On TTE-representations and Borel complexity

by

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“Wir müssen wissen, wir werden wissen.”

David Hilbert

Abstract

We give a summary of the TTE-approach to computable anlysis, as background for a discussion about Borel complexity on represented spaces. We study the hyperspace $\mathcal{A}(X)$ of closed subsets of a separable metric space X, and consider the representations ψ_-, ψ_+ and ψ of this space, corresponding to the upper Fell topology, lower Fell topology and Fell topology, respectively. All of these representations are Borel equivalent, and admits Borel measurable liftings of the Cantor derivative, if X is compact. However, if X is an uncountable Polish space, the map $A \mapsto A_P$ sending a closed subset to its perfect component, which corresponds to the transfinitely iterated Cantor derivative, does not have a Borel measurable lifting relative to any of these representations. Finally, we study a representation ϕ of the Borel algebra $\mathcal{B}(X)$ on a topological space X, reflecting the way the Borel sets are generated from the open sets. We show that complementation, binary union and countable union all have computable liftings relative to ϕ, and we find conditions ensuring that the dual of a continuous function has a continuous lifting. Background from descriptive set theory is provided in an appendix.

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Contents

Abstract **ii**

Acknowledgements **iii**

Symbols **v**

1 Introduction **1**

2 Computable analysis via TTE-representations **3**
2.1 A two-step approach . . . 3
2.2 Infinite computations . . . 4
2.3 Computing on names . . . 7

3 Borel complexity of the Cantor derivative **19**
3.1 Representations of the hyperspace of closed sets . . . 19
3.2 Borel measurable liftings of the identity and the Cantor derivative . . . 20
3.3 Borel non-measurability of the transfinitely iterated derivative . . . 28

4 Computable and continuous operations on $\mathcal{B}(X)$ **37**
4.1 A representation of the Borel algebra . . . 37
4.2 Computable liftings of complement and union . . . 39
4.3 Continuous lifting of the dual of Borel functions . . . 40

A Some definitions and results from descriptive set theory **43**

Bibliography **49**

Symbols

Σ	A finite alphabet with $\{0,1\} \subset \Sigma$.
Σ^*	The set of finite strings over Σ.
Σ^ω	The set of infinite strings over Σ.
wu, wp	Concatenation of $w \in \Sigma^*$ with $u \in \Sigma^*$ or $p \in \Sigma^\omega$.
$w \sqsubseteq u, w \sqsubseteq p$	$w \in \Sigma^*$ is a prefix of $u \in \Sigma^*$ or $p \in \Sigma^\omega$.
$w \lhd u, w \lhd p$	$w \in \Sigma^*$ is a subword of $u \in \Sigma^*$ or $p \in \Sigma^\omega$.
$f : X \to Y$	A function from X into Y, which might not be total.
$< \cdot >$	A tupling function, which might depend on context.

Chapter 1

Introduction

In Chapter 2 we give a summary of the TTE-approach to computable anlysis. First we introduce Type-2 machines, capable of handling infinite input and output. Then we transfer the resulting computability concepts to other spaces by naming systems, that is surjective functions from the set of finite or infinite string over an alphabet Σ onto other spaces.

We define several levels of equivalence between naming systems, ensuring that the computability, continuity or Borel measurability induced by equivalent systems are the same. Of these classes, we consider the class of admissible representations, which is maximal under continuous reductions, to be the most natural one. This chapter is essentially a summary of the presentation given by K. Weihrauch, see [1] for more information.

The introduction of naming systems in Chapter 2 provides the context for our discussion of complexity on represented spaces in Chapter 3. We study the hyperspace $\mathcal{A}(X)$ of closed subsets of a separable metric space X, and consider the representations ψ_-, ψ_+ and ψ of this space, corresponding to necessary negative information about the elements of $\mathcal{A}(X)$, full positive information about the elements of $\mathcal{A}(X)$ and both, respectively. The main source for this chapter is the article [2] by V. Brattka and G. Gherardi. The main difference is that we are not particularly concerned with the effective Borel hierarchy, only the classical Borel hierarchy. This makes sense, considering that our main result is negative.

We construct Borel measurable liftings of the identity and the Cantor derivative with respect to these representations. In particular, if X is compact, all of these representations are Borel equivalent, and admits Borel measurable liftings of the Cantor derivative.

However, if X is an uncountable Polish space, then the transfinitely iterated Cantor derivative does not have a Borel measurable lifting relative to any of these representations. The transfinitely iterated derivative is the map $A \mapsto A_P$ sending a closed subset to its unique perfect component, which exists by the Cantor-Bendixson theorem. The non-existence of Borel measurable liftings of this map is the main result of this chapter, and in fact of this thesis. It illustrates a theorem from reversed mathematics, which says that Cantor-Bendixson is equivalent to Π^1_1-comprehension modulo elementary second order number theory.

Finally, in Chapter 4 we construct a representation ϕ of the Borel algebra $\mathcal{B}(X)$ on a topological space X, reflecting the way the Borel sets are generated from the open sets. We show that complementation, binary union and countable union all have computable liftings relative to ϕ, and ask for conditions ensuring that the dual of a Borel measurable function has a continuous lifting. It turns out that this will hold for continuous functions, if we start with only countably many names for the basis elements.

Background from descriptive set theory is provided in Appendix A, and is mostly from Moschovakis book [3].

Chapter 2

Computable analysis via TTE-representations

2.1 A two-step approach

The TTE (Type Two Enumeration) approach to computable analysis cosists of two steps:

1. Introduce a concept of computability on infinite strings of symbols.
2. Name elements of other sets by infinite strings and consider the induced computability.

For countable sets like the rationals, naming by finite strings of symbols would suffice, but for uncountable sets the first step is essential. We handle this by generalizing Turing machines to Type-2 machines with, possibly, infinite input and output. Computations on Type-2 machines might go on forever, but we will ensure that any initial seqment of the output can be obtained in a finite number of steps from an initial segment of the input.

The second step introduces computability concepts on arbitrary named spaces. However, the induced computability depends on the chosen naming system. Even if we consider naming systems to be equivalent when they give rise to the same computability concepts, we will in general obtain multiple equivalence classes. Among these, we will christen the class of admissible representations as the "natural" one. We will consider second countable T_0 spaces, and define this class directly by a canonical representative. As we shall see, there is an equivalent characterization, in terms of maximality under continuous reduction. This equivalent characterization can be extended to other topological spaces, but all spaces we consider will be second countable and T_0, hence we have little need for this [4].

2.2 Infinite computations

A Type-2 machine takes a number of finite or infinite strings of symbols as input, and gives a single finite or infinte string of symbols as output. In a successful computation, the machine first reads a finite portion of the input, while doing all the necessary book-keeping on designated work tapes, then, after a while, the machine writes some finite portion of the output, only to start over again from the top, reading some more input. This goes on until the machine halts, in the case of finite output, or forever, in the case of infinite output. In the latter case it is important to note that the machine will always continue to write, ensuring that some infinite string is being produced on the output tape.

Definition 2.1 (Type-2 machine)**.** A *k-ary Type-2 machine* M consists of:

1. A Turing machine over a input/output alphabet Σ with k one-way, read-only input tapes, a finite number of two-way work tapes, and a single one-way, write-only output tape.
2. A *type specification* $(Y_1, \ldots, Y_k, Y_0)$, where $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ for each $i = 0, 1, \ldots, k$, specifying whether the input/output on tape i is a finite or infinite string of symbols.

The type specification allows us to interpret the behaviour of the machine differently, depending on what kind of output we expect, which is important for the definition of the function $f_M : Y_1 \times \cdots \times Y_k \to Y_0$ computed by a machine M.

Definition 2.2 (Function computed by a Type-2 machine)**.** Let M be a k-ary Type-2 machine with type specification $(Y_1, \ldots, Y_k, Y_0)$. We define the *function*

$$f_M : Y_1 \times \cdots \times Y_k \to Y_0$$

computed by the machine M by:

1. $Y_0 = \Sigma^*$:
   $f_M(y_1, \ldots, y_k) = y_0$ iff M halts on input $(y_1, \ldots, y_k)$ after a finite number of steps, with y_0 on the output tape. Otherwise, $(y_1, \ldots, y_k) \notin \mathrm{dom}(f_M)$.

2. $Y_0 = \Sigma^\omega$:
   $f_M(y_1, \ldots, y_k) = y_0$ iff M computes forever on input $(y_1, \ldots, y_k)$ and writes y_0 on the output tape. Otherwise, $(y_1, \ldots, y_k) \notin \mathrm{dom}(f_M)$.

Note that this definition requires every finite portion of an infinite output to be produced within finitely many steps, because otherwise the output will not be written in ω steps, wich is what "forever" means here.

Definition 2.3 (Computable string function)**.** A string function $f : Y_1 \times \cdots \times Y_k \to Y_0$ is *computable* iff $f = f_M$ for some Type-2 machine M.

We give Σ and Σ^* the discrete topologies, and we give Σ^ω the topology induced by the basis

$$\{B_w\}_{w \in \Sigma^*} \quad , \quad B_w = \{p \in \Sigma^\omega : w \sqsubseteq p\}.$$

Unless otherwise specified, products are given the usual product topologies.

Theorem 2.4 (Computable implies continuous)**.** *Any computable function,*

$$f : Y_1 \times \cdots \times Y_k \to Y_0,$$

where $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ *for* $i = 0, 1, \ldots, k$*, is continuous.*

Proof. Let M be a Type-2 machine computing f. If $(y_0, \ldots, y_k) \in \mathrm{dom}(f)$, then, by definition, M writes any initial segment of $f(y_1, \ldots, y_k)$ in a finite number of steps. Consequently, each finite initial segment of the output of f can only depend on a finite initial portions of the inputs. With respect to the chosen topologies, this is equivalent to continuity of f, even in the trivial case of finite input. □

Proposition 2.5 (Computable extension of composition)**.** *Suppose* $f : Y_1 \times \cdots \times Y_k \to Y_0$ *and* $g : Y_0 \to Y$ *are computable functions. Then their composition* $g \circ f$ *has a computable extension* h*, such that* $g \circ f$ *is exactly the restriction of* h *to* $\mathrm{dom}(f)$*. If* $Y_0 = \Sigma^*$ *or* $Y = \Sigma^\omega$*, the composition* $g \circ f$ *itself is computable.*

Proof. Let M_f and M_g be Type-2 machines computing f and g, respectively. If $Y_0 = \Sigma^*$, we might construct a machine M computing $g \circ f$, simply by first running M_f, then M_g. Unfortunately, this does not work if $Y_0 = \Sigma^\omega$, because M_f computes forever, meaning that M_g never gets started. We remedy this by letting M_g make one step each time M_f writes a symbol, before M_f is allowed to continue. The cost of this approach is that the resulting machine M may, in general, terminate on some inputs not in the domain of f. However, if $Y = \Sigma^\omega$, this cannot happen, since M_g only makes finitely many steps if M_f does not produce an infinite sequence. □

For a set $A \subset \Sigma^*$, we say that A is decidable if its characteristic function is computable, and we say that A is semidecidable, if A is the domain of a computable function. It is a wellknown fact that according to this definition $A \subset \Sigma^*$ is decidable iff both A and $\Sigma^* \setminus A$ are semidecidable. This is the motivation behind the following definition.

Definition 2.6 (Semidecidable and decidable subsets)**.** Let $Y = Y_1 \times \cdots \times Y_k$, where $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ for $i = 0, 1, \ldots, k$. A set $A \subset Y$ is *semidecidable* in Y if A is the domain of a computable function with finite output. If $A \subset Z \subset Y$, we say that A is *semidecidable in* Z if A is the intersection of Z and a semidecidable set. In any case, A is *decidable* if both A and its complement are semidecidable.

Note that while the semidecidable sets of $Z \subset Y$ are exactly the intersections of semidecidable subset of Y with Z, the intersections of decidable subset of Y with Z are merely a subcollection of the decidable sets of $Z \subset Y$. This is because the restrictions of two sets to Z might be complements of each other, even though the original two sets was not.

Proposition 2.7 (Semidecidable set are open)**.** *Let* $Y = Y_1 \times \cdots \times Y_k$*, with* $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ *for* $i = 0, 1, \ldots, k$*, and let* $A \subset Z \subset Y$*. If* A *is semidecidable in* Z*, then* A *is open in* Z*, and if* A *is decidable in* Z*, then* A *is clopen in* Z[1]*.*

Proof. Follows directly from the definition of semidecidable, since the topology of Σ^* is discrete, and computable functions are continuous. □

Proposition 2.8 (Inverse images of computable functions preserves effectiveness)**.** *Suppose* $f : Y_1 \times \cdots \times Y_k \to Y_0$*, with* $Y_i \in \{\Sigma^*, \Sigma^\omega\}$*, is computable, and suppose* $A \subset Z \subset Y_0$*. If* A *is (semi)decidable in* Z*, then* $f^{-1}(A)$ *is (semi)decidable in* $f^{-1}(Z)$*.*

Proof. Let $A = \mathrm{dom}(g) \cap Z$, where $g : Y_0 \to \Sigma^*$ is a computable function. Let h be a computable extension of $g \circ f$, such that $\mathrm{dom}(g \circ f) = \mathrm{dom}(h) \cap \mathrm{dom}(f)$. Then

$$\begin{aligned} f^{-1}(A) &= f^{-1}(\mathrm{dom}(g)) \cap f^{-1}(Z) = \mathrm{dom}(g \circ f) \cap f^{-1}(Z) \\ &= \mathrm{dom}(h) \cap \mathrm{dom}(f) \cap f^{-1}(Z) = \mathrm{dom}(h) \cap f^{-1}(Z). \end{aligned}$$

Inverse images preserves complements, so the result for decidability follows immediately. □

We now define a function that allows us to unambiguously code multiple finite strings by a single finite or infinite string.

[1] We say that A is clopen in Z if both A and $Z \setminus A$ are open in Z.

Definition 2.9 (Wrapping function)**.** Define the wrapping function $\iota : \Sigma^* \to \Sigma^*$ by

$$\iota(a_1 a_2 \dots a_n) = 110a_1 0a_2 0 \dots 0a_n 011.$$

This definition ensures that $\iota(v) \lhd \iota(u) \Rightarrow v = u$ and that any combined suffix of $\iota(v)$ and prefix of $\iota(u)$ must be one of $\lambda, 1, 11, \iota(v)$.

2.3 Computing on names

We can transfer the computability concepts defined via Type-2 machines to any set X that is not to large, by naming the elements of X by either finite or infinite strings, and then interpreting computations on names as computations on the corresponding elements of X.

Definition 2.10 (Naming systems)**.** A *naming system* for a set X is a surjective function $\psi : Y \to X$, where $Y \in \{\Sigma^*, \Sigma^\omega\}$. We call ψ a *notation* if $Y = \Sigma^*$, and a *representation* if $Y = \Sigma^\omega$.
If $\psi(y) = x$, we say that y is a ψ-*name* for x.

Definition 2.11 (Naming systems for $\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$)**.** We let $\nu_\mathbb{N} : 2^* \to \mathbb{N}$ to be the usual binary notation of the natural numbers.
We also define a binary representation $\delta_{\mathbb{N}^\mathbb{N}} : 2^\omega \to \mathbb{N}^\mathbb{N}$ of Baire space, by

$$p \in \mathrm{dom}(\delta_{\mathbb{N}^\mathbb{N}}) \Leftrightarrow p(i) = 1 \text{ for infinitely many } i\text{'s},$$

and

$$\delta_{\mathbb{N}^\mathbb{N}}(0^{i_0} 1 0^{i_1} 1 0^{i_2} 1 \dots) = (i_0, i_1, i_2, \dots),$$

where each i_n is a natural number.

We can transfer all kinds of concepts from Σ^* or Σ^ω to any named set X.

Definition 2.12 (Classes of subsets of named spaces)**.** Let $\psi : Y \to X$ be a naming system, and suppose $A \subset X$. Then

1. A is ψ-*decidable* iff the set of names for A is a decidable subset of $\mathrm{dom}(\psi)$.
2. A is ψ-*open* iff the set of names for A is an open subset of $\mathrm{dom}(\psi)$.
3. A is ψ-*Borel* iff the set of names for A is a Borel subset of $\mathrm{dom}(\psi)$.

Let τ_ψ denote the collection of all ψ-open subsets of X. It is trivial to check that τ_ψ is a topology, the *final topology* of ψ on X. As we will see later, the ψ-Borel subsets of X are exactly the members of the Borel algebra generated by τ_ψ.

Definition 2.13 (Liftings of functions between named spaces)**.** Let $\psi : Y \to X$ and $\psi' : Y' \to X'$ be two naming systems. A (ψ, ψ')*-lifting* of a function $f : X \to X'$ is a function $F : Y \to Y'$ such that $\psi' \circ F(y) = f \circ \psi(y)$ for all $y \in \mathrm{dom}(f \circ \psi)$.

Definition 2.14 (Classes of functions on named spaces)**.** Let $\psi : Y \to X$ and $\psi' : Y' \to X'$ be two naming systems.

1. A function $f : Y \to Y'$ is (ψ, ψ')-*computable* iff it has a computable (ψ, ψ')-lifting.
2. A function $f : Y \to Y'$ is (ψ, ψ')-*continuous* iff it has a continuous (ψ, ψ')-lifting.
3. A function $f : Y \to Y'$ is (ψ, ψ')-*Borel measurable* iff it has a Borel measurable (ψ, ψ')-lifting.

It is now natural to ask: Is (ψ, ψ')-continuity and (ψ, ψ')-Borel measurability the same as continuity and Borel measurability with respect to the induced topologies τ_ψ and $\tau_{\psi'}$? We will answer this question for the admissible representations at the end of this section.

Definition 2.15 (Computable functions on $\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$)**.** A function $f : \mathbb{N} \to \mathbb{N}$ is *computable* iff it is $(\nu_\mathbb{N}, \nu_\mathbb{N})$-computable.
A function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is *computable* iff it is $(\delta_{\mathbb{N}^\mathbb{N}}, \delta_{\mathbb{N}^\mathbb{N}})$-computable.

For functions from $\mathbb{N}$ to $\mathbb{N}$ this is of course nothing else than the usual definition in terms of Turing machines.

We now turn our attention to relationships between different naming systems[2].

Definition 2.16 (Translations between naming systems)**.** Let $\psi : Y \to X$ and $\psi' : Y' \to X'$ be two naming systems[3]. We say that $F : Y \to Y'$ *translates* ψ to ψ' iff for any ψ-name of $x \in X$, F gives a ψ'-name of x, that is

$$\forall y \in \mathrm{dom}(\psi).\ (\psi' \circ F)(y) = \psi(y).$$

[2]Here we focus on naming systems, but the definitions of translation and reduction are the same for arbitrary functions.

[3]We do not require $X = X'$, but see the comment after the definition

Note that, assuming the Axiom of choice, $\psi : Y \to X$ can be translated to $\psi' : Y' \to X'$ iff $X \subset X'$, and that a translation of ψ to ψ' is the same as a (ψ, ψ')-lifting of the identity on X.

Definition 2.17 (Levels of reducibility and equivalence)**.** Let $\psi : Y \to X$ and $\psi' : Y' \to X'$ be two naming systems.

1. ψ is *reducible* to ψ', $\psi \leq \psi'$, iff ψ can be translated to ψ' by a computable function.
2. ψ is *continuously reducible* to ψ', $\psi \leq_\tau \psi'$, iff ψ can be translated to ψ' by a continuous function.
3. ψ is *Borel reducible* to ψ', $\psi \leq_{\mathbf{B}} \psi'$, iff ψ can be translated to ψ' by a Borel measurable function.
4. $\psi \equiv \psi' \Leftrightarrow \psi \leq \psi' \wedge \psi' \leq \psi$.
5. $\psi \equiv_\tau \psi' \Leftrightarrow \psi \leq_\tau \psi' \wedge \psi' \leq_\tau \psi$.
6. $\psi \equiv_{\mathbf{B}} \psi' \Leftrightarrow \psi \leq_{\mathbf{B}} \psi' \wedge \psi' \leq_{\mathbf{B}} \psi$.

The composition of computable functions has a computable extension, the composition of continuous functions is continuous, and the composition of Borel measurable functions is Borel measurable. Furthermore, the identity function is computable, hence continuous, and of course Borel measurable. Thus $\leq, \leq_\tau, \leq_{\mathbf{B}}$ are pre-orders, and $\equiv, \equiv_\tau, \equiv_{\mathbf{B}}$ are equivalence relations on the class[4] of naming systems. The next result tells us that these equivalences between naming systems correspond exactly to equivalence of induced concepts.

Theorem 2.18 (Equivalent naming systems induces the same concepts)**.** *Let $\psi : Y \to X$, $\psi' : Y' \to X$ be two naming systems.*

1. *The naming systems ψ, ψ' induces the same computability iff $\psi \equiv \psi'$.*
2. *The naming systems ψ, ψ' induces the same continuity iff $\psi \equiv_\tau \psi'$.*
3. *The naming systems ψ, ψ' induces the same Borel measurability iff $\psi \equiv_{\mathbf{B}} \psi'$.*

Proof. (⇐):
Functions: Suppose $\psi_0 : Y_0 \to X_0$ and $\psi'_0 : Y'_0 \to X_0$ are naming systems for some space X_0. Furthermore, suppose

$$\psi' \leq \psi \;\wedge\; \psi_0 \leq \psi'_0$$

[4]Even though we refer to class-relations, any restriction to a set contained in the class of naming systems will be a relation in the usual sense, i.e. a set of ordered pairs.

Suppose $f : X \to X_0$ is (ψ, ψ_0)-computable. Choose a computable (ψ, ψ_0)-lifting $F : Y \to Y_0$ of f. Let $\rho : Y' \to Y$ be a computable reduction of ψ' to ψ, and let $\rho_0 : Y_0 \to Y_0'$ be a computable reduction of ψ_0 to ψ_0'. Then $\rho_0 \circ F \circ \rho$ has a computable extension F', which is a (ψ', ψ_0')-lifting of $f = \mathrm{id}_{X_0} \circ f \circ \mathrm{id}_X$. So f is (ψ', ψ_0')-computable. The reverse direction follows by symmetry, so if

$$\psi \equiv \psi' \wedge \psi_0 \equiv \psi_0',$$

then any function $f : X \to X_0$ is (ψ, ψ_0)-computable iff it is (ψ', ψ_0')-computable.
Subsets: Suppose $A \subset Y$ is ψ-decidable. Choose a computable reduction $\rho' : Y' \to Y$ of ψ' to ψ. Then $\psi^{-1}(A) \subset \mathrm{dom}(\psi)$ is decidable, that is, $\psi^{-1}(A) = B \cap \mathrm{dom}(\psi)$ for some decidable $B \subset X$. Hence

$$\begin{aligned}(\psi')^{-1}(A) &= (\rho')^{-1}(B \cap \mathrm{dom}(\psi)) \cap \mathrm{dom}(\psi') \\ &= (\rho')^{-1}(B) \cap (\rho')^{-1}(\mathrm{dom}(\psi)) \cap \mathrm{dom}(\psi') \\ &= (\rho')^{-1}(B) \cap \mathrm{dom}(\psi')\end{aligned}$$

is decidable in $\mathrm{dom}(\psi')$, since inverse images of computable functions preserves decidability. Consequently, A is ψ'-decidable.
(⇒):
Suppose ψ and ψ induces the same computability. Then in particular, since $\mathrm{id}_X : X \to X$ is (ψ, ψ)-computable, it is (ψ, ψ')-computable and (ψ', ψ)-computable. Thus $\psi \equiv \psi'$.

The other cases are proved in the same way, the crucial properties being closure under composition, relativization of concepts to subsets and preservation under inverse images.
□

There will in general be many non-eqivalent naming systems. We now define the admissible representations, which are maximal with respect to continuity, and which will ensure that the notions of (ψ, ψ')-continuity and (ψ, ψ')-Borel measurability correspond to the usual notions of continuity and Borel measurability.

Definition 2.19 (Effective and computable spaces)**.** An *effective topological space* is a triple (X, σ, ν), where $\nu : \Sigma^* \to \sigma$ is a notation for a covering $\sigma \subset \mathcal{P}(X)$ of X, such that the topology induced by σ as a subbasis is T_0, or equivalently

$$x = y \Leftrightarrow \{A \in \sigma : x \in A\} = \{A \in \sigma : y \in A\}.$$

We say that X is a *computable space* iff the relation

$$u, v \in \mathrm{dom}(\nu) \wedge \nu(u) = \nu(v)$$

is semidecidable.

The elements of $A \in \sigma$ are often called *atomic properties*. In this terminology the T_0-requirement amounts to saying that the elements of X are identified by their atomic properties. Note also that the basis induced by σ is countable, since, by assumption, σ has a notation and thus is countable, so any effective topological space is second countable.

Definition 2.20 (Standard representation of an effective space)**.** Let (X, σ, ν) be an effective topological space. The *standard representation* of X induced by ν is the function $\delta_\nu : \Sigma^\omega \to X$ defined by

$$p \in \mathrm{dom}(\delta_\nu) \Rightarrow \{w : \iota(w) \lhd p\} \subset \mathrm{dom}(\nu)^5$$
$$\delta_\nu(p) = x \Leftrightarrow \{A \in \sigma : x \in A\} = \{\nu(w) : \iota(w) \lhd p\},$$

and $p \notin \mathrm{dom}(\delta_\nu)$ otherwise.

The standard representation of an effective space X will be the canonical element from the class of admissible represtations for X.

Proposition 2.21 (The standard representation is continuous and open)**.** *Let* (X, σ, ν) *be an effective topological space. The standard representation* δ_ν *is continuous and open with respect to the topology induced by* σ*.*

Proof. If A is an element of the subbasis σ, then

$$\delta_\nu^{-1}(A) = \{p : \iota(w) \lhd p \text{ for some } \nu\text{-name } w \text{ of } A\}.$$

For every element $p \in \delta_\nu^{-1}(A)$, any initial segment of p coding at least one ν-name of A will give a neighbourhood of p contained in $\delta_\nu^{-1}(A)$.
It is a bit more complicated to prove that δ_ν is open. For a general word w, note that $w = vv'$, where v is either the empty word or ends in 11, and there is no $u \in \Sigma^*$ such that $\iota(u) \lhd v'$. In any case, if we let $w' = w0^411 = vv'0^411$, then $\iota(u) \lhd w'$ iff $\iota(u) \lhd w$ iff $\iota(u) \lhd v$. Thus, any $x \in X$ named by $vp \in vv'\Sigma^\omega = w\Sigma^\omega$ also have a name $w'p \in w'\Sigma^\omega$, so

$$\delta(w\Sigma^\omega) = \delta(w'\Sigma^\omega) = \{x : x \in \iota(u) \text{ for all } \iota(u) \lhd w'\} = \bigcap_{\iota(u) \lhd w'} \nu(u),$$

which is of course open. □

Corollary 2.22 (Standard representation induces the topology of an effective space). *Let (X, σ, ν) be an effective topological space. The topology of X generateds by σ is exactly the final topology of the standard representation δ_ν .*

Note that since the standard representation is surjective, continuous and open, it is in particular a *quotient map* from its domain onto X, and the topology of X is in fact nothing else than the unique *quotient topology* induced by this quotient map.

Proposition 2.23 (The standard representation is maximal w.r.t $\leq_\tau$). *Let (X, σ, ν) be an effective topological space. Then any continuous naming system $\psi : Y \to X$ is continuously reducible to the standard representation δ_ν.*

Proof. The result is trivial if ψ is a notation. Otherwise, for each ψ-name p, we need to code all $w \in \Sigma^*$ such that $\psi(p) \in \nu(w)$ into a corresponding δ_ν-name. Note that

$$\psi(p) \in \nu(w) \Leftrightarrow \psi(p_{<n}\Sigma^\omega) \subset \nu(w) \text{ for some } n \in \mathbb{N}.$$

Let $w_0, w_1, \dots$ be a list of $\mathrm{dom}(\nu)$. Then we can define a continuous translation of ψ to δ_ν by

$$f(p) = h(0)h(1)\dots,$$

where

$$h(< i, n >) = \begin{cases} \iota(w_i) & \text{if } \psi(p_{<n}\Sigma^\omega) \subset \nu(w_i), \\ 11 & \text{otherwise.} \end{cases}$$

□

Using this lemma once for each standard representation, we immediately obtain the following corollary.

Corollary 2.24 (Topology of a space determines the standard representation). *If (X, σ, ν) and (X, σ', ν') are effective topological spaces such that the topologies induced by σ and σ' are the same, then the standard representations induced by ν and ν' are equivalent, $\delta_\nu \equiv_\tau \delta_{\nu'}$.*

In other words, the standard representation of a second countable T_0-space X depend only on the topology of X. Hence, as long as a topology of X is already given, we will, from now on, just write δ_X for the standard representation of X. More importantly, we can now define the admissible naming systems.

Definition 2.25 (Admissible naming systems). A naming system $\psi : Y \to X$ of some second countable T_0-space X is *admissible* iff it is continuously equivalent to the standard representation δ_X of X.

Lemma 2.26 (Admissible implies continuous)**.** *Every admissible naming system $\psi : Y \to X$ of some second countable T_0-space X is continuous.*

Proof. Suppose $B \subset X$ is open. Let $g : Y \to \Sigma^\omega$ be a continuous translation of ψ to the standard representation $\delta_X : \Sigma^\omega \to X$ of X. Choose an open set $A \subset \Sigma^\omega$ such that $\delta_X^{-1}(B) = A \cap \mathrm{dom}(\delta_X)$. Then

$$\psi^{-1}(B) = g^{-1}(\delta_X^{-1}(B)) \cap \mathrm{dom}(\psi) = g^{-1}(A \cap \mathrm{dom}(\delta_X)) \cap \mathrm{dom}(\psi) = g^{-1}(A) \cap \mathrm{dom}(\psi),$$

which is open in $\mathrm{dom}(\psi)$. □

Corollary 2.27 (Alternative characterization of admissible naming systems)**.** *Let X be a second countable T_0-space. A naming system $\psi : Y \to X$ is admissible iff it is continuous and a maximal naming system for X with respect to $\leq_\tau$.*

Proof. This follows trivially from assumptions by the known properties of $\leq_\tau$ and the previous lemma. □

Proposition 2.28 (A space admits a notation iff it is discrete)**.** *Let X be a second countable T_0 space. Then X has an admissible notation iff every notation for X is admissible iff X is discrete.*

Proof. If X has an admissible notation, then using the Axiom of choice and the fact that any function from a discrete space is continuous, we see that any notation of X is admissible. But obviously, since X has the final topology of any admissible naming system, this means that X is discrete. Suppose X is discrete. Then $\sigma = \{\{x_0\}, \{x_1\}, \ldots\}$ is a subbasis for X. Fix a notation $\mu : \Sigma^* \to \sigma$ of σ. Then (X, σ, μ) is an effective topological space. Define a notation $\nu : \Sigma^* \to X$ of X by

$$\mu(w) = \{\nu(w)\}, \quad \text{for all } w \in \Sigma^*.$$

Let $\delta_X : \Sigma^\omega \to X$ be the standard representation of X. We can define a continuous reduction $\rho : \Sigma^\omega \to \Sigma^*$ of δ_X to ν by

$$\rho(p) = w, \text{ where } w \text{ is the first word such that } \iota(w) \lhd p\ .$$

Hence ν is an admissible notation for X. □

Example 2.1 (Admissible naming systems for $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$)**.** *The binary notation $\nu_{\mathbb{N}}$ for natural numbers is admissible.*

The binary representation $\delta_{\mathbb{N}^{\mathbb{N}}}$ of Baire space is continuously reducible to the standard representation of $\mathbb{N}^{\mathbb{N}}$, and hence in particular admissible.

Proof. That $\nu_{\mathbb{N}}$ is admissible follows from the fact that $\mathbb{N}$ is discrete. □

We now want to introduce naming by elements of $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$ as equivalent, but more practical, variants of naming by Σ^* and Σ^ω, respectively. We will temporarily introduce parallel terminology, but after we have seen that naming by $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$ in a natural way is equivalent to naming by finite and infinite strings over a finite alphabet, we will just use the language we are already familiar with.

Definition 2.29 ($\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-naming systems)**.** We call any surjective function $\psi : Z \to X$, where $Z \in \{\mathbb{N}, \mathbb{N}^{\mathbb{N}}\}$, a $\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-*naming system* for X. We call ψ a $\mathbb{N}$-*notation* if $Z = \mathbb{N}$, and a $\mathbb{N}^{\mathbb{N}}$-*representation* if $Z = \mathbb{N}^{\mathbb{N}}$.

Definition 2.30 ($\mathbb{N}$-notated effective and computable spaces)**.** A $\mathbb{N}$-*notated effective topological space* is a triple $(X, \sigma, \tilde{\nu})$, where $\tilde{\nu} : \mathbb{N} \to \sigma$ is a notation for a covering $\sigma \subset \mathcal{P}(X)$ of X, such that the topology induced by σ as a subbasis is T_0, or equivalently

$$x = y \Leftrightarrow \{A \in \sigma : x \in A\} = \{A \in \sigma : y \in A\}.$$

We say that X is a $\mathbb{N}$-*notated computable space* iff the relation

$$i, j \in \mathrm{dom}(\tilde{\nu}) \wedge \tilde{\nu}(i) = \tilde{\nu}(j)$$

is semidecidable.

Definition 2.31 (Standard $\mathbb{N}^{\mathbb{N}}$-representation of a $\mathbb{N}$-notated effective space)**.** Let $(X, \sigma, \tilde{\nu})$ be a $\mathbb{N}$-notated effective topological space. The *standard* $\mathbb{N}^{\mathbb{N}}$*-representation* of X induced by $\tilde{\nu}$ is the function $\tilde{\delta}_{\tilde{\nu}} : \mathbb{N}^{\mathbb{N}} \to X$ defined by

$$\alpha \in \mathrm{dom}(\tilde{\delta}_{\tilde{\nu}}) \Rightarrow \{i : i + 1 \in \mathrm{range}(\alpha)\} \subset \mathrm{dom}(\tilde{\nu})$$
$$\tilde{\delta}_{\tilde{\nu}}(\alpha) = x \Leftrightarrow \{A \in \sigma : x \in A\} = \{\tilde{\nu}(i) : i + 1 \in \mathrm{range}(\alpha)\},$$

and $\alpha \notin \mathrm{dom}(\tilde{\delta}_{\tilde{\nu}})$ otherwise.

Note that unless we are interested in the computability part of the last two definitions, the function $\tilde{\nu}$ is superfluous, and we might just work with a general enumerated subbasis instead. We will do this whenever it is convenient.
In any case, the results 2.21-2.24 goes through for $\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-naming systems, with only trivial modifications of the proofs. So in particular, the standard $\mathbb{N}^{\mathbb{N}}$-representation of a second countable T_0 space depends only on the topology, and we are justified in making the following definition.

Definition 2.32 (Admissible $\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-naming systems)**.** A $\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-naming system $\psi : Z \to X$ of some second countable T_0-space X is *admissible* iff it is continuously equivalent to the standard $\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-representation $\tilde{\delta}_X$ of X.

Again, admissible $\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-naming systems are continuous, and again, we obtain an alternative characterization in terms of maximality under continuous reductions.

Corollary 2.33 (Alternative characterization of admissible $\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-naming systems)**.** *Let X be a second countable T_0-space. An $\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-naming system $\psi : Z \to X$ is admissible iff it is continuous and a maximal $\mathbb{N}/\mathbb{N}^{\mathbb{N}}$-naming system for X with respect to $\leq_\tau$.*

We are now at the point where we can tie all of this together. To see that the topological concepts related to admissible naming systems are the same, we only need to show that the standard representation δ_X and the standard $\mathbb{N}^{\mathbb{N}}$-representation $\tilde{\delta}_X$ are continuously equivalent.

Proposition 2.34 ($\tilde{\delta}_X$ is continuously equivalent to δ_X)**.** *Suppose X is a second countable T_0 space, with standard representation δ_X and standard $\mathbb{N}^{\mathbb{N}}$-representation $\tilde{\delta}_X$. Then $\delta_X \equiv_\tau \tilde{\delta}_X$.*

Proof. □

Note that this discussion also indicates that it does not matter which finite alphabet Σ we use, as long as Σ contains two or more symbols.
From now on we will no longer distinguish between naming by $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$, and naming by finite or infinite sequences of symbols over a finite alphabet. In practice, we will only work with $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$, because it is so much easier.

Example 2.2 (Admissible representation of $\ell^2(\mathbb{R})$)**.** *Fix an enumeration $\{q_0, q_1, \ldots\}$ of the rationals, and for each $s \in \mathbb{N}$, define $\gamma_s = (q_{(s)_1}, q_{(s)_2}, \ldots, q_{(s)_{|s|-1}}, 0, 0, \ldots)$. Then $\{\gamma_s\}_{s\in\mathbb{N}}$ is a countable dense subset of $\ell^2 = \ell^2(\mathbb{R})$, and $\sigma = \{B_s\}$ is a (sub)basis for the topology of ℓ^2, where $B_s = B(\gamma_s, q_{(s)_0})$. Hence, the representation $\delta_{\ell^2} : \mathbb{N}^{\mathbb{N}} \to \ell^2$ defined by*

$$\delta_{\ell^2}(\alpha) = \gamma \Leftrightarrow \{s : \gamma \in B_s\} = \{s : s+1 \in \text{range}(\alpha)\}$$

is the standard representation of ℓ^2, and hence admissible. Furthermore, if we let $\nu : \mathbb{N} \to \sigma$ be the obvious notation defined by $\nu(s) = B_s$, then (ℓ^2, σ, ν) is a computable topological space.

Proof. First, lets check that $\{\gamma\}_{s\in\mathbb{N}}$ is dense in ℓ^2. Note that for any $\gamma \in \ell^2$,

$$\begin{aligned}\lim_{n\to\infty}(\sum_{i=n}^{\infty}|\gamma(i)|^2) &= \lim_{n\to\infty}(\|\gamma\|^2 - \sum_{i=0}^{n-1}|\gamma(i)|^2)\\ &= \|\gamma\|^2 - \lim_{n\to\infty}(\sum_{i=0}^{n-1}|\gamma(i)|^2)\\ &= \|\gamma\|^2 - \|\gamma\|^2 = 0.\end{aligned}$$

Hence, given any $\epsilon > 0$, we can choose an n such that $\sum_{i=n}^{\infty}|\gamma(i)|^2 < \frac{\epsilon^2}{2}$. Furthermore, since the rationals are dense in the reals, we can choose $s_i \in \mathbb{N}$ such that

$$|q_{s_i} - \alpha(i)|^2 < \frac{\epsilon^2}{2n}$$

for $0 \leq i \leq n-1$ Then, if we let $s =< 0, s_0, s_1, \ldots, s_{n-1} >$, we have

$$\begin{aligned}\|\gamma - \gamma_s\|^2 &= \sum_{i=0}^{n-1}|\gamma(i) - \gamma_s(i)|^2 + \sum_{i=n}^{\infty}|\gamma(i)|^2\\ &< \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2.\end{aligned}$$

Second, we should also note that the relation

$$\begin{aligned}s, t \in \mathrm{dom}(\nu) \wedge \nu(s) = \nu(t) &\Leftrightarrow \nu(s) = \nu(t)\\ &\Leftrightarrow (|s| = |t| \wedge \forall i < |s|.\,(s)_i = (t)_i) \vee ((s)_0 \leq 0 \wedge (t)_0 \leq 0)\end{aligned}$$

is semidecidable, and actually even primitive recursive. The crucial observation here is that $B_s = B_t$ only if they have the same center and the same radius, or otherwise, if they are both empty. The rest is obviously true. □

Proposition 2.35 (Borel measurable right inverse of admissible representations)**.** *[2] Every admissible representation of a second countable T_0 space X has a $\underset{\sim}{\mathbf{\Delta}}^0_2$-measurable right inverse.*

Proof. It suffices to consider the standard representation $\delta_X : \mathbb{N}^{\mathbb{N}} \to X$. Since we are not interested in effectivity at the moment, we might just assume that $\{B_0, B_1, \ldots\}$ is a countable subbasis for X, and that δ_X is defined by

$$\delta_X(\alpha) = x \Leftrightarrow \{i : x \in B_i\} = \{i : i+1 \in \mathrm{range}(\alpha)\}.$$

We can define a right inverse of $e_X : X \to \mathbb{N}^{\mathbb{N}}$ by

$$e_X(x)(i) = \begin{cases} i+1, & \text{if } x \in B_i, \\ 0, & \text{otherwise.} \end{cases}$$

To analyse the complexity of e_X, it suffices to consider inverse images of subbasis elements of $\mathbb{N}^{\mathbb{N}}$, which are of the form $B_{(i,j)} = \{\alpha : \alpha(i) = j\}$. There are three cases to consider.
$j = 0$: Then $e_X^{-1}(B_{(i,0)}) = X \setminus B_i$
$j = i+1$: Then $e_X^{-1}(B_{(i,i+1)}) = B_i$
$j \neq 0 \wedge j \neq i+1$: Then $e_X^{-1}(B_{(i,j)}) = \emptyset$, by definition of e_X.
In conclusion, e_X is Borel measurable, and even $\utilde{\mathbf{\Delta}}^0_2$. Furthermore, $e_X(X) \subset \mathrm{dom}(\delta_X)$, so for any $B \subset \mathbb{N}^{\mathbb{N}}$,

$$e_X^{-1}(B \cap \mathrm{dom}(\delta_X)) = e_X^{-1}(B).$$

Hence, since $e_X : X \to \mathbb{N}^{\mathbb{N}}$ is Borel, the restriction $e'_X : X \to \mathrm{dom}(\delta_X)$ is also Borel. In general, we may restrict the codomain of a Borel function to any set containing the range of the function, and still get a Borel function. □

Note that the right inverse defined in the proof above is Borel measurable with respect to the Borel algebra $\mathcal{B}(X)$ built up from the open subsets of X.

Proposition 2.36 (Admissible representations induce the usual Borel structure)**.** *The ψ-Borel subsets of X are exactly the members of the Borel algebra $\mathcal{B}(X)$ generated by τ_ψ.*

Proof. Suppose $A \subset X$ is in $\mathcal{B}(X)$. Then since inverse images of Borel sets under Borel functions are Borel, and ψ is continuous, $\psi^{-1}(A) \in \mathcal{B}(\mathrm{dom}(\psi))$, so A is ψ-Borel.
On the other hand, if $A \subset X$ is ψ-Borel, then by definition, $\psi^{-1}(A) \in \mathcal{B}(\mathrm{dom}(\psi))$. Let $e : X \to \mathrm{dom}(\psi)$ be a Borel measurable right inverse of ψ. Then $A = e^{-1}(\psi^{-1}(A)) \in \mathcal{B}(X)$, since e is Borel mesurable. □

The main result of this section tells us that admissible naming systems of second countable T_0 spaces induces exactly the same continuity and Borel measurability as the usual topological notions, supporting the claim that the class of admissible naming systems is particularly natural.

Theorem 2.37. *Suppose $\psi : Y \to X$ and $\psi : Y' \to X'$ are admissible naming systems of second countable T_0-spaces X and X'.*

1. *A function $f : X \to X'$ is continuous iff it is (ψ, ψ')-continuous.*

2. *A function $f : X \to X'$ is Borel iff it is (ψ, ψ')-Borel.[2]*

Proof. 1.): (⇒): Suppose $f : X \to X'$ is continuous. Then $f \circ \psi$ is a continuous function into X'. Hence there exists a continuous reduction $F : Y \to Y'$ of $f \circ \psi$ to ψ'. But then, for every $\alpha \in \mathrm{dom}(f \circ \psi)$,

$$(f \circ \psi)(\alpha) = (\psi' \circ F)(\alpha),$$

so F is a continuous (ψ, ψ')-lifting of f, and f is (ψ, ψ')-continuous.
(⇐): Suppose that $f : X \to X'$ is (ψ, ψ')-continuous, and let $F : Y \to Y'$ be a continuous (ψ, ψ')-lifting of F. Then for every $\alpha \in \mathrm{dom}(f \circ \psi)$,

$$(f \circ \psi)(\alpha) = (\psi' \circ F)(\alpha),$$

so $f \circ \psi$ is continuous, which means that f is continuous, since X has the final topology of ψ.
2.): (⇒): Suppose $f : X \to X'$ is Borel measurable. Let $e' : X' \to Y'$ be a Borel measurable right inverse of ψ'. Then $F = e' \circ f \circ \psi$ is Borel measurable (ψ, ψ')-lifting of f.
(⇐): Suppose $f : X \to X'$ is (ψ, ψ')-Borel, and let $F : Y \to Y'$ be a Borel measurable lifting of F. Let $e : X \to Y$ be a Borel measurable right inveres of ψ. Then $f = \psi' \circ F \circ e$ is Borel measurable. □

Chapter 3

Borel complexity of the Cantor derivative

3.1 Representations of the hyperspace of closed sets

Suppose X is a separable metric space, with a countable dense subset $\{r_0, r_1, \ldots\}$. Assume further that $\{q_0, q_1, \ldots\}$ is an enumeration of the rationals. We define $B_s = B(r_{(s)_0}, q_{(s)_1})$, that is, the open ball with center $r_{(s)_0}$ and radius $q_{(s)_1}$, and we let $\overline{B}_s$ be the corresponding closed ball. Of course, $\{B_s\}_{s\in\mathbb{N}}$ is a countable basis for X. Note that relations between basis elements can be coded as relations between their indexes, which, as subsets of discrete spaces, are trivially $\utilde{\mathbf{\Delta}}^0_1$. We will freely abuse notation, and write things like

$$s \subset t \Leftrightarrow B_s \subset B_t,$$

whenever the intended meaning is clear from the context.

We will be interested in the hyperspace $\mathcal{A}(X)$ of closed subsets of X. We will give three different representations of this set.

Definition 3.1. Define a representation $\psi_- : \mathbb{N}^\mathbb{N} \to \mathcal{A}(X)$ by

$$\psi_-(\alpha) = X \setminus (\textstyle\bigcup_{s+1\in \text{range}(\alpha)} B_s).$$

The representation ψ_- identifies the elements of X by sufficient negative information, and is admissible with respect to the *upper Fell topology*, which is generated by the subbasis of all sets of the form

$$K^- = \{A \in \mathcal{A}(X) : A \cap K = \emptyset\},$$

where K is a compact subset of X.

Definition 3.2. Define a representation $\psi_+ : \mathbb{N}^\mathbb{N} \to \mathcal{A}(X)$ by

$$\psi_+(\alpha) = A \Leftrightarrow \{B_s : A \cap B_s\} = \{B_s : s + 1 \in \text{range}(\alpha)\}.$$

The representation ψ_+ identifies the elements of X by full positive information, and is admissible with respect to the *lower Fell topology*, with subbasis consisting of all sets of the form

$$U^+ = \{A \in \mathcal{A}(X) : A \cap U \neq \emptyset\},$$

where U is an open subset of X.

Definition 3.3. Define $\psi = \psi_- \wedge \psi_+$, that is,

$$\psi(\alpha) = A \;\Leftrightarrow\; \psi_-(\alpha_-) = A \,\wedge\, \psi_+(\alpha_+) = A,$$

where $\alpha_-(i) = \alpha(2i)$ and $\alpha_+(i) = \alpha(2i + 1)$ for all $i \in \mathbb{N}$.

The representation ψ identifies the elements of X by both sufficient negative information and full positive information. It is admissible with respect to the *Fell topology*, wich has a subbasis consisting of all sets of the form K^-, where $K \subset X$ is compact, and all sets of the form U^+, where $U \subset X$ is open.

3.2 Borel measurable liftings of the identity and the Cantor derivative

It might be useful to know when the representations ψ_-, ψ_+ and ψ are Borel equivalent. Hence, we will first try to construct Borel reductions between these representations, that is, Borel measurable liftings of the identity with respect to different representations. This should also give some idea of the problems involved in passing between these representations.

Lemma 3.4. *Suppose X is a separable metric space. The identity function on the hyperspace of closed subsets of X,* $\text{id} : \mathcal{A}(X) \to \mathcal{A}(X)$*, has a*

1. *$\utilde{\boldsymbol{\Delta}}^0_1$-measurable (ψ, ψ_-)-lifting* I^-,

2. $\underset{\sim}{\mathbf{\Delta}}^0_1$*-measurable* (ψ, ψ_+)*-lifting* I^+,

3. $\underset{\sim}{\mathbf{\Delta}}^0_2$*-measurable* (ψ_+, ψ_-)*-lifting* I^-_+,

4. $\underset{\sim}{\mathbf{\Delta}}^0_2$*-measurable* (ψ_+, ψ)*-lifting* I_+,

5. $\underset{\sim}{\mathbf{\Delta}}^0_3$*-measurable* (ψ_-, ψ_+)*-lifting* I^+_-*, provided* X *is compact,*

6. $\underset{\sim}{\mathbf{\Delta}}^0_3$*-measurable* (ψ_-, ψ)*-lifting* I_-*, provided* X *is compact.*

Proof. 1.) Define $\mathrm{I}^- : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\mathrm{I}^-(\alpha)(s) = \alpha(2s)$, for all $\alpha \in \mathbb{N}^\mathbb{N}$ and $s \in \mathbb{N}$. The inverse image of a subasis element $B_{(i,j)} = \{\alpha : \alpha(i) = j\}$ is the subasis element $B_{(2i,j)}$, which is clopen.

2.) Define $\mathrm{I}^+ : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\mathrm{I}^+(\alpha)(s) = \alpha(2s+1)$, for all $\alpha \in \mathbb{N}^\mathbb{N}$ and $s \in \mathbb{N}$. The inverse image of a subasis element $B_{(i,j)}$ is the subasis element $B_{(2i+1,j)}$, which is clopen.

3.) Define $\mathrm{I}^-_+ : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, for all $\alpha \in \mathbb{N}^\mathbb{N}$ and $s \in \mathbb{N}$, by

$$\mathrm{I}^-_+(\alpha)(s) = \begin{cases} 0, & \text{if } \exists i.\alpha(i) = s+1, \\ s+1 & \text{if } \neg\exists i.\alpha(i) = s+1. \end{cases}$$

To analyse the Borel complexity of I^-_+, it suffices to consider subbasis elements of $\mathbb{N}^\mathbb{N}$, of the form $B_{(s,j)} = \{\alpha : \alpha(s) = j\}$. There are three cases to consider.
$j = 0$: Then $(\mathrm{I}^-_+)^{-1}(B_{(s,0)}) = \{\alpha : \exists i.\alpha(i) = s+1\}$ is $\underset{\sim}{\mathbf{\Sigma}}^0_1$, since the evaluation map $(\alpha, i) \mapsto \alpha(i)$ is continuous.
$j = s+1$: Then $(\mathrm{I}^-_+)^{-1}(B_{(s,s+1)}) = \{\alpha : \neg\exists i.\alpha(i) = s+1\}$ is $\underset{\sim}{\mathbf{\Pi}}^0_1$.
$j \neq 0 \wedge j \neq s+1$: Then $(\mathrm{I}^-_+)^{-1}(B_{(i,j)}) = \emptyset$, by definition of I^-_+.
In conclusion, I^-_+ is $\underset{\sim}{\mathbf{\Delta}}^0_2$-measurable.

4.) Define $\mathrm{I}_+ : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, for all $\alpha \in \mathbb{N}^\mathbb{N}$ and $s \in \mathbb{N}$, by

$$\mathrm{I}_+(\alpha)(s) = \begin{cases} \mathrm{I}^-_+(\alpha)(i), & \text{if } s = 2i, \\ \alpha(i), & \text{if } s = 2i+1. \end{cases}$$

It suffices to consider subbasis elements $B_{(s,j)} = \{\alpha : \alpha(s) = j\}$. There are two cases.
$s = 2i$: Then $(\mathrm{I}_+)^{-1}(B_{(2i,j)}) = (\mathrm{I}^-_+)^{-1}(B_{(i,j)})$ is $\underset{\sim}{\mathbf{\Delta}}^0_2$, by 3.).
$s = 2i+1$: Then $(\mathrm{I}_+)^{-1}(B_{(2i+1,j)}) = B_{(i,j)}$ which is $\underset{\sim}{\mathbf{\Sigma}}^0_1$.
In conclusion, I_+ is $\underset{\sim}{\mathbf{\Delta}}^0_2$-measurable.

5.) This case is complicated by the fact that we do not have full information to begin with. However, assuming that X is compact, we can remedy this. Define a relation

$$\begin{aligned}
\textsc{Disjoint}_-(\alpha, s) &\Leftrightarrow B_s \cap \psi_-(\alpha) \neq \emptyset, \\
&\Leftrightarrow B(r_{(s)_0}, q_{(s)_1}) \subset \bigcup_i B_{\alpha_*(i)} \\
&\Leftrightarrow \forall t.[((t)_0 = (s)_0 \wedge q_{(t)_1} < q_{(s)_1}) \rightarrow \overline{B}_t \subset \bigcup_i B_{\alpha_*(i)}] \\
&\Leftrightarrow \forall t.[((t)_0 = (s)_0 \wedge q_{(t)_1} < q_{(s)_1}) \rightarrow \exists m.(\overline{B}_t \subset \bigcup_{i<m} B_{\alpha_*(i)})],
\end{aligned}$$

where

$$\alpha_*(i) = \begin{cases} <0,0>, & \text{if } i = 0, \\ i-1, & \text{otherwise,} \end{cases}$$

so in particular $B_{\alpha_*(0)} = \emptyset$. The relation $\textsc{Disjoint}_- \subset \mathbb{N}^\mathbb{N} \times \mathbb{N}$ is $\underset{\sim}{\mathbf{\Pi}}^0_2$, since the relation

$$(t, w) \Leftrightarrow \overline{B}_t \subset (B_{(w)_0} \cup (B_{(w)_1} \cup \ldots \cup B_{(w)_{|w|-1}})$$

is trivially $\underset{\sim}{\mathbf{\Delta}}^0_1$, and the map $(\alpha, m) \mapsto <\alpha(0), \alpha(1), \ldots, \alpha(m-1)>$ is continuous. Now we can define $\mathrm{I}^+_- : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ by

$$\mathrm{I}^+_-(\alpha)(s) = \begin{cases} s+1, & \text{if } \neg\textsc{Disjoint}(\alpha, s), \\ 0, & \text{if } \textsc{Disjoint}(\alpha, s), \end{cases}$$

for all $\alpha \in \mathbb{N}^\mathbb{N}$ and $s \in \mathbb{N}$. Again we consider subbasis elements $B_{(s,j)}$ of $\mathbb{N}^\mathbb{N}$. There are three cases.

$j = s+1$: Then $(\mathrm{I}^+_-)^{-1}(B_{(s,s+1)}) = \{\alpha : \neg\textsc{Disjoint}(\alpha, s)\}$ is $\underset{\sim}{\mathbf{\Sigma}}^0_2$.

$j = 0$: Then $(\mathrm{I}^+_-)^{-1}(B_{(s,0)}) = \{\alpha : \textsc{Disjoint}(\alpha, s)\}$ is $\underset{\sim}{\mathbf{\Pi}}^0_2$.

$j \neq 0 \wedge j \neq s+1$: Then $(\mathrm{I}^+_-)^{-1}(B_{(i,j)}) = \emptyset$.

In conclusion, I^-_+ is $\underset{\sim}{\mathbf{\Delta}}^0_3$-measurable.

6.) Suppose X is compact. Then we can define $\mathrm{I}_- : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$, for all $\alpha \in \mathbb{N}^\mathbb{N}$ and $s \in \mathbb{N}$, by

$$\mathrm{I}_-(\alpha)(s) = \begin{cases} \alpha(i), & \text{if } s = 2i, \\ \mathrm{I}^+_-(\alpha)(i), & \text{if } s = 2i+1. \end{cases}$$

Consider inverse images of subbasis elements $B_{(s,j)} = \{\alpha : \alpha(s) = j\}$. There are two cases.

$s = 2i$: Then $(\mathrm{I}_-)^{-1}(B_{(2i,j)}) = B_{(i,j)}$ which is $\underset{\sim}{\mathbf{\Sigma}}^0_1$.

$s = 2i+1$: Then $(\mathrm{I}_-)^{-1}(B_{(2i+1,j)}) = (\mathrm{I}^+_-)^{-1}(B_{(i,j)})$ is $\underset{\sim}{\mathbf{\Delta}}^0_3$, by 5.).

In conclusion, I_+ is $\underset{\sim}{\mathbf{\Delta}}^0_3$-measurable. □

The Cantor derivative is the the function $d : \mathcal{A}(X) \to \mathcal{A}(X)$ mapping a closed set A to the set A' of all limit points of A. In other words, the Cantor derivative strips a closed set of all its isolated points. However, this does not exclude the possibility that A' might have isolated points which was not isolated in A. To get a perfect subset of A, that is, a closed subset without isolated points, we have to iterate the Cantor derivative more than countably many times. We will discuss this in more detail in the next section. Now we construct Borel-measurable liftings of the Cantor derivative, relative to the representations ψ_-, ψ_+ and ψ.
In general, we must at least assume that the space X is σ-compact, as the Cantor derivative is not Borel measurable relative to (ψ, ψ_+) unless X is σ-compact [2].

Proposition 3.5. *The Cantor derivative $d : \mathcal{A}(X) \to \mathcal{A}(X)$ has a*

1. *$\underset{\sim}{\mathbf{\Delta}}^0_2$-measurable (ψ_+, ψ_-)-lifting F^-_+,*
2. *$\underset{\sim}{\mathbf{\Delta}}^0_4$-measurable (ψ_+, ψ_+)-lifting F^+_+, provided X is compact,*
3. *$\underset{\sim}{\mathbf{\Delta}}^0_4$-measurable (ψ_+, ψ)-lifting F_+, provided X is compact,*
4. *$\underset{\sim}{\mathbf{\Delta}}^0_3$-measurable (ψ_-, ψ_-)-lifting F^-_-, provided X is compact,*
5. *$\underset{\sim}{\mathbf{\Delta}}^0_6$-measurable (ψ_-, ψ_+)-lifting F^+_-, provided X is compact,*
6. *$\underset{\sim}{\mathbf{\Delta}}^0_6$-measurable (ψ_-, ψ)-lifting F_-, provided X is compact,*
7. *$\underset{\sim}{\mathbf{\Delta}}^0_2$-measurable (ψ, ψ_-)-lifting F^-,*
8. *$\underset{\sim}{\mathbf{\Delta}}^0_4$-measurable (ψ, ψ_+)-lifting F^+, provided X is compact,*
9. *$\underset{\sim}{\mathbf{\Delta}}^0_4$-measurable (ψ, ψ)-lifting F, provided X is compact.*

Proof. 1.) Suppose α is a ψ_+-name for the closed subset $\psi_+(\alpha) \subset X$. We want to find a ψ_--name $F^-_+(\alpha)$ for $\psi_+(\alpha)' \subset X$. Since ψ_- is a representation by only sufficient negative information and $\{B_s\}_{s\in\mathbb{N}}$ is a basis for X, we only need to make sure that every number $s+1$ for which $B_s \cap \psi_+(\alpha)$ is empty or a singleton is included in $\mathrm{range}(F^-_+(\alpha))$. Remembering that X is Hausdorff, we can conveniently define the useful relation $\textsc{MaxOne}_+ \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ by

$$\begin{aligned}\textsc{MaxOne}_+(\alpha, s) \;&\Leftrightarrow B_s \cap \psi_+(\alpha) \text{ contains at most one point.}\\ &\Leftrightarrow \text{ There are no two basis elements intersecting}\\ &\quad\; \psi_+(\alpha) \text{ which are contained in } B_s \text{ and disjoint from each other.}\\ &\Leftrightarrow \text{ There are no two numbers } n+1, m+1 \in \text{range}(\alpha)\\ &\quad\;\text{such that } B_n \subset B_s \wedge B_m \subset B_s \wedge B_n \cap B_m = \emptyset.\\ &\Leftrightarrow \neg\exists m.\exists n.[(\exists i.\alpha(i) = n+1) \wedge (\exists i.\alpha(i) = m+1)\\ &\qquad\qquad \wedge (n \subset s) \wedge (m \subset s) \wedge (n \cap m = \emptyset)].\end{aligned}$$

Note that the evaluation map $\phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ defined by $\phi(\alpha, i) = \alpha(i)$ is continuous, since

$$\phi^{-1}(\{j\}) = \{(\alpha, i) : \alpha(i) = j\} = \bigcup_{i \in \mathbb{N}} B_{(i,j)} \times \{i\},$$

where each $B_{(i,j)} = \{\alpha : \alpha(i) = j\}$ is a subbasis element of $\mathbb{N}^{\mathbb{N}}$. Thus it is easy to see that the relation $\textsc{MaxOne}_+$ is $\utilde{\mathbf{\Pi}}^0_1$. Now we can simply define a (ψ_+, ψ_-)-lifting $F_+^- : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ of the Cantor derivative by

$$F_+^-(\alpha)(s) = \begin{cases} s+1, & \text{if } \textsc{MaxOne}_+(\alpha, s), \\ 0, & \text{if } \neg\textsc{MaxOne}_+(\alpha, s). \end{cases}$$

We now want to analyse the Borel complexity of F_+^-. There are three cases to consider.
$j = 0$: Then

$$(F_+^-)^{-1}(B_{(s,0)}) = \{\alpha : \neg\textsc{MaxOne}_+(\alpha, s)\},$$

which is $\utilde{\mathbf{\Sigma}}^0_1$, since $\alpha \mapsto (\alpha, s)$ is continuous for each fixed number s.
$j = s+1$: Then

$$(F_+^-)^{-1}(B_{(s,s+1)}) = \{\alpha : \textsc{MaxOne}_+(\alpha, s)\},$$

which is $\utilde{\mathbf{\Pi}}^0_1$, since $\alpha \mapsto (\alpha, s)$ is continuous for each fixed number s.
$j \neq 0 \wedge j \neq s+1$: Then $(F_+^-)^{-1}(B_{(i,j)}) = \emptyset$, by definition of $F(^-_+)$.
In conclusion, F_+^- is $\utilde{\mathbf{\Delta}}^0_2$-measurable.

2.) Suppose α is a ψ_+-name for the closed subset $\psi_+(\alpha) \subset X$. We want to find a ψ_+-name $F_+^+(\alpha)$ for $\psi_+(\alpha)' \subset X$. The new challenge, compared to the problem of constructing a (ψ_+, ψ_-)-lifting, is that because ψ_+ is a representation by full positive information, we now have to remove *every* number $s+1$ for which $B_s \cap \psi_+(\alpha)$ is discrete from range$(F_+^+(\alpha))$. However, this challenge is simplified by the assumption that X is compact. Since $\psi_+(\alpha)$ is closed, it is compact, and

$$B_s \cap \psi_+(\alpha) \text{ is discrete } \Leftrightarrow B_s \cap \psi_+(\alpha) \text{ is finite,}$$

where we have also used the fact that X is T_0. Hence it will be convenient to define a relation $\textsc{Finite}_+ \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ by

$$\begin{aligned}
\textsc{Finite}_+(\alpha, s) &\Leftrightarrow B_s \cap \psi_+(\alpha) \text{ is a finite set.} \\
&\Leftrightarrow B_s \cap \psi_+(\alpha) \text{ is exhausted by some finite number of singletons.} \\
&\Leftrightarrow \text{There is a } t, \text{ such that for any } n > t, \text{ if } B_n \cap \psi_+(\alpha) \neq \emptyset \text{ and } B_n \subset B_s, \\
&\qquad \text{then there is an } m < t \text{ such that } |B_m \cap \psi_+(\alpha)| = 1, \text{ and } B_m \subset B_n \\
&\Leftrightarrow \exists t.\forall n.[(n > t \wedge \exists i.\alpha(i) = n + 1 \wedge n \subset s) \\
&\qquad \rightarrow \exists m \leq t.(\exists j.\alpha(j) = m + 1 \wedge \textsc{MaxOne}_+(\alpha, m) \wedge m \subset n)]
\end{aligned}$$

It is easy to check that the relation $\textsc{Finite}_+$ is $\utilde{\mathbf{\Sigma}}^0_3$, and we can now define a (ψ_+, ψ_+)-lifting $F_+^+ : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ of the Cantor derivative by

$$F_+^+(\alpha)(s) = \begin{cases} 0, & \text{if } \textsc{Finite}_+(\alpha, s), \\ s + 1, & \text{if } \neg\textsc{Finite}_+(\alpha, s). \end{cases}$$

To analyse the Borel complexity of F_+^+ it is enough to consider subbasis elements $B_{(s,j)}$. There are three cases.

$j = 0$: Then

$$(F_+^+)^{-1}(B_{(s,0)}) = \{\alpha : \textsc{Finite}_+(\alpha, s)\},$$

which is of course $\utilde{\mathbf{\Sigma}}^0_3$.

$j = s + 1$: Then

$$(F_+^+)^{-1}(B_{(s,s+1)}) = \{\alpha : \neg\textsc{Finite}_+(\alpha, s)\},$$

which is easily seen to be $\utilde{\mathbf{\Pi}}^0_3$.

$j \neq 0 \wedge j \neq s + 1$: Then $F_+^+)^{-1}(B_{(i,j)}) = \emptyset$ by definition.

Consequently, F_+^+ is $\utilde{\mathbf{\Delta}}^0_4$-measurable.

3.) Define $F_+ : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$, for all $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $s \in \mathbb{N}$, by

$$F_+(\alpha)(s) = \begin{cases} F_+^-(\alpha)(i), & \text{if } s = 2i, \\ F_+^+(\alpha)(i), & \text{if } s = 2i + 1. \end{cases}$$

There are two cases to consider.

$s = 2i$: Then $(F_+)^{-1}(B_{(2i,j)}) = (F_+^-)^{-1}(B_{(i,j)})$ is $\utilde{\mathbf{\Delta}}^0_2$, by 1.).

$s = 2i+1$: Then $(F_+)^{-1}(B_{(2i+1,j)}) = (F_+^+)^{-1}(B_{(i,j)})$ is $\underset{\sim}{\mathbf{\Delta}}^0_4$ by 2.).
In conclusion, F_+ is $\underset{\sim}{\mathbf{\Delta}}^0_4$-measurable.

4.) It suffices to ensure that every number $s+1$ for which $B_s \cap \psi_-(\alpha)$ is empty or a singleton is included in range$(F_-^-(\alpha))$. If we assume that X is compact, then we can define a relation $\textsc{MaxOne}_- \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ by

$$\begin{aligned}
\textsc{MaxOne}_-(\alpha, s) &\Leftrightarrow B_s \cap \psi_-(\alpha) \text{ contains at most one point} \\
&\Leftrightarrow \text{ There are no two basis elements intersecting} \\
&\qquad \psi_-(\alpha) \text{ which are contained in } B_s \text{ and disjoint from each other} \\
&\Leftrightarrow \neg\exists m.\exists n.[\neg\textsc{Disjoint}_-(\alpha, m) \wedge \neg\textsc{Disjoint}_-(\alpha, n) \\
&\qquad\qquad \wedge (n \subset s) \wedge (m \subset s) \wedge (n \cap m = \emptyset)],
\end{aligned}$$

where $\textsc{Disjoint}_-$ is the $\underset{\sim}{\mathbf{\Pi}}^0_2$-relation defined in the proof of 3.5. Hence $\textsc{MaxOne}_-$ is also $\underset{\sim}{\mathbf{\Pi}}^0_2$. We can define a (ψ_-, ψ_-)-lifting $F_-^- : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ of the Cantor derivative by

$$F_-^-(\alpha)(s) = \begin{cases} s+1, & \text{if } \textsc{MaxOne}_-(\alpha, s), \\ 0, & \text{if } \neg\textsc{MaxOne}_-(\alpha, s). \end{cases}$$

There are three cases to consider.
$j = 0$: Then

$$(F_-^-)^{-1}(B_{(s,0)}) = \{\alpha : \neg\textsc{MaxOne}_-(\alpha, s)\},$$

which is $\underset{\sim}{\mathbf{\Sigma}}^0_2$.
$j = s+1$: Then

$$(F_-^-)^{-1}(B_{(s,s+1)}) = \{\alpha : \textsc{MaxOne}_-(\alpha, s)\},$$

which is $\underset{\sim}{\mathbf{\Pi}}^0_2$.
$j \neq 0 \wedge j \neq s+1$: Then $(F_-^-)^{-1}(B_{(i,j)}) = \emptyset$, by definition.
In conclusion, F_-^- is $\underset{\sim}{\mathbf{\Delta}}^0_3$-measurable.

5.) If we assume that X is compact, it suffices to remove every $s+1$ such that $B_s \cap \psi_-(\alpha)$ is finite from range$(F_-^+(\alpha))$. We define a relation $\textsc{Finite}_- \subset \mathbb{N}^\mathbb{N} \times \mathbb{N}$ by

$$\begin{aligned}
\textsc{Finite}_-(\alpha, s) &\Leftrightarrow B_s \cap \psi_+(\alpha) \text{ is exhausted by some finite number of singletons.}\\
&\Leftrightarrow \text{There is a } t \text{, such that for any } n > t \text{, if } B_n \cap \psi_-(\alpha) \neq \emptyset \text{ and } B_n \subset B_s,\\
&\qquad\qquad \text{then there is an } m < t \text{ such that } |B_m \cap \psi_-(\alpha)| = 1 \text{, and } B_m \subset B_n\\
&\Leftrightarrow \exists t. \forall n. [(n > t \wedge \neg\textsc{Disjoint}(\alpha, n) \wedge n \subset s)\\
&\qquad\qquad \rightarrow \exists m \leq t. (\neg\textsc{Disjoint}(\alpha, m) \wedge \textsc{MaxOne}_-(\alpha, m) \wedge m \subset n)]
\end{aligned}$$

Since $\textsc{Disjoint}_-$ and $\textsc{MaxOne}_-$ are both $\mathbf{\Pi}^0_2$, the relation $\textsc{Finite}_-$ is $\mathbf{\Sigma}^0_5$. We can define a (ψ_-, ψ_+)-lifting $F_-^+ : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ of the Cantor derivative by

$$F_-^+(\alpha)(s) = \begin{cases} 0, & \text{if } \textsc{Finite}_-(\alpha, s), \\ s+1, & \text{if } \neg\textsc{Finite}_-(\alpha, s). \end{cases}$$

There are three cases to consider.
$j = 0$: Then

$$(F_-^+)^{-1}(B_{(s,0)}) = \{\alpha : \textsc{Finite}_-(\alpha, s)\},$$

which is $\mathbf{\Sigma}^0_5$.
$j = s+1$: Then

$$(F_-^+)^{-1}(B_{(s,s+1)}) = \{\alpha : \neg\textsc{Finite}_-(\alpha, s)\},$$

which is $\mathbf{\Pi}^0_5$.
$j \neq 0 \wedge j \neq s+1$: Then $F_-^+)^{-1}(B_{(i,j)}) = \emptyset$ by definition.
Consequently, F_-^+ is $\mathbf{\Delta}^0_6$-measurable.

6.) Define $F_- : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, for all $\alpha \in \mathbb{N}^\mathbb{N}$ and $s \in \mathbb{N}$, by

$$F_-(\alpha)(s) = \begin{cases} F_-^-(\alpha)(i), & \text{if } s = 2i, \\ F_-^+(\alpha)(i), & \text{if } s = 2i+1. \end{cases}$$

There are two cases to consider.
$s = 2i$: Then $(F_-)^{-1}(B_{(2i,j)}) = (F_-^-)^{-1}(B_{(i,j)})$ is $\mathbf{\Delta}^0_3$ by 4.).
$s = 2i+1$: Then $(F_-)^{-1}(B_{(2i+1,j)}) = (F_-^+)^{-1}(B_{(i,j)})$ is $\mathbf{\Delta}^0_6$ by 5.).
In conclusion, F_- is $\mathbf{\Delta}^0_6$-measurable.

7.) Just define a (ψ, ψ_-)-lifting $F^- : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ of the Cantor derivative by $F^- = F_+^- \circ \mathrm{I}^+$. Then, since I^+ is continuous, and F_+^- is $\mathbf{\Delta}^0_2$-measurable, so is F^-.

8.) Just define a (ψ, ψ_+)-lifting $F^+ : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ of the Cantor derivative by $F^+ = F^+_+ \circ \mathrm{I}^+$. Then, since I^+ is continuous, and F^+_+ is $\underset{\sim}{\mathbf{\Delta}}^0_4$-measurable, so is F^-.

9.) Define $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, for all $\alpha \in \mathbb{N}^\mathbb{N}$ and $s \in \mathbb{N}$, by

$$F(\alpha)(s) = \begin{cases} F^-(\alpha)(i), & \text{if } s = 2i, \\ F^+(\alpha)(i), & \text{if } s = 2i+1. \end{cases}$$

There are two cases to consider.
$s = 2i$: Then $F^{-1}(B_{(2i,j)}) = (F^-)^{-1}(B_{(i,j)})$ is $\underset{\sim}{\mathbf{\Delta}}^0_2$, by 8.).
$s = 2i+1$: Then $F^{-1}(B_{(2i+1,j)}) = F^+)^{-1}(B_{(i,j)})$ is $\underset{\sim}{\mathbf{\Sigma}}^0_4$, by 9.).
In conclusion, F_- is $\underset{\sim}{\mathbf{\Delta}}^0_4$-measurable.

□

3.3 Borel non-measurability of the transfinitely iterated derivative

Suppose X is a Polish space, that is, a separable completely metrizable space. We say that X is perfect if it has no isolated points, and we say that a subset $Y \subset X$ is perfect if it is closed and has no isolated points in the subspace topology. Every perfect subset Y of X is itself a perfect Polish space, since the restriction of any complete metric on X is a complete metric on Y. Since each perfect Polish space is Borel isomorphic to Baire space, this means, in particular, that any perfect $Y \subset X$ has cardinality $|Y| = |\mathbb{N}^\mathbb{N}| = \aleph_1$. By the Cantor-Bendixon theorem, every closed subset $A \subset X$ of a Polish space has a unique decompositon

$$A = A_P \cup A_S,$$

where A_P is perfect and A_S is countable. In particular, the continuum hypothesis holds for closed subsets of Polish spaces[1]. Since the decomposition of a closed set into a perfect part and a scattered part is uniqe, we are justified in making the following definition.

Definition 3.6. Suppose X is a Polish space. The *perfection map* on $\mathcal{A}(X)$ is the function $P : \mathcal{A}(X) \to \mathcal{A}(X)$ defined by

$$P(A) = A_P,$$

[1]Via the notion of κ-Suslin sets one can prove that every $\underset{\sim}{\mathbf{\Sigma}}^1_1$ subset of a Polish space contains a perfect subset, confirming the continuum hypothesis for analytic subsets of Polish spaces.

for all $A \in \mathcal{A}(x)$. In other words, P maps every closed subset to its perfect component.

In this section, we will prove that the perfection map does not have a Borel measurable lifting relative to any of the representations ψ_-, ψ_+ or ψ, unless X is countable, in which case $P(A) = \emptyset$ for any closed $A \subset X$.
The next result, which is suggested as an exercise in [3], sheds some light on why this is to be expected, and also gives a proof of the Cantor-Bendixson theorem.

Proposition 3.7. *Define the iterated Cantor derivatives $d^\eta : \mathcal{A}(X) \to \mathcal{A}(X)$ by the recursion*

1. $d^0(A) = A$,
2. $d^{\eta+1}(A) = d(d^\eta(A))$,
3. $d^\eta = \bigcap_{\xi<\eta} d^\xi(A)$, *if η is a limit ordinal.*

Then $P = d^{\aleph_1}$.

Proof. (Proposition 3.7 and the Cantor-Bendixson theorem). If $A \subset X$ is closed, define

1. $A_0 = A$,
2. $A_{\eta+1} = d(A_\eta)$,
3. $A_\eta = \bigcap_{\xi<\eta} A_\xi$, if η is a limit ordinal.

Suppose $C \subset X$ is closed. Any limit point of $d(C)$ is in particular a limit point of C, and hence a member of $d(C)$. Thus, since intersections of closed sets are closed, an easy induction on η shows that each A_η is closed. So it suffices to prove that for any closed subset A of X, there exists a countable ordinal λ, such that

(i) $A \setminus A_\lambda$ is countable.

(ii) A_λ has no isolated points.

(i): Let $Y \subset X$. For any isolated point $y \in Y$, there exists a basis element B, such that $\{y\} = Y \cap B$. Since X is second countable, this means that Y contains at most countably many isolated points. Consequently, $|A_\eta \setminus A_{\eta+1}| = \aleph_0$ for each ordinal η. Because countable unions of countable sets are countable, this implies that $A \setminus A_\lambda$ is countable for every countable ordinal λ.
(ii): Assume, for contradiction, that A_η contains at least one isolated point for each

countable ordinal η. Then, if η is countable, $A_{\eta'} \setminus A_\eta \neq \emptyset$ for every $\eta' < \eta$. Fix a countable basis σ for the topology of X. For each $\eta < \aleph_1$, choose x_η such that $x_\eta \in A_{\eta'}$ for all $\eta' < \eta$, but $x_\eta \notin A_\eta$. Then, since $x_\eta \notin \overline{A}_\eta = A_\eta$, there exists a basis element $B_\eta \in \sigma$, such that $B_\eta \cap A_\eta = \emptyset$, but $x_\eta \in B_\eta$, implying $B_\eta \cap A_{\eta'} \neq \emptyset$ for all $\eta' < \eta$. Consequently, $B_\eta \neq B_{\eta'}$ if $\eta \neq \eta'$, contradicting the assumption that σ contains only countably many elements.

Set $A_P = d^{\aleph_1}(A)$ and $A_S = A \setminus A_P$, so $A = A_P \cup A_S$. To prove uniqueness, suppose also that $A = A'_P \cup A'_S$, where A'_P is perfect, A'_S is countable, and $A'_P \cap A'_S = \emptyset$. No point of A removed in the process of iterating the Cantor derivative can be a member of any perfect subset of A, since isolated points are preserved by subspaces. Hence $d^{\aleph_1}(A) = A_P$ is the largest perfect subset of A, and therefore $A'_P \subset A_P$.

If $A'_S \subset A_S$ as well, uniqueness follows. To prove this, it suffices to show that $A_P \setminus A'_P$ is either empty or uncountable. Fix some complete metric on the perfect Polish space X_P, and suppose $A_P \setminus A'_P \neq \emptyset$. Since A_P is a perfect metric space, any open subset of A_P contains two disjoint, non-empty closed balls. Using this fact repeatedly, we can assign to each finite sequence $w \in 2^*$ a closed ball $\overline{B}_w = \overline{B}(x_w, \epsilon_w) \subset A_P \setminus A'_P$ such that

1. $w \sqsubseteq v \Rightarrow B_w \supset \overline{B}_v$,
2. $w \not\sqsubseteq v \wedge v \not\sqsubseteq w \Rightarrow \overline{B}_w \cap \overline{B}_v = \emptyset$,
3. $\epsilon_w < 2^{-|w|}$,

for all $w, v \in 2^*$. This gives a Cauchy sequence corresponding to each $\alpha \in 2^{\mathbb{N}}$, and each of these sequences will have distinct limits[2]. Hence $|A_P \setminus A'_P| \geq |2^{\mathbb{N}}| > \aleph_0$. □

We have seen that even in the least complicated cases, it is difficult to find simple liftings of the Cantor derivative. Moreover, it has actually been proven that $d^n : \mathcal{A}(2^{\mathbb{N}}) \to \mathcal{A}(2^{\mathbb{N}})$, where $n \in \mathbb{N}$, does not have a $\underset{\sim}{\mathbf{\Sigma}}^0_{2n}$-measurable (ψ, ψ)-lifting [2]. Thus, as long as we restrict our attention to the Cantor space, one possible approach might be to extend this result to transfinite ordinals η, and then prove that $P \neq d^\eta$ for all $\eta < \aleph_1$ [3]. We will not adopt this strategy, partly because we would also have to show that the composition $d_1^{\aleph}$ is in fact more complex than its components. However, it is a wortwhile exercise to check that $P \neq d^n$ for any $n \in \mathbb{N}$, especially since the demonstration of this fact gives one of the key ideas for our subsequent approach.

Proposition 3.8. *Let $P : \mathcal{A}(2^{\mathbb{N}}) \to \mathcal{A}(2^{\mathbb{N}})$ be the perfection map, and let $d^n : \mathcal{A}(2^{\mathbb{N}}) \to \mathcal{A}(2^{\mathbb{N}})$ be the n times iterated Cantor derivative. Then $P \neq d^n$ for all $n \in \mathbb{N}$.*

[2] Actually, this construction gives a continuous injection of $2^{\mathbb{N}}$ into any perfect Polish space.
[3] This is not true in general, recall that the perfection map is trivial for countable spaces.

Proof. It suffices to show that for any n, there exists a countable closed subset $C_n \subset 2^{\mathbb{N}}$, such that $d^n(C_n) \neq \emptyset$. Define recursively:

$$C_0 = S_0,$$
$$C_{n+1} = S_{n+1} \cup C_n,$$

where $S_n = \{\alpha : \alpha(i) = 1 \text{ for exactly } n \text{ indices } i \in \mathbb{N}\}$. Each $\alpha \in S_n$ is the limit of $\{\beta \in S_{n+1} : \forall i.[(\alpha(i) = 1) \to (\beta(i) = 1)]\}$. On the other hand, every $\beta \in S_{n+1}$ is an isolated point of C_{n+1}, since no other sequence in C_{n+1} agrees with β any further than to the last index i such that $\beta(i+1) = 1$. Hence $d(C_{n+1}) = C_n$ for all $n \in \mathbb{N}$, and consequently $d^n(C_n) = C_0 \neq \emptyset$. □

The idea we extract from this proof is that the set $P^{-1}(\{\emptyset\})$ might be very complicated. Perhaps it wont even be Borel. Then if $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a lifting of the perfection map, maybe the subsets $F^{-1}(\psi_-^{-1}(\{\emptyset\}))$, $F^{-1}(\psi_+^{-1}(\{\emptyset\}))$ and $F^{-1}(\psi^{-1}(\{\emptyset\}))$ of $\text{dom}(F)$ wont be Borel either[4]. Because

$$\{\emptyset\} = \mathcal{A}(2^{\mathbb{N}}) \setminus (2^{\mathbb{N}})^+ = (2^{\mathbb{N}})^-$$

is Borel for each of these three representations, that would prove the non-existence of a Borel measurable lifting for the perfection map on $\mathcal{A}(2^{\mathbb{N}})$.
However, we promised to settle the question for all Polish spaces, not just Cantor space. The next result tells us that neither this restriction, nor the particular choice of representations among ψ_-, ψ_+ and ψ, cause any loss of generality.

Lemma 3.9. *Suppose the perfection map $P_{\mathcal{A}(X)} : \mathcal{A}(X) \to \mathcal{A}(X)$ has a Borel measurable lifting with respect to some combination of the representations ψ_-, ψ_+ or ψ, for some uncountable Polish space X. Then $P_{\mathcal{A}(2^{\mathbb{N}})} : \mathcal{A}(2^{\mathbb{N}}) \to \mathcal{A}(2^{\mathbb{N}})$ has a Borel measurable lifting with respect to any combinations of these representations.*

Proof. Since X is uncountable, by Cantor-Bendixson, X has a non-empty perfect subspace X_P. Then there exists a continuous injection $\pi : 2^{\mathbb{N}} \to X_P$[5]. Since Cantor space is compact and continuous maps preserves compactness, $Y = \pi(2^{\mathbb{N}})$ is a compact subspace of X_P. But X_P is Hausdorff, so Y is closed in X_P, and therefore also in X, since X_P is closed. Consequently, $\mathcal{A}(Y) = \{A \in \mathcal{A}(X) : A \subset Y\}$, and there is an obvious bijective

[4] Actually, since, for example, $F^{-1}(\psi_-^{-1}(\{\emptyset\})) \cap \text{dom}(\psi_-) = \psi_-(F^{-1}(\{\emptyset\}))$, Borel sets are well behaved with respect to subspaces, and every admissible representation has a Borel measurable right inverse, this would follow.

[5] See the proof of proposition 3.7.

correspondence $\Phi : \mathcal{A}(2^{\mathbb{N}}) \to \mathcal{A}(Y)$, defined by

$$\Phi(A) = \pi(A).$$

We will show that this function is actually a Borel isomorphism for all the relevant topologies. Since both $2^{\mathbb{N}}$ and Y are compact, the representations ψ_-, ψ_+ and ψ of these spaces are all Borel equivalent, as we saw in section 3.2. Consequently, it suffices to consider the lower Fell topology induced by ψ_+, with subbasis elements of the form $U^+ = \{A : U \cap A \neq \emptyset\}$, where U is open. Since

$$\begin{aligned} \Phi(A) \in U^+ &\Leftrightarrow \pi(A) \cap U \neq \emptyset \\ &\Leftrightarrow A \cap \pi^{-1}(U) \neq \emptyset \\ &\Leftrightarrow A \in \pi^{-1}(U)^+, \end{aligned}$$

Φ is a homeomorphism, and thus, in particular, a Borel isomorphism. If $P_{\mathcal{A}(X)} : \mathcal{A}(X) \to \mathcal{A}(X)$ is Borel measurable relative to some of the Fell topologies, then so is the restriction $P_{\mathcal{A}(Y)} : \mathcal{A}(Y) \to \mathcal{A}(Y)$. Then, by the discussion above, $P_{\mathcal{A}(2^{\mathbb{N}})} = \Phi^{-1} \circ P_{\mathcal{A}(Y)} \circ \Phi$ is Borel measurable with respect to any combination of the Fell topologies. □

This lemma tells us that if the perfection map $P_{\mathcal{A}(2^{\mathbb{N}})} : \mathcal{A}(2^{\mathbb{N}}) \to \mathcal{A}(2^{\mathbb{N}})$ does not have a Borel measurable lifting relative to, say, (ψ_+, ψ_+), then $P_{\mathcal{A}(X)} : \mathcal{A}(X) \to \mathcal{A}(X)$ does not have a Borel measurable lifting relative to any combination of ψ_-, ψ_+ and ψ, for any uncountable Polish space X.
Hence, it suffices to show that if $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a (ψ_+, ψ_+)-lifting of the perfection map, then $F^{-1}(\psi_+^{-1}(\{\emptyset\})) \subset \mathrm{dom}(F)$ is not Borel, and consequently, F is not Borel. But how do we show that a set is not Borel? Well, because $\mathcal{B}(X) \restriction \mathbb{N}^{\mathbb{N}} \subsetneq \utilde{\mathbf{\Pi}}^1_1 \restriction \mathbb{N}^{\mathbb{N}}$ and $\mathcal{B}(X)$ is closed under Borel substitutions, it suffices to show that $F^{-1}(\psi_+^{-1}(\{\emptyset\}))$ is $\utilde{\mathbf{\Pi}}^1_1$-complete with respect to Borel reductions, that is, any $\utilde{\mathbf{\Pi}}^1_1$-set is the inverse image of $F^{-1}(\psi_+^{-1}(\{\emptyset\}))$ under a Borel measurable function. Furthermore, it is enough to show that some set which we already know to be $\utilde{\mathbf{\Pi}}^1_1$-complete is Borel reducible to $F^{-1}(\psi_+^{-1}(\{\emptyset\})) \subset \mathrm{dom}(F)$. The next result gives such a set [5].

Proposition 3.10. *Suppose a bijective enumeration $\mathbb{Q} \cap \mathrm{I} = \{q_0, q_1, \ldots\}$ of the rational unit interval is given. Identify $\mathcal{P}(\mathbb{Q} \cap \mathrm{I})$ with the Cantor space $2^{\mathbb{N}}$ via the correspondence*

$$Y \leftrightarrow \alpha_Y,$$

where $Y \subset \mathbb{Q} \cap \mathrm{I}$, $\alpha_Y \in 2^{\mathbb{N}}$, *and* α_Y *is defined by*

$$\alpha_Y(i) = \begin{cases} 1, & \textit{if } q_i \in Y, \\ 0, & \textit{if } q_i \notin Y. \end{cases}$$

Define

$$\mathrm{WO} = \{\alpha \in 2^{\mathbb{N}} : \textit{ The restriction of } <_{\mathbb{R}} \textit{ to } Y_\alpha \textit{ is a well-ordering.}\},$$
$$\mathrm{IF} = 2^{\mathbb{N}} \setminus \mathrm{WO} = \{\alpha \in 2^{\mathbb{N}} : \textit{ The restriction of } <_{\mathbb{R}} \textit{ to } Y_\alpha \textit{ is not a well-ordering.}\}.$$

The set WO *is* $\underset{\sim}{\mathbf{\Pi}}^1_1$*-complete, and the set* IF *is* $\underset{\sim}{\mathbf{\Sigma}}^1_1$*-complete.*

We will allow ourself to use notation for sets to write things like $\alpha \subset \beta \Leftrightarrow Y_\alpha \subset Y_\beta$ whenever we find this convenient. Recall that

$$F^{-1}(\psi_+^{-1}(\{\emptyset\})) \cap \mathrm{dom}(\psi_+) = \psi_+^{-1}(P^{-1}(\{\emptyset\})) \subset \mathrm{dom}(F).$$

We will try to construct a Borel reduction of IF to $F^{-1}(\psi_+^{-1}(\{\emptyset\}))$ by passing through $P^{-1}(\{\emptyset\})$. First we need to describe this set.

Lemma 3.11. *Let* $P : \mathcal{A}(2^{\mathbb{N}}) \to \mathcal{A}(2^{\mathbb{N}})$ *be the perfection map. Then*

$$P^{-1}(\{\emptyset\}) = \{A \in \mathcal{A}(2^{\mathbb{N}}) : |A| \leq \aleph_0\}.$$

That is, $P^{-1}(\{\emptyset\})$ *consists exactly of the countable closed subsets of Cantor space.*

Proof. By Cantor-Bendixson, every closed $A \subset 2^{\mathbb{N}}$ can be written as a disjoint union $A = A_P \cup A_S$ of one perfect and one countable subset in a unique way. If A is countable, then $A = \emptyset \cup A$, and by uniqueness, $A_P = \emptyset$. On the other hand, if A is uncountable, then $A_P \neq \emptyset$, since otherwise we would have $A_S = A$, contradicting countability of A_S. Alternatively, $P(A) = A_P \neq \emptyset$ because $A_P = d^\lambda(A)$ for some countable ordinal λ, meaning that we have only removed countably many points of A. □

So we want to construct a function $\rho_0 : 2^{\mathbb{N}} \to \mathcal{A}(2^{\mathbb{N}})$ such that $\rho_0(\alpha) \in P^{-1}(\{\emptyset\})$ when $\alpha \in \mathrm{IF}$, and $\rho_0(\alpha) \in \mathcal{A}(2^{\mathbb{N}}) \setminus P^{-1}(\{\emptyset\})$ when $\alpha \in \mathrm{WO}$.
In other words, we need to assign some countable closed subset of Cantor space to every well-ordered subset of the rational unit interval, and some uncountable closed subset of Cantor space to every ill-founded subset of the rational unit interval.

Lemma 3.12. *Define* $\rho_0 : 2^{\mathbb{N}} \to \mathcal{A}(2^{\mathbb{N}})$ *by*

$$\rho_0(\alpha) = \mathcal{S}_\alpha,$$

where

$$\mathcal{S}_\alpha = \{\beta : \beta \subset \alpha \wedge \textsc{DecSeqRep}(\beta)\},$$

with

$$\textsc{DecSeqRep}(\alpha) \Leftrightarrow \forall i. \forall j. [(\alpha(i) = 1 \wedge \alpha(j) = 1 \wedge i < j) \rightarrow q_i > q_j].$$

Then $|\mathcal{S}_{\alpha_Y}| \leq \aleph_0$ *if* Y *is a well-ordered subset of the rational unit interval, and* $|S_{\alpha_Y}| > \aleph_0$ *if* Y *is an ill-founded subset of the rational unit interval. Hence* $\rho_0(\mathrm{WO}) \subset P^{-1}(\{\emptyset\})$, *and* $\rho_0(\mathrm{IF}) \subset \mathcal{A}(2^{\mathbb{N}}) \setminus P^{-1}(\{\emptyset\})$.

Proof. Suppose $Y \subset \mathbb{Q} \cap \mathrm{I}$. Note that

$$\alpha_Y \in \mathrm{IF} \Leftrightarrow Y \text{ contains an infinite decreasing sequence.}$$

Since every subset of a decreasing sequence is again a decreasing sequence, and since there are only countably many finite sequences from a countable set, it follows that

$$\alpha_Y \in \mathrm{IF} \Leftrightarrow Y \text{ contains more than countably many decreasing sequences.}$$

So each $Y \in \mathcal{P}(\mathbb{Q} \cap \mathrm{I})$, or $\alpha_Y \in 2^{\mathbb{N}}$, naturally corresponds to an uncountable subset of $\mathcal{P}(\mathbb{Q} \cap \mathrm{I}) \cong 2^{\mathbb{N}}$ if $Y \in \mathrm{IF}$, and to a countable subset if $Y \in \mathrm{WO}$.
This seems to be exactly what we asked for. There are, however, two problems to overcome. Firstly, we would like to specify the decreasing sequences of $Y \subset \mathbb{Q} \cap \mathrm{I}$, hence in particular, we wish for a formula expressing that $\alpha \in 2^{\mathbb{N}}$ codes a decreasing sequence. Secondly, we have no guarantee that the subset of $2^{\mathbb{N}}$ consisting of codes for decreasing sequences in Y will be closed. We could try to fix this by taking closures, but we risk accidentally passing from a countable set to an uncountable set in the process. Furthermore, it is possible that taking closures would significantly increase the complexity of the function we are trying to construct. It is hard to tell how severe these problems are, but they might turn out to be less serious if, in solving the first problem, we take care to keep the set S_{α_Y} assigned to α_Y small, and the assignment simple.
Hence, we might optimistically try to solve the first problem by defining a relation

$$\textsc{DecSeqRep}(\alpha) \Leftrightarrow \forall i. \forall j. [(\alpha(i) = 1 \wedge \alpha(j) = 1 \wedge i < j) \rightarrow q_i > q_j].$$

Intuitively, if $\textsc{DecSeqRep}(\alpha_Y)$, then the representation α_Y of Y explicitly lists the elements of Y in decreasing order. This is not true in general for the representation α_Y of a decreasing sequence Y,

$$Y \text{ is a decreasing sequence of rationals} \not\Rightarrow \textsc{DecSeqRep}(\alpha_Y).$$

But of course

$$\textsc{DecSeqRep}(\alpha_Y) \Rightarrow Y \text{ is a decreasing sequence of rationals,}$$

and

$$\textsc{DecSeqRep}(\alpha) \wedge \beta \subset \alpha \Rightarrow \textsc{DecSeqRep}(\beta).$$

Furthermore, for every infinite decreasing sequence Y, it is easy to explicitly define by recursion an infinite decreasing subsequence $Z \subset Y$ such that $\textsc{DecSeqRep}(\alpha_Z)$. Hence,

$$\alpha_Y \in \mathrm{IF} \Leftrightarrow \mathcal{S}_{\alpha_Y} = \{\beta : \beta \subset \alpha \wedge \textsc{DecSeqRep}(\beta)\} \text{ is uncountable.}$$

But is $\mathcal{S}_{\alpha_Y}$ closed? Note that

$$2^{\mathbb{N}} \setminus \mathcal{S}_{\alpha_Y} = \{\beta : \beta \not\subset \alpha_Y\} \cup \{\beta : \neg\textsc{DecSeqRep}(\beta)\}$$

Suppose $\beta \not\subset \alpha_Y$, and let n be the first index such that $\beta(n) = 1$ and $\alpha_Y(n) = 0$. Then $\{\gamma : \gamma_{\leq n} = \beta_{\leq n}\}$ is a neighbourhood of β contained in $\{\beta : \beta \not\subset \alpha_Y\}$, so $\{\beta : \beta \not\subset \alpha_Y\}$ is open.

Suppose $\neg\textsc{DecSeqRep}(\beta)$. Let n be the least index such that

$$\beta(n) = 1 \wedge \exists i < n.(\beta(i) = 1 \wedge q_i < q_n).$$

Then $\{\gamma : \gamma_{\leq n} = \beta_{\leq n}\}$ is a neighbourhood of β contained in $\{\beta : \neg\textsc{DecSeqRep}(\beta)\}$, so $\{\beta : \neg\textsc{DecSeqRep}(\beta)\}$ is open. So by a stroke of good fortune $\mathcal{S}_Y$ is actually closed, and we can just define $\rho_0 : 2^{\mathbb{N}} \to \mathcal{A}(2^{\mathbb{N}})$ by

$$\rho_0(\alpha_Y) = \mathcal{S}_Y.$$

□

Lemma 3.13. *Suppose* $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *is a* (ψ_+, ψ_+)*-lifting of the perfection map* $P : \mathcal{A}(2^{\mathbb{N}}) \to \mathcal{A}(2^{\mathbb{N}})$*. Then there is a Borel measurable function* $\rho : 2^{\mathbb{N}} \to \mathrm{dom}(F)$ *such that*

$$\rho(\mathrm{IF}) \subset F^{-1}(\psi_+^{-1}(\{\emptyset\}))$$
$$\rho(2^{\mathbb{N}} \setminus \mathrm{IF}) \subset \mathrm{dom}(F) \setminus F^{-1}(\psi_+^{-1}(\{\emptyset\}))$$

Consequently, F *is not Borel measurable.*

Proof. Let $e_{\mathcal{A}(2^{\mathbb{N}})} : \mathcal{A}(2^{\mathbb{N}}) \to \mathbb{N}^{\mathbb{N}}$ be the Borel measurable right inverse of the standard representation $\psi_+ = \delta_{\mathcal{A}(2^{\mathbb{N}})}$ of $\mathcal{A}(2^{\mathbb{N}})$ with the lower Fell topology which we discussed in section 2.3, that is,

$$e_{\mathcal{A}(2^{\mathbb{N}})}(A)(n) = \begin{cases} n+1, & \text{if } A \in B_n^+, \\ 0, & \text{otherwise.} \end{cases}$$

Define $\rho = e_{\mathcal{A}(2^{\mathbb{N}})} \circ \rho_0$. We need to show that that ρ is Borel measurable, or equivalently, that the graph $\mathcal{G}_\rho$ of ρ is $\underset{\sim}{\mathbf{\Sigma}}^1_1$. Since

$$\begin{aligned} \mathcal{G}_\rho(\alpha,\gamma) &\Leftrightarrow e_{\mathcal{A}(2^{\mathbb{N}})}(\mathcal{S}_\alpha) = \gamma \\ &\Leftrightarrow \forall n.[(\mathcal{S}_\alpha \in B_n^+ \wedge \gamma(n) = n+1) \vee (\mathcal{S}_\alpha \notin B_n^+ \wedge \gamma(n) = 0)], \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_\alpha \in B_n^+ &\Leftrightarrow \exists \beta . \mathcal{S}_\alpha(\beta) \wedge B_n(\beta) \\ &\Leftrightarrow \exists \beta . \beta \subset \alpha \wedge \textsc{DecSeqRep}(\beta) \wedge B_n(\beta) \end{aligned}$$

is Borel, $\mathcal{G}_\rho$ is Borel. The result follows. □

We have now arrived at the main result of this thesis.

Theorem 3.14. *Suppose X is a Polish space.*
If X is countable, then the perfection map $P : \mathcal{A}(X) \to \mathcal{A}(X)$ is constant, $P(A) = \emptyset$ for all $A \in \mathcal{A}(X)$, and in particular, P has a continuous (δ,δ')-lifting for any representations δ, δ' of $\mathcal{A}(X)$[6]*.*
If X is uncountable, then P has no Borel measurable (δ,δ')-lifting for any representations $\delta, \delta' \in \{\psi_-, \psi_+, \psi\}$.

[6]If $\emptyset$ has a computable δ'-name, then P even has a computable lifting when X is countable

Chapter 4

Computable and continuous operations on $\mathcal{B}(X)$

4.1 A representation of the Borel algebra

Suppose X is a topological space. Lets assume that we are already given a representation

$$\psi : \mathbb{N}^{\mathbb{N}} \to \tau_X$$

of the open subsets of X, or if X is second countable, a representation

$$\psi : \mathbb{N}^{\mathbb{N}} \to \sigma_X$$

of a countable basis for X, which we might assume have countable domain. We want to construct a representation of $\mathcal{B}(X)$ that reflects the way the Borel algebra is built up recursively from the open sets by the operations of complement and countable union. Hence, in the case where X is second countable, and we have a representation of a basis, we will in any case get a representation of the open sets. Thus we can forget about this distinction for the moment. It will however be important in the last section of this chapter.

First we define some functions which will be useful to us later.

Definition 4.1. We define:

1. The tupling function $\Diamond : (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$\Diamond(\alpha_0, \alpha_1, \ldots) =< \alpha_0, \alpha_1, \ldots >= \beta,$$

where $\beta(< n, i >) = \alpha_n(i)$.

2. The shift map $* : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$*(\alpha) = \alpha^* = (\alpha(1), \alpha(2), \ldots).$$

3. The function $+_n : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$+_n(\alpha) = (\alpha(0) + n, \alpha(1) + n, \ldots)$$

4. The negation map $\urcorner : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$\urcorner(\alpha) = (1, \alpha(0), \alpha(1), \ldots)$$

5. The countable union map $\bigsqcup^{\omega} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$\bigsqcup^{\omega}(\alpha) = (2, \alpha(0), \alpha(1), \ldots)$$

6. The binary union map $\sqcup : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ by

$$\sqcup(\alpha, \beta) = \bigsqcup^{\omega}(\gamma)$$

   where $\gamma =< \alpha, \beta, 3^{\omega}, 3^{\omega}, \ldots >$.

Note that all of these functions are continuous and injective, and that the tupling function $\Diamond$ is actually a homeomorphism between $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$. This demonstrates a peculiar feature of Baire space, namely, Baire space is homeomorphic to the countable product of Baire space with itself.

Now we define a new representation $\psi' : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ of the open subsets of X, by

$$\mathrm{dom}(\psi') = +_4(\mathrm{dom}(\psi)), \quad \psi'(\alpha) = \psi(+_4^{-1}(\alpha))$$

This leaves the numbers $0, 1, 2, 3$ free to convey extra information, or no information at all. We are ready to define a representation for $\mathcal{B}(X)$.

Definition 4.2. We define a representation $\phi : \mathbb{N}^{\mathbb{N}} \to \mathcal{B}(X)$ of the Borel algebra on X. The domain of ϕ is defined recursively by

1. $\mathrm{dom}(\psi') \subset \mathrm{dom}(\phi)$,

2. $\alpha \in \text{dom}(\phi) \Rightarrow \urcorner(\alpha) \in \text{dom}(\phi)$,

3. $\alpha_1, \alpha_2, \ldots \in \text{dom}(\phi) \Rightarrow \bigsqcup^{\omega}(\Diamond(\alpha_1, \alpha_2, \ldots)) \in \text{dom}(\phi)$.

The value of ϕ is defined by recursion on complexity of α by

1. $\psi(\urcorner(\alpha)) = \neg\psi(\alpha)$,

2. $\psi(\bigsqcup^{\omega}(\alpha)) = \bigcup_{n<\omega} \phi(\pi_n(\Diamond^{-1}(\alpha)))$,

3. $\psi(\alpha) = \psi'(\alpha)$, if $\alpha(0) \notin \{1, 2\}$.

4.2 Computable liftings of complement and union

We now prove that the topological operations of complementation and union are (ϕ, ϕ)-computable.

Proposition 4.3. *Complementation* $\neg : \mathcal{B}(X) \to \mathcal{B}(X)$ *has a computable* (ϕ, ϕ)*-lifting.*

Proof. The function $\urcorner : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a lifting of complementation. We say that a function on $\mathbb{N}^{\mathbb{N}}$ is computable iff it is $(\delta_{\mathbb{N}^{\mathbb{N}}}, \delta_{\mathbb{N}^{\mathbb{N}}})$-computable. A computable $(\delta_{\mathbb{N}^{\mathbb{N}}}, \delta_{\mathbb{N}^{\mathbb{N}}})$-lifting is given by $p \mapsto 0^1 1p$. □

Proposition 4.4. *Binary union* $\cup : \mathcal{B}(X) \times \mathcal{B}(X) \to \mathcal{B}(X)$ *has a computable* (ϕ, ϕ)*-lifting.*

Of course, $\sqcup : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a lifting of binary union. A $(\delta_{\mathbb{N}^{\mathbb{N}}}, \delta_{\mathbb{N}^{\mathbb{N}}})$-computable lifting of $\sqcup$ is given by

$$(p, q) \mapsto 0^2 1 0^{\gamma(0)} 1 0^{\gamma(1)} 1 \ldots,$$

where the value

$$\gamma(s) = \begin{cases} \alpha((s)_2), & \text{if } (s)_1 = 0, \\ \beta((s)_2), & \text{if } (s)_1 = 1, \\ 3 & \text{otherwise.} \end{cases}$$

is clearly computable by a Type-2 machine which takes as input $\delta_{\mathbb{N}^{\mathbb{N}}}$-names p, q for α, β respectively, since $(\cdot)_1$ and $(\cdot)_2$ are computable.

Before we can discuss computability of countable union, we must agree on a representation for $\mathcal{B}(X)$. The following definition should come as no surprise.

Definition 4.5. Define a representation $\phi^\omega : \mathbb{N}^\mathbb{N} \to \mathcal{B}(X)^\mathbb{N}$ by

$$\alpha \in \mathrm{dom}(\phi^\omega) \Leftrightarrow \alpha(0) = 0 \wedge \alpha^* \in \Diamond(\{(\alpha_0, \alpha_1, \ldots) : \forall i.\alpha_i \in \mathrm{dom}(\phi)\}),$$
$$\phi^\omega((0, <\alpha_0, \alpha_1, \ldots>)) = (\phi(\alpha_0), \phi(\alpha_1), \ldots).$$

Proposition 4.6. *Countable union $\bigcup^\omega : \mathcal{B}^\mathbb{N} \to \mathcal{B}(X)$ has a computable (ϕ^ω, ϕ)-lifting.*

Proof. Of course, $\bigsqcup^\omega \circ *$ is a (ϕ^ω, ϕ)-lifting for countable union. A computable $(\delta_{\mathbb{N}^\mathbb{N}}, \delta_{\mathbb{N}_\mathbb{N}})$-lifting for $\bigsqcup^\omega \circ *$ is given by $0^{i_0}10^{i_1}10^{i_2}1\ldots \mapsto 0^1 10^{i_1}10^{i_1}1\ldots$. □

4.3 Continuous lifting of the dual of Borel functions

Let X and Y be topological spaces with representations $\psi_X : \mathbb{N}^\mathbb{N} \to \sigma_X$ and $\psi_Y : \mathbb{N}^\mathbb{N} \to \sigma_Y$, where σ_X and σ_Y are bases for X and Y, respectively. Suppose $f : X \to Y$ is a Borel measurable function. Then we can define a dual

$$\hat{f} : \mathcal{B}(Y) \to \mathcal{B}(X)$$

by

$$\hat{f}(B) = f^{-1}(B),$$

that is, $\hat{f}$ is the operation of taking inverse images of f.

We want to explore the possibility of finding conditions such that:

$$f \text{ is Borel continuous} \Rightarrow \hat{f} \text{ has a continuous } (\phi_Y, \phi_X)\text{-lifting.}$$

Note that if we knew how to lift the operation $\hat{f}$ on the the basis elements of Y, we would automatically know how to lift the operation on all of $\mathcal{B}(Y)$. Because $\mathcal{B}(Y)$ is the closure of σ_Y under complements and countable unions, $\hat{f}$ is the unique extension of $\hat{f} \restriction \sigma_Y$ to $\mathcal{B}(X)$ that preserves these operations. Suppose $\hat{f}$ is defined for all $B \in \sigma_Y$. Then we can define $\hat{f}$ recursively by

1. $\hat{f}(\neg B) = \neg \hat{f}(B)$,

2. $\hat{f}(\bigcup_i B_i) = \bigcup_i \hat{f}(B_i)$.

Note that this might cause us to define $\hat{f}$ on the same input many times, but these definitions must always agree, so this is not a problem.

Similarly, suppose we have defined a lifting $\hat{F}$ for every $\alpha \in \mathrm{dom}(\psi_Y)$, that is, for every ϕ_Y-name of a basis element. Then we can define recursively

1. $\hat{F}(\urcorner(\alpha)) = \urcorner(\hat{f}(\alpha))$,

2. $\hat{F}(\bigsqcup^{\omega}(\alpha)) = \bigsqcup^{\omega} \hat{F}(\alpha)$.

A function $G : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous if and only if, for any $\alpha \in \mathrm{dom}(G)$, and any $m \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that, for any $\beta \in \mathrm{dom}(G)$, if β agrees with α up to n, then $G(\alpha)$ agrees with $G(\beta)$ up to m. Hence it is obvious that if we start with a continuous function $G = \hat{F} \restriction \mathrm{dom}(\psi_Y)$, then the extension defined by the recursion above will also be continuous.

The next result gives conditions which ensure that a weaker version of the claim will hold.

Proposition 4.7. *Suppose X and Y are topological spaces, and $\psi_X : \mathbb{N}^{\mathbb{N}} \to \sigma_X$, $\psi_Y : \mathbb{N}^{\mathbb{N}} \to \sigma_Y$ are representations of bases for X, Y. Suppose the domains of ψ_X and ψ_Y are countable, so in particular, X and Y are second countable. Then if $f : X \to Y$ is continuous, the dual $\hat{f} : \mathcal{B}(Y) \to \mathcal{B}(X)$ has a continuous (ϕ_Y, ϕ_X)-lifting.*

Proof. Let $\{\alpha_i^X\}$ and $\{\alpha_j^Y\}$ be enumeratons of $\mathrm{dom}(\psi_X)$ and $\mathrm{dom}(\psi_Y)$, respectively. Write $\sigma_X = \{B_i^X\}$ and $\sigma_Y = \{B_j^Y\}$ for the corresponding bases, that is $\psi_X(\alpha_i^X) = B_i^X$ and $\psi_Y(\alpha_j^Y) = B_j^Y$. Suppose $f : X \to Y$ is continuous. Then for any basis element $B_j^Y \subset Y$,

$$\hat{f}(B_j^Y) = \bigcup \{B_i^X : f(B_i^X) \subset B_j^Y\}$$

Define a relation $\textsc{ImageInc}_f \subset \mathbb{N} \times \mathbb{N}$ by

$$\textsc{ImageInc}_f(i,j) \Leftrightarrow f(B_i^X) \subset B_j^Y.$$

For $\alpha_j^Y \in \mathrm{dom}(\psi_Y)$, we might define

$$\hat{F}(\alpha_j^Y) = \bigsqcup^{\omega}(\beta^j),$$

where

$$\beta^j = <\beta_0^j, \beta_1^j, \dots>$$

with

$$\beta_i^j = \begin{cases} \alpha_i^X, & \text{if } \textsc{ImageInc}_f(i,j), \\ 3^{\omega}, & \text{if } \neg\textsc{ImageInc}_f(i,j). \end{cases}$$

Then $\hat{F} \upharpoonright \mathrm{dom}(\psi_Y)$ is trivially continuous, since it has discrete domain, and hence the unique extension $\hat{F}$ is continuous. □

The proof of this proposition hinges upon the fact that we only need to consider countably many names, which enable us to shamelessly code any relation between basis elements by trivial functions and relations. It is therefore unlikely that we can extend this result to the higher Borel classes, since we immediately will have to consider uncountably many names.

Appendix A

Some definitions and results from descriptive set theory

We start by discussing the Borel structure on arbitrary topological spaces, before we restrict our attention to Polish spaces, that is, separable spaces which are completely metrizable. Thus until otherwise stated, all definitions and results are with respect to general topological spaces.

Definition A.1. Suppose X is a topological space. The *Borel algebra* $\mathcal{B}(X)$ on X is the smallest collection of subsets of X, containing the open sets, that is closed under complement and countable union. The members of $\mathcal{B}(X)$ are called the *Borel sets* of X.

Definition A.2. Let X be a topological space. The *Borel class of order η restricted to X*, denoted by $\utilde{\mathbf{\Sigma}}^0_\eta \restriction X$, is defined recursively as follows for all ordinals $\eta < \aleph_1$:

$$\begin{aligned} \utilde{\mathbf{\Sigma}}^0_1 \restriction X = & \text{ all open subsets of } X. \\ \utilde{\mathbf{\Sigma}}^0_\eta \restriction X = & \text{ all sets of the form } \bigcup_{i\in\mathbb{N}} X \setminus P_i, \text{where each } P_i \text{ is a in lower Borel class.} \end{aligned}$$

The *Borel class of order η*, $\utilde{\mathbf{\Sigma}}^0_\eta$, is the proper class of all sets wich are elements of $\utilde{\mathbf{\Sigma}}^0_\eta \restriction Y$ for some topological space Y. The *dual Borel class of order η*, denoted by $\utilde{\mathbf{\Pi}}^0_\eta$, is the class of all complements of members of $\utilde{\mathbf{\Sigma}}^0_\eta$, and the *ambiguous Borel class of order η* is their intersection, $\utilde{\mathbf{\Delta}}^0_\eta = \utilde{\mathbf{\Sigma}}^0_\eta \cap \utilde{\mathbf{\Pi}}^0_\eta$.

This gives a quick and economical way of inductively defining the class $\utilde{\mathbf{\Sigma}}^0_\eta$ for any countable ordinal η, and thereby, as the next result shows, $\mathcal{B}(X)$, for any topological space X.

Proposition A.3. *Suppose X is a topological space. The Borel algebra $\mathcal{B}(X)$ on X is exactly the collection of all Borel subsets of X of countable order.*

Proof. [1] Set $\mathbf{B} = \{A \subset X : A \in \underset{\sim}{\mathbf{\Sigma}}^0_\eta \restriction X \text{ for some } \eta < \aleph_1\}$. Of course, $\mathbf{B}$ contains the open sets, and if $A \in \underset{\sim}{\mathbf{\Sigma}}^0_\eta$, then $X \setminus A \in \underset{\sim}{\mathbf{\Sigma}}^0_{\eta+1}$, so $\mathbf{B}$ is closed under complements. Furthermore, for each $j \in \mathbb{N}$, if $A_j \in \underset{\sim}{\mathbf{\Sigma}}^0_{\eta_j}$, then $X \setminus A_j \in \underset{\sim}{\mathbf{\Sigma}}^0_{\eta_j+1}$, so

$$\bigcup_{j\in\mathbb{N}} A_j = \bigcup_{j\in\mathbb{N}} X \setminus (X \setminus A_j)$$

is in $\underset{\sim}{\mathbf{\Sigma}}^0_\eta$, where $\eta = \sup(\{\eta_j + 2 : j \in \mathbb{N}\})$. Thus $\mathbf{B}$ is also closed under countable unions, and hence $\mathcal{B}(X) \subset \mathbf{B}$. On the other hand, a trivial induction on η shows that for any $\eta < \aleph_1$, if $A \in \underset{\sim}{\mathbf{\Sigma}}^0_\eta$, then $A \in \mathcal{B}(X)$. □

This recursive characterization of $\mathcal{B}(X)$ enables us to use proofs by transfinite induction to obtain a number of useful facts about Borel sets and functions.

Proposition A.4. *Suppose X is a topological space, and $Y \subset X$. Assume that Y is given the subspace topology. Then $\mathcal{B}(Y) = \{B \cap Y : B \in \mathcal{B}(X)\}$.*

Proof. Let $\mathbf{B}_Y = \{B \cap Y : B \in \mathcal{B}(X)\}$. Then $\mathbf{B}_Y$ contains the open sets of Y, and if $A = B \cap Y$, then $Y \setminus A = (X \setminus B) \cap Y$, so $\mathbf{B}_Y$ is obviously closed under complements. Furthermore, if $A_i = B_i \cap Y$ for each $i \in \mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} A_i = (\bigcup_{i\in\mathbb{N}} B_i) \cap Y$. Hence $\mathbf{B}_Y$ is also closed under countable unions, and thus $\mathcal{B}(X) \subset \mathbf{B}_Y$. On the other hand, an easy induction on η shows that any set of the form $B \cap Y$, with $B \in \Sigma^0_\eta \restriction X$, is a member of $B \in \Sigma^0_\eta \restriction Y$. Consequently, $\mathbf{B}_Y \subset \mathcal{B}(Y)$. □

Definition A.5. Suppose X and Y are topological spaces. A function $f : X \to Y$ is *Borel measurable* if $f^{-1}(U)$ is Borel whenever U is open, and $\underset{\sim}{\mathbf{\Sigma}}^0_\eta$*-measurable* if $f^{-1}(U)$ is in $\underset{\sim}{\mathbf{\Sigma}}^0_\eta$ whenever U is open.

Lemma A.6. *The inverse image of a Borel set under a Borel measurable function is again Borel.*

Proof. Let $f : X \to Y$ be a Borel measurable function, and let B be a Borel set of Y. If B is open, then $f^{-1}(B)$ is a Borel set of X by definition. Suppose the result is known for any Borel set of order $\eta' < \eta$. If $B \in \underset{\sim}{\mathbf{\Sigma}}^0_\eta$, then by definition, there exists $\eta_i < \eta$ and $P_i \in \underset{\sim}{\mathbf{\Sigma}}^0_{\eta_i} \restriction Y$, such that $B = \bigcup_{i\in\mathbb{N}} Y \setminus P_i$. Consequently, since inverse images preserves unions and complements,

$$f^{-1}(B) = f^{-1}(\bigcup_{i\in\mathbb{N}} Y \setminus P_i) = \bigcup_{i\in\mathbb{N}} X \setminus f^{-1}(P_i).$$

[1] Moschovakis [3] gives this proof in the setting of Polish spaces, but the same proof goes through for general topological spaces.

It follows from the induction hypothesis that $f^{-1}(B) \in \mathcal{B}(X)$, by closure of this collection under complements and countable unions. □

Corollary A.7. *The composition of Borel measurable functions is Borel measurable.*

We now restrict our attention to Polish spaces. So from now on all definitions and results refer to Polish spaces, their subsets and classes of these subsets.

We summarize some useful facts about the Borel and Lusin classes of subsets of Polish spaces from Moschovakis "Descriptive Set Theory" [3], to make our presentation as self-contained as possible, but omit the proofs.

We first introduce some operators on sets and classes of sets, which will allow us to use logical notation to define and discuss classes of sets. Let us agree to think of a set as a property of its members and to use the notation

$$A(x) \Leftrightarrow x \in A,$$

whenever this is convenient.

Definition A.8. The *complementation* operator $\neg$ is defined by

$$\neg A = X \setminus A,$$

whenever $A \subset X$. If Γ is some class of sets, then we define

$$\neg\Gamma = \{\neg B : B \in \Gamma\}.$$

Strictly speaking, there is one complementation operator $\neg_X$ for each set X, but the subscript is cumbersome and adds little in terms of clarity, so we will never bother to write it down.

Definition A.9. The operation of *projection along* Y, denoted $\exists^Y$, is defined by

$$\exists^Y A = \{x \in X : \exists y. A(x, y), \}$$

whenever $A \subset X \times Y$. If Γ is some class of sets, then we define

$$\exists^Y \Gamma = \{\exists^Y B : B \in \Gamma \wedge B \subset X \times Y \text{ for some set } X\}.$$

The other logical symbols also have corresponding operators, defined in a similar fashion. For Polish spaces we usually define the Borel pointclasses of finite order recursively by

$$\begin{aligned} \underset{\sim}{\mathbf{\Sigma}}^0_1 &= \text{ all open subsets} \\ \underset{\sim}{\mathbf{\Sigma}}^0_{n+1} &= \exists^{\mathbb{N}} \neg \underset{\sim}{\mathbf{\Sigma}}^0_n, \end{aligned}$$

rather than using the equivalent definition above. We could also define the Borel classes of countable order in similar language. Hence, the following definition of the Lusin classes of subsets of Polish spaces is a quite natural extension of the Borel hierarchy.

Definition A.10. The *Lusin class of order* n, denoted by $\underset{\sim}{\mathbf{\Sigma}}^1_n$, is defined recursively for all $n \in \mathbb{N}$ as follows:

$$\begin{aligned} \underset{\sim}{\mathbf{\Sigma}}^1_1 &= \exists^{\mathbb{N}^{\mathbb{N}}} \neg \underset{\sim}{\mathbf{\Sigma}}^0_1 \\ \underset{\sim}{\mathbf{\Sigma}}^1_{n+1} &= \exists^{\mathbb{N}^{\mathbb{N}}} \neg \underset{\sim}{\mathbf{\Sigma}}^1_n, \end{aligned}$$

where $\underset{\sim}{\mathbf{\Sigma}}^0_1$ and all other classes of sets are understood to be restricted to subsets of Polish spaces. The *dual* and *ambiguous Lusin classes of finite order* are defined by $\underset{\sim}{\mathbf{\Pi}}^1_n = \neg \underset{\sim}{\mathbf{\Sigma}}^1_n$ and $\underset{\sim}{\mathbf{\Delta}}^1_n = \underset{\sim}{\mathbf{\Sigma}}^1_n \cap \underset{\sim}{\mathbf{\Pi}}^1_n$, respectively.

Theorem A.11. *Each Borel class, dual Borel class and ambiguous Borel class is closed under finite unions and intersections, and inverse images of continuous functions. Furthermore, each Borel class is closed under countable unions, each dual Borel class is closed under countable intersections, and each ambiguous Borel class is closed under complements.*
The following diagram of inclusions holds among the Borel classes:

$$\begin{array}{ccccccccccc} & & \underset{\sim}{\mathbf{\Sigma}}^0_1 & & \cdots & & \underset{\sim}{\mathbf{\Sigma}}^0_\eta & & & & \cdots \\ & \subseteq & & \supseteq & \subseteq & \supseteq & \subseteq & \supseteq & & \subseteq & \\ \underset{\sim}{\mathbf{\Delta}}^0_1 & & & & \underset{\sim}{\mathbf{\Delta}}^0_2 \;\cdots & & \underset{\sim}{\mathbf{\Delta}}^0_\eta & & & \underset{\sim}{\mathbf{\Delta}}^0_{\eta+1} & \cdots \\ & \subseteq & & \supseteq & \subseteq & \supseteq & \subseteq & \supseteq & & \subseteq & \\ & & \underset{\sim}{\mathbf{\Pi}}^0_1 & & \cdots & & \underset{\sim}{\mathbf{\Pi}}^0_\eta & & & & \cdots \end{array}$$

If we restrict these classes to one fixed perfect Polish space, then this diagram holds with strict inclusions.

Theorem A.12. *Each Lusin class, dual Lusin class and ambiguous Lusin class is closed under countable unions and intersections, and inverse images of continuous functions. Furthermore, each Lusin class is closed under* $\exists^Y$*, each dual Borel class is closed under* $\forall^Y$*, and each ambiguous Borel class is closed under complements, where* Y *is any perfect*

Polish space.
The following diagram of inclusions holds among the Lusin classes:

$$\begin{array}{ccccccccccccc}
 & & \utilde{\boldsymbol{\Sigma}}^1_1 & & & & \utilde{\boldsymbol{\Sigma}}^1_2 & & & & \utilde{\boldsymbol{\Sigma}}^1_3 & & \cdots \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} & & \rotatebox{45}{$\subseteq$} & & & \\
\utilde{\boldsymbol{\Delta}}^1_1 & & & & \utilde{\boldsymbol{\Delta}}^1_2 & & & & \utilde{\boldsymbol{\Delta}}^1_2 & & \cdots & & \\
 & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\supseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\supseteq$} & & \rotatebox{-45}{$\subseteq$} & & & \\
 & & \utilde{\boldsymbol{\Pi}}^1_1 & & & & \utilde{\boldsymbol{\Pi}}^1_2 & & & & \utilde{\boldsymbol{\Pi}}^1_3 & & \cdots
\end{array}$$

If we restrict these classes to one fixed perfect Polish space, then this diagram holds with strict inclusions.
In particular, any Borel set is $\utilde{\boldsymbol{\Delta}}^1_1$.

Actually, $\utilde{\boldsymbol{\Delta}}^1_1$ is exactly the Borel sets, which follows from the more general Suslin theorem, see Moschovakis for a proof of this [3]. Note that since inverse images of Borel sets under Borel functions are again Borel, and $\mathcal{B}(\mathbb{N}^{\mathbb{N}}) = \utilde{\boldsymbol{\Delta}}^1_1 \upharpoonright \mathbb{N}^{\mathbb{N}} \subsetneq \utilde{\boldsymbol{\Sigma}}^1_1 \upharpoonright \mathbb{N}^{\mathbb{N}}$, any subset of Baire space which is $\utilde{\boldsymbol{\Sigma}}^1_1$-complete with respecct to Borel reductions is not Borel. This observation will be important when we want to prove that a function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is not Borel measurable.

We now restrict our attention even further, to finite products of $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$, and define the effective Borel classes, starting with the semidecidable subsets[2] instead of the open sets.

Definition A.13. We say that $Y_1 \times \ldots \times Y_k$, with the product topology, is of *type* 1 if each Y_i is either $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$, with $Y_i = \mathbb{N}^{\mathbb{N}}$ for at least one $1 \leq i \leq k$. If $Y_i = \mathbb{N}$ for each $1 \leq i \leq k$, then we say that $Y_1 \times \cdots \times Y_k$ is of *type* 0.

From now on, all definitions and results will refer to spaces of type 1 or 0. Note that these spaces might be considered to be Polish spaces.

Definition A.14. The *effective Borel class of order* n, denoted Σ^0_n, is defined recursively for all $n \in \mathbb{N}$ as follows

$$\begin{aligned}
\Sigma^0_1 &= \text{ all semidecidable sets} \\
\Sigma^0_{n+1} &= \exists^{\mathbb{N}} \neg \Sigma^0_n,
\end{aligned}$$

[2]See Chapter 1.

where Σ^0_1 and all other classes of sets are understood to be restricted to subsets of spaces of type 1 or 0. The *dual* and *ambiguous effective Borel classes* are defined by $\Pi^0_n = \neg\Sigma^0_n$ and $\Delta^0_n = \Sigma^0_\eta \cap \Pi^0_n$, respectively.

Definition A.15. The *effective Lusin class of order* n, denoted by Σ^1_n, is defined recursively for all $n \in \mathbb{N}$ as follows:

$$\Sigma^1_1 = \exists^{\mathbb{N}^\mathbb{N}} \neg\Sigma^0_1$$
$$\Sigma^1_{n+1} = \exists^{\mathbb{N}^\mathbb{N}} \neg\Sigma^1_n,$$

where Σ^0_1 and all other classes of sets are understood to be restricted to subsets of spaces of type 1 or 0. The *dual* and *ambiguous effective Lusin classes* are defined by $\Pi^1_n = \neg\Sigma^1_n$ and $\Delta^1_n = \Sigma^1_\eta \cap \Pi^1_n$, respectively.

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