ON SOLVING THE $\bar{\partial}$-EQUATION ON LAMINATIONS OF RIEMANN SURFACES

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1. INTRODUCTION

Solving the $\bar{\partial}$-equation, called the Cauchy-Riemann equations, is among the more interesting problems in complex analysis, and many questions about this subject still remain. In this paper, we shall focus on the somewhat generalised *tangential Cauchy-Riemann equations*, $\bar{\partial}_b$, that appear on laminations. We will show how the equations can be solved on some example spaces by using the methodologies B. Deroin demonstrated in [1], and by simplifying the tangential Cauchy-Riemann equations to the standard Cauchy-Riemann equations over each leaf of the lamination. In particular, the result we will prove is the following:

Theorem 1.1. *Let X be a compact hyperbolic Riemann surface lamination with CR line bundle $L \to X$, and assume that L is equipped with a positive metric σ. Then there exists an integer $k_0 \in \mathbb{N}$ such that for all integers $k \geq k_0$ and for any smooth $(0,1)$-form v with coefficients in $L^{\otimes k}$, there exists a smooth section u of $L^{\otimes s}$ with $\bar{\partial}_b u = v$.*

This result is not unknown; indeed, similar results were shown in [2]. However, we shall employ a different method, which hopefully may be generalised to show similar or stronger results. Furthermore, as we develop the method for our proof, we are able to solve $\bar{\partial}_b$ on some example laminations where the leaves may be *parabolic* rather than hyperbolic; that is, they are covered by $\mathbb{C}$ rather than $\mathbb{D}$.

In Section 2 we shall proceed to give a formal definition of laminations. In Section 3, we shall introduce the notions of curvature and bounded geometry. In Section 4, we shall show some geometric comparison results, and how they can be used to show that certain sums converge. In Section 5, some results of Deroin will be demonstrated in order to showcase some of the origin of our methods. In Section 6, we will show that if a $(0,1)$-form is transversally continuous, then so are its minimal solutions. In Section 7, we will use our results to demonstrate how to solve $\bar{\partial}_b$ on some simpler laminations. Finally, we will prove the main result in Section 8.

As a final note: After commencing with this thesis, it was discovered that in the newly published [3], Ohsawa solves the $\bar{\partial}_b$-solution on laminations in a very general setting. However, this thesis uses very different methods than what was used in Ohsawa's paper.

2. LAMINATIONS

We shall focus our work on a particular kind of topological spaces, called *laminations*, which locally can be viewed as a stack of disks. More precisely, we have the following definition:

Definition 2.1. Let X be a topological space, and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X, associated with a family of homeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{D} \times T_\alpha$, where T_α is a metrizable topological space. The maps $\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1}$ are locally on the form $\phi_{\beta\alpha}(z_\alpha, t_\alpha) = (z_{\beta\alpha}(z_\alpha, t_\alpha), t_{\beta\alpha}(t_\alpha))$, where $z_{\beta\alpha}$ is a holomorphic function of z_α for each fixed t_α.

We call the U_α flow boxes, and we call the sets $\mathcal{L}_{\alpha,t_\alpha} := \phi_\alpha^{-1}(\mathbb{D} \times \{t_\alpha\})$ plaques. Any nonempty subset $\mathcal{L} \subset X$ is called a leaf of the lamination if, whenever $x \in \mathcal{L} \cap U_\alpha$ for some α, then $\mathcal{L}$ contains the plaque in U_α containing x and $\mathcal{L}$ is minimal with respect to this condition. The set X is then a disjoint union of leaves and for every x, the leaf $\mathcal{L}_x$ through x consists of all points in X which can be joined to x by a curve which is locally contained in a plaque. A basis for a topology on a leaf $\mathcal{L}$ is given by proclaiming that each plaque in $\mathcal{L}$ is an open set, and that each set $U \cap \mathcal{L}$ is open, where U is an open subset of X. Then each leaf is a Hausdorff topological space, and each leaf has a natural structure of a Riemann surface inherited from the maps ϕ_α.

Let $L \to X$ be a complex line bundle. We will call L a *holomorphic line bundle* if it is defined by transition functions $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$, where $f_{\alpha\beta}$ is holomorphic along every plaque. A section u of L will be said to be *smooth* if it is continuous and smooth along every leaf. A *weight* σ on L will be a family of continuous functions σ_α on U_α, smooth along every plaque, such that $\sigma_\alpha = \sigma_\beta + 2\log|f_{\alpha\beta}|$ on $U_\alpha \cap U_\beta$, and such that all partial derivatives of each σ_α are continuous and smooth along every plaque. We say that σ is *positive* if each σ_α is strictly subharmonic along every leaf.

On Riemann surface laminations, the concept of tangent bundles and cotangent bundles only makes sense along the leaves. However, this still gives us a natural definition of $(0,1)$-forms with coefficients in L, as well as the $\bar{\partial}$-operator acting on sections along leaves; we denote this operator by $\bar{\partial}_b$. We say that a $(0,1)$-form is *smooth* if it is continuous and smooth along every plaque when viewed in local coordinates.

Inspired by [1] and [2], we will attempt to solve the $\bar{\partial}_b$-equations by pulling back the problem to line bundles over the universal covers of each leaf, solve the $\bar{\partial}$-equations there with some additional detail, and then push the solutions back to the lamination.

3. Hermitian holomorphic line bundles

The solutions presented in this paper will depend on some ideas developed by B. Deroin; we shall outline these ideas as Deroin presented them in [1], and then proceed to adapt these ideas to our goal of solving $\bar{\partial}_b$-equations.

In [1], Deroin considers certain manifolds M, constructs a particular metric g, and then develops a method of extending families of so-called 1-*jets* of holomorphic sections into one single bounded holomorphic section, by assuring that these 1-jets are sufficiently "spread out" and then showing that the sum of their averages over non-intersecting balls converges. In doing so, he is able to construct a holomorphic locally bilipshitz immersion $\pi : (M, g) \to \mathbb{C}P^N$ into projective space.

Our objective is not the same as Deroin's; however, many of the methods we use are the same or at least quite similar to the ones used by Deroin. In this section we shall introduce the basic concepts tied to hermitian holomorphic line bundles, that is, holomorphic line bundles equipped with a hermitian metric.

For all our future purposes, we shall need to define the concept of the *curvature* of a holomorphic line bundle over a complex manifold.

3.1. **Hermitian metrics.**

Definition 3.1. Let M be a complex manifold with smooth Hermitian metric $\omega = e^{-\psi}\frac{\sqrt{-1}}{2}dz \wedge d\bar{z}$, and let $L \to M$ be a holomorphic line bundle, equipped with a Hermitian metric $e^{-\sigma}$. The *curvature* of $e^{-\sigma}$ is the $(1,1)$-form defined by

$$\Omega_\sigma := dd^c\sigma = \sqrt{-1}\partial\bar{\partial}\sigma$$

If Ω_σ is strictly positive on the restriction to any complex line, it is simply said to be *strictly positive.*

A Hermitian metric is called *kählerian* if its fundamental form ω is closed, that is, $d\omega = 0$. We define the *kählerian metric assoiciated with* $dd^c\sigma$, called g, by the formula

$$g(u, v) := 2dd^c\sigma(u, \sqrt{-1}v)$$

We shall repeatedly refer to d_g; this will be the distance derived from g. The following definition can be found in [5]:

Definition 3.2. Letting $e_1, ..., e_n \in T_pM$ be an orthonormal basis, we can define a Riemannian volume form $d\text{vol}$ relative to g by

$$d\text{vol}(v_1, ..., v_n) = \det(g(v_i, e_j))$$

We will usually denote this volume form by $\mathrm{d}v_g$.

We shall mostly be working on the space of L^2 functions relative to whichever line bundle we are currently working on. This space is defined as follows:

Definition 3.3. Let M be a complex manifold with smooth Hermitian metric $\omega = e^{-\psi}\frac{\sqrt{-1}}{2}dz \wedge d\bar{z}$, and let $L \to M$ be a holomorphic line bundle, equipped with a Hermitian metric $e^{-\sigma}$. The $L^2(L, \sigma)$-norm on L is defined by

$$\|u\|_{L^2(L,\sigma)} := \sqrt{\int_M |u|^2 e^{-\sigma} dv_g}$$

Whenever σ is obvious from the context, we shall usually just write $\|\cdot\|_2$.

We shall frequently demand that the Ricci curvature of L is bounded from below by some real number c. By this, we shall actually mean that the Gaussian curvature of (M, g) is bounded from below by c; that is,

$$\frac{dd^c\psi}{dv_g} \geq c$$

3.2. **Bounded Geometry.** We shall now proceed to define the notions of *bounded geometry* and *radius* for a manifold. These are conditions on the metric and line bundle over M, and grant important information; information we will use frequently.

Definition 3.4. We define the *radius* $r(|\cdot|)$ of a hermitian metric $|\cdot|$ of L as the supremum of the set of real numbers $r \geq 0$ such that, for every point $x \in M$ there exists a biholomorphism

$$z : B_g(x, r) \to U_x$$

to an open set $U_x \subset \mathbb{C}$ sending x to 0 and such that z is 2-bilipschitz when we equip U_x with the standard euclidian metric on $\mathbb{C}^n$. In particular, this means that g is complete.

If $r(|\cdot|) > 0$ and the Ricci curvature of g is uniformly bounded from below on M, we say that $|\cdot|$ is of *bounded geometry*.

As an additional remark, Deroin requires that the radius is greater than or equal to r only if there for all x exists some holomorphic section $s : B_g(x, r) \to L$ satisfying, for all $y \in B_g(x, r)$,

$$e^{-2d_g(x,y)^2} \leq |s(y)| \leq e^{-d_g(x,y)^2/2}$$

This section shall only be used, and mentioned, in the next lemma, as well as Section 5; for all other purposes, we shall refrain from discussing it.

Example 3.5. Consider $\mathbb{D}$ with the standard Poincaré metric $e^{-2\log(1-|z|)^2}$, with line bundle taken to some power k; in this case, $g = \frac{kdzd\bar{z}}{(1-|z|^2)^2}$. The Ricci curvature of g is uniformly bounded from below on M, as $dd^c\psi = \frac{-4}{(1-|z|^2)^2}dz \wedge d\bar{z}$. We assert that the radius r is greater than $\sqrt{k}$: for we can define the map

$$z_0 : B_g(0, \sqrt{k}) \to U_0 \subset \mathbb{C}$$

by

$$z_0(x) := \sqrt{k}x$$

This map is biholomoprhic and 2-bilipschitz; a simple permutation allows us to make a similar map for any point $p \in \mathbb{D}$.

Lemma 3.6. *Let M be any* compact Riemann surface *with smooth Hermitian metric $\omega = e^{-\psi}\frac{\sqrt{-1}}{2}dz \wedge d\bar{z}$, and let $L \to M$ be a holomorphic line bundle, equipped with a Hermitian metric $e^{-\sigma}$. Then M has bounded geometry.*

Proof. The Ricci curvature is bounded from below, because we can consider the expression $\frac{dd^c\psi}{dv_g}$ as a continuous function, and by the compactness of X, this function is bounded.

The radius is strictly positive; for, by the definition of a Riemann surface, there exists around any point $x \in X$ a neighbourhood $V_x \subset X$ such that we have a coordinate system $\zeta_x : V_x \to U_x \subset \mathbb{C}$, with $0 \in U_x$. For some subset $U'_x \subset U_x$, $\zeta_x^{-1}|_{U'_x}$ is 2-bilipshictz and biholomorphic. Now we can take a subset V'_x of $\zeta_x^{-1}(U'_x)$ such that there exists some r_x such that, for all $y \in V'_x$, there exists a ball $B_g(y, r) \subset \zeta_x^{-1}(U'_x)$. Since M is compact, cover M by $\{V'_x\}_{x\in M}$ and take a finite subcover $\{V'_{x_j}\}_{1\leq j\leq m}$; now the radius of M is bounded below by $\min(\{r_{x_j}\}_{1\leq j\leq m}) > 0$. □

For many applications, we will require specific lower bounds on the Ricci curvature and radii. The next lemma tells us that, as long as *some* bounds exist, we can obtain the required bounds by substituting our line bundle for some power of itself.

Lemma 3.7. *If $L \to M$ is a hermitian holomorphic line bundle of strictly positive curvature and bounded geometry, with Ricci curvature bounded from below by $-c$, then the radii of the powers $|\cdot|^{\otimes k}$ satisfy, for all $k \geq 0$,*

$$r(|\cdot|^{\otimes k}) \geq \sqrt{k}r(|\cdot|)$$

Additionally, the Ricci curvature of the metrics g_k induced by $|\cdot|^{\otimes k}$ tend uniformly towards 0 as k tends to infinity.

Proof. For each integer $k \geq 1$, consider k-th powers; the metrics σ_k of $L^{\otimes k}$, and kählerian metrics $g_k = kg$ on M. At all points $x \in M$, the coordinates $z_k = \sqrt{k}z$ effect a 2-bilipschitz biholomorphism of $B_{g_k}(x, r\sqrt{k})$ in the open $\sqrt{k}U_x \subset \mathbb{C}$, and we also have sections $s^k : V_x \to L^{\otimes k}$ satisfying

$$e^{-2d_{g_k}(x,y)^2} \leq |s^k(y)| \leq e^{-d_{g_k}(x,y)^2/2}$$

for each $y \in B_{g_k}(x, r\sqrt{k})$. Then the radius $r(|\cdot|^{\otimes k})$ must be larger than $\sqrt{k}r(|\cdot|)$. Additionally, the Ricci curvature of the metric g_k tends uniformly towards 0 as k tends to infinity, as

$$\frac{dd^c\psi}{dv_{g_k}} = \frac{dd^c\psi}{kdv_g}$$

□

3.3. **Curvature and $\bar{\partial}$-solutions.** We want to use the above definitions to solve the $\bar{\partial}_b$-equations with some extra degree of detail based on manifolds satisfying conditions tied to the above definitions. The following, which can be found in [4], is due to Hörmander:

Theorem 3.8. *Let M be a Riemann surface with smooth Hermitian metric $\omega = e^{-\psi}\frac{\sqrt{-1}}{2}dz \wedge d\bar{z}$, and let $L \to M$ be a holomorphic line bundle equipped with a metric $e^{-\sigma}$ whose curvature satisfies*

$$dd^c(\sigma + \psi) \geq c \cdot dv_g$$

for some strictly positive constant c. If v is a smooth, $L^2(\sigma)$ $(0,1)$-form taking values in L, then there exists a smooth, $L^2(\sigma)$ section u of L such that

$$|v|_{L^2(\sigma)} \leq \frac{1}{c}|u|_{L^2(\sigma)}$$

In the next section, we will show that, if the Ricci curvature and radius are bounded in a certain way, then sufficiently nice sums on the form $\sum e^{-d_g(x,\cdot)}$ converge. As such, we would like the ability to solve the Cauchy-Riemann equations in $L^2(e^{d_g(x,\cdot)}|\cdot|)$-norm; this, combined with some later results, shall allow us to find $L^2(\sigma)$ solutions.

However, the problem with this approach is twofold: In order to use Theorem 3.8, we would need to be able to estimate $dd^c(d_g(x,\cdot))$ - and this is not necessarily well defined, because $d_g(x,\cdot)$ is not necessarily a smooth function; and even if we could estimate it, there is no guarantee that the positivity condition in Theorem 3.8 is satisfied for the line bundle we are currently working on.

We can, however, find a smooth function ϕ_x which closely approximates $d_g(x,\cdot)$ and use $|\cdot|'_x := e^{\phi_x}|\cdot|$ instead; this, combined with taking an if necessary even larger power of the line bundle, will allow us to find actual $L^2(e^{d_g(x,\cdot)}|\cdot|)$-minimal solutions. First, a definition, and then a lemma taken almost word-for-word from Deroin's article.

Definition 3.9. Consider a manifold (M,g). For any real number $\delta > 0$, we say that a subset Ξ of M is *δ-separated* if, for any pair of points $\xi_1, \xi_2 \in \Xi$, $\xi_1 \neq \xi_2$,

$$d_g(\xi_1,\xi_2) \geq \delta$$

Lemma 3.10. *Let $0 < \epsilon < \delta$ be two real numbers. There exists a finite number of δ-separated subsets $T_1, \cdots, T_k$ such that the union of these sets is ϵ-dense in M.*

Proof. If a subset is ϵ-separated and maximal with respect to inclusion, then it is clearly also ϵ-dense in M. So let us take any such subset T. We then let $T_1 \subset T$ be some δ-separated subset of T which is also maximal with respect to inclusion. We then inductively choose $T_j \subset T \setminus \bigcup_{i=1}^{j-1} T_i$ as δ-separated subsets, maximal with respect to inclusion.

We claim that there must exist some $l \in \mathbb{N}$ such that $\bigcup_{i=1}^{l} T_i = T$. For suppose there exists a $t \in T$ such that $t \notin T_m$ for $m = 1, \cdots, l$. As each T_m is maximal, there exists a $t_m \in T_m \cap B_g(t,\delta)$ for each m. As $T_{m_1} \cap T_{m_2} = \emptyset$ for each $m_1 \neq m_2$, $t_{m_1} \neq t_{m_2}$. Thus, there are at least $l+1$ points of T in $B_g(t,\delta)$, and the $l+1$ balls of radius $\epsilon/2$ centered in these points are all disjoint. With these conditions, $l+1$ is bounded, dependent only on the Ricci curvature of g. □

This shall allow us to define the ϕ_x mentioned above:

Lemma 3.11. *Consider (M,g). There exists positive real numbers c_0, r_0 so the if Ricci curvature is bounded from below by $-c_0$ and so that the radius is greater than r_0, then for every $x \in M$ there exists a smooth function ϕ_x such that*

$$|\phi_x(\cdot) - d_g(x,\cdot)|_\infty \leq 1$$

and such that

$$dd^c\phi_x \geq -A \cdot dv_g$$

for some positive constant A not dependent on x.

Proof. By Lemma 3.10, there exists an integer n and a family of $3r_0$-separated sets $\{\Xi_j\}_{1\leq j\leq n} \in M$ such that their union is r_0-dense.

Fix any $x_0 \in M$ and any j, $1 \leq j \leq n$. For each $\xi \in \Xi_j$, define $\phi_\xi := d_g(x_0, \xi)$.

Now for every $B_g(\xi, r_0)$ we want to define α_ξ as a smooth cut-off with $|d(d\alpha_\xi)| \leq A$ for some positive constant A, and with $d(d\alpha_\xi) = 0$ outside $B_g(2\xi, r_0)$. In Euclidean space, this is the case for functions of the type $\tilde{\alpha}_\xi := e^{1/(r_0^2-|x-\xi|^2)}$. As such, we can define a function $\eta : \mathbb{C}^n \to [0,1]$ which is 0 outside $B_{eucl}(0, 1/2)$, which is smooth and with $|d(d\eta)| \leq A$. Then we define $\alpha'_\xi := \eta(z_\xi(y)/2)$ on $B_g(\xi, r_0)$ and 0 everywhere else.

Since by the definition of $\{B_g(\xi, r_0)\}_{1\leq j\leq n, \xi\in\Xi_j}$ each point of M can be in no more than n of these sets, we can modify the α'_ξ to α_ξ so that $\sum_{1\leq j\leq n}\sum_{\xi\in\Xi_j} \alpha_\xi(y) = 1$ for all $y \in M$.

Thus $\phi(y) := \sum_{1\leq j\leq n}\sum_{\xi\in\Xi_j} \alpha_\xi(y)\phi_\xi(y)$ is our required function. □

We shall now combine Theorem 3.8 together with the ϕ_x we created, and then use the fact that $e^{\phi_x}|\cdot|$ and $e^{d_g(x,\cdot)}|\cdot|$ are uniformly comparable to find actual $L^2(e^{d_g(x,\cdot)}|\cdot|)$ solutions.

Lemma 3.12. *Assume that M is a manifold with smooth hermitian metric $\omega = e^{-\psi}\frac{\sqrt{-1}}{2}dz \wedge d\bar{z}$, and assume $L \to M$ is a line bundle with positive metric $e^{-\sigma}$. Then there exists positive real numbers c_0, r_0 such that if the Ricci curvature is bounded from below by $-c_0$ and if that the geometry is bounded with radius greater than r_0, then there exists a positive constant C and an integer k_0 such that, for any $x \in M$, the following is true:*

Consider the norm $|\cdot|_x := e^{d_g(x,\cdot)}|\cdot|$. For any $k \geq k_0$ there exists a positive constant C such that for any smooth, $L^2(|\cdot|_x)$ $(0,1)$-form v taking coefficients in $L^{\otimes k}$, there exists a smooth, $L^2(|\cdot|_x)$ section u of $L^{\otimes k}$ such that $\bar{\partial}u = v$ and such that

$$|u|_x \leq C|v|_x$$

Proof. Assume that c_0 and r_0 satisfy the conditions in the previous lemma. Consider the norm $|\cdot|'_x := e^{\phi_x}|\cdot|$, as introduced earlier. Note that it is uniformly equivalent to $|\cdot|_x$, with estimate

$$e^{-r_0}|\cdot|'_x \leq |\cdot|_x \leq e^{r_0}|\cdot|'_x$$

It is also smooth, with curvature

$$dd^c(\phi + \psi) = dd^c\phi + dd^c\psi \geq -(A + c_0) \cdot dv_g$$

This is all independent of the line bundle. Thus if we take a k large enough so that, when regarding $(L^{\otimes k}, \sigma_k)$, we have $dd^c\sigma_k \geq (A + c_0 +$

$c) \cdot dv_{g_k}$ for some $c > 0$, then

$$dd^c(\sigma_k + \psi + \phi_x) \geq c \cdot dv_{g_k}$$

and we can use Theorem 3.8 to find a positive constant C, independent of v, and a smooth, $L^2(|\cdot|'_x)$ section u' satisfying $\bar{\partial}u' = v$ and

$$\begin{aligned} |u'|'_x &\leq C|v|'_x \\ &\leq e^{r_0}C|v|_x \end{aligned}$$

But u' is also $L^2(|\cdot|_x)$, since

$$|u'|_x \leq e^{2r_0}C|v|_x$$

Thus there exist smooth, $L^2(|\cdot|_x)$ sections u of $L^{\otimes k}$ solving $\bar{\partial}u = v$, and the $L^2(|\cdot|_x)$-minimal of these (found by subtracting the projection onto the holomorphic sections) satisfies the required inequality, since some solution does. □

4. Volume comparison

As stated in the previous section, we would want to show that, given sufficient conditions on the Ricci curvature and radius of a line bundle, some sums of type $\sum_z e^{-d_g(x,z)}$ converge.

We will achieve this through geometrical observations; by comparing the volume of balls in M with balls in simpler spaces, we shall be able to convert the problem of convergent sums into a much simpler problem of convergent integrals over euclidean space.

4.1. Constant-Curvature Space Form Comparison.

For our proof, the following result, whose proof can be found for example in [5], will be crucial:

Lemma 4.1. *Let (M, g) be a complete n-dimensional Riemannian manifold whose Ricci curvature is bounded from below by $(n-1)\cdot k$ for some real number k. Then for any $p \in M$,*

$$r \to \frac{volB(p,r)}{v(n,k,r)}$$

is a non-increasing function which tends to 1 as $r \to 0$; here $v(n,k,r)$ denotes the volume of a ball of radius r in the constant-curvature space form S^n_k - for $k<0$, this has Ricci curvature $\frac{1}{\sqrt{-k}}$.

Here a *constant-curvature space form M* is a complete Riemannian manifold of constant Ricci curvature. This gives us the following corollary:

Lemma 4.2. *For any real number $\alpha > 0$, there exists a real number $k_0 < 0$ and a constant $C > 0$ depending only on α such that if the Ricci curvature of (M, σ) is bounded from below by $(n-1)\cdot k_0$, then*

$$volB_g(p,r) \leq Ce^{\alpha r/2}$$

for all points $p \in M$.

Proof. From [6] it is known that if $k < 0$, then, by setting $R = 1/\sqrt{-k}$, we have

$$v(n,k,r) = \pi R^3 \sinh(2r/R) - 2\pi R^2 r$$

and so, by the definition of $\sinh$, it follows that

$$v(n,k,r) \leq \pi R^3 e^{2r/R}$$

It follows that if we have $k_0 = -\alpha^2/16$, then

$$v(n,k_0,r) \leq 4\pi\alpha^{-1}e^{\alpha r/2}$$

and so our proof goes through by demanding $C \geq 4\pi\alpha^{-1}$, as, by the previous lemma,

$$\text{vol}B_g(p,r) \leq v(n,k_0,r)$$

□

Example 4.3. With the terminology above, consider $M = \mathbb{D}$ with where $g = \frac{kdzd\bar{z}}{(1-|z|^2)^2}$ for some constant $k > 0$. This means that the length of any curve $\gamma : [0,1] \to \mathbb{D}$ is given by

$$\int_0^1 |\gamma'(t)|\sqrt{\frac{k}{|(1-|\gamma(t)|^2)^2|}}dt$$

and the distance between two points is given by the infimum of these integrals for curves beginning in one of the points and ending in the other. In particular, for any $z = |z|e^{i\theta} \in \mathbb{D}$, this comes from the curve $\gamma(t) = t|z|e^{i\theta}$ - refer for example to [7] - and so

$$\begin{aligned} d_g(0,z) &= \int_0^1 \frac{|z|\sqrt{k}}{1-t^2|z|^2}dt \\ &= \frac{\sqrt{k}}{2}\log\frac{1+|z|}{1-|z|} \end{aligned}$$

Note that if $d_g(0,z) = r$, then by a simple calculation we have that $|z| = \frac{e^{2r/k}-1}{e^{2r/k}+1}$; we shorthand this as r'. Now the area of a ball of radius r centered at 0 is given by

$$\begin{aligned} \text{vol}B_g(0,r) &= \int_{B_g(0,r)} dv_g \\ &= \int_{B_g(0,r)} \frac{k}{(1-|z|^2)^2}dzd\bar{z} \\ &= k\int_0^{r'}\int_0^{2\pi}\frac{\rho}{(1-\rho^2)^2}d\theta d\rho \\ &= k\int_0^{r'}\frac{2\pi\rho}{(1-\rho^2)^2}d\rho \\ &= k\pi(\frac{1}{1-r'}-1) = k\pi(e^{2r/k}-1) \end{aligned}$$

and this is obviously smaller than $k\pi e^{2r/k}$ on $\mathbb{D}$.

4.2. **Sum Convergence.** It turns out that this comparison result is all we need to show that sums of the type $\sum e^{-d_g(\xi,p)}$ converge over any δ-separated subset for all $p \in M$:

Lemma 4.4. *Let $L \to M$ be a holomorphic line bundle with metric σ, and let g be the associated kählerian metric. Then there exists constant $c_0 > 0$, $r_0 > 0$ such that if the Ricci curvature of g is uniformly bounded from below by $-c_0$ and the radius r is greater than r_0, then for every real number $\delta > 0$, every δ-separated subset Ξ of M and every $p \in M$, the sum*

$$\sum_{\xi \in \Xi} e^{-d_g(\xi,p)}$$

converges.

Proof. As Ξ is δ-separated, we have that

$$\begin{aligned}\sum_{\xi \in \Xi} e^{-d_g(\xi,p)} &\leq \sum_{\xi \in \Xi} E(\delta) \int_{B_g(\xi,\delta/2)} e^{-d_g(x,p)}\, dv_g(x) \\ &\leq E(\delta) \int_M e^{-d_g(x,p)}\, dv_g(x)\end{aligned}$$

where $E(\delta) := e^{\delta}/(\inf_{y \in M} \mathrm{vol}(B_g(y,\delta/2)))$ - the denominator is never 0 by Definition 3.4, since if $\delta/2 \leq r_0$ we have a 2-bilipschitz map from $B_g(y,\delta/2)$ into an open set in $\mathbb{C}^n$, and if $\delta/2 > r_0$ then we clearly have $\mathrm{vol} B_g(y,\delta/2) \geq \mathrm{vol} B_g(y,\delta/2) > 0$.

Let us assume that c_0 is small enough and r_0 large enough to satisfy the conditions of Lemma 4.2 with $\alpha = 1$. Now define $A_1 := B_g(p,1)$, and, by induction, $A_n := B_g(p,n) \backslash A_{n-1}$ for all $n \geq 2$. As $\mathrm{vol} A_n \leq \mathrm{vol} B_g(p,n)$, we can use Lemma 4.2 to transfer the problem to $\mathbb{R}^+$:

$$\begin{aligned}\int_M e^{-d_g(x,p)} dv_g(x) &\leq \sum_{n \in \mathbb{N}} e^{-n+1} \mathrm{vol} A_n \\ &\leq e \sum_{n \in \mathbb{N}} e^{-n/2} \\ &\leq C \int_0^{\infty} e^{-y/2}\, \mathrm{d}_{\mathrm{eucl}} y \\ &\leq C < \infty\end{aligned}$$

where C is a constant that depends only on δ. Note in particular that by the same logic as above, we can, for any real number $\epsilon > 0$, find a real number $R_0 > 0$ such that $\sum_{\xi(0) \in M \backslash B_g(p,R_0)} e^{-d_g(\xi,p)} \leq \epsilon$ for all $p \in M$. □

Example 4.5. Let us again consider $\mathbb{D}$ with $g = \frac{k dz d\bar{z}}{(1-|z|^2)^2}$, and assume $k \geq 4$. In Example 4.3, we showed that $d_g(0,z) = \frac{k}{2} \log\left(\frac{1+|z|}{1-|z|}\right)$. Thus, for any δ-separated subset of $\mathbb{D}$ and any $x \in \mathbb{D}$, there exists positive constants C, C', C'' depending only on δ such that

$$\begin{aligned}
\sum_{\xi\in\Xi} e^{-d_g(x,\xi)} &\leq E(\delta)\int_{\mathbb{D}} e^{-d_g(x,\xi)}\frac{k}{(1-|z|^2)^2}dzd\bar{z}\\
&\leq C\sum_{n\in\mathbb{N}} e^{-n+1}\int_{B_g(0,n)}\frac{k}{(1-|z|^2)^2}dzd\bar{z}\\
&\leq C'\sum_{n\in\mathbb{N}} e^{-n}\cdot\frac{k\pi}{2}e^{2n/k}\\
&= kC'\sum_{n\in\mathbb{N}} e^{-n/2}\\
&\leq kC''\int_0^\infty e^{-x/2}dx < \infty
\end{aligned}$$

Corollary 4.6. *Let X be a compact Riemann surface of genus greater than or equal to 1, and let $L \to X$ be a positive holomorphic line bundle with metric σ; suppose that there exist positive constants c and r such that the Ricci curvature is bounded from below by $-c$ and the geometry is bounded with radius greater than r. Let $f : M \to X$ be the universal covering of X, where M is either $\mathbb{C}$ or $\mathbb{D}$, and let Γ be the Deck group of f. Then there exists some positive constant δ such that for all $x \in M$, the set $\{\varphi(x)\}_{\varphi\in\Gamma}$ is δ-separated with respect to the d_g derived from σ.*

Proof. This follows from the fact that every $\varphi \in \Gamma$ is an automorphism on M, together with choosing an open set $U \subset M$ such that $U \cap \varphi(U) \neq \emptyset$ only if $\varphi = id$. □

5. Extension of jets

In this section, we shall give the proof of one of the theorems from [1]; although the result itself shall not prove important to us, we shall adopt some part of the methodology.

Theorem 5.1. *Let (M, g) be a complete Hermitian manifold, and $L \to M$ a holomorphic line bundle admitting a Hermitian metric of bounded geometry such that the curvature Ω satisfies all inequalities on the form*

$$\frac{1}{C}g(u) \leq \Omega(u, \sqrt{-1}u) \leq Cg(u)$$

for some constant $C \geq 1$ independent of u. Then there exists an integer N and a bilipschitz, locally holomorphic immersion $\pi : (M, g) \to \mathbb{C}P^N$.

5.1. **Jets.** Deroin's method relies on using the conditions on the curvature to create local holomorphic sections with certain estimates. Using this to create holomorphic sections extending so-called *1-jets*, he arrives at his result. Our main interest shall be the method of extension, rather than the result, but for completion we shall demonstrate as much as possible.

Definition 5.2. In Euclidean space, the concept of a k-jet of a smooth function f in the point x is well-defined as the space of functions whose values, as well as derivatives up to and including the k-th order, agree at x. It can be shown that, by use of coordinate system, this notion can be extended to jets of holomorphic sections on complex manifolds; this notion is well-defined and independent on the choice of coordinate system.

One of the most important tools in Deroin's proof is the Gårding inequality, which can be found in [8]:

Lemma 5.3. *If $|\cdot|$ is a strictly positive metric of bounded geometry with radius $r(|\cdot|) = r$, then we have the* uniform Gårding inequalities*: For each holomorphic section $\tau : B_g(y, r) \to E$, we have*

$$|J_1\tau(y)| \leq C(r)\sqrt{\int_{B_g(y,r)} |\tau(z)|^2 dv_g(z)}$$

where $J_1\tau(y)$ is the 1-jet of τ at y, and where $C(r)$ depends only on r and decreases as r increases. In the same manner, we conclude that any holomorphic section $h : M \to L$ of $L^2_g(|\cdot|_{x,\alpha})$ satisfies the estimate

$$|h(y)| \leq C|h|_{x,\alpha,2}e^{-\alpha d(x,y)}$$

for every $y \in M$, where C is a constant depending only on r.

5.2. **Extension.** Deroin uses the notation $|\cdot|_{x,\alpha} := e^{\alpha d_g(x,\cdot)}|\cdot|$, which we shall adopt. The main result for extending jets is the following:

Lemma 5.4. *There exist real numbers $\alpha > 0$, $r_0 > 0$ and $C > 0$ such that if $E \to M$ is a holomorphic line bundle equipped with a metric $|\cdot|$ of strictly positive curvature and of bounded geometry with radius satisfying $r(|\cdot|) \geq r_0$, and for which the Ricci curvature of g is uniformly bounded from below by $-\frac{1}{4}$, then each 1-jet j of a holomorphic section of M in E in a point x extends to a holomorphic section $h : M \to E$ of $L^2_g(|\cdot|_{x,\alpha})$ of norm less than $C|j|$.*

In order to prove this, we shall need some intermediate results. From Lemma 3.12, we have the following corollary, telling us that sections that are "almost holomorphic" in a certain sense can be approximated by actual holomorphic sections.

Corollary 5.5. *Suppose s is a smooth, $L^2_g(|\cdot|')$ section of M in L, and suppose as well that $|\bar{\partial}s|'_2 \leq \delta$ for some constant δ. By Lemma 3.12, there exists a constant C, independent of s, and a smooth, $L^2_g(|\cdot|')$ section u of M in L satisfying the estimates $\bar{\partial}(s-u) = 0$ and $|u|'_2 \leq C|\bar{\partial}s|'_2$ for some universal constant C'. The section $h = s - u$ is holomorphic, and we have that $|s-h|'_2 \leq C\delta$.*

The following lemma is due to [9]:

Lemma 5.6. *Let $L \to M$ be a holomorphic line bundle and let $|\cdot|$ be a metric of strictly positive curvature and of radius $r := r(|\cdot|) > 0$. Then the following is true: Let x be a point in M, and let j be a 1-jet of a holomorphic section of L centered in x. Then there exists a smooth, compactly supported section $\bar{s} : M \to L$ which passes through j and which is holomorphic on $B_g(x, r/3)$, such that there exists some universal constant C with*

$$|\bar{s}|_{x,\alpha,2} \leq C|j| \text{ and } |\bar{\partial}\bar{s}|_{x,\alpha,2} \leq \frac{C|j|}{r}$$

Proof. Let $s : B_g(x, r) \to L$ be the section defined in Definition 3.4, and let P be a polynomial that is linear in the coordinates z of Definition 3.4 centered in x such that $j = J_1(Ps)(x)$. We define the section $\bar{s} := \phi s$, where $\phi : B_g(x, r) \to \mathbb{R}$ is a smooth function which is identically 1 on $B_g(x, r/3)$ and identically 0 outside of $B_g(x, 2r/3)$. We also demand that ϕ takes values between 0 and 1 and that there exists some universal constant C such that

$$|d\phi|_g \leq C/r$$

We can for example construct this function by considering a function $\psi : B_{\text{eucl}}(0, \frac{1}{2}) \to \mathbb{R}$ with the same properties we demanded from ϕ;

we could then define $\phi := \psi(z(y)/2r)$. We now have that

$$|\bar{\partial}\bar{s}|_{x,\alpha,2} = |\bar{\partial}(\phi)Ps|_{x,\alpha,2} \leq \frac{C}{r}|Ps|_{x,\alpha,2}$$

By the linearity of P, we have that

$$|P(y)| \leq C|j|(1 + d(x,y))$$

As such we have

$$|Ps|_{x,\alpha,2} \leq C\sqrt{\int_{B_g(x,r)} (1 + d(x,y))^2 e^{\alpha d(x,y) - d(x,y)^2}\, \mathrm{d}v_g(y)}$$

This integral converges, as the integral

$$\int_{\mathbb{C}^n} (1 + 2|z|)^2 e^{2\alpha|z| - |z|^2/2}\, \mathrm{d}v_{\mathrm{eucl}}(z)$$

in $\mathbb{C}^n$ also converges, and this is greater (bar multiplication by some constant) than our last term by the fact that there exists a holomorphic, 2-bilipschitz coordinate chart on $B_g(x,r)$, as stated in Definition 3.4. □

We are now ready to prove Lemma 5.4:

Proof of Lemma 5.4. Let $|\cdot|$ be a metric of L of strictly positive curvature and of bounded geometry, for which the Ricci curvature of the associated kählerian metric g is bounded from below by $-\frac{1}{4}$, and for which the radius $r(|\cdot|)$ is greater than 1. Let x be a point of M, and let j be a 1-jet in x of some holomorphic section of M in L. Consider the section $\bar{s}$ from Lemma 5.6 relative to j, and permute it by way of Corollary 5.5 to some holomorphic section $h : M \to L$ in $L^2_g(|\cdot|_{x,\alpha})$; h satisfies

$$|h - \bar{s}|_{x,\alpha,2} \leq \frac{C|j|}{r}$$

The section $h - \bar{s}$ is holomorphic on the ball $B_g(x, r/3)$; the Gårding inequality gives us

$$|J_1 h - j| \leq \frac{C|j|}{r} \tag{1}$$

for some universal constant C. We choose r so that $C/r = \frac{1}{2}$, where C is the constant from 1. By Lemma 5.6 we have that

$$|\bar{s}|_{x,\alpha,2} \leq C|j|$$

and so we have

$$|h|_{x,\alpha,2} \leq \frac{C|j|}{r}$$

Let $H(\cdot)$ be a rule which assigns to any jet the appropriate holomorphic section as constructed above (that is, $H(j) = h$), define $h_1 := H(j) = h$, and define, by induction, $h_{q+1} := H(j - J_1(\sum_{i=1}^{i=q} h_i))$ for $q \geq 2$. For all $q \geq 1$ we now have that

$$|J_1(\sum_{i=1}^{i=q} h_i) - j| \leq (\frac{1}{2})^q |j|, \qquad |h_{q+1}|_{x,\alpha,2} \leq C(\frac{1}{2})^q |j|$$

As such, the series $\sum_{i=1}^{i=q} h_i$ converges to a holomorphic section $h' : M \to L$ in $L^2_g(|\cdot|_{x,\alpha})$ which passes through j such that

$$|h'|_{x,\alpha,2} \leq C|j|$$

where C is a universal constant. □

5.3. **Fuchsian series.** Let $L \to M$ be a holomorphic line bundle equipped with a metric $|\cdot|$ of strictly positive curvature and of bounded geometry. Let g be the kählerian metric associated with the curvature form of $|\cdot|$. Let us assume that the Ricci curvature of g is uniformly bounded from below by $-\frac{1}{4}$, and that the radius $r(|\cdot|)$ of $|\cdot|$ is greater than some real number r_0.

With these assumptions, every 1-jet j of a holomorphic section in $L^2(|\cdot|_{x,\alpha})$ extends to some holomorphic section in $L^2(|\cdot|_{x,\alpha})$ of norm less than $C|j|$, where $C > 0$ is a universal constant (see the preceding lemma). Let us by $m(j) : M \to E$ denote the extension of lowest norm.

Definition 5.7. Recall that we say that a subset $T \subset M$ is *δ-separated* for some real number $\delta > 0$ if for each pair of distinct points of T, the distance between them is at least δ.

So let $T \subset M$ be a δ-separated subset, and let $j = \{j_t\}_{t \in T}$ be a family of 1-jets of holomorphic sections of M in E defined on T. The series

$$\sigma(j) := \sum_{t \in T} m(j_t)$$

is the *fuchsian series* associated with j.

We shall show that these series indeed do converge, and, what is more, that they define an extension of the entire family $\{j_t\}_{t \in T}$.

Lemma 5.8. *There exists a constant $c_0 > 0$ such that if the Ricci curvature of g is uniformly bounded from below by $-c_0$, then the fuchsian series $\sigma(j)$ converges uniformly on every compact subset of M, towards a holomorphic section of E, satisfying*

$$|\sigma(j)|_{\infty,M} \leq C(\delta)|j|_{\infty,T}$$

for every δ-separated subset of T and every bounded family j of 1-jets defined on T. In addition, we have

$$|J_1\sigma(j) - j|_{\infty,T} \leq D(\delta)|j|_{\infty,T}$$

where D tends to 0 as δ tends to infinity. In particular, there exists a δ_0 such that, as long as $\delta \geq \delta_0$, every bounded family $\{j_t\}_{t\in T}$ of 1-jets of holomorphic sections on T extend to a bounded holomorphic section $\sigma : M \to E$ satisfying

$$|\sigma|_\infty \leq C|j|_{\infty,T}$$

where C is a universal constant.

Proof. By the Gårding inequalities and by the result we derived by way of Hörmander, we have

$$\begin{aligned}|\sigma(j)(\cdot)| &\leq \sum_{t\in T}|m(j_t)(\cdot)| \\ &\leq \sum_{t\in T} C_1|m(j_t)|_{x,\alpha,2}e^{-\alpha d(x,\cdot)} \\ &\leq \sum_{t\in T} C_1C_2|j_t|_{x,\alpha,2}e^{-\alpha d(x,\cdot)}\end{aligned}$$

But we also that have that, for every $y \in M$,

$$\begin{aligned}\sum_{t\in T} e^{-\alpha d(t,y)} &\leq E(\delta)\sum_{t\in T}\int_{B_g(t,\delta/2)} e^{-\alpha d(z,y)}\,\mathrm{d}v_g(z) \\ &\leq E(\delta)\int_M e^{-\alpha d(z,y)}\,\mathrm{d}v_g(z)\end{aligned}$$

where $E(\delta) = e^{\alpha\delta}/\nu(\delta/2)$ and $\nu(\delta) = \inf_{x\in M}\mathrm{vol}(B_g(x,\delta))$. By Lemma 4.2 for every $r \geq 1$, there exists a universal constant C such that

$$vol(B_g(x,r)) \leq Ce^{\alpha r/2}$$

and that this means that there exists a universal constant such that the integrals

$$\int_M e^{-\alpha d(z,y)}\,\mathrm{d}v_g(z)$$

are all bounded by this constant. This proves the first inequality.

Now assume that $\delta \geq 2$. Again, by the Gårding inequality, we have that, for all $y \in T$,

$$\begin{aligned}|J_1\sigma(y) - j_y| &\leq \sum_{t\in T, t\neq y}|J_1 m(j_t)(y)| \\ &\leq C|j|_\infty \sum_{t\in T, t\neq y} e^{-\alpha d(t,y)}\end{aligned}$$

so we have

$$|J_1\sigma(y) - j_y| \leq C|j|_\infty \frac{e^\alpha}{\nu(1)} \int_{M-B_g(y,\delta-1)} e^{-\alpha d(z,y)}\,\mathrm{d}v_g(z)$$

As we have that the functions

$$f_\delta : y \in M \mapsto \int_{M-B_g(y,\delta-1)} e^{-\alpha d(z,y)}\,\mathrm{d}v_g(z)$$

converge uniformly to 0 as δ tends to infinity, the second inequality has been shown.

For the third part of the lemma, we start by defining $\sigma_1 = \sigma(j)$, and then recursively, $\sigma_{q+1} = \sigma(j - J_1(\sigma_1 + \cdots + \sigma_q))$ for each integer $q > 1$. For every q, the previous results give the inequalities

$$|j - J_1(\sigma_1 + \cdots + \sigma_q)|_{\infty,T} \leq |j|_\infty (\frac{1}{2})^q,$$
$$|\sigma_q|_{\infty,M} \leq C(\delta_0)|j|_\infty (\frac{1}{2})^{q-1}$$

The series $\sum_q \sigma_q$ converges uniformly on M towards a holomorphic section $\sigma : M \to E$ bounded by $C|j|_\infty$ and extending the family of jets j. The constant C is universal, as we can choose C to be decreasing with δ. □

5.4. **Immersion Into Projective Space.** We will now prove Theorem 5.1. By the observations of Lemma 3.7, we can, for any $r_0, c_0 > 0$, assume that the geometry of (M, g) with line bundle L is bounded with radius greater than r_0 and Ricci curvature bounded from below by $-c_0$. We will construct an immersion on the form $\pi = [\tau_0 : \cdots : \tau_N] : M \to \mathbb{C}P^N$ for some $N \geq n$, where n is the dimension of M.

Lemma 5.9. *There exists a real number $\epsilon > 0$ such that for any $\delta > 0$ and any δ-separated subset $T \subset M$, there exists a meromorphic function $\pi_T : M \to \mathbb{C}P^n$ on the form $\pi_T = [\tau_0 : \cdots : \tau_n]$, where the τ_l are holomorphic sections of L which are bounded by some universal constant, and which are well defined on the ϵ-neighbourhood T_ϵ of T, and such that there exists some constant $C > 0$ with $\pi_T^*\Omega \geq C\Omega$.*

Proof. In each point $t \in T$, define the 1-jets $j_0(t), ..., j_n(t)$ of holomorphic sections of T_tM in L by

$$j_0(t) = J_1(s), j_1(t) = J_1(z_1 s), ..., j_n(t) = J_1(z_n s)$$

in the coordinates z centred at t from Definition 3.4, where s is the section from the same definition. These jets are bounded by 1, and by 5.8, they can be extended to holomorphic sections $\tau_0, ..., \tau_n : M \to L$

such that the norm is bounded by some constant C depending only on δ.

The quotient τ_0/s is a holomorphic function taking values in $\mathbb{C}$, defined on the ball $B_g(t, r_0)$ and bounded by $C\exp(2r_0^2)$. By Schwarz's lemma, $|\tau_0| \geq 1/2$ on the union of balls $B_g(t, \epsilon_1)$ centred at a point $t \in T$ of radius $\epsilon_1 > 0$, with ϵ_1 depending only on r_0 and δ_0.

Additionally, the function

$$f = (\frac{\tau_1}{\tau_0}, ..., \frac{\tau_n}{\tau_0})$$

is a holomorphic function $B_g(t, r) \to \mathbb{C}^n$, bounded by a constant depending only on C, δ and r_0 and such that the derivative in 0, when viewed in the coordinates z, is the identity. Another use of Schwarz's lemma gives us that $\|df_x\| \geq 1/2$ for each point $x \in B_g(t, \epsilon_2)$, where ϵ_2 is a real number depending only on r_0 and δ_0. Thus setting $\epsilon = \min(\epsilon_1, \epsilon_2)$ gives us the required result. $\square$

Now recall Lemma 3.10; it allowed us to, for any two real numbers $0 < \epsilon < \delta$ to choose a finite family $\{T_m\}_{1 \leq m \leq l}$ of δ-separated subsets $T_m \subset M$ such that their union was ϵ-dense.

So for each T_m, construct the $\tau_{m,0}, ..., \tau_{m,n}$ from the lemma above, and define

$$\pi = [\tau_{m,j}]_{1 \leq m \leq l, 0 \leq j \leq n} : M \to \mathbb{C}P^N$$

where $N = (n + 1)l - 1$. Composition with the projections $p_m : \mathbb{C}P^N \to \mathbb{C}P^n$ we find meromorphic functions $p_m \circ \pi = \pi_{T_m}$. As such we have $\pi^*\Omega \geq C\Omega$ for some strictly positive constant C.

In addition, for each point $x \in M$ we have that at least one of the sections $\tau_{m,j}$ is of norm greater than $1/2$, and they are all bounded by some universal constant. As such, π is lipschitz. This π thus satisfies all the conditions of, and proves, Theorem 5.1.

As an example, $\pi : X \to \mathbb{C}P^N$ exists when X is a Riemann surface, by the logic above combined with Lemma 3.6.

6. Continuity of solutions

In this section, we shall use the results from Sections 3 and 4 to show that when we consider similar-looking $(0,1)$-forms on similar-looking manifolds with similar-looking line bundles, their L^2-minimal solutions, too, will be quite similar.

Let $\{M_j\}_{j=1,2}$ be two Riemann surfaces, and let $\{p_j : L_j \to M_j, j = 1,2\}$ be two holomorphic line bundles, with $|\cdot|_j$ smooth hermitian metrics on L_j of strictly positive curvature and of bounded geometry for $j = 1,2$. Suppose that for $j = 1,2$ there exists positive real numbers r_0, c_0 so that the radius $r(|\cdot|_j)$ is greater than r_0 and that the Ricci curvature of the kählerian metric g_j, induced by $|\cdot|_j$ on M_j, is bounded from below by $-c_0$.

Now by the results of Sections 3 and 4, we can first take a power k_1 so that, when considering $g' := g_{k_1}$, δ-separated sums of the type $\sum e^{-d_{g'}}$ converge, and then a k_0 such that for all $k \geq k_0$ and considering $L_j^{\otimes k}$ and defining $|\cdot|_x = e^{d_{g'}(x,\cdot)}|\cdot|$, we can find smooth, $L^2(|\cdot|_x)$-minimal solutions u_j for smooth, $L^2(|\cdot|_x)$ $(0,1)$-forms v_j taking coefficients in $L_j^{\otimes k}$.

For any $R > 0$, consider domains D_j and points $x_j \in D_j$ with $B(x_j, R) \subset D_j \subset M_j$. We now suppose there exists a line bundle isomorphism $\mu_R : p_1^{-1}(D_1) \to p_2^{-1}(D_2)$ and some real number $0 \leq \epsilon < 1$ such that

- μ_R is $(1+\epsilon)$-bilipschitz,
- $e^{-\epsilon}|\cdot|_2 \leq \mu_{R*}|\cdot|_1 \leq e^{\epsilon}|\cdot|_2$ and $e^{-\epsilon}|\cdot|_1 \leq \mu_R^*|\cdot|_2 \leq e^{\epsilon}|\cdot|_1$
- For every $x \in M$, every $L^2(|\cdot|_{\mu(x)})$ section ω_2 with coefficients in L_2, and every smooth, $L^2(|\cdot|_x)$ $(0,1)$-form τ_1 with coefficients in L_1, we have

$$|\tau_1 - \bar{\partial}\mu_R^*\omega_2|_1' \leq |\tau_1 - \mu_R^*\bar{\partial}\omega_2|_1' + \epsilon$$

  By letting 1 and 2 swap place, we also require

$$|\tau_2 - \bar{\partial}\mu_{R*}\omega_1|_2' \leq |\tau_2 - \mu_{R*}\bar{\partial}\omega_1|_2' + \epsilon$$

Here

$$|\cdot|_1' = \|\, e^{d_{g_1}(\xi,\cdot)}|\cdot|\,\|_{L^2(L_1,\sigma_1)}$$
$$|\cdot|_2' = \|\, e^{d_{g_2}(\mu(\xi),\cdot)}|\cdot|\,\|_{L^2(L_2,\sigma_2)}$$

Whenever the base point ξ as well as μ is obvious through context, we shall use these norms with no change in notation.

We shall henceforth suppress the R in μ_R, to avoid cluttering the notation too greatly; however, μ should still very much be considered to be defined through R.

The main result of this section will show that, for such norms centred sufficiently close to 0, closeness between $(0,1)$-forms translates to closeness between their norm-minimal solutions. However, if they are centred far from the origin, we shall be able to use the remark at the end of Lemma 4.4, so this shall prove unproblematic.

Before we begin the proof, let us fix some $\xi \in \Xi \cap B_{g_2}(0,R)$, and define a smooth function $\psi : M_2 \to [0,1]$ which is identically 1 on $B_{g_2}(\mu(\xi),R)$, 0 outside $B_{g_2}(\mu(\xi),2R)$, and such that $|d\psi| \leq C/R$, where C is a constant which depends only on r_0.

Consider as well for each for our chosen ξ and the constructed ψ the family $\mathcal{H}$ of smooth, $L^2(|\cdot|_2')$ sections h' such that $\bar{\partial}h' = \psi v_2$; as v_2 and ψ are both smooth, v_2 is $L^2(|\cdot|_2')$, and $L^2(|\psi v_2|_2') \leq L^2(|v_2|_2')$, we have by Hörmander that $\mathcal{H}$ is non-empty.

Proposition 6.1. *Assume the above discussion. Then there exists some integer k_0 such that for all integers $k \geq k_0$, the following is true:*

Consider $\xi \in M_1 \cap B_1(0,R)$, some $\delta > 0$ and μ relative to some $R > 0$. Then there exists a function $\gamma(R,\epsilon)$ which tends to 0 as R tends to infinity and ϵ tends to 0, such that the following is true:

Let us assume that we are given a smooth $(0,1)$-form v_1 taking coefficients in $L_1^{\otimes k}$ with support on some ball $B_{g_1}(\xi,\delta)$. Let u_1 be the $L^2(|\cdot|_\xi')$-minimal solution of $\bar{\partial}u = v_1$. Additionally, let us assume that we are given a smooth $(0,1)$-form v_2 taking coefficients in $L_2^{\otimes k}$ with support on some ball $B_{g_2}(\mu(\xi),\delta)$. Let u_2 be the $L^2(|\cdot|_2')$-minimal solution of $\bar{\partial}u = v_2$. Then

$$|\mu_* u_1 - u_2|_2' \leq$$
$$C(\gamma(R,\epsilon)(\max\{|v_1|_1', |v_2|_2'\}) + \max(|v_1 - \mu^* v_2|_1', |v_2 - \mu_* v_1|_2'))$$

Proof. We begin by asserting that

$$|\psi v_2 - \bar{\partial}(\psi\mu_* u_1)|_2' \leq \Delta_1$$

for every t, where

$$\Delta_1 := |v_2 - \bar{\partial}\mu_* u_1|_2' + (C/R)|\mu_* u_1|_2'$$

This follows from

$$\begin{aligned} |\psi v_2 - \bar{\partial}(\psi\mu_* u_1)|_2' &\leq |\psi v_2 - \psi\bar{\partial}\mu_* u_1|_2' + |(\bar{\partial}\psi)\mu_* u_1|_2' \\ &\leq |v_2 - \bar{\partial}\mu_* u_1|_2' + (C/R)|\mu_* u_1|_2' \end{aligned}$$

By Hörmander, there must then exist some $h_2 \in \mathcal{H}$ such that $h_2 - \psi\mu_* u_1$ is the $L^2(|\cdot|_2')$-minimal solution of $\bar{\partial}u = \psi v_2 - \bar{\partial}(\psi\mu_* u_1)$; that is, there exists a h_2 such that

$$|h_2 - \psi\mu_* u_1|_2' \leq C\Delta_1 \qquad (*)$$

We want to show that h_2 is also sufficiently close to u_2. But, as v_2 is compactly supported, then clearly we have can assume that R is large enough that $\bar{\partial} h_2 = v_2$. In particular, this means that we have

$$|h_2|_2' \geq |u_2|_2'$$

since u_2 is $L^2(|\cdot|_{\xi_2})$-minimal and both h_2 and u_2 are solutions for v_2. Combining this with $(*)$ we have

$$\begin{aligned} C\Delta_1 &\geq |h_2 - \psi\mu_* u_1|_2' \\ &\geq |h_2|_2' - |\psi\mu_* u_1|_2' \\ &\geq |u_2|_2' - |\psi\mu_* u_1|_2' \end{aligned}$$

which yields

$$|\psi\mu_* u_1|_2' \geq |u_2|_2' - C\Delta_1$$

By the bilipschitz-property of μ, we also have that there exists some function strictly positive $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\epsilon\to 0} \nu(1+\epsilon) = 1$ so that

$$|\psi\mu_* u_1|_2' \leq |\mu_* u_1|_2' \leq \nu(1+\epsilon)|u_1|_1'$$

This yields

$$|u_2|_2' - C\Delta_1 \leq |\psi\mu_* u_1|_2' \leq \nu(1+\epsilon)|u_1|_1'$$

As the conditions on μ are symmetric, we can repeat the exact same construction, modified for $\mu^* u_2 - u_2$, in order to receive

$$|u_1|_1' - C\Delta_2 \leq \nu(1+\epsilon)|u_2|_2'$$

where

$$\Delta_2 := |v_1 - \bar{\partial}\mu^* u_2|_1' + (1/R)|\mu^* u_2|_1'$$

We can thus re-use $(*)$ to receive

$$\begin{aligned} |h_2|_2' &\leq |\psi\mu_* u_1|_2' + C\Delta_1 \\ &\leq \nu(1+\epsilon)|u_1|_1' + C\Delta_1 \\ &\leq \nu^2(1+\epsilon)|u_2|_2' + C\nu(1+\epsilon)\Delta_2 + C\Delta_1 \end{aligned}$$

$h_2 - u_2$ is holomorphic; as u_2 is $L^2(|\cdot|_2')$-minimal, $h_2 - u_2$ is orthogonal to u_2. As such we have

$$\begin{aligned} |h_2 - u_2|_2' &= \sqrt{(|h_2|_2')^2 - (|u_2|_2')^2} \\ &\leq \big((\nu^4(1+\epsilon) - 1)|u_2|_2'^2 \\ &\quad + 2C\nu^2(1+\epsilon)(\nu(1+\epsilon)\Delta_2 + \Delta_1)|u_2|_2' \\ &\quad + (\nu(1+\epsilon)\Delta_2 + \Delta_1)^2\big)^{1/2} \end{aligned}$$

which yields our proof, as we have that

$$|\mu_* u_1 - u_2|_2' \leq |\psi \mu_* u_1 - u_2|_2' + |(1-\psi)\mu_* u_1|_2'$$

□

7. Solutions on Laminations

In this section, we shall use the methods developed in the previous sections to solve the $\bar{\partial}_b$-equation on compact Riemann surface laminations. We shall begin by showing how these methods apply on the rather simple case of the lamination simply itself being a compact Riemann surface, and then proceed to give examples for more complex cases.

7.1. **Case of a Single Riemann Surface.** Let X be a compact Riemann surface of genus greater than or equal to 1, and let $f : M \to X$ be the universal covering; by [12] M is either $\mathbb{C}$ in the case that the genus of X is 1, or $\mathbb{D}$ otherwise. Furthermore, denote by Γ the Deck group of f.

It is of course well known that one can solve $\bar{\partial}$ here; however, the simplicity of the setting shall allow us to demonstrate the method which we will generalise to use on more complicated cases.

So suppose we have a holomorphic line bundle $L \to X$ with hermitian metric σ such that there exist real numbers c_0, $r_0 > 0$ such that the Ricci curvature is bounded from below by $-c_0$ and the geometry is bounded with radius greater than r_0. We would like to show that if we pull back the line bundle to M, we would still have these same properties.

As before, we use the norm $|\cdot|_p = e^{d_{g'}(p,\cdot)}|\cdot|$ for all $p \in M$.

Lemma 7.1. *Assume that there exists positive constants c_0, r_0 such that (L, σ) has Ricci curvature bounded from below by $-c_0$ and has radius greater than r_0. Then with $\tilde{L} := f^*L$ and $\tilde{\sigma} := f^*\sigma$, the same statement is true for $\tilde{L}$.*

Proof. We have that the curvature $\tilde{\Omega}$ of $L \to M$ has the same lower bound as the curvature Ω of $L \to X$, since

$$dd^c\tilde{\sigma} = dd^c f^*\sigma = f^*dd^c\sigma \geq c_0 \cdot f^*dv_g = c_0 \cdot dv_{\tilde{g}}$$

As for the radius, recall the definition in 3.4 - pushing forward with f^* allows us to fulfill the same condition on M. □

We will use this to solve the $\bar{\partial}$ equation on X:

Theorem 7.2. *Suppose we are given X as above, with a positive holomorphic line bundle $L \to X$. Then there exists an integer k_0 such that, given any $k \geq k_0$ and any smooth $(0, 1)$-form v with coefficients in $L^{\otimes k}$, there exists a smooth section u of $L^{\otimes k}$ such that $\bar{\partial}_b u = v$.*

Proof. We begin by determining k_0. Consider, for any k, the positive, holomorphic line bundle $\tilde{L}^{\otimes k} \to M$ defined by $\tilde{L}^{\otimes k} := f^* L^{\otimes k}$, with hermitian metric $\tilde{\sigma}_k := f^*\sigma_k$. By Lemma 7.1, this line bundle has the same lower bound on Ricci curvature and same positive radius as $L^{\otimes k}$. By Lemma 4.2, there exists a k_1 and a constant $C > 0$ such that $\mathrm{vol} B_{g'}(0, r) \leq Ce^{r/2}$, where $g' := g_{k_1}$ is the kählerian metric derived from $(\tilde{L}^{\otimes k_1}, \tilde{\sigma}_{k_1})$ (the lack of $\tilde{}$ on g should not prove confusing). As such, we know from Lemma 4.4 that $\sum_{\xi\in\Xi} e^{-d_{g'}(x,\xi)}$ converges for all $x \in M$, all $\delta > 0$ and all δ-separated sets $\Xi \subset M$. For all $\varphi \in \Gamma$, we can use this together with Lemma 3.12 to find a k_0 such that we can find smooth, $L^2(e^{d_{g'}(\varphi(0),\cdot)}|\cdot|)$ solutions for smooth, $L^2(e^{d_{g'}(\varphi(0),\cdot)}|\cdot|)$ $(0, 1)$-forms with coefficients in $\tilde{L}^{\otimes k}$ for all $k \geq k_0$; combining this with the above and with Corollary 4.6, we have that $\sum_{\varphi\in\Gamma} e^{-d_{g'}(0,\varphi(0))}$ converges. This shall prove to be what we need.

Let $\{U_j\}_{1\leq j\leq n}$ be a cover of X by simply connected open sets. Letting $\{\alpha_j\}_{1\leq j\leq n}$ be a partition of unity relative to $\{U_j\}_{1\leq j\leq n}$, we can limit ourselves to solving $\bar{\partial}_b u_j = v_j := \alpha_j \cdot v$ for each j. Focusing on one such j, we drop the subscript.

We shall write $\tilde{v} := f^*v$. Let U_{id} be a pre-image of f^*U; for simplicity, we will assume that $0 \in U_{id}$. Define $U_\varphi := \varphi(U_{id})$ for all $\varphi \in \Gamma$, and let $\tilde{v}_\varphi := \tilde{v}|_{U_\varphi}$.

Under these conditions, we can let $\tilde{u}_\varphi$ be the $L^2(|\cdot|_{\varphi(0)})$-minimal smooth solution of $\tilde{v}_\varphi$ as described in Lemma 3.12. Define $\tilde{u} = \sum_{\varphi\in\Gamma} \tilde{u}_\varphi$. This sum converges with $\tilde{u} \in L^2(\tilde{\sigma}_k)$, since, for any $r > 0$, there exists constants C, C', $C'' > 0$ such that

$$\begin{aligned}
\sum_\varphi \|\tilde{u}_\varphi\|_{L^2(L|_{\mathbb{D}_r},\sigma)} &\leq C \sum_\varphi e^{-d_g(0,\varphi(0))} |\tilde{u}_\varphi|_{\varphi(0)} \\
&\leq C'(\sup_\varphi |\tilde{v}_\varphi|_{\varphi(0)}) \sum_\varphi e^{-d_g(0,\varphi(0))} \\
&\leq C''|\tilde{v}|_{L^2(\tilde{L},\tilde{\sigma})} < \infty
\end{aligned}$$

It remains to show that $\tilde{u}$ is actually well defined. By the definition of f, this equates to showing that $\tilde{u} = \varphi^*\tilde{u}$ for all $\varphi \in \Gamma$; it will be sufficient to show that $\tilde{u}_{id} = \varphi^*\tilde{u}_\varphi$ for all $\varphi \in \Gamma$. Since by the definition of $\tilde{v}$ we have $\tilde{v}_{id} = \varphi^*\tilde{v}_\varphi$, and $\bar{\partial}\varphi^*\tilde{u}_\varphi = \varphi^*\bar{\partial}\tilde{u}_\varphi = \varphi^*\tilde{v}_\varphi$, it remains to show that $\tilde{u}_{id}$ and $\varphi^*\tilde{u}_\varphi$ are the same solution; that is, that they are both $L^2(|\cdot|_0)$-minimal.

This is by definition true for $\tilde{u}_{id}$. For $\varphi^*\tilde{u}_\varphi$, this equates to saying that, for any holomorphic section h of $L^{\otimes k}$, we have

$$\int_M (\varphi^*\tilde{u}_\varphi)\overline{h}e^{-\sigma_k+d_{g'}(0,\cdot)}dv_{g_k} = 0$$

But we have that

$$\begin{aligned}\int_M (\varphi^*\tilde{u}_\varphi)\overline{h}e^{-\sigma_k+d_{g'}(0,\cdot)}dv_{g_k} &= \int_{\varphi(M)} \varphi_*[(\varphi^*\tilde{u}_\varphi)\overline{h}e^{-\sigma_k+d_{g'}(0,\cdot)}dv_{g_k}] \\ &= \int_M \tilde{u}_\varphi\overline{\varphi_*h}e^{-\sigma_k+d_{g'}(\varphi(0),\cdot)}dv_{g_k} = 0\end{aligned}$$

as φ is an automorphism, so $\varphi(M) = M$, σ_k and dv_{g_k} are Γ-invariant by definition, φ_*h is still holomorphic, and $\tilde{u}_\varphi$ is $L^2(|\cdot|_{\varphi(0)})$-minimal.

Pushing $\tilde{u}$ forward to X by defining $u := f_*\tilde{u}$, we now have a $L^2(\sigma_k)$ section u solving $\bar{\partial}_b u = v$. □

7.2. **Suspensions.** For our first example, we shall work on so-called suspensions; these can be thought of as being similar to tori, but twisted in a sense that will be described below.

Definition 7.3. Let X be a Riemann surface, of genus one or greater; as above, it is covered by either $\mathbb{D}$ or $\mathbb{C}$, and we will again denote it by M. Let Γ be the Deck-group associated with the covering $f : M \to X$.

Let T be a compact smooth manifold, and assume that we are given a homomorphism $\gamma : \Gamma \to Diff(T)$. Let $\tilde{\Gamma}$ be the group of diffeomorphisms of $M \times T$ consisting of elements $\tilde{\varphi} := (\varphi, \gamma(\varphi))$ for $\varphi \in \Gamma$. We consider the quotient $Y := (M \times T)/\tilde{\Gamma}$, and denote the quotient map by $\pi : M \times T \to Y$. We shall call the structure $M \times T \xrightarrow{\pi} (M \times T)/\tilde{\Gamma}$ a *suspension* over X.

Y has the structure of a Riemann surface lamination, constructed as follows: Let $U \subset M$ be a domain with $\varphi(U) \cap U \neq \emptyset, \varphi \in \Gamma \Rightarrow \varphi = id$. Letting $\tilde{U} := \{[(z,t)] | z \in U, t \in T\}$, we define the coordinate chart $\Phi_{\tilde{U}} : \tilde{U} \to U \times T$ by $[(z,t)] \to (z,t)$.

If $\tilde{V}$ is another chart with $\tilde{U} \cap \tilde{V} \neq \emptyset$, then we must have points $(z_1, t_1) \in M \times T$, $(z_2, t_2) \in M \times T$ such that $[(z_1, t_1)] = [(z_2, t_2)]$; that is, there must exist some $\varphi \in \Gamma$ such that $z_2 = \varphi(z_1)$ and $t_2 = \gamma(\varphi)(t_1)$. As such, the transition between $\Phi_{\tilde{U}}(\tilde{U} \cap \tilde{V})$ and $\Phi_{\tilde{V}}(\tilde{U} \cap \tilde{V})$ are given by $(z,t) \mapsto (\varphi(z), \gamma(\varphi)(t))$. Furthermore, the map $F : Y \to X$ given by $[(z,t)] \mapsto f([z])$ is a natural projection, and $F^{-1}(x)$ is diffeomorphic to T for all x.

Suppose that we are given a line bundle $L \to X$ with metric σ; we define $L^* := F^*L$ with metric $\sigma^* = F^*\sigma$.

Theorem 7.4. *Let $M \times T \xrightarrow{\pi} Y = (M \times T)/\tilde{\Gamma}$ be a suspension over a compact Riemann surface X, and assume we are given a line bundle $L \to X$ with metric σ. Then there exists an integer k_0 such that, given any $k \geq k_0$ and any smooth $(0,1)$-form v on Y with coefficients in $L^{*\otimes k}$, there exists a continuous section $u : Y \to L^{*\otimes k}$, smooth on every leaf, satisfying $\bar{\partial}_b u = v$.*

Proof. We must again determine how large a k_0 we need; but this process shall prove almost identical to the one used in the previous example; note in particular that Lemma 7.1 easily generalises to saying that for the covering of each leaf, we have bounded geometry with the same estimates on Ricci curvature and radius as on the leaves.

Let us work on the M's. By Lemma 4.2, there exists a k_1 and a constant $C > 0$ such that $\mathrm{vol} B_{g'}(0, r) \leq Ce^{r/2}$, where $g' := g_{k_1}$ is the kählerian metric derived from $(L^{\otimes k_1}, \sigma_{k_1})$. As such, $\sum_{\xi \in \Xi} e^{-d_{g'}(x,\xi)}$ converges for all $x \in M$, all $\delta > 0$ and all δ-separated sets $\Xi \subset M$. For all $\varphi \in \Gamma$, we can use this together with Lemma 3.12 to find a k_0 such that we can find smooth, $L^2(e^{d_{g'}(0,\varphi(0))}|\cdot|)$ solutions for smooth, $L^2(e^{d_{g'}(0,\varphi(0)}|\cdot|)$ $(0,1)$-forms with coefficients in $\tilde{L}^{\otimes k}$ for all $k \geq k_0$; combining this with the above and with Lemma 4.6, we have that $\sum_{\varphi \in \Gamma} e^{-d_{g'}(0,\varphi(0))}$ converges.

Let $\{U_j\}_{1 \leq j \leq n}$ be a cover of X by simply connected open sets. Letting $\{\alpha_j\}_{1 \leq j \leq n}$ be a partition of unity relative to $\{U_j\}_{1 \leq j \leq n}$, we can limit ourselves to solving $\bar{\partial}_b u_j = v_j := (\alpha_j \circ F) \cdot v$ for each j. Focusing on one such j, we drop the subscript.

Let $\tilde{L}$ denote the line bundle $\tilde{L} := \pi^* L^*$ with metric $\tilde{\sigma} := \pi^*\sigma$. By pulling back v to $M \times T$ by $\tilde{v} := \pi^* v$ and defining $\tilde{v}_t := \tilde{v}(\cdot, t)$, we can simply use the same method as in 7.2 to find each $\tilde{u}_{\varphi,t}$ and $\tilde{u}_t := \sum_{\varphi \in \Gamma} \tilde{u}_{\varphi,t}$.

It remains to show that when we regard it as a function of both z and t, $\tilde{u}$ is continuous and well-defined. For the continuity part, assume that $R > r$; we use the results in Section 6, combined with

$$\begin{aligned}
&\|\tilde{u}_{t_1} - \tilde{u}_{t_2}\|_{L^2(\tilde{L}|_{B_g(0,r)},\tilde{\sigma})} \\
&\leq C(r) \sum_{\varphi \in \Gamma} e^{-d'_g(\varphi(0),0)} |\tilde{u}_{\varphi,t_1} - \tilde{u}_{\varphi,t_2}|_{\varphi(0)} \\
&\leq C(r) \sum_{\varphi(0) \in M \setminus B_g(0,R)} e^{-d'_g(\varphi(0),0)} \big(|\tilde{u}_{\varphi,t_1}|_{\varphi(0)} + |\tilde{u}_{\varphi,t_2}|_{\varphi(0)} \big) \\
&\quad + C(r) \sum_{\varphi(0) \in B_g(0,R)} e^{-d'_g(\varphi(0),0)} |\tilde{u}_{\varphi,t_1} - \tilde{u}_{\varphi,t_2}|_{\varphi(0)}
\end{aligned}$$

for some constant $C(r)$ depending only on r. The first term tends to 0 as R tends to infinity by the remark at the end of Lemma 4.4, since both the norm terms are bounded by the fact that

$$\begin{aligned}
|\tilde{u}_{\varphi,t_1}|_{\varphi(0)} + |\tilde{u}_{\varphi,t_2}|_{\varphi(0)} &\leq C(|\tilde{v}_{\varphi,t_1}|_{\varphi(0)} + |\tilde{v}_{\varphi,t_2}|_{\varphi(0)}) \\
&\leq C'(\|\tilde{v}_{t_1}\|_2 + \|\tilde{v}_{t_2}\|_2) < \infty
\end{aligned}$$

while the third term tends to 0 by the convergence of the sum and by the results of Section 6.

To be well defined, we need to have that $\tilde{u}_t = \varphi^* \tilde{u}_{\gamma(\varphi)(t)}$. It will be sufficient to show that $\tilde{u}_{id,t} = \varphi^* \tilde{u}_{\varphi,\gamma(\varphi)(t)}$ for all $\varphi \in \Gamma$. Since by the definition of $\tilde{v}$ we have $\tilde{v}_{id,t} = \varphi^* \tilde{v}_{\varphi,\gamma(\varphi)(t)}$, and $\bar{\partial} \varphi^* \tilde{u}_{\varphi,\gamma(\varphi)(t)} = \varphi^* \bar{\partial} \tilde{u}_{\varphi,\gamma(\varphi)(t)} = \varphi^* \tilde{v}_{\varphi,\gamma(\varphi)(t)}$, it remains to show that $\tilde{u}_{id,t}$ and $\varphi^* \tilde{u}_{\varphi,\gamma(\varphi)(t)}$ are the same solution; that is, that they are both $L^2(|\cdot|_0)$-minimal.

This is by definition true for $\tilde{u}_{id,t}$. For $\varphi^* \tilde{u}_{\varphi,\gamma(\varphi)(t)}$, this equates to saying that, for any holomorphic section h of $\tilde{L}^{\otimes k}$, we have

$$\int_M (\varphi^* \tilde{u}_{\varphi,\gamma(\varphi)(t)}) \overline{h} e^{-\sigma_k + d_{g'}(0,\cdot)} dv_{g_k} = 0$$

But we have that

$$\begin{aligned}
\int_M (\varphi^* \tilde{u}_{\varphi,\gamma(\varphi)(t)}) \overline{h} e^{-\sigma_k + d_{g'}(0,\cdot)} dv_{g_k} &= \int_{\varphi(M)} \varphi_* [(\varphi^* \tilde{u}_{\varphi,\gamma(\varphi)(t)}) \overline{h} e^{-\sigma_k + d_{g'}(0,\cdot)} dv_{g_k}] \\
&= \int_M \tilde{u}_{\varphi,\gamma(\varphi)(t)} \overline{\varphi_* h} e^{-\sigma_k + d_{g'}(\varphi(0),\cdot)} dv_{g_k} = 0
\end{aligned}$$

as φ is an automorphism, so $\varphi(M) = M$, σ_k and dv_{g_k} are Γ-invariant by definition, since $\varphi_* \sigma_k = \varphi_*(\pi^* \sigma_k^*) = \pi^* \sigma_k^* = \sigma_k$, $\varphi_* h$ is still holomorphic, and $\tilde{u}_{\varphi,\gamma(\varphi)(t)}$ is $L^2(|\cdot|_{\varphi(0)})$-minimal.

Pushing $\tilde{u}$ forward to X by defining $u := f_* \tilde{u}$, we now have a continuous section u solving $\bar{\partial}_b u = v$. □

7.3. **The "tower structure".** In our second example, we shall show that even for structures that might immediately appear quite different from the case of suspensions, the method for solving $\bar{\partial}_b$ remains much the same.

Definition 7.5. Let X_1 be a compact Riemann surface of genus greater than or equal to one, and let $f_1 : M \to X_1$ be a universal covering map, with Deck-group Γ; here M is either $\mathbb{C}$ if the genus of X_1 is one, or $\mathbb{D}$ otherwise. Further, we have a sequence of compact Riemann surfaces $\{X_j\}_{j\in\mathbb{N}}$ such that for each $j \geq 2$ there exists a $\pi_{j-1} : X_j \to X_{j-1}$ where π_{j-1} is an *unbranched covering* of X_{j-1} by X_j; that is, π_{j-1} is a covering that is also an immersion and globally m_j to 1, where m_j is some integer depending on j.

We now define X_∞ by letting it be the set of all points $(x_1, x_2, ...)$ of $\prod_j X_j$ such that $\pi_{j-1}(x_j) = x_{j-1}$. The basis for the topology on X_∞ is given as follows: Given a point $x = (x_1, x_2, ...)$ of X_∞ and an open disc U_{x_1} around x_1 in X_1, the basis consists of all sets

$$U_x^n = \big\{\{y_1, y_2, ...\} \,|\, \pi_1 \circ \pi_2 \circ \cdots \circ \pi_j(y_{j+1}) \in U_{x_1}, \text{with}\, y_1, y_2, ..., y_n \\ \text{uniquely determined by the lifting of} x_1 \text{ to } x_2 \text{ to } \cdots \text{ to } x_n\big\}$$

We call the construction $X_1 \overset{\pi_1}{\longleftarrow} X_2 \overset{\pi_2}{\longleftarrow} \cdots$ and X_∞ a *tower* over X_1.

We shall show that there exists a compact topological space T and a continuous surjective map $f_\infty : M \times T \to X_\infty$. Take $y_1 := f_1(0)$; we then define $T := \{(x_1, x_2, ...) \in X_\infty \,|\, x_1 = y_1\}$.

This can be used to define the map f_∞. For take any $\bar{x} = (x_1, x_2, ...) \in T$. Then we can define $f_2^{\bar{x}} : M \to X_2$ as the uniquely determined lifting of f_1 by π_1 such that $f_2^{\bar{x}}(0) = x_2$; inductively, let $f_j^{\bar{x}} : M \to X_j$ be the uniquely determined lifting of $f_{j-1}^{\bar{x}}$ by π_{j-1} such that $f_j^{\bar{x}}(0) = x_j$, and so on for all $j \geq 2$. We now set $f_\infty(z, \bar{x}) := (f_1(z), f_2^{\bar{x}}(z), ...)$.

Furthermore, given any $\bar{x} \in T$, consider, for any $\varphi \in \Gamma$, the point $\bar{x}' = (f_1^{\bar{x}}(\varphi(0)), f_2^{\bar{x}}(\varphi(0)), ...) \in T$. This relation means that there must exist some homomorphism $\lambda : \Gamma \to Homeo(T)$ such that $f_\infty(z, \bar{x}) = f_\infty(\varphi^{-1}(z), \lambda(\varphi)(\bar{x}))$ for all $z \in M$ and all $\bar{x} \in T$.

We will define coordinate charts on X_∞ in the following manner:

Let $U \subset M$ be a domain such that $\varphi(U) \cap U \neq \emptyset$, $\varphi \in \Gamma \Rightarrow \varphi = id$. We let $\tilde{U} := \{(f_1^{\bar{x}}(z), f_2^{\bar{x}}(z), ...) | z \in U, t \in T\}$, and define $\Phi_{\tilde{U}} : \tilde{U} \to U \times T$ by $(f_1^{\bar{x}}(z), f_2^{\bar{x}}(z), ...) \mapsto (z, \bar{x})$.

Suppose $\tilde{V}$ is another chart with $\tilde{U} \cap \tilde{V} \neq \emptyset$. Then there must be a point $(z, \bar{x}) \in U \times T$ and a point $(z', \bar{x}') \in V \times T$ with $f_\infty(z, \bar{x}) = f_\infty(z', \bar{x}')$. This means that, for some $\varphi \in \Gamma$, $z' = \varphi(z)$ and $\bar{x}' = \lambda(\varphi)(\bar{x})$. As such, the transition between $\Phi_{\tilde{U}}(\tilde{U} \cap \tilde{V})$ and $\Phi_{\tilde{V}}(\tilde{U} \cap \tilde{V})$

is given by $(z, \bar{x}) \mapsto (\varphi(z), \lambda(\varphi)(\bar{x}))$. Now X_∞ has the structure of a Riemann surface lamination, with leaves given by $f_\infty(M \times \{t\})$. The map $F : X_\infty \to X_1$ given by $(x_1, x_2, ...) \mapsto x_1$ is a natural projection, and each fibre $X_\infty^{x_1} := F^{-1}(x_1)$ is homeomorphic to T.

Suppose $L_1 \to X_1$ is a positive holomorphic line bundle with metric σ_1. Now $L_\infty := F^* L_1$ is a positive holomorphic line bundle with metric $\sigma_\infty := F^*\sigma_1$. Given a smooth, $L^2(\sigma_\infty)$ $(0,1)$-form v on X_∞ taking coefficients in L_∞, we would like to find a smooth, $L^2(\sigma_\infty)$ section solving $\bar{\partial}_b u = v$.

In order to do this, we will push the problem over to $M \times T$ and solve it there, by use of Theorem 3.8, and then pull the solution back to X_∞. However, in order to do this, we would need to work on a line bundle over $M \times T$, with a corresponding metric, satisfying the requirements of Theorem 3.8; that is, viewing the restriction of the line bundle to every $M \times \{t\}$, we would want the geometry to be bounded with radius greater than 1, and the Ricci curvature to be bounded from below by $-1/4$ - however, by exactly the same logic as previously, this will be the case as long as we have such conditions on L_1.

The theorem we want to prove is the following:

Theorem 7.6. *Assume that we are given* $X_1 \overset{\pi_1}{\longleftarrow} X_2 \overset{\pi_2}{\longleftarrow} \cdots$ *and* X_∞ *as above, as well as a line bundle* $L_1 \to X_1$ *with metric* σ_1*. Let* L_∞ *be as above. Then there exists an integer* k_0 *such that, given any* $k \geq k_0$ *and any smooth* $(0,1)$*-form* v *with coefficients in* $L_\infty^{\otimes k}$*, there exists a smooth section* u *of* X_∞ *in* $L_\infty^{\otimes k}$ *such that* $\bar{\partial}_b u = v$*.*

Proof. We start by determining k; but this is done exactly as in the previous example.

Let $\{U_j\}_{1\leq j\leq n}$ be a cover of X_1 by simply connected open sets. Letting $\{\alpha_j\}_{1\leq j\leq n}$ be a partition of unity relative to $\{U_j\}_{1\leq j\leq n}$, we can limit ourselves to solving $\bar{\partial}_b u_j = v_j := (\alpha_j \circ F) \cdot v$ for each j. Focusing on one such j, we drop the subscript.

We shall write $\tilde{v} := f_\infty^* v$, and $\tilde{v}_t := \tilde{v}(\cdot, t)$. Let U_{id} be a pre-image of f^*U; for simplicity, we will assume that $0 \in U_{id}$. Define $U_\varphi := \varphi(U_{id})$ for all $\varphi \in \Gamma$, and let $\tilde{v}_{\varphi,t} := \tilde{v}_t|_{U_\varphi}$. Again, we can just repeat the process from the simple case on each leaf.

Now we can show that $\tilde{u}_t$ is continuous as a function of t, using the results in Section 6; once again, this shall prove quite similar to the previous example, as

$$\begin{aligned} &\|\tilde{u}_{t_1} - \tilde{u}_{t_2}\|_{L^2(\tilde{L}|_{B_g(0,r)},\tilde{\sigma})} \\ &\leq C(r) \sum_{\varphi\in\Gamma} e^{-d'_g(\varphi(0),0)} |\tilde{u}_{\varphi,t_1} - \tilde{u}_{\varphi,t_2}|_{\varphi(0)} \\ &\leq C(r) \sum_{\varphi(0)\in M\setminus B_g(0,R)} e^{-d'_g(\varphi(0),0)} \big(|\tilde{u}_{\varphi,t_1}|_{\varphi(0)} + |\tilde{u}_{\varphi,t_2}|_{\varphi(0)}\big) \\ &\quad + C(r) \sum_{\varphi(0)\in B_g(0,R)} e^{-d'_g(\varphi(0),0)} |\tilde{u}_{\varphi,t_1} - \tilde{u}_{\varphi,t_2}|_{\varphi(0)} \end{aligned}$$

and this tends to 0 as R tends to infinity and t_1 and t_2 are chosen sufficiently close by the exact same logic as in the previous example.

To be well defined, we need to have that $\tilde{u}_t = \varphi^*\tilde{u}_{\lambda(\varphi)(t)}$. It will be sufficient to show that $\tilde{u}_{id,t} = \varphi^*\tilde{u}_{\varphi,\lambda(\varphi)(t)}$ for all $\varphi \in \Gamma$. Since by the definition of $\tilde{v}$ we have $\tilde{v}_{id,t} = \varphi^*\tilde{v}_{\varphi,\lambda(\varphi)(t)}$, and $\bar{\partial}\varphi^*\tilde{u}_{\varphi,\lambda(\varphi)(t)} = \varphi^*\bar{\partial}\tilde{u}_{\varphi,\lambda(\varphi)(t)} = \varphi^*\tilde{v}_{\varphi,\lambda(\varphi)(t)}$, it remains to show that $\tilde{u}_{id,t}$ and $\varphi^*\tilde{u}_{\varphi,\lambda(\varphi)(t)}$ are the same solution; that is, that they are both $L^2(|\cdot|_0)$-minimal.

This is by definition true for $\tilde{u}_{id,t}$. For $\varphi^*\tilde{u}_{\varphi,\lambda(\varphi)(t)}$, this equates to saying that, for any holomorphic section h of $\tilde{L}^{\otimes k}$, we have

$$\int_M (\varphi^*\tilde{u}_{\varphi,\lambda(\varphi)(t)})\overline{h}e^{-\sigma_k+d_{g'}(0,\cdot)}dv_{g_k} = 0$$

But we have that

$$\begin{aligned} \int_M (\varphi^*\tilde{u}_{\varphi,\lambda(\varphi)(t)})\overline{h}e^{-\sigma_k+d_{g'}(0,\cdot)}dv_{g_k} &= \int_{\varphi(M)} \varphi_*[(\varphi^*\tilde{u}_{\varphi,\lambda(\varphi)(t)})\overline{h}e^{-\sigma_k+d_{g'}(0,\cdot)}dv_{g_k}] \\ &= \int_M \tilde{u}_{\varphi,\lambda(\varphi)(t)}\overline{\varphi_*h}e^{-\sigma_k+d_{g'}(\varphi(0),\cdot)}dv_{g_k} = 0 \end{aligned}$$

as φ is an automorphism, so $\varphi(M) = M$, σ_k and dv_{g_k} are Γ-invariant by definition, φ_*h is still holomorphic, and $\tilde{u}_{\varphi,\lambda(\varphi)(t)}$ is $L^2(|\cdot|_{\varphi(0)})$-minimal.

All that remains is to regard $\tilde{u}_t$ as a function of z and t and define $u := f_{\infty *}\tilde{u}$; it is now well defined, continuous, and solves $\bar{\partial}_b u = v$. □

We notice that after breaking down the structure of the two examples, the methods of solution are essentially identical. This shall give us a good indication as to how to work even in more general cases.

8. Hyperbolic Laminations

In the previous section, we demonstrated how to solve the Cauchy-Riemann equations on some simpler laminations. However, we did not utilise the full power of Section 6; if we could actually construct the μ-s, we could work on more advanced cases. In this section, we would thus like to prove the main theorem.

In both our previous examples, we worked on line bundles originating on a simple Riemann surface. The main point of interest in this section is to be able to work on line bundles originating on the lamination itself.

8.1. Line Bundle Isomorphisms.

Another point is that in our previous examples, the covering map was the same for each leaf. However, this might not always be the case. However, as long as all leaves are *hyperbolic*, that is, as long as each leaf is covered by $\mathbb{D}$, the following result, whose proof can be found in [11], states that the covering maps indeed vary with some form of regularity.

Theorem 8.1. *Let X be a compact Riemann surface lamination, and assume that all leaves in X are hyperbolic. Then, if x_j is a sequence of points in X converging to a point $x_0 \in X$, if v_j is a sequence of tangent vectors at the points x_j converging to a nonzero tangent vector v_0 at x_0, and if $f_j : \mathbb{D} \to \mathcal{L}_{x_j}$ are the universal covering maps with $f_j(0) = x_j$, $f_j'(0) = \lambda_j \cdot v_j$ for some real numbers $\lambda_j > 0$, then the sequence f_j converges uniformly on compacts to the universal covering map $f_0 : \mathbb{D} \to \mathcal{L}_x$ with $f(0) = x$ and $f'(0) = \lambda_0 \cdot v_0$ for $\lambda_0 > 0$.*

By mimicking the methods from the previous section, it becomes easy to find solutions on the universal cover of each leaf, and show that these solutions are well defined when pushed forward to the lamination. Demonstrating continuity, however, relies on the existence of the μ-s. This will follow from the following:

Proposition 8.2. *Suppose X is a compact hyperbolic manifold, and let $L^* \to X$ be a positive holomorphic line bundle. Let $L \to \mathbb{D} \times T$ be defined by $L|_{\mathbb{D}\times\{t\}} := f_t^* L^*$. Then for any $0 < r < 1$ there exists an isomorphism $\Phi_r : L(r) \to (\mathbb{D}_r \times T) \times \mathbb{C}$, where $L(r) := L|_{\mathbb{D}_r \times T}$.*

Proof. As X is compact, we can find a finite covering $\{V_\alpha\}$ of simply connected open sets, along with the lamination-defining homeomorphisms $\phi_\alpha : V_\alpha \to \mathbb{D} \times T_\alpha$, such that the line bundle is trivial over each V_j.

For any α, $\phi_\alpha^* V_\alpha$ can be written as a locally finite covering $\{U_j\}$ of $\mathbb{D} \times T_\alpha$, with L trivial on the restriction to each U_j - working on any one of the α, we henceforth drop the subscript.

Now we view the line bundle L as the covering $\{U_j\}$ and a family of functions $f_{ij}(z,t) \in \mathcal{O}^*(U_i \cap U_j)$ satisfying the condition $f_{ij} \cdot f_{jk} \cdot f_{ki} = 1$ on $U_i \cap U_j \cap U_k$ for all i, j, k.

As the line bundle is trivial over U_j, and as *any* line bundle over $\mathbb{D}$ is trivial (see for example [12]), we can choose branches $g_{ij}(z,t) := \log f_{ij}(z,t)$, smooth in z and continuous in t, such that we have $g_{ij} + g_{jk} + g_{ki} = 0$.

Now we construct a smooth partition of unity $\{\alpha_j\}$ relative to $\{U_j\}$. By the local finiteness of the covering, we can repeat the construction from Lemma 3.11 to create α_j that are smooth in z and continuous in t. So define

$$g_j(z,t) := \sum_k \alpha_k(z,t) \cdot \log f_{kj}(z,t)$$

Clearly we have

$$g_j - g_i := \sum_k \alpha_k \cdot (\log f_{kj} - \log f_{ki}) = \sum_k \alpha_k \cdot \log f_{ij} = g_{ij}$$

And, as $\{U_j\}$ is locally finite, $g_j(z,t)$ is smooth in z and continuous in t.

Now define $\omega := \bar{\partial}_b g_j$; this is well defined, as, since $f_{ij}(z,t) \in \mathcal{O}^*(U_i \cap U_j)$, we have, on $U_j \cap U_i$,

$$\bar{\partial}_b(g_j - g_i) = \bar{\partial} \log f_{ij} = 0$$

Thus ω is smooth in z and continuous in t.

Now define $\tilde{\chi}_r(z) : \mathbb{D} \to \mathbb{R}$ as a smooth function identically 1 on $\bar{\mathbb{D}}_r$, with compact support, and $\chi_r(z,t) : \mathbb{D} \times T \to \mathbb{R}$ by $\chi_r(z,t) := \tilde{\chi}_r(z)$.

From [13], we know that the fact that there exists some function $h_R(z,t)$ with $\bar{\partial} h = \chi_R \cdot \omega$, on the form

$$h_R(z,t) := -\frac{1}{2\pi i} \int_{\mathbb{D}} \frac{\chi_R \cdot \omega}{\zeta - z} \wedge d\zeta$$

In particular, $\bar{\partial} h_R = \omega$ on $\mathbb{D}_R$, and so $g_j - h_R$ is holomorphic on $U_j \cap \mathbb{D}_R$ for every j. Furthermore we see from the above formula that h_R is smooth in z and continous in t.

Thus defining $f_j := e^{g_j - h_R}$ on $U_j \cap \mathbb{D}_R$, we have that on $(U_j \cap U_i \cap \mathbb{D}_R) \times T$,

$$f_i^{-1} f_j = e^{g_j - g_i} = e^{g_{ij}} = f_{ij}$$

In particular, this means that we have found a splitting of the line bundle $L|_{\mathbb{D}_R\times T}$, and this means the required Φ_R exists by standard line bundle theory. □

Thus the required μ_R from M_1 to M_2 would be the $\Phi_{R,t_2}^{-1}\circ\Phi_{R,t_1}$.

8.2. **Proof of Theorem 1.1.**

Proof. Since X is compact, we can take a finite covering $\{V_\alpha\}$ of open sets. Consider, for any given α, the flow-box

$$\phi_\alpha = (z_\alpha, t_\alpha) : V_\alpha \to \mathbb{D}\times T_\alpha$$

and let T_0 denote the transversal $\phi_\alpha^{-1}(\{0\}\times T)$. For each $t\in T_0$, let $\mathcal{L}_t$ be the leaf through t, and let $f_t:\mathbb{D}\to\mathcal{L}_t$ be a universal covering map with $f_t(0)=t$.

For each t, consider $L_t := f_t^*L$ with metric $\sigma_t := f_t^*\sigma$. We begin by determining k_0; this will prove to be almost identical to doing this in the case of a single Riemann surface, for by a simple generalisation of Lemma 7.1, bounds on the geometry of the leaves of X give us similar bounds on the geometry of each $\mathbb{D}\times\{t\}$. Thus, there exist k_1 and k_0 such that, for all t, and all $k\geq k_0$, we know from Lemma 4.4 that $\sum_{\xi\in\Xi} e^{-d_{g'}(x,\xi)}$ converges for all $x\in\mathbb{D}$, all $\delta>0$ and all δ-separated sets $\Xi\subset\mathbb{D}$, where $g'_t := g_{t,k_1}$ is the kählerian metric derived from $(L_t^{\otimes k_1}, \sigma_{t,k_1})$. Letting Γ_t be the Deck group of f_t, then for any $\varphi\in\Gamma_t$, we can use this together with Lemma 3.12 to find a k_0 such that we can find smooth, $L^2(e^{d_{g'}(\varphi(0),\cdot)}|\cdot|)$ solutions for smooth, $L^2(e^{d_{g'}(\varphi(0),\cdot}|\cdot|)$ $(0,1)$-forms with coefficients in $L_t^{\otimes k}$ for all $k\geq k_0$, and again by Corollary 4.6, we have that $\sum_{\varphi\in\Gamma_t} e^{-d_{g'}(0,\varphi(0))}$ converges.

Thus, we can on $\mathbb{D}\times\{t\}$ look at the Deck group Γ_t and construct $u_t := \sum_{\varphi\in\Gamma_t} u_{\varphi,t}$; as in the case of a single Riemann surface, this sum converges. We would want to define $u := (f_t)_*(\sum_{\varphi\in\Gamma_t} u_{\varphi,t})$ on each $\mathcal{L}_t$. It remains to check that this notion is well defined, and that it is continuous.

To check that it is well defined, consider any two points $t_1, t_2\in T_0$ both contained in the leaf $\mathcal{L}_t$ (they may be the same point). We would want to show that if there is some $\gamma\in Aut_{hol}\mathbb{D}$ such that $f_{t_2}=f_{t_1}\circ\gamma$, then $u_{t_2}=\gamma^*u_{t_1}$.

For any $\varphi\in\Gamma_t$, we have that $v_{\varphi,t_2}=\gamma^*v_{\gamma(\varphi),t_1}$, and would want to show that $u_{\varphi,t_2}=\gamma^*u_{\gamma(\varphi),t_1}$. Thus since both u_{φ,t_2} and $\varphi^*u_{\gamma(\varphi),t_2}$ are solutions for v_{φ,t_2}, we would want to show that they are both $L^2(L_t,|\cdot|_{\varphi(0)})$-minimal. This follows from the fact that, for any holomorphic section h of L_{t_2}, we have (dropping the index k and writing

1 and 2 in place of t_1 and t_2 for ease of notation)

$$\begin{aligned}\int_{\mathbb{D}}(\gamma^* u_{\gamma(\varphi),1})\overline{h}e^{-\sigma_2+d_{g_2'}(\varphi(0),\cdot)}dv_{g_2} &= \int_{\gamma(\mathbb{D})}\gamma_*[(\gamma^* u_{\gamma(\varphi),1})\overline{h}e^{-\sigma_2+d_{g_2'}(\varphi(0),\cdot)}dv_{g_2}] \\ &= \int_{\mathbb{D}} u_{\gamma(\varphi),1}\overline{\gamma_* h}e^{-\sigma_1+d_{g_1'}(\gamma\circ\varphi((0),\cdot))}dv_{g_1} = 0\end{aligned}$$

since $u_{\gamma(\varphi),1}$ is $L^2(e^{d_{g_1'}(\gamma\circ\varphi((0),\cdot))}|\cdot|)$-minimal, and because $\gamma(\mathbb{D}) = \mathbb{D}$, $\gamma_* h$ is holomorphic, and, as $L_2 = f_2^* L = \gamma^*(f_1^* L)$, we have $\gamma_*\sigma_2 = \sigma_1$ and $\gamma_* dv_{g_2} = dv_{g_1}$, and $\gamma_* d_{g_2'}(\varphi((0),\cdot)) = d_{g_1'}(\gamma\circ\varphi((0),\cdot))$.

As for continuity, this proof works just like in the two examples we worked on, although μ is now as described above, rather than being completely trivial; again we use the convention of dropping k and writing 1 and 2 instead of t_1 and t_2, for simplicity. So as long as we at least have $R > r$,

$$\begin{aligned}&\|u_2 - \mu_{R*}u_1\|_{L^2(L_2|_{B_{g_2}(0,r)},\sigma_2)} \\ &\le C(r)\sum_{\varphi\in\Gamma_2} e^{-d'_{g_2}(\varphi(0),0)}|u_2 - \mu_{R*}u_1|_{\varphi(0)} \\ &\le C(r)\sum_{\varphi(0)\in\mathbb{D}\setminus B_{g_2}(0,R/8)} e^{-d'_{g_2}(\varphi(0),0)}\left(|u_2|_{\varphi(0)} + |\mu_{R*}u_1|_{\varphi(0)}\right) \\ &\quad + C(r)\sum_{\varphi(0)\in B_{g_2}(0,R/8)} e^{-d'_{g_2}(\varphi(0),0)}|u_2 - \mu_{R*}u_1|_{\varphi(0)}\end{aligned}$$

for some constant $C(r)$ depending only on r; and by the exact same logic as in our two examples, coupled with the fact that when t_1 is sufficiently close to t_2, μ_R is 2-bilipschitz, this can be made arbitrarily small by making R sufficiently large and t_2 and t_1 sufficiently close. □

8.3. **Closing Observations.** Although we were able to find a somewhat generalised result, the method has further potential. In particular, the μ-s of Section 6 allow for the leaves to not have the same universal covering space, as opposed to only not having the same covering map. Furthermore, the concepts of bounded goemetry and convergence of δ-separated sums makes sense even on more complicated spaces than $\mathbb{C}$ or $\mathbb{D}$.

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