | \mathcal{G}_t]. \tag{3.42}$$

3.4.2 Case II: Standard information on the time-distortion

In order to find a BSDE corresponding to (3.30) for the information flow $\mathbb{F}$, we must abandon the BSDE type in (2.6), i.e.

$$\begin{aligned} Y(t) = \xi &+ \int_t^T g(s, \lambda(s), Y(s), Z(s), U(s,z))ds - \int_t^T Z(s)dW(s) \\ &- \int_t^T \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz). \end{aligned}$$

Instead we consider a general BSDE be on the form

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))d\langle\mu\rangle(s) - \int_t^T Z(s)d\mu(s) - N(T) + N(t), \quad (3.43)$$

as in [3],where

(i) $(\mu(t),\ 0 \le t \le T)$ is a square integrable, càdlàg $(\mathbb{F}, \mathbb{P})-$martingale,

(ii) $\langle\mu\rangle$ is its conditional quadratic variation (See 4.III.[11]), and

(iii) N is a square integrable $(\mathbb{F}, \mathbb{P})-$martingale orthogonal to μ, and $N(0) = 0$.

The solution of such a BSDE is $(Y(t), Z(t), N(t))$, and the theory about existence and uniqueness of such a solution for Lipschitz generators can be found in Theorem 2.1, [3]. By the same definition of standard parameters, and the same assumptions on the parameters in the set of admissible portfolios and equivalent measures as before, there exists a unique solution to (3.43). See [3] for $\beta-$standard parameters with $\beta = 0$.

Definition 3.9 (Doléans exponential) *The Doléans exponential $\mathcal{E}(X)$ of a semimartingale X is a unique solution of the Doléans equation*

$$\mathcal{E}(X)(t) = 1 + \int_0^t \mathcal{E}(X)(s^-)dX(s).$$

The solution for any semimartingale X is

$$\begin{aligned} \mathcal{E}(X)(t) = &\exp\Big(X(t) - X(0) - \frac{1}{2}[X](t)\Big) \\ &\cdot \prod_{s<t}(1 + \Delta X(s)) \exp\Big(- \Delta X(s) + \frac{1}{2}\Delta X^2(s)\Big), \quad t > 0. \end{aligned}$$

When X is a local martingale, then $\mathcal{E}(X)$ is a local martingale.

We use the Doléans exponential as a Radon-Nikodym density process, and need $\mathcal{E}(X)$ to be a positive uniformly integrable martingale. If $\Delta X(t) > -1$ almost surely for all t, then the process $\mathcal{E}(X)$ is positive. A generalized Novikov condition found in [10] ensure that $\mathcal{E}(X)$ is uniformly integrable.

The next result is Lemma 2.2 in [3]:

Lemma 3.10 *Let a,b,c be predictable bounded processes, let $\mathcal{E}$ be the Doléans exponential of the martingale $(\int_0^t b(s)d\mu(s),\ 0 \leq t)$, and define*

$$\psi(t) = \exp\left(\int_0^t a(s)d\langle\mu\rangle(s)\right), \qquad \Psi(t) = \psi(t)\mathcal{E}(t). \tag{3.44}$$

Suppose that

(i) $\mathcal{E}$ *is a positive uniformly integrable martingale;*

(ii) $\mathbb{E}[(\sup_{t\in[0,T]}\psi(t))^2\mathcal{E}^2(T)] < \infty$.

If the linear backward equation

$$\begin{aligned} dY(t) &= -(a(t)Y(t) + b(t)Z(t) + c(t))d\langle\mu\rangle(s) + Z(t)d\mu(t) + dN(t) \\ Y(T) &= \xi, \end{aligned} \tag{3.45}$$

has solution (Y, Z, N) in $\mathbb{S}_2 \times \mathcal{I}_{\mathbb{F}} \times \mathcal{L}_2$, where $\mathcal{L}_2$ is the space of L_2-bounded $(\mathbb{F}, \mathbb{P})-$martingales, then Y is given by

$$Y(t) = \mathbb{E}\left[\xi\frac{\Psi(T)}{\Psi(t)} + \int_t^T \frac{\Psi(s)}{\Psi(t)}c(s)d\langle\mu\rangle(s)\middle|\mathcal{F}_t\right], \quad 0 \leq t \leq T. \tag{3.46}$$

Proof: See proof of Lemma 2.2 in [3].

Remark. Whenever expression (3.46) makes sense, it is a solution of the linear equation (3.45). $\diamond$

The next theorem is a comparison theorem from [3] (Theorem 2.2, [3]):

Theorem 3.11 *Consider two linear BSDEs of the form (3.45), for $i = 1, 2$*

$$Y_i(t) = \xi_i + \int_t^T f_i(s, Y_i(s), Z_i(s))d\langle\mu\rangle(s) - \int_t^T Z_i(s)d\mu(s) - N_i(T) + N_i(t).$$

Define $\delta Y(t) = Y_1(t) - Y_2(t)$ and $\delta Z(t) = Z_1(t) - Z_2(t)$, and let

$$\begin{aligned} \Delta_Y f_1(t) &= \frac{f_1(t, Y_1(t), Z_1(t)) - f_1(t, Y_2(t), Z_1(t))}{\delta Y(t)}\mathbb{1}_{\delta Y(t)\neq 0}, \\ \Delta_Z f_1(t) &= \frac{f_1(t, Y_2(t), Z_1(t)) - f_1(t, Y_2(t), Z_2(t))}{\delta Z(t)}\mathbb{1}_{\delta Z(t)\neq 0}, \\ \delta_2 f(t) &= f_1(t, Y_2(t), Z_2(t)) - f_2(t, Y_2(t), Z_2(t)), \end{aligned}$$

and assume $\Delta_Y f_1(t)$ and $\Delta_Z f_1(t)$ verify the conditions (i) and (ii) in Lemma 3.10. Assume that $\xi_1 \geq \xi_2$ and, for any t, $\delta_2 f(\omega, t) \geq 0$ $\mathbb{P}-$a.s. Then, for any t, $Y_1(t) \geq Y_2(t)$ $\mathbb{P}-$a.s.

Proof: The process $\delta Y(t)$ solves the following linear BSDE:

$$d\delta Y(t) = -[\Delta_Y f_1(t)\delta Y(t) + \Delta_Z f_1(t)\delta Z(t) + \delta f(t)]d\langle\mu\rangle(t) \\ - \delta Z(t)d\mu(t) - d\delta N(t).$$

According to Lemma 3.10, the solution of this BSDE is

$$\Psi(t)\delta Y(t) = \mathbb{E}\left[\Psi(T)(\xi_1 - \xi_2) + \int_t^T \delta_2 f(t)\Psi(s)d\langle\mu\rangle(s)\middle|\mathcal{F}_t\right].$$

By the assumptions $\xi_1 \geq \xi_2$ and, for any t, $\delta_2 f(\omega, t) \geq 0$ $\mathbb{P}-$a.s., the expression inside the conditional expectation is non-negative. The result $Y_1(t) \geq Y_2(t)$ $\mathbb{P}-$a.s., for any t, follows. ■

In order to apply this theory, we must adapt to the given setup. First we must define the $(\mathbb{F}, \mathbb{P})-$martingale random field M, and its conditional quadratic variation $\langle M\rangle$: Define M by

$$M(dt, dz) = \delta_{\{0\}}(dz)dW(t) + \mathbb{1}_{\mathbb{R}_0}(z)\tilde{H}(dt, dz),$$

where $\delta_{\{0\}}$ is the Dirac delta with point mass at zero, and its conditional quadratic variation by

$$\langle M\rangle(dt, dz) = \delta_{\{0\}}(dz)dt + \mathbb{1}_{\mathbb{R}_0}(z)\nu(dz)\lambda(t)dt.$$

Moreover, by the Girsanov theorem we define the $(\mathbb{F}, \mathbb{Q})-$martingale random field $\widetilde{M}$ by

$$\widetilde{M}(dt, dz) = M(dt, dz) - b(t, z)d\langle M\rangle(dt, dz). \tag{3.47}$$

Note that $\langle M\rangle = \langle\widetilde{M}\rangle$, since $\langle M\rangle$ has no quadratic variation.

We write the expression

$$Y^{\pi,\theta}(t) = \mathbb{E}_{\mathbb{Q}}\Bigg[e^{-\int_t^T r(s)ds}F - \int_t^T e^{-\int_t^s r(u)du}\pi(s)(\mu(s) - r(s))ds \\ - \int_t^T e^{-\int_t^s r(u)du}\pi(s)\sigma(s)dW(s) \\ - \int_t^T \int_{\mathbb{R}_0} e^{-\int_t^s r(u)du}\pi(s)\gamma(s, z)\tilde{H}(ds, dz)\Bigg|\mathcal{F}_t\Bigg]\Bigg\}, \tag{3.48}$$

with the new notation. Recall the definition of $\psi(t)$ and $\Psi(t)$ from (3.44). Here $\mathcal{E}(t) = Z(t)$, i.e. the Radon-Nikodym density process defined in Section 3.1 and $\psi(t) = e^{\int_0^t r(s)ds}$. Then we have

$$\begin{aligned}
&\psi(t)Y^{\pi,\theta}(t) \\
&\quad = \mathbb{E}_{\mathbb{Q}}\Big[\psi(T)F - \int_t^T \psi(s)\pi(s)\Big[(\mu(s) - r(s))ds - \sigma(s)dW(s) \\
&\qquad - \int_{\mathbb{R}_0} \gamma(s,z)\tilde{H}(ds,dz)\Big]\Big|\mathcal{F}_t\Big]\Big\} \\
&\quad = \mathbb{E}_{\mathbb{Q}}\Big[\psi(T)F - \int_t^T \psi(s)\pi(s)\Big[(\mu(s) - r(s))ds + \sigma(s)\{dW^\theta(s) + \theta_0(s)ds\} \\
&\qquad + \int_{\mathbb{R}_0} \gamma(s,z)\{\tilde{H}^\theta(ds,dz) + \theta_1(s,z)\nu(dz)\lambda(s)ds\}\Big]\Big|\mathcal{F}_t\Big]\Big\}
\end{aligned}$$

By use of the martingale property, we get

$$\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}}\Big[\psi(T)F - \int_t^T \psi(s)\pi(s)\Big[\{(\mu(s) - r(s)) + \sigma(s)\theta_0(s)\}ds \\
&\quad + \int_{\mathbb{R}_0} \gamma(s,z)\theta_1(s,z)\nu(dz)\lambda(s)ds\}\Big]\Big|\mathcal{F}_t\Big]\Big\} \\
&= \mathbb{E}_{\mathbb{Q}}\Big[\psi(T)F - \int_t^T \int_{\mathbb{R}} \psi(s)\pi(s)\Big[\{(\mu(s) - r(s)) + \sigma(s)\theta_0(s)\}\mathbb{1}_{\{0\}}(z) \\
&\quad + \gamma(s,z)\theta_1(s,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\Big[\delta_{\{0\}}(dz)dt + \mathbb{1}_{\mathbb{R}_0}(z)\nu(dz)\lambda(t)dt\Big]\Big|\mathcal{F}_t\Big]\Big\} \\
&= \mathbb{E}_{\mathbb{Q}}\Big[\psi(T)F - \int_t^T \int_{\mathbb{R}} \psi(s)\pi(s)\Big[\{(\mu(s) - r(s)) + \sigma(s)\theta_0(s)\}\mathbb{1}_{\{0\}}(z) \\
&\quad + \gamma(s,z)\theta_1(s,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\langle M\rangle(ds,dz)\Big|\mathcal{F}_t\Big]\Big\}. \qquad (3.49)
\end{aligned}$$

Hence, after the measure change we have $Y^{\pi,\theta}(t)$ on the same form as (3.46):

$$\begin{aligned}
Y^{\pi,\theta}(t) = \mathbb{E}\Big[\frac{\Psi(T)}{\Psi(t)}F - \int_t^T \int_{\mathbb{R}} \frac{\Psi(s)}{\Psi(t)}\pi(s)\Big[\{(\mu(s) - r(s)) + \sigma(s)\theta_0(s)\}\mathbb{1}_{\{0\}}(z) \\
+\gamma(s,z)\theta_1(s,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\langle M\rangle(ds,dz)\Big|\mathcal{F}_t\Big]\Big\}.
\end{aligned}$$

By Lemma 3.10, this is the solution of the BSDE

$$
\begin{aligned}
dY^{\pi,\theta}(t) =& - \int_{\mathbb{R}} \Big\{ - r(t)\mathbb{1}_{\{0\}}(z)Y(t) + \theta_0(t)\mathbb{1}_{\{0\}}(z)\bar{Z}(t) \\
& + \theta_1(t,z)\mathbb{1}_{\mathbb{R}_0}(z)\bar{U}(t,z) - \pi(t)\Big[\{(\mu(t) - r(t)) \\
& + \sigma(t)\theta_0(t)\}\mathbb{1}_{\{0\}}(z) + \gamma(t,z)\theta_1(t,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\Big\} d\langle M\rangle(t) \\
& + \int_{\mathbb{R}} \bar{Z}(t)\mathbb{1}_{\{0\}}(z) + \bar{U}(t,z)\mathbb{1}_{\mathbb{R}_0}(z)M(dt,dz) + dN(t) \\
Y^{\pi,\theta}(T) =& F.
\end{aligned} \tag{3.50}
$$

Moreover, going back to (3.49), we obtain by direct derivation of the BSDE the following:

$$
\begin{aligned}
\psi(t)Y^{\pi,\theta}(t) =& \mathbb{E}_{\mathbb{Q}}\Big[\psi(T)F - \int_t^T \int_{\mathbb{R}} \psi(s)\pi(s)\Big[\{(\mu(s) - r(s)) + \sigma(s)\theta_0(s)\}\mathbb{1}_{\{0\}}(z) \\
& + \gamma(s,z)\theta_1(s,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\langle M\rangle(ds,dz)\Big|\mathcal{F}_t\Big]\Big\} \qquad (3.51) \\
=& \mathbb{E}_{\mathbb{Q}}\Big[\psi(T)F - \int_0^T \int_{\mathbb{R}} \psi(s)\pi(s)\Big[\{(\mu(s) - r(s)) + \sigma(s)\theta_0(s)\}\mathbb{1}_{\{0\}}(z) \\
& + \gamma(s,z)\theta_1(s,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\langle M\rangle(ds,dz)\Big|\mathcal{F}_t\Big]\Big\} \\
& + \int_0^t \int_{\mathbb{R}} \psi(s)\pi(s)\Big[\{(\mu(s) - r(s)) + \sigma(s)\theta_0(s)\}\mathbb{1}_{\{0\}}(z) \\
& + \gamma(s,z)\theta_1(s,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\langle M\rangle(ds,dz) \\
=& \mathbb{E}_{\mathbb{Q}}\Big[\xi_0 + \int_0^T \int_{\mathbb{R}} Z(t)\mathbb{1}_{\{0\}}(z) + U(t,z)\mathbb{1}_{\mathbb{R}_0}(z)\widetilde{M}(dt,dz)\Big|\mathcal{F}_t\Big] \qquad (3.52) \\
& + \int_0^t \int_{\mathbb{R}} \psi(s)\pi(s)\Big[\{(\mu(s) - r(s)) + \sigma(s)\theta_0(s)\}\mathbb{1}_{\{0\}}(z) \\
& + \gamma(s,z)\theta_1(s,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\langle M\rangle(ds,dz) \\
=& \mathbb{E}_{\mathbb{Q}}[\xi_0|\mathcal{F}_t] + \int_0^t \int_{\mathbb{R}} Z(s)\mathbb{1}_{\{0\}}(z) + U(s,z)\mathbb{1}_{\mathbb{R}_0}(z)\widetilde{M}(ds,dz) \\
& + \int_0^t \int_{\mathbb{R}} \psi(s)\pi(s)\Big[\{(\mu(s) - r(s)) + \sigma(s)\theta_0(s)\}\mathbb{1}_{\{0\}}(z) \\
& + \gamma(s,z)\theta_1(s,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\langle M\rangle(ds,dz). \qquad (3.53)
\end{aligned}
$$

In (3.52) we used the integral representation with respect to the $\mathbb{F}-$filtration. It includes the unknown $L_2(\mathbb{Q})-$variable ξ_0. This variable is orthogonal to the stochastic integrals. The differential form of (3.53) is

$$\begin{aligned}
d(\psi(t)Y^{\pi,\theta}(t)) = &\int_{\mathbb{R}} Z(t)\mathbb{1}_{\{0\}}(z) + U(t,z)\mathbb{1}_{\mathbb{R}_0}(z)\widetilde{M}(dt,dz) \\
&+ \int_{\mathbb{R}} \psi(t)\pi(t)\Big[\{(\mu(t)-r(t)) + \sigma(t)\theta_0(t)\}\mathbb{1}_{\{0\}}(z) \\
&+ \gamma(t,z)\theta_1(t,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\langle M\rangle(dt,dz) + d\mathbb{E}_{\mathbb{Q}}[\xi_0|\mathcal{F}_t].
\end{aligned}$$

Recall the definition of $\gamma(t)$ from (3.44), and the definition of $\widetilde{M}$ from (3.47). Hence,

$$\begin{aligned}
dY^{\pi,\theta}(t) = &- \int_{\mathbb{R}} \Big\{ - r(t)\mathbb{1}_{\{0\}}(z)Y(t) + \theta_0(t)\mathbb{1}_{\{0\}}(z)e^{\int_0^t r(s)ds}Z(t) \\
&+ \theta_1(t,z)\mathbb{1}_{\mathbb{R}_0}(z)e^{\int_0^t r(s)ds}U(t,z) - \pi(t)\Big[\{(\mu(t)-r(t)) \\
&+ \sigma(t)\theta_0(t)\}\mathbb{1}_{\{0\}}(z) + \gamma(t,z)\theta_1(t,z)\mathbb{1}_{\mathbb{R}_0}(z)\}\Big]\Big\}\langle M\rangle(dt,dz) \\
&+ \int_{\mathbb{R}} e^{\int_0^t r(s)ds}Z(t)\mathbb{1}_{\{0\}}(z) + e^{\int_0^t r(s)ds}U(t,z)\mathbb{1}_{\mathbb{R}_0}(z)M(dt,dz) \\
&+ e^{\int_0^t r(s)ds}d\mathbb{E}_{\mathbb{Q}}[\xi_0|\mathcal{F}_t] \\
Y^{\pi,\theta}(T) = &F,
\end{aligned}$$

or equivalently

$$\begin{aligned}
dY^{\pi,\theta}(t) = &- \int_{\mathbb{R}} \Big[\Big\{ - r(t)Y(t) - \pi(t)(\mu(t)-r(t)) \\
&+ \theta_0(t)\{e^{\int_0^t r(s)ds}Z(t) - \pi(t)\sigma(t)\}\Big\}\mathbb{1}_{\{0\}}(z) \\
&+ \Big\{e^{\int_0^t r(s)ds}U(t,z) - \pi(t)\gamma(t,z)\Big\}\theta_1(t,z)\mathbb{1}_{\mathbb{R}_0}(z)\Big]\langle M\rangle(dt,dz) \\
&+ \int_{\mathbb{R}} \Big[e^{\int_0^t r(s)ds}Z(t)\mathbb{1}_{\{0\}}(z) + e^{\int_0^t r(s)ds}U(t,z)\mathbb{1}_{\mathbb{R}_0}(z)\Big]M(dt,dz) \qquad (3.54) \\
&+ e^{\int_0^t r(s)ds}d\mathbb{E}_{\mathbb{Q}}[\xi_0|\mathcal{F}_t] \\
Y^{\pi,\theta}(T) = &F.
\end{aligned}$$

Now compare (3.50) and (3.54). We see that

$$\bar{Z}(t) = e^{\int_0^t r(s)ds} Z(t)$$
$$\bar{U}(t,z) = e^{\int_0^t r(s)ds} U(t,z),$$
$$\text{and} \qquad N(t) = \int_{(0,t]} e^{\int_0^s r(u)du} d\mathbb{E}_{\mathbb{Q}}[\xi_0|\mathcal{F}_s].$$

Note that we have explicitly given the integration limits of the stochastic integral. This is because we want to make the reader aware that the integration does not contain zero, i.e. the integration respects the condition that $N(0) = 0$.

Recall that $N(t)$ is a $(\mathbb{F}, \mathbb{P})-$martingale and orthogonal to M. $\int_{(0,t]} e^{\int_0^s r(u)du} d\mathbb{E}_{\mathbb{Q}}[\xi_0|\mathcal{F}_s]$ is an $(\mathbb{F}, \mathbb{Q})-$martingale and orthogonal to $\widetilde{M}$ by construction. This anticipates the next result, that $N(t)$ is an $(\mathbb{F}, \mathbb{Q})-$martingale and strongly orthogonal to $\widetilde{M}$.

Lemma 3.12 *Let the Doléans exponential $\mathcal{E}(t)$ in Lemma 3.10 be the Radon-Nikodym density process $Z(t)$ defined in Section 3.1. Then the process $N(t)$ from (3.45) is an $(\mathbb{F}, \mathbb{Q})-$martingale and strongly orthogonal to $\widetilde{M}$.*

Proof: ♣ $N(t)$ is orthogonal to $\widetilde{M}$ by

$$\begin{aligned}
\langle N, \widetilde{M}\rangle &= \langle N(\cdot), \int_0^\cdot \int_{\mathbb{R}} M(dt,dz) - \int_0^\cdot \int_{\mathbb{R}} b(t,z) d\langle M\rangle(dt,dz)\rangle \\
&= \langle N(\cdot), \int_0^\cdot \int_{\mathbb{R}} M(dt,dz)\rangle - \langle N(\cdot), \int_0^\cdot \int_{\mathbb{R}} b(t,z) d\langle M\rangle(dt,dz)\rangle \\
&= 0,
\end{aligned}$$

because $\langle N, M\rangle = 0$ by construction, and $\langle N(\cdot), \int_0^\cdot \int_{\mathbb{R}} b(t,z) d\langle M\rangle(dt,dz)\rangle = 0$ since $N(t)$ is a martingale and $\langle M\rangle$ has no quadratic variation.

For $N(t)$ to be an $(\mathbb{F}, \mathbb{Q})-$martingale, it must satisfy

$$\mathbb{E}_{\mathbb{Q}}[N(T)|\mathcal{F}_t] = \frac{\mathbb{E}[Z(T)N(T)|\mathcal{F}_t]}{Z(t)} = N(t),$$

i.e. if $Z(t)N(t)$ is an $(\mathbb{F}, \mathbb{P})-$martingale, then $N(t)$ is an $(\mathbb{F}, \mathbb{Q})-$martingale.

The dynamics of $Z(t)N(t)$ are

$$d(Z(t)N(t)) = Z(t)dN(t) + N(t)dZ(t) + dZ(t)dN(t).$$

Recall that the Doléans exponential $Z(t)$ is the solution of the Doléans equation

$$\mathcal{E}(X)(t) = 1 + \int_0^t \mathcal{E}(X)(s^-)dX(s),$$

for the $(\mathbb{F}, \mathbb{P})-$martingale $X : X(t) = \int_0^t \int_{\mathbb{R}} b(s,z) M(ds, dz)$. Thus

$$dZ(t) = Z(t^-) \int_{\mathbb{R}} b(t,z) M(dt, dz).$$

Hence, by $\mathbb{P}-$orthogonality between M and N, the dynamics of $Z(t)N(t)$ are

$$d(Z(t)N(t)) = Z(t)dN(t) + N(t)dZ(t),$$

and this is an $(\mathbb{F}, \mathbb{P})-$martingale by the square-integrability of Z and N. ♣

The unique optimal solution

From (3.54) we see that the generator f (cf. (3.43)) of the BSDE with the solution (3.48) is

$$\begin{aligned}\int_0^T f(t)d\langle\mu\rangle(t) = \int_0^T \int_{\mathbb{R}} \Big\{ &- r(t)Y(t) + \pi(t)(\mu(t) - r(t)) \\ &+ \theta_0(t)\{e^{\int_0^t r(s)ds} Z(t) - \pi(t)\sigma(t)\}\Big\} \mathbb{1}_{\{0\}}(z) \\ &+ \Big\{ e^{\int_0^t r(s)ds} U(t,z) - \pi(t)\gamma(t,z)\Big\}\theta_1(t,z)\mathbb{1}_{\mathbb{R}_0}(z)\langle M\rangle(dt,dz).\end{aligned}$$

We recognize the generator from the case of information flow $\mathbb{G}$, and by the comparison theorem, Theorem 3.11, and the optimization theorem, Theorem 3.7, the same conditions on the parameters will be required for the optimal solution here as in the case of information flow $\mathbb{G}$. Note, however, that even though the optimization conditions are the same, the measurability assumptions on the control parameters are different. This may lead to different solutions.

In addition, the price process in the case of $\mathbb{F}-$information contains an orthogonal process and not an orthogonal random variable, like the price process in the case of $\mathbb{G}-$information.

Summary

The unique optimal solution of the problem (3.18), when $\mathcal{M}_t = \mathcal{F}_t$, is to find $(\hat{\pi}, \hat{\theta}) \in \Pi_{\mathbb{F}} \times \mathcal{Q}_{\mathbb{F}}$ so that the following conditions hold

$$e^{\int_0^t r(s)ds}\hat{Z}(t) - \hat{\pi}(t)\sigma(t) = 0, \tag{3.55}$$

$$\int_{\mathbb{R}_0} (e^{\int_0^t r(s)ds}\hat{U}(t,z) - \hat{\pi}(t)\gamma(t,z))\nu(dz)\lambda(t) = 0, \tag{3.56}$$

$$(\mu(t) - r(t)) + \sigma(t)\hat{\theta}_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\hat{\theta}_1(t,z)\nu(dz)\lambda(t) = 0. \tag{3.57}$$

The dynamics of the price process are then

$$\begin{aligned} d\hat{Y}(t) &= r(t)\hat{Y}(t)dt + e^{\int_0^t r(s)ds}\hat{Z}(t)dW^{\hat{\theta}}(t) + \int_{\mathbb{R}_0} e^{\int_0^t r(s)ds}\hat{U}(t,z)\tilde{H}^{\hat{\theta}}(dt,dz) \\ &\quad + e^{\int_0^t r(s)ds} d\mathbb{E}_{\hat{\mathbb{Q}}}[\hat{\xi}_0|\mathcal{F}_t], \\ \hat{Y}(T) &= F, \end{aligned}$$

where $(\hat{Z}, \hat{U}, \hat{\xi}_0) \in \mathcal{I}_{\mathbb{F}} \times L_2(\mathbb{Q})$ are the solutions of

$$e^{-\int_0^T r(t)dt}F = \hat{\xi}_0 + \int_0^T \hat{Z}(s)dW^{\hat{\theta}}(s) + \int_0^T \int_{\mathbb{R}_0} \hat{U}(s,z)\tilde{H}^{\hat{\theta}}(ds,dz). \tag{3.58}$$

Moreover, the optimal $\hat{\theta}$ is a martingale measure to the price process, since we have from (3.51) that

$$e^{-\int_0^t r(s)ds}\hat{Y}(t) = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(s)ds}\hat{Y}(T)|\mathcal{F}_t]. \tag{3.59}$$

3.5 Solution via the maximum principle

This section is greatly inspired by the calculations in [1]. The processes involved in [1] and in this thesis are different. Nevertheless, a great part of the theory can be adapted to the setup for this thesis. We do the calculations for completeness of the thesis and to rule out the possible ambiguities in the the paper and this thesis.

Recall the theory presented in Section 2.4, in particular the optimization theorems via the maximum principle: Theorem 2.15 and Theorem 2.16. We use these theorems to solve the optimization problem (3.18) at time zero.

In addition to the assumptions on the parameters in the admissible set of portfolios, Definition 3.6, and the set of admissible equivalent probability measures, Definition 3.5, we need the following assumptions. The assumptions come from the paper [8].

In the method of the maximum principle, the admissible controls $(\pi, \theta) \in \Pi \times \mathcal{Q}$ must in addition satisfy:

$$\mathbb{E}\left[\int_0^T |f(s, X(s^-), \lambda(s), \pi(s), \theta(s))|^2 ds + |l(X(T))| + |\partial_x l(X(T))|^2\right] < \infty,$$

where f and l are the functions in the performance criterion, and X is the state process in (2.7). Moreover, for some $K_1 > 0$ we have

$$\begin{aligned} |\partial_x b(t, X(t^-), \lambda(t), \pi(t), \theta(t))| &\leq K_1 \quad d\mathbb{P} \times dt - a.e., \\ |\partial_x \sigma(t, X(t^-), \lambda(t), \pi(t), \theta(t))| &\leq K_1 \quad d\mathbb{P} \times dt - a.e., \\ \int_{\mathbb{R}_0} (\partial_x \gamma(t, X(t^-), \lambda(t), \pi(t), \theta(t), z))^2 \nu(dz) \sqrt{\lambda(t)} &\leq K_1 \quad d\mathbb{P} \times dt - a.e., \end{aligned}$$

where b, σ and γ are the functions inside the state process (2.7). In addition, we have the following sufficient conditions to ensure existence of a strong solution of the state process (2.7): For some $K_2 > 0$ we have

$$\begin{aligned} |\sigma(t, x, \lambda(t), \pi(t), \theta(t)) - \sigma(t, x', \lambda(t), \pi(t), \theta(t))| &\leq K_2 |x - x'| \quad \mathbb{P} - a.s. \\ |\gamma(t, x, \lambda(t), \pi(t), \theta(t), z) - \gamma(t, x', \lambda(t), \pi(t), \theta(t), z)| &\leq K_2 |z| |x - x'| \quad \mathbb{P} - a.s. \\ |b(t, x, \lambda(t), \pi(t), \theta(t)) - b(t, x', \lambda(t), \pi(t), \theta(t))| &\leq K_2 |x - x'| \quad \mathbb{P} - a.s. \\ \int_0^T |\sigma(t, 1, \lambda(t), \pi(t), \theta(t))|^2 dt &< \infty \quad \mathbb{P} - a.s. \\ \int_0^T \int_{\mathbb{R}_0} |\gamma(t, 1, \lambda(t), \pi(t), \theta(t), z)|^2 \nu(dz) \lambda(t) dt &< \infty \quad \mathbb{P} - a.s. \\ \int_0^T |b(t, 1, \lambda(t), \pi(t), \theta(t))|^2 dt &< \infty \quad \mathbb{P} - a.s. \end{aligned}$$

See the references connected to these assumptions in [8] for the justification of the existence of a strong solution. We still call the sets by the same names $\Pi_{\mathbb{M}}$ and $\mathcal{Q}_{\mathbb{M}}$, for $\mathbb{M} = \mathbb{F}, \mathbb{G}$.

The financial market is defined by the Definition 3.3. However, in the calculations via the maximum principle we let π be the *proportion* of the wealth invested in the risky asset, so the corresponding dynamics are

$$\begin{aligned} dX^{\pi}(t) &= X^{\pi}(t^-)\Bigg[(r(t) + \pi(t)(\mu(t) - r(t)))dt + \pi(t)\sigma(t)dW(t) \\ &\qquad + \int_{\mathbb{R}_0} \pi(t)\gamma(t,z)\tilde{H}(dt,dz)\Bigg], \qquad (3.60) \\ X^{\pi}(0) &= x > 0. \end{aligned}$$

For admissible portfolios (Definition 3.6), the solution to the dynamics of the value process (3.60) is given by

$$\begin{aligned} X^{\pi}(t) = x \cdot \exp\Bigg[\int_0^t \Bigg\{ & r(t) + \pi(t)(\mu(t) - r(t)) - \frac{1}{2}\pi^2(t)\sigma^2(t) \\ & + \int_{\mathbb{R}_0} [\ln(1 + \pi(s)\gamma(s,z)) - \pi(s)\gamma(s,z)]\nu(dz)\lambda(s)\Bigg\} ds \\ & + \int_0^t \pi(s)\sigma(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \ln(1 + \pi(s)\gamma(s,z))\tilde{H}(ds,dz)\Bigg]. \end{aligned}$$

Recall the dynamics of the Radon-Nikodym density process Z defined in (3.1) in Section 3.1. Define the 2-dimensional state process $Y(t)$ as follows:

$$\begin{aligned} dY(t) &= \begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} dZ(t) \\ dX^{\pi}(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ X^{\pi}(t^-)[r(t) + \pi(t)(\mu(t) - r(t))] \end{bmatrix} dt \\ &\quad + \begin{bmatrix} Z(t^-)\theta_0(t) \\ X^{\pi}(t^-)\pi(t)\sigma(t) \end{bmatrix} dW(t) + \begin{bmatrix} Z(t^-)\int_{\mathbb{R}_0}\theta_1(t,z) \\ X^{\pi}(t^-)\int_{\mathbb{R}_0}\pi(t)\gamma(t,z) \end{bmatrix} \tilde{H}(dt,dz). \end{aligned}$$

We get the following Hamiltonian:

$$\begin{aligned} &\mathcal{H}(t, y_1, y_2, \theta, \pi, p, q, r) \\ &\quad = y_2\{r(t) + \pi(t)(\mu(t) - r(t))\}p_2(t) + y_1\theta_0(t)q_1(t) + y_2\pi(t)\sigma(t)q_2(t) \\ &\qquad + \int_{\mathbb{R}_0} y_1\theta_1(t,z)r_1(t,z) + y_2\pi(t)\gamma(t,z)r_2(t,z)\nu(dz)\lambda(t). \qquad (3.61) \end{aligned}$$

Remark. r is the first is the short interest rate, while r_1, r_2 are the solutions to the adjoint equations defined below. $\diamond$

The adjoint equations are

$$
\begin{aligned}
dp_1(t) &= \{\theta_0(t)q_1(t) + \int_{\mathbb{R}_0} \theta_1(t,z)r_1(t,z)\nu(dz)\lambda(t)\}dt + q_1(t)dW(t) \\
&\qquad + \int_{\mathbb{R}_0} r_1(t,z)\tilde{H}(ds,dz) \\
p_1(T) &= \nabla_{y_1} U(Y_2(T))
\end{aligned}
\tag{3.62}
$$

and

$$
\begin{aligned}
dp_2(t) &= \Big\{[r(t) + \pi(t)(\mu(t) - r(t))]p_2(t) + \pi(t)\sigma(t)q_2(t) \\
&\quad + \int_{\mathbb{R}_0} \pi(t)\gamma(t,z)r_2(t,z)\nu(dz)\lambda(t)\Big\}dt + q_2(t)dW(t) + \int_{\mathbb{R}_0} r_2(t,z)\tilde{H}(ds,dz)
\end{aligned}
\tag{3.63}
$$

$$
p_2(T) = \nabla_{y_2} U(Y_2(T))
$$

3.5.1 Case I: Knowledge of the time-distortion

First, we look at the following performance criterion

$$J(\pi,\theta) = \mathbb{E}_{\mathbb{Q}}\left[X^{\pi}(0) - e^{-\int_0^T r(s)ds}X^{\pi}(T) + Fe^{-\int_0^T r(s)ds}\middle|\mathcal{F}_T^{\Lambda}\right],$$

and the optimization problem

$$\operatorname*{ess\,inf}_{\pi\in\Pi_{\mathbb{G}}}\left\{\operatorname*{ess\,sup}_{\theta\in\mathcal{Q}_{\mathbb{G}}} J(\pi,\theta)\right\}. \tag{3.64}$$

From the theory in Section 2.4, we see that the optimization problem corresponds to an $f=0$ and a utility function $U:[0,\infty)\to[-\infty,\infty)$ given by

$$U(X(T)) = X(0) - e^{-\int_0^T r(s)ds}X(T) + Fe^{-\int_0^T r(s)ds}.$$

From Theorem 2.15 and the first equality in (2.13), we have the following optimization condition by minimizing the Hamiltonian $\mathcal{H}(t,y_1,y_2,\lambda,\pi,\theta,p,q,r)$ from (3.61) over all $\pi\in\Pi_{\mathbb{G}}$:

$$(\mu(t)-r(t))\hat{p}_2(t) + \sigma(t)\hat{q}_2(t) + \int_{\mathbb{R}_0}\gamma(t,z)\hat{r}_2(t,z)\nu(dz)\lambda(t) = 0. \tag{3.65}$$

From the second equality in (2.13), we get a second and a third optimization condition by maximizing the Hamiltonian $\mathcal{H}(t,y_1,y_2,\lambda,\pi,\theta,p,q,r)$ over all $\theta\in\mathcal{Q}_{\mathbb{G}}$:

$$\hat{Z}(t)\hat{q}_1(t) = 0, \tag{3.66}$$

and

$$\int_{\mathbb{R}_0}\hat{Z}(t)\hat{r}_1(t,z)\nu(dz)\lambda(t) = 0.$$

We guess for a solution of $\hat{p}_1(t)$ of the form

$$\hat{p}_1(t) = U(f(t)\hat{X}(t)),$$

with f a differentiable function. We Utilize Itô's lemma on $\hat{p}_1(t)$, and get

$$
\begin{aligned}
d\hat{p}_1(t) &= f'(t)\hat{X}(t)U'(f(t)\hat{X}(t))dt \\
&\quad + \hat{X}(t)[(r(t) - (\mu(t) - r(t))\hat{\pi}(t))dt \\
&\quad + \sigma(t)\hat{\pi}(t)dW(t)]f(t)U'(f(t)\hat{X}(t)) \\
&\quad + \frac{1}{2}f^2(t)\hat{X}^2(t)\sigma^2(t)\hat{\pi}^2(t)U''(f(t)\hat{X}(t))dt \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t)) \\
&\qquad - f(t)\hat{X}(t)\hat{\pi}(t)\gamma(t,z)U'(f(t)\hat{X}(t))\}\nu(dz)\lambda(t)dt \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t))\}\tilde{H}(dt,dz) \\
&= \Big\{ f'(t)\hat{X}(t)U'(f(t)\hat{X}(t)) + \frac{1}{2}f^2(t)\hat{X}^2(t)\sigma^2(t)\hat{\pi}^2(t)U''(f(t)\hat{X}(t)) \\
&\quad + \hat{X}(t)(r(t) - (\mu(t) - r(t))\hat{\pi}(t))f(t)U'(f(t)\hat{X}(t)) \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t)) \\
&\qquad - f(t)\hat{X}(t)\hat{\pi}(t)\gamma(t,z)U'(f(t)\hat{X}(t))\}\nu(dz)\lambda(t) \Big\} dt \\
&\quad + \hat{X}(t)f(t)\sigma(t)\hat{\pi}(t)U'(f(t)\hat{X}(t))dW(t) \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t))\}\tilde{H}(dt,dz).
\end{aligned}
$$

Comparing this with (3.62), we get

$$
\hat{q}_1(t) = \hat{X}(t)f(t)\sigma(t)\hat{\pi}(t)U'(f(t)\hat{X}(t)), \tag{3.67}
$$

$$
\hat{r}_1(t,z) = U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t)),
$$

and

$$
\begin{aligned}
& f'(t)\hat{X}(t)U'(f(t)\hat{X}_2(t)) + \frac{1}{2}f^2(t)\hat{X}^2(t)\sigma^2(t)\hat{\pi}^2(t)U''(f(t)\hat{X}(t)) \\
&\quad + \hat{X}(t)(r(t) - (\mu(t) - r(t))\hat{\pi}(t))f(t)U'(f(t)\hat{X}(t)) \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t)) \\
&\qquad - f(t)\hat{X}(t)\hat{\pi}(t)\gamma(t,z)U'(f(t)\hat{X}(t))\}\nu(dz)\lambda(t) \\
&= \hat{\theta}_0(t)\hat{q}_1(t) + \int_{\mathbb{R}_0} \hat{\theta}_1(t,z)\hat{r}_1(t,z)\nu(dz)\lambda(t).
\end{aligned}
$$

Substituting (3.67) into condition (3.66), we obtain

$$\hat{Z}(t)\hat{X}(t)f(t)\sigma(t)\hat{\pi}(t)U'(f(t)\hat{X}(t)) = 0,$$

or

$$\hat{\pi}(t) = 0.$$

Now, we try a process $\hat{p}_2(t)$ of the form

$$\hat{p}_2(t) = \hat{Z}(t)f(t)U'(f(t)\hat{X}(t)).$$

We Utilize Itô's lemma on $\hat{p}_2(t)$, and get

$$\begin{aligned}
d\hat{p}_2(t) &= \hat{Z}(t)f'(t)U'(f(t)\hat{X}(t)) + \hat{Z}(t)f(t)dU'(f(t)\hat{X}(t)) \\
&\quad + f(t)U'(f(t)\hat{X}(t))d\hat{Z}(t) \\
&= \hat{Z}(t)\{f'(t)U'(f(t)\hat{X}(t)) + f(t)f'(t)\hat{X}(t)U''(f(t)\hat{X}(t)) \\
&\quad + f^2(t)\hat{X}(t)r(t)U''(f(t)\hat{X}(t))\}dt \\
&\quad + \hat{Z}(t)\theta_0(t)f(t)U'(f(t)\hat{X}(t))dW(t) \\
&\quad + \int_{\mathbb{R}_0} \hat{Z}(t)f(t)\theta_1(t,z)U'(f(t)\hat{X}(t))\tilde{H}(dt,dz).
\end{aligned}$$

Comparing with (3.63), we get

$$\begin{aligned}
\hat{q}_2(t) &= \hat{Z}(t)\theta_0(t)f(t)U'(f(t)\hat{X}(t)), \\
\hat{r}_2(t,z) &= \hat{Z}(t)f(t)\theta_1(t,z)U'(f(t)\hat{X}(t)),
\end{aligned}$$

and

$$\begin{aligned}
&f'(t)U'(f(t)\hat{X}(t)) + f(t)f'(t)\hat{X}(t)U''(f(t)\hat{X}(t)) \\
&\qquad - f^2(t)\hat{X}(t)r(t)U''(f(t)\hat{X}(t)) = r(t)f(t)U'(f(t)\hat{X}(t)).
\end{aligned} \tag{3.68}$$

Inserting $\hat{p}_2(t)$, $\hat{q}_2(t)$ and $\hat{r}_2(t,z)$ into (3.65) yields

$$\begin{aligned}
&(\mu(t) \; - r(t))\hat{Z}(t)f(t)U'(f(t)\hat{X}(t)) + \sigma(t)\hat{Z}(t)\theta_0(t)f(t)U'(f(t)\hat{X}(t)) \\
&\qquad + \int_{\mathbb{R}_0} \gamma(t,z)\hat{Z}(t)f(t)\theta_1(t,z)U'(f(t)\hat{X}(t))\nu(dz)\lambda(t) = 0,
\end{aligned}$$

or

$$(\mu(t) \; - r(t)) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\theta_1(t,z)\nu(dz)\lambda(t) = 0.$$

From (3.68), we get

$$[U'(f(t)\hat{X}(t)) + f(t)\hat{X}(t)U''(f(t)\hat{X}(t))][f'(t) - r(t)f(t)] = 0,$$

or

$$f'(t) - r(t)f(t) = 0,$$

i.e.

$$f(t) = \exp\Big(- \int_t^T r(s)ds \Big).$$

Summary

The unique optimal solution of the problem (3.64) is to find $(\hat{\pi}, \hat{\theta}) \in \Pi_{\mathbb{G}} \times \mathcal{Q}_{\mathbb{G}}$ so that the following conditions hold

$$\hat{\pi} = 0, \tag{3.69}$$

$$(\mu(t) - r(t)) + \sigma(t)\hat{\theta}_0(t) + \int_{\mathbb{R}_0} \gamma(t, z)\hat{\theta}_1(t, z)\nu(dz)\lambda(t) = 0. \tag{3.70}$$

The performance criterion corresponds to the price process at time $t = 0$, and here the solution of (3.64) is

$$\operatorname*{ess\,inf}_{\pi \in \Pi_{\mathbb{G}}} \left\{ \operatorname*{ess\,sup}_{\theta \in \mathcal{Q}_{\mathbb{G}}} J(\pi, \theta) \right\} = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(s)ds} F | \mathcal{F}_T^{\Lambda}].$$

Recall from the derivation of this criterion, that $J(\pi, \theta) = Y(0)$. Hence, the optimal solution of the optimization problem (3.18) at time $t = 0$ by the maximum principle is that $(\hat{\pi}, \hat{\theta})$ satisfies (3.69) and (3.70), and

$$\hat{Y}(0) = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(s)ds} F | \mathcal{F}_T^{\Lambda}]. \tag{3.71}$$

Moreover, the optimal $\hat{\theta}$ is a martingale measure to the price process, since from (3.71) we have that

$$\hat{Y}(0) = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(s)ds} \hat{Y}(T) | \mathcal{F}_T^{\Lambda}].$$

3.5.2 Case II: Standard information on the time-distortion

Now, we look at a new performance criterion

$$J(\pi, \theta) = \mathbb{E}_{\mathbb{Q}}\left[X^{\pi}(0) - e^{-\int_0^T r(s)ds} X^{\pi}(T) + Fe^{-\int_0^T r(s)ds}\right],$$

with a new optimization problem

$$\inf_{\pi \in \Pi_{\mathbb{F}}} \left\{ \sup_{\theta \in \mathcal{Q}_{\mathbb{F}}} J(\pi, \theta) \right\}. \tag{3.72}$$

Note that for this problem, we do not deal with random variables. Therefore, we find the infimum and not the essential infimum, and likewise for the supremum. From the theory in Section 2.4, we see that the optimization problem corresponds to an $f = 0$ and a utility function $U : [0, \infty) \to [-\infty, \infty)$ given by

$$U(X(T)) = X(0) - e^{-\int_0^T r(s)ds} X(T) + Fe^{-\int_0^T r(s)ds}.$$

From Theorem 2.16 and the first equality in (2.22), we have the following optimization condition by minimizing the Hamiltonian $\mathcal{H}^{\mathbb{F}}(t, y_1, y_2, \lambda, \pi, \theta, p, q, r)$ from (3.61) over all $\pi \in \Pi_{\mathbb{F}}$:

$$(\mu(t) - r(t))\mathbb{E}[\hat{p}_2(t)|\mathcal{F}_t] + \sigma(t)\mathbb{E}[q_2(t)|\mathcal{F}_t] + \int_{\mathbb{R}_0} \gamma(t, z)\mathbb{E}[\hat{r}_2(t, z)|\mathcal{F}_t]\lambda(t)\nu(dz) = 0. \tag{3.73}$$

From the second equality in (2.22), we get a second and a third optimization condition by maximizing the Hamiltonian $\mathcal{H}^{\mathbb{F}}(t, y_1, y_2, \lambda, \pi, \theta, p, q, r)$ over all $\theta \in \mathcal{Q}_{\mathbb{F}}$:

$$\mathbb{E}[\hat{Z}(t)\hat{q}_1(t)|\mathcal{F}_t] = 0, \tag{3.74}$$

and

$$\int_{\mathbb{R}_0} \mathbb{E}[\hat{Z}(t)\hat{r}_1(t, z)|\mathcal{F}_t]\nu(dz)\lambda(t) = 0.$$

We guess on a solution of $\hat{p}_1(t)$ of the form

$$\hat{p}_1(t) = U(f(t)\hat{X}(t)),$$

with f a differentiable function. We use Itô's lemma on $\hat{p}_1(t)$, and get

$$\begin{aligned}
d\hat{p}_1(t) &= f'(t)\hat{X}(t)U'(f(t)\hat{X}(t))dt \\
&\quad + \hat{X}(t)[(r(t) - (\mu(t) - r(t))\hat{\pi}(t))dt \\
&\quad + \sigma(t)\hat{\pi}(t)dW(t)]f(t)U'(f(t)\hat{X}(t)) \\
&\quad + \frac{1}{2}f^2(t)\hat{X}^2(t)\sigma^2(t)\hat{\pi}^2(t)U''(f(t)\hat{X}(t))dt \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t)) \\
&\qquad - f(t)\hat{X}(t)\hat{\pi}(t)\gamma(t,z)U'(f(t)\hat{X}(t))\}\nu(dz)\lambda(t)dt \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t))\}\tilde{H}(dt,dz) \\
&= \Bigg\{ f'(t)\hat{X}(t)U'(f(t)\hat{X}(t)) + \frac{1}{2}f^2(t)\hat{X}^2(t)\sigma^2(t)\hat{\pi}^2(t)U''(f(t)\hat{X}(t)) \\
&\quad + \hat{X}(t)(r(t) - (\mu(t) - r(t))\hat{\pi}(t))f(t)U'(f(t)\hat{X}(t)) \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t)) \\
&\qquad - f(t)\hat{X}(t)\hat{\pi}(t)\gamma(t,z)U'(f(t)\hat{X}(t))\}\nu(dz)\lambda(t) \Bigg\} dt \qquad (3.75) \\
&\quad + \hat{X}(t)f(t)\sigma(t)\hat{\pi}(t)U'(f(t)\hat{X}(t))dW(t) \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t))\}\tilde{H}(dt,dz).
\end{aligned}$$

Comparing (3.75) with (3.62), we get

$$\hat{q}_1(t) = \hat{X}(t)f(t)\sigma(t)\hat{\pi}(t)U'(f(t)\hat{X}(t)), \qquad (3.76)$$

$$\hat{r}_1(t,z) = U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t)),$$

and

$$\begin{aligned}
&f'(t)\hat{X}(t)U'(f(t)\hat{X}_2(t)) + \frac{1}{2}f^2(t)\hat{X}^2(t)\sigma^2(t)\hat{\pi}^2(t)U''(f(t)\hat{X}(t)) \\
&\quad + \hat{X}(t)(r(t) - (\mu(t) - r(t))\hat{\pi}(t))f(t)U'(f(t)\hat{X}(t)) \\
&\quad + \int_{\mathbb{R}_0} \{U(\hat{X}(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}(t)) \\
&\qquad - f(t)\hat{X}(t)\hat{\pi}(t)\gamma(t,z)U'(f(t)\hat{X}(t))\}\nu(dz)\lambda(t) \\
&= \hat{\theta}_0(t)\hat{q}_1(t) + \int_{\mathbb{R}_0} \hat{\theta}_1(t,z)\hat{r}_1(t,z)\nu(dz)\lambda(t).
\end{aligned}$$

Substituting (3.76) into condition (3.74), we obtain

$$\begin{aligned}\mathbb{E}[\hat{Z}(t)\hat{X}(t)f(t)\sigma(t)\hat{\pi}(t)U'(f(t)\hat{X}(t))|\mathcal{F}_t] = 0\\ \hat{\pi}(t)\hat{X}(t)f(t)\sigma(t)U'(f(t)\hat{X}(t))\hat{Z}(t) = 0,\end{aligned}$$

or

$$\hat{\pi}(t) = 0.$$

Now, we try a process $\hat{p}_2(t)$ of the form

$$\hat{p}_2(t) = \hat{Z}(t)f(t)U'(f(t)\hat{X}(t)).$$

We use Itô's lemma on $\hat{p}_2(t)$, and get

$$\begin{aligned}d\hat{p}_2(t) &= \hat{Z}(t)f'(t)U'(f(t)\hat{X}(t)) + \hat{Z}(t)f(t)dU'(f(t)\hat{X}(t))\\ &\quad + f(t)U'(f(t)\hat{X}(t))d\hat{Z}(t)\\ &= \hat{Z}(t)\{f'(t)U'(f(t)\hat{X}(t)) + f(t)f'(t)\hat{X}(t)U''(f(t)\hat{X}(t))\\ &\quad + f^2(t)\hat{X}(t)r(t)U''(f(t)\hat{X}(t))\}dt\\ &\quad + \hat{Z}(t)\hat{\theta}_0(t)f(t)U'(f(t)\hat{X}(t))dW(t)\\ &\quad + \int_{\mathbb{R}_0}\hat{Z}(t)f(t)\hat{\theta}_1(t,z)U'(f(t)\hat{X}(t))\tilde{H}(dt,dz).\end{aligned}$$

Comparing this and (3.63), we see that

$$\hat{q}_2(t) = \hat{Z}(t)\hat{\theta}_0(t)f(t)U'(f(t)\hat{X}(t)),$$

$$\hat{r}_2(t,z) = \hat{Z}(t)f(t)\hat{\theta}_1(t,z)U'(f(t)\hat{X}(t)),$$

and

$$\begin{aligned}&f'(t)U'(f(t)\hat{X}(t)) + f(t)f'(t)\hat{X}(t)U''(f(t)\hat{X}(t))\\ &\quad - f^2(t)\hat{X}(t)r(t)U''(f(t)\hat{X}(t)) = r(t)f(t)U'(f(t)\hat{X}(t)).\end{aligned} \tag{3.77}$$

Inserting $\hat{p}_2(t)$, $\hat{q}_2(t)$ and $\hat{r}_2(t,z)$ in (3.73) yields

$$\begin{aligned}&(\mu(t) - r(t))\mathbb{E}[\hat{Z}(t)f(t)U'(f(t)\hat{X}(t))|\mathcal{F}_t] + \sigma(t)\mathbb{E}[\hat{Z}(t)\theta_0(t)f(t)U'(f(t)\hat{X}(t))|\mathcal{F}_t]\\ &\quad + \int_{\mathbb{R}_0}\gamma(t,z)\mathbb{E}[\hat{Z}(t)f(t)\theta_1(t,z)U'(f(t)\hat{X}(t))|\mathcal{F}_t]\lambda(t)\nu(dz) = 0,\end{aligned}$$

or

$$(\mu(t) - r(t)) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}_0}\gamma(t,z)\theta_1(t,z)\lambda(t)\nu(dz) = 0.$$

From (3.77), we get

$$[U'(f(t)\hat{X}(t)) + f(t)\hat{X}(t)U''(f(t)\hat{X}(t))][f'(t) - r(t)f(t)] = 0,$$

or

$$f'(t) - r(t)f(t) = 0,$$

i.e.

$$f(t) = \exp\left(- \int_t^T r(s)ds \right).$$

Summary

The unique optimal solution of the problem (3.72) is to find $(\hat{\pi}, \hat{\theta}) \in \Pi_{\mathbb{F}} \times \mathcal{Q}_{\mathbb{F}}$ so that the following conditions hold

$$\hat{\pi} = 0, \tag{3.78}$$

$$(\mu(t) - r(t)) + \sigma(t)\hat{\theta}_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\hat{\theta}_1(t,z)\nu(dz)\lambda(t) = 0. \tag{3.79}$$

The performance criterion corresponds to the price process at time $t = 0$, and here the solution of (3.72) is

$$\inf_{\pi \in \Pi_{\mathbb{F}}} \left\{ \sup_{\theta \in \mathcal{Q}_{\mathbb{F}}} J(\pi, \theta) \right\} = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(s)ds} F].$$

Recall from the derivation of this criterion, that $J(\pi, \theta) = Y(0)$. Hence, the optimal solution of the optimization problem (3.18) at time $t = 0$ by the maximum principle is that $(\hat{\pi}, \hat{\theta})$ satisfies (3.78) and (3.79), and

$$\hat{Y}(0) = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(s)ds} F]. \tag{3.80}$$

Moreover, the optimal $\hat{\theta}$ is a martingale measure to the price process, since from (3.80) we have that

$$\hat{Y}(0) = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(s)ds} \hat{Y}(T)].$$

3.6 Analysis and comparison of the solutions ♣

We have solved the problem (3.18) for both the filtrations $\mathbb{G}$ and $\mathbb{G}$ via the maximum principle and via BSDEs. In this section, we make a comparison of the solutions, and analyze whether the continuous evaluation via BSDEs makes a difference to the capital requirement for the worst case scenario hedge.

With this in mind, we give a toy example to point out for which contingent claims the solutions of the two methods coincide.

Recall the results on page 58, 68, 75, and 80.

3.6.1 Example: $e^{\int_0^T r(t)dt}F$ is $\mathcal{F}_T^\Lambda-$measurable

In this example, we let $e^{\int_0^T r(t)dt}F$ be $\mathcal{F}_T^\Lambda-$measurable. The purpose of this example is not to be realistic, but to show the differences and similarities of the solutions of (3.18) by the two methods studied.

The solution of the optimization via BSDEs with information flow $\mathbb{F}$ contains an unknown process, and it is difficult to find an explicit expression of the solution for a general contingent claim. By assuming that $e^{\int_0^T r(t)dt}F$ is $\mathcal{F}_T^\Lambda-$measurable, we know what this unknown process is. With this in mind, we analyze the integral representation of $L_2(\mathbb{Q})-$variables.

Let $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{Q})$, then by the integral representations for the information flows $\mathbb{F}$ and $\mathbb{G}$, we have:

$$\xi = \xi_0 + \int_0^T Z(t)dW^\theta(t) + \int_0^T \int_{\mathbb{R}_0} U(t,z)\tilde{H}^\theta(dt,dz), \quad (Z,U) \in \mathcal{I}_\mathbb{F},$$

$$\xi = \mathbb{E}_\mathbb{Q}[\xi|\mathcal{F}_T^\Lambda] + \int_0^T \bar{Z}(t)dW^\theta(t) + \int_0^T \int_{\mathbb{R}_0} \bar{U}(t,z)\tilde{H}^\theta(dt,dz), \quad (\bar{Z},\bar{U}) \in \mathcal{I}_\mathbb{G}.$$

Call the spaces spanned by the stochastic integrals for

$$\mathbb{X}_\mathbb{F} = \left\{ \int_0^T Z(t)dW^\theta(t) + \int_0^T \int_{\mathbb{R}_0} U(t,z)\tilde{H}^\theta(dt,dz), \quad (Z,U) \in \mathcal{I}_\mathbb{F} \right\},$$

$$\mathbb{X}_\mathbb{G} = \left\{ \int_0^T Z(t)dW^\theta(t) + \int_0^T \int_{\mathbb{R}_0} U(t,z)\tilde{H}^\theta(dt,dz), \quad (Z,U) \in \mathcal{I}_\mathbb{G} \right\}.$$

Clearly, $\mathbb{X}_\mathbb{F} \subset \mathbb{X}_\mathbb{G}$ by the definition of the integrands in Definition 2.7. Let $\mathbb{X}^C$ be the complement of $\mathbb{X}$ in $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{Q})$. The complement, is the remains of the space $L_2(\Omega, \mathcal{F}_T, \mathbb{Q})$ which is not covered by the stochastic integrals. Since

$$\mathbb{X}_\mathbb{F} \cup \mathbb{X}_\mathbb{F}^C = \mathbb{X}_\mathbb{G} \cup \mathbb{X}_\mathbb{G}^C = L_2(\Omega, \mathcal{F}_T, \mathbb{Q}),$$

we have that $\mathbb{X}_\mathbb{G}^C \subset \mathbb{X}_\mathbb{F}^C$, i.e. the complement of the space $\mathbb{X}_\mathbb{G}$ is a subset of the complement of $\mathbb{X}_\mathbb{F}$. We know that $\mathbb{X}_\mathbb{G}^C$ is the set of $\mathcal{F}_T^\Lambda-$measurable random variables by the integral representation (2.4). By the arguments above, we conclude that $\mathbb{X}_\mathbb{F}^C$ contains the $\mathcal{F}_T^\Lambda-$measurable random variables.

Hence, when $e^{\int_0^T r(t)dt}F$ is an $\mathcal{F}_T^{\Lambda}-$measurable random variable, then $\xi_0 = e^{-\int_0^T r(t)dt}F$. The integrands below are indexed by $\mathbb{F}$ or $\mathbb{G}$, which indicates of which problem they are a solution. We can conclude the following in succeeding order:

I. By $\mathbb{E}_{\hat{\mathbb{Q}}}[e^{\int_0^T r(t)dt}F|\mathcal{F}_T^{\Lambda}] = e^{\int_0^T r(t)dt}F$, the equation in (3.41) yields $\hat{Z}_{\mathbb{F}}(t) = \hat{U}_{\mathbb{F}}(t,z) = 0$;

II. By I and (3.38) (or (3.39)), the optimal investment strategy is $\hat{\pi}(t) = 0$, and by (3.42)
$$\hat{Y}_{\mathbb{G}}(t) = e^{-\int_t^T r(s)ds}F.$$

III. Moreover, by II:
$$\hat{Y}_{\mathbb{G}}(0) = e^{-\int_0^T r(t)dt}F.$$

IV. The optimal parameters of the measure change $(\hat{\theta}_0, \hat{\theta}_1)$ are still a solution of (3.40).

And

I. By $\xi_0 = e^{-\int_0^T r(t)dt}F$, the equation in (3.58) yields $\hat{Z}_{\mathbb{F}}(t) = \hat{U}_{\mathbb{F}}(t,z) = 0$;

II. By I and (3.55) (or (3.56)), the optimal investment strategy is $\hat{\pi}(t) = 0$, and by (3.59)
$$\hat{Y}_{\mathbb{F}}(t) = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_t^T r(t)dt}F|\mathcal{F}_t]$$

III. Moreover, by II:
$$\hat{Y}_{\mathbb{F}}(0) = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_0^T r(t)dt}F].$$

IV. The optimal parameters of the measure change $(\hat{\theta}_0, \hat{\theta}_1)$ are still a solution of (3.57).

Hence, we see that the optimal price process solved via the BSDEs at $t = 0$ coincides with the optimal price process solved via the maximum principle when the contingent claim $e^{\int_0^T r(t)dt}F$ is $\mathcal{F}_T^{\Lambda}-$measurable, for both the case of information flow $\mathbb{F}$ and $\mathbb{G}$. ♣

In the example (3.6.1) we see that for special cases of contingent claims F and the short rate r, the solution $(\hat{Y}, \hat{\pi}, \hat{\theta})$ of the problem in (3.18) via the maximum principle evaluated at the initial time, is the same as the solution via BSDEs continuously evaluated for $t \in [0, T]$. This holds for both the information flow $\mathbb{F}$ and $\mathbb{G}$.

The reason the solutions coincide, is because when $e^{-\int_0^T r(t)dt}F$ is $\mathcal{F}_T^{\Lambda}-$measurable, there is not possible to hedge the risk by investing in the stock. The stock is strongly orthogonal to the $\mathcal{F}_T^{\Lambda}-$measurable elements in $L_2(\hat{\mathbb{Q}})$. Since $e^{\int_0^t r(s)ds}S_1(t)$ is a $(\mathbb{G}, \hat{\mathbb{Q}})-$martingale, we have:

$$\mathbb{E}_{\hat{\mathbb{Q}}}[e^{\int_0^T r(s)ds}F \cdot S_1(T)|\mathcal{G}_t] = F \cdot \mathbb{E}_{\hat{\mathbb{Q}}}[e^{\int_0^T r(s)ds}S_1(T)|\mathcal{G}_t] = Fe^{\int_0^t r(s)ds}S_1(t), \quad t \in [0, T].$$

Strong orthogonality follows from Definition D.8. Therefore, the optimal strategy is to put all the money in the risk-free asset, and wait until time of maturity.

The results in the toy example are, however, not valid for general contingent claims. In general, the optimal investment will be the solution of (3.55) and (3.56) (optionally (3.38) and (3.39)). And whenever $e^{-\int_0^T r(t)dt}F$ is not totally contained in $\mathbb{X}_{\mathbb{F}}^C$ (optionally $\mathbb{X}_{\mathbb{G}}^C$), then the solution is $\hat{\pi}(t) \neq 0$. Hence, the solution $\hat{\pi}(t) = 0$ cannot in general be the solution to the optimization problem (3.18), and the strategy will over-hedge the contingent claim with respect to the preferences of the agent. What we mean by over-hedge is explained below.

Let π be different from $\hat{\pi}$. By the solution in Sections 3.4.1 and 3.4.2, we know that π will generate a value process which is greater or equal to the value process of $\hat{\pi}$, i.e.

$$X^{\pi}(t) \geq X^{\hat{\pi}}(t), \quad \mathbb{P} \times dt - a.e.$$

since the solutions are the infimum over the admissible sets of portfolios. Thus, the solution $\pi(t) = 0$ will in general over-hedge the contingent claim, and make the agent assume a bigger potential loss than the optimal solution via BSDEs. We may say that the maximum principle gives a more conservative solution than the BSDEs. The solution of the maximum principle tells the agent to invest all the money in the risk-free bond, and just sit an wait for the bad economy to come. The solution of the BSDEs suggest that the agent in some cases invests in the risky asset, and to lower the capital requirement. The author thinks this solution has more credibility to it. One has to try to create profit, even though the requirements to stay solvent are high. To go and hide in a dark room, throw the key away, and sit and wait for a crack in the market, seems a bit too extreme, even for an ambiguity averse agent.

Over to the optimal solution $\hat{\theta}(t)$ of "the market". As we can see from the summary pages, the solution is the same for the optimization via the maximum principle as for the optimization via BSDEs. They both have the solution that $\hat{\theta}$

should satisfy

$$(\mu(t) - r(t)) + \sigma(t)\hat{\theta}_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\hat{\theta}_1(t,z)\nu(dz)\lambda(t) = 0.$$

We have previously claimed that this is an equivalent martingale measure, and for completeness we prove that this indeed is true, i.e. the discounted asset is a $\mathbb{Q}-$martingale:

$$\begin{aligned}
d\left(e^{-\int_0^t r(s)ds}S_1(t)\right) &= -r(t)e^{-\int_0^t r(s)ds}S_1(t)dt + e^{-\int_0^t r(s)ds}dS_1(t) \\
&= e^{-\int_0^t r(s)ds}S_1(t)\Bigg[(\mu(t) - r(t))dt + \sigma(t)dW(t) \\
&\quad + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{H}(dt,dz)\Bigg] \\
&= e^{-\int_0^t r(s)ds}S_1(t)\Bigg[(\mu(t) - r(t))dt + \sigma(t)\Big\{dW^{\hat{\theta}}(t) + \hat{\theta}_0(t)dt\Big\} \\
&\quad + \int_{\mathbb{R}_0} \gamma(t,z)\Big\{\tilde{H}^{\hat{\theta}}(dt,dz) + \hat{\theta}_1(dt,dz)\Big\}\Bigg] \\
&= e^{-\int_0^t r(s)ds}S_1(t)\Bigg[\sigma(t)dW^{\hat{\theta}}(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{H}^{\hat{\theta}}(dt,dz)\Bigg],
\end{aligned}$$

since

$$(\mu(t) - r(t)) + \sigma(t)\hat{\theta}_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\hat{\theta}_1(t,z)\nu(dz)\lambda(t) = 0.$$

Because the optimal probability distribution induced by $\hat{\theta}$ is an equivalent martingale measure, the optimal price process $\hat{Y}$ corresponds to the *dynamic superhedging price* of the portfolio $\hat{\pi}$. The dynamic superhedging price has the conditional risk measures of the form:

$$\rho_t(X) = \operatorname*{ess\,sup}_{\theta \in \text{ EMM}} \mathbb{E}_{\mathbb{Q}}[-X|\mathcal{F}_t],$$

where EMM is the set of equivalent martingale measures to the probability measure for which the financial market is defined. The optimal price process $\hat{Y}$ corresponds to the *dynamic superhedging price* because the set of EMMs are contained in the set of equivalent measures $\mathcal{Q}$, and therefore finding the superhedging price is a subproblem of (3.18).

Superhedging strategies are not popular in financial markets. Firstly, they are difficult to compute. Secondly, superhedging prices are in general not arbitrage-free and they are usually too high to be accepted by buyers. However, under the assumptions that the essential supremum and infimum *are attained* in Theorem 3.7, the optimal price process have the representation

$$\hat{Y}(t) = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-\int_t^T r(s)ds}F|\mathcal{F}_t],$$

and is not just a limit of elements of this form. So the prices under $\hat{\mathbb{Q}}$ are arbitrage-free.

This said, the question of existence of arbitrage opportunities is not a goal to answer in our application. The optimal solution will serve as a suggestion to the sufficient capital required to withstand extreme scenarios.

3.7 Further research

The theory in Section 3.4.2 opens up for more general processes than what this thesis entails. We have worked with DSPRFs that are centered by a compensator that is absolutely continuous with respect to the Lebesgue measure. This was utilized in the solution via BSDEs in Section 3.4.1 in the case of future insight of the time-distortion process. However, the solution via BSDEs in Section 3.4.2 in the case of standard information of the time-distortion required no such dependency, thus random fields with compensators of more general form may be studied in this theory.

Ambit fields are examples of generalizations of the random fields treated in the present thesis. We suggest a representation of ambit fields via stochastic integrals of meta-time changed Lévy random fields. Meta-time change can be regarded as a stochastic perturbation of the time-space by a random measure (introduced only from a statistical point of view by Barndorff-Nielsen and Pedersen, 2012.) It is not certain that this extension is applicable in the theory mentioned above, since general ambit fields are not semimartingales, and specific integration theory has to be devised.

The motivation of this generalization is to model environmental factors representing the exogenous turbulence and environmental risk factors, e.g. in energy price dynamics, that can not be captured by time-perturbation alone. Hedging of environmental risk is crucial in derivative pricing in energy markets.

Appendix

A Dynamic risk measures via g-expectations

A dynamic risk measure is a functional that quantifies the riskiness of a financial position continuously over a specified period of time. This is useful since the information flow is continuous, and investors should, in theory, continuously update their portfolio. The theory in this section is obtained from [5].

The definition of a dynamic risk measure is as follows:

Definition A.1 *A family* $(\rho_t)_{0\le t\le T}$ *of mappings* $\rho_t : L_2(\Omega, \mathcal{F}_T, \mathbb{P}) \to L_2(\Omega, \mathcal{F}_t, \mathbb{P})$ *such that*

$$\rho_T(\xi) = -\xi, \qquad \text{for } \xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$$

is called a dynamic risk measure.

The measure $\rho_t(\xi)$ quantifies at time t the risk of the financial position ξ which is going to be liquidated at time T. Since the manager is not interested in losing any money in expectation, we call the risk of ξ acceptable at time t if $\rho_t(\xi) \le 0$.

Risk measures are adopted to finance in order to measure risk in a reasonable manner, and certain properties should be satisfied in order to concur with reality. We introduce the key properties, sometimes called the axioms of risk measures, for the risk measure to agree with financial practice and views of investors:

- **Convexity**

  $$\rho_t(c\xi_1 + (1-c)\xi_2) \le c\rho_t(\xi_1) + (1-c)\rho(\xi_2),$$
  $$0 \le t \le T, \quad c \in [0,1], \quad \xi_1, \xi_2 \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$$

  This implies that diversification reduces the risk.

- **Monotonicity**

  $$\xi_1 \ge \xi_2 \;\Rightarrow\; \rho_t(\xi_1) \le \rho_t(\xi_2), \quad 0 \le t \le T, \quad \xi_1, \xi_2 \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$$

  This reflects the common rule that if a position has higher or equal pay-off in all scenarios, then it is less risky.

- **Cash invariance**

  $$\rho_t(c) = -c, \quad 0 \le t \le T, \quad c \in \mathbb{R}.$$

  The risk of a constant pay-off should be minus the pay-off, or in order to protect ourselves from this risk, we need to reserve c money units to be able to pay the liability at time $t = T$. This property can be exchanged by "cash sub-additivity" which respects the time-value of money. This property is defined as the non-decreasing mapping $c \mapsto \rho_t(\xi + c) + c$ on $\mathbb{R}$, $0 \le t \le T$, $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$. This basically says, that in order to cover for a constant liability c, we need less than c. This agrees with putting money in a risk-free bank.

- **Translation invariance**

$$\rho_t(\xi_1 + \xi_2) = \rho_t(\xi_1) - \xi_2,$$
$$0 \le t \le T, \quad \xi_1 \in L_2(\Omega, \mathcal{F}_T, \mathbb{P}), \quad \xi_2 \in L_2(\Omega, \mathcal{F}_t, \mathbb{P}).$$

  The riskiness of a position is only affected by uncertainty of this position, and components that are determined by the current information should be treated as constants. This is also a property that suggests that $\rho_t(\xi)$ can be thought of as the price of the position ξ that makes it acceptable. Indeed, we have

$$\rho_t(\xi + \rho_t(\xi)) = \rho_t(\xi) - \rho_t(\xi) = 0.$$

  So the risk is measured to be zero. Therefore, a risk measure is also called a capital requirement, which is a correct term when working with replication of liabilities.

- **Sub-linearity: sub-additivity and positively homogeneity**

$$\rho_t(\xi_1 + \xi_2) \le \rho_t(\xi_1) + \rho_t(\xi_2), \quad \rho_t(c\xi) = c\rho_t(\xi),$$
$$0 \le t \le T, \quad c > 0, \quad \xi_1, \xi_2, \xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$$

  Sub-additivity is a property that supports that diversification is encouraged, while positive homogeneity says that the risk of a position should be proportional to the volume. Positive homogeneity property does not include liquidity risk in the market.

- **Time-consistency**

$$\rho_s(\xi) = \rho_s(-\rho_t(\xi)), \quad 0 \le s \le t \le T, \quad \xi_1, \xi_2 \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$$

  Quantifying the risk of the position directly from time $t = T$ should yield the same result as quantifying an intermediate quantification of the position.

In this thesis we will model dynamic risk measures by BSDEs, and this is done through $g-$expectations. Recall the general representation of a BSDE from (2.6). The generator g decides the properties of the dynamic risk measure. The definition of a $g-$expectation, in the setup of this thesis, is as follows:

Definition A.2 *Let*

(i) $(\mu(t),\ 0 \le t \le T)$ *is a square integrable, càdlàg* $(\mathbb{F}, \mathbb{P})-$*martingale,*

(ii) $\langle \mu \rangle$ *is its dual predictable projection, and*

(iii) N *is a square integrable* $(\mathbb{F}, \mathbb{P})-$*martingale orthogonal to* μ*, and* $N(0) = 0$.

Then the $g-$*expectation* $\mathcal{E}_g(\xi)(t) := Y(t)$ *is the solution of the BSDE*

$$\begin{aligned} dY(t) &= -g(t, Y(t), Z(t))d\langle\mu\rangle(t) + Z(t)d\mu(t) + dN(t) \\ Y(T) &= \xi \end{aligned}$$

The $g-$expectation studied in this thesis is of the form

$$\mathcal{E}_g(\xi)(t) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-\xi|\mathcal{M}_t],$$

where $\mathcal{M}_t$ is either $\mathcal{F}_t$ or $\mathcal{G}_t$ (see Section 2.1.) The dynamic risk measure is defined by this $g-$expectation

$$\rho_t(\xi) = \mathcal{E}_g(\xi)(t).$$

B Proofs of results in Section 2.2

In section 2.2 we only presented the main results of the integral (2.4) and martingale representation (2.5) for easy and quick reading. Here in the appendices we will elaborate for completeness. The results and proofs are taken from [8] and [12].

To ease the notation, define the operator I:

Definition B.1 *Define the integral operator* $I : \mathcal{I}_{\mathbb{G}} \to L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ *by*

$$I(\theta) := \int_0^T \theta_0(t)dW(t) + \int_0^T \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{H}(dt,dz).$$

Define the norm on $\mathcal{I}_{\mathbb{G}}$ *by*

$$\|\theta\|_{\mathcal{I}_{\mathbb{G}}} = \left(\mathbb{E}\Big[\int_0^T {\theta_0}^2(t)dt + \int_0^T \int_{\mathbb{R}_0} {\theta_1}^2(t,z)\nu(dz)\lambda(t)dt\Big]\right)^{1/2}$$

Remark. Note that for all $\theta \in \mathcal{I}_{\mathbb{G}}$ we have $\|\theta\|_{\mathcal{I}_{\mathbb{G}}} < \infty$, and that the operator $I : \mathcal{I}_{\mathbb{G}} \to L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ is isomorphic, i.e.

$$\left(\mathbb{E}[I(\theta)^2]\right)^{1/2} = \|I(\theta)\|_{L_2(\mathbb{P})} = \|\theta\|_{\mathcal{I}_{\mathbb{G}}}. \quad \diamond \tag{B.1}$$

First we find a total subset of $L_2(\Omega, \mathcal{F}_T, \mathbb{P})$.

$$\mathcal{K} := \{\theta \in \mathcal{I}_{\mathbb{G}} \mid \theta_0 \text{ and } \theta_1 \text{ are } \mathcal{F}_T^\Lambda\text{-measurable, } \theta_1 \text{ is bounded } \mathbb{P} \times dt\text{-a.e.,} \\ \text{and } I(\theta) \text{ is a bounded random variable}\}$$

The following lemmas and their proofs can be found in [8].

Lemma B.2 *For any* $\theta \in \mathcal{K}$ *we have*

$$e^{I(\theta)} \in L_2(\Omega, \mathcal{F}_T, \mathbb{P}), \tag{B.2}$$

and

$$\frac{e^{I(\theta)}}{\mathbb{E}[e^{I(\theta)}|\mathcal{F}_T^\Lambda]} \in L_2(\Omega, \mathcal{F}_T, \mathbb{P}). \tag{B.3}$$

Furthermore, the random variables $\{e^{I(\theta)},\ \theta \in \mathcal{K}\}$ *form a total subset of* $L_2(\Omega, \mathcal{F}_T, \mathbb{P})$.

Proof: (B.2) and (B.3) are proved in [Lemma 4, Lemma 6 and Lemma 9 in [12]]. We prove that $\{e^{I(\theta)},\ \theta \in \mathcal{K}\}$ form a total subset of $L_2(\Omega, \mathcal{F}_T, \mathbb{P})$:

If $\theta \in \mathcal{K}$, then by (B.2), we have $e^{I(\theta)} \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$. Let $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ be such that

$$\mathbb{E}[\xi e^{I(\theta)}] = 0, \quad \text{for all } \theta \in \mathcal{K}.$$

By Proposition 2.6 we deduce that

$$\mathbb{E}[\xi e^{\sum_{k=1}^n z_k I(\theta_k)}] = 0, \quad \text{for all } \theta_1, ..., \theta_n \in \mathcal{K},\ z_1, ...z_n \in \mathbb{R},\ n \geq 1. \tag{B.4}$$

Fix $n \geq 1$ and $\theta_1, ..., \theta_n \in \mathcal{K}$. Then (B.4) says that the Laplace transform of the signed measure

$$\tau(B) = \mathbb{E}[\xi \mathbb{1}_B(I(\theta_1), ..., I(\theta_n))],$$

for $B \in \mathcal{B}_{\mathbb{R}}$, is identically zero on $\mathbb{R}$. Consequently, the measure τ is zero, which implies $\mathbb{E}[\xi \mathbb{1}_G] = 0$ for any $G \in \mathcal{F}_T$, so $\xi = 0$. ■

Next we show that the elements in the total subset has the integral representation stated in Theorem 2.8.

Lemma B.3 *Assume $\theta \in \mathcal{K}$. Define, for $t \in [0, T]$,*

$$\zeta(t) = \exp\left\{ \int_0^t \theta_0(s) dW(s) + \int_0^t \int_{\mathbb{R}_0} \theta_1(s, z) \tilde{H}(ds, dz) \right\}.$$

Then the following representation holds:

$$\begin{aligned}\zeta(T) = \mathbb{E}[\zeta(T)|\mathcal{F}_T^{\Lambda}] + &\int_0^T \left[\mathbb{E}[\frac{\zeta(T)}{\zeta(s^-)}|\mathcal{F}_T^{\Lambda}] \zeta(s^-) \theta_0(s) \right] dW(s) \\ &+ \int_0^T \int_{\mathbb{R}_0} \left[\mathbb{E}[\frac{\zeta(T)}{\zeta(s^-)}|\mathcal{F}_T^{\Lambda}] \zeta(s^-)(e^{\theta_1(s,z)-1}) \right] \tilde{H}(ds, dz)\end{aligned} \tag{B.5}$$

Remark. The integrands in (B.5) are $\mathbb{G}-$predictable. ◇

Proof: Let

$$\begin{aligned}Y(t) &= \frac{\zeta(t)}{\mathbb{E}[\zeta(t)|\mathcal{F}_T^{\Lambda}]} \\ &= \exp\left\{ \int_0^t \theta_0(s) dW(s) + \int_0^t \int_{\mathbb{R}_0} \theta_1(s, z) \tilde{H}(ds, dz) - \int_0^t \frac{1}{2} \theta_0(s)^2 ds \right. \\ &\qquad \left. - \int_0^t \int_{\mathbb{R}_0} [e^{\theta_1(s,z)} - 1 - \theta_1(s, z)] \nu(dz) \lambda(s) ds \right\},\end{aligned} \tag{B.6}$$

by Lévy-Khintchine. Note that $Y(t)$ and $\zeta(t)$ are elements of $L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ by Lemma B.2. By Itô's formula

$$\begin{aligned}dY(t) &= Y(t^-) \Big(\theta_0(t) dW(t) + \int_{\mathbb{R}_0} (e^{\theta_1(t,z)} - 1) \tilde{H}(dt, dz) \Big) \\ Y(0) &= 1.\end{aligned} \tag{B.7}$$

Combining (B.6) and (B.7) the above equalities yields

$$\begin{aligned}
\zeta(T) &= \mathbb{E}[\zeta(T)|\mathcal{F}_T^\Lambda]Y(T) \\
&= \mathbb{E}[\zeta(T)|\mathcal{F}_T^\Lambda]\Big(1 + \int_0^T Y(s^-)\theta_0(s)dW(s) \\
&\quad + \int_0^T \int_{\mathbb{R}_0} Y(s^-)(e^{\theta_1(s,z)} - 1)\tilde{H}(ds,dz)\Big) \\
&= \mathbb{E}[\zeta(T)|\mathcal{F}_T^\Lambda] + \int_0^T \mathbb{E}[\zeta(T)|\mathcal{F}_T^\Lambda]Y(s^-)\theta_0(s)dW(s) \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[\zeta(T)|\mathcal{F}_T^\Lambda]Y(s^-)(e^{\theta_1(s,z)} - 1)\tilde{H}(ds,dz) \\
&= \mathbb{E}[\zeta(T)|\mathcal{F}_T^\Lambda] + \int_0^T \mathbb{E}[\frac{\zeta(T)}{\zeta(s^-)}|\mathcal{F}_T^\Lambda]\zeta(s^-)\theta_0(s)dW(s) \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[\frac{\zeta(T)}{\zeta(s^-)}|\mathcal{F}_T^\Lambda]\zeta(s^-)(e^{\theta_1(s,z)} - 1)\tilde{H}(ds,dz),
\end{aligned}$$

where we used Proposition 2.6 and the equations

$$Y(s)\mathbb{E}[\zeta(T)|\mathcal{F}_T^\Lambda] = Y(s)\mathbb{E}[\zeta(s)|\mathcal{F}_T^\Lambda]\mathbb{E}[\frac{\zeta(T)}{\zeta(s)}|\mathcal{F}_T^\Lambda] = \zeta(s)\mathbb{E}[\frac{\zeta(T)}{\zeta(s)}|\mathcal{F}_T^\Lambda]. \quad \blacksquare$$

Now that we have shown that the elements in the total set in $L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ have the integral representation in Theorem 2.8, we are ready to prove that all elements in $L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ has this representation.

Proof of Theorem 2.8: At first let $\xi = \zeta(T)$, where

$$\zeta(T) = e^{I(\theta)}, \quad \theta \in \mathcal{K}.$$

From Lemma B.3 the integral representation (2.4) holds in this case.

Consider a general $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$. Then ξ can be approximated by a sequence of linear combinations of the form (2.4) by Lemma B.2. Let $\{\xi_n\}_{n\geq 1}$ be such a sequence. Then, by (B.1), we have

$$\begin{aligned}
\mathbb{E}[(\xi_n - \xi_m)^2] = \mathbb{E}\Big[&(\mathbb{E}[\xi_n - \xi_m|\mathcal{F}_T^\Lambda])^2 + \int_0^T ({\theta_0}^{(n)}(s) - {\theta_0}^{(m)}(s))^2 ds \\
&+ \int_0^T \int_{\mathbb{R}_0} ({\theta_1}^{(n)}(s,z) - {\theta_1}^{(m)}(s,z))^2 \Lambda(ds,dz)\Big].
\end{aligned}$$

Thus $\{\theta^{(n)}\}_{n\geq 1}$ is a Cauchy-sequence in $\mathcal{I}_{\mathbb{G}}$, which proves existence. To prove uniqueness, suppose

$$\begin{aligned}
\xi &= \mathbb{E}[\xi|\mathcal{F}_T^\Lambda] + I(\theta) \\
&= \mathbb{E}[\xi|\mathcal{F}_T^\Lambda] + I(\Theta),
\end{aligned}$$

for $\Theta \in \mathcal{I}_{\mathbb{G}}$. Then, using (B.1),

$$\mathbb{E}\bigg[\int_0^T ({\theta_0}^{\theta}(s) - {\theta_0}^{\Theta}(s))^2 ds + \int_0^T \int_{\mathbb{R}_0} ({\theta_1}^{\theta}(s,z) - {\theta_1}^{\Theta}(s,z))^2 \Lambda(ds,dz)\bigg] = 0. \quad \blacksquare$$

Proof of Theorem 2.9: The proof is a modification of the proof of Theorem 4.3.4 in [13]. By Theorem 2.8 applied to $T = t$ and $\xi = M(t)$, we have that for all $t \in [0,T]$ there exists a unique couple $({\theta_0}^{(t)}(s,\omega), {\theta_1}^{(t)}(s,z,\omega))$ such that

$$\begin{aligned} M(t) &= \mathbb{E}[M(t)|\mathcal{F}_T^{\Lambda}] + \bigg(\int_0^t {\theta_0}^{(t)}(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} {\theta_1}^{(t)}(s,z)\tilde{H}(ds,dz)\bigg) \\ &= \mathbb{E}[M(T)|\mathcal{F}_T^{\Lambda}] + \bigg(\int_0^t {\theta_0}^{(t)}(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} {\theta_1}^{(t)}(s,z)\tilde{H}(ds,dz)\bigg), \end{aligned}$$

since

$$\mathbb{E}[M(T)|\mathcal{F}_T^{\Lambda}] = \mathbb{E}[\mathbb{E}[M(T)|\mathcal{G}_t]|\mathcal{F}_T^{\Lambda}] = \mathbb{E}[M(t)|\mathcal{F}_T^{\Lambda}].$$

Now assume that $0 \le t_1 < t_2$. Then

$$\begin{aligned} M(t_1) &= \mathbb{E}[M(t_2)|\mathcal{G}_{t_1}] \\ &= \mathbb{E}[M(T)|\mathcal{F}_T^{\Lambda}] + \mathbb{E}\bigg[\int_0^{t_2} {\theta_0}^{(t_2)}(s)dW(s) + \int_0^{t_2}\int_{\mathbb{R}_0} {\theta_1}^{(t_2)}(s,z)\tilde{H}(ds,dz)\bigg|\mathcal{G}_{t_1}\bigg] \\ &= \mathbb{E}[M(T)|\mathcal{F}_T^{\Lambda}] + \int_0^{t_1} {\theta_0}^{(t_2)}(s)dW(s) + \int_0^{t_1}\int_{\mathbb{R}_0} {\theta_1}^{(t_2)}(s,z)\tilde{H}(ds,dz). \end{aligned} \tag{B.8}$$

But we also have

$$M(t_1) = \mathbb{E}[M(T)|\mathcal{F}_T^{\Lambda}] + \int_0^{t_1} {\theta_0}^{(t_1)}(s)dW(s) + \int_0^{t_1}\int_{\mathbb{R}_0} {\theta_1}^{(t_1)}(s,z)\tilde{H}(ds,dz). \tag{B.9}$$

Hence, comparing the equations (B.8) and (B.9), we get that

$$\begin{aligned} 0 &= \mathbb{E}\bigg[\bigg(\int_0^{t_1} {\theta_0}^{(t_1)}(s) - {\theta_0}^{(t_2)}(s)dW(s) \\ &\quad + \int_0^{t_1}\int_{\mathbb{R}_0} {\theta_1}^{(t_1)}(s,z) - {\theta_1}^{(t_2)}(s,z)\tilde{H}(ds,dz)\bigg)^2\bigg] \\ &= \mathbb{E}\bigg[\int_0^{t_1} ({\theta_0}^{(t_1)}(s) - {\theta_0}^{(t_2)}(s))^2 ds + \int_0^{t_1}\int_{\mathbb{R}_0} ({\theta_1}^{(t_1)}(s,z) - {\theta_1}^{(t_2)}(s,z))^2 \Lambda(ds,dz)\bigg], \end{aligned}$$

and therefore

$$\begin{aligned} &{\theta_0}^{(t_1)}(\omega,t) = {\theta_0}^{(t_2)}(\omega,t) \quad \text{for a.a. } (\omega,t) \in \Omega \times [0,t_1], \\ &{\theta_1}^{(t_1)}(\omega,t,z) = {\theta_1}^{(t_2)}(\omega,t,z) \quad \text{for a.a. } (\omega,t,z) \in \Omega \times [0,t_1] \times \mathbb{R}_0. \end{aligned}$$

Thus, define

$$\begin{aligned}
&\theta_0(\omega,t) = {\theta_0}^{(T)}(\omega,t) \quad \text{for a.a. } (\omega,t) \in \Omega \times [0,T],\\
&\theta_1(\omega,t,z) = {\theta_1}^{(T)}(\omega,t,z) \quad \text{for a.a. } (\omega,t,z) \in \Omega \times [0,T] \times \mathbb{R}_0,
\end{aligned}$$

and then we get for all $t \in [0,T]$

$$\begin{aligned}
M(t) &= \mathbb{E}[M(T)|\mathcal{F}_T^\Lambda] + \int_0^t {\theta_0}^{(T)}(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} {\theta_1}^{(T)}(s,z)\tilde{H}(ds,dz)\\
&= \mathbb{E}[M(T)|\mathcal{F}_T^\Lambda] + \int_0^t \theta_0(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \theta_1(s,z)\tilde{H}(ds,dz). \qquad \blacksquare
\end{aligned}$$

C BSDEs: existence and uniqueness

This theory is taken from [8], with minor changes due to different use of stochastic integrator. In [8] the stochastic integrator is $\mu(dt,dz) = \delta_{\{0\}}(z)dB(t) + \mathbb{1}_{\mathbb{R}_0}\tilde{H}(dt,dz)$, while in this thesis the stochastic integrators are divided into $dW(t)$ and $\tilde{H}(dt,dz)$, separately. This change affects only some coefficients in the calculations.

We adopt the setup from Section 2.3, and answer questions of existence and uniqueness of the solution of the backward stochastic differential equation

$$\begin{aligned} Y(t) = \xi + \int_t^T g(s,\lambda(s),Y(s),Z(s),U(s,\cdot))ds - \int_t^T Z(s)dW(s) \\ - \int_t^T \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz), \quad t \in [0,T]. \end{aligned} \tag{C.1}$$

So, given terminal condition ξ and generator g, a solution is given by the $\mathbb{G}-$adapted processes (Y,Z,U) on $(\Omega,\mathcal{F}_T,\mathbb{P})$ satisfying (C.1).

Recall Definition 2.7 of the space $\mathcal{I}_{\mathbb{G}}$, Definition 2.10 of the space $\mathbb{S}_2$, and the definition of standard parameters, i.e.

Definition C.1 *The couple (ξ,g) are standard parameters, with respect to $\mathbb{G}$, for a BSDE on $(\Omega,\mathcal{F}_T,\mathbb{P})$ if $\xi \in L_2(\Omega,\mathcal{F}_T,\mathbb{P})$ and $g : \Omega\times[0,T]\times[0,\infty)\times\mathbb{R}^2\times\mathcal{R} \to \mathbb{R}$ satisfies for some $K_g > 0$:*

(i) $g(\cdot,\cdot,\lambda,Y,Z,U(\cdot))$ is $\mathbb{G}$-adapted for all $\lambda \in \mathcal{L}$, $Y \in \mathbb{S}_2(\mathbb{P})$, $(Z,U) \in \mathcal{I}$;

(ii) For all $\lambda \in \mathcal{L}$ we have $g(\cdot,\cdot,\lambda(\cdot),0,0,0)$ $\mathbb{G}$-predictable and

$$\mathbb{E}\left[\int_0^T g^2(t,\lambda(t),0,0,0)dt\right] < \infty;$$

(iii) For all $\lambda \in [0,\infty), y_1,y_2,z_1,z_2 \in \mathbb{R}$ and $u_1,u_2 \in \mathcal{R}$ we have

$$\begin{aligned} &|g(t,\lambda,y_1,z_1,u_1) - g(t,\lambda,y_2,z_2,u_2)| \leq \\ &\quad K_g\left(|y_1-y_2|+|z_1-z_2|+\sqrt{\int_{\mathbb{R}_0}|u_1(x)-u_2(x)|^2\nu(dx)\lambda}\right). \end{aligned}$$

Moreover, recall the fundamental inequality $(a_1+a_2+...+a_n)^2 \leq n(a_1^2+a_2^2+...+a_n^2)$ for any $n \in \mathbb{N}$ and $a_1,a_2,...,a_n \in \mathbb{R}$.

Lemma C.2 *Consider (Y,Z,U), $(Y',Z',U') \in \mathbb{S}_2 \times \mathcal{I}_{\mathbb{G}}$. Let $g : \Omega\times[0,T]\times[0,\infty)\times\mathbb{R}^2\times\mathcal{R}$ satisfy (ii) and (iii) in Definition C.1. Then, for any $t \in [0,T]$,* *we have*

$$\mathbb{E}\bigg[\bigg(\int_t^T g(s,\lambda(s),Y(s),Z(s),U(s,\cdot)) - g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))ds\bigg)^2\bigg]$$
$$\leq 3K_g^2(T-t)\mathbb{E}\bigg[(T-t)\sup_{t\leq r\leq T}|Y(r)-Y'(r)|^2 + \int_t^T |Z(s)-Z'(s)|^2 ds$$
$$+ \int_t^T \int_{\mathbb{R}_0} |U(s,z)-U'(s,z)|^2 \Lambda(ds,dz)\bigg] \tag{C.2}$$

and

$$\mathbb{E}\bigg[\bigg(\int_t^T |g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))|ds\bigg)^2\bigg]$$
$$\leq (T-t)\mathbb{E}\bigg[2\int_t^T |g(s,\lambda(s),0,0,0)|ds$$
$$+ 6K_g^2\bigg((T-t)\sup_{t\leq r\leq T}|Y'(r)|^2 + \int_t^T |Z'(s)|^2 ds + \int_t^T \int_{\mathbb{R}_0} |U'(s,z)|^2 \Lambda(ds,dz)\bigg)\bigg] \tag{C.3}$$

Proof: Fix $t \in [0,T]$. From the Lipschitz conditions in Definition C.1 (iii), we have:

$$\mathbb{E}\bigg[\bigg(\int_t^T g(s,\lambda(s),Y(s),Z(s),U(s,\cdot)) - g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))ds\bigg)^2\bigg]$$
$$\leq K_g^2\mathbb{E}\bigg[\bigg(\int_t^T |Y(s)-Y'(s)| + |Z(s)-Z'(s)|$$
$$+ \sqrt{\int_{\mathbb{R}_0} |U(s,z)-U'(s,z)|^2 \nu(dz)\lambda(s)ds}\bigg)^2\bigg]$$
$$\leq 3K_g^2(T-t)\mathbb{E}\bigg[\bigg(\int_t^T |Y(s)-Y'(s)|^2 + |Z(s)-Z'(s)|^2 ds$$
$$+ \int_t^T \int_{\mathbb{R}_0} |U(s,z)-U'(s,z)|^2 \Lambda(ds,dz)\bigg)^2\bigg]$$
$$\leq 3K_g^2(T-t)\mathbb{E}\bigg[(T-t)\sup_{t\leq r\leq T}|Y(r)-Y'(r)|^2 + \int_t^T |Z(s)-Z'(s)|^2 ds$$
$$+ \int_t^T \int_{\mathbb{R}_0} |U(s,z)-U'(s,z)|^2 \Lambda(ds,dz)\bigg)^2\bigg].$$

Moreover, from (ii) in Definition C.1, we can prove the second inequality:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^T |g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))|ds\Big)^2\Big] \\
&\qquad \le (T-t)\mathbb{E}\Big[\int_t^T |g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))|^2 ds\Big] \\
&\qquad \le (T-t)\mathbb{E}\Big[\int_t^T \Big(|g(s,\lambda(s),0,0,0)| \\
&\qquad\qquad + |g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot)) - g(s,\lambda(s),0,0,0)|\Big)^2 ds\Big] \\
&\qquad \le 2(T-t)\mathbb{E}\Big[\int_t^T |g(s,\lambda(s),0,0,0)|^2 \\
&\qquad\qquad + |g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot)) - g(s,\lambda(s),0,0,0)|^2 ds\Big].
\end{aligned}$$

The result follows from (C.2) by proceeding as in the proof of (C.3). ■

Lemma C.3 *Consider $Y' \in \mathbb{S}_2$, $(Z,U),(Z',U') \in \mathcal{I}_{\mathbb{G}}$ and let (ξ, g) be standard parameters. Define a stochastic process $Y(t)$, $t \in [0,T]$, by*

$$\begin{aligned}
Y(t) = \xi &+ \int_t^T g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))ds - \int_t^T Z(s)dW(s) \\
&- \int_t^T \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz).
\end{aligned} \tag{C.4}$$

Then $Y \in \mathbb{S}_2$. In particular, we have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\le r\le T}|Y(r)|^2\Big] \le \mathbb{E}\Big[&4\xi^2 + 4\Big(\int_t^T |g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))|ds\Big)^2 \\
&+ 40\Big(\int_t^T |Z(s)|^2 ds + \int_t^T \int_{\mathbb{R}_0} |U(s,z)|^2 \Lambda(ds,dz)\Big)\Big].
\end{aligned} \tag{C.5}$$

Proof: Directly from (C.4), taking the square, we have

$$\begin{aligned}
|Y(t)|^2 \le 4\xi^2 &+ 4\Big(\int_t^T |g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))|ds\Big)^2 \\
&+ 4\Big(\int_t^T Z(s)dW(s)\Big)^2 + 4\Big(\int_t^T \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz)\Big)^2.
\end{aligned}$$

Using this representation for $Y(t)$, we see that

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\le r\le T}|Y(r)|^2\Big] \le \mathbb{E}\Big[4\xi^2 + 4\Big(\int_t^T |g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))|ds\Big)^2\Big] \\ +\mathbb{E}\Big[\sup_{t\le r\le T} 4\Big(\int_r^T Z(s)dW(s)\Big)^2 \\ + \sup_{t\le r\le T} 4\Big(\int_r^T\int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz)\Big)^2\Big].\end{aligned}$$

We have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\le r\le T}\Big(\int_r^T Z(s)dW(s)\Big)^2\Big] \\ &= \mathbb{E}\Big[\sup_{t\le r\le T}\Big(\int_t^T Z(s)dW(s) - \int_t^r Z(s)dW(s)\Big)^2\Big] \\ &\le \mathbb{E}\Big[2\Big(\int_t^T Z(s)dW(s)\Big)^2 + 2\sup_{t\le r\le T}\Big(\int_t^r Z(s)dW(s)\Big)^2\Big] \\ &\le 10\mathbb{E}\Big[\int_t^T |Z(s)|^2 ds\Big],\end{aligned}$$

where the last inequality follows from Doob's martingale inequality, i.e.
for a continuous martingale $M(t)$, with $p>1,\ T>0$, and $\mathbb{E}[|M(T)|^p]<\infty$, we have

$$\mathbb{E}[\sup_{0\le t\le T}|M(t)|^p] \le \Big(\frac{p}{p-1}\Big)^p \mathbb{E}[|M(T)|^p].$$

The same calculation holds for

$$\mathbb{E}\Big[\sup_{t\le r\le T}\Big(\int_r^T\int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz)\Big)^2\Big],$$

with the obvious modifications. Equation (C.5) follows, and we conclude that $Y\in\mathbb{S}_2$ by (C.3) since $Y'\in\mathbb{S}_2$, $(Z,U),(Z',U')\in\mathcal{I}_{\mathbb{G}}$ and (ξ,g) are standard parameters. ■

Now let (ξ,g) be standard parameters, and define the mapping

$$\Theta:\mathbb{S}_2\times\mathcal{I}_{\mathbb{G}}\to\mathbb{S}_2\times\mathcal{I}_{\mathbb{G}},\quad \Theta(Y',Z',U') := (Y,Z,U) \tag{C.6}$$

as follows. Let (Z,U) be the unique element in $\mathcal{I}_{\mathbb{G}}$ that provides the stochastic integral representation

$$M(t) = M(0) + \int_0^t Z(s)dW(s) + \int_0^t\int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz),\quad t\in[0,T],$$

of the martingale

$$M(t) = \mathbb{E}\Big[\xi + \int_0^T g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))ds\Big|\mathcal{G}_t\Big], \quad t \in [0,T].$$

The component Y in (C.6) is defined by

$$Y(t)) = \mathbb{E}\Big[\xi + \int_t^T g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))ds\Big|\mathcal{G}_t\Big], \quad t \in [0,T].$$

Note that

$$\begin{aligned}Y(t) &= M(t) - \int_0^t g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))ds \\ &= M(0) + \int_0^t Z(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz) \\ &\quad - \int_0^t g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))ds.\end{aligned}$$

Since $Y(T) = \xi$, we also have $Y(t) = \xi - Y(T) + Y(t)$ so that

$$\begin{aligned}Y(t) &= \xi + \int_t^T g(s,\lambda(s),Y'(s),Z'(s),U'(s,\cdot))ds \\ &\quad - \int_t^T Z(s)dW(s) - \int_0^t \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz).\end{aligned} \tag{C.7}$$

Thus, $Y \in \mathbb{S}_2$ by Lemma C.3 and the mapping (C.6) is well-defined.

We use the mapping Θ to prove that the BSDE of type (C.1) admits a unique solution for the given standard parameters (ξ, g).

Lemma C.4 *Consider $(Y_1', Z_1', U_1'), (Y_2', Z_2', U_2') \in \mathbb{S}_2 \times \mathcal{I}_{\mathbb{G}}$ and define $(Y_1, Z_1, U_1) = \Theta(Y_1', Z_1', U_1')$ and $(Y_2, Z_2, U_2) = \Theta(Y_2', Z_2', U_2')$. Set $\bar{Y}' = Y_1' - Y_2'$, $\bar{Z}' = Z_1' - Z_2'$, $\bar{U}' = U_1' - U_2'$, and similar for Y_i, Z_i, U_i, $i = 1,2$. There exists a $K > 0$ such that*

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\le r\le T}|\bar{Y}(r)|^2 + \int_t^T |\bar{Z}(s)|^2 ds + \int_t^T \int_{\mathbb{R}_0} |\bar{U}(s,z)|^2 \Lambda(ds,dz)\Big] \\ &\quad \le K(T-t)\mathbb{E}\Big[(T-t)\sup_{t\le r\le T}|\bar{Y}'(r)|^2 + \int_t^T |\bar{Z}'(s)|^2 ds \\ &\qquad + \int_t^T \int_{\mathbb{R}_0} |\bar{U}'(s,z)|^2 \Lambda(ds,dz)\Big], \quad t \in [0,T].\end{aligned} \tag{C.8}$$

Proof: By the representation of Y in (C.7) we have, for any $t \in [0,T]$, that

$$\bar{Y}(t) = \int_t^T g(s,\lambda(s),Y_1'(s),Z_1'(s),U_1'(s,\cdot)) - g(s,\lambda(s),Y_2'(s),Z_2'(s),U_2'(s,\cdot))ds \\ - \int_t^T \bar{Z}(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \bar{U}(s,z)\tilde{H}(ds,dz).$$

Since

$$\mathbb{E}\Big[\bar{Y}(t)\int_t^T \bar{Z}(s)dW(s)\Big] = \mathbb{E}\Big[\bar{Y}(t)\mathbb{E}\Big[\int_t^T \bar{Z}(s)dW(s)\Big|\mathcal{G}_t\Big]\Big] = 0,$$

and

$$\mathbb{E}\Big[\bar{Y}(t)\int_t^T \int_{\mathbb{R}_0} \bar{U}(s,z)\tilde{H}(ds,dz)\Big] = \mathbb{E}\Big[\bar{Y}(t)\mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} \bar{U}(s,z)\tilde{H}(ds,dz)\Big|\mathcal{G}_t\Big]\Big] = 0,$$

and that W and $\tilde{H}$ are conditionally orthogonal, we have

$$\mathbb{E}\Big[\Big(\bar{Y}(t) + \int_t^T \bar{Z}(s)dW(s) + \int_t^T \int_{\mathbb{R}_0} \bar{U}(s,z)\tilde{H}(ds,dz)\Big)^2\Big] \\ = \mathbb{E}\Big[|\bar{Y}(t)|^2 + \int_t^T |\bar{Z}(s)|^2 ds + \int_t^T \int_{\mathbb{R}_0} |\bar{U}(s,z)|^2 \Lambda(ds,dz)\Big] \\ = \mathbb{E}\Big[\Big(\int_t^T g(s,\lambda(s),Y_1'(s),Z_1'(s),U_1'(s,\cdot)) - g(s,\lambda(s),Y_2'(s),Z_2'(s),U_2'(s,\cdot))ds\Big)^2\Big]$$

We apply (C.2) and obtain

$$\mathbb{E}\Big[\int_t^T |\bar{Z}(s)|^2 ds + \int_t^T \int_{\mathbb{R}_0} |\bar{U}(s,z)|^2 \Lambda(ds,dz)\Big] \\ \leq \mathbb{E}\Big[|\bar{Y}(t)|^2 + \int_t^T |\bar{Z}(s)|^2 ds + \int_t^T \int_{\mathbb{R}_0} |\bar{U}(s,z)|^2 \Lambda(ds,dz)\Big] \\ \leq 3K_g^2(T-t)\mathbb{E}\Big[(T-t)\sup_{t\leq r\leq T}|\bar{Y}'(r)|^2 + \int_t^T |\bar{Z}'(s)|^2 ds \\ + \int_t^T \int_{\mathbb{R}_0} |\bar{U}'(s,z)|^2 \Lambda(ds,dz)\Big]. \tag{C.9}$$

By (C.2), (C.5) and (C.9) we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\le r\le T}|\bar{Y}(r)|^2\Big]\\
&\le \mathbb{E}\Big[0+4\Big(\int_t^T |g(s,\lambda(s),Y_1'(s),Z_1'(s),U_1'(s,\cdot))\\
&\quad - g(s,\lambda(s),Y_2'(s),Z_2'(s),U_2'(s,\cdot))|ds\Big)^2\\
&\quad + 40\Big(\int_t^T |\bar{Z}(s)|^2 ds + \int_t^T\int_{\mathbb{R}_0} |\bar{U}(s,z)|^2\Lambda(ds,dz)\Big)\Big]\\
&\le 12K_g^2(T-t)\mathbb{E}\Big[(T-t)\sup_{t\le r\le T}|\bar{Y}'(r)|^2 + \int_t^T |\bar{Z}'(s)|^2 ds\\
&\quad + \int_t^T\int_{\mathbb{R}_0}|\bar{U}'(s,z)|^2\Lambda(ds,dz)\Big]\\
&\quad + 120K_g^2(T-t)\mathbb{E}\Big[(T-t)\sup_{t\le r\le T}|\bar{Y}'(r)|^2 + \int_t^T |\bar{Z}'(s)|^2 ds\\
&\quad + \int_t^T\int_{\mathbb{R}_0}|\bar{U}'(s,z)|^2\Lambda(ds,dz)\Big]\\
&= (12+120)K_g^2(T-t)^2\mathbb{E}\Big[\sup_{t\le r\le T}|\bar{Y}'(r)|^2\Big]\\
&\quad + (12+120)K_g^2(T-t)\mathbb{E}\Big[\int_t^T |\bar{Z}'(s)|^2 ds + \int_t^T\int_{\mathbb{R}_0}|\bar{U}'(s,z)|^2\Lambda(ds,dz)\Big].
\end{aligned} \tag{C.10}$$

Combining (C.9) and (C.10) gives (C.8). ■

Theorem C.5 *Let (ξ, g) be standard parameters. Then there exists a uniqiue solution $(Y, Z, U) \in \mathbb{S}_2 \times \mathcal{I}_{\mathbb{G}}$ such that*

$$\begin{aligned}
Y(t) = \xi &+ \int_t^T g(s,\lambda(s),Y(s),Z(s),U(s,\cdot))ds - \int_t^T Z(s)dW(s)\\
&- \int_t^T\int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz).
\end{aligned} \tag{C.11}$$

Proof: In this proof we apply the Banach fixed point theorem, that states that every contraction mapping on a non-empty complete metric space (a Hilbert space holds this property) has a unique fixed point.

Choose $t_1 \in [0,T)$ such that $\max\{K(T-t_1)^2, K(T-t_1)\} < 1$, where K is the same as in Lemma C.4. Denote $\mathbb{S}_2(u,v)$ as the space consisting of the elements of $\mathbb{S}_2$ equipped with the norm $\|Y\|^2_{\mathbb{S}_2(u,v)} = \mathbb{E}[\sup_{u\le r\le v}|Y(r)|^2]$, and $\mathcal{I}(u,v)$ as the space of elements of $\mathcal{I}_{\mathbb{G}}$ equipped with the norm $\|\theta_0\|^2_{\mathcal{I}(u,v)} = \mathbb{E}[\int_u^v |Z(s)|^2 ds + \int_u^v \int_{\mathbb{R}_0} |U(s,z)|^2 \Lambda(ds,dz)]$. From (C.8), Θ is a contraction on $\mathbb{S}_2(t_1,T) \times \mathcal{I}(t_1,T)$, and thus there exists a unique $(Y_1, Z_1, U_1) \in \mathbb{S}_2(t_1,T) \times \mathcal{I}(t_1,T)$ such that $\Theta(Y_1,Z_1,U_1) = (Y_1,Z_1,U_1)$ on $[t_1,T]$, i.e.

$$\begin{aligned} Y_1(t) = \xi + \int_t^T g(s,\lambda(s),Y_1(s),Z_1(s),U_1(s,\cdot))ds - \int_t^T Z_1(s)dW(s) \\ - \int_t^T \int_{\mathbb{R}_0} U_1(s,z)\tilde{H}(ds,dz), \quad t \in [t_1,T]. \end{aligned}$$

Now consider a modification of Θ, the mapping $\tilde{\Theta}(Y',Z',U') = (Y,Z,U)$, $(Y',Z',U') \in \mathbb{S}_2(t_2,t_1) \times \mathcal{I}(t_2,t_1)$, with standard parameters $(Y_1(t_1), g)$, which are defined as follows. Here t_2 is such that $\max\{K(t_1-t_2)^2, K(t_1-t_2)\} < 1$. The components (Z,U) are determined by the unique element in $\mathcal{I}(t_2,t_1)$ from the martingale representation (2.5) of the $\mathbb{G}-$martingale

$$M(t) = \mathbb{E}\left[Y_1(t_1) + \int_0^{t_1} g(s,\lambda(s),Y'(s),Z'(s),U'(s,z))ds \middle| \mathcal{G}_t\right].$$

By the martingale representation theorem $M(t)$ is uniquely described by

$$\begin{aligned} M(t) &= \mathbb{E}[M(t_1)|\mathcal{F}_T^\Lambda] + \int_0^t Z(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz) \\ &= \mathbb{E}[M(t_1)|\mathcal{F}_T^\Lambda] + \int_0^{t_2} Z(s)dW(s) + \int_{t_2}^t Z(s)dW(s) \\ &\quad + \int_0^{t_2} \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz) + \int_{t_2}^t \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz) \\ &= M(t_2) + \int_{t_2}^t Z(s)dW(s) + \int_{t_2}^t \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz). \end{aligned}$$

The component Y is obtained in a similar manner as for Θ, i.e.

$$Y(t) = \mathbb{E}\left[Y_1(t_1) + \int_t^{t_1} g(s,\lambda(s),Y'(s),Z'(s),U'(s,z))ds \middle| \mathcal{G}_t\right], \quad t \in [t_2,t_1].$$

For argumentation for this, look at the derivation of Θ above.

Following the same argumentation as above, we conclude that Θ is a contraction on $\mathbb{S}_2(t_2,t_1) \times \mathcal{I}(t_2,t_1)$ so that there exists a unique element $(Y_2,Z_2,U_2) \in \mathbb{S}_2(t_2,t_1) \times \mathcal{I}(t_2,t_1)$ such that $\tilde{\Theta}(Y_2,Z_2,U_2) = (Y_2,Z_2,U_2)$. Then we have

$$\begin{aligned} Y_2(t) = Y_1(t_1) + \int_t^T g(s,\lambda(s),Y_2(s),Z_2(s),U_2(s,z))ds - \int_t^T Z_2(s)dW(s) \\ - \int_t^T \int_{\mathbb{R}_0} U_2(s,z)\tilde{H}(ds,dz), \quad t \in [t_2,t_1]. \end{aligned}$$

Now consider

$$Y(t) = Y_1(t)\mathbb{1}_{t_1<t\le T}(t) + Y_2(t)\mathbb{1}_{t_2<t\le t_1}(t), \quad t \in [t_2, T]$$
$$Z(t) = Z_1(t)\mathbb{1}_{t_1<t\le T}(t) + Z_2(t)\mathbb{1}_{t_2<t\le t_1}(t), \quad t \in [t_2, T]$$
$$U(t,\cdot) = U_1(t,\cdot)\mathbb{1}_{t_1<t\le T}(t) + U_2(t,\cdot)\mathbb{1}_{t_2<t\le t_1}(t), \quad t \in [t_2, T].$$

We can see that

$$\begin{aligned} Y(t) = \xi + \int_t^T g(s,\lambda(s),Y(s),Z(s),U(s,z))ds - \int_t^T Z(s)dW(s) \\ - \int_t^T \int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz), \quad t \in [t_1,T]. \end{aligned} \tag{C.12}$$

In fact, clearly (C.12) holds for $t \in [t_2, T]$. Assume $t \in (t_2, t_1]$, then

$$\begin{aligned} Y(t) &= = Y_1(t_1) + \int_t^{t_1} g(s,\lambda(s),Y_2(s),Z_2(s),U_2(s,z))ds - \int_t^{t_1} Z_2(s)dW(s) \\ &\quad - \int_t^{t_1}\int_{\mathbb{R}_0} U_2(s,z)\tilde{H}(ds,dz) \\ &= \xi + \int_{t_1}^T g(s,\lambda(s),Y_1(s),Z_1(s),U_1(s,\cdot))ds - \int_{t_1}^T Z_1(s)dW(s) \\ &\quad - \int_{t_1}^T\int_{\mathbb{R}_0} U_1(s,z)\tilde{H}(ds,dz) \\ &\quad + \int_t^{t_1} g(s,\lambda(s),Y_2(s),Z_2(s),U_2(s,z))ds - \int_t^{t_1} Z_2(s)dW(s) \\ &\quad - \int_t^{t_1}\int_{\mathbb{R}_0} U_2(s,z)\tilde{H}(ds,dz) \\ &= \xi + \int_t^T g(s,\lambda(s),Y(s),Z(s),U(s,\cdot))ds - \int_t^T Z(s)dW(s) \\ &\quad - \int_t^T\int_{\mathbb{R}_0} U(s,z)\tilde{H}(ds,dz). \end{aligned}$$

Hence, $(Y,Z,U) \in \mathbb{S}_2(t_2,T) \times \mathcal{I}(t_2,T)$. Proceed iteratively. Eventually, there is a step n such that $\max\{K(t_n - t_{n+1})^2, K(t_n - t_{n+1})\} < 1$ for $t_{n+1} = 0$. Then we conclude and have found a unique solution $(Y,Z,U) \in \mathbb{S}_2(0,T) \times \mathcal{I}(0,T) = \mathbb{S}_2 \times \mathcal{I}_{\mathbb{G}}$ such that (C.11) holds. ■

D Local martingales and quadratic variation

The theory in this section is from the book by Philip Protter [11]. Much of the theory in [11] is formulated for semimartingales, but from the corollary of Theorem 26.III in [11], we know that every local martingale is a semimartingale.

The process X is locally integrable if there exist a sequence of stopping times $(T_n)_{n\geq 1}$ increasing to ∞ a.s. such that $\mathbb{E}[|X(T_n)\mathbb{1}_{T_n>0}|] < \infty$ for each n.

A family of random variables $(U_\alpha)_{\alpha\in A}$ is uniformly integrable if

$$\lim_{n\to\infty} \sup_{\alpha} \int_{\{|U_\alpha|\geq n\}} |U_\alpha| dP = 0.$$

Definition D.1 (Local martingale) *An adapted, càdlàg process X is a local martingale if there exists a sequence of increasing stopping times, T_n, with $\lim_{n\to\infty} T_n = \infty$ a.s. such that $X(t \wedge T_n)\mathbb{1}_{\{T_n>0\}}$ is a uniformly integrable martingale for each n.*

Definition D.2 (Quadratic variation of a semimartingale) *Let X, Y be semimartingales. The quadratic variation process of X, denoted $[X,X] = ([X,X]_t)_{t\geq 0}$, is defined by*

$$[X,X] = X^2 - 2\int X_- dX.$$

The quadratic covariation of X, Y, also called the bracket process of X, Y, is defined by

$$[X,Y] = XY - \int X_- dY - \int Y_- dX.$$

The operation $(X,Y) \mapsto [X,Y]$ is bilinear and symmetric, hence we have the polarization identity

$$[X,Y] = \frac{1}{2}\{[X+Y, X+Y] - [X,X] - [Y,Y]\}.$$

Definition D.3 (Conditional quadratic variation of a semimartingale) *Let X, Y be semimartingales such that their quadratic variation processes, also called sharp bracket processes, are locally integrable. The conditional quadratic variation process of X, denoted $\langle X,X\rangle = (\langle X,X\rangle_t)_{t\geq 0}$, exists and is defined to be the compensator of $[X,X]$, i.e. $[X,X] - \langle X,X\rangle$ is a local martingale. The operation $(X,Y) \mapsto \langle X,Y\rangle$ is bilinear and symmetric, hence we have the polarization identity*

$$\langle X,Y\rangle = \frac{1}{2}\{\langle X+Y, X+Y\rangle - \langle X,X\rangle - \langle Y,Y\rangle\}.$$

While $[X,X]$, $[Y,Y]$ and $[X,Y]$ remain invariant with a change to an equivalent probability measure, the conditional quadratic variation in general change with a change to an equivalent probability measure and may even no longer exist.

Remark. For a continuous semimartingale M, we have that $\langle M\rangle = [M]$. $\diamond$

Remark. $[M,N] - \langle M,N\rangle$ is a local martingale. $\diamond$

Example

An easy example shoes the differences between the bracket processes $[,]$ and the sharp bracket processes $\langle , \rangle$. Let $N(t)$ be a Poisson process of intensity λ. Then

$$[N](t) = \sum_{s<t} (\Delta N(s))^2 = \sum_{s<t} \Delta N(s) = N(t), \quad \text{not predictable,}$$

while

$$\langle N \rangle(t) = \lambda t, \quad \text{predictable, deterministic even.}$$

Moreover,

$$[N](t) - \langle N \rangle(t) = N(t) - \lambda t, \quad \text{a local martingale.}$$

Theorem D.4 (Theorem 51.I.[11]) *Let X be a local martingale such that $\mathbb{E}[\sup_{s\leq t}|X(s)|] < \infty$ for every $t \geq 0$. Then X is a martingale. If $\mathbb{E}[\sup_t|X(t)|] < \infty$, then X is a uniformly integrable martingale.*

Corollary D.5 (Corollary to Theorem 27.II.[11]) *Let M be a local martingale. Then M is a martingale with $\mathbb{E}[M^2(t)] < \infty$ for all t if and only if $\mathbb{E}[[M,M](t)] < \infty$ for all t. If $\mathbb{E}[[M,M](t)] < \infty$, then $\mathbb{E}[M^2(t)] = \mathbb{E}[[M,M](t)]$.*

The following theory is from Section 3.IV.[11]:

Definition D.6 *$\mathbb{M}_2 \subset L_2$ consists of L_2-martingales M such that $\mathbb{E}[\sup_t|M(t)|] < \infty$ and $M(0) = 0$. $\mathbb{M}_2$ can be endowed with a norm*

$$\|M\| = \mathbb{E}[M^2(T)]^{1/2} = \mathbb{E}[[M,M](T)]^{1/2},$$

and it is a Hilbert space.

Definition D.7 ((Weak) orthogonality) *$N, M \in \mathbb{M}_2$ are (weakly) orthogonal if $\mathbb{E}[M(T)N(T)] = 0$.*

Definition D.8 (Strong orthogonality) *$N, M \in \mathbb{M}_2$ are strongly orthogonal if $L = MN$ is a (uniformly integrable) martingale. Or, equivalently $N, M \in \mathbb{M}_2$ are strongly orthogonal if and only if $[M, N]$ is a uniformly integrable martingale.*

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