PRICING OF SPREAD OPTIONS IN ENERGY MARKETS WITH NON-ZERO STRIKES

by

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Abstract

The main scope of this thesis is to implement a structured numerical analysis to check the exactness and applicability of the famous Kirk formula (1995) [12] and the newer Bjerksund-Stensland formula (2011) [24] widely used by energy markets practitioners while pricing and hedging (bivariate and trivariate) spread options when the strike price is different from zero.

This research found that by varying volatilities, drifts, correlations, strikes, exercise times, heating rates and initial price of emission-certificates, these two analytical approximations have limitations for pricing and hedging spread options. Notably the more recent Bjerksund-Stensland formula, which is supposed to be an improvement on the Kirk formula, is not better to provide reliable result in three-dimensional trading markets. This is important, as energy markets often are three-dimentional. It will be shown mathematically with numerical experiments that both approximations provide acceptable results for pricing bivariate spread options with respect to positive strike prices. But their performances are unsatisfactory for negative strike prices. Furthermore, neither of them performed well to price trivariate spread options. And both performed poorly in hedging trivariate spread options.

Although using a closed-form formula is very attractive for practitioners, this research proposes that it is safer to keep using the slower Monte Carlo numerical method, until future researches perfect existing closed-form formulas or discover a new one.

Frequently Used Notation

When a numbered equation/figure/table is referred to, it will be by chapter and number, e.g. equation (3.1) will be referred to the first equation in Chapter 3, Table (4.5) the fifth table in Chapter 4. Results, however, will simply be referred to as e.g. Definition 1. References are referred to by only numbers, e.g. [7] will be the seventh source from the complete list of sources in the bibliography.

Some symbols are standard for the entire thesis, most of which are listed below. Vectors and matrices are generally captured in bold font.

Ω	Sample space, a subset of $\mathbb{R}$.
$\mathcal{F}$	σ-algebra.
$\mathcal{F}_t$	Filtration with time t.
$X(t,\omega)$	(A path of) stochastic process with time t and state ω.
$M(t)$	Martingale with time t.
$\phi(\cdot)$	Probability density function of the standard normal distribution[1].
$\Phi(\cdot)$	Cumulative distribution function of the standard normal distribution.
$\mathcal{P}[\cdot]$	Probability measure.
$\mathcal{Q}[\cdot]$	Risk-neutral probability measure, or equivalent martingale measure.
$\mathrm{E}[\cdot]$	Expected value with respect to $\mathcal{P}$.
$\mathrm{E}_{\mathcal{Q}}[\cdot]$	Expected value with respect to $\mathcal{Q}$.
$\max\{\cdot\}$	The greatest element in a set.
$\{\cdot\}^+$	The positive part of a set.
$\mathbb{1}_A$	The indicator function[2] of a subset A of a set X.
Δ	The delta-hedge parameter.

[1] This is described by $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}x^2\right\}$.

[2] The indicator function is defined as $\mathbb{1}_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$.

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Chapter 1

Introduction and literature review

At the core of their trading activities, energy markets practitioners, in Europe and beyond, use extensively spread options to valuate commodities price differences over time and places. Spread options are defined as an option written on the price difference between two commodities. In this thesis we will focus on call options, with a strike price K, having a payoff function at exercise time T given as

$$\max\{P(T) - hG(T) - K, 0\} \tag{1.1}$$

where $P(T)$ and $G(T)$ are the prices of two energy commodities at the time of exercise T. Typically in a given energy market, P may be the price of electricity and G the price of gas, and h the heating rate, converting gas into electricity. The producer's income is a European call option, depending on the differnce $P(T) - hG(T)$, or, on the *spread* between electricity and gas. Such a spread option provides the value of operating a gas-fired power plant, with fixed operation cost of K.

There have been numerous articles on the topic of pricing such spread options when the strike price is zero, but very few have claimed to find an approximation that can apply to a non-zero strike price for all circumstances. In general, there is no satisfactory analytical formula for the price of a spread call option when $K \neq 0$. If $K = 0$ one can price it by using the famous Margrabe formula [34] (see also Theorem 13 in Chapter 2). And there are two, supposedly very efficient, approximations formulas suggested by Kirk [12], and Bjerksund-Stensland [24]. Both are variants on the Margrabe formula.

In recent years, power plants emitting CO_2 must also pay for their emissions. Articles in the media, such as Bloomberg on May 2nd, 2014 [38], have pointed out that for the first time, the amount (throughout April 2014) of carbon dioxide in the atmosphere averaged more than 400 parts per million, a highly symbolic threshold. EU power plants have increasingly been obliged to purchase additional emission permits in the market. This means that an additional cost of carbon emission has been introduced into the operation of a coal or gas power plant, and we can view this as a trivariate spread option

$$\max\{P(T) - hG(T) - C(T), 0\} \tag{1.2}$$

with $C(T)$ being the price of a certificate to emit a certain amount of CO_2 at the time of exercise T. Hence, we end up with a spread option on three assets, which can be viewed as a spread between power (electricity) and total production cost, being the sum of gas and emission price.

The main purpose of this thesis is to price such spread options using the famous Kirk approximation (1995) and the newer Bjerksund-Stensland approximation (2011), and to implement a structured numerical analysis to check the exactness and applicability of them in pricing and hedging (bivariate and trivariate) spread options in energy market when the strike price differs from zero.

This research found that by varying volatilities, drifts, correlations, strikes, exercise times, heating rates and initial price of emission-certificate, these two analytical approximations have shortcomings for being able to price and hedge trivariate spread options. Notably, this present research found that the more recent Bjerksund-Stensland formula, which is supposed to be an improvement on the Kirk formula, is not better to provide reliable result in three dimensional trading markets. This is important, because energy markets often need to use three-dimentional calculations. Although the intention to discover a closed-form formula can be very attractive, the project proposes that it would be better, for the time being, for the practitioners to keep using well proven numerical methods, such as the Monte Carlo method, even if its disadvantage is to be computationally slow.

It will be shown mathematically with numerical experiments that those two approximations provide acceptable results for pricing bivariate spread options with respect to positive strike prices. But their performances are unsatisfactory for negative strike prices. Furthermore, neither of them performed well to price trivariate spread options. And both performed poorly in hedging trivariate spread options. Therefore, this research suggests that practitioners may decide to focus on numerical methods until other better closed-form formulas are attested.

1.1 Energy market

An Energy Market (EM) is a market where the commodities being sold and bought are energy sources, such as electricity or natural gas. An EM involves both physical and financial elements. The physical market contains natural resources, infrastructure, institutions and market participants involved in producing energy and delivering it to consumers. For its part, the financial market includes the trading of financial products derived from physically traded commodities, and they are used for price hedging and risk management. Financial products do not involve the delivery of energy, since retail consumers usually have few options for storing it. Instead, financial markets trade paper and money. As for the end-users of energy, limited by individual storage capacity, they cannot adapt purchases with price fluctuations.

History. From the 19th century, until the second part of the 20th century, a few petroleum companies [13] had tight control on most aspects of the energy market. With physical control of much of the world's known reserves, extraction, transportation and trade, they dominated the energy market and therefore were often described as a cartel with monopolistic characteristics. From the middle of the 20th century onward, newly established sovereign states, holding vast energy reserves, joined forces to counter balance in the energy market, the overwhelming power of the petrol companies. This led to the creation of the Organization of the Petroleum Exporting Countries (OPEC), which until today plays an important role in the supply side of the energy market. The new reality of OPEC had been painfully demonstrated to leading economies during the 1973 oil crisis. Throughout the 20th century, the energy market has been steadily modernized and liberalized but also regulated. In different parts of the world, national and regional authorities have sought to protect consumers' rights, and have moved to curb oligopolies. Those institutions include the Australian Energy Market Commission, the Energy Market Authority in Singapore and the Energy Community in Europe. Member states of the European Union (EU) are required to liberalize their energy markets [21], like the Nord Pool Spot for Nordic countries and the European Energy Exchange AG (EEX) in Germany. Since 2003, energy markets have been investigated, as sharp increase in oil price was thought to be linked to extreme speculation. In 2008, it was the turn of the petroleum importing nations to organize their own conferences to voice their concerns about the impact of the energy market on their own economies [41].

More about Nord Pool Spot: The Nordic electricity exchange Nord Pool Spot runs the largest electrical energy market in the world [36], offering both day-ahead and intraday markets to its participants. Nord Pool Spot covers Norway, Finland, Sweden, Denmark, Estonia and Lithuania. Nord Pool Spot is an exchange primarily servicing stakeholders (producers, retailers, end-users, traders and brokers) at the wholesale market for electricity. Some 330 companies from 20 countries trade on the exchange [42]. At times some areas have surplus of power while others run deficit. Should there be insufficient transmission capacity on the grid between two areas, bottlenecks occur and price differences arise. Due to the bottlenecks, the Nord Pool Spot exchange area is divided into a number of bidding areas [42]. The Nordic transmission system operators (TSO's) decide the number of bidding areas within its boundaries. Eastern Denmark and Western Denmark are always treated as two different bidding areas. Sweden has been divided into four bidding areas since 2011. Norway currently (December 2013) has five bidding areas, while Finland, Estonia and Lithuania are treated as three different bidding areas since June 2013. Two Nordic commercial players situated in different bidding areas cannot trade elctricity with each other, because Nord Pool Spot handles all the trading capacity on the cross-border links, on behalf of the Nordic TSO's. As the power markets gradually are becoming more integrated, Nord Pool Spot also keeps close relations with other exchanges in Europe. Today, market coupling exists between the Nordic, German and Central Western European exchange areas. For example, new cables connecting the Nord Pool Spot to the German EEX market are planned to be built.

Energy Union. Energy supply and demand may depend highly on political and economic stability. Given the current crisis with Russia regarding Ukraine, Poland's Prime Minister Donald Tusk has recently pressed his EU partners to envision the establishment of an energy union as a mean of reducing European EM's dependency on Russian gas (read more in [39] and [43]). PM Tusk said that it was crucial to find a way that would take three important European interests into account: energy independence, reasonable energy prices, and climate challenge facing Europe. German Chancellor Angela Merkel expressed her support during discussions with PM Tusk in Berlin April 24th, 2014, but added that "more details have to be elaborated"[45].

Green energy. Increasingly renewable and sustainable energy are playing a greater role in the energy market. Green energy has benefited from alarming forecasts about global warming, technological innovation, government subsidies, prices, tax incentives, and at times the perceived potential for profits by investors. A report from `EnergyUnion.eu` predicts that by the year 2030, green energy could provide 35% of the world's energy needs [37], given the political will to promote its large scale deployment in all sectors at a global level. Future of renewable energy development will strongly depend on political choices by national governments, EU decision making processes and international agreements. But it should also be noted that until now, governments' energy policies and subsidies have been necessary for green energy to be competitive in the energy market.

Weather. It is worth mentioning that weather is a significant factor affecting energy demand and causing seasonal fluctuations of energy prices. Cold weather and short days drive winter demand in northern regions, causing the occurrence of seasonal load during the long period of darkness. Meanwhile, the heat waves in southern regions typically contribute to peak consumption of air conditioners. Unexpected changes in the weather can also have extreme short-term effects on energy usage. Raising the energy demand, even a single day, can dramatically affect energy prices. The difficulty to predict consumers demand appropriately is one of the major concerns for energy suppliers.

1.2 Financial derivatives in energy market

As aforementioned, financial markets differ from physical markets in that no physical storage or delivery of energy occurs. Financial traders may use longer-term contracts, in a way which requires (or provides) no physical delivery, but a financial payout instead. Indeed, in addition to trading physical electricity and natural gas, there is a significant market for electricity derivatives and gas derivatives. Derivatives are financial instruments whose values are derived from some physical or financial fundamentals, known as the *underlying assets* specified by the contract, and in this case, the price of electricity and the price of natural gas. Traditionally, most derivatives are traded over-the-counter (OTC) or on off-exchange markets.

Description of some key trading mechanisms and concepts:

Short selling is the selling of contracts a trader does not own at present, on the assumption that the trader will buy offsetting contracts prior to the contracts' expiration. Short selling is one of the ways market participants can trade future financially - they sell the future, and buy it before the contract expires so the contracts net out and the trader faces no delivery obligation. This can be done on an exchange or or other markets that allows for bidirectional trading.

A **position** is the net holdings of a participant in a market. A trader's position in a specific instrument is combined purchases and sales of that contract. A trader's overall position is the combination of all positions in all contracts the trader owns.

Time: Each contract has a number of time elements. The trade date is the date on which the contract is written. The expiration day is the last day for a contract, after which it is no longer available to be bought or sold; it is often the same day as the settlement day.

Price: The price paid for a contract set by the market which is usually known at the time the contract is bought or sold.

1.3 Option contracts

An option contract conveys a right (but not an obligation) to buy or sell something. It comes in two forms: the right to buy or the right to sell at a pre-set price at or before a specified date. The buyer buys the right - the option - to buy or sell in the future. The seller (or writer) sells the obligation to buy or sell if the buyer exercises his or her right, i.e. the option. In terms of terminology in mathematical finance, the price of an option is the discounted expected value of the payoff of the option at its exercise time, where the expectation is taken under the risk-neutral probability measure, cf. Section 3.2.

The price paid to buy the option is simply known as the option's *price*. An option is a *call option* if it gives the buyer the right to buy, while it is called a *put option* if it gives the buyer the right to sell. Deciding to buy or sell the underlying commodity is known as *exercising* the option. The price at which the option may be exercised is called the *strike price* (or just *strike*). At the time of exercise, traders may either exercise their call (put) options if the actual price is higher (lower) than the strike price, or not exercise their call (put) options if the actual price is lower (higher) than the strike price. In the latter case, the traders' cost will simply be the price of the option, i.e. the contract, if we ignore other expenses. There are many different styles of option classifications. A *European option* may be exercised only at the pre-defined expiration date of the option. An *American option* on the other hand may be exercised at any time before the expiration date. Option contracts traded on futures exchanges are mainly American-style, whereas those traded on the OTC markets are mainly European. Nearly all stock and equity options are American options, while indexes are generally represented by European options. Commodity options can be of either style. An *Asian option* (or *average option*) is an option of which the payoff is determined by the average underlying price over a pre-specified time interval. It can protect investors from the volatility risk of the market.

Traders buy and sell options depending on their objectives. Broadly speaking, market participants trade to accomplish any of the following objectives. First, they provide a risk management tool akin to insurance. Second, traders may use options traded on exchanges or electronic trading platforms to speculate. Finally, traders may use options to boost their trading income or to reduce the volatility of their returns.

Different types of options in energy markets. The use of spread options is widespread in spite of the fact that the development of pricing and hedging techniques has not been followed at the same pace. In energy markets one finds an abundance of different spread options. We refer to Carmona and Durrleman (2003) [25] for their extensive survey on the matter.

For example, in agricultural futures markets, there are *location spreads* that based on the prices of the same commodity (*quality spreads*) at two different locations, and *calendar spreads* that based on the prices of the same commodity at two different points in time, such as, for example, the *soybean calendar spreads*. In currency and fixed income markets, we find *bond spreads*, *cross-currency spreads*, and spreads based on differences between two interest rates, two yields, two maturities, and so on. In energy markets, spreads are typically used as a way to quantify the cost of production of refined products from the non-refined raw material. *Crack spread* is a term used for the difference between the price of crude oil and petroleum products extracted from it. The New York Mercantile Exchange (NYMEX) offers the only exchange-traded options on energy spreads: the heating oil/crude oil and gasoline/crude oil crack spread options. *Spark spread* is referred to as the difference between the market price of electricity and gas, while *dark spread* as the difference between the market price of electricity and coal. *Clean spark spread* (or *spark green spread*) represents the net revenue a generator makes from selling electricity, having bought gas and the required number of carbon allowances, while *clean dark spread* (or *dark green spread*) refers to an analogous indicator for coal-fired generation of electricity. The difference between the spark green spread and the dark green spread is known as the *climate spread*.

1.4 Monte Carlo method

The name Monte Carlo comes originally from a city in Monaco, which is famous for its gambling casinos. During the late 1930s and 1940s, many computer simulations were performed to estimate the probability that the chain reaction needed for the hydrogen bomb to work successfully. The physicists and astronomers involved in this work were big fans of gambling, so they gave the simulations the code name Monte Carlo. See the full story in [33].

In computing, the Monte Carlo algorithm is a broad class of randomized algorithm. One of its main advantages is that it can be easily implemented for any type of probability distributions and for almost all kinds of financial derivatives, and it is most powerful when it is highly complex or (in most of the cases) impossible to obtain a closed-form expression, or infeasible to apply a deterministic algorithm, as often used in physics and mathematic. This algorithm depends on repeated random samplings with deterministic running time to obtain required numerical results. The magnitude of the random samples is usually suggested to be at least 3000 (VOSE Software RISK SOFTWARE SPECIALISTS) or 5000 (FMRIB Software Library version 5.0). The higher the number, the closer the p-value will be to the p-value that would be found by systematically examining all possible permutations. Practitioners will try different numbers of permutations up to several millions. However, the main weakness of the Monte Carlo approach is that it takes a massive amount of time for analyzing, computing and plotting the data. Hence, practitioners will keep increasing the permutations until the result stabilizes at a certain precision, say, two decimal places. This is exactly the approach we have adopted in Chapter 3 and Chapter 4.

In most of the cases, the terms *Monte Carlo algorithms* and *Monte Carlo methods* (or *Monte Carlo experiments*) refer to the same concept. Generally speaking, they are techniques that can be used to solve mathematical and statistical problems, mainly in three problems classes: optimization, numerical integration and generation of draws from a probability distribution. While in other cases, Monte Carlo methods refer to the methods based on the Monte Carlo algorithm. Some people use the term Monte Carlo method *only* when they want to differentiate between the different methods used in pricing financial derivatives. In this project, we will adapt the term Monte Carlo method with respect to both the algorithm and the method based on the algorithm, for pricing and hedging the spread options. Apart from Monte Carlo methods, other methods [18] in pricing derivatives may include, but are not limited to:

- Finite Difference Method (see also [3] and [2])
- Risk-Neutral Valuation (see also [11] and [22])
- Transform methods, for example the fast Fourier-transformation (see also [19], [23] and [35])
- (Approximated or exact) Analytic Method. On rare occasions, mathematicians discover closed-form formulas like the Margrabe formula, the Black-Scholes formula [5], the analytical formula for asian options ([14] and [32]), or the Kirk formula and the Bjerksund-Stensland formula being considered in later chapters.

In mathematical finance, a Monte Carlo option model uses Monte Carlo methods to calculate the value of an option with multiple sources of uncertainty or with complicated features [16]. The most significant advantage of the Monte Carlo method is that it is flexible and relatively easy to be implemented for numerical evaluation of nearly all derivatives, which is also the main reason that we in this project are using this method. The Monte Carlo computation for this project will be to:

1. Generate a large number of random numbers from standard normal distribution.
2. Calculate the underlyings of the option for each path.
3. Calculate the associated exercise value, i.e. the payoff, of the option for each path.
4. Take the average of these payoffs.
5. Discount the average value back to the initial time.

This result is the value of the option.

This thesis has been divided into 7 chapters. Chapter 1 and 2 are introductory chapters, providing some background on financial derivatives in the energy market and some results on stochastic analysis in continuous time. In Chapter 3, the bivariate geometric Brownian motion and the trivariate geometric Brownian motion are introduced, as the models for spread options and trivariate spread options, respectively. Chapter 4 presents a structured numerical analysis for pricing spread options in the light of comparing the Kirk formula in Section 4.1 and the Bjerksund-Stensland formula in Section 4.2, with numerical results found using the Monte Carlo method in different scenarios in Section 4.3. By using up to $6,000,000$ permutations in Monte Carlo, when Bjerksund and Stensland [24] only used $100,000$, this research looked at the parameters pairwisely to investigate the effect on pricing:

- By varying combinations of volatilities, this research found in Section 4.3.1 that the Bjerksund-Stensland formula did slightly better than the Kirk formula for both positive and negative strike prices. Both formulas produced very small and close relative errors.
- By varying combinations of strike price and correlation, this research found in Section 4.3.2 that the Bjerksund-Stensland formula was better than the Kirk formula, especially for large positive values of strike prices and correlation coefficients.
- By varying combinations of drift (here interest rate) and the time to exercise, this research found in Section 4.3.3 that neither seemed to have very large relative errors for positive strike prices; as per negative strike prices, the Bjerksund-Stensland formula achieved slightly smaller relative errors than the Kirk formula, nevertheless still far more for the acceptable standard to edge risk.

A study of forward prices in energy market is outlined at the end of this chapter, see Section 4.4 , since the method can be applied to forward contracts as well.

Similar analysis for pricing of trivariate spread options using the Monte Carlo method is applied in Chapter 5, where we first derived the updated formula corresponding to the Kirk formula in Section 5.1 and the Bjerksund-Stensland formula in Section 5.2 , then this research implemented Monte Carlo simulation using those revisited formulas with 4,000,000 permutations in Section 5.3. This time large variations showed up in pricing the trivariate spread options. Here we looked at two scenarios in details:

- When the heating rate was changed, as in Section 5.3.1, both of the two approximations produced large (but still close) relative errors, up to 32.74% when the heating rate was $h = 0.5$. And the relative error was decreasing when the heating rate was increasing from $h = 0.5$ to $h = 1.0$.

- When the initial price of emission-certificate of carbon dioxide, $C(0)$, was changed, as in Section 5.3.2, large relative errors occurred again. As $C(0)$ became higher, the relative price of power $\frac{P(0)}{C(0)}$ and that of gas $\frac{G(0)}{C(0)}$ became lower, causing the relative error to increase a little at first, then decrease to about 5.5% when $h = 1.0$ and to about 24.6% when $h = 0.8$, which was still too large to be accepted.

In Chapter 6 we derived delta-hedge parameters for the Kirk formula and those for the Bjerksund-Stensland formula for bivariate spread options in Section 6.1 and for trivariate spread options in Section 6.2, where mathematical and numerical comparisons were presented in Section 6.3, showing that both performed very well (with an accuracy of three decimals of places) in hedging bivariate spread options, whereas significantly inaccurate in hedging trivariate spread options.

The conclusion and some ideas for further work are discussed in Chapter 7. Most programming codes for this thesis have been written in R and MATLAB, as listed in the appendices.

Chapter 2

Some Preliminaries on Stochastic Analysis

We start with some basic definitions which can be of great importance to this project.

Definition 1. If Ω is a given set, then a **σ-algebra** $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of Ω with the following properties:

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$, where $F^C = \Omega \setminus \mathcal{F}$ is the complement of F in Ω

(iii) $A_1, A_2, ... \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The pair $(\Omega, \mathcal{F})$ is called a **measurable space**. A **probability measure** $\mathcal{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathcal{P} : \mathcal{F} \longrightarrow [0, 1]$ such that

(a) $\mathcal{P}(\emptyset) = 0$, $\mathcal{P}(\Omega) = 1$

(b) if $A_1, A_2, ... \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$), then

$$\mathcal{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathcal{P}(A_i)$$

The triple $(\Omega, \mathcal{F}, \mathcal{P})$ is called a **probability space**. The subsets F of Ω which belong to $\mathcal{F}$ are called **$\mathcal{F}$-measurable sets**. If $(\Omega, \mathcal{F}, \mathcal{P})$ is a given probability space, then a function $Y : \Omega \longrightarrow \mathbb{R}^n$ ic called **$\mathcal{F}$-measurable** if

$$Y^{-1}(U) := \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F}$$

for all open sets $U \in \mathbb{R}^n$. If $X := \Omega \longrightarrow \mathbb{R}^n$ is any function, then **the σ-algebra $\mathcal{H}_X$ generated by** X is the smallest σ-algebra on Ω containing all the sets

$$X^{-1}(U); \qquad U \subseteq \mathbb{R}^n \text{ open.}$$

Definition 2. A **random variable** X is a function $X(\omega) : \Omega \longrightarrow \mathbb{R}$. The **cumulative distribution function** of X is defined as the probability that X is less than or equal to the real number x, that is,

$$F_X(x) = \mathcal{P}(X \leq x) \tag{2.1}$$

The ϵ**-quantile** of a random variable X is defined as the number q_ϵ such that

$$F_X(q_\epsilon) = \mathcal{P}(X \leq q_\epsilon) = \epsilon; \qquad \epsilon \in [0, 1] \tag{2.2}$$

The **probability density function** of a random variable X is defined as the derivative of the (continuous) cumulative distribution function F_X, that is,

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{2.3}$$

The **expectation** (or **expected value**, **mean**, **first moment**) of a (continuous) random variable X with probability density function $f_X(x)$ is given by

$$\mu = \mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{2.4}$$

Furthermore, the expectation of a measurable function $h(X) : \mathbb{R} \longrightarrow \mathbb{R}$ is given by the inner product of f and h, that is,

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{2.5}$$

Note that equation (2.5) is valid (finite) only if the integral converges absolutely. The **variance** of a (continuous) random variable X with probability density function $f_X(x)$ is defined by

$$\sigma^2 = \mathrm{Var}[X] = \mathrm{E}\left[(X - \mu)^2\right] = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx \tag{2.6}$$

Note that a continuous distribution may not have a valid (finite) variance no matter its expectation is valid or not. This is because the integral in equation (2.6) diverges. A shortcut formula for calculating the variance is given by

$$\sigma^2 = \mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx - \mu^2 \tag{2.7}$$

The **standard deviation** of a random variable X is defined as the square root of its variance, that is,

$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}[X]} \tag{2.8}$$

Definition 3. Given two random variables X and Y, the **joint probability distribution** is defined by a joint cumulative distribution function as

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y}\mathcal{P}(X \leq x, Y \leq y) \tag{2.9}$$

The two random variables X and Y are said to be **independent** if and only if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \tag{2.10}$$

In probability theory and statistics, a measure of (linear) dependence between two random variables X and Y is the **covariance**,

$$\text{Cov}[X, Y] = \text{E}\left[(X - \text{E}[X])\,(Y - \text{E}[Y])\right] \tag{2.11}$$

By using the linearity property of expectations, this can be simplified to

$$\text{Cov}[X, Y] = \text{E}[XY] - \text{E}[X]\text{E}[Y] \tag{2.12}$$

Another popoular measure of (linear) dependence between two random variables X and Y is the **(population) correlation coefficient**, which is the normalized covariance

$$\text{corr}(X, Y) = \frac{\text{Cov}[X, Y]}{\text{sd}(X)\text{sd}(Y)} \tag{2.13}$$

Note that when X and Y independent, then $\text{E}[XY] = \text{E}[X]\text{E}[Y]$, implying $\text{Cov}[X, Y] = 0$ and therefore $\text{corr}(X, Y) = 0$.

Given n random variables $X_1, ..., X_n$, their **correlation matrix** is an $n \times n$ matrix, of which the i, j-th entry is equal to the correlation coefficient $\text{corr}(X_i, X_j)$. Because of the fact that $\text{corr}\,(X_i, X_j) = \text{corr}\,(X_j, X_i)$, the correlation matrix is always symmetric, with its diagonal elements always equal to 1, since they are the correlations of the random variables with themselves. The correlation matrix serves as a tool for describing the correlations among n random variables.

Definition 4. If Y is a **normal** (or **Gaussian)** random variable, and its probability density is denoted by ϕ, then the **normal (or Gaussian) distribution** is given by

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{2.14}$$

where μ is the mean or expectation (and is also the mode and median) of the distribution, and σ is the standard deviation of the distribution. When $\mu = 0$ and $\sigma = 1$, the distribution is called the **standard normal distribution** or the **unit normal distribution**. If Z denotes the standard normal variable, then the **standard normal distribution** is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{2.15}$$

Then the variable $X = e^{\mu+\sigma Z}$ is called a **log-normal** variable, where the parameters μ and σ are respectively the mean and standard deviation of the variable's natural logarithm. The probability density function of a log-normal distribution is given by

$$f_X(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}, \qquad x > 0 \tag{2.16}$$

Theorem 5. (The central limit theorem) Suppose $\{X_1, ..., X_n\}$ is a sequence of n independent and identically distributed random variables from a distribution of expetation $\mathrm{E}[X_i] = \mu$ and variance $\mathrm{Var}[X_i] = \sigma^2 < \infty$, for $1 \le i \le n$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ denote the sample average. Then as n approaches infinity, the random variable $\sqrt{n}\left(\bar{X}_n - \mu\right)$ converge in distribution to a normal distribution $\mathcal{N}(0, \sigma^2)$,

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2) \tag{2.17}$$

The usefulness of this theorem is that the distribution of $\bar{X}_n$ approaches a normal $\mathcal{N}(\mu, \frac{\sigma^2}{n})$ regardless of the shape of the distribition of the individual X_i's. Hence, this theorem will be appealed in our study of Monte Carlo methods.

Definition 6. Mathematically, a **(1-dimensional) Brownian motion (BM)** $B(t)$ is an almost surely time-continuous stochastic process characterized by the following conditions:

- $B(0) = 0$.
- $B(t)$ has independent increments, i.e., the stochastic variable $B(t) - B(s)$ is independent with $B(v) - B(u)$, for $0 \le s < t \le u < v$.
- $B(t)$ has normal increments, i.e. the variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, for $0 \le s < t$.
- $B(t)$ has stationary increments, i.e. the distribution of $B(t) - B(s)$ only depends on $t - s$, not t or s.

Definition 7. Let B_t be a (1-dimensional) Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$. Assume there exist two adapted processes $Y(t)$ and $Z(t)$. A **(1-dimensional) Itô process** is a stochastic process $X(t)$ on $(\Omega, \mathcal{F}, \mathcal{P})$ of the form

$$X(t) = X(0) + \int_0^t Y(s)\mathrm{d}B(s) + \int_0^t Z(s)\mathrm{d}s \tag{2.18}$$

where

$$\mathcal{P}\left[\int_0^t Y^2(s)\mathrm{d}B_s < \infty \text{ for all } t \ge 0\right] = 1 \tag{2.19}$$

$$\mathcal{P}\left[\int_0^t |Z(s)|\,\mathrm{d}s < \infty \text{ for all } t \ge 0\right] = 1 \tag{2.20}$$

If $X(t)$ is an Itô process of the integral form like equation (2.18), then it is sometimes written in the shorter differential form

$$\mathrm{d}X(t) = Y(t)\mathrm{d}B(t) + Z(t)\mathrm{d}t \tag{2.21}$$

Theorem 8. (The Itô formula) Let $g(t, x)$ be a function which is once continuously differentiable in $t \in [0, \infty]$ and twice continuously differentiable in $x \in \mathbb{R}$. Let $X(t)$ be an Itô process. Then $Y(t) = g(t, X(t))$ is again an Itô process, and the **(1-dimensional) Itô formula** is given by

$$\mathrm{d}g(t, X(t)) = \frac{\partial g}{\partial t}(t, X(t))\mathrm{d}t + \frac{\partial g}{\partial x}(t, X(t))\mathrm{d}X(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(\mathrm{d}X(t))^2 \tag{2.22}$$

where $(\mathrm{d}X(t))^2 = \mathrm{d}X(t) \cdot \mathrm{d}X(t)$ is computed according to the **Itô rules**:

$$\mathrm{d}t \cdot \mathrm{d}t = dt \cdot \mathrm{d}B(t) = \mathrm{d}B(t) \cdot \mathrm{d}t = 0, \qquad \mathrm{d}B(t) \cdot \mathrm{d}B(t) = \mathrm{d}t \tag{2.23}$$

The Itô formula can be expanded to the n-dimentional case. Introduce m independent Brownian motions $B_1(t), ..., B_m(t)$ and assume that $X_1(t), ..., X_n(t)$ are n Itô processes with dynamics

$$\begin{aligned}
\mathrm{d}X_1(t) &= Y_{11}(t)\mathrm{d}B_1(t) + \cdots + Y_{1m}(t)\mathrm{d}B_m(t) + Z_1(t)\mathrm{d}t \\
\cdots \quad & \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
\cdots \quad & \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
\cdots \quad & \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
\mathrm{d}X_n(t) &= Y_{n1}(t)\mathrm{d}B_1(t) + \cdots + Y_{nm}(t)\mathrm{d}B_m(t) + Z_n(t)\mathrm{d}t
\end{aligned}$$

where $Y_{ij}(t)$'s and $Z_i(t)$'s are adapted Itô processes, for $i = 1, ..., n$ and $j = 1, ..., m$. If we denote the vector $\mathbf{X}(t) = (X_1(t), ..., X_n(t))'$ and let $\mathbf{g}(t, \mathbf{x}) = (g_1(t, \mathbf{x}, ..., g_p(t, \mathbf{x})))'$ be a vector-valued function which is once continuously differentiable in $t \in [0, \infty]$ and twice continuously differentiable in $\mathbf{x} \in \mathbb{R}^n$, then the **multi-dimensional Itô formula** is given by considering each coordinate process, that is, for $k = 1, ..., p$,

$$\begin{aligned}
\mathrm{d}g_k(t, \mathbf{X}(t)) =& \frac{\partial g_k}{\partial t}(t, \mathbf{X}(t))\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(t, \mathbf{X}(t))\mathrm{d}X_i(t) \\
&+ \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, \mathbf{X}(t))\mathrm{d}X_i(t)\mathrm{d}X_j(t)
\end{aligned} \tag{2.24}$$

where $\mathrm{d}X_i(t)\mathrm{d}X_j(t)$ is computed according to the multi-dimensional version of the Itô rules

$$\mathrm{d}t \cdot \mathrm{d}t = dt \cdot \mathrm{d}B_i(t) = \mathrm{d}B_i(t) \cdot \mathrm{d}t = 0, \qquad \mathrm{d}B_i(t) \cdot \mathrm{d}B_j(t) = \delta_{ij}\mathrm{d}t \tag{2.25}$$

with the Kronecker delta δ_{ij} defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.26}$$

Definition 9. A **(1-dimensional) geometric Brownian motion (GBM)**, also known as **(1-dimensional) exponential Brownian motion**, is an almost surely time-continuous stochastic process in which the logarithm of the randomness follows a Brownian motion with a drift. Technically, a stochastic process $S(t)$ is said to follow a GBM if it satisfies the following stochastic differencial equation (SDE):

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}B(t), \qquad t > 0 \tag{2.27}$$

where $B(t)$ is a Brownian motion, μ is the percentage drift and σ is the percentage volatility. For an arbitrary initial value $S(0)$, this SDE has an analytic solution

$$S(t) = S(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right\}, \qquad t > 0 \tag{2.28}$$

The (1-dimensional) GBM can be extended to the multi-dimensional case, since a three-dimensional (trivariate) option price model is required as an assumption in the later chapters of this thesis.

Suppose $\{S_1(t), ..., S_n(t)\}$ is a sequence of n adapted Itô processes. Let $B_1(t), ..., B_m(t)$ be m independent Brownian motions. For each component $i = 1, ..., n$, $S_i(t)$ is defined through the SDE:

$$\mathrm{d}S_i(t) = \alpha_i S_i(t)\mathrm{d}t + S_i(t)\sum_{j=1}^{m}\sigma_{ij}\mathrm{d}B_j(t) \tag{2.29}$$

This is called a **multi-dimensional GBM**.

It is possible to demonstrate that the explicit solution, $S_i(t)$, of this SDE is given by

$$S_i(t) = S_i(0)\exp\left\{\left(\alpha_i - \frac{1}{2}\sum_{j=1}^{m}\sigma_{ij}^2\right)t + \sum_{j=1}^{m}\sigma_{ij}B_j(t)\right\} \tag{2.30}$$

Note that the number of independent Brownian motions m can be less than, equal to or larger than the number of Itô processes n. In mathematical finance, $S_i(t)$ can apply to the value of the i-th asset at time t, with parameter α_i as the drift and parameters σ_{ij} as the volaitlity which describes the correlation among the log-returns of the assets.

Theorem 10. As it has been stated in Definition 9, the $S_i(t)$'s in equation (2.29) are assumed to denote the values of the assets in a market, where the number of independent stochastic noises (here Brownian motions) m and the number of assets n can be either equal or unequal. However, the market is said to be **complete** if and only if $m \leq n$. In contrast with complete markets, there exist **incomplete** markets, which is often the case for energy markets, since there usually will be a lot of different sources of noises. Note that if all claims in a market can be hedged, then the market achieves completeness. This is why we are also interested in finding hedging portfolios for spread options. More about this in chapter 6.

Definition 11. A **filtration** (on $(\Omega, \mathcal{F})$) is a family $\mathcal{M}_t = \{\mathcal{M}_t\}_{t\geq 0}$ of σ-algebras $\mathcal{M}_t \subseteq \mathcal{F}$ such that

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subseteq \mathcal{M}_t$$

i.e., $\{\mathcal{M}_t\}$ is increasing. An n-dimensional stochastic process $\{M(t)\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ is called a **martingale** with respect to the filtration $\{\mathcal{M}_t\}_{t\geq 0}$ and with respect to the probability measure $\mathcal{P}$ if the following conditions are fulfilled:

(i) M(t) is $\mathcal{M}_t$-measurable for all t.

(ii) $\mathrm{E}\left[|M(t)|\right] < \infty$ for all t.

(iii) $\mathrm{E}\left[M(t) \mid M(s)\right] = M(s)$ for all $s \leq t$.

Theorem 12. (The Girsanov theorem I) Let $Y(t) \in \mathbb{R}$ be an Itô process of the form

$$\mathrm{d}Y(t) = a(t,\omega)\mathrm{d}t + \mathrm{d}B(t); \qquad t \leq T,\ Y(0) = 0 \tag{2.31}$$

where $T \leq \infty$ is a given constant and $B(t)$ is a Brownian motion. Put

$$M(t,\omega) = \exp\left\{-\int_0^t a(s,\omega)\mathrm{d}B(s) - \frac{1}{2}\int_0^t a^2(s,\omega)\mathrm{d}s\right\}; \qquad 0 \leq t \leq T \tag{2.32}$$

where

$$\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T a^2(s,\omega)\mathrm{d}s\right)\right] < \infty \tag{2.33}$$

Assume that $\{M(t,\omega)\}_{t\leq T}$ is a martingale with respect to a filtration $\mathcal{F}_t$ and a probability measure $\mathcal{P}$. Define the measure $\mathcal{Q}$ on $\mathcal{F}_t$ by

$$d\mathcal{Q}(\omega) = M(T,\omega)d\mathcal{P}(\omega) \tag{2.34}$$

Then $\mathcal{Q}$ is a probability measure on $\mathcal{F}_T$ and $Y(t)$ is a Brownian motion with respect to $\mathcal{Q}$, for $0 \leq t \leq T$.

Theorem 13. (The Margrabe formula) In mathematical finance, **the Margrabe formula** applies to an option to exchange one risky asset for another at maturity. Suppose there exist only two risky assets in the market. Their prices, $S_1(t)$ and $S_2(t)$ respectively, at time $t < T$, are assumed to follow GBM, where T is the exercising time. Each of the assets has a constant continuous dividend yield q_i, for $i = 1, 2$. The option, C, gives the buyer the right but not the obligation to exchange the second asset for the first at time T. We wish to price the payoff of the option

$$C(T) = \max\left\{S_1(T) - S_2(T), 0\right\} \tag{2.35}$$

If the volatilities of S_i's are σ_i for $i = 1, 2$, then $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$ is constant, where ρ is the correlation coefficient of the Brownian motions of the S_i's. The Margrabe formula states that the right price for the option at time 0 is

$$C(0) = e^{-q_1T}S_1(0)\Phi(d_1) - e^{-q_2T}S_2(0)\Phi(d_2) \tag{2.36}$$

where

$$d_1 = \frac{\ln\left(\frac{S_1(0)}{S_2(0)}\right) + (q_2 - q_1 + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{2.37}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{2.38}$$

Theorem 14. (The Cholesky decomposition) In linear algebra, **the Cholesky decomposition** or **the Cholesky factorization** is a decomposition of a Hermitian positive-definite matrix $\mathbf{A}$ into the form

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T \tag{2.39}$$

where $\mathbf{L}$ is a lower triangular matrix with real and strictly positive diagonal entries, and $\mathbf{L}^T$ denotes the conjugate transpose of $\mathbf{L}$. Every Hermitian positive-definite matrix (and thus also every real-valued symmetric positive-definite matrix) has a unique Cholesky decomposition.

Definition 15. Absolute error is the amount of physical error in an approximation, while **relative error** gives an indication of how good an approximation is relative to the size of the thing being measured. Denote $\tilde{p}$ the simulated result by Monte Carlo method and $\hat{p}$ the estimated value by a closed-form formula. Then the relative error of the estimated value is given by

$$\epsilon_{\hat{p}} = \left|\frac{\tilde{p} - \hat{p}}{\tilde{p}}\right| \cdot 100\% = \left(\left|\frac{\tilde{p}}{\hat{p}}\right| - 1\right) \cdot 100\% \tag{2.40}$$

Chapter 3

Model: Trivariate Geometric Brownian Motion

3.1 Spot price dynamics

The payoff of a trivariate spread option is given by

$$\max\{P(T) - hG(T) - C(T), 0\} \tag{3.1}$$

where P is the price of power, G is the price of gas, which is used as a fuel for producing power, C is the price of a certificate to emit a certain amount of carbon dioxide (CO_2), and the constant h is the heating rate, which converts gas into the energy equivalent of electricity.

The three price dynamics, assumed to be GBM, are given by the differential form

$$\mathrm{d}P(t) = \mu_P(t)P(t)\mathrm{d}t + \sigma_P(t)P(t)\mathrm{d}W_P(t) \tag{3.2}$$
$$\mathrm{d}G(t) = \mu_G(t)G(t)\mathrm{d}t + \sigma_G(t)G(t)\mathrm{d}W_G(t) \tag{3.3}$$
$$\mathrm{d}C(t) = \mu_C(t)C(t)\mathrm{d}t + \sigma_C(t)C(t)\mathrm{d}W_C(t) \tag{3.4}$$

where $\mu_i(t)$ and $\sigma_i(t)$ are time-dependent, deterministic drift and volatility respectively for $i = P, G, C$ and $\mathbf{W}(t) = (W_C(t), W_P(t), W_G(t))$ is a correlated three-dimentional BM, of which the correlation matrix is given by

$$\boldsymbol{\rho}(t) = \begin{pmatrix} 1 & \rho_{CP}(t) & \rho_{CG}(t) \\ \rho_{CP}(t) & 1 & \rho_{PG}(t) \\ \rho_{CG}(t) & \rho_{PG}(t) & 1 \end{pmatrix} \tag{3.5}$$

for time-dependent correlations $\rho_{CP}(t), \rho_{CG}(t), \rho_{PG}(t) \in (-1, 1)$.

In order to ensure that $\mathbf{W}(t)$ is a well-defined trivariate BM, we must have that

$$\rho_{CP}^2(t) + \rho_{CG}^2(t) + \rho_{PG}^2(t) \geq 1 + 2\rho_{CP}(t)\rho_{CG}(t)\rho_{PG}(t) \tag{3.6}$$

It will be of convenience to solve the stochastic differential equations (3.2) to (3.4) for the price dynamics explicitly. We solve (3.2) to get $P(t)$ by applying Theorem 5 (The Itô formula) on $\ln P(t)$,

$$\begin{aligned} d\left(\ln P(t)\right) &= \frac{1}{P(t)}\mathrm{d}P(t) + \frac{1}{2}\left(-\frac{1}{(P(t))^2}\right)\left(\mathrm{d}P(t)\right)^2 \\ &= \frac{\left(\mu_P(t)P(t)\mathrm{d}t + \sigma_P(t)P(t)\mathrm{d}W_P(t)\right)}{P(t)} - \frac{1}{2}\frac{\left(\sigma_P(t)P(t)\right)^2\mathrm{d}t}{(P(t))^2} \\ &= \left(\mu_P(t) - \frac{1}{2}\sigma_P^2(t)\right)\mathrm{d}t + \sigma_P(t)\mathrm{d}W_P(t) \end{aligned} \tag{3.7}$$

This can be written on the integral form

$$\ln P(t) - \ln P(0) = \int_0^t \left(\mu_P(s) - \frac{1}{2}\sigma_P^2(s)\right)\mathrm{d}s + \int_0^t \sigma_P(s)\mathrm{d}W_P(s)$$

Hence, we get the solution

$$P(t) = P(0)\exp\left\{\int_0^t \left(\mu_P(s) - \frac{1}{2}\sigma_P^2(s)\right)\mathrm{d}s + \int_0^t \sigma_P(s)\mathrm{d}W_P(s)\right\} \tag{3.8}$$

Similarly, we can solve (3.3) and (3.4) explicitly for $G(t)$ and $C(t)$ and get

$$G(t) = G(0)\exp\left\{\int_0^t \left(\mu_G(s) - \frac{1}{2}\sigma_G^2(s)\right)\mathrm{d}s + \int_0^t \sigma_G(s)\mathrm{d}W_G(s)\right\} \tag{3.9}$$

$$C(t) = C(0)\exp\left\{\int_0^t \left(\mu_C(s) - \frac{1}{2}\sigma_C^2(s)\right)\mathrm{d}s + \int_0^t \sigma_C(s)\mathrm{d}W_C(s)\right\} \tag{3.10}$$

3.2 Trivariate spread option

The price at time t of a trivariate spread option is given by the risk-neutral expectation of the discounted payoff of the option at the exercise time T, conditioned on the filtration $\mathcal{F}_t$, that is,

$$V(t,T) = e^{-r(T-t)}\mathrm{E}_{\mathcal{Q}}\left[\max\left\{P(T) - hG(t) - C(T), 0\right\} \mid \mathcal{F}_t\right] \tag{3.11}$$

where $\mathcal{Q}$ denotes the risk-neutral probability measure.

Since $\boldsymbol{\rho}(t)$ is a real-valued symmetric positive-definite matrix given by (3.5), by Theorem 10 (The Cholesky decomposition) there exists a 3-by-3 lower triangular matrix $\mathbf{L}(t)$ such that $\boldsymbol{\rho}(t) = \mathbf{L}(t)\mathbf{L}(t)^T$. Such a matrix $\mathbf{L}(t)$ is given as

$$\mathbf{L}(t) = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{CP}(t) & \sqrt{1-\rho_{CP}^2(t)} & 0 \\ \rho_{CG}(t) & \frac{\rho_{PG}(t)-\rho_{CP}(t)\rho_{CG}(t)}{\sqrt{1-\rho_{CP}^2(t)}} & \sqrt{1-\rho_{PG}^2(t) - \frac{(\rho_{PG}(t)-\rho_{CP}(t)\rho_{CG}(t))^2}{1-\rho_{CP}^2(t)}} \end{pmatrix} \tag{3.12}$$

We see that to ensure that $\mathbf{L}(t)$ in (3.12) has real and strictly positive diagonal entries, we must have that

$$1-\rho_{CP}^2(t)>0 \tag{3.13}$$

and

$$1-\rho_{PG}^2(t)-\frac{\left(\rho_{PG}(t)-\rho_{CP}(t)\rho_{CG}(t)\right)^2}{1-\rho_{CP}^2(t)}>0 \tag{3.14}$$

which lead the condition (3.6). For the sake of convenience, we denote

$$a(t)=\frac{\rho_{PG}(t)-\rho_{CP}(t)\rho_{CG}(t)}{\sqrt{1-\rho_{CP}^2(t)}} \tag{3.15}$$

$$b(t)=\sqrt{1-\rho_{PG}^2(t)-\frac{\left(\rho_{PG}(t)-\rho_{CP}(t)\rho_{CG}(t)\right)^2}{1-\rho_{CP}^2(t)}} \tag{3.16}$$

for the rest of this project.

Define a new three-demensional, independent Brownian motion given by

$$\mathbf{U}(t)=(W_C(t),U_1(t),U_2(t)) \tag{3.17}$$

Then immediately we have that $\mathbf{W}(t)=\mathbf{L}(t)\mathbf{U}(t)$, or in terms of the components,

$$\mathrm{d}W_C(t)=\mathrm{d}W_C(t)$$

$$\mathrm{d}W_P(t)=\rho_{CP}(t)\mathrm{d}W_C(t)+\sqrt{1-\rho_{CP}^2(t)}dU_1(t) \tag{3.18}$$

$$\mathrm{d}W_G(t)=\rho_{CG}(t)\mathrm{d}W_C(t)+a(t)dU_1(t)+b(t)dU_2(t) \tag{3.19}$$

The explicit price dynamics for $P(t)$ and $G(t)$ now become

$$P(t)=P(0)\exp\left\{\int_0^t\left(\mu_P(s)-\frac{1}{2}\sigma_P^2(s)\right)\mathrm{d}s+\int_0^t\sigma_P(s)\rho_{CP}(s)\mathrm{d}W_C(s)+\int_0^t\sigma_P(s)\sqrt{1-\rho_{CP}^2(s)}dU_1(s)\right\} \tag{3.20}$$

$$G(t)=G(0)\exp\left\{\int_0^t\left(\mu_G(s)-\frac{1}{2}\sigma_G^2(s)\right)\mathrm{d}s+\int_0^t\sigma_G(s)\rho_{CG}(s)\mathrm{d}W_C(s)+\int_0^t\sigma_G(s)a(s)dU_1(s)+\int_0^t\sigma_G(s)b(s)dU_2(s)\right\} \tag{3.21}$$

Thus the price of the spread option (3.11) has become

$$\begin{aligned}
V(t,T) &= e^{-r(T-t)}\mathrm{E}_{\mathcal{Q}}\left[C(T)\cdot\max\left\{\frac{P(T)}{C(T)} - h\frac{G(T)}{C(T)} - 1, 0\right\}\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\mathrm{E}_{\mathcal{Q}}\left[C(t)\exp\left\{\int_t^T\left[\mu_C(s) - \frac{1}{2}\sigma_C^2(s)\right]\mathrm{d}s + \int_t^T\sigma_C(s)\mathrm{d}W_C(s)\right\}\right. \\
&\qquad \left.\cdot\max\left\{\frac{P(T)}{C(T)} - h\frac{G(T)}{C(T)} - 1, 0\right\}\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}C(t)\exp\left\{\int_t^T\mu_C(s)\mathrm{d}s\right\} \\
&\qquad \cdot\mathrm{E}_{\mathcal{Q}}\left[\exp\left\{\int_t^T -\frac{1}{2}\sigma_C^2(s)\mathrm{d}s + \int_t^T\sigma_C(s)\mathrm{d}W_c(s)\right\}\right. \\
&\qquad \left.\cdot\max\left\{\frac{P(T)}{C(T)} - h\frac{G(T)}{C(T)} - 1, 0\right\}\middle|\mathcal{F}_t\right] \qquad (3.22)
\end{aligned}$$

Moreover, let us define

$$M(t) = \exp\left\{\int_0^t -\frac{1}{2}\sigma_C^2(s)\mathrm{d}s + \int_0^t\sigma_C(s)\mathrm{d}W_C(s)\right\} \qquad (3.23)$$

Applying Theorem 5 (The Itô formula) on $M(t)$, we obtain

$$\begin{aligned}
\mathrm{d}M(t) &= M(t)\left(-\frac{1}{2}\sigma_C^2(t)\mathrm{d}t + \sigma_C(t)\mathrm{d}W_C(t)\right) + \frac{1}{2}\cdot M(t)\cdot\sigma_C^2(t)\mathrm{d}t \\
&= \sigma_C(t)M(t)\mathrm{d}W_C(t) \qquad (3.24)
\end{aligned}$$

Hence, we claim that $M(t)$ is a martingale with respect to $\mathcal{F}_t$. By Theorem 8 (The Girsanov theorem I), there exists a probability measure $\tilde{\mathcal{Q}}$ on the σ-algebra $\mathcal{F}_t$ satisfying

$$\left.\frac{d\tilde{\mathcal{Q}}}{d\mathcal{Q}}\right|_{\mathcal{F}_t} = M(t) \qquad (3.25)$$

where $\tilde{\mathcal{Q}}(A) = \mathrm{E}_{\mathcal{Q}}[M(T)\cdot\mathbb{1}_A]$, and $\tilde{\mathcal{Q}}$ is absolutely continuous with respect to the restriction of $\mathcal{Q}$ to $\mathcal{F}_T$ for fixed $T > 0$. Then we claim that

$$\mathrm{d}\tilde{W}_C(t) = -\sigma_C(t)\mathrm{d}t + \mathrm{d}W_C(t) \qquad (3.26)$$

is a BM under $\tilde{\mathcal{Q}}$. As a consequence of this, we could denote

$$\tilde{\mu}_P(s) = \mu_P(s) - \mu_C(s) \qquad (3.27)$$

$$\tilde{\mu}_G(s) = \mu_G(s) - \mu_C(s) \qquad (3.28)$$

$$\tilde{\sigma}_P^2(s) = \sigma_P^2(s) + \sigma_C^2(s) - 2\sigma_P(s)\sigma_C(s)\rho_{CP}(s) \qquad (3.29)$$

$$\tilde{\sigma}_G^2(s) = \sigma_G^2(s) + \sigma_C^2(s) - 2\sigma_G(s)\sigma_C(s)\rho_{CG}(s) \qquad (3.30)$$

as two updated drifts and two updated volatilities under the probability measure $\tilde{\mathcal{Q}}$.

Now we are ready to derive the dynamics for $P(T)$ and $G(T)$ regarding $C(T)$ respectively.

$$
\begin{aligned}
\tilde{P}(T) &= \frac{P(T)}{C(T)} \\
&= \frac{P(t)}{C(t)} \exp\left\{ \int_t^T \left[\mu_P(s) - \mu_C(s) - \frac{1}{2}\sigma_P^2(s) + \frac{1}{2}\sigma_C^2(s) \right] \mathrm{d}s \right. \\
&\quad \left. + \int_t^T \left[\sigma_P(s)\rho_{CP}(s) - \sigma_C(s) \right] \mathrm{d}W_C(s) + \int_t^T \sigma_P(s)\sqrt{1-\rho_{CP}^2(s)} dU_1(s) \right\} \\
&= \frac{P(t)}{C(t)} \exp\left\{ \int_t^T \left[\mu_P(s) - \mu_C(s) - \frac{1}{2}\sigma_P^2(s) + \frac{1}{2}\sigma_C^2(s) \right] \mathrm{d}s \right. \\
&\quad + \int_t^T \left[\sigma_P(s)\rho_{CP}(s) - \sigma_C(s) \right] \left[d\tilde{W}_C(s) + \sigma_C(s)\mathrm{d}s \right] \\
&\quad \left. + \int_t^T \sigma_P(s)\sqrt{1-\rho_{CP}^2(s)} dU_1(s) \right\} \\
&= \frac{P(t)}{C(t)} \exp\left\{ \int_t^T \left[\tilde{\mu}_P(s) - \frac{1}{2}\tilde{\sigma}_P^2(s) \right] \mathrm{d}s \right. \\
&\quad \left. + \int_t^T \left[\sigma_P(s)\rho_{CP}(s) - \sigma_C(s) \right] \mathrm{d}\tilde{W}_C(s) + \int_t^T \sigma_P(s)\sqrt{1-\rho_{CP}^2(s)} dU_1(s) \right\}
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
\tilde{G}(T) &= \frac{G(T)}{C(T)} \\
&= \frac{G(t)}{C(t)} \exp\left\{ \int_t^T \left[\mu_G(s) - \mu_C(s) - \frac{1}{2}\sigma_G^2(s) + \frac{1}{2}\sigma_C^2(s) \right] \mathrm{d}s \right. \\
&\quad + \int_t^T \left[\sigma_G(s)\rho_{CG}(s) - \sigma_C(s) \right] \mathrm{d}W_C(s) \\
&\quad \left. + \int_t^T \sigma_G(s)a(s)dU_1(s) + \int_t^T \sigma_G(s)b(s)dU_2(s) \right\} \\
&= \frac{G(t)}{C(t)} \exp\left\{ \int_t^T \left[\mu_G(s) - \mu_C(s) - \frac{1}{2}\sigma_G^2(s) + \frac{1}{2}\sigma_C^2(s) \right] \mathrm{d}s \right. \\
&\quad + \int_t^T \left[\sigma_G(s)\rho_{CG}(s) - \sigma_C(s) \right] \left[\mathrm{d}\tilde{W}_C(s) + \sigma_C(s)\mathrm{d}s \right] \\
&\quad \left. + \int_t^T \sigma_G(s)a(s)dU_1(s) + \int_t^T \sigma_G(s)b(s)dU_2 s \right\} \\
&= \frac{G(t)}{C(t)} \exp\left\{ \int_t^T \left[\tilde{\mu}_G(s) - \frac{1}{2}\tilde{\sigma}_G^2(s) \right] \mathrm{d}s \right. \\
&\quad + \int_t^T \left[\sigma_G(s)\rho_{CG}(s) - \sigma_C(s) \right] \mathrm{d}\tilde{W}_C(s) \\
&\quad \left. + \int_t^T \sigma_G(s)a(s)dU_1(s) + \int_t^T \sigma_G(s)b(s)dU_2(s) \right\} \qquad (3.32)
\end{aligned}
$$

Since the stochastic processes $\left(\tilde{W}_C, U_1, U_2\right)$ are independent from each other, the sum of the last two stochastic integrals in (3.31) will be normally distributed with mean 0 and variance $\int_t^T \tilde{\sigma}_P^2(s)\mathrm{d}s$, while the sum of the last three stochastic integrals in (3.32) will be normally distributed with mean 0 and variance $\int_t^T \tilde{\sigma}_G^2(s)\mathrm{d}s$, by the properties of BM, with $\tilde{\sigma}_P^2(s)$ and $\tilde{\sigma}_G^2(s)$ given as in equation (3.29) and (3.30), respectively. Furthermore, we could also find the covariance between the two sums of the stochastic integrals in $\tilde{P}(T)$ and $\tilde{G}(T)$ according to the Itô rules

$$
\begin{aligned}
&\mathrm{E}_{\mathcal{Q}}\left[\left(\int_t^T \left[\sigma_P(s)\rho_{CP}(s)-\sigma_C(s)\right]\mathrm{d}\tilde{W}_C(s)+\int_t^T \sigma_P(s)\sqrt{1-\rho_{CP}^2(s)}dU_1(s)\right)\right.\\
&\quad\left.\cdot\left(\int_t^T \left[\sigma_G(s)\rho_{CG}(s)-\sigma_C(s)\right]\mathrm{d}\tilde{W}_C(s)+\int_t^T \sigma_G(s)a(s)dU_1(s)+\int_t^T \sigma_G(s)b(s)dU_2(s)\right)\right]\\
&\quad=\int_t^T \left[\sigma_P(s)\rho_{CP}(s)-\sigma_C(s)\right]\left[\sigma_G(s)\rho_{CG}(s)-\sigma_C(s)\right]\mathrm{d}s\\
&\qquad+\int_t^T \sigma_P(s)\sqrt{1-\rho_{CP}^2(s)}\sigma_G(s)a(s)\mathrm{d}s\\
&\quad=\int_t^T \left[\sigma_P(s)\sigma_G(s)\rho_{PG}(s)-\sigma_P(s)\sigma_C(s)\rho_{CP}(s)-\sigma_C(s)\sigma_G(s)\rho_{CG}(s)+\sigma_C^2(s)\right]\mathrm{d}s
\end{aligned}
\tag{3.33}
$$

Hence, the spread option price (3.22) can be expressed as

$$
\begin{aligned}
V(t,T)&=C(t)\exp\left\{\int_t^T (\mu_C(s)-r)\,\mathrm{d}s\right\}\mathrm{E}_{\tilde{\mathcal{Q}}}\left[\max\left\{\tilde{P}(T)-h\tilde{G}(T)-1,0\right\}\Big|\,\tilde{\mathcal{F}}_t\right]\\
&=g(t,T,C(t))\mathrm{E}_{\tilde{\mathcal{Q}}}\left[\max\left\{\tilde{P}(T)-h\tilde{G}(T)-1,0\right\}\Big|\,\tilde{\mathcal{F}}_t\right]
\end{aligned}
\tag{3.34}
$$

where

$$
g(t,T,C(t))=C(t)\exp\left\{\int_t^T (\mu_C(s)-r)\,\mathrm{d}s\right\}
\tag{3.35}
$$

and $\left(\tilde{P}(T),\tilde{G}(T)\right)$ is a bivariate GBM with respect to the filtration $\tilde{\mathcal{F}}_t$ which consists of $\tilde{P}(t)$ and $\tilde{G}(t)$. We see from equation (3.34) that this is actually the price of a trivariate spread option with a non-zero strike price of $K=1$.

Chapter 4

Pricing of Bivariate Spread Options

In the paper published by Bjerksund and Stensland (2011) [24], the numerical example given show that their approximation is really excellent. However, it lists only a limited amount of examples. We would like to make a structured numerical analysis of their method compared to the method of Kirk, to check the validity of the approximation. This study will use GBM with constant parameters, as described in the previous chapter.

Consider a European call option on the price spread $P - hG$ with heating rate h, strike $K \geq 0$ and exercising time T. The payoff of the call option at time T is given by

$$V(T) = \{P(T) - hG(T) - K\}^{+} \tag{4.1}$$

The call value at time 0 can be represented by

$$V(r, 0, K) = e^{-rT}\mathrm{E}_{\tilde{\mathcal{Q}}}\left[\max\left\{P(T) - hG(T) - K, 0\right\}\right] \tag{4.2}$$

where the expectation is taken with respect to the risk-neutral probability measure $\tilde{\mathcal{Q}}$, with r the riskless interest rate.

4.1 The Kirk formula

Kirk (1995) [12] suggests the following formula for pricing the spread call option:

$$c_K(0, T) = P(0)\Phi(d_{K,1}) - hG(0)\Phi(d_{K,2}) - e^{-rT}K\Phi(d_{K,2}) \tag{4.3}$$

where

$$d_{K,1} = \frac{\ln\left(\frac{P(0)}{hG(0)+Ke^{-rT}}\right) + \frac{1}{2}\sigma_K^2 T}{\sigma_K\sqrt{T}} \tag{4.4}$$

$$d_{K,2} = d_{K,1} - \sigma_K\sqrt{T} \tag{4.5}$$

$$\sigma_K = \sqrt{\sigma_1^2 - 2\frac{hG(0)}{hG(0)+Ke^{-rT}}\rho\sigma_1\sigma_2 + \left(\frac{hG(0)}{hG(0)+Ke^{-rT}}\right)^2\sigma_2^2} \tag{4.6}$$

with σ_1 and σ_2 being the volatilities for $P(t)$ and $G(t)$ respectively.

4.2 The Bjerksund-Stensland formula

Bjerksund and Stensland (2011) [24] propose an alternative closed-form formula of pricing the spread call option:

$$c_{BS}(0,T) = P(0)\Phi(d_{BS,1}) - hG(0)\Phi(d_{BS,2}) - e^{-rT}K\Phi(d_{BS,3}) \tag{4.7}$$

where

$$d_{BS,1} = \frac{\ln\left(\frac{P(0)}{a}\right) + \left(r + \frac{1}{2}\sigma_1^2 - b\rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2\right)T}{\sigma_{BS}\sqrt{T}} \tag{4.8}$$

$$d_{BS,2} = \frac{\ln\left(\frac{P(0)}{a}\right) + \left(r - \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2 - b\sigma_2^2\right)T}{\sigma_{BS}\sqrt{T}} \tag{4.9}$$

$$d_{BS,3} = \frac{\ln\left(\frac{P(0)}{a}\right) + \left(r - \frac{1}{2}\sigma_1^2 + \frac{1}{2}b^2\sigma_2^2\right)T}{\sigma_{BS}\sqrt{T}} \tag{4.10}$$

$$\sigma_{BS} = \sqrt{\sigma_1^2 - 2b\rho\sigma_1\sigma_2 + b^2\sigma_2^2} \tag{4.11}$$

with the constants a and b defined as

$$a = hG(0)e^{rT} + K \tag{4.12}$$

$$b = \frac{hG(0)e^{rT}}{a} \tag{4.13}$$

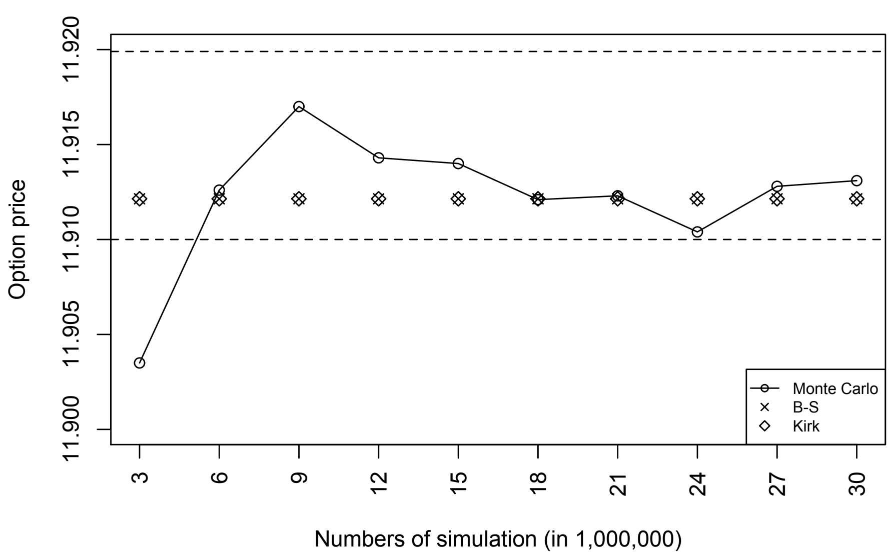
4.3 Numerical results

As aforementioned, we are providing a structured numerical analysis of the Bjerksund-Stensland formula (4.7) compared to the Kirk formula (4.3) to check the validity of the approximations. Note that when the strike is $K = 0$, both of the two formulas become the Margrabe formula (2.36) , which is an accurate closed-form formula for pricing the exchange option proposed by William Margrabe (1978) [34]. We are going to run a simulation by Monte Carlo method as a benchmark with increasing permutations to see how many trials are necessary to ensure a given precision of the price, say, a precision of two decimal places.

We adopt the numerical example in Carmona and Durrleman (2003a) [25], where the annual riskless interest rate is $r = 0.05$ and the time horizon is $T = 1$ (year). The spot prices are respectively $P(0) = 110$ and $G(0) = 100$, with the annualized volatilitie $\sigma_1 = 0.10$ and $\sigma_2 = 0.15$. Inserting these values, both Bjerksund-Stensland (4.7) and Kirk (4.3) produce a price equal to 11.9121. Figure 4.1 shows the results from Monte Carlo method together with this exact price, indicating that at least 6,000,000 permutations are necessary in order to obtain two decimal places of accuracy. We will therefore adopt this number of trials for simulation for the rest of this chapter.

In this section, we will compare the accuracy of the Kirk approximation (4.3) and the Bjerksund-Stenland approximation (4.7) along with the results simulated by Monte Carlo method. This means that we are going to vary volatilities, drifts, correlation, strikes and exercise times to investigate the effect, if any, on pricing the spread option.

Figure 4.1: *Monte Carlo benchmark for spread option, with two approximations*



4.3.1 Different combinations of volatilities

We start with different combinations of volatilities σ_1 and σ_2 when other parameters are constants as given above. The results of the valuation of the spread option in the case of a positive strike price $K = 10$ and a negative one $K = -10$ are shown in Table 4.1 and Table 4.2 respectively. For each σ_2 value, the first row (in italics) lists the result obtained by Monte Carlo method with $6,000,000$ permutations, which is used as a benchmark for the true spread option price. The second row gives the result from the Bjerksund-Stensland formula (4.7), with its relative error given in the third row (in bold). The fourth row provides the result from the Kirk formula (4.3), with its relative error given in the fifth row (in bold). We see that the Bjerksund-Stensland approximation performs better in both Table 4.1 and 4.2, in the light of that it produces smaller relative error in 50 (out of 81) cells in the case of a positive strike $K = 10$, and in 65 (out of 81) cells in the case of a negative one $K = -10$. The relative errors for both approximations when $K = 10$ and $K = -10$ are plotted in Figure 4.2, where we see that the relative errors obtained by the two approximations are very small. Note that the units of the vertical axes (relative error) are different in each of the four subfigures.

4.3.2 Different combinations of strikes and correlations

We then consider different combinations of strike price K and correlation ρ. The results of the simulation are presented in Table 4.3. For each K value, the first row (in italics) lists the result obtained by Monte Carlo method with $6,000,000$ permutations, which is used as a benchmark for the true spread option price. The second row gives the result from the Bjerksund-Stensland formula (4.7), with its relative error given in the third row (in bold). The fourth row provides the result from the Kirk formula (4.3), with its relative error given in the fifth row (in bold). We see that the Bjerksund-Stensland approximation performs better, in the light of that it produces smaller relative error in 73 (out of 81) cells in Table 4.3. In addition it shows that the Kirk formula tends to overprice the spread option when the strike is further away from zero, which is in consistent with the results of Bjerksund and Stensland [24]. The relative errors for both approximations are plotted in Figure 4.3, from which we can clearly articulate that the Bjerksund-Stensland approximation beats the Kirk approximation, especially for large positive values of strike price and correlation. Note that the units of the vertical axes (relative error) are different in each of the two subfigures.

4.3.3 Different combinations of drifts and exercise times

The last scenario in this section is to consider the effect on option price by different combinations of drift (here: risk-free interest rate) r and time to exercise T. The results are listed in Table 4.4 when the strike is positive ($K = 10$) and in Table 4.5 when the strike is negative ($K = -10$). For each T value, the first row (in italics) shows the result obtained by Monte Carlo method with $6,000,000$ permutations, which is used as a benchmark for the true spread option price. The second row gives the result from the Bjerksund-Stensland formula (4.7), with its relative error given in the third row (in bold). The fourth row provides the result from the Kirk formula (4.3), with its relative error given in the fifth row (in bold). We see that in the case of a positive strike $K = 10$, neither of the approximations does a satisfying work since they both produce very large deviations. In spite of this, the Bjerksund-Stensland approximation achieves smaller relative errors in 25 (out of 30) cells. As for a negative strike $K = -10$ however, both approximations perform very well when the time to exercise is short, but still, the Bjerksund-Stenslandand approximation obtains smaller relative error in 28 (out of 30) cells. The relative errors for both approximations when $K = 10$ and $K = -10$ are plotted in Figure 4.4, where we see that the relative errors for $K = 10$ are incredibly large. Note that the vertical axes (relative error) are different in each of the four subfigures.

4.4 A study of forward prices

So far we have considered the dynamics only for option prices. However, in many relevant situations, it might be practical to apply the approximations for forward price dynamics. We will first look at the method for a complete market, then move for an incomplete one.

Let t denote time, T the exercise time for option, and τ the delivery time for forward. Note that it is reasonable to assume $0 \leq T \leq \tau$. Let S_P and S_G denote the spot prices for power and gas, and f_P and f_G the forward prices for power and gas, respectively. Some standard assumptions are necessary to capture the fact, such as:

- Forward prices are lognormal distributed.
- Forward prices are equal to the expected future spot prices, that is,

$$f_P(t,\tau) = \mathrm{E}_{\mathcal{Q}}[S_P(\tau)|\mathcal{F}_t] = S_P(t)e^{r(\tau-t)} \tag{4.14}$$

$$f_G(t,\tau) = \mathrm{E}_{\mathcal{Q}}[S_G(\tau)|\mathcal{F}_t] = S_G(t)e^{r(\tau-t)} \tag{4.15}$$

  where r is the risk-free interest rate. Hence, forward prices are martingales under the objective probability measure $\mathcal{Q}$.

- Spot prices converge to forward prices as $t \to \tau$, that is,

$$S_P(\tau) = \lim_{t\to\tau} f_P(t,\tau) = f_P(\tau,\tau) \tag{4.16}$$

$$S_G(\tau) = \lim_{t\to\tau} f_G(t,\tau) = f_G(\tau,\tau) \tag{4.17}$$

- Spot volatility $\Sigma_i(\cdot) : [0,T] \to R$ for $i = P, G$ is a deterministic function proportional to the forward price level. It is a falling curve with exercising time, or equivalently, an increasing curve as the forward approaches its delivery time.

From equations (4.14) and (4.15), the payoff of the call option (4.1) at time T is now given by

$$\begin{aligned} V(T,\tau) &= \{f_P(T,\tau) - h \cdot f_G(T,\tau) - K)^+\} \\ &= \left\{S_P(T)e^{r(\tau-T)} - h \cdot S_G(T)e^{r(\tau-T)} - K, 0\right\}^+ \\ &= e^{r(\tau-T)}\left\{S_P(T) - h \cdot S_G(T) - Ke^{-r(\tau-T)}, 0\right\}^+ \end{aligned} \tag{4.18}$$

Therefore we have derived a spread option of the forwards in complete market. Thus we can apply the abovementioned theory about the Kirk formula and the Bjerksund-Stensland formula on (4.18).

Now let's move the focus to incomplete market. Assume that the forward prices follow the Itô stochastic differential equations

$$df_P(t,\tau) = f_P(t,\tau)\Sigma_P(t,\tau)\mathrm{d}B_P(t) \tag{4.19}$$

$$df_G(t,\tau) = f_G(t,\tau)\Sigma_G(t,\tau)\mathrm{d}B_G(t) \tag{4.20}$$

where $\Sigma_P(t)$ and $\Sigma_G(t)$ are bounded, deterministic functions, and $B_P(t)$ and $B_G(t)$ are BM to the $\tau-$maturity forward prices on probability space $(\Omega, \mathcal{F}, \mathcal{Q})$ along with the standard filtration $\mathcal{F}_t : t \in [0,T]$.

The solution of equation (4.19) can be found by applying Theorem 5 (The Itô formula)

$$d\left(\ln f_P(t,\tau)\right) = -\frac{1}{2}\Sigma_P^2(t,\tau)\mathrm{d}t + \Sigma_P(t,\tau)\mathrm{d}B_P(t) \tag{4.21}$$

Then integrating both sides

$$\ln\left(\frac{f_P(t,\tau)}{f_P(0,\tau)}\right) = -\frac{1}{2}\int_0^t \Sigma_P^2(s,\tau)\mathrm{d}s + \int_0^t \Sigma_P(s,\tau)\mathrm{d}B_P(s) \tag{4.22}$$

or equivalently

$$f_P(t,\tau) = f_P(0,\tau)\exp\left\{-\frac{1}{2}\int_0^t \Sigma_P^2(s,\tau)\mathrm{d}s + \int_0^t \Sigma_P(s,\tau)\mathrm{d}B_P(s)\right\} \tag{4.23}$$

Similarly, we can solve the stochastic differential equation (4.20) and get

$$f_G(t,\tau) = f_G(0,\tau)\exp\left\{-\frac{1}{2}\int_0^t \Sigma_G^2(s,\tau)\mathrm{d}s + \int_0^t \Sigma_G(s,\tau)\mathrm{d}B_G(s)\right\} \tag{4.24}$$

A typical case in energy market is the occurrence of the *Samuelson effect* in the volatility structure of the forward price, in the sense that the volatility is a decreasing function with time to maturity $\tau - t$. This requires a model with time-dependent volatility function, as we have seen so far. The volatilities $\Sigma_i(t,\tau)$ for $i = P, G$ can therefore be captured in two ways:

(i) by a two-parameter exponential model

$$\Sigma_i(t,\tau) = \sigma_{i,0}e^{-\alpha_i(\tau-t)} \tag{4.25}$$

(ii) by a three-parameter model

$$\Sigma_i'(t,\tau) = \sigma_{i,1} + \sigma_{i,2}e^{-\alpha_i(\tau-t)} \tag{4.26}$$

where $\alpha_i, \sigma_{i,0}, \sigma_{i,1}, \sigma_{i,2}$ are strictly positive constants.

Table 4.1: *Simulated spread option values for different volatilities when* $K = 10$

σ_2 \ σ_1	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
	4.4475	*5.9688*	*7.8267*	*9.8215*	*11.8691*	*13.9560*	*16.0841*	*18.1464*	*20.2742*
	4.4482	5.9757	7.8304	9.8207	11.8736	13.9571	16.0550	18.1579	20.2599
0.10	**0.02%**	**0.12%**	**0.05%**	**0.01%**	**0.04%**	**0.01%**	**0.18%**	**0.06%**	**0.07%**
	4.4481	5.9758	7.8304	9.8207	11.8737	13.8573	16.0552	18.1582	20.2604
	0.01%	**0.12%**	**0.05%**	**0.01%**	**0.04%**	**0.71%**	**0.18%**	**0.07%**	**0.07%**
	5.6167	*6.5498*	*7.9976*	*9.7292*	*11.6269*	*13.5719*	*15.5696*	*17.6068*	*19.6780*
	5.6182	6.5433	7.9909	9.7218	11.6038	13.5694	15.5827	17.6228	19.6773
0.15	**0.03%**	**0.10%**	**0.08%**	**0.08%**	**0.20%**	**0.02%**	**0.08%**	**0.09%**	**0.00%**
	5.6181	6.5435	7.9910	9.7218	11.6038	13.5695	15.5829	17.6232	19.6778
	0.02%	**0.10%**	**0.08%**	**0.08%**	**0.20%**	**0.02%**	**0.09%**	**0.09%**	**0.00%**
	7.1798	*7.6166*	*8.6334*	*10.0316*	*11.6702*	*13.4718*	*15.3907*	*17.2898*	*19.2875*
	7.1748	7.6237	8.6354	10.0350	11.6781	13.4708	15.3564	17.3001	19.2802
0.20	**0.07%**	**0.09%**	**0.02%**	**0.03%**	**0.07%**	**0.01%**	**0.22%**	**0.06%**	**0.04%**
	7.1750	7.6243	8.6360	10.0354	11.6782	13.4708	15.3564	17.3003	19.2806
	0.07%	**0.10%**	**0.03%**	**0.04%**	**0.07%**	**0.01%**	**0.22%**	**0.06%**	**0.04%**
	8.9073	*9.0289*	*9.6556*	*10.7088*	*12.0861*	*13.6650*	*15.3757*	*17.2364*	*19.0498*
	8.9093	9.0274	9.6635	10.7226	12.0894	13.6674	15.3872	17.2022	19.0810
0.25	**0.02%**	**0.02%**	**0.08%**	**0.13%**	**0.03%**	**0.02%**	**0.07%**	**0.20%**	**0.16%**
	8.9105	9.0290	9.6653	10.7240	12.0903	13.6678	15.3873	17.2022	19.0811
	0.04%	**0.00%**	**0.10%**	**0.14%**	**0.03%**	**0.02%**	**0.08%**	**0.20%**	**0.16%**
	10.7290	*10.6180*	*10.9570*	*11.7247*	*12.8215*	*14.1501*	*15.6554*	*17.3257*	*19.0998*
	10.7304	10.6214	10.9633	11.7157	12.8035	14.1457	15.6731	17.3327	19.0860
0.30	**0.01%**	**0.03%**	**0.06%**	**0.08%**	**0.14%**	**0.03%**	**0.11%**	**0.04%**	**0.07%**
	10.733	10.6248	10.9669	11.7191	12.8062	14.1475	15.6741	17.3331	19.0861
	0.04%	**0.06%**	**0.09%**	**0.05%**	**0.12%**	**0.02%**	**0.12%**	**0.04%**	**0.07%**
	12.6007	*12.3287*	*12.4509*	*12.9504*	*13.7728*	*14.8885*	*16.1897*	*17.7088*	*19.2982*
	12.5964	12.3274	12.4457	12.9408	13.7707	14.8768	16.1991	17.6856	19.2944
0.35	**0.03%**	**0.01%**	**0.04%**	**0.07%**	**0.02%**	**0.08%**	**0.06%**	**0.13%**	**0.02%**
	12.6029	12.3336	12.4520	12.9470	13.7763	14.8812	16.2023	17.6875	19.2954
	0.02%	**0.04%**	**0.01%**	**0.03%**	**0.03%**	**0.05%**	**0.08%**	**0.12%**	**0.01%**
	14.4946	*14.1055*	*14.0600*	*14.3431*	*14.9526*	*15.8320*	*16.9410*	*18.2485*	*19.6988*
	14.4858	14.1006	14.0488	14.3349	14.9391	15.8233	16.9412	18.2466	19.6987
0.40	**0.06%**	**0.03%**	**0.08%**	**0.06%**	**0.09%**	**0.05%**	**0.00%**	**0.01%**	**0.00%**
	14.4969	14.1108	14.0588	14.3449	14.9486	15.8317	16.9480	18.2516	19.7021
	0.02%	**0.04%**	**0.01%**	**0.01%**	**0.03%**	**0.00%**	**0.04%**	**0.02%**	**0.02%**
	16.4031	*15.9055*	*15.7330*	*15.8495*	*16.2617*	*16.9465*	*17.8670*	*18.9954*	*20.3020*
	15.3864	15.9144	15.7314	15.8497	16.2621	16.9467	17.8703	18.9955	20.2858
0.45	**6.20%**	**0.06%**	**0.01%**	**0.00%**	**0.00%**	**0.00%**	**0.02%**	**0.00%**	**0.08%**
	16.4038	15.9301	15.7469	15.8647	16.2768	16.9603	17.8822	19.0053	20.2932
	0.00%	**0.15%**	**0.09%**	**0.10%**	**0.09%**	**0.08%**	**0.09%**	**0.05%**	**0.04%**
	18.2973	*17.7502*	*17.4678*	*17.5595*	*17.7098*	*18.1994*	*18.9773*	*19.9060*	*21.0548*
	18.2907	17.7524	17.4675	17.4501	17.7018	18.2111	18.9564	19.9091	21.0385
0.50	**0.04%**	**0.01%**	**0.00%**	**0.62%**	**0.05%**	**0.06%**	**0.11%**	**0.02%**	**0.08%**
	18.3163	17.7754	17.4894	17.4716	17.7229	18.2314	18.9750	19.9254	21.0520
	0.10%	**0.14%**	**0.12%**	**0.50%**	**0.07%**	**0.18%**	**0.01%**	**0.10%**	**0.01%**

Table 4.2: *Simulated spread option values for different volatilities when* $K = -10$

σ_2 \ σ_1	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
	19.6470	*19.9493*	*20.6338*	*21.6763*	*22.9721*	*24.4380*	*26.0132*	*27.6970*	*29.4087*
	19.6429	19.9517	20.6392	21.6791	22.9753	24.4437	26.0265	27.6855	29.3954
0.10	**0.02%**	**0.01%**	**0.03%**	**0.01%**	**0.01%**	**0.02%**	**0.05%**	**0.04%**	**0.05%**
	19.6453	19.9525	20.6392	21.6791	22.9755	24.4441	26.0270	27.6859	29.3958
	0.01%	**0.02%**	**0.03%**	**0.01%**	**0.01%**	**0.02%**	**0.05%**	**0.04%**	**0.04%**
	20.0878	*20.3232*	*20.8827*	*21.7635*	*22.8961*	*24.2449*	*25.7081*	*27.2763*	*28.9076*
	20.0837	20.3315	20.8930	21.7664	22.9006	24.2307	25.7011	27.2703	28.9084
0.15	**0.02%**	**0.04%**	**0.05%**	**0.01%**	**0.02%**	**0.06%**	**0.03%**	**0.02%**	**0.00%**
	20.1003	20.3406	20.8967	21.7673	22.9007	24.2308	25.7013	27.2706	28.9088
	0.06%	**0.09%**	**0.07%**	**0.02%**	**0.02%**	**0.06%**	**0.03%**	**0.02%**	**0.00%**
	20.9910	*21.1031*	*21.5209*	*22.2177*	*23.1793*	*24.3435*	*25.6654*	*27.1080*	*28.6441*
	21.0052	21.1064	21.5141	22.2133	23.1699	24.3368	25.6660	27.1169	28.6573
0.20	**0.07%**	**0.02%**	**0.03%**	**0.02%**	**0.04%**	**0.03%**	**0.00%**	**0.03%**	**0.05%**
	21.0456	21.1347	21.5305	22.2211	23.1729	24.3376	25.6662	27.1169	28.6574
	0.26%	**0.15%**	**0.04%**	**0.02%**	**0.03%**	**0.02%**	**0.00%**	**0.03%**	**0.05%**
	22.3062	*22.2283*	*22.4641*	*23.0085*	*23.7595*	*24.7452*	*25.9290*	*27.2488*	*28.6434*
	22.3098	22.2447	22.4772	22.9925	23.7632	24.7527	25.9212	27.2314	28.6512
0.25	**0.02%**	**0.07%**	**0.06%**	**0.07%**	**0.02%**	**0.03%**	**0.03%**	**0.06%**	**0.03%**
	22.3753	22.2977	22.5146	23.0154	23.7754	24.7583	25.9234	27.2320	28.6513
	0.31%	**0.31%**	**0.22%**	**0.03%**	**0.07%**	**0.05%**	**0.02%**	**0.06%**	**0.03%**
	23.8697	*23.6612*	*23.7229*	*24.0681*	*24.6436*	*25.4301*	*26.4542*	*27.6324*	*28.8836*
	23.8657	23.6560	23.7222	24.0580	24.6442	25.4527	26.4511	27.6067	28.8894
0.30	**0.02%**	**0.02%**	**0.00%**	**0.04%**	**0.00%**	**0.09%**	**0.01%**	**0.09%**	**0.02%**
	23.9543	23.7341	23.7842	24.1022	24.6725	25.4690	26.4596	27.6106	28.8910
	0.35%	**0.31%**	**0.26%**	**0.14%**	**0.12%**	**0.15%**	**0.02%**	**0.08%**	**0.03%**
	25.5834	*25.2699*	*25.1897*	*25.3759*	*25.7852*	*26.4127*	*27.2174*	*28.2337*	*29.3337*
	25.5806	25.2558	25.1795	25.3535	25.7671	26.4008	27.2292	28.2250	29.3612
0.35	**0.01%**	**0.06%**	**0.04%**	**0.09%**	**0.07%**	**0.05%**	**0.04%**	**0.03%**	**0.09%**
	25.6896	25.3569	25.2663	25.4217	25.8163	26.4333	27.2490	28.2360	29.3668
	0.42%	**0.34%**	**0.30%**	**0.18%**	**0.12%**	**0.08%**	**0.12%**	**0.01%**	**0.11%**
	27.4088	*26.9680*	*26.7779*	*26.8564*	*27.0740*	*27.5500*	*28.2175*	*29.0435*	*30.0542*
	27.3968	26.9824	26.7897	26.8249	27.0846	27.5567	28.2228	29.0610	30.0484
0.40	**0.04%**	**0.05%**	**0.04%**	**0.12%**	**0.04%**	**0.02%**	**0.02%**	**0.06%**	**0.02%**
	27.5236	27.1041	26.8993	26.9171	27.1567	27.6091	28.2582	29.0832	30.0612
	0.42%	**0.50%**	**0.45%**	**0.23%**	**0.31%**	**0.21%**	**0.14%**	**0.14%**	**0.02%**
	29.2726	*28.7682*	*28.5038*	*28.4362*	*28.5572*	*28.8715*	*29.3814*	*30.0898*	*30.9446*
	29.2782	28.7944	28.5078	28.4271	28.5537	28.8812	29.3974	30.0859	30.9277
0.45	**0.02%**	**0.09%**	**0.01%**	**0.03%**	**0.01%**	**0.03%**	**0.05%**	**0.01%**	**0.05%**
	29.4200	28.9336	28.6376	28.5415	28.6485	28.9551	29.4514	30.1228	30.9512
	0.50%	**0.57%**	**0.47%**	**0.37%**	**0.32%**	**0.29%**	**0.24%**	**0.11%**	**0.02%**
	31.2070	*30.6628*	*30.3171*	*30.1178*	*30.1425*	*30.3534*	*30.7257*	*31.2820*	*31.9959*
	31.2017	30.6635	30.3015	30.1250	30.1383	30.3393	30.7207	31.2706	31.9743
0.50	**0.02%**	**0.00%**	**0.05%**	**0.02%**	**0.01%**	**0.05%**	**0.02%**	**0.04%**	**0.07%**
	31.3558	30.8175	30.4486	30.2591	30.2544	30.4345	30.7944	31.3243	32.0111
	0.48%	**0.50%**	**0.43%**	**0.47%**	**0.37%**	**0.27%**	**0.22%**	**0.14%**	**0.05%**

Table 4.3: *Simulated spread option values for different strikes and correlations*

K \ ρ	-0.9	-0.7	-0.5	-0.3	0	0.3	0.5	0.7	0.9
	30.7623	*30.4916*	*30.2424*	*29.9940*	*29.6293*	*29.3220*	*29.1687*	*29.0574*	*29.0304*
	30.7583	30.4924	30.2315	29.9785	29.6239	29.3204	29.1648	29.0637	29.0261
-20	**0.01%**	**0.00%**	**0.04%**	**0.05%**	**0.02%**	**0.01%**	**0.01%**	**0.02%**	**0.01%**
	30.8205	30.5511	30.2863	30.0289	29.6664	29.3527	29.1885	29.0772	29.0295
	0.19%	**0.20%**	**0.15%**	**0.12%**	**0.13%**	**0.10%**	**0.07%**	**0.07%**	**0.00%**
	26.7082	*26.3680*	*26.0472*	*25.7216*	*25.2536*	*24.8196*	*24.5565*	*24.3759*	*24.2831*
	26.6975	26.3731	26.0478	25.7240	25.2495	24.8097	24.5582	24.3689	24.2766
-15	**0.04%**	**0.02%**	**0.00%**	**0.01%**	**0.02%**	**0.04%**	**0.01%**	**0.03%**	**0.03%**
	26.7407	26.4144	26.0872	25.7611	25.2824	24.8370	24.5803	24.3837	24.2816
	0.12%	**0.18%**	**0.15%**	**0.15%**	**0.11%**	**0.07%**	**0.10%**	**0.03%**	**0.01%**
	22.8555	*22.4531*	*22.0614*	*21.6924*	*21.0783*	*20.4667*	*20.0853*	*19.7555*	*19.5465*
	22.8512	22.4678	22.0762	21.6773	21.0696	20.4651	20.0837	19.7540	19.5445
-10	**0.02%**	**0.07%**	**0.07%**	**0.07%**	**0.04%**	**0.01%**	**0.01%**	**0.01%**	**0.01%**
	22.8760	22.4918	22.0995	21.6998	21.0905	20.4839	20.1003	19.7672	19.5510
	0.09%	**0.17%**	**0.17%**	**0.03%**	**0.06%**	**0.08%**	**0.07%**	**0.06%**	**0.02%**
	19.2588	*18.8240*	*18.3699*	*17.8913*	*17.1485*	*16.3603*	*15.8273*	*15.2985*	*14.8727*
	19.2585	18.8202	18.3658	17.8941	17.1516	16.3675	15.8291	15.3023	14.8700
-5	**0.00%**	**0.02%**	**0.02%**	**0.02%**	**0.02%**	**0.04%**	**0.01%**	**0.02%**	**0.02%**
	19.2681	18.8297	18.3751	17.9032	17.1605	16.3760	15.8374	15.3100	14.8756
	0.05%	**0.03%**	**0.03%**	**0.07%**	**0.07%**	**0.10%**	**0.06%**	**0.08%**	**0.02%**
	15.9556	*15.4759*	*14.9663*	*14.4247*	*13.5767*	*12.6153*	*11.9130*	*11.1484*	*10.3651*
	15.9563	15.4723	14.9649	14.4304	13.5677	12.6131	11.9121	11.1520	10.3653
0	**0.00%**	**0.02%**	**0.01%**	**0.04%**	**0.07%**	**0.02%**	**0.01%**	**0.03%**	**0.00%**
	15.9563	15.4723	14.9649	14.4304	13.5677	12.6131	11.9121	11.1520	10.3653
	0.00%	**0.02%**	**0.01%**	**0.04%**	**0.07%**	**0.02%**	**0.01%**	**0.03%**	**0.00%**
	12.9753	*12.4520*	*11.9148*	*11.3412*	*10.3852*	*9.3012*	*8.4701*	*7.4929*	*6.2863*
	12.9761	12.4606	11.9160	11.3369	10.3867	9.3017	8.4675	7.4945	6.2867
5	**0.01%**	**0.07%**	**0.01%**	**0.04%**	**0.01%**	**0.01%**	**0.03%**	**0.02%**	**0.01%**
	12.9736	12.4579	11.9132	11.3340	10.3837	9.2984	8.4639	7.4902	6.2800
	0.01%	**0.05%**	**0.01%**	**0.06%**	**0.01%**	**0.03%**	**0.07%**	**0.04%**	**0.10%**
	10.3419	*9.8202*	*9.2478*	*8.6480*	*7.6612*	*6.5188*	*5.6191*	*4.5371*	*3.0572*
	10.3410	9.8118	9.2509	8.6522	7.6626	6.5171	5.6182	4.5338	3.0574
10	**0.01%**	**0.09%**	**0.03%**	**0.05%**	**0.02%**	**0.03%**	**0.02%**	**0.07%**	**0.33%**
	10.3431	9.8135	9.2523	8.6533	7.6633	6.5173	5.6181	4.5334	3.0560
	0.01%	**0.07%**	**0.05%**	**0.06%**	**0.03%**	**0.02%**	**0.02%**	**0.08%**	**0.04%**
	8.0631	*7.5388*	*6.9904*	*6.3974*	*5.4231*	*4.3052*	*3.4358*	*2.4048*	*1.0599*
	8.0629	7.5399	6.9865	6.3966	5.4249	4.3066	3.4370	2.4038	1.0592
15	**0.00%**	**0.01%**	**0.06%**	**0.01%**	**0.03%**	**0.03%**	**0.03%**	**0.04%**	**0.07%**
	8.0757	7.5520	6.9978	6.4072	5.4345	4.3155	3.4457	2.4131	1.0730
	0.16%	**0.18%**	**0.11%**	**0.15%**	**0.21%**	**0.24%**	**0.29%**	**0.35%**	**1.24%**
	6.1370	*5.6450*	*5.1619*	*4.5666*	*3.6719*	*2.6632*	*1.9156*	*1.0875*	*0.2261*
	6.1412	5.6443	5.1216	4.5688	3.6709	2.6654	1.9157	1.0872	0.2249
20	**0.07%**	**0.01%**	**0.78%**	**0.05%**	**0.03%**	**0.08%**	**0.01%**	**0.03%**	**0.53%**
	6.1684	5.6700	5.1458	4.5916	3.6915	2.6840	1.9330	1.1036	0.2402
	0.51%	**0.44%**	**0.31%**	**0.55%**	**0.53%**	**0.78%**	**0.91%**	**1.48%**	**6.24%**

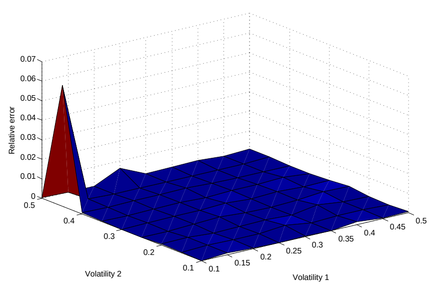
Table 4.4: *Simulated spread option values for different drifts and exercise times when $K = 10$*

T \ ρ	0.00	0.03	0.05	0.07	0.10
1 month	*0.4606*	*0.4736*	*0.4826*	*0.4914*	*0.5046*
	1.5137	1.5443	1.5613	1.5755	1.5915
	228.64%	**226.08%**	**223.52%**	**220.61%**	**215.40%**
	1.5493	1.5619	1.5703	1.5788	1.5915
	236.37%	**229.79%**	**225.38%**	**221.29%**	**215.40%**
2 months	*0.6519*	*0.6778*	*0.6957*	*0.7142*	*0.7415*
	2.0914	2.1666	2.2075	2.2407	2.2756
	220.82%	**219.65%**	**217.31%**	**213.74%**	**206.89%**
	2.1909	2.2162	2.2331	2.2501	2.2755
	236.08%	**226.97%**	**220.99%**	**215.05%**	**206.88%**
3 months	*0.7988*	*0.8375*	*0.8656*	*0.8927*	*0.9346*
	2.5026	2.6306	2.6997	2.7549	2.8103
	213.29%	**214.10%**	**211.89%**	**208.60%**	**200.70%**
	2.6832	2.7212	2.7466	2.7720	2.8102
	235.90%	**224.92%**	**217.31%**	**210.52%**	**200.68%**
6 months	*1.1298*	*1.2101*	*1.2666*	*1.3225*	*1.4074*
	3.3014	3.6181	3.7894	3.9234	4.0480
	192.21%	**198.99%**	**199.18%**	**196.67%**	**187.62%**
	3.7940	3.8704	3.9212	3.9719	4.0477
	235.81%	**219.84%**	**209.58%**	**200.33%**	**187.60%**
9 months	*1.3828*	*1.5089*	*1.5937*	*1.6786*	*1.8105*
	3.7726	4.3047	4.5965	4.8235	5.0255
	172.82%	**185.29%**	**188.42%**	**187.35%**	**177.58%**
	4.6460	4.7608	4.8358	4.9124	5.0251
	235.98%	**215.51%**	**203.43%**	**192.65%**	**177.55%**
12 months	*1.5964*	*1.7664*	*1.8797*	*1.9967*	*2.1706*
	4.0655	4.8260	5.2508	5.5816	5.8673
	154.67%	**173.21%**	**179.34%**	**179.54%**	**170.31%**
	5.3639	5.5171	5.6181	5.7182	5.8666
	236.00%	**212.34%**	**198.88%**	**186.38%**	**170.28%**

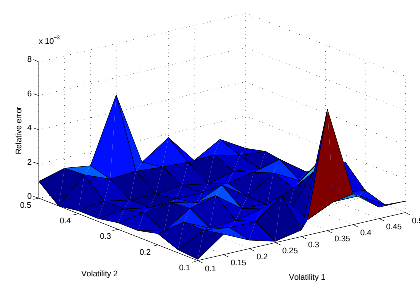
Table 4.5: *Simulated spread option values for different drifts and exercise times when $K = -10$*

T \ ρ	0.00	0.03	0.05	0.07	0.10
1 month	*19.9997*	*19.9744*	*19.9592*	*19.9419*	*19.9166*
	20.0000	19.9750	19.9584	19.9418	19.9170
	0.00%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
	20.0000	19.9750	19.0584	19.9418	19.9170
	0.00%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
2 months	*20.0002*	*19.9498*	*19.9171*	*19.8832*	*19.8342*
	19.9997	19.9503	19.9174	19.8845	19.8353
	0.00%	**0.00%**	**0.00%**	**0.01%**	**0.01%**
	20.0005	19.9507	19.9176	19.8846	19.8353
	0.00%	**0.00%**	**0.00%**	**0.01%**	**0.01%**
3 months	*20.0010*	*19.9244*	*19.8769*	*19.8266*	*19.7526*
	19.9962	19.9272	19.8799	19.8320	19.7595
	0.02%	**0.01%**	**0.02%**	**0.03%**	**0.03%**
	20.0062	19.9317	19.8823	19.8332	19.7600
	0.03%	**0.04%**	**0.03%**	**0.03%**	**0.04%**
6 months	*20.0004*	*19.8482*	*19.7533*	*19.6556*	*19.5111*
	19.9268	19.8774	19.8194	19.7465	19.6172
	0.37%	**0.15%**	**0.33%**	**0.46%**	**0.54%**
	20.0982	19.9527	19.8570	19.7623	19.6220
	0.49%	**0.53%**	**0.52%**	**0.54%**	**0.57%**
9 months	*20.0019*	*19.7776*	*19.6343*	*19.4862*	*19.2787*
	19.7342	19.8370	19.8246	19.7618	19.5981
	1.34%	**0.30%**	**0.97%**	**1.41%**	**1.66%**
	20.2942	20.0827	19.9445	19.8085	19.6086
	1.46%	**1.54%**	**1.58%**	**1.65%**	**1.71%**
12 months	*19.9999*	*19.7090*	*19.5165*	*19.3241*	*19.0503*
	19.4218	19.7787	19.8590	19.8366	19.6574
	2.89%	**0.35%**	**1.75%**	**2.65%**	**3.19%**
	20.5515	20.2779	20.1003	19.9267	19.6733
	2.76%	**2.89%**	**2.99%**	**3.12%**	**3.27%**

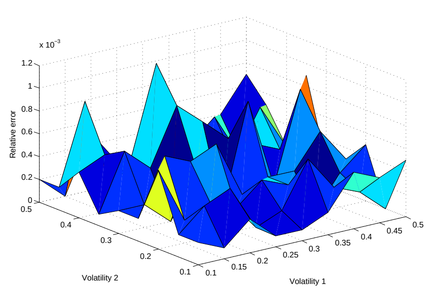
Figure 4.2: *Relative errors for various volatilities*



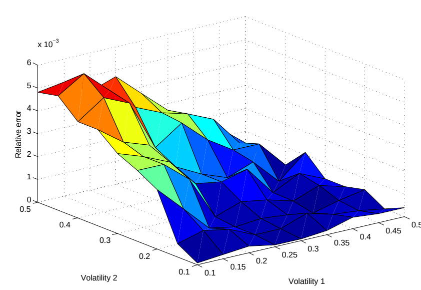
(a) *Bjerksund-Stensland,* $K = 10$



(b) *Kirk,* $K = 10$

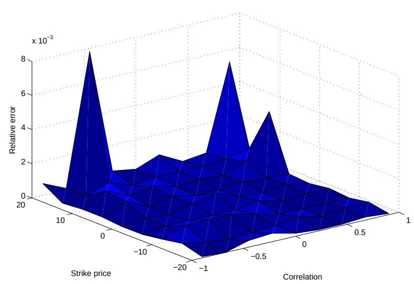


(c) *Bjerksund-Stensland,* $K = -10$

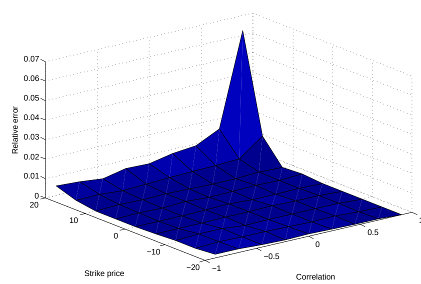


(d) *Kirk,* $K = -10$

Figure 4.3: *Relative errors for variours strike prices and correlations*

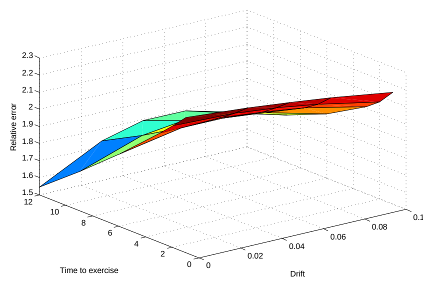


(a) *Bjerksund-Stensland*

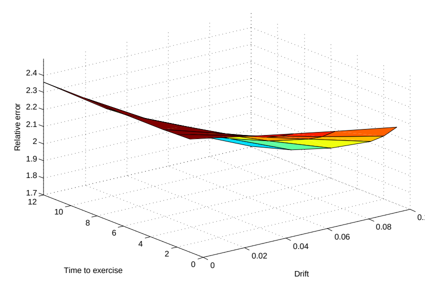


(b) *Kirk*

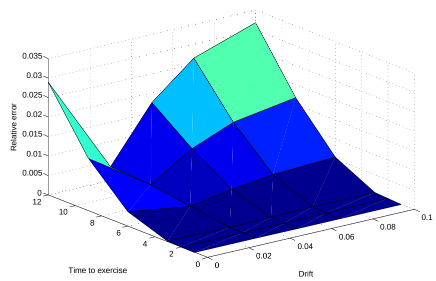
Figure 4.4: *Relative errors for variours drifts and exercising times*



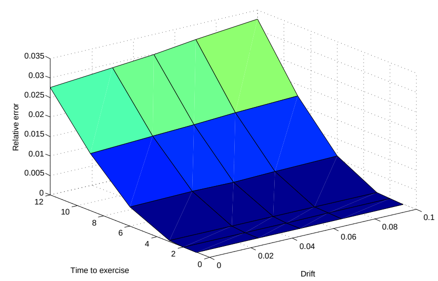
(a) *Bjerksund-Stensland,* $K = 10$



(b) *Kirk,* $K = 10$



(c) *Bjerksund-Stensland,* $K = -10$



(d) *Kirk,* $K = -10$

Chapter 5

Pricing of Trivariate Spread Options

The true price of a trivariate spread option is given in (3.1). Assuming time-independent drifts, the option price (3.34) at the initial time $t = 0$ will become

$$V(0,T) = C(0)e^{\mu_C T}\left(e^{-rT}\mathrm{E}_{\tilde{\mathbb{Q}}}\left[\max\left\{\tilde{P}(T) - h\tilde{G}(T) - 1, 0\right\}\right]\right) \tag{5.1}$$

Hence, the numerical analysis study in Chapter 4 could be put into a context of a trivariate spread option, where the strike price is 1. In this chapter we would like to analyse the exactness and applicability of the method applied in the previous chapter.

5.1 The Kirk formula revisited

As a consequence of (5.1), the updated Kirk formula for pricing trivariate spread options at time $t = 0$ is given by

$$\begin{aligned}\tilde{c}_K(0,T) &= C(0)e^{\mu_C T}\left(\tilde{P}(0)\Phi(\tilde{d}_{K,1}) - h\tilde{G}(0)\Phi(\tilde{d}_{K,2}) - e^{-rT}\cdot 1\cdot\Phi(\tilde{d}_{K,2})\right)\\ &= e^{\mu_C T}\left(P(0)\Phi(\tilde{d}_{K,1}) - hG(0)\Phi(\tilde{d}_{K,2}) - e^{-rT}C(0)\Phi(\tilde{d}_{K,2})\right)\end{aligned} \tag{5.2}$$

where

$$\tilde{d}_{K,1} = \frac{\ln\left(\frac{\tilde{P}(0)}{h\tilde{G}(0)+1\cdot e^{-rT}}\right) + \frac{1}{2}\tilde{\sigma}_K^2 T}{\tilde{\sigma}_K\sqrt{T}} \tag{5.3}$$

$$\tilde{d}_{K,2} = \tilde{d}_{K,1} - \tilde{\sigma}_K\sqrt{T} \tag{5.4}$$

$$\tilde{\sigma}_K = \sqrt{\sigma_P^2 - 2\frac{h\tilde{G}(0)}{h\tilde{G}(0)+1\cdot e^{-rT}}\rho_{PG}\sigma_P\sigma_G + \left(\frac{h\tilde{G}(0)}{h\tilde{G}(0)+1\cdot e^{-rT}}\right)^2\sigma_G^2} \tag{5.5}$$

5.2 The Bjerksund-Stensland formula revisited

Similarly, by applying (5.1), the updated Bjerksund-Stensland formula for pricing trivariate spread options at time $t = 0$ is given by

$$\begin{aligned} \tilde{c}_{BS}(0,T) &= C(0)e^{\mu_C T}\left(\tilde{P}(0)\Phi(\tilde{d}_{BS,1}) - h\tilde{G}(0)\Phi(\tilde{d}_{BS,2}) - e^{-rT}\cdot 1\cdot\Phi(\tilde{d}_{BS,3})\right) \\ &= e^{\mu_C T}\left(P(0)\Phi(\tilde{d}_{BS,1}) - hG(0)\Phi(\tilde{d}_{BS,2}) - e^{-rT}C(0)\Phi(\tilde{d}_{BS,3})\right) \end{aligned} \tag{5.6}$$

where

$$\tilde{d}_{BS,1} = \frac{\ln\left(\frac{\tilde{P}(0)}{\tilde{a}}\right) + \left(r + \frac{1}{2}\sigma_P^2 - \tilde{b}\rho_{PG}\sigma_P\sigma_G + \frac{1}{2}\tilde{b}^2\sigma_G^2\right)T}{\tilde{\sigma}_{BS}\sqrt{T}} \tag{5.7}$$

$$\tilde{d}_{BS,2} = \frac{\ln\left(\frac{\tilde{P}(0)}{\tilde{a}}\right) + \left(r - \frac{1}{2}\sigma_P^2 + \rho_{PG}\sigma_P\sigma_G + \frac{1}{2}\tilde{b}^2\sigma_G^2 - \tilde{b}\sigma_G^2\right)T}{\tilde{\sigma}_{BS}\sqrt{T}} \tag{5.8}$$

$$\tilde{d}_{BS,3} = \frac{\ln\left(\frac{\tilde{P}(0)}{\tilde{a}}\right) + \left(r - \frac{1}{2}\sigma_P^2 + \frac{1}{2}\tilde{b}^2\sigma_G^2\right)T}{\tilde{\sigma}_{BS}\sqrt{T}} \tag{5.9}$$

$$\tilde{\sigma}_{BS} = \sqrt{\sigma_P^2 - 2\tilde{b}\rho_{PG}\sigma_P\sigma_G + \tilde{b}^2\sigma_G^2} \tag{5.10}$$

with the constants $\tilde{a}$ and $\tilde{b}$ defined as

$$\tilde{a} = h\tilde{G}(0)e^{rT} + 1 \tag{5.11}$$

$$\tilde{b} = \frac{h\tilde{G}(0)e^{rT}}{\tilde{a}} \tag{5.12}$$

5.3 Numerical results

Now we are ready to construct a structured numerical analysis comparing the revisted Kirk formula (5.2) and the revisited Bjerksund-Stensland formula (5.6) in pricing of trivariate spread options along with the true values simulated by Monte Carlo method.

Assume the correlation matrix for the spot prices of power $P(t)$, gas $G(t)$ and CO_2 certificate $C(t)$ is given by

$$\boldsymbol{\rho}(t) = \begin{pmatrix} \rho_{CC} & \rho_{CP} & \rho_{CG} \\ \rho_{PC} & \rho_{PP} & \rho_{PG} \\ \rho_{GC} & \rho_{GP} & \rho_{GG} \end{pmatrix} = \begin{pmatrix} 1 & -0.7 & 0.8 \\ -0.7 & 1 & 0.1 \\ 0.8 & 0.1 & 1 \end{pmatrix}$$

then the conditions (3.6), (3.13) and (3.14) are all fulfilled, such that the lower triangular matrix (3.12) is given by

$$L(t) = \begin{pmatrix} 1 & 0 & 0 \\ -0.7 & 0.7141428 & 0 \\ 0.8 & 0.9241849 & 0.3686222 \end{pmatrix}$$

The initial values of the electricity price, the gas price and the price of emission-certificate of CO_2 are assumed to be $P(0) = 110$, $G(0) = 100$ and $C(0) = 70$.

Other relevant values are assumed to have the values

$$\begin{aligned} \mu_P &= 0.20 \qquad \sigma_P = 0.10 \\ \mu_G &= 0.15 \qquad \sigma_G = 0.15 \\ \mu_C &= 0.05 \qquad \sigma_C = 0.15 \end{aligned}$$

such that

$$\begin{aligned} \tilde{\mu}_P &= \mu_P - \mu_C = 0.15 \\ \tilde{\mu}_G &= \mu_G - \mu_C = 0.10 \\ \tilde{\sigma}_P &= \sqrt{\sigma_P^2 + \sigma_C^2 - 2\sigma_P\sigma_C\rho_{CP}} = 0.2313007 \\ \tilde{\sigma}_G &= \sqrt{\sigma_G^2 + \sigma_C^2 - 2\sigma_G\sigma_C\rho_{CG}} = 0.0948683 \end{aligned}$$

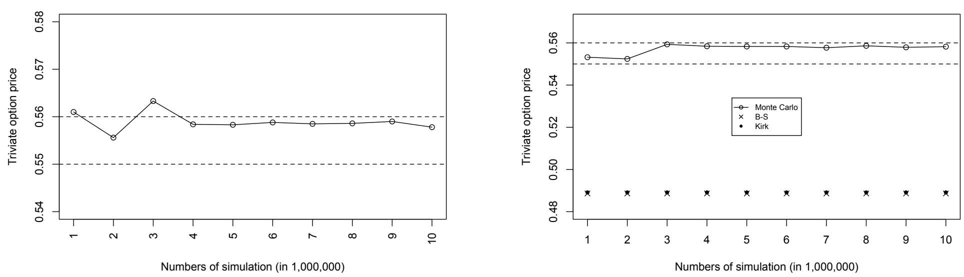
The strike price is assumed to be $K = 1$ in order to be able to fit into the expression of the trivariate option price (3.34). Assume the heating rate to be $h = 1$ for this moment (we are going to vary this parameter later), the exercise time $T = 1$ (year), and the interest rate $r = 0.05$. Applying these values into Monte Carlo method for equations (3.15) to (3.23) and (3.27) to (3.34) with increasing permutations, we wish to find out how many trials are necessary to ensure a given precision of the price. Figure 5.1(a) shows the results from Monte Carlo method from $1,000,000$ to $10,000,000$ permutations, from which we see that $4,000,000$ permutations are necessary to ensure two decimal places of accuracy.

However, when the results simulated by Monte Carlo method are plotted together with the approximated trivariate option price according to the revisited Kirk formula (5.2) and the revisited Bjerksund-Stensland formula (5.6), large variations occur, as it is shown in Figure 5.1(b). Note that the intervals of the vertical axes (trivariate option price) are different in each of the two subfigures. Given the heating rate $h = 1$, it is worth mentioning that Monte Carlo method with $4,000,000$ permutations produces a trivariate option value of 0.5584, while the revisited Kirk formula (5.2) and the revisited Bjerksund-Stensland formula (5.6) gives 0.4891 and 0.4886 respectively, with a relative error of 12.41% and

12.50%; both are relatively large.

In the rest of this section, we will compare the accuracy of the revisited Kirk formula and the revisited Bjerksund-Stensland formula along with the results simulated by Monte Carlo method. In Subsection 5.3.1 we will look at six different values of heating rates, while in Subsection 5.3.2 we will vary the initial price of $C(0)$ with respect to the initial price of power $P(0)$ and the initial price of gas $G(0)$. The purpose is to investigate their effect, if they had any, on pricing the trivariate spread option.

Figure 5.1: *Monte-Carlo benchmark for trivariate spread option*



(a) *Monte-Carlo* (b) *Monte-Carlo and approximations*

5.3.1 Various heating rates

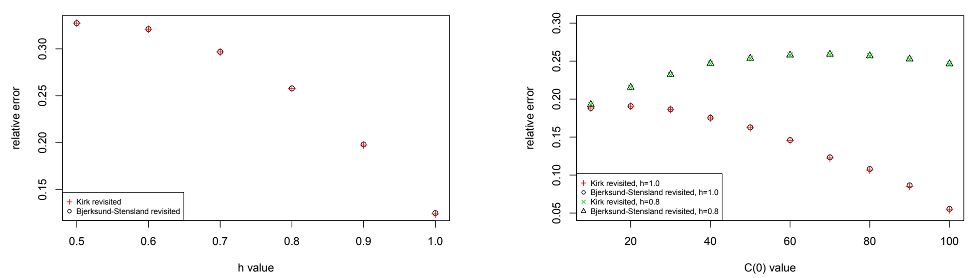
In this subsection we will discuss the effect on pricing of trivariate spread option caused by different values of heating rates. The price of the trivariate option is produced according to the two abovementioned approximation methods. The results are shown in Table 5.1, where the first column lists six different heating rates, the second column (in italics) gives the results from Monte Carlo method, the third column gives the approximated prices of trivariate option produced by the revisited Bjerksund-Stensland formula (5.6), the fourth column gives the approximated prices of trivaraite options produced by the revisited Kirk formula (5.2), and the last two columns (in bold) give the relative errors due to those two formulas, respectively.

We see that as the heating rate decreases, both the Monte Carlo method and the two approximation methods are increasing, but at the same time, both relative errors are increasing as well, indicating large variations and uncertainty for both of the methods. In other words, as Figure 5.2(a) indicates, relative errors produced by two approximations are quite close, and they are decreasing when the values of heating rate are growing towards 1.0.

Table 5.1: *Simulated trivariate spread option values for different heating rates*

h	MC	B-S	Kirk	error_{BS}	error_K
1.0	*0.5584*	0.4886	0.4891	**12.50%**	**12.41%**
0.9	*1.1118*	0.8916	0.8922	**19.81%**	**19.75%**
0.8	*2.1527*	1.5976	1.5982	**25.79%**	**25.76%**
0.7	*3.9786*	2.7976	2.7980	**29.68%**	**29.67%**
0.6	*7.0114*	4.7608	4.7611	**32.10%**	**32.09%**
0.5	11.6325	7.8244	7.8245	**32.74%**	**32.74%**

Figure 5.2: *Relative errors for pricing the trivariate spread option*



(a) *various heating rates* (b) *various $C(0)$'s*

5.3.2 Various $C(0)$ values regarding $P(0)$ and $G(0)$

Now we would like to look into the effect on pricing of trivariate spread option when the initial value of the price of emission-certificates of carbon dioxide $C(0)$ is varied comparing to the initial value of the power price $P(0)$ and that of the gas price $G(0)$. Given $P(0) = 110$ and $G(0) = 100$, two values of heating rate, $h = 1.0$ and $h = 0.8$, are chosen. And for each of them, the initial price of emission-certificates of carbon dioxide $C(0)$ is varying from 10 to 100 with a step of 10. The results are shown in Table 5.2(a) and Table 5.2(b) where the first column gives ten initial values of $C(0)$, the second and third columns give the relative value of $P(0)$ and $G(0)$ relatively to each $C(0)$, the five columns to the right give respectively the results simulated by Monte Carlo method (in italics), the approximated prices of trivariate spread options produced by the revisted Bjerksund-Stensland formula (5.6) and that by the revisited Kirk formula (5.2) along with their relative errors (in bold), for each value of $C(0)$.

The tables imply that when $C(0)$ gets larger, the relative value $\frac{P(0)}{C(0)}$ and $\frac{G(0)}{C(0)}$ become smaller and closed to the strike price $K = 1$, causing both the simulated option price and the two approximated option prices to decrease. We can also conclude from Table 5.2(a) that when $h = 1.0$, the relative errors due to the two approximation methods become smaller as $C(0)$ beomes larger, while from Table 5.2(b) we see that when $h = 0.8$, the relative errors still remainl large. These phenomena are also captured in Figure 5.2(b), that the relative errors for both approximation methods tend to decrease for a higher value of heating rate, but tend to increase for some time and then stay high for a lower one.

Table 5.2: *Simulated trivariate spread option values for different $C(0)$ comparing to $P(0)$ and $G(0)$*

(a) *when* $h = 1.0$

$C(0)$	$\frac{P(0)}{C(0)}$	$\frac{G(0)}{C(0)}$	MC	B-S	Kirk	error_{BS}	error_K
10	11.00	10.00	*13.8255*	11.2216	11.2217	**18.83%**	**18.83%**
20	5.50	5.00	*8.9355*	7.2304	7.2311	**19.08%**	**19.07%**
30	3.67	3.33	*5.5026*	4.4766	4.4777	**18.65%**	**18.63%**
40	2.75	2.50	*3.2471*	2.6771	2.6783	**17.55%**	**17.52%**
50	2.20	2.00	*1.8565*	1.5543	1.5553	**16.28%**	**16.22%**
60	1.83	1.67	*1.0310*	0.8804	0.8812	**14.61%**	**14.53%**
70	1.57	1.43	*0.5574*	0.4886	0.4891	**12.34%**	**12.25%**
80	1.38	1.25	*0.2990*	0.2667	0.2671	**10.80%**	**10.67%**
90	1.22	1.11	*0.1573*	0.1437	0.1439	**8.65%**	**8.52%**
100	1.10	1.00	*0.0811*	0.0766	0.0767	**5.55%**	**5.43%**

(b) *when* $h = 0.8$

$C(0)$	$\frac{P(0)}{C(0)}$	$\frac{G(0)}{C(0)}$	MC	B-S	Kirk	error_{BS}	error_K
10	11.00	10.00	*29.9682*	24.1874	24.1868	**19.29%**	**19.29%**
20	5.50	5.00	*21.7978*	17.1022	17.1018	**21.54%**	**21.54%**
30	3.67	3.33	*15.0126*	11.5241	11.5242	**23.24%**	**23.24%**
40	2.75	2.50	*9.8607*	7.4259	7.4265	**24.69%**	**24.69%**
50	2.20	2.00	*6.1612*	4.5981	4.5989	**25.37%**	**25.36%**
60	1.83	1.67	*3.7065*	2.7504	2.7511	**25.80%**	**25.78%**
70	1.57	1.43	*2.1560*	1.5976	1.5982	**25.90%**	**25.87%**
80	1.38	1.25	*1.2189*	0.9056	0.9060	**25.70%**	**25.67%**
90	1.22	1.11	*0.6733*	0.5031	0.5033	**25.28%**	**25.25%**
100	1.10	1.00	*0.3649*	0.2750	0.2751	**24.64%**	**24.61%**

Chapter 6

Hedging of Bivariate and Trivariate Spread Options

In most of the liberalized markets, traders decide what products to trade, how to trade them and in which combinations. Their strategies could be originated from a number of considerations: to get cash; to buy or sell the ownership of physical products (here electricity or natural gas); or to hedge.

Hedging risk may sound like a cautious approach to investing, but it is often the most aggressive investors (speculators) who hedge to manage the risk of their physical positions. By reducing the risk in one part of a portfolio, an investor can often take more risk elsewhere, increasing her absolute returns while putting less capital at risk in each individual investment.

In this chapter, we first derive the delta-hedge parameters for bivariate spread options under the Kirk formula (4.3) and the Bjerksund-Stensland formula (4.7) in Section 6.1, and for trivariate spread options under the revisited Kirk formula (5.2) and the revisited Bjerksund-Stensland formula (5.6) in Section 6.2. Numerical results using Monte Carlo method are presented in Section 6.3, where it is shown that both of the two formulas perform very well in hedging bivariate spread options but poorly for the trivariate spread options.

6.1 Delta-hedge parameters for bivariate spread options

As mentioned in Section 4.3, for spread options, when $K = 0$, both the Kirk formula and the Bjerksund-Stensland formula converge to the same one, the Margrabe formula, which returns the true price for the spread option at time 0,

$$V_M(0,T) = P(0)\Phi(d_1) - G(0)\Phi(d_2) \tag{6.1}$$

where

$$d_1 = \frac{\ln\left(\frac{P(0)}{G(0)}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \tag{6.2}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{6.3}$$

$$\sigma = \sqrt{\sigma_P^2 - 2\rho_{PG}\sigma_P\sigma_G + \sigma_G^2} \tag{6.4}$$

We differentiate (6.1) with respect to $P(0)$ and $G(0)$ to get the two delta-hedge parameters:

$$\Delta_{M,1} = \frac{\partial V_M(0,T)}{\partial P(0)} = \Phi(d_1) + \frac{P(0)\phi(d_1) - G(0)\phi(d_2)}{P(0)\sigma\sqrt{T}} \tag{6.5}$$

$$\Delta_{M,2} = \frac{\partial V_M(0,T)}{\partial G(0)} = -\Phi(d_2) - \frac{P(0)\phi(d_1) - G(0)\phi(d_2)}{G(0)\sigma\sqrt{T}} \tag{6.6}$$

We are going to compare (6.5) and (6.6) with the true delta parameters of the bivariate spread option by inserting (3.8) and (3.9) into the conditional expectation,

$$\begin{aligned} V_b(0,T) &= e^{-rT}\mathrm{E}_{\mathcal{Q}}\left[\max\{P(T) - G(T), 0\}\right] \\ &= e^{-rT}\mathrm{E}_{\mathcal{Q}}\left[\max\left\{P(0)e^{X_P} - G(0)e^{X_G}, 0\right\}\right] \end{aligned} \tag{6.7}$$

where

$$X_P = \int_0^T \left(\mu_P(s) - \frac{1}{2}\sigma_P^2(s)\right)\mathrm{d}s + \int_0^T \sigma_P(s)\mathrm{d}W_P(s) \tag{6.8}$$

$$X_G = \int_0^T \left(\mu_G(s) - \frac{1}{2}\sigma_G^2(s)\right)\mathrm{d}s + \int_0^T \sigma_G(s)\mathrm{d}W_G(s) \tag{6.9}$$

The delta-hedge parameters are obtained by diffentiating (6.7) with respect to $P(0)$ and $G(0)$ respectively:

$$\Delta_{b,1} = \frac{\partial V_b(0,T)}{\partial P(0)} = e^{-rT}\mathrm{E}_{\mathcal{Q}}\left[\mathbb{1}\left\{P(0)e^{X_P} - G(0)e^{X_G} > 0\right\} \cdot e^{X_P}\right] \tag{6.10}$$

$$\Delta_{b,1} = \frac{\partial V_b(0,T)}{\partial G(0)} = -e^{-rT}\mathrm{E}_{\mathcal{Q}}\left[\mathbb{1}\left\{P(0)e^{X_P} - G(0)e^{X_G} > 0\right\} \cdot e^{X_G}\right] \tag{6.11}$$

6.2 Delta-hedge parameters for trivariate spread options

The updated Kirk formula for trivariate spread options is given by (5.2). Assuming $\tilde{\sigma}_K$ to be constant, we obtain three delta-hedge parameters from the Kirk approximation,

$$
\begin{aligned}
\Delta_{K,1} &= \frac{\partial \tilde{c}_K(0,T)}{\partial P(0)} \\
&= e^{\mu_C T}\left[\Phi(\tilde{d}_{K,1}) + \frac{P(0)\phi(\tilde{d}_{K,1}) - hG(0)\phi(\tilde{d}_{K,2}) - e^{-rT}C(0)\phi(\tilde{d}_{K,2})}{P(0)\tilde{\sigma}_K\sqrt{T}}\right]
\end{aligned} \tag{6.12}
$$

$$
\begin{aligned}
\Delta_{K,2} &= \frac{\partial \tilde{c}_K(0,T)}{\partial G(0)} \\
&= -e^{\mu_C T}\left[h\Phi(\tilde{d}_{K,2}) + \frac{P(0)\phi(\tilde{d}_{K,1}) - hG(0)\phi(\tilde{d}_{K,2}) - e^{-rT}C(0)\phi(\tilde{d}_{K,2})}{\left(h\tilde{G}(0) + e^{-rT}\right)\tilde{\sigma}_K\sqrt{T}}\right]
\end{aligned} \tag{6.13}
$$

$$
\begin{aligned}
\Delta_{K,3} &= \frac{\partial \tilde{c}_K(0,T)}{\partial C(0)} \\
&= -e^{\mu_C T}\left[e^{-rT}\Phi(\tilde{d}_{K,2}) + \frac{P(0)\phi(\tilde{d}_{K,1}) - hG(0)\phi(\tilde{d}_{K,2}) - e^{-rT}C(0)\phi(\tilde{d}_{K,2})}{\tilde{\sigma}_K\sqrt{T}}\cdot\right. \\
&\qquad\left.\left(\frac{1}{C(0)} - \frac{1}{h\tilde{G}(0) + e^{-rT}}\cdot\frac{h\tilde{G(0)}}{C(0)}\right)\right]
\end{aligned} \tag{6.14}
$$

with $\tilde{d}_{K,1}$, $\tilde{d}_{K,2}$ and $\tilde{\sigma}_K$ given as in equation (5.3), (5.4) and (5.5), respectively.

The updated Bjerksund-Stensland formula for trivariate spread options is given by (5.6). Assuming $\tilde{a}$, $\tilde{b}$ and $\tilde{\sigma}_{BS}$ to be constants, we obtain three delta-hedge parameters from the Bjerksund-Stensland approximation,

$$\begin{aligned}\Delta_{BS,1} &= \frac{\partial \tilde{c}_{BS}(0,T)}{\partial P(0)} \\ &= e^{\mu_C T}\left[\Phi(\tilde{d}_{BS,1}) + \frac{P(0)\phi(\tilde{d}_{BS,1}) - hG(0)\phi(\tilde{d}_{BS,2}) - e^{-rT}C(0)\phi(\tilde{d}_{BS,3})}{P(0)\tilde{\sigma}_{BS}\sqrt{T}}\right]\end{aligned} \tag{6.15}$$

$$\begin{aligned}\Delta_{BS,2} &= \frac{\partial \tilde{c}_{BS}(0,T)}{\partial G(0)} \\ &= -e^{\mu_C T}\left[h\Phi(\tilde{d}_{BS,2}) + \frac{P(0)\phi(\tilde{d}_{BS,1}) - hG(0)\phi(\tilde{d}_{BS,2}) - e^{-rT}C(0)\phi(\tilde{d}_{BS,3})}{(hG(0)e^{rT}+1)\tilde{\sigma}_{BS}\sqrt{T}}\right]\end{aligned} \tag{6.16}$$

$$\begin{aligned}\Delta_{BS,3} &= \frac{\partial \tilde{c}_{BS}(0,T)}{\partial C(0)} \\ &= -e^{\mu_C T}\left[e^{-rT}\Phi(\tilde{d}_{BS,3}) + \frac{P(0)\phi(\tilde{d}_{BS,1}) - hG(0)\phi(\tilde{d}_{BS,2}) - e^{-rT}C(0)\phi(\tilde{d}_{BS,3})}{\tilde{\sigma}_K\sqrt{T}}\cdot\right. \\ &\qquad \left.\left(\frac{1}{C(0)} - \frac{1}{h\tilde{G}(0)+e^{-rT}}\cdot\frac{h\tilde{G(0)}}{C(0)}\right)\right]\end{aligned} \tag{6.17}$$

with $\tilde{d}_{BS,1}$, $\tilde{d}_{BS,2}$, $\tilde{d}_{BS,3}$, $\tilde{\sigma}_{BS}$, $\tilde{a}$ and $\tilde{b}$ given as in equations (5.7) to (5.12).

We would like to compare the three delta-hedge parameters (6.12), (6.13) and (6.14) obtained from the Kirk formula with those (6.15), (6.16) and (6.17) from the Bjerksund-Stensland formula together with the ones obtained directly from the conditional expectation (3.1). At $t = 0$, the price of the trivariate spread option is given as

$$V_t(0,T) = e^{-rT}\mathrm{E}_{\mathcal{Q}}\left[\max\left\{P(T) - G(t) - C(0), 0\right\}\right] \tag{6.18}$$

Applying (3.8) and (3.9), we may rewrite (6.18) as

$$V_t(0,T) = e^{-rT}\mathrm{E}_{\mathcal{Q}}\left[\max\left\{P(0)e^{X_P} - G(0)e^{X_G} - C(0)e^{X_C}, 0\right\}\right] \tag{6.19}$$

where X_P and X_G are given as in (6.8) and (6.9), and X_C is given as

$$X_C = \int_0^T \left(\mu_C(s) - \frac{1}{2}\sigma_C^2(s)\right)\mathrm{d}s + \int_0^T \sigma_C(s)\mathrm{d}W_C(s) \tag{6.20}$$

Therefore, we could differentiate (6.19) with respect to $P(0)$, $G(0)$ and $C(0)$ to get the true delta-hedge parameters

$$\begin{aligned}
\Delta_{t,1} &= \frac{\partial V_t(0,T)}{\partial P(0)} \\
&= e^{-rT}\mathrm{E}_{\mathcal{Q}}\left[\mathbb{1}\left\{P(0)e^{X_P} - hG(0)e^{X_G} - C(0)e^{X_C} > 0\right\} \cdot e^{X_P}\right] \quad (6.21)\\
\Delta_{t,2} &= \frac{\partial V_t(0,T)}{\partial G(0)} \\
&= -e^{-rT}\mathrm{E}_{\mathcal{Q}}\left[\mathbb{1}\left\{P(0)e^{X_P} - G(0)e^{X_G} - C(0)e^{X_C} > 0\right\} \cdot e^{X_G}\right] \quad (6.22)\\
\Delta_{t,3} &= \frac{\partial V_t(0,T)}{\partial C(0)} \\
&= -e^{-rT}\mathrm{E}_{\mathcal{Q}}\left[\mathbb{1}\left\{P(0)e^{X_P} - G(0)e^{X_G} - C(0)e^{X_C} > 0\right\} \cdot e^{X_C}\right] \quad (6.23)
\end{aligned}$$

6.3 Numerical results

Applying the same values as in the numerical analysis for bivariate spread option in Section 4.3, i.e. $P(0) = 110$, $G(0) = 100$, $\sigma_P = 0.10$, $\sigma_G = 0.15$, $\rho_{PG} = 0.1$, $T = 1$, $r = 0.05$ and $h = 1$, we get the results in Table 6.1(a), showing the delta-hedge parameters approximated by the Kirk or the Bjerksund-Stensland formula as well as the true parameters simulated by Monte Carlo method (using $4,000,000$ permutations). We see that the results from the approximations are quite satisfying, with an accuracy of three decimal places.

Applying the same values as in the numerical analysis for trivariate spread option in Section 5.3, i.e. $P(0) = 110$, $G(0) = 100$, $C(0) = 70$, $\sigma_P = 0.10$, $\sigma_G = 0.15$, $\sigma_C = 0.15$, $\rho_{PG} = 0.1$, $\rho_{CG} = 0.8$, $\rho_{CP} = -0.7$, $T = 1$, $r = 0.05$ and $h = 1$, we get the results in Table 6.1(b), showing the delta-hedge parameters approximated by the revisited Kirk formula and the revisited Bjerksund-Stensland formula as well as the true parameters simulated by Monte Carlo method (using $4,000,000$ permutations). This time we see that, although the two approximations produce close results, both of them differ greatly from the results simulated by Monte Carlo method, with relative errors larger than 130%.

Table 6.1: *Delta-hedge parameters*

(a) *for bivariate spread option*

	Δ_1	Δ_2
Kirk or Bjerksund-Stensland	0.7392	-0.6805
Monte Carlo	*0.7394*	*-0.6807*

(b) *for trivariate spread options*

	Δ_1	Δ_2	Δ_3
Kirk	0.0501	-0.0302	-0.0287
Bjerksund-Stensland	0.0501	-0.0286	-0.0323
Monte Carlo	*0.0206*	*-0.0126*	*-0.0121*

Chapter 7

Conclusions and Discussions

7.1 Conclusions

Pricing of spread options with non-zero strikes

Taking into account of the fact that Energy Market practitioners are currently making use of alternatives to the Monte Carlo method, we deemed it important to examine the exactness of the two most popular closed-form formulas. We compared the Bjerksund-Stensland formula and the Kirk formula in pricing and hedging spread options in a trivariate financial world consisting of the price of power (electricity) P, the price of gas G and the price of emission-certificate C. We used trivariate geometric Brownian motions for modelling the dynamics for the three price dynamics, and we applied Monte Carlo method as our benchmark for simulating the accurate results. In this research, $6,000,000$ permutations were used for the bivariate case and $4,000,000$ for the trivariate one.

We made a structured numerical analysis with respect to six parameters (two volatilities, strike price, correlation coefficient, interest rate and time to exercise) for bivariate spread options. When volatilities were varied, we found that the Bjerksund-Stensland formula was slightly better than the Kirk formula, in the sense that the Bjerksund-Stensland formula produced smaller relative errors in 50 out of 81 combinations for a positive strike price and 65 out of 81 combinations for a negative one. When strike prices and correlation coefficients were varied, we found that the Bjerksund-Stensland formula turned out to be better as well, as it produced smaller relative errors in 73 out of 81 combinations of different strike prices and correlations, especially for large positive values of strike prices and correlation coefficients close to 1.0. However, when the interest rate and time of exercise were varied, we found that neither the Bjerksund-Stensland formula, nor the Kirk formula were satisfying when negative strike prices were given, even though the Bjerksund-Stensland formula did achieve slightly smaller relative errors than the Kirk formula in 28 out of 30 combinations; but both formula performed very pooly when positive strike prices were given.

We have proved in Chapter 3 (equation 3.34) that pricing spread options in a trivariate financial world consisting of P, G, C can be tranformed to pricing spread options in a bivariate world consisting of P, G, 1, where 1 represents the non-zero strike price $K = 1$. Hence, we are able to adapt a similar analysis to a three-dimensional case, as it is often the case in EM operations. Different values of heating rate and initial emission-certificate price were investigated for their effect on pricing trivariate spread options. When heating rates were varied, we found that both of the two formulas produced large relative errors, up to 32.74% when the heating rate $h = 0.5$ was given, and that the relative errors tended to decrease when the heating rates increased from $h = 0.5$ to $h = 1.0$. Then when the initial price of emission-certificate $C(0)$ were varied, the relative price of power $P(0)/C(0)$ and that of gas $G(0)/C(0)$ were also varied, and large relative errors occurred. Therefore we conclude that the Bjerksund-Stensland formula does not fit well in pricing trivariate spread options with non-zero strikes. Although the wish to rely on quicker closed-form formulas is understandable, we believe that it would be better for practitioners, for the time being, to keep using other well tried numerical methods, such as Monte Carlo method, even if its disadvantage is to be computationally slow.

Hedging of spread options with non-zero strikes

For the spread options and trivariate spread options, we derived the delta-hedge parameters with respect to the price of electricity, the price of gas, and the price of emission-certificate. Starting from the same initial values as in the paper of Bjerksund and Stensland (2011) [24], we compared the values of delta-hedge parameters derived from the Kirk formula and Bjerksund-Stensland formula, together with the results simulated by the Monte Carlo method with $4,000,000$ permutations.

For bivariate spread options, since the value of K does not exist any more (or just consider $K = 0$), both the Kirk formula and the Bjerksund-Stensland formula transform into the Margrabe formula, from which we obtained satisfactory delta-hedge parameters with respect to $P(0)$ and $G(0)$, with an accuracy of three decimal places. For trivariate spread options, however, the delta-hedge parameters with respect to $P(0)$, $G(0)$ and $C(0)$ produced by both formulas were far from the results simulated by the Monte Carlo method, with relative errors of more than 130%. Hence, we conclude that the Bjerksund-Stensland formula is indeed as good as the Kirk formula in hedging bivariate spread options. Nonetheless, for hedging trivariate spread options, neither of them should be used by real investors in energy markets. Numerical methods, such as Monte Carlo, remain today a better tool.

7.2 Outlook

In this research we focused on Monte Carlo simulation to valuate spread call option. For example, it takes about 3'10", 3'15" and 1'15" respectively to generate the prices for spread option (with an accuracy of two decimal places) pairwisely with respect to two volatilities, strike price and correlation coefficients, interest rate and exercise time of the spread call option, using 6,000,000 permutations in Monte Carlo algorithms (see appendices A.2 - A.4) implemented in R (version 2.15.1 GUI 1.52 Leopard build 32-bit) in this research. From this point onward, it may be possible to introduce some improvements to the Monte Carlo algorithms which could possibly produce quicker and precise results, for example through the variance reduction method [10]. One may also want to look at other numerical methods, such as the Fourier transform method (see also [19], [23] and [35]).

As previously mentioned, it has always been appealing to find a closed-form formula, which is both quick and robust in pricing and hedging spread options in energy markets when the strike price differs from zero. Future works may look at the possibilities of modifying the existing closed-form formulas (such as the Kirk formula, the Bjerksund-Stensland formula, or others), or creating a new mathematical closed-form formula.

Appendix A

R scripts

All the R codes used for computation and simulation in this project are listed as the following.

A.1 R codes for the benchmark of pricing bivariate spread option

```
# Initial values of parameters:
s1=110; s2=100;
sigma1=0.10; sigma2=0.15;
K=0; rho=0.5; delta.t=1; r=0.05

# Monte Carlo method:
MC=3e+06
call.price.benchmark=rep(0,10)
for(k in 1:10)
{
  N=MC*k
  eps1=rnorm(N)
  eps2=rho*eps1+sqrt(1-rho^2)*rnorm(N)
  S1=s1*exp((r-sigma1^2/2)*delta.t+sigma1*sqrt(delta.t)*eps1)
  S2=s2*exp((r-sigma2^2/2)*delta.t+sigma2*sqrt(delta.t)*eps2)
  call.price=exp(-r*delta.t)*pmax(S1-S2-K,0)
  call.price.benchmark[k]=mean(call.price)
}
print(round(call.price.benchmark,4))

# The Bjerksund-Stensland formula:
BS<-function(s1,s2,K,sigma1,sigma2,rho,delta.t,r)
```

```
{
  a=s2*exp(r*delta.t)+K
  b=s2*exp(r*delta.t)/a
  sigma=sqrt(sigma1^2-2*b*rho*sigma1*sigma2+b^2*sigma2^2)
  d1=(log(s1/a)+(r+sigma1^2/2-b*rho*sigma1*sigma2+b^2*sigma2^2/2)*delta.t)/(sigma*sqrt(delta.t))
  d2=(log(s1/a)+(r-sigma1^2/2+rho*sigma1*sigma2+b^2*sigma2^2/2- b*sigma2^2)*delta.t)/(sigma*sqrt(delta.t))
  d3=(log(s1/a)+(r-sigma1^2/2+b^2*sigma2^2/2)*delta.t)/(sigma*sqrt(delta.t))
  c=s1*pnorm(d1)-s2*pnorm(d2)-exp(-r*delta.t)*K*pnorm(d3)
}
testBS=BS(s1,s2,K,sigma1,sigma2,rho,delta.t,r)
print(round(testBS,4))

# The Kirk formula:
Kirk<-function(s1,s2,K,sigma1,sigma2,rho,delta.t,r)
{
  sigma.k=sqrt(sigma1^2-2*s2*exp(r*delta.t)/(s2*exp(r*delta.t)+K)*rho*sigma1*sigma2+(s2*exp(r*delta.t)/(s2*exp(r*delta.t)+K)*sigma2)^2)
  d1=(r*delta.t+log(s1/(s2*exp(r*delta.t)+K))+sigma.k^2/2*delta.t)/(sigma.k*sqrt(delta.t))
  d2=d1-sigma.k*sqrt(delta.t)
  c=s1*pnorm(d1)-s2*pnorm(d2)-exp(-r*delta.t)*K*pnorm(d2)
}
testKirk=Kirk(s1,s2,K,sigma1,sigma2,rho,delta.t,r)
print(round(testKirk,4))

# Plot all the results in one graph:
xx=(1:10)*MC
yy=round(as.numeric(call.price.benchmark),4)
plot(xx,yy,"o",xlab="Numbers of simulation (in 1,000,000)",ylab="Bivariate option price",ylim=c(11.90,11.92),xaxt="n")
axis(1,at=(1:10)*MC,las=2,labels=c("3","6","9","12","15","18","21","24","27","30"))
abline(h=11.9100,lty=2)
abline(h=11.9199,lty=2)
points(xx,rep(testBS,10),pch=4)
points(xx,rep(testKirk,10),pch=5)
legend("bottomright",c("Monte Carlo","B-S","Kirk"),cex=0.7,lty=c(1,0,0),pch=c(1,4,5))
```

A.2 R codes for pricing bivariate spread option for various volatilities

```
# Monte Carlo method:
sigma1.list=c("0.10","0.15","0.20","0.25","0.30","0.35","0.40","0.45","0.50")
sigma2.list=c("0.10","0.15","0.20","0.25","0.30","0.35","0.40","0.45","0.50")
vol=matrix(rep(0,81),nrow=9,dimnames=list(sigma1.list,sigma2.list))
N=6e+06
K=10
for(i in 1:9)
{
        sigma2=i*0.05+0.05
        for(j in 1:9)
        {
                sigma1=j*0.05+0.05
                eps1=rnorm(N)
                eps2=rho*eps1+sqrt(1-rho^2)*rnorm(N)
                S1=s1*exp((r-sigma1^2/2)*delta.t+sigma1*sqrt(delta.t)*eps1)
                S2=s2*exp((r-sigma2^2/2)*delta.t+sigma2*sqrt(delta.t)*eps2)
                call.price=pmax(S1-S2-K,0)*exp(-r*delta.t)
                vol[i,j]=mean(call.price)
        }
}
print(round(vol,4))

# The Bjerksund-Stensland formula:
vol.BS=matrix(rep(0,81),nrow=9,dimnames=list(sigma1.list,sigma2.list))
BS.vol<-function(s1,s2,K,rho,delta.t,r)
{
        a=s2*exp(r*delta.t)+K
        b=s2*exp(r*delta.t)/a
        for(i in 1:9)
        {
                sigma2=i*0.05+0.05
                for(j in 1:9)
                {
```

```
			sigma1=j*0.05+0.05
			sigma=sqrt(sigma1^2-2*b*rho*sigma1*sigma2+b^2*sigma2^2)
			d1=(log(s1/a)+(r+sigma1^2/2-b*rho*sigma1*sigma2+b^2*sigma2^2/2)*delta.t)/(sigma*sqrt(delta.t))
			d2=(log(s1/a)+(r-sigma1^2/2+rho*sigma1*sigma2+b^2*sigma2^2/2-b*sigma2^2)*delta.t)/(sigma*sqrt(delta.t))
			d3=(log(s1/a)+(r-sigma1^2/2+b^2*sigma2^2/2)*delta.t)/(sigma*sqrt(delta.t))
			c=s1*pnorm(d1)-s2*pnorm(d2)-exp(-r*delta.t)*K*pnorm(d3)
			vol.BS[i,j]=c
		}
	}
	return(vol.BS)
}
testBS.vol=BS.vol(s1=110,s2=100,K=10,rho=0.5,delta.t=1,r=0.05)
testBS.vol=BS.vol(s1=110,s2=100,K=-10,rho=0.5,delta.t=1,r=0.05)
print(round(testBS.vol,4))

# The Kirk formula:
vol.Kirk=matrix(rep(0,81),nrow=9,dimnames=list(sigma1.list,sigma2.list))
Kirk.vol<-function(s1,s2,K,rho,delta.t,r)
{
	for(i in 1:9)
	{
		sigma2=i*0.05+0.05
		for(j in 1:9)
		{
			sigma1=j*0.05+0.05
			sigma.k=sqrt(sigma1^2-2*s2*exp(r*delta.t)/(s2*exp(r*delta.t)+K)*rho*sigma1*sigma2+(s2*exp(r*delta.t)/(s2*exp(r*delta.t)+K)*sigma2)^2)
			d1=(r*delta.t+log(s1/(s2*exp(r*delta.t)+K))+sigma.k^2/2*delta.t)/(sigma.k*sqrt(delta.t))
			d2=d1-sigma.k*sqrt(delta.t)
			c=s1*pnorm(d1)-s2*pnorm(d2)-exp(-r*delta.t)*K*pnorm(d2)
```

```
                    vol.Kirk[i,j]=c
             }
      }
      return(vol.Kirk)
}
testKirk.vol=Kirk.vol(s1=110,s2=100,K=10,rho=0.5,delta.t=1,r=0.05)
testKirk.vol=Kirk.vol(s1=110,s2=100,K=-10,rho=0.5,delta.t=1,r
   =0.05)
print(round(testKirk.vol,4))
```

A.3 R codes for pricing bivariate spread option for various strike prices and correlations

```
# Monte Carlo method:
K.list=c("-20","-15","-10","-5","0","5","10","15","20")
rho.list=c("-0.9","-0.7","-0.5","-0.3","0.0","0.3","0.5","0.7","
   0.9")
K.num=seq(-20,20,5)
rho.num=c(-0.9,-0.7,-0.5,-0.3,0,0.3,0.5,0.7,0.9)
K.rho<-matrix(rep(0,81),nrow=9,dimnames=list(K.list,rho.list))
N=6e+06
for(i in 1:9)
{
      K=K.num[i]
      for(j in 1:9)
      {
             rho=rho.num[j]
             eps1=rnorm(N)
             eps2=rho*eps1+sqrt(1-rho^2)*rnorm(N)
             S1=s1*exp((r-sigma1^2/2)*delta.t+sigma1*sqrt(delta.t
                )*eps1)
             S2=s2*exp((r-sigma2^2/2)*delta.t+sigma2*sqrt(delta.t
                )*eps2)
             call.price=pmax(S1-S2-K,0)*exp(-r*delta.t)
             K.rho[i,j]=mean(call.price)
             print(round(K.rho,4))
      }
}
print(round(K.rho,4))

# The Bjerksund-Stensland formula:
K.rho.BS<-matrix(rep(0,81),nrow=9,dimnames=list(K.list,rho.list))
```

```
BS.K.rho<-function(s1,s2,sigma1,sigma2,delta.t,r)
{
	for(i in 1:9)
	{
		K=K.num[i]
		a=s2*exp(r*delta.t)+K
		b=s2*exp(r*delta.t)/a
		for(j in 1:9)
		{
			rho=rho.num[j]
			sigma=sqrt(sigma1^2-2*b*rho*sigma1*sigma2+b^2
			   *sigma2^2)
			d1=(log(s1/a)+(r+sigma1^2/2-b*rho*sigma1*
			   sigma2+b^2*sigma2^2/2)*delta.t)/(sigma*
			   sqrt(delta.t))
			d2=(log(s1/a)+(r-sigma1^2/2+rho*sigma1*sigma2
			   +b^2*sigma2^2/2-b*sigma2^2)*delta.t)/(
			   sigma*sqrt(delta.t))
			d3=(log(s1/a)+(r-sigma1^2/2+b^2*sigma2^2/2)*
			   delta.t)/(sigma*sqrt(delta.t))
			c=s1*pnorm(d1)-s2*pnorm(d2)-exp(-r*delta.t)*K
			   *pnorm(d3)
			K.rho.BS[i,j]=c
		}
	}
	return(K.rho.BS)
}
testBS.K.rho=BS.K.rho(s1=110,s2=100,sigma1=0.10,sigma2=0.15,delta.
   t=1,r=0.05)
print(round(testBS.K.rho,4))

# The Kirk formula:
K.rho.Kirk<-matrix(rep(0,81),nrow=9,dimnames=list(K.list,rho.list)
   )
Kirk.K.rho<-function(s1,s2,sigma1,sigma2,delta.t,r)
{
	for(i in 1:9)
	{
		K=K.num[i]
		for(j in 1:9)
		{
			rho=rho.num[j]
```

```
            sigma=sqrt(sigma1^2-2*s2*exp(r*delta.t)/(s2*exp(r*delta.t)+K)*rho*sigma1*sigma2+(s2*exp(r*delta.t)/(s2*exp(r*delta.t)+K)*sigma2)^2)
            d1=(r*delta.t+log(s1/(s2*exp(r*delta.t)+K))+sigma^2/2*delta.t)/(sigma*sqrt(delta.t))
            d2=d1-sigma*sqrt(delta.t)
            c=s1*pnorm(d1)-s2*pnorm(d2)-exp(-r*delta.t)*K*pnorm(d2)
            K.rho.Kirk[i,j]=c
        }
    }
    return(K.rho.Kirk)
}
testKirk.K.rho=Kirk.K.rho(s1=110,s2=100,sigma1=0.10,sigma2=0.15,delta.t=1,r=0.05)
print(round(testKirk.K.rho,4))
```

A.4 R codes for pricing bivariate spread option for various drifts and exercising times

```
# Monte Carlo method:
time.list=c("1 month","2 months","3 months","6 months","9 months","12 months")
r.list=c("0.00","0.03","0.05","0.07","0.10")
time.num=c(1/12,2/12,3/12,6/12,9/12,1)
r.num=c(0,0.03,0.05,0.07,0.1)
r.time<-matrix(rep(0,30),nrow=6,dimnames=list(time.list,r.list))
N=6e+06
K=-10
for(i in 1:6)
{
    delta.t=time.num[i]
    for(j in 1:5)
    {
        r=r.num[j]
        eps1=rnorm(N)
        eps2=rho*eps1+sqrt(1-rho^2)*rnorm(N)
        S1=s1*exp((r-sigma1^2/2)*delta.t+sigma1*sqrt(delta.t)*eps1)
        S2=s2*exp((r-sigma2^2/2)*delta.t+sigma2*sqrt(delta.t)*eps2)
```

```
                call.price=pmax(S1-S2-K,0)*exp(-r*delta.t)
                r.time[i,j]=mean(call.price)
                print(round(r.time,4))
        }
}
print(round(r.time,4))

# The Bjerksund-Stensland formula:
r.T.BS<-matrix(rep(0,30),nrow=6,dimnames=list(time.list,r.list))
BS.r.T<-function(s1,s2,sigma1,sigma2,K,rho)
{
        for(i in 1:6)
        {
                delta.t=time.num[i]
                a=s2*exp(r*delta.t)+K
                b=s2*exp(r*delta.t)/a
                sigma=sqrt(sigma1^2-2*b*rho*sigma1*sigma2+b^2*sigma2
                    ^2)
                for(j in 1:5)
                {
                        r=r.num[j]
                        d1=(log(s1/a)+(r+sigma1^2/2-b*rho*sigma1*
                            sigma2+b^2*sigma2^2/2)*delta.t)/(sigma*
                            sqrt(delta.t))
                        d2=(log(s1/a)+(r-sigma1^2/2+rho*sigma1*sigma2
                            +b^2*sigma2^2/2-b*sigma2^2)*delta.t)/(
                            sigma*sqrt(delta.t))
                        d3=(log(s1/a)+(r-sigma1^2/2+b^2*sigma2^2/2)*
                            delta.t)/(sigma*sqrt(delta.t))
                        c=s1*pnorm(d1)-s2*pnorm(d2)-exp(-r*delta.t)*K
                            *pnorm(d3)
                        r.T.BS[i,j]=c
                }
        }
        return(r.T.BS)
}
testBS.r.T=BS.r.T(s1=110,s2=100,sigma1=0.10,sigma2=0.15,K=10,rho
    =0.5)
print(round(testBS.r.T,4))

# The Kirk formula:
r.T.Kirk<-matrix(rep(0,30),nrow=6,dimnames=list(time.list,r.list))
Kirk.r.T<-function(s1,s2,sigma1,sigma2,K,rho)
```

```
{
        for(i in 1:6)
        {
                delta.t=time.num[i]
                for(j in 1:5)
                {
                        r=r.num[j]
                        sigma=sqrt(sigma1^2-2*s2*exp(r*delta.t)/(s2*
                           exp(r*delta.t)+K)*rho*sigma1*sigma2+(s2*
                           exp(r*delta.t)/(s2*exp(r*delta.t)+K)*
                           sigma2)^2)
                        d1=(r*delta.t+log(s1/(s2*exp(r*delta.t)+K))+
                           sigma^2/2*delta.t)/(sigma*sqrt(delta.t))
                        d2=d1-sigma*sqrt(delta.t)
                        c=s1*pnorm(d1)-s2*pnorm(d2)-exp(-r*delta.t)*K
                           *pnorm(d2)
                        r.T.Kirk[i,j]=c
                }
        }
        return(r.T.Kirk)
}
testKirk.r.T=Kirk.r.T(s1=110,s2=100,sigma1=0.10,sigma2=0.15,K=-10,
   rho=0.5)
print(round(testKirk.r.T,4))
```

A.5 R codes for the benchmark of pricing trivariate spread option

```
# Initial values of parameters:
rhoPG=0.1; rhoCG=0.8; rhoCP=-0.7

# Check condition (3.6):
rhoCP^2+rhoCG^2+rhoPG^2-(1+2*rhoCP*rhoCG*rhoPG)
# Check condition (3.13):
1-rhoCP^2
# Check condition (3.14):
1-rhoPG^2-(rhoPG-rhoCP*rhoCG)^2/(1-rhoCP^2)

rho.matrix=matrix(c(1,rhoCP,rhoCG,rhoCP,1,rhoPG,rhoCG,rhoPG,1),
   byrow=T,nrow=3,dimnames=list(c("C","P","G"),c("C","P","G")))
H.matrix=matrix(c(1,0,0,rhoCP,sqrt(1-rhoCP^2),0,rhoCG,(rhoPG-rhoCP
   *rhoCG)/sqrt(1-rhoCP^2),sqrt(1-rhoPG^2-(rhoPG-rhoCP*rhoCG)^2/
```

```
    (1-rhoCP^2))),byrow=T,nrow=3)

muP=0.20; muG=0.15; muC=0.05
muP.tilde=muP-muC
muG.tilde=muG-muC

sigmaP=0.10; sigmaG=0.15; sigmaC=0.15
sigmaP.tilde=sqrt(sigmaP^2+sigmaC^2-2*rhoCP*sigmaC*sigmaP)
sigmaG.tilde=sqrt(sigmaG^2+sigmaC^2-2*rhoCG*sigmaC*sigmaG)

p0=110; g0=100; c0=70
K=1; delta.t=1; h=1; r=0.05

# Monte Carlo method:
MC=1e+06
TriPrice.benchmark=rep(0,10)
for(k in 1:10)
{
        N=MC*k
        WC=sqrt(delta.t)*rnorm(N)
        U1=sqrt(delta.t)*rnorm(N)
        U2=sqrt(delta.t)*rnorm(N)
        a=(rhoPG-rhoCP*rhoCG)/sqrt(1-rhoCP^2)
        b=sqrt(1-rhoPG^2-(rhoPG-rhoCP*rhoCG)^2/(1-rhoCP^2))
        WP=rhoCP*WC+sqrt(1-rhoCP^2)*U1
        WG=rhoCG*WC+a*U1+b*U2
        C=c0*exp((muC-sigmaC^2/2)*delta.t+sigmaC*WC)
        WC.tilde=-sigmaC*delta.t+WC
        P.tilde=p0/c0*exp((muP.tilde-sigmaP.tilde^2/2)*delta.t+(
           sigmaP*rhoCP-sigmaC)*WC.tilde+sigmaP*sqrt(1-rhoCP^2)*U1)
        G.tilde=g0/c0*exp((muG.tilde-sigmaG.tilde^2/2)*delta.t+(
           sigmaG*rhoCG-sigmaC)*WC.tilde+sigmaG*a*U1+sigmaG*b*U2)
        Martingale=exp(-sigmaC^2/2*delta.t+sigmaC*WC)
        TriPrice.benchmark[k]=c0*exp((muC-r)*delta.t)*mean(
           Martingale*pmax(P.tilde-h*G.tilde-1,0))
}
print(round(TriPrice.benchmark,4))

# The revisited Bjerksund-Stensland formula:
BS.tri<-function(p0,g0,c0,K,sigma1,sigma2,rho,delta.t,r,muC,h)
{
        P=p0/c0
        G=g0/c0
```

```
        a=h*G*exp(r*delta.t)+K
        b=h*G*exp(r*delta.t)/a
        sigma.bs=sqrt(sigma1^2-2*b*rho*sigma1*sigma2+b^2*sigma2^2)
        d1=(log(P/a)+(r+sigma1^2/2-b*rho*sigma1*sigma2+b^2*sigma2^2
           /2)*delta.t)/(sigma.bs*sqrt(delta.t))
        d2=(log(P/a)+(r-sigma1^2/2+rho*sigma1*sigma2+b^2*sigma2^2/
           2-b*sigma2^2)*delta.t)/(sigma.bs*sqrt(delta.t))
        d3=(log(P/a)+(r-sigma1^2/2+b^2*sigma2^2/2)*delta.t)/(sigma.
           bs*sqrt(delta.t))
        c=c0*exp(muC*delta.t)*(P*pnorm(d1)-h*G*pnorm(d2)-exp(-r*
           delta.t)*K*pnorm(d3))
}
testBS.tri=BS.tri(p0=110,g0=100,c0=10,K=1,sigma1=sigmaP.tilde,
   sigma2=sigmaG.tilde,rho=rhoPG,delta.t=1,r=0.05,muC=0.05,h=1)
print(round(testBS.tri,4))

# The revisited Kirk formula:
Kirk.tri<-function(p0,g0,c0,K,sigma1,sigma2,rho,delta.t,r,muC,h)
{
        P=p0/c0
        G=g0/c0
        sigma.k=sqrt(sigma1^2-2*h*G*exp(r*delta.t)/(h*G*exp(r*delta
           .t)+K)*rho*sigma1*sigma2+(h*G*exp(r*delta.t)/(h*G*exp(r*
           delta.t)+K)*sigma2)^2)
        d1=(r*delta.t+log(P/(h*G*exp(r*delta.t)+K))+sigma.k^2/2*
           delta.t)/(sigma.k*sqrt(delta.t))
        d2=d1-sigma.k*sqrt(delta.t)
        c=c0*exp(muC*delta.t)*(P*pnorm(d1)-h*G*pnorm(d2)-exp(-r*
           delta.t)*K*pnorm(d2))
}
testKirk.tri=Kirk.tri(p0=110,g0=100,c0=10,K=1,sigma1=sigmaP.tilde,
   sigma2=sigmaG.tilde,rho=rhoPG,delta.t=1,r=0.05,muC=0.05,h=1)
print(round(testKirk.tri,4))

# Plot all the results in one graph:
xx=(1:10)*MC
yy=round(as.numeric(TriPrice.benchmark),4)
plot(xx,yy,"o",xlab="Numbers of simulation (in 1,000,000)",ylab="
   Triviate option price",ylim=c(0.48,0.57),xaxt="n")
axis(1,at=(1:10)*MC,las=1,labels=1:10)
abline(h=0.5500,lty=2)
abline(h=0.5600,lty=2)
points(xx,rep(testBS.tri,10),pch=4)
```

```
points(xx,rep(testKirk.tri,10),pch=20)
legend("center",c("Monte Carlo","B-S","Kirk"),cex=0.7,lty=c(1,0,0),pch=c(1,4,20))
```

A.6 R codes for differentiating delta-hedge parameters for bivariate spread option

```
# Initial values of parameters:
p0=110; g0=100; c0=70
K=1; delta.t=1; h=1; r=0.05
rhoPG=0.1; rhoCG=0.8; rhoCP=-0.7
muP=0.20; muG=0.15; muC=0.05
muP.tilde=muP-muC
muG.tilde=muG-muC
sigmaP=0.10; sigmaG=0.15; sigmaC=0.15
sigmaP.tilde=sqrt(sigmaP^2+sigmaC^2-2*rhoCP*sigmaC*sigmaP)
sigmaG.tilde=sqrt(sigmaG^2+sigmaC^2-2*rhoCG*sigmaC*sigmaG)

# Monte Carlo method:
bi.true<-function(N,p0,g0,sigmaP,sigmaG,rhoPG,delta.t,r)
{
       WP=sqrt(delta.t)*rnorm(N)
       WG=rhoPG*WP+sqrt(1-rhoPG^2)*sqrt(delta.t)*rnorm(N)
       XP=(r-sigmaP^2/2)*delta.t+sigmaP*WP
       XG=(r-sigmaG^2/2)*delta.t+sigmaG*WG
       indicator=(p0*exp(XP)-g0*exp(XG)>0)
       delta1=exp(-r*delta.t)*mean(indicator*exp(XP))
       delta2=-exp(-r*delta.t)*mean(indicator*exp(XG))
       delta=c(delta1,delta2)
}
bi.delta.true=bi.true(N=4e+06,p0=110,g0=100,sigmaP=0.10,sigmaG=0.15,rhoPG=0.1,delta.t=1,r=0.05)
print(round(bi.delta.true,4))

# The Kirk and Bjerksund-Stensland formula (same):
bi<-function(N,p0,g0,sigmaP,sigmaG,rhoPG,delta.t,r)
{
       sigma=sqrt(sigmaP^2-2*rhoPG*sigmaP*sigmaG+sigmaG^2)
       d1=(log(p0/g0)+sigma^2*delta.t/2)/(sigma*sqrt(delta.t))
       d2=d1-sigma*sqrt(delta.t)
       delta1=pnorm(d1)+(p0*dnorm(d1)-g0*dnorm(d2))/(p0*sigma*sqrt(delta.t))
```

```
        delta2=-pnorm(d2)-(p0*dnorm(d1)-g0*dnorm(d2))/(g0*sigma*sqrt(delta.t))
        delta=c(delta1,delta2)
}
bi.delta=bi(p0=110,g0=100,sigmaP=0.10,sigmaG=0.15,rhoPG=0.1,delta.t=1,r=0.05)
print(round(bi.delta,4))
```

A.7 R codes for differentiating delta-hedge parameters for trivariate spread option

```
# Monte Carlo method:
tri.true<-function(N,p0,g0,c0,sigmaP,sigmaG,sigmaC,rhoPG,rhoCP,rhoCG,delta.t,r,h)
{
        WC=sqrt(delta.t)*rnorm(N)
        U1=sqrt(delta.t)*rnorm(N)
        U2=sqrt(delta.t)*rnorm(N)
        a=(rhoPG-rhoCP*rhoCG)/sqrt(1-rhoCP^2)
        b=sqrt(1-rhoPG^2-(rhoPG-rhoCP*rhoCG)^2/(1-rhoCP^2))
        WP=rhoCP*WC+sqrt(1-rhoCP^2)*U1
        WG=rhoCG*WC+a*U1+b*U2
        XP=(r-sigmaP^2/2)*delta.t+sigmaP*WP
        XG=(r-sigmaG^2/2)*delta.t+sigmaG*WG
        XC=(r-sigmaC^2/2)*delta.t+sigmaC*WC
        indicator=(p0*exp(XP)-h*g0*exp(XG)-c0*exp(XC)>0)
        delta1=c0*exp((muC-r)*delta.t)*mean(indicator*exp(XP))
        delta2=-c0*exp((muC-r)*delta.t)*mean(indicator*h*exp(XG))
        delta3=-exp(-r*delta.t)*mean(indicator*exp(XC))
        delta=c(delta1,delta2,delta3)
}
tri.delta.true=tri.true(N=4e+04,p0=110,g0=100,c0=70,sigmaP=0.10,sigmaG=0.15,sigmaC=0.15,rhoPG=0.1,rhoCG=0.8,rhoCP=-0.7,delta.t=1,r=0.05,h=1)
print(round(tri.delta.true,4))

# The revisited Kirk formula:
delta.K<-function(p0,g0,c0,sigmaP,sigmaG,rhoPG,delta.t,r,muC,h)
{
        P=p0/c0; G=g0/c0
        sigma.k=sqrt(sigmaP^2-2*h*G*exp(r*delta.t)/(h*G*exp(r*delta.t)+1)*rhoPG*sigmaP*sigmaG+(h*G*exp(r*delta.t)/(h*G*exp(
```

```
        r*delta.t)+1)*sigmaG)^2)
        d1=(log(P/(h*G+exp(-r*delta.t)))+sigma.k^2/2*delta.t)/(sigma.k*sqrt(delta.t))
        d2=d1-sigma.k*sqrt(delta.t)
        delta1=exp(muC*delta.t)*(pnorm(d1)+(p0*dnorm(d1)-h*g0*dnorm(d2)-exp(-r*delta.t)*c0*dnorm(d2))/(p0*sigma.k*sqrt(delta.t)))
        delta2=exp(muC*delta.t)*(-h*pnorm(d2)-(p0*dnorm(d1)-h*g0*dnorm(d2)-exp(-r*delta.t)*c0*dnorm(d2))/((h*G+exp(-r*delta.t))*sigma.k*sqrt(delta.t)))
        delta3=exp(muC*delta.t)*(-exp(-r*delta.t)*pnorm(d2)-(p0*dnorm(d1)-h*g0*dnorm(d2)-exp(-r*delta.t)*c0*dnorm(d2))*(1/c0-(1/(h*G+exp(-r*delta.t)))*(h*G/c0))/(sigma.k*sqrt(delta.t)))
        delta=c(delta1,delta2,delta3)
}
delta.K=delta.K(p0=110,g0=100,c0=70,sigmaP=sigmaP.tilde,sigmaG=sigmaG.tilde,rhoPG=rhoPG,delta.t=1,r=0.05,muC=0.05,h=1)
print(round(delta.K,4))

# The revisited Bjerksund-Stensland formula:
delta.BS<-function(p0,g0,c0,sigmaP,sigmaG,rhoPG,delta.t,r,muC,h)
{
        P=p0/c0
        G=g0/c0
        a=h*G*exp(r*delta.t)+1
        b=h*G*exp(r*delta.t)/a
        sigma.bs=sqrt(sigmaP^2-2*b*rhoPG*sigmaP*sigmaG+b^2*sigmaG^2)
        d1=(log(P/a)+(r+sigmaP^2/2-b*rhoPG*sigmaP*sigmaG+b^2*sigmaG^2/2)*delta.t)/(sigma.bs*sqrt(delta.t))
        d2=(log(P/a)+(r-sigmaP^2/2+rhoPG*sigmaP*sigmaG+b^2*sigmaG^2/2-b*sigmaG^2)*delta.t)/(sigma.bs*sqrt(delta.t))
        d3=(log(P/a)+(r-sigmaP^2/2+b^2*sigmaG^2/2)*delta.t)/(sigma.bs*sqrt(delta.t))
        delta1=exp(muC*delta.t)*(pnorm(d1)+(p0*dnorm(d1)-h*g0*dnorm(d2)-exp(-r*delta.t)*c0*dnorm(d3))/(p0*sigma.bs*sqrt(delta.t)))
        delta2=exp(muC*delta.t)*(-h*pnorm(d2)-(p0*dnorm(d1)-h*g0*dnorm(d2)-exp(-r*delta.t)*c0*dnorm(d3))/((h*G*exp(r*delta.t)+1)*sigma.bs*sqrt(delta.t)))
        delta3=exp(muC*delta.t)*(-exp(r*delta.t)*pnorm(d3)-(p0*dnorm(d1)-h*g0*dnorm(d2)-exp(r*delta.t)*c0*dnorm(d3))*(1
```

```
        /c0-1/(h*G*exp(r*delta.t)+1)*h*G*exp(r*delta.t)/c0)/(
        sigma.bs*sqrt(delta.t)))
    delta=c(delta1,delta2,delta3)
}
delta.BS=delta.BS(p0=110,g0=100,c0=70,sigmaP=sigmaP.tilde,sigmaG=
   sigmaG.tilde,rhoPG=rhoPG,delta.t=1,r=0.05,muC=0.05,h=1)
print(round(delta.BS,4))
```

A.8 R codes for plotting relative errors for pricing trivariate spread options

```
# Plot the relative errors for various h:
h=seq(1,0.5,by=-0.1)
BS.h=c(12.50,19.81,25.79,29.68,32.10,32.74)/100
K.h=c(12.41,19.75,25.76,29.67,32.09,32.74)/100
plot(h,BS.h,pch=1,xlab="h value",ylab="relative error")
points(h,K.h,pch=3,col=2)
legend("bottomleft",cex=0.7,c("the Kirk formula","the Bjerksund-
   Stensland formula"),pch=c(3,1),col=c(2,1))

# Plot the relative errors for various relative C(0):
C_0=seq(10,100,by=10)
BS.h_1.0=c
   (18.83,19.08,18.65,17.55,16.28,14.61,12.34,10.80,8.65,5.55)/100
K.h_1.0=c
   (18.83,19.07,18.63,17.52,16.22,14.53,12.25,10.67,8.52,5.43)/100
BS.h_0.8=c
   (19.29,21.54,23.24,24.69,25.37,25.80,25.90,25.70,25.28,24.64)/
   100
K.h_0.8=c
   (19.29,21.54,23.24,24.69,25.36,25.78,25.87,25.67,25.25,24.61)/
   100
plot(C_0,BS.h_1.0,pch=1,col=1,xlab="C(0) value",ylab="relative
   error",ylim=c(0.05,0.3))
points(C_0,K.h_1.0,pch=3,col=2)
points(C_0,BS.h_0.8,pch=2,col=1)
points(C_0,K.h_0.8,pch=4,col=3)
legend("bottomleft",cex=0.7,c("the Kirk formula, h=1.0","the
   Bjerksund-Stensland formula, h=1.0","the Kirk formula, h=0.8","
   the Bjerksund-Stensland formula, h=0.8"),pch=c(3,1,4,2),col=c
   (2,1,3,1))
```

Appendix B

MATLAB scripts

All the MATLAB codes used for plotting in this project are listed as the following.

B.1 MATLAB codes for plotting relative errors for pricing spread options with various volatilities

```
% Define some reasonable volatility values:
x=linspace(0.10,0.50,9);
y=linspace(0.10,0.50,9);
[xx,yy]=meshgrid(x,y);

% a) The Bjerksund-Stensland formula, when K=10:
BS_vol_plus10=[0.02/100,0.12/100,0.05/100,0.01/100,0.04/100,0.01/100,0.18/100,0.06/100,0.07/100,
0.03/100,0.10/100,0.08/100,0.08/100,0.20/100,0.02/100,0.08/100,0.09/100,0.00/100,
0.07/100,0.09/100,0.02/100,0.03/100,0.07/100,0.01/100,0.22/100,0.06/100,0.04/100,
0.02/100,0.02/100,0.08/100,0.13/100,0.03/100,0.02/100,0.07/100,0.20/100,0.16/100,
0.01/100,0.03/100,0.06/100,0.08/100,0.14/100,0.03/100,0.11/100,0.04/100,0.07/100,
0.03/100,0.01/100,0.04/100,0.07/100,0.02/100,0.08/100,0.06/100,0.13/100,0.02/100,
0.06/100,0.03/100,0.08/100,0.06/100,0.09/100,0.05/100,0.00/100,0.01/100,0.00/100,
6.20/100,0.06/100,0.01/100,0.00/100,0.00/100,0.00/100,0.02/100,0.00/100,0.08/100,
```

```
0.04/100,0.01/100,0.00/100,0.62/100,0.05/100,0.06/100,0.11/
    100,0.02/100,0.08/100];
figure(1)
surf(xx,yy,BS_vol_plus10);
xlabel('Volatility 1');
ylabel('Volatility 2');
zlabel('Relative error');

% b) The Kirk formula, when K=10:
Kirk_vol_plus10=[0.01/100,0.12/100,0.05/100,0.01/100,0.04/100,0.71
    /100,0.18/100,0.07/100,0.07/100,
0.02/100,0.10/100,0.08/100,0.08/100,0.20/100,0.02/100,0.09/
    100,0.09/100,0.00/100,
0.07/100,0.10/100,0.03/100,0.04/100,0.07/100,0.01/100,0.22/
    100,0.06/100,0.04/100,
0.04/100,0.00/100,0.10/100,0.14/100,0.03/100,0.02/100,0.08/
    100,0.20/100,0.16/100,
0.04/100,0.06/100,0.09/100,0.05/100,0.12/100,0.02/100,0.12/
    100,0.04/100,0.07/100,
0.02/100,0.04/100,0.01/100,0.03/100,0.03/100,0.05/100,0.08/
    100,0.12/100,0.01/100,
0.02/100,0.04/100,0.01/100,0.01/100,0.03/100,0.00/100,0.04/
    100,0.02/100,0.02/100,
0.00/100,0.15/100,0.09/100,0.10/100,0.09/100,0.08/100,0.09/
    100,0.05/100,0.04/100,
0.10/100,0.14/100,0.12/100,0.50/100,0.07/100,0.18/100,0.01/
    100,0.10/100,0.01/100];
figure(2)
surf(xx,yy,Kirk_vol_plus10);
xlabel('Volatility 1');
ylabel('Volatility 2');
zlabel('Relative error');

% c) The Bjerksund-Stensland formula, when K=-10:
BS_vol_minus10=[0.02/100,0.01/100,0.03/100,0.01/100,0.01/100,0.02/
    100,0.05/100,0.04/100,0.05/100,
0.02/100,0.04/100,0.05/100,0.01/100,0.02/100,0.06/100,0.03/
    100,0.02/100,0.00/100,
0.07/100,0.02/100,0.03/100,0.02/100,0.04/100,0.03/100,0.00/
    100,0.03/100,0.05/100,
0.02/100,0.07/100,0.06/100,0.07/100,0.02/100,0.03/100,0.03/
    100,0.06/100,0.03/100,
```

```
0.02/100,0.02/100,0.00/100,0.04/100,0.00/100,0.09/100,0.01/100,0.09/100,0.02/100,
0.01/100,0.06/100,0.04/100,0.09/100,0.07/100,0.05/100,0.04/100,0.03/100,0.09/100,
0.04/100,0.05/100,0.04/100,0.12/100,0.04/100,0.02/100,0.02/100,0.06/100,0.02/100,
0.02/100,0.09/100,0.01/100,0.03/100,0.01/100,0.03/100,0.05/100,0.01/100,0.05/100,
0.02/100,0.00/100,0.05/100,0.02/100,0.01/100,0.05/100,0.02/100,0.04/100,0.07/100];
figure(3)
surf(xx,yy,BS_vol_minus10);
xlabel('Volatility 1');
ylabel('Volatility 2');
zlabel('Relative error');

% d) The Kirk formula, when K=-10:
Kirk_vol_minus10=[0.01/100,0.02/100,0.03/100,0.01/100,0.01/100,0.02/100,0.05/100,0.04/100,0.04/100,
0.06/100,0.09/100,0.07/100,0.02/100,0.02/100,0.06/100,0.03/100,0.02/100,0.00/100,
0.26/100,0.15/100,0.04/100,0.02/100,0.03/100,0.02/100,0.00/100,0.03/100,0.05/100,
0.31/100,0.31/100,0.22/100,0.03/100,0.07/100,0.05/100,0.02/100,0.06/100,0.03/100,
0.35/100,0.31/100,0.26/100,0.14/100,0.12/100,0.15/100,0.02/100,0.08/100,0.03/100,
0.42/100,0.34/100,0.30/100,0.18/100,0.12/100,0.08/100,0.12/100,0.01/100,0.11/100,
0.42/100,0.50/100,0.45/100,0.23/100,0.31/100,0.21/100,0.14/100,0.14/100,0.02/100,
0.50/100,0.57/100,0.47/100,0.37/100,0.32/100,0.29/100,0.24/100,0.11/100,0.02/100,
0.48/100,0.50/100,0.43/100,0.47/100,0.37/100,0.27/100,0.22/100,0.14/100,0.05/100];
figure(4)
surf(xx,yy,Kirk_vol_minus10);
xlabel('Volatility 1');
ylabel('Volatility 2');
zlabel('Relative error');
```

B.2 MATLAB codes for plotting relative errors for pricing spread options with various strike prices and correlation coefficients

```
% Define some reasonable strike prices and correlations:
x=linspace(-0.9,0.9,9);
y=linspace(-20,20,9);
[xx,yy]=meshgrid(x,y);

% a) The Bjerksund-Stensland formula:
BS_k_rho=[0.01/100,0.00/100,0.04/100,0.05/100,0.02/100,0.01/100,0.01/100,0.02/100,0.01/100,
0.04/100,0.02/100,0.00/100,0.01/100,0.02/100,0.04/100,0.01/100,0.03/100,0.03/100,
0.02/100,0.07/100,0.07/100,0.07/100,0.04/100,0.01/100,0.01/100,0.01/100,0.01/100,
0.00/100,0.02/100,0.02/100,0.02/100,0.02/100,0.04/100,0.01/100,0.02/100,0.02/100,
0.00/100,0.02/100,0.01/100,0.04/100,0.07/100,0.02/100,0.01/100,0.03/100,0.00/100,
0.01/100,0.07/100,0.01/100,0.04/100,0.01/100,0.01/100,0.03/100,0.02/100,0.01/100,
0.01/100,0.09/100,0.03/100,0.05/100,0.02/100,0.03/100,0.02/100,0.07/100,0.33/100,
0.00/100,0.01/100,0.06/100,0.01/100,0.03/100,0.03/100,0.03/100,0.04/100,0.07/100,
0.07/100,0.01/100,0.78/100,0.05/100,0.03/100,0.08/100,0.01/100,0.03/100,0.53/100];
figure(5)
surf(xx,yy,BS_k_rho);
xlabel('Correlation');
ylabel('Strike price');
zlabel('Relative error');

% b) The Kirk formula:
Kirk_k_rho=[0.19/100,0.20/100,0.15/100,0.12/100,0.13/100,0.10/100,0.07/100,0.07/100,0.00/100,
0.12/100,0.18/100,0.15/100,0.15/100,0.11/100,0.07/100,0.10/100,0.03/100,0.01/100,
0.09/100,0.17/100,0.17/100,0.03/100,0.06/100,0.08/100,0.07/100,0.06/100,0.02/100,
```

```
0.05/100,0.03/100,0.03/100,0.07/100,0.07/100,0.10/100,0.06/100,0.08/100,0.02/100,
0.00/100,0.02/100,0.01/100,0.04/100,0.07/100,0.02/100,0.01/100,0.03/100,0.00/100,
0.01/100,0.05/100,0.01/100,0.06/100,0.01/100,0.03/100,0.07/100,0.04/100,0.10/100,
0.01/100,0.07/100,0.05/100,0.06/100,0.03/100,0.02/100,0.02/100,0.08/100,0.04/100,
0.16/100,0.18/100,0.11/100,0.15/100,0.21/100,0.24/100,0.29/100,0.35/100,1.24/100,
0.51/100,0.44/100,0.31/100,0.55/100,0.53/100,0.78/100,0.91/100,1.48/100,6.24/100];
figure(6)
surf(xx,yy,Kirk_k_rho);
xlabel('Correlation');
ylabel('Strike price');
zlabel('Relative error');
```

B.3 MATLAB codes for plotting relative errors for pricing spread options with various drifts and exercising times

```
% Define some reasonable drifts ane exercising times:
x=[0,0.03,0.05,0.07,0.10];
y=[1,2,3,6,9,12];
[xx,yy]=meshgrid(x,y);

% a) The Bjerksund-Stensland formula, when K=10:
BS_r_T_plus10=[228.64/100,226.08/100,223.52/100,220.61/100,215.40/100,
220.82/100,219.65/100,217.31/100,213.74/100,206.89/100,
213.29/100,214.10/100,211.89/100,208.60/100,200.70/100,
192.21/100,198.99/100,199.18/100,196.67/100,187.62/100,
172.82/100,185.29/100,188.42/100,187.35/100,177.58/100,
154.67/100,173.21/100,179.34/100,179.54/100,170.31/100];
figure(7)
surf(xx,yy,BS_r_T_plus10);
xlabel('Drift');
ylabel('Time to exercise');
zlabel('Relative error');

% b) The Kirk formula, when K=10:
```

```
Kirk_r_T_plus10=[236.37/100,229.79/100,225.38/100,221.29/100,215.40/100,
236.08/100,226.97/100,220.99/100,215.05/100,206.88/100,
235.90/100,224.92/100,217.31/100,210.52/100,200.68/100,
235.81/100,219.84/100,209.58/100,200.33/100,187.60/100,
235.98/100,215.51/100,203.43/100,192.65/100,177.55/100,
236.00/100,212.34/100,198.88/100,186.38/100,170.28/100];
figure(8)
surf(xx,yy,Kirk_r_T_plus10);
xlabel('Drift');
ylabel('Time to exercise');
zlabel('Relative error');

% c) The Bjerksund-Stensland formula, when K=-10:
BS_r_T_minus10=[0.00/100,0.00/100,0.00/100,0.00/100,0.00/100,
0.00/100,0.00/100,0.00/100,0.01/100,0.01/100,
0.02/100,0.01/100,0.02/100,0.03/100,0.03/100,
0.37/100,0.15/100,0.33/100,0.46/100,0.54/100,
1.34/100,0.30/100,0.97/100,1.41/100,1.66/100,
2.89/100,0.35/100,1.75/100,2.65/100,3.19/100];
figure(9)
surf(xx,yy,BS_r_T_minus10);
xlabel('Drift');
ylabel('Time to exercise');
zlabel('Relative error');

% d) The Kirk formula, when K=-10:%
Kirk_r_T_minus10=[0.00/100,0.00/100,0.00/100,0.00/100,0.00/100
0.00/100,0.00/100,0.00/100,0.01/100,0.01/100
0.03/100,0.04/100,0.03/100,0.03/100,0.04/100
0.49/100,0.53/100,0.52/100,0.54/100,0.57/100
1.46/100,1.54/100,1.58/100,1.65/100,1.71/100
2.76/100,2.89/100,2.99/100,3.12/100,3.27/100];
figure(10)
surf(xx,yy,Kirk_r_T_minus10);
xlabel('Drift');
ylabel('Time to exercise');
zlabel('Relative error');
```

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