Reduced form modeling of emission derivatives using Levy processes.

by

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Contents

Chapter 1

Introduction

The European Union emission trading scheme (EU ETS) is an attempt at reducing emission of greenhouse gases such as CO_2. Trading schemes is widely considered as the most cost efficient way to reduce emissions. Simply put, a trading scheme works as follows: The regulator sets a cap on the total amount of emissions allowed in a certain market and then allocates emission allowances to the market participants. The allowances can then be traded freely during the so called trading period, until they are handed in to the regulator at the end of the period. Producers that have emitted more than what is covered by its allowances must either pay a penalty or buy unused allowances from someone else.

Economic theory suggests that the trading of allowances lets the market reach the target level of emissions at the lowest possible total cost[14]. With that said, it should be noted that there is some controversy regarding the effectiveness of the present scheme [5] [14].

The topic of this thesis, however, is not the efficiency of the ETS, but rather the pricing of the emission allowances. Following the conventions of Carmona et al. [5] we will not look at the spot prices of the allowances but rather on futures contracts on these. We will also consider standard European call options on these allowances.

In this thesis we take a reduced form approach based on the paper by Carmona et al. [5]. They develop the following expression:

$$F(t,T) = \pi E(1_N|\mathcal{F}_t) \tag{1.0.1}$$

Here $F(t,T)$ is the price at time t of a futures contract on an emission allowance, and the constant π stands for the penalty which must be paid upon non-compliance. The function 1_N is the indicator function and $N = \{\omega \in \Omega \mid Q_T \geq \gamma\}$ is the event that the total emission Q_T in the market is higher than the total amount of allowances γ. This expression will be the

starting point for this thesis. In order to understand how the expression 1.0.1 is derived we will review the equilibrium model of Carmona et. al in chapter 2.

In this thesis we want to express equation 1.0.1 more explicitly. In order to do this we must decide on how to model the underlying process Q_t. We will model Q_t directly as a Levy process. In chapter 3 we will discuss this choice and show some consequences of this choice. We will also prove a result regarding bounded sub-martingales with binary final value (such as $F(t,T)$). This result is new, to the best of my knowledge, and it will have some interesting consequences in chapter 6.

Then we go on to develop expressions for the risk-neutral price of emission futures when the underlying process is a Levy process in chapter 4.3.2. We will use an approach based on the Fourier transform which has been successfully applied to option pricing [7]. Using the Esscher transform, as described in [3], we will also find an expression for the risk premium.

In chapter 5 we will try to find explicit price expressions analytically for some examples of Levy processes. We will also look at some numerical simulations of the price processes.

As already mentioned we are interested in pricing options on emission futures. With other approaches, such as in [4] and [6] it is possible to get quite explicit expressions for this price. In chapter 6 we will try to find a corresponding expression. We will also try to analytically approximating the option risk-neutral price.

Chapter 2

Origins of price expression

As mentioned in the introduction, the starting point for this thesis is the expression 1.0.1. In this chapter we will take a closer look at where this expression comes from. We will begin by reviewing the equilibrium model developed in the paper [5]. We then follow the lines of the same paper and deduce necessary conditions for the existence of an equilibrium, which leads to the price expression 1.0.1.

2.1 Equilibrium model

In this section we will introduce a large number of new variables. The table 2.1 is included to help the reader keep an overview over these variables.

The model in [5] describes a market where the demand is inelastic, and a finite amount of firms are all trying to maximize their expected profits. The inelasticity means that demand is independent of the price, which might seem unrealistic, but might not be so far from the truth in energy markets. Energy producers are also the largest polluters, and thus energy markets is the main object of emission trading schemes.

In this model, the firms are assumed to be risk neutral. That is, their goal is to maximize expected profit, regardless of risk. It is argued in the paper that this assumption does not change the theoretical result, but simplifies the model a great deal. The firms produce different goods using different technologies. To each pair of good and technology used to produce the good, there are different factors that the firm needs to take in to consideration. Such factors include the price of the good, the marginal cost and the amount of pollution emitted. Since an ETS is implemented, there is a fixed amount of emission allowances on the market in every trading period. These allowances can be traded among the firms, so if it's cheaper for a firm to buy an allowance

Table 2.1: Overview of constants

$i \in I$	Firm in the set of firms I.
$k \in K$	Good in the set of goods K.
$j \in J$	Technology in the set of technologies J.
$\xi^{i,j,k}$	Quantity of good k produced by firm i with technology j.
S_t^k	Price of good k at time t.
$C_t^{i,j,k}$	Production cost of good k at time t, for firm i with technology j.
$\kappa^{i,j,k}$	Production capacity of good k for firm i using technology i .
Λ_i	The amount of emissions allowances given to firm i at time $t = 0$.
θ_t^i	The amount of emission allowances held by firm i at time t.
A_t	The price at time t of an forward contract on an emission allowance t.
$e^{i,j,k}$	The emission of firm i for producing a unit of good k using technology j.
Δ^i	The uncontrollable emissions of firm i.
π_i	Price of penalty.
Π_i	Total penalty paid by firm i.
E_i	Total emissions by firm i.
Υ_i	Total number of allowances owned by firm i, $\Upsilon_i = \theta_T^i + \Lambda_i$.
ρ_i	Profit from production for firm i.
Θ_i	Profit for firm i from emission trading.
P_i	Total profit of firm i. $P_i = \Theta_i + \rho_i - \Pi_i$.
D_t^k	Demand for good k at time t.

than reducing their emissions thats what they will do.

In the end of the trading period firms have to pay a penalty for all emissions exceeding their allowances. In this model banking is not allowed, so that in the end of the trading period, unused allowances are worthless. This is natural since in this model we are only considering one trading period.[1]. For simplicity some more assumptions are made. First it is assumed that no new technologies (i.e more carbon efficient power production) are invented during the trading period. Second, the model does not consider abatement. That means firms cannot get negative emissions by developing clean energy projects. This differs from other common approaches such as in [15] where they allow participants to reduce their emissions by abatement. Abatement accordingly is an important factor in such models. It can be questioned whether these assumptions are realistic, but the hope is that as long as the model gives a good overall picture, one can somehow compensate for missing details at a later stage.

There is a practical convention that should be mentioned: All cash flows and values are expressed in time T currency, where T is the end of the trading period. This way, we do not have to bother with discount factors.

The expression 1.0.1 comes out of the necessary conditions for the existence of a market equilibrium. In order to deduce these conditions we need to know what is meant by market equilibrium. Loosely speaking, the market (with ETS implemented) is said to be in an equilibrium state if the following holds:

1. Every firm in the market is satisfied with their strategy.

2. The demand for every good is met.

3. The total amount of emission contracts bought is at all times equal the total amount of emission contracts sold.

This is very intuitive but not so useful for proofs of any kind, so let us have a closer look at each of these points. In order to do that we need a mathematical framework: Throughout the rest of the thesis we let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t = 1, 2, \ldots, T\}, P)$ be a filtered probability space. Ω contains all possible scenarios from time 0 to T. The σ-algebra $\mathcal{F}$ defines which events have a probability, and the probability measure P assigns a probability to all the events in $\mathcal{F}$. The filtration $\mathbb{F}$ is the set of σ-algebras $\mathcal{F}_t$ that represents the information available at time t. With this in mind, let us start with the first condition.

[1]The model can be extended to multi period trading, and indeed it is in [5].

Condition 1

What does it mean that the firms are satisfied with their strategies? Since firms are assumed to be risk neutral we say that that firms are satisfied when they have maximized their expected profits. Profit is revenue minus costs, so it is the expected value of this difference the firms are trying to maximize. In this model there are three factors for a firm to consider when creating their strategy. There is the production of goods, and the trading of emission allowances. Both of these have a direct impact on the profit of the firms. In addition the firms has to keep in mind that they will have to pay a penalty π per unit of emission exceeding their allowances. For an arbitrary firm i in the market, let us look at these three factors separately:

Production

In this model there are a finite set of goods K. In addition, for each good $k \in K$, there is a finite set of technologies $J^{i,k}$ that firm i can use to produce good k. Firm i needs to decide on which quantity $\xi_t^{i,j,k}$, of good k, to produce using technology j. To decide this, firm i needs the price S_t^k of good k, and the production cost $C_t^{i,j,k}$ of producing good k using technology j, both at time t. With these quantities, an expression for the profit ρ_i of firm i due to production is as follows:

$$\rho_i = \sum_{t=1}^{T} \sum_{k \in K} \sum_{j \in J^k} (S_t^k - C_t^{i,j,k}) \xi_t^{i,j,k} \tag{2.1.1}$$

We call the process $(\xi^{i,j,k})_{t=0}^T$ the production strategy of firm i. In reality, there are limits to how much a firm can produce of a certain good. For instance a power company only has a certain number of wind power stations, and thus the amount of energy it can produce using wind is limited. To mirror this, we introduce capacities $\kappa^{i,j,k}$ associated with each firm, technology and good. We let the production strategy of each firm $(\xi^{i,j,k})_{t=0}^T$ be constrained by

$$\xi_t^{i,j,k} < \kappa^{i,j,k} \tag{2.1.2}$$

Allowance trading

We assume that all firms are endowed with a certain number Λ_i of emission allowances. The firms are allowed to trade the allowances, and furthermore they are allowed to trade forward contracts on these. A forward contract (also called futures contract, or just futures) is a contract between two parties. One party agrees to deliver a certain amount of the the so called un-

derlying, which is emission allowances in our case, at a future date for a price agreed upon when the contract is made. For simplicity, we will only consider trading actual contracts in time T, and in the rest of the time look at forward contracts. This is because the spot price and the forward price will only differ by a discount factor if we assume no arbitrage.

To find an expression for the profit (or loss) due to trading, we need to introduce some more notation. Let A_t denote the price at time t of a forward contract granting one unit of emission at time T. Let θ_t^i denote the amount of contracts firm i is holding at time t. If θ_t^i is positive, firm i will receive $|\theta_t^i|$ allowances at time T, and if it's negative it has to give away the same amount. Now the amount that firm i gains from holding θ_t^i contracts through the time from t to $t+1$ is given:

$$((A_T - A_t) - (A_T - A_{t+1}))\theta_t^i = (A_{t+1} - A_t)\theta_t^i$$

In the end of the trading period, at time T, the actual allowances has to be bought in order to fulfill the contracts. Let θ_T^i denote the quantity of allowances bought at time T. This θ_T^i differs from θ_t^i due to the fact that a firm cannot sell more physical allowances than it actually has. Thus if we denote the amount of allowances allocated to firm i by Λ_i then we have:

$$\Lambda_i - \theta_T^i \geq 0 \text{ so } \theta_T^i \geq -\Lambda_i \tag{2.1.3}$$

Now we have all we need to express the total trading profit Θ_i of firm i. It is given as follows:

$$\Theta_i = \sum_{t=1}^{T-1}(A_{t+1} - A_t)\theta_t^i - \theta_T^i A_T \tag{2.1.4}$$

We call the process $(\theta^i)_{t=0}^T$ the trading strategy of firm i.

Penalty

At the end of the compliance period the companies has to pay a penalty π for each unit of emission exceeding the number of allowances they hold. The amount of emissions E_i for a business is dependent on their production strategy. Say that when firm i is producing good k with technology j, they will emit $e^{i,j,k}$. In addition we assume there are some uncontrollable emissions for each firm, denoted Δ^i. These emissions can be explained by the firms' desire (for good reputation, or to comply with regulations) to always produce enough to satisfy the demand. Therefore it has to overproduce a little, but it is impossible to know how much. This will cause some (uncontrollable)

amount to go to waste, and will also cause uncontrollable extra emissions Δ^i. The total number of allowances Υ the firm has is given by $\Lambda^i + \theta^i_T$. The firm only pays if the emissions exceed their allowances, they do not gain anything for having unused allowances. Therefore the penalty paid Π_i can be expressed as $\pi(E_i - \Upsilon_i)^+$, that is

$$\begin{aligned} \Pi_i &= \pi(\Delta^i + \sum_{t=1}^{T-1} \sum_{k\in K} \sum_{j\in J^k} (\xi^{i,j,k}_t e^{i,j,k}_t) - \Lambda^i - \theta^i_T)^+ && (2.1.5) \\ &= \pi(E_i - \Lambda^i - \theta^i_T)^+ && (2.1.6) \end{aligned}$$

It turns out that we need to make some assumptions about Λ_i and Δ^i. We make the following assumptions about Λ_i:

Assumption 2.1.1.

$$\Lambda_i \in L^1(\mathcal{F}_T) \tag{2.1.7}$$

and

$$P(\Lambda_i > 0) = 1 \tag{2.1.8}$$

The first one is for technical reasons, and the second is because there is no point in looking at a cap and trade system without any allowances. Also for technical reasons we make the following assumption about Δ^i:

Assumption 2.1.2.

$$\Delta^i \in L^1(\mathcal{F}_T) \tag{2.1.9}$$

And the $\mathcal{F}_{T-1}$ conditional distribution of the total uncontrolled emissions $\Delta = \sum_{i\in I} \Delta^i$ has almost surely no point mass.

Now that we have expressions for profits from trading in (2.1.4) and production in (2.1.1), and an expression for emission penalty (2.1.5), we can set up the firms total profit P_i function as

$$\begin{aligned} P^{A,S}_i(\xi^i, \theta^i) &= \rho_i + \Theta_i - \Pi_i && (2.1.10) \\ &= \sum_{t=1}^{T} \sum_{k\in K} \sum_{j\in J^k} (S^k_t - C^{i,j,k}_t)\xi^{i,j,k}_t && (2.1.11) \\ &+ \sum_{t=1}^{T-1} (A_{t+1} - A_t)\theta^i_t - \theta^i_T A_T && (2.1.12) \\ &- \pi(\Delta^i + \sum_{t=1}^{T-1} \sum_{k\in K} \sum_{j\in J^k} (\xi^{i,j,k}_t e^{i,j,k}_t) - \Lambda_i - \theta^i_T)^+ && (2.1.13) \end{aligned}$$

Notice that P_i is a function of ξ^i and θ^i with parameters A and S. We consider ξ^i and θ^i to be the firms trading and production strategies respectively. These are what the firm can change, the parameters A and S are the price processes of allowances and products, and are outside of the firms' control. Nevertheless they are important to take into consideration.

With the expression 2.1.10 we can say that the price processes A^* and S^* satisfy the first condition of an equilibrium if all firms have strategies (θ^{*i}, ξ^{*i}) such that

$$E(P_i^{A^*,S^*}(\xi^{*i}, \theta^{*i})) \geq E(P_i^{A,S}(\xi^i, \theta^i)) \tag{2.1.14}$$

for all other strategies (θ^i, ξ^i). However, we are by no means guaranteed that this expression is defined, so it would be good if we made some assumptions about our processes at this point.

First, since both the trading strategy θ_t^i and the production $\xi_t^{i,j,k}$ only are based on information up to time t, it is natural to assume that they are $\mathcal{F}_t$-adapted. Second we assume that the processes $C_t^{i,j,k}$, $S_t^{i,j,k}$ and A_t are $\mathcal{F}_t$-adapted integrable processes, and that $C_t^{i,j,k}$ and $S_t^{i,j,k}$ are nonnegative.

Condition 2

This condition is easy to understand; the demand for each good has to be met for there be an equilibrium. Otherwise someone will be willing to pay the price needed for one of the producers to produce one more unit of good. To express this condition more precisely, we introduce some notation. For each good $k \in K$ we denote the demand for that good, at time t by D_t^k. Then using the notation above, condition 2 can be expressed as

$$\sum_{i \in I, j \in J^{i,k}} \xi_t^{i,j,k} = D_t^k$$

Condition 3

Using the notation above we can express this simply as

$$\sum_{i \in I} \theta_t^i = 0 \ \forall t \in \{0, 1, \ldots, T\}$$

Thus to recap, we have defined an equilibrium in the market to be as follows:

Definition 2.1.3. *We say that the market is in an equilibrium, with equilibrium price processes (A^*, S^*), if each firm i have strategies (θ^{*i}, ξ^{*i}) satisfying the following three conditions:*

(i) Given the price processes (A^*, S^*), firm i cannot achieve any greater expected profit by changing strategy:

$$E(P_i^{A^*,S^*}(\theta^{*i}, \xi^{*i})) \geq E(P_i^{A^*,S^*}(\xi^i, \theta^i))$$

(ii) The demand for every good is met at all times:

$$\sum_{i\in I, j\in J^{i,k}} \xi_t^{*i,j,k} = D_t^k$$

(iii) At all times the amount of forwards sold equals the amount of forwards bought:

$$\sum_{i\in I} \theta_t^{*i} = 0 \quad \forall t \in \{0, 1, \ldots, T-1\} P\text{-}a.s$$

2.2 Condition for equilibrium

Above we have modeled a market with an emission cap implemented and defined an equilibrium. We will now see that in order for this type of equilibrium to exist, there are some necessary conditions on the equilibrium price of allowances A_t^*. This is what leads to the price expression (1.0.1). The following proof is an elaboration of the proof given in [5] with some of the finer points explained in more detail. First, let us take a closer look at the first condition. By expanding $P_i^{A^*,S^*}(\theta^{*i}, \xi^{*i})$, we see that it is the same as

$$\begin{gathered} E(\rho_i^{S^*}(\xi^{*i}) + \Theta_i^{A^*}(\theta^{*i}) - \pi(E_i(\xi^{*i}) - \Lambda^i - \theta_T^{*i})^+) \geq \\ E(\rho_i^{S^*}(\xi^i) + \Theta_i^{A^*}(\theta^i) - \pi(E_i(\xi^i) - \Lambda^i - \theta_T^i)^+) \end{gathered}$$

for all strategies[2] (θ^i, ξ^i). Since the strategies θ^i and ξ^i are independent, and none of the summands in the expectations above depend on both, we see that the strategies must be optimized separately.

$$E(\Theta_i^{A^*}(\theta^{*i}) + \pi\ \theta_T^{*i}) \geq E(\Theta_i^{A^*}(\theta^i) + \pi\ \theta_T^i) \tag{2.2.1}$$

Which is the same as

$$E\left(\sum_{t=1}^{T-1}(A_{t+1}^* - A_t^*)\theta_t^{*i} - \theta_T^{*i}A_T^* + \pi\ \theta_T^{*i}\right) \geq E\left(\sum_{t=1}^{T-1}(A_{t+1}^* - A_t^*)\theta_t^i - \theta_T^i A_T^* + \pi\ \theta_T^i\right) \tag{2.2.2}$$

[2] In the original paper they define what is an admissible strategy. We will not discuss this since it is a distraction from getting an expression for the price.

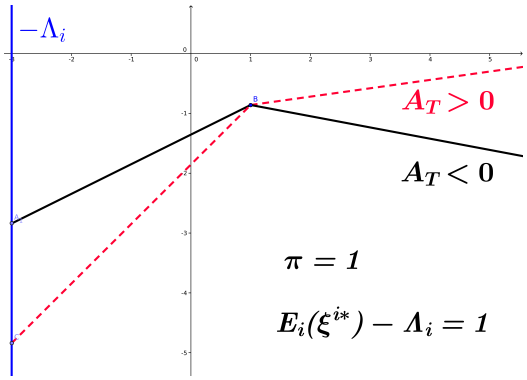

Figure 2.1: Example plot of $f_i(z)$. The black and the dotted line is example paths of f_i when $A_T < 0$ and $A_T > 0$ respectively.

From this we can deduce that $E(A^*_{t+1}|\mathcal{F}_t) = A^*_t$ for all t. To see this, assume the opposite: assume that we can find a time t such that $E(A^*_{t+1}|\mathcal{F}_t) \neq A^*_t$. Then we can find and a set $A \in \mathcal{F}_t$ such that $E(A^*_{t+1}1_A|\mathcal{F}_t) > 1_A A^*_t$ (also for $<$). Now using this, we make a new strategy θ^{i*} by letting it be equal to $\hat{\theta}^i$ for all times except t and setting $\hat{\theta}^i{}_t = \theta^{i*}_t + 1_A$. We see that $\hat{\theta}^i$ outperforms θ^{i*}, which is a contradiction with (2.2.2). Thus we must have $E(A^*_{t+1}|\mathcal{F}_t) = A^*_t$ for all t. This is identical to saying that A^* is a martingale with respect to the filtration $\mathbb{F}$.

From inequality 2.2.2 and condition 1 it can be seen that all the summands depending on θ^{i*}_T can be grouped together. We then optimize θ^{i*}_T separately and deduce:

$$E(-\theta^{*i}_T A^*_T - \pi(E_i(\xi^{*i}) - \Lambda^i - \theta^{*i}_T)^+) \geq E(-\theta^i_T A^*_T - \pi(E_i(\xi^i) - \Lambda^i - \theta^i_T)^+) \quad (2.2.3)$$

Now consider the function (fig. 2.1)

$$f_i(z) = -zA^*_T - \pi(E_i(\xi^{*i}) - \Lambda^i - z)^+$$

By looking at 2.2.3 we see that θ^{*i}_T must maximize $f_i(z)$ for $z \in [-\Lambda^i, \infty)$ (The lower bound is due to the constraint on θ^i_T in (2.1.3)). We easily see that it is either affine or continuous piecewise affine. This leads us to conclude some interesting facts: Since θ^{*i}_T must maximize $f_i(z)$ for $z \in [-\Lambda^i, \infty)$, and θ^{*i}_T must be finite, we conclude that $A^*_T \geq 0$. Also, we must have $A^*_T \leq \pi$ since otherwise $f_i(z)$ is maximized on by $z = \theta^{*i}_T = -\Lambda^i$, but then $\sum_{i\in I}(\theta^{*i}) = -\Lambda \neq 0$ a.s. (from assumption (2.1.8))which breaks the third condition of the equilibrium. Hence we know that $A^*_T \in [0, \pi]$ almost surely If we look on f_i we see that $A^*_T \in (0, \pi] \Rightarrow \theta^{*i}_T \leq E_i - \Lambda^i \ \forall i$ a.s. since θ^{*i}_T maximizes f_i.

Hence

$$\begin{aligned}
\{\omega|A_T^*(\omega) \in (0,\pi]\} &\subseteq \bigcap_{i\in I}\{\omega|\theta_T^{*i}(\omega) \leq E_i(\omega) - \Lambda^i(\omega)\} \\
&\subseteq \{\omega|\sum_{i\in I}\theta_T^{*i}(\omega) \leq \sum_{i\in I}(E_i(\omega) - \Lambda^i(\omega))\} \\
\text{so } \{\omega|A_T^*(\omega) \in (0,\pi]\} &\cap \{\omega|\sum_{i\in I}(E_i(\omega) - \Lambda^i(\omega)) < 0\} \subseteq \{\omega|\sum_{i\in I}\theta_T^{*i}(\omega) < 0\}
\end{aligned}$$

and by condition three in the equilibrium, we get

$$\{\omega|A_T^*(\omega) = 0\} \supseteq \{\omega|\sum_{i\in I}(E_i(\omega) - \Lambda^i(\omega)) < 0\}$$

Using that $A_T^* \in [0,\pi) \Rightarrow \theta_T^{*i} \geq E_i - \Lambda^i \ \forall i$ a.s. a similar argument leads to

$$\{\omega|A_T^*(\omega) = \pi\} \supseteq \{\omega|\sum_{i\in I}(E_i(\omega) - \Lambda^i(\omega)) > 0\}$$

If we denote $\sum_{i\in I}\Lambda_i$ by Λ and the sum $\sum_{i\in I}E_i$ by E we can define the set N' as follows.

$$N' = \{\omega|E(\omega) > \Lambda(\omega)\}$$

Then, what we have just seen above means that

$$A_T^* = \pi 1_{N'}$$

Using the martingale property of A^* we get a nice expression for A_t^*.

$$A_t^* = E(A_T^*|\mathcal{F}_t) = E(\pi 1_{N'}|\mathcal{F}_t) = \pi E(1_{N'}|\mathcal{F}_t) = \pi P(E > \Lambda|\mathcal{F}_t)$$

By assumption 2 $P(E = \Lambda|\mathcal{F}_t) = 0$ so $P(E \geq \Lambda|\mathcal{F}_t) = P(E > \Lambda|\mathcal{F}_t)$. Thus if we let $N = \{\omega|E(\omega) \geq \Lambda(\omega)\}$ we get

$$A_t^* = \pi P(E > \Lambda|\mathcal{F}_t) = \pi E(1_N|\mathcal{F}_t)$$

Which is the same as the price expression (1.0.1), since $Q(\omega,T) = E(\omega)$ and $\gamma = \Lambda$.

Chapter 3

Choice of underlying process

In this section we will look closer at the underlying process Q_t. The underlying process Q_t can be understood as the total emission in the market up to time t. In this thesis we suggest modeling Q_t directly as Levy process (with jumps). If we think about Q_t as the actual total emissions in the market, upwards jumps in Q_t can be justified by considering the following scenario: Suppose we have an energy company A with two production options. The first option is hydro power and the second is coal. As long as the demand is lower than the production capacity for hydro power A will use this option since it is cheaper, and A will have zero emissions. As soon as the demand rises above the capacity (i.e. due to cold weather) A has to start its coal plant and thereby causing a jump in the emissions. Strictly speaking the jump will be in the emission rate and not the total emission. One can either accept this inaccuracy, or think of Q_t not as the actual total emission, but as the market's perception of the total emission. Assuming that information is not continuously available, a jump in the emission rate will then cause an information shock. With this idea, one should allow Q_t to have both upwards and downwards jumps. In [4] they seem to support the idea of thinking about the underlying as the market's perception of the total emission. They also use jump processes to model the underlying in order to capture information shocks.

In [6] they argue that these information shocks, and correspondingly the jumps, will become less common and a less important feature as the emission market matures (this seems to be supported by figure 3). They argue that most of the vital information, such as energy consumption and production, is publicly accessible. They also argue that there is a growing number of firms monitoring the market and providing better analyses as time progresses. These two factors will make information more readily available at all times and prevent shocks. Nevertheless jumps has been an important

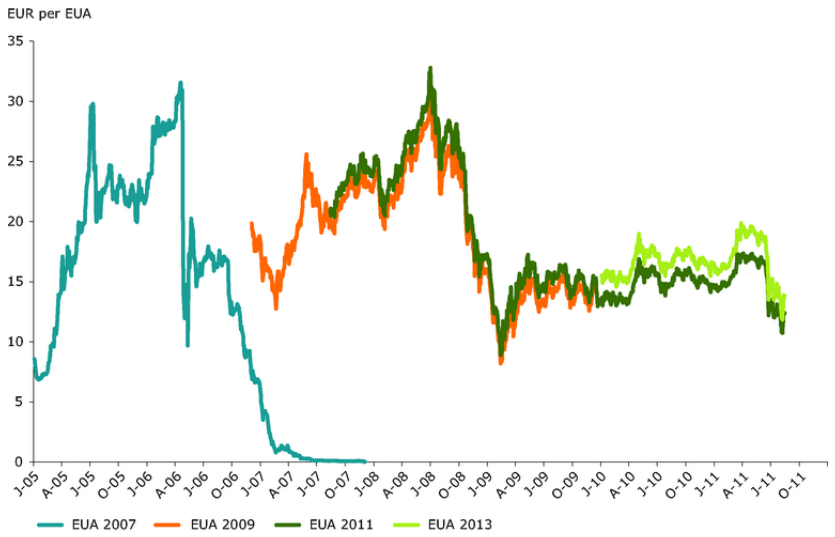

Figure 3.1: Historical prices of EUAs[1].

feature historically. Figure 3 shows the historical prices of EUAs. We can clearly see the importance of jumps, especially downwards (A-06, O-08 and J-11). Even though figure 3 shows jumpy behaviour, that is not the main argument for including jumps in our model. Jump processes tend to better model the large risk of drastic events that are so often present in financial markets [9]. Therefore we will use jump processes to model the underlying. More specifically we will use a special class of jump processes called Levy processes. We will give a brief introduction to these processes in section 3.2.

A common approach to model the underlying is to model the emission rate instead of total emission. Then one calculates the total emission as an integral of the rate over time. This approach is in contrast to our direct modeling approach. For reference we will include an example of how such an approach would work, and what results can be derived. The following is a synopsis of some of the paper [13] by Grüll and Kiesel.

3.1 Example of reduced form model

The authors base their work on the same paper by Carmona [5] as we do, and is thus an interesting comparison. As proposed by the paper of Chesney and Taschini [8], the authors model the emission rate q_t as a geometric Brownian

[1]Source: http://www.eea.europa.eu/data-and-maps/figures/eua-future-prices-200520132011

motion

$$q_t = q_0 \, e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]}$$

Then the cumulative emission Q_t is given

$$Q_t = \int_0^t q_s ds$$

Since this integral has no closed form density, the authors discuss different approximations. In their numerical analysis, they find that the linear approximation used by Chesney and Taschini [8] differs from the two moment-matching approaches introduced in their paper, which in turn give rather similar results. The approach they focus most of their attention at is the log normal moment matching approach. A description of this approach is as follows:

Let Q_{t_1,t_2} denote the stochastic variable $\int_{t_1}^{t_2} q_s ds = Q_{t_2} - Q_{t_1}$, then Q_{t_1,t_2} can be approximated by $Q^L_{t_1,t_2}$ which is a log normally distributed variable so that the first two moments of $Q^L_{t_1,t_2}$ matches those of Q_{t_1,t_2}. In order to formulate this more precisely, an expression for the moments of Q_{t_1,t_2} is needed. In the paper by Milevsky and Posner [19] the first two moments of Q_{t_1,t_2}, $m1$ and m_2 are expressed analytically. We leave out these expressions since the purpose here is not to dive into details, but give an overview of the approach in [13].

Now, using the expression for the moments in [19] one can compute $Q^L_{t_1,t_2}$ to:

$$Q^L_{t,T} = q_t e^{\ln(\frac{\alpha_\tau^2}{\sqrt{2\beta_\tau}}) + \sqrt{\ln(\frac{2\beta_\tau}{\alpha_\tau^2})}Z}$$

where $Z \sim N(0,1)$ and $\tau = T - t$. Let S^L_t denote the futures price in this approach. Then using the price formula from Carmona, it is clear that when $Q_t > \gamma$ the price $S^L_t = \pi$ since $(Q_{T-t} > \gamma - Q_t | \mathbb{F}_t) = 1$. Remember that π is the penalty for non-compliance. When $Q_t < \gamma$ one can compute S^L_t as

follows:

$$S_t^L = \mathbb{P}(Q_{T-t}^L > \gamma - Q_t|\mathbb{F}_t) \tag{3.1.1}$$

$$= \mathbb{P}(q_t e^{\ln(\frac{\alpha_\tau^2}{\sqrt{2\beta_\tau}})+\sqrt{\ln(\frac{2\beta_\tau}{\alpha_\tau^2})}Z} > \gamma - Q_t|\mathbb{F}_t) \tag{3.1.2}$$

$$= \mathbb{P}(\ln\left(\frac{\alpha_\tau^2}{\sqrt{2\beta_\tau}}\right) + \sqrt{\ln\left(\frac{2\beta_\tau}{\alpha_\tau^2}\right)}Z > \ln\left(\frac{\gamma - Q_t}{q_t}\right)|\mathbb{F}_t) \tag{3.1.3}$$

$$= \mathbb{P}\left(Z > \frac{\ln\left(\frac{\gamma-Q_t}{q_t}\right) - \ln\left(\frac{\alpha_\tau^2}{\sqrt{2\beta_\tau}}\right)}{\sqrt{\ln\left(\frac{2\beta_\tau}{\alpha_\tau^2}\right)}}|\mathbb{F}_t\right) \tag{3.1.4}$$

$$= \Phi\left(\frac{-\ln\left(\frac{\gamma-Q_t}{q_t}\right) + 2\ln\left(\alpha_\tau\right) - \frac{1}{2}\ln\left(\sqrt{2\beta_\tau}\right)}{\sqrt{\ln\left(2\beta_\tau\right) - 2\ \ln(\alpha_\tau)}}\right) \tag{3.1.5}$$

which is a very neat expression. We see that this price expression allows for doing many useful calculations relatively easy. Using this nice price formula, and the fact that they have an explicit expression for the emission rate q_t, the authors use Ito's formula to find an expression for the dynamics dS_t^L of the price. This is again be used to explicitly express how sensitive the futures price is to changes in the underlying.

It is clear that this approach has some very strong points. The tractable expressions for the S_t^L and its dynamics, and that we can compute the sensitivity on the underlying, are definitely among them. On the other hand there are some drawbacks as well. The approximation of Q_t does not capture the tail behaviour (probability for dramatic events). This tail behaviour, had it been accounted for, might have had considerable influence on the price. Furthermore, because the estimation is on the underlying, and not the price, it is hard to say just how much the resulting price estimate differs from the "real" price suggested by the model.

3.2 Levy processes

As mentioned, we are going to model Q_t directly as a Levy process. In this section we will give a brief overview over these processes. All the results regarding Levy processes in general is from the book on Levy processes by Cont and Tankov [9] and will be given without proof.

Let's start with the definition of a Levy process. Suppose $(S_t)_{t\geq 0}$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$. Then $(S_t)_{t\geq 0}$ is a Levy

process if it is a cadlag process with $S_0 = 0$ and satisfies the following three conditions:

1. **Independent increments** For any increasing sequence $t_0, t_1, \ldots$, the random variables $S_{t_0}, S_{t_1} - S_{t_0}, \ldots, S_{t_k} - S_{t_{k-1}}$ are independent.

2. **Stationary increments** The distribution of $S_{t+u} - S_t$ is independent of t.

3. **Stochastic continuity** For all times t and all $\epsilon > 0$, we have:

$$\lim_{\delta \to 0} P(S_{t+\delta} - S_t \geq \epsilon) = 0$$

It can be deduced It is a famous result called the Levy–Ito decomposition, which states that all levy processes can be represented in the following way:

$$S_t = \gamma t + B_t + S_t^l + \lim_{\epsilon \to 0} S_t^\epsilon \tag{3.2.1}$$

Where γt is linear drift, B_t is Brownian motion, S_t^l is a compound Poisson process and S_t^ϵ is a square integrable pure jump martingale:

$$S_t^\epsilon = \int_{|s| \leq 1 \times x \in [0,t]} s[J_S(ds \times dx) - \nu(ds \times dx)] \tag{3.2.2}$$

Here J_S is the jump measure of S with intensity $\nu(ds \times dx) = \nu(ds)dx$, where ν is the so called Levy measure of S. Intuitively the jump measure $J_S([0,t] \times B)$ counts the number of jumps in the time interval $[0,t]$ with jump size in B, for a given path of S. The Levy measure $\nu(B)$ can be understood as the expected number of (non-zero) jumps in the unit interval with jump size in B.

Simply put, this means all Levy process consists of a linear drift, a Brownian motion and a jump process. The reason for this relatively complicated last term S_t^ϵ, is a problem with convergence due to the nature of the jumps of Levy processes. While it is true that for any $\delta \in \mathbb{R}$, every Levy process S_t has only finitely many jumps larger than δ, S_t may have infinitely many small jumps. The sum of these small jumps does not necessarily converge. Therefore one has to split the jump process into a part with large jumps[2], and a part with small jumps. For the part with small jumps one subtracts the intensity inside the sum, making sure that it converges almost surely, and also making it a martingale in the process.

[2]For instance those larger than 1

A class of Levy processes that are especially nice are the so called processes of finite variation. These processes are made up of linear combinations of increasing and decreasing processes.[3] For processes of finite variation we have:

$$\int_{|s|\leq 1} s\nu(ds) < \infty \tag{3.2.3}$$

thus we do not have to split the jump process apart. These processes cannot have any Brownian motion part since they are neither increasing nor decreasing. Thus any finite variation process can be written as:

$$\gamma t + S_t^l = \gamma t + \sum_{x\in[0,t]}^{\Delta S\neq 0} \Delta S_x$$

that is, a linear drift and the sum of its jumps. A subordinator is an increasing process, and thus a special case of finite variation processes with only positive jumps and positive drift.

The characteristic function $\Phi(u)_X$ of a stochastic variable X is defined as $E(e^{iXu})$. The characteristic function will be useful to us since it is always defined, and fully characterizes the stochastic variable. Furthermore it is often available in closed form. As a direct consequence of the first two properties of Levy processes, it follows that the characteristic function $\Phi_{S_t}(u)$ of a Levy process $(S_t)_{t\geq 0}$ can be written as follows:

$$\Phi_{S_t}(u) = E(e^{iuS_t}) = e^{\psi(u)t} \tag{3.2.4}$$

for some continuous function $\psi : \mathbb{R}^d \to \mathbb{R}$. Thus, if we know the characteristic function of S_t for any time, we know it for all times. When this is combined with the Levy–Ito decomposition we get a way to represent Levy processes called Lévy–Khintchine representation:

$$\psi(u) = -\frac{1}{2}u^2 A + i\gamma u + \int_{-\infty}^{\infty} (e^{ixu} - 1 - iux_{|x|\leq 1}\nu(dx))$$

where A is the correlation matrix for a d-dimensional Brownian motion, γ is a drift vector and ν is the Levy measure of the process. The result is that Levy processes are completely characterized by the triplet (γ, A, ν) which is called the characteristic triplet. Since finite variation processes don't have any Brownian motion, their characteristic triplet is always $(\gamma, 0, \nu)$ with ν satisfying 3.2.3. Subordinators additionally satisfy $\nu((-\infty, 0]) = 0$ and $\gamma \geq$

[3] Since we work in 1 dimension

0. One final property of Levy processes that might be of interest is the Markov property. For any $\mathcal{F}_t$-adapted Levy process S_t we have:

$$E(f(S_t)|\mathcal{F}_x) = E(f(S_t)|\sigma(S_x)) \tag{3.2.5}$$

for $f : \mathbb{R}^n \to \mathbb{R}$ being bounded and measurable, and $t \geq x$.

3.3 Early observations

Now that we have had a quick look at Levy processes, we shall see if we can already use our knowledge in relation to our price expression. Before we do this, we will introduce a new convention that we will use throughout the rest of the thesis: From now on, when we talk about the risk-neutral futures price, we will mean the normalized $F(t,T)$ from expression 1.0.1. We will denote this price by A_t:

$$A_t = E(1_{Q_T \geq \gamma}|\mathcal{F}_t) \tag{3.3.1}$$

This is done for practical reasons. The disadvantage of carrying the symbol π around would be apparent in chapter 4.3.2 where we use the Fourier transform, which involves the real number π. With this convention we can go on.

First, it will be interesting to know whether or not making the underlying Q_t a Levy process will make the derivative A_t a Levy process as well. This would make calculations with A_t easier, but it would make the model less realistic. Intuitively it seems very wrong to model the price process with independent increments. The price is not going to rise sharply right after a steep fall, people are afraid and will be cautious. Also intuitively it seems wrong with stationary increments; the price should become more sensitive to changes in the underlying and thus more volatile as $t \to T$. It turns out we don't have to worry:

Proposition 3.3.1. *The price process A_t is not a Levy process.*

Proof. Since $A_t = E(1_{Q_T > \gamma}|\mathcal{F}_t) = P(Q_T > \gamma|\mathcal{F}_t)$ it is bounded above and below. This implies non-independent increments since $A_{t_2} - A_{t_1} \leq 1 - A_{t_1}$ thus not independent of A_{t_1}. □

Let us continue the line of thought in the last proof. It might seem intuitive that since the increase is bounded by $1 - A_t$, and the expected increase $E(A_{t+h} - A_t|\mathcal{F}_t) = 0$, the probability of an increase must rise as A_t grows. We can write

$$E(A_{t+h} - A_t|\mathcal{F}_t) = d_1 P(A_{t+h} > A_t|\mathcal{F}_t) + d_2 P(A_{t+h} \leq A_t|\mathcal{F}_t) = 0 \tag{3.3.2}$$

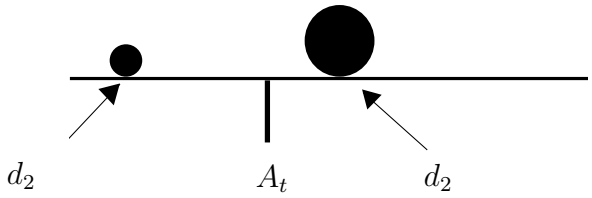

Figure 3.2: This figure brings to mind two children playing on a seesaw, which is a fitting analogy. The figure shows the seesaw in balance.

where d_1 and d_2 is the center of gravity given increase and decrease respectively. Then the above mentioned intuition translates to d_2 growing as A_t grows. Consider figure 3.2 to get a picture of what is going on. Moving A_t is like moving the balance point; according to the intuition the sizes of d_1 and d_2 must change corresponding to the movement of A_t in order for the "seesaw" to remain in balance. However we are forgetting that the position of d_1 and d_2 is allowed to shift depending on $\mathcal{F}_{t+h}$. Technically there is an issue with the measurability of d_1 and d_2 reflecting that we don't know where they will be at time $t + h$.

The intuition is not completely wasted though: Suppose we force d_1 and d_2 to stay at their respective ends of the "seesaw". Then changing their size is the only way to keep the "seesaw" in balance as A_t moves. We formulate this more properly:

Theorem 3.3.2. *Let $(\Omega, \mathcal{F}, P)$ be a measure space. Let X_t be a sub-martingale with respect to the filtration $\mathcal{F}_t$, where $t \in [0, T]$. Assume X_t is bounded $a \leq X_t \leq b$ for some $a, b \in \mathbb{R}$. Further assume that X_T is binary, that is $X_T = a$ or $X_T = b$ a.s. Then*

$$P(X_T = b|\mathcal{F}_t) \geq \frac{X_t - a}{b - a} \text{ a.s.}$$

with equality when X_t is a martingale.

Proof. Since X_t is a sub-martingale we have $E(X_T|\mathcal{F}_t) \geq X_t$. But since X_T is binary we can write

$$E(X_T|\mathcal{F}_t) = E(X_T 1_{X_T=a}(\omega) + X_T 1_{X_T=b}(\omega)|\mathcal{F}_t) \tag{3.3.3}$$
$$= E(X_T 1_{X_T=a}(\omega)|\mathcal{F}_t) + E(X_T 1_{X_T=b}(\omega)|\mathcal{F}_t) \tag{3.3.4}$$

Now we can use that $X_T 1_{X_T=a}(\omega) = a1_{X_T=a}(\omega)$ a.s. and likewise on the other side, so equation 3.3.4 can be written:

$$aE(1_{X_T=a}(\omega)|\mathcal{F}_t) + bE(1_{X_T=b}(\omega)|\mathcal{F}_t) \tag{3.3.5}$$
$$= aP(X_T = a|\mathcal{F}_t) + bP(X_T = b|\mathcal{F}_t) \tag{3.3.6}$$
$$= a(1 - P(X_T = b|\mathcal{F}_t)) + bP(X_T = b|\mathcal{F}_t) \tag{3.3.7}$$
$$= a + (b - a)P(X_T = b|\mathcal{F}_t) \tag{3.3.8}$$

Now recalling the sub-martingale property the lemma is proven. □

If we apply this to our price process A_t we get the following:

Corollary 3.3.3. $P(A_T = 1|\mathcal{F}_t) = A_t$

There is one more property of A_t that can be deduced at this point. There is a connection between the jumps of the underlying Q_t and the price A_t:

Proposition 3.3.4. *Any jumps in the underlying Q_t will cause the price A_t to jump in the same direction.*

Proof. We will look at positive jumps: Let t_1+ denote $\lim_{t\to t_1, t>t_1}$ and let t_1- be defined respectively. Suppose the underlying Q_t has a positive jump in time t_1, that is $Q_{t_1+} = Q_{t_1-} + \delta$ for some $\delta > 0$. Recall that $A_t = P(Q_T > \gamma|\mathcal{F}_t) = P(Q_T > \gamma|Q_t)$ since Q_t has the Markov property by being a Levy process. We have:

$$\begin{aligned} A_{t_1+} &= P(Q_T > \gamma|Q_{t_1+}) \\ &= P(Q_{T-t_1} > \gamma - (Q_{t_1-} + \delta)) \\ &\geq P(Q_{T-t_1} > \gamma - Q_{t_1-}) \\ &= P(Q_T > \gamma|Q_{t_1-}) = A_{t_1-} \end{aligned}$$

Where we have used the stationary increments property of Levy processes. Using a similar argument for negative jumps the lemma follows. □

Chapter 4

Price expressions

In this chapter we will derive two price expressions for A_t when Q_t is a Levy process. A common approach, used amongst others by [6] and [14], is to express the price as the solution to a stochastic differential equation (SDE). Since we model the underlying as a Levy process, the dynamics of the underlying and therefore the SDE, can get rather ugly. On the other hand, as mentioned earlier, we know that characteristic functions are always defined and often known in closed form. Therefore we will use an approach along the lines of the one developed in the paper of Carr et al. [7]. In this paper the Fourier transform is used to get an expression for the price of a call option. This expression ends up as an integral of a function depending on the characteristic function of the underlying.

We will first find an expression using a method from [7]. This expression allows the fast Fourier transform (FFT) which makes numerical approximation efficient. The first expression seems more complicated than necessary so we will go on to find a second price expression. Finally we will use the Esscher transform as in [3] to find an expression for the risk premium and the market dynamics of the emission futures.

This chapter contains much complex analysis, all of which can be found in [20].

4.1 The first price expression

We begin by finding the Fourier transform of the indicator function $1_N(\omega)$. Recall that for a function $f : \mathbb{R} \to \mathbb{C}$ the Fourier transform $\hat{f}$ of f is given as follows:

$$\hat{f}(y) = \int_{\mathbb{R}} e^{-ixy} f(x) dx \tag{4.1.1}$$

and that as long as the integral is defined we can write f as:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) e^{ixy} dy \tag{4.1.2}$$

We want to use this on $1_N(\omega) = 1_{\{\omega | Q(\omega,T) > \gamma\}}(\omega)$ in expression (1.0.1). Since $Q : \Omega \times \mathbb{R} \to \mathbb{R}$ we can look at $f(x) = 1_{\{x>\gamma\}}(x)$ and substitute Q for x later. There is one problem with this approach. The integral $\int_{\mathbb{R}} 1_{\{x>\gamma\}}(x)dx = \int_{\gamma}^{\infty} dx$ which does not converge, therefore f is not integrable over $\mathbb{R}$. In order to take the Fourier transform of f, we use the technique described in [7]. We introduce the function $g(x) = e^{-\alpha x} f(x)$ for some $\alpha > 0$. It is easily seen that g is integrable; we find the Fourier transform of g:

$$\begin{aligned} \hat{g}(y) = \int_{\mathbb{R}} e^{-ixy} g(x) dx &= \int_{\mathbb{R}} e^{-(\alpha+iy)x} f(x) dx \\ &= \int_{\mathbb{R}} e^{-(\alpha+iy)x} 1_{\{x>\gamma\}} dx \\ &= \int_{\gamma}^{\infty} e^{-(\alpha+iy)x} dx \\ &= e^{-\gamma\alpha} \frac{e^{-i\gamma y}}{(\alpha + iy)} \end{aligned}$$

Since γ and α are positive we can use Jordan's Lemma to see that $\hat{g}(y)$ integrates to 0 when integrating over $\mathbb{R}$. Thus $\hat{g}(y)$ is integrable, and we can try to use the inversion theorem. We will find the inverse formally and call it $\tilde{f}$, and then check if $\tilde{f} = f$. To that end, we first find the formal inverse $\tilde{g}$:

$$\tilde{g}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(y) e^{ixy} dy = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-\gamma(\alpha+iy)}}{(\alpha + iy)} \, e^{ixy} dy \tag{4.1.3}$$

$$= \frac{e^{-\alpha\gamma}}{2\pi} \int_{\mathbb{R}} \frac{e^{(x-\gamma)iy}}{(\alpha + iy)} dy \tag{4.1.4}$$

Since $f(x) = e^{\alpha x} g(x)$ we get the following expression of $\tilde{f}$:

$$\tilde{f}(x) = \frac{e^{\alpha(x-\gamma)}}{2\pi} \int_{\mathbb{R}} \frac{e^{(x-\gamma)iy}}{(\alpha + iy)} dy \tag{4.1.5}$$

Now let us check that $\tilde{f} = f$. We will look at the three cases, $x < \gamma$, $x = \gamma$ and $x > \gamma$ one at a time.

- $x = \gamma$**:** In this case the integrand becomes $1/(\alpha + iy)$ so then

$$\int_{\mathbb{R}} \frac{1}{(\alpha + iy)} dy = \int_0^{\infty} \frac{1}{(\alpha + iy)} dy + \int_{-\infty}^{0} \frac{1}{(\alpha + iy)} dy$$

This does not converge as an improper integral, but we can define $\tilde{f}$ to be the principal value

$$\tilde{f}(x) = \frac{1}{2\pi} p.v \int_{\mathbb{R}} \hat{f}(y) e^{ixy} dy$$

Then it converges and

$$\begin{aligned} p.v \int_{\mathbb{R}} \frac{1}{(\alpha + iy)} dy &= \lim_{y \to \infty} -i \left(\log(|\alpha + iy| - \log(|\alpha - iy|) \right) \\ &\quad + \lim_{y \to \infty} \left(\text{Arg}(\alpha + iy) - \text{Arg}(\alpha - iy) \right) \\ &= \frac{\pi}{2} - -\frac{\pi}{2} = \pi \end{aligned}$$

then

$$\tilde{f}(x) = \frac{1}{2\pi} \pi = \frac{1}{2}$$

- $x < \gamma$**:** For $x < \gamma$ we get a negative exponent in the integrand. Thus the contour integral along a semicircle in the positive half plane will diverge as the radius grows beyond bounds. In the negative half plane though, it will vanish, by Jordan's Lemma. Thus, if we continue to consider the Cauchy Principal value we can find it by integrating along the contour $[-\rho, \rho] + e^{-\rho i \theta}$ where $0 \leq \theta \geq \pi$. Since the integrand has no singularities in the lower half plane, the integral is zero. Thus for $x < \gamma$ we have $\hat{f} = f$.

- $x > \gamma$**:** For the case $x > \gamma$ we can use the same argument, except this time we must use the semicircle in the upper half-plane, and there we have a singularity in $y = -\alpha i$. We find the residue and use Cauchy's residue theorem:

$$\begin{aligned} \tilde{f}(x) &= 2\pi i \left(\frac{e^{\alpha(x-\gamma)}}{2\pi} \right) \text{Res}(\text{Integrand} : -\alpha i) \\ &= i \left(e^{\alpha(x-\gamma)} \right) \left(\lim_{y \to -ai} -i e^{(x-\gamma)iy} \right) \\ &= i \left(e^{\alpha(x-\gamma)} \right) \left(-i e^{-\alpha(x-\gamma)} \right) \\ &= 1 \end{aligned}$$

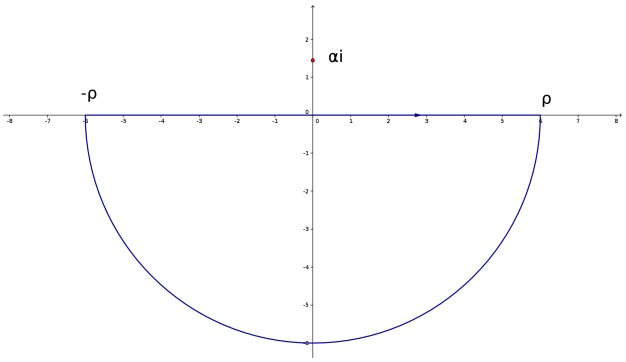

Figure 4.1: Contour for $x < \gamma$

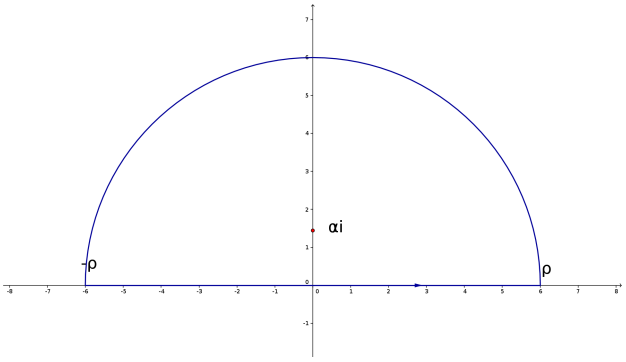

Figure 4.2: Contour for $x > \gamma$

Thus $\tilde{f} = f$ for $x > \gamma$.

We can conclude that our $\tilde{f} = f$ almost everywhere, but because we had to use the principal value we get $\tilde{f} = 1/2f$ in the discontinuity at $x = \gamma$.

What consequences does it have that our transformed $\tilde{f}$ differs from the original in the point $x = \gamma$? Recall that the reason we are interested in f is because we want to find $E(1_N|\mathcal{F}_t)$, that is

$$\int_\Omega 1_{Q_T \geq \gamma}(\omega) dP(\omega)_t$$

And as long as

$$P_t(Q_T = \gamma) = 0 \tag{4.1.6}$$

we have that

$$\int_\Omega f(Q_T) - \tilde{f}(Q_T) dP(\omega)_t = \frac{1}{2} P_t(Q_T = \gamma) = 0$$

Therefore we can use $\tilde{f}$ as long as 4.1.6 is satisfied, which will be assumed throughout the rest of this section.

Now, making the necessary assumptions about the law of Q_T, we can substitute Q_T for x in (4.1.5) to get an expression for the price :

$$A_t = E\left(\frac{e^{\alpha(Q_T-\gamma)}}{2\pi} p.v \int_{\mathbb{R}} \frac{e^{(Q_T-\gamma)iy}}{(\alpha+iy)} dy|\mathcal{F}_t\right)$$

4.1.1 Some technical difficulties

In order to continue we would like to pull out the integral outside of the conditional expectation. At first it seems that this will be alright straight from Fubini. But our expectation is conditional, which makes it a stochastic variable, thus we are required to look more carefully to confirm the original hunch.

Lemma 4.1.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then for any $(\mathcal{F} \otimes \mathcal{B})$ measurable $X : \Omega \times \mathbb{R} \to \mathbb{C}$ such that $\int_{\mathbb{R}} X(\omega, y)dy < \infty$ for all ω, we have that if either*

$$E(\int_{\mathbb{R}} X(\omega, y)dy|\mathcal{F}_t) \tag{4.1.7}$$

or

$$\int_{\mathbb{R}} E(X(\omega, y)|\mathcal{F}_t)dy \tag{4.1.8}$$

is finite, they are both finite and they are almost surely equal.

Proof. The lemma is a consequence of Fubini as follows: By definition

$$E(\int_{\mathbb{R}} X(\omega, y)dy|\mathcal{F}_t)$$

is the a.s unique $\mathcal{F}_t$ measurable stochastic variable satisfying that:

$$\int_H E(\int_{\mathbb{R}} X(\omega, y)dy|\mathcal{F}_t)dP = \int_H \int_{\mathbb{R}} X(\omega, y)dydP$$

for all $H \in \mathcal{F}_t$. We also have that $\int_{\mathbb{R}} E(X(\omega, y)|\mathcal{F}_t)dy$ is the is the a.s unique $\mathcal{F}_t$ measurable stochastic variable satisfying that

$$\int_{\mathbb{R}} \int_H E(X(\omega, y)|\mathcal{F}_t)dPdy = \int_{\mathbb{R}} \int_H X(\omega, y)dPdy$$

for all $H \in \mathcal{F}_t$. Thus, by Fubini we have that if $\int_H \int_{\mathbb{R}} X(\omega, y)dydP$ or $\int_{\mathbb{R}} \int_H X(\omega, y)dPdy$ is finite, then so is the other, and they are equal.

Thus, since $E(\int_{\mathbb{R}} X(\omega, y)dy|\mathcal{F}_t)$ and $\int_{\mathbb{R}} E(X(\omega, y)|\mathcal{F}_t)dy$ are both the a.s unique stochastic variables to have the same particular property, they must be a.s the same. □

Another point that requires some attention is the fact that we are using the Cauchy principal value. On this point Lemma 4.1.1 is not enough for our needs, but we can make the following addition:

Lemma 4.1.2. *Lemma 4.1.1 holds even when the integral is interpreted as the Cauchy principal value.*

Proof. By definition p.v $\int_{\mathbb{R}} X(\omega, y)dy = \lim_{R\to\infty} \int_{\mathbb{R}} 1_{[-R,R]}X(\omega, y)dy$. Now let us make a sequence of ever increasing R. Let $R_n = 2^n$, then we can find an N so that for all $n > N$[1] lemma 4.1.1 holds for $1_{[-R_n,R_n]}X(\omega, y)$. This means that:

$$E(\int_{\mathbb{R}} 1_{[-R_n,R_n]}X(\omega, y)dy|\mathcal{F}_t) = \int_{\mathbb{R}} E(1_{[-R_n,R_n]}X(\omega, y)|\mathcal{F}_t)dy \text{ a.s} \quad (4.1.9)$$

that is (by recycling the argument in the proof of lemma 4.1.1)

$$E(\int_{-R_n}^{R_n} X(\omega, y)dy|\mathcal{F}_t) = \int_{-R_n}^{R_n} E(X(\omega, y)|\mathcal{F}_t)dy \text{ a.s} \quad (4.1.10)$$

Suppose the limit $\lim_{n\to\infty}$ exists and is finite on either side of the equality. Then since the nth element is equal (a.s) for all $n > N$ the limits must be equal (a.s). This limit is of course the Cauchy principal value, which leads to the conclusion:

$$E(p.v \int_{\mathbb{R}} X(\omega, y)dy|\mathcal{F}_t) = p.v \int_{\mathbb{R}} E(X(\omega, y)|\mathcal{F}_t)dy \text{ a.s} \quad (4.1.11)$$

□

4.1.2 Final steps

With these technical difficulties out of the way we are ready to proceed. For simpler notation, we will often skip writing $p.v$, just knowing that if the usual integral does not make sense we are talking about the Cauchy principal value.

We note that the conditions for the lemma are satisfied. Since measurable functions are closed under composition, addition and multiplication, we see that our integrand satisfies measurability requirement. The requirement that

[1] This precaution is added it in case there are examples where p.v $\int_{\mathbb{R}} X(\omega, y)dy < \infty$ but $\int_{\mathbb{R}} 1_{[-R,R]}X(\omega, y)dy \not< \infty$ for all finite R

the integral is finite is what we just discussed above. So using lemma 4.1.2:

$$A_t = E\left(\frac{e^{\alpha(Q_T-\gamma)}}{2\pi}\int_{\mathbb{R}}\frac{e^{(Q_T-\gamma)iy}}{(\alpha+iy)}dy|\mathcal{F}_t\right) \tag{4.1.12}$$

$$= E\left(\int_{\mathbb{R}}\frac{e^{\alpha(Q_T-\gamma)+(Q_T-\gamma)iy}}{2\pi(\alpha+iy)}dy|\mathcal{F}_t\right) \tag{4.1.13}$$

$$= \int_{\mathbb{R}} E\left(\frac{e^{\alpha(Q_T-\gamma)+(Q_T-\gamma)iy}}{2\pi(\alpha+iy)}|\mathcal{F}_t\right)dy \text{ a.s} \tag{4.1.14}$$

$$= \int_{\mathbb{R}} E\left(\frac{e^{(\alpha+iy)}e^{(Q_T-\gamma)}}{2\pi(\alpha+iy)}|\mathcal{F}_t\right)dy \tag{4.1.15}$$

$$= \int_{\mathbb{R}} \frac{e^{-\gamma(\alpha+iy)}}{2\pi(\alpha+iy)}E\left(e^{(\alpha+iy)Q_T}|\mathcal{F}_t\right)dy \tag{4.1.16}$$

We can now use the independent increments and stationary increments properties of Levy processes to look closer on the conditional expectation:

$$E\left(e^{(\alpha+iy)Q_T}|\mathcal{F}_t\right) = E\left(e^{(\alpha+iy)(Q_T-Q_t)+Q_t(\alpha+iy)}|\mathcal{F}_t\right) \tag{4.1.17}$$

$$= E\left(e^{(\alpha+iy)(Q_T-Q_t)}|\mathcal{F}_t\right)E\left(e^{(\alpha+iy)Q_t}|\mathcal{F}_t\right) \tag{4.1.18}$$

$$= E\left(e^{(\alpha+iy)Q_{(T-t)}}\right)e^{(\alpha+iy)Q_t(\omega)} \tag{4.1.19}$$

As we see, we have gotten rid of the conditional expectation and we are closer to a expression involving the characteristic function. We now plug (4.1.19) in to (4.1.16) and get:

$$A_t = \int_{\mathbb{R}}\frac{e^{-\gamma(\alpha+iy)}}{2\pi(\alpha+iy)}E\left(e^{(\alpha+iy)Q_{(T-t)}}\right)e^{(\alpha+iy)Q_t}dy \tag{4.1.20}$$

$$= \left(\frac{e^{\alpha(Q_t-\gamma)}}{2\pi}\int_{\mathbb{R}}\frac{e^{(Q_t-\gamma)iy}}{(\alpha+iy)}E\left(e^{(\alpha+iy)Q_{(T-t)}}\right)dy\right) \tag{4.1.21}$$

$$\tag{4.1.22}$$

Now, if we let $b = y - ia$, then $ib = a + iy$ and:

$$E\left(e^{(\alpha+iy)Q_{(T-t)}}\right) = E\left(e^{ibQ_{(T-t)}}\right) \tag{4.1.23}$$

$$= \Phi_{Q_{(T-t)}}(b) \tag{4.1.24}$$

$$= e^{(T-t)\psi_Q(b)} \tag{4.1.25}$$

Pulling everything together, we can now express the price using the characteristic function as follows:

Proposition 4.1.3. *Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t = 1, 2, \ldots, T\}, P)$ be a filtered probability space. Let $Q(\omega, t)$ denote the total emissions in the market at time t, and let γ be the total allowances allocated. Suppose that the forward allowance price A_t is given*

$$A_t = E(1_N|\mathcal{F}_t)$$

where $N = \{\omega : Q(\omega, T) \geq \gamma\}$ Then, if Q_T is a Levy process satisfying $P(Q_T = \gamma) = 0$, we have

$$A_t = \frac{e^{\alpha(Q_t-\gamma)}}{2\pi} \int_{\mathbb{R}} \frac{e^{(Q_t-\gamma)iy}}{(\alpha + iy)} e^{(T-t)\psi_Q(y-ia)} dy \ a.s$$

where ψ_Q is the characteristic exponent of Q.

Remark 4.1.4. *Note that*

$$A_t = e^{\alpha(Q_t-\gamma)} \mathcal{F}^{-1} \left[\frac{e^{(T-t)\psi_Q(y-ia)}}{(\alpha + iy)} \right]_{(Q_t-\gamma)}$$

where

$$\mathcal{F}^{-1}[f]_x = \frac{1}{2\pi} \int_{\mathbb{R}} f(y) e^{ixy} dy$$

denotes the inverse Fourier transform of f in x. Thus, depending the on characteristic function of Q_t, the inverse FFT might work well as a way to solve this integral numerically.

4.2 Second price expression

The price expression in proposition 4.1.3 is nice enough, but the expression seems to be more cumbersome than necessary due to the variable α. The variable α is just a "trick variable" introduced to make $1_{x>\gamma}$ integrable. It seems unnecessary to carry it around; it is clouding the expression and it will always be canceled out in the end. It would be nice to find a Fourier transform straight from our indicator function, without any trick variables. To do this, we have to find another Fourier transform of the indicator function $1_{\omega>\gamma}(\omega)$. It turns out

$$\pi\delta(\omega) + p.v\frac{e^{-\gamma i\omega}}{i\omega}$$

is such a Fourier transform in a sense, but not quite. It is a not a function but a distribution; it is not a Fourier transform in a classical sense. With this in mind, we try our luck with so called distributions and see if we cannot make some sense out of it. We start with a quick review of distributions.

4.2.1 Distributions

Distributions can be seen as continuous linear functionals on an underlying vector space of so called test functions. The space of test functions is the set of functions used to decide whether or not two distributions are in fact the same. To clarify this, let the test space be denoted T, and let $f : T \to \mathbb{R}$ and $g : T \to \mathbb{R}$ be functionals. Then $f = g$ iff $f(\phi) = g(\phi) \forall \phi \in T$ [11]. Different test spaces can be chosen for different purposes [18], and we will choose with Fourier transforms in mind. In [16] an intuitive space of Gaussian test functions is developed, but it seems that the most standard choice is the Schwartz space Ş. We will go with the standard choice and when we talk of a distribution we will mean a continuous linear functional on the Schwartz space Ş. Since we are interested in distributions in connection with Fourier transforms, we need to know what is meant by the Fourier transform of a distribution. Let f be a classically transformable function, then for all $\phi \in$ Ş the following identity hold:

$$\int_{-\infty}^{\infty} \mathcal{F}(f)|_x \phi(x) dx = \int_{-\infty}^{\infty} f(y) \mathcal{F}(\phi)|_y dy$$

where $\mathcal{F}(f)|_x = \int_{-\infty}^{\infty} f(y) e^{-ixy} dy$ is the Fourier transform of f. This can be used to define the Fourier transform of a distribution in the following way. Let ψ be a distribution, then $\mathcal{F}(\psi)|_x$ is defined to be the distribution φ that satisfies:

$$\int_{-\infty}^{\infty} \varphi(x) \phi(x) dx = \int_{-\infty}^{\infty} \psi(y) \mathcal{F}(\phi)|_y dy$$

for all $\phi \in$ Ş. The inverse is defined analogously.

4.2.2 Deduction

The distribution

$$g(\omega) = \pi \delta(\omega) + p.v \frac{e^{-\gamma i \omega}}{i\omega}$$

takes a test function ϕ to

$$p.v \int \phi(\omega) \left(\delta(\omega) + \frac{e^{-\gamma i \omega}}{i\omega} \right) d\omega = 2\pi \phi(0) + p.v \int \phi(\omega) \frac{e^{-\gamma i \omega}}{i\omega} d\omega$$

To see that this is indeed the Fourier transform we are looking for, we try to inverse transform it and see if we get our original function. We need:

$$p.v \int_{-\infty}^{\infty} (\pi \delta(\omega) + \frac{e^{-\gamma i \omega}}{i\omega}) \mathcal{F}^{-1}(\phi)|_\omega d\omega = \int_{-\infty}^{\infty} \phi(y) 1_{y \geq \gamma}(y) \, dy$$

We use the fact that ϕ is classically transformable:

$$\begin{aligned}
&p.v\int_{-\infty}^{\infty}(\pi\delta(\omega)+\frac{e^{-\gamma i\omega}}{i\omega})\mathcal{F}^{-1}(\phi)|_{\omega}d\omega\\
&=p.v\int_{-\infty}^{\infty}(\pi\delta(\omega)+\frac{e^{-\gamma i\omega}}{i\omega})\int_{-\infty}^{\infty}\phi(y)\frac{e^{i\omega y}}{2\pi}dy\ d\omega\\
&=\int_{-\infty}^{\infty}p.v\int_{-\infty}^{\infty}(\pi\delta(\omega)+\frac{e^{-\gamma i\omega}}{i\omega})\phi(y)\frac{e^{i\omega y}}{2\pi}d\omega\ dy\\
&=\int_{-\infty}^{\infty}\phi(y)(\frac{1}{2}+p.v\int_{-\infty}^{\infty}\frac{e^{(y-\gamma)i\omega}}{2\pi i\omega}d\omega)\ dy\\
&=\int_{-\infty}^{\infty}\phi(y)1_{y\geq\gamma}(y)\ dy
\end{aligned}$$

since

$$p.v\int_{-\infty}^{\infty}\frac{e^{(x-\gamma)i\omega}}{2\pi i\omega}d\omega=\begin{cases}\frac{1}{2} & x\geq\gamma\\ -\frac{1}{2} & x<\gamma\end{cases}$$

This shows that $\pi\delta(\omega)+p.v\frac{e^{-\gamma i\omega}}{i\omega}$ is indeed the Fourier transform of $1_{y\geq\gamma}(y)$ in the sense described above, and we can write: [2]

$$1_{y\geq\gamma}(y)=p.v\int_{-\infty}^{\infty}\frac{e^{\omega yi}}{2\pi}\left(\pi\delta(\omega)+\frac{e^{-\gamma i\omega}}{i\omega}\right)d\omega$$

Thus if we are careful to remember that this is not a function in the normal sense, we might try to proceed as above to find a nice expression for the price. Let us substitute our process Q_T for y and calculate:

$$\begin{aligned}
F(t,T)=&E\left(p.v\int\frac{e^{iQ_T\omega}}{2\pi}\left(\delta(\omega)+\frac{e^{-\gamma i\omega}}{i\omega}\right)d\omega|\mathcal{F}_t\right) &(4.2.1)\\
=&\left(\frac{1}{2}+E\left(p.v\int\frac{e^{(Q_T-\gamma)i\omega}}{2\pi i\omega}d\omega|\mathcal{F}_t\right)\right) &(4.2.2)\\
=&\left(\frac{1}{2}+p.v\int\frac{e^{-\gamma i\omega}}{2\pi i\omega}E\left(e^{Q_Ti\omega}|\mathcal{F}_t\right)d\omega\right)\text{a.s} &(4.2.3)
\end{aligned}$$

where we have used lemma 4.1.2 in the last equation. Now as we have seen $E\left(e^{Q_Ti\omega}|\mathcal{F}_t\right)$ can be written $e^{i\omega Q_t}\Phi(\omega)_{Q_{T-t}}=e^{i\omega Q_t+(T-t)\psi_Q(\omega)}$. Thus we get the following expression:

[2] This is really the point, all we want is an integral expression to use with lemma 4.1.2.

Proposition 4.2.1. *Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t = 1, 2, \ldots, T\}, P)$ be a filtered probability space. Let $Q(\omega, t)$ denote the total emissions in the market at time t, and let γ be the total allowances allocated. Suppose that the forward allowance price $F(t, T)$ is given:*

$$F(t,T) = E(1_N|\mathcal{F}_t)$$

where $N = \{\omega : Q(\omega, T) \geq \gamma\}$ Then we have

$$F(t,T) = \left(\frac{1}{2} + p.v \int \frac{e^{(Q_t-\gamma)i\omega+(T-t)\psi_Q(\omega)}}{2\pi i\omega} d\omega\right) a.s$$

where ψ_Q is the characteristic exponent of Q.

This expression is somewhat simpler, and does not contain the "trick variable" α. We might hope that this makes it easier to find analytical solutions.

In the paper Carr et al. [7], an expression for the probability for a call option being 'in the money' is stated. This expression is very similar to the expression we have just found. This makes sense when we think about what it means for an option to be 'in the money', which is when the underlying is greater than some threshold. Therefore we might think of our forward price as the probability of an option on the total emission Q_T, with strike price γ to be in the money. This option would be in the money precisely when $Q_T \geq \gamma$.

In the paper [7] it is argued that this price expression does not allow the use of FFT since it is singular when $\omega = 0$. This is the reason why the other method is developed. Therefore we conclude that 4.2.1 does not allow the use of the FFT. It is still useful as a simpler and less clouded expression for attempts at analytical solutions and proofs. It may also be used with other numerical integration methods.

4.3 Market dynamics

In the derivation of the two price expressions above, one of the fundamental assumptions was that all the firms are risk neutral. In reality most firms are not risk neutral but rather risk averse. Thus we should not expect the above model to reflect observed data. Let $\bar{A}_t$ denote the market dynamics. Then the problem can be rephrased as

$$\bar{A}_t \neq E(1_N|\mathcal{F}_t)$$

A common solution to this problem is to find another probability measure $\bar{P}$, which is (somewhat confusingly) called the risk neutral pricing measure. It is a requirement of $\bar{P}$ that it satisfies[3]:

$$\bar{A}_t = E_{\bar{P}}(1_N|\mathcal{F}_t) \tag{4.3.1}$$

In this section we will derive the pricing measure and the market dynamics. We will begin by looking at the theory of pricing measures, and then go on to use the Esscher transform to find it.

4.3.1 Pricing measure

We want to give a brief overview of the idea behind pricing measures. In order to do that we need to discuss some of the common expressions in finance. We will not dive into technical details when it comes to defining these, but rather give an intuitive idea.

The first question that needs to be answered about the pricing measure is whether or not it exists. The second is where to start looking. Luckily for us, both of these questions are answered by the fundamental theorem of asset pricing [9]. Assuming an arbitrage free market, the fundamental theorem of asset pricing tells us that there must exist a measure $\bar{P}$ satisfying requirement 4.3.1. Furthermore it states that this measure is equivalent to the objective measure P, and that all tradable assets must be martingales with respect to $\bar{P}$. There are some new expressions here, we will give a short explanation:

That the market is arbitrage free, means that there is no way of investing so that there is zero chance of loss and a positive chance of gain. That is, there are no risk free profit opportunities. In practice, arbitrages occur, but they will disappear quickly once they are discovered. Therefore this assumption is generally considered to be quite realistic.

When we talk about the market we mean the collection all the assets. It can be thought of as an n-dimensional stochastic process where n is the number of assets [22]. When we talk about assets we mean the underlying processes in the market; the sources of randomness if you will. In our model the only asset is Q_t. In a model of the stock market it could be all the individual stocks.

The fact that all tradable assets must be martingales with respect to $\bar{P}$ in order to avoid arbitrage is illustrated by example 4.3.1.

[3]Recall that all prices are in time T currency, so we do not need to think about discount factors.

Example 4.3.1. *Let S_t be the price of a tradable asset, say a bar of gold. Let $\Pi_t(S_T) = E_{\bar{P}}(S_T|\mathcal{F}_t)$. Suppose that $\Pi_t(S_T) < S_t$, for instance $\Pi_t(S_T) = 90\$$ and $S_t = 100\$$. Then, supposing we have 10 bars to start with, we can sell one bar of gold for 100\$, buy a contract guaranteeing 1 bar at time T for 90\$. Thus at time T we will have 10 bars and 10\$. Thus a guarantee of making 10\$ with no chance of loss, which is an arbitrage opportunity.*

Now if instead $\Pi_t(S_T) > S_t$, for instance $\Pi_t(S_T) = 100\$$ and $S_t = 90\$$. Then we buy a bar of gold and enter a contract to deliver a bar of gold at time T in return for 100\$ now. Thus at time T we will have made 10\$ with no risk, same as above, and again we have arbitrage.

So we know that a risk-neutral pricing exist and that it is equivalent with the objective measure P. This gives us an idea about where we should start looking.

The next question is: Are there more than one measure satisfying these conditions? The second fundamental theorem of asset pricing states that for complete markets there is only one suitable pricing measure, and thus the price is uniquely determined. Loosely speaking, a market is complete if for every derivative on the market one can construct a portfolio of tradable assets that replicates the behavior of that derivative. At any given time, the replicating portfolio will have the same value as the derivative a.s. This gives a way of pricing the derivative: By finding the smallest price of a replicating portfolio.

The assumption that the market is complete is generally considered more unrealistic than the absence of arbitrage. In our model this assumption can easily be shown to be false: In a complete market all assets must be tradable, and in our model the underlying process is the total emission, which is not tradable. Hence, as we model it, the emission market is incomplete.

The fact that the total emissions is not tradable means that even if $E_{\bar{P}}(Q_T|\mathbb{F}_t) \neq Q_t$, we don not get an arbitrage since we cannot buy or sell the underlying like we did in example 4.3.1. Therefore there is no martingale requirement for non-tradable assets, and as pointed out in [3], any equivalent measure will do[4] as a pricing measure.

How then are we to calculate the prices in our model? If any equivalent measure will do, how can we say that one price is better than another? One way is to use so called market price of risk. Suppose we find a way of changing P into an equivalent measure that is a function of $\theta \in \mathbb{R}$. Then we let $\bar{A}_t(\theta) = E_{\bar{P}(\theta)}(1_{Q_t>\gamma}|\mathcal{F}_t)$ model the actual observed forward prices in the market. We can then compute $\bar{A}_t(\theta) - A_t$. We can think about this as a

[4]It will do in terms of satisfying the requirements of the fundamental theorem of asset pricing

way of quantifying the price of risk in the market. The difference $\bar{A}_t(\theta) - A_t$ is called the risk premium and θ is called the market price of risk. From the difference in the observed data and our predicted price A_t we estimate θ. This estimate is called the implied market price of risk. The implied market price of risk is used to decide on a pricing measure which again is used to model $\bar{A}_t$. We can argue that $\bar{P}(\theta)$ is better than other equivalent measures since it has been calibrated to data.

So now we know that all we have to do is to perform a measure change in a way that the resulting measure is a function of some θ, and is equivalent to P. Before we go on, let us quickly review what this means mathematically.

4.3.2 Measure change

We have mentioned that $\bar{P}$ is equivalent to P. Mathematically we say that two measures μ and ν, defined on the same space $\mathcal{X}$ and the same sigma algebra $\mathcal{A}$, are equivalent, if for any $A \in \mathcal{A}$ we have $\mu(A) = 0 \Leftrightarrow \nu(A) = 0$.

Now recall that the Radon Nikodym theorem gives us that as long as μ and ν are σ-finite, we can find measurable positive functions $\frac{d\nu}{d\mu}$ and $\frac{d\mu}{d\nu}$ called the Radon-Nikodym derivatives so that

$$\mu(A) = \int_{\mathcal{X}} 1_{\mathcal{A}} \frac{d\mu}{d\nu} d\nu$$

and

$$\nu(A) = \int_{\mathcal{X}} 1_{\mathcal{A}} \frac{d\nu}{d\mu} d\mu$$

for any $A \in \mathcal{A}$. In our case with $\mathcal{X} = \Omega$ and $P(\Omega) = 1 = \bar{P}(\Omega)$ so we trivially have σ-finiteness. Hence we must have a stochastic variable $\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) > 0$ such that:

$$\bar{P}(A) = \int_A \frac{d\bar{P}}{dP}(\omega) dP$$

Notice that since $\bar{P} = 1$ we have:

$$\bar{P}(\Omega) = \int_\Omega \frac{d\bar{P}}{dP}(\omega) dP = 1$$

which is to say: $E_P(\frac{d\bar{P}}{dP}) = 1$. Therefore, finding an equivalent measure is to find a suitable Radon-Nikodym derivative.

4.3.3 Esscher transform

We have already mentioned that we will use the Esscher transform to make the measure change. This is mostly for practical reasons; under the Esscher

transform the underlying Q_t will remain a Levy process with regards to the new measure $\bar{P}$. The Esscher transform works as follows: Let Q_t be a Levy process with characteristic triplet (A, b, ν) such that $\int_{|x|\geq 1} e^{\theta x}\nu(dx) < \infty$. Then the Esscher transform X_t of Q_t is a new Levy process with characteristic triplet $(0, \bar{b}, \bar{\nu})$ such that :

$$\bar{b} = b + \int_{|x|<1} x(e^{\theta x} - 1)\nu(dx) \tag{4.3.2}$$

$$\bar{\nu}(dx) = e^{\theta x}\nu(dx) \tag{4.3.3}$$

$$\tag{4.3.4}$$

where $\theta \in \mathbb{R}$. The new process $X_t(\theta)$ defines a new measure $\bar{P}(\theta)$ on Ω such that Q_t is still a Levy process with respect to $\bar{P}(\theta)$ [9]. Furthermore the Radon- Nikodym derivative of $\bar{P}$ with respect to P is given as follows [9]:

$$\frac{d\bar{P}(\theta)}{dP}\Big|_{\mathcal{F}_t} = \frac{e^{\theta Q_t}}{E(e^{\theta Q_t})}$$

So in light of section 4.3.2, we know that:

$$\bar{P}[\theta]|_{\mathcal{F}_t}(A) = \int_A \frac{e^{\theta Q_t}}{E(e^{\theta Q_t})}(\omega)dP$$

This is what we need. We can now express:

$$\bar{A}_t(\theta) = E_{\bar{P}(\theta)}(1_{Q_T>\gamma}|\mathcal{F}_t) = E(\frac{e^{\theta Q_t}}{E(e^{\theta Q_t})}1_{Q_T>\gamma}|\mathcal{F}_t)$$

In order to calculate $\bar{A}_t(\theta, T)$ we can use the Fourier transform technique we used above. For simpler notation we will suppress the dependence on θ in $\bar{A}_t$ and $\bar{P}$. With this in mind we compute:

$$\bar{A}_t = E_{\bar{P}}\left(\frac{e^{\alpha(Q_T-\gamma)}}{2\pi}\int_{\mathbb{R}}\frac{e^{(Q_T-\gamma)iy}}{(\alpha+iy)}dy|\mathcal{F}_t\right) \tag{4.3.5}$$

$$= \int_{\mathbb{R}} E_{\bar{P}}\left(\frac{e^{\alpha(Q_T-\gamma)}}{2\pi}\frac{e^{(Q_T-\gamma)iy}}{(\alpha+iy)}|\mathcal{F}_t\right)dy \tag{4.3.6}$$

$$= \int_{\mathbb{R}} \frac{e^{-\gamma(\alpha+iy)}}{2\pi(\alpha+iy)}E_{\bar{P}}\left(e^{(\alpha+iy)Q_T}|\mathcal{F}_t\right)dy \tag{4.3.7}$$

Since Q_t is a levy process with respect to $\bar{P}$ we can compute:

$$E_{\bar{P}}\left(e^{(\alpha+iy)Q_T}|\mathcal{F}_t\right) = E_{\bar{P}}\left(e^{(\alpha+iy)(Q_T-Q_t)}|\mathcal{F}_t\right)E_{\bar{P}}\left(e^{(\alpha+iy)Q_t}|\mathcal{F}_t\right) \tag{4.3.8}$$

$$= E_{\bar{P}}\left(e^{(\alpha+iy)(Q_T-Q_t)}\right)e^{(\alpha+iy)Q_t} \tag{4.3.9}$$

$$= \frac{E\left(e^{(\alpha+\theta+iy)Q_{T-t}}\right)}{E(e^{\theta Q_{T-t}})}e^{(\alpha+iy)Q_t} \tag{4.3.10}$$

$$\tag{4.3.11}$$

We can express this in terms of the characteristic function of Q. We have

$$E(e^{(\alpha+\theta+iy)Q_{T-t}}) = E(e^{i(y-i(\alpha+\theta))Q_{T-t}}) = e^{(T-t)\psi_Q(y-i(\alpha+\theta))}$$

and

$$E(e^{\theta Q_{T-t}}) = e^{(T-t)\psi_Q(-i\theta)}$$

The final calculations give:

$$\bar{A}_t(T) = \int_{\mathbb{R}} \frac{e^{-\gamma(\alpha+iy)}}{2\pi(\alpha+iy)} \frac{e^{(T-t)\psi_Q(y-i(\alpha+\theta))}}{e^{(T-t)\psi_Q(-i\theta)}} e^{(\alpha+iy)(Q_t)} dy \tag{4.3.12}$$

$$= \int_{\mathbb{R}} \frac{e^{-\gamma(\alpha+iy)}}{2\pi(\alpha+iy)} e^{(T-t)(\psi_Q(y-i(\alpha+\theta))-\psi_Q(-i\theta))} e^{(\alpha+iy)(Q_t)} dy \tag{4.3.13}$$

$$= \frac{e^{\alpha(Q_t-\gamma)}}{2\pi} \int_{\mathbb{R}} \frac{e^{iy(Q_t-\gamma)}}{(\alpha+iy)} e^{(T-t)(\psi_Q(y-i(\alpha+\theta))-\psi_Q(-i\theta))} dy \tag{4.3.14}$$

We summarize this to a result:

Proposition 4.3.2. *Under the assumptions of proposition 4.1.3 the market dynamics $\bar{A}_t$ is given:*

$$\bar{A}_t = \frac{e^{\alpha(Q_t-\gamma)}}{2\pi} \int_{\mathbb{R}} \frac{e^{iy(Q_t-\gamma)}}{(\alpha+iy)} e^{(T-t)(\psi_Q(y-i(\alpha+\theta))-\psi_Q(-i\theta))} dy \tag{4.3.15}$$

where θ is the market price of risk.

We can easily use the same method for the second price expression 4.2.1, then we get:

$$\bar{A}_t(T) \quad = (\tfrac{1}{2} + p.v \int \tfrac{e^{-\gamma i\omega}}{2\pi i\omega} E_{\bar{\mathbb{P}}} \left(e^{Q_T i\omega} | \mathcal{F}_t\right) d\omega \tag{4.3.16}$$

Doing similar computations as above we end up with the following proposition:

Proposition 4.3.3. *Under the assumptions of proposition 4.2.1 the market dynamics $\bar{A}_t$ is given:*

$$\bar{A}_t = \frac{1}{2} + p.v \int \frac{e^{(Q_t-\gamma)i\omega+(T-t)(\psi_Q(\omega-i\theta)-\psi_Q(-i\theta))}}{2\pi i\omega} d\omega \tag{4.3.17}$$

where θ is the market price of risk.

Remark 4.3.4. *We can add the ψ_Q's in 4.3.15 and in 4.3.17. This is possible since they have the same levy measure. If we try to do so for 4.3.17, assuming Q_t is a subordinator, we see that*

$$\psi_Q(\omega - i\theta) - \psi_Q(-i\theta)) \tag{4.3.18}$$

$$= bi(\omega - i\theta + i\theta) + \int_0^\infty (e^{ix(\omega - i\theta)} - 1) - (e^{ix(-i\theta)} - 1)\nu(dx) \tag{4.3.19}$$

$$= i\omega + \int_0^\infty e^{ix(\omega - i\theta)} - e^{x\theta}\nu(dx) \tag{4.3.20}$$

$$= i\omega + \int_0^\infty e^{x\theta}(e^{ix\omega} - 1)\nu(dx) \tag{4.3.21}$$

$$\tag{4.3.22}$$

which is just $\psi^E(\omega)$, the characteristic exponent with Esscher transformed Levy measure, which makes sense. So in this case we could have arrived at $\bar{A}_t$ using the Esscher transformed Levy measure instead of the Radon-Nikodym derivative.

4.4 Findings

In this chapter we have arrived at relatively explicit price expressions for general Levy processes in 4.1.3 and 4.2.1. We have also derived expressions for the market dynamics in 4.3.3 and 4.3.2. We see that our price expressions provide great flexibility since they hold for general Levy processes. It is reasonable to believe that for some Levy processes, it should be possible to use expression 4.3.3 or 4.3.2, and achieve a good fit to the empirical market data.

In the article by Grüll et al. they were able to use Ito's formula to get an expression for the dynamics dS_t^L of the price expression [13]. This is used to model the sensitivity of the price to the underlying. We have not discussed it in this thesis, but there is a version of Ito's formula for general Levy processes [9]. It may be possible to use this to derive an expression for the sensitivity corresponding to the expression in [13]. If the resulting dynamics is not tractable, one could possibly estimate it instead. This would be an interesting task for future work.

Chapter 5

Examples

In this chapter we will look at some examples where we model Q_t as some specific Levy process. For all of the examples we will include attempts at analytical solutions. Even if the attempts are not successful, they are included to show what approaches have been tried, and why they have failed. In the last two examples we have also made numerical approximations.

The examples considered are the compound Poisson, the inverse Gaussian and the Meixner process.

5.1 Compound Poisson

One of the simplest Levy-processes is the compound Poisson (C.P) process. The C.P process is also a fundamental part of many Levy processes, and all discretized paths of jump processes can be seen as a realization of a C.P process. It us unlikely that the C.P process is a good way to model the total emissions, but it is included because the properties just mentioned.

The C.P process is a process with exponentially distributed jump times, and jump sizes following some probability distribution. The compound Poisson can be thought of as the sum of a Poisson distributed number of independent identically distributed (i.i.d) stochastic variables. More precisely we can define it as follows:

Definition 5.1.1. *A Compound Poisson process X_t with jump intensity λ and jump size distribution f is a Levy processes such that*

$$X_t = \sum_{i}^{N_t} Y_i$$

Where Y_i are i.i.d stochastic variables with probability distribution f and N_t is

a Poisson distributed stochastic variable with parameter λ independent from Y_i for all i.

The characteristic function of a Compound Poisson process is connected with its jump size distribution in a very nice and simple way [9].

Theorem 5.1.2. *Let X_t be a Compound Poisson process with jump intensity λ and jump size distribution f. Then the characteristic function $\Phi_{X_t}(z)$ is given as follows*

$$\Phi_{X_t}(z) = e^{\lambda t(\hat{f}(z)-1)} = e^{\lambda t \int_{\mathbb{R}}(e^{izx}-1)f(dx)}$$

where $\hat{f}$ denotes the characteristic function of the i.i.d Y_i with distribution f.

This property of the Compound Poisson makes it easy to plug it into our model. All we have to do is find a jump size distribution with tractable characteristic function. The exponential distribution seems like the best choice.

5.1.1 Computation

We will model the total emissions Q_t with a compound Poisson process with intensity λ and exponentially distributed jumps. So $X_t = \sum_i^{N_t} Y_i$ where Y_i are exponentially distributed with intensity $\beta > 0$. Then

$$\Phi_{Y_i}(z) = \frac{\beta}{\beta - iz}$$

So by theorem 5.1.2:

$$\Phi_{Q_t}(z) = e^{\lambda t\left(\frac{\beta}{\beta-iz}-1\right)} = e^{\lambda t\left(\frac{iz}{\beta-iz}\right)}$$

For easier notation we introduce two new variables: let $h = T - t$ and $q = Q_t - \gamma$. If we put this in to price expression 4.2.1 we get that the price is given:

$$A_t = \frac{1}{2} + \text{p.v} \int_{\mathbb{R}} \frac{e^{qiy}}{2\pi iy} e^{\lambda h\left(\frac{iy}{\beta-iy}\right)} dy \tag{5.1.1}$$

Let us see what we can infer about the price from this expression. In the special case when $h = 0$ the exponent is only qiy and the integral can be easily computed using residue theory. Recall that $h = 0$ means that the trading period is over and that the price must be either 1 or 0. By computing the residues we confirm that for $q < 0$ $A_t = 0$ and for $q > 0$ $A_t = 1$. For all

other cases we need to do some more work. It seems unlikely to find an anti derivative so instead we will try a contour integral approach.

We see that the integrand has 2 singularities $s_1 = 0$ and $s_2 = -i\beta$. Notice that s_1 is a simple pole while s_2 is an essential singularity. Now let C_R be the circle in $\mathbb{C}$ with radius R and parameterization $y = Re^{i\theta}$ for $\theta \in [0, 2\pi]$. If we look at $\frac{e^{\lambda h\left(\frac{iy}{\beta - iy}\right)}}{iy}$ we see that:

$$\lim_{R\to\infty} \left| \frac{e^{\lambda h\left(\frac{iRe^{i\theta}}{\beta - iRe^{i\theta}}\right)}}{iRe^{i\theta}} \right| = 0 \tag{5.1.2}$$

for all $\theta \in [0, 2\pi]$. Thus we can use Jordan's lemma to compute 5.1.1. Using this, and that there are no singularities in the upper half plane we compute the integral for $q > 0$ and find $A_t = 1$, which is reassuring. In order to find A_t when $q < 0$, if we want to make use of Jordan's lemma, we compute the contour integral in the lower half plane. Thus we must find the residue at s_2, unfortunately this is easier said than done. What follows is an unsuccessful attempt that may shed some light on the difficulties.

Since s_2 is an essential singularity, the best bet is to find the Laurent series of the integrand around s_2. To do this, we rewrite the integrand and make the substitution $x = y + i\beta$:

$$\frac{e^{qiy+\lambda h\left(\frac{iy}{\beta - iy}\right)}}{iy} = \frac{-ie^{q\beta - \lambda h + iqx + \left(\frac{i\lambda h\beta}{x}\right)}}{x - i\beta} \tag{5.1.3}$$

For simpler notation we let $A = q\beta - \lambda h, B = iq, C = i\lambda h\beta$. Then the integrand can be rewritten to the infinite triple sum:

$$\frac{-ie^{A+Bx+Cx^{-1}}}{x - i\beta} = \frac{e^A}{\beta} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^n x^k x^{-j} B^k C^j}{(i\beta)^n k! j!} \tag{5.1.4}$$

Since we are looking for the residue, we only need the coefficient of x^{-1}, so we set $j = n + k + 1$ and find that the residue is given by:

$$\frac{e^A}{\beta} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{B^k C^{(n+k+1)}}{(i\beta)^n k!(n + k + 1)!} \tag{5.1.5}$$

$$= \frac{e^A}{\beta} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{B^{(n-k)} C^{(n+1)}}{(i\beta)^k (n - k)!(n + 1!)} \tag{5.1.6}$$

$$= \frac{e^A}{\beta} \sum_{n=0}^{\infty} \frac{C^{(n+1)} \Gamma(n + 1, i\beta B)}{(n + 1)! n! (i\beta)^n} \tag{5.1.7}$$

Which again can be written as an integral. It can be seen that the sum 5.1.7 converges since:

$$\left|\frac{B^{(n-k)}C^{(n+1)}}{(i\beta)^k(n-k)!(n+1!)}\right| \leq \left|\frac{B^{(n-k)}C^{(n+1)}}{(i\beta)^k(n+1!)}\right| \tag{5.1.8}$$

and

$$\frac{e^A}{\beta}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{B^{(n-k)}C^{(n+1)}}{(i\beta)^k(n+1!)} = \frac{be^{-\frac{iC}{b}}\left(-1+e^{\frac{i}{b+B}C}\right)}{i+bB} \tag{5.1.9}$$

for $bB \neq i$, but it is hard to find an explicit expression. It seems like numerical approximation is a better approach, but then other more complex models are more interesting.

5.2 Inverse Gaussian

In this section we will be using some well known facts about the inverse Gaussian process as well as some complex analysis. This material is taken from [9] and [20] respectively.

The inverse Gaussian (I.G) process is an example of a subordinator. The I.G has Levy-measure:

$$\nu(dx) = \frac{ce^{-\lambda x}}{x^{3/2}}1_{x>0}dx$$

where changing c will change the intensity of all jumps, and changing λ will change the decay rate of big jumps. In order to use our formula, we need to know the characteristic function of the I.G. It can be found from its Laplace transform:

$$E(e^{uQ_t}) = e^{-2ct\sqrt{\pi}(\sqrt{\lambda-u}-\sqrt{\lambda})}$$

We substitute $u = zi$ and find that the characteristic function for Q_t is given:

$$\Phi_{Q_t}(z) = e^{t\psi}, \psi = -2c\sqrt{\pi}(\sqrt{\lambda - z} - \sqrt{\lambda})$$

Now, plugging this in to the price formula 4.1.3 we get:

$$A_t = \frac{e^{\alpha(Q_t-\gamma)}}{2\pi}\int_{\mathbb{R}}\frac{e^{(Q_t-\gamma)iy}}{(\alpha+iy)}e^{(T-t)(-2c\sqrt{\pi}(\sqrt{\lambda-\alpha-iy}-\sqrt{\lambda}))}dy \tag{5.2.1}$$

5.2.1 Attempt at analytical solution

In order to make sure that we have a well defined function, we take $\sqrt{z}$ to mean the $e^{\log|z|+i\arg(z)}$. Here $\arg(z)$ is taken to mean the principal value

argument with branch cut along the negative real axis. This works since for y and $\beta \in \mathbb{R}, \beta - yi$ is never a negative real number. In fact, in order for $\beta - yi$ to be on the negative real axis, we need:

$$y \in \{z | z = ai, a \in \mathbb{R} \wedge a < -\beta\} \tag{5.2.2}$$

where $\beta = \lambda - \alpha$. Now let us try to evaluate the integral. First observe that the integrand has a singularity of order 1 at $y = \alpha i$. Second, observe that our integrand has a branch cut in the set described in 5.2.2. This is the along the negative imaginary axis from $-\beta$ to $-\infty$.

Now observe that

$$\lim_{|y|\to\infty} e^{Aiy - B\sqrt{C - iy}} = \lim_{|y|\to\infty} e^{Aiy} \tag{5.2.3}$$

so for $(Q_t - \gamma) > 0$ and $\mathrm{Im}(y) > 0$ the integrand will vanish as $|y| \to \infty$. Therefore we have:

$$\lim_{\rho\to\infty} \int_{C_\rho^+} \frac{e^{(Q_t-\gamma)iy}}{(\alpha + iy)} e^{(T-t)(-2c\sqrt{\pi}(\sqrt{\lambda - \alpha - iy} - \sqrt{\lambda}))} dy = 0 \tag{5.2.4}$$

for $(Q_t - \gamma) > 0$. Here C_ρ is the half circle in the upper half plane with radius ρ. So for $(Q_t - \gamma) > 0$ we solve the integral using residue theory. This gives us $A_t = 1$, which is as it should be. Also if $T - t = 0$ the branch cut in the lower half plane disappears. So when $(Q_t - \gamma) < 0$ we have $A_t = 0$ since there are no singularities in the lower half plane. In all the other cases, it seems a little harder to solve the integral analytically.

It seems unlikely to find an anti derivative so the contour integration approach seems like the best bet. But the branch cut gives us some trouble. One could propose to integrate along negative semicircle and then up and down on each side along the branch cut. This will not work since for $\alpha > \lambda$ we get $-\beta > 0$ causing the branch cut will go through the origin and dividing the proposed contour into two parts. Then the contour that follows the branch cut will then double over itself, and it will no longer be simple closed.

Let us see what happens if we divide the integral into two parts, the positive and negative line. Then we can create two contours, integrate them, and add them together. Figure 5.1 illustrates the contours. In order to do this let

$$\kappa^+ = \{\lim_{\epsilon\to 0}(xi + \epsilon) | x \in \mathbb{R} \wedge x < -\beta, \epsilon \in \mathbb{R}^+\}$$

and

$$\kappa^- = \{\lim_{\epsilon\to 0}(xi - \epsilon) | x \in \mathbb{R} \wedge x < -\beta, \epsilon \in \mathbb{R}^+\}$$

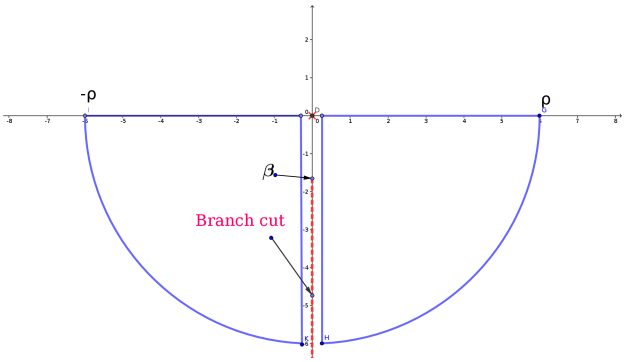

Figure 5.1: The contour is split in the middle.

The two sets κ^+ and κ^- is the branch cut approached from the two different sides. Let us look at how the integrand behaves on these sets, for convenience we will denote the integrand by f. We see that f is the same on both sets except for the square root part, this is because:

$$\forall y \in \kappa^+, \exists x \in \mathbb{R}, x < -\beta \text{ st:} \tag{5.2.5}$$

$$\sqrt{\beta - iy} = \lim_{\epsilon \to 0} \sqrt{|\beta - i(xi + \epsilon)|}\ e^{i\frac{\arg(\beta - i(xi+\epsilon))}{2}} \tag{5.2.6}$$

$$= -i\sqrt{|\beta + x|} \tag{5.2.7}$$

and

$$\forall y \in \kappa^-, \exists x \in \mathbb{R}, x < -\beta \text{ st.} \tag{5.2.8}$$

$$\sqrt{\beta - iy} = \lim_{\epsilon \to 0} \sqrt{|\beta - i(xi - \epsilon)|}\ e^{i\frac{\arg(\beta - i(xi-\epsilon))}{2}} \tag{5.2.9}$$

$$= i\sqrt{|\beta + x|} \tag{5.2.10}$$

So we see that f is different on each side of the branch cut, as should be expected. Let us denote the integrand f by f_1 when integrating along κ^+ and f_2 when integrating along κ^-. Now let C_ρ^- denote the semicircle in the negative half of $\mathbb{C}$ with radius ρ. Recall that for $Q_t - \gamma < 0$ the contour integral along C_ρ^- will vanish as $\rho \to \infty$, except at the branch cut. Therefore, all that is needed to solve the integral in 5.2.1 is to find:

$$\int_{-\rho i}^{-\beta i} f_2(y)dy - \int_{-\rho i}^{-\beta i} f_1(y)dy$$

now for both f_1 and f_2 we can do the following substitution. Set $z = yi$, then

$$\int_{-\rho i}^{-\beta i} f_x(y)dy = \int_{\beta}^{\rho} i f_x(z)dz, \text{ where } z = yi$$

For convenience we set $q = (Q_t - \gamma)$ and $K = 2hc\sqrt{\pi}$. So what we want to find can be expressed as follows:

$$\lim_{\rho\to\infty}\int_\beta^\rho if_2(z)dz - \int_\beta^\rho if_1(z)dz, \text{ where } z = yi \tag{5.2.11}$$

$$= \lim_{\rho\to\infty}\int_\beta^\rho i\frac{e^{\overbrace{qz + K\sqrt{\lambda}}^{a} - i\overbrace{K\sqrt{|\beta - z|}}^{\theta}} - e^{\overbrace{qz + K\sqrt{\lambda}}^{a} + i\overbrace{K\sqrt{|\beta - z|}}^{\theta}}}{\alpha + z}dz \tag{5.2.12}$$

$$= \lim_{\rho\to\infty}\int_\beta^\rho i\frac{e^{a-i\theta} - e^{a+i\theta}}{\alpha + z}dz \tag{5.2.13}$$

$$= \lim_{\rho\to\infty}\int_\beta^\rho i\frac{e^a(\cos(-\theta) + i\sin(-\theta) - \cos(\theta) - i\sin(\theta))}{\alpha + z}dz \tag{5.2.14}$$

$$= \lim_{\rho\to\infty}\int_\beta^\rho 2\frac{e^a\sin(\theta)}{\alpha + z}dz \tag{5.2.15}$$

$$= \lim_{\rho\to\infty}\int_\beta^\rho 2\frac{e^{qz+K\sqrt{\lambda}}\sin(K\sqrt{|\beta - z|})}{\alpha + z}dz \tag{5.2.16}$$

$$= \lim_{\rho\to\infty}\int_0^\rho 2\frac{e^{q(z+\lambda-\alpha)+2hc\sqrt{\pi\lambda}}\sin(2hc\sqrt{\pi z})}{\lambda + z}dz \tag{5.2.17}$$

where we have substituted back for β in the final equality. If we multiply with $\frac{e^{\alpha q}}{2\pi}$ we get the following expression for the price:

$$A_t = \frac{e^{q\lambda+2(T-t)c\sqrt{\pi\lambda}}}{\pi}\int_0^\infty \frac{e^{(Q_t-\gamma)z}\sin(2(T-t)c\sqrt{\pi z})}{\lambda + z}dz \tag{5.2.18}$$

This seems to be as long as this approach takes us. It might well be possible to solve 5.2.1 analytically but some original approach is needed.

In the next section we will try to simulate the price process numerically.

5.2.2 Simulation

In this section we will simulate the price processes when the underlying is modeled as an inverse Gaussian process. We have chosen to use a straight forward numerical integration approach. It is appropriate to mention how an inverse FFT approach would work.

The inverse FFT is a fast way of computing the sum $\sum_{j=0}^N e^{\frac{i2\pi(j-1)(k-1)}{N}}x(j)$ for $k = 1, 2...N$ [7]. The sum can be used to approximate the integral in expression 4.1.3 in analogy to the method described in the paper [7] by Carr

et al. Then using the inverse FFT would give $A_t(k)$ for N equally spaced values of k. This could be thought of as a discretization of the function $A_t(x)$, where $x \in [0, L]$.The upper limit L would have to be chosen with consideration to Q_T.

In order to approximate $A_t(Q_t)$ for some realization of Q_t, one could use linear interpolation on the two closest k. This method might work very well, but may need some more analysis with regards to efficient implementation, and also with regards to the accuracy of the approximation.

For our purposes the straight forward numerical integration approach will be fast enough. The simulations are done in Python using the numerical integration routines in the numpy and scipy packages. These routines provide estimates of the error due to numerical integration. They also warn when these estimates seem to be inaccurate.

Simulation of the inverse Gaussian

Recall that since Q_t is a levy process we have

$$Q_{t_n} = Q_{t_n} - Q_{t_{n-1}} + Q_{t_{n-1}} - Q_{t_{n-2}} + Q_{t_{n-2}} - + \cdots - + Q_{t_0} \tag{5.2.19}$$

$$\sim Q_{t_n - t_{n-1}} + Q_{t_{n-1} - t_{n-2}} + \cdots + Q_{t_1 - t_0} + Q_{t_0} \tag{5.2.20}$$

That is, the sum $Q_{t_n - t_{n-1}} + Q_{t_{n-1} - t_{n-2}} + \cdots + Q_{t_1 - t_0} + Q_{t_0}$ have the same distribution as Q_{t_n}. Therefore we shall simulate Q_t by partitioning t in to n parts, simulate every Q_{dt_n} using the marginal probability distribution

$$\frac{c \cdot dt_n}{x^3/2} e^{2c \cdot dt_n \sqrt{\pi\lambda} - \lambda x - \pi c^2 \cdot dt_n^2 / x}$$

and then add them all together.

Numerical challenges

There are some problems with numerically integrating the integral 5.2.1. Therefore we will use the expression 5.2.18 that we ended up with after our attempt at an analytical solution, which behaves better numerically. There are still some challenges though; when the product $2(T-t)c\sqrt{\pi}$ is large the sine function oscillates rapidly. Too rapid oscillations causes problems with the numerical integration. Therefore we will use the scaling property of stable processes described in [9]. We have that $Q_t(\lambda, c) \sim R^2 \cdot Q_{t/R}(R^2\lambda, c)$. This gives a more flexible price formula where R can be adjusted to minimize numerical errors.

$$F(t,T) = \frac{e^{(R^2 Q_{t/R} - \gamma)\lambda R^2 + 2(T-t)c\sqrt{\pi\lambda}}}{\pi} \int_0^\infty \frac{e^{(R^2 Q_{t/R} - \gamma)z} \sin(2\frac{(T-t)}{R} c\sqrt{\pi z})}{R^2\lambda + z} dz \tag{5.2.21}$$

Another problem that arises when integrating numerically is that when γ is big, $e^{(R^2Q_{t/R}-\gamma)}$is rounded off to zero. To try to mitigate this we can make the substitution $u = e^z$ in the integral. This gives the following price formula:

$$F(t,T) = \frac{e^{(R^2Q_{t/R}-\gamma)\lambda R^2+2(T-t)c\sqrt{\pi\lambda}}}{\pi}\int_1^\infty \frac{u^{(R^2Q_{t/R}-\gamma)}\sin[2\frac{(T-t)}{R}c\sqrt{\pi\ln(u)}]}{R^2\lambda u+\ln(u)u}du \tag{5.2.22}$$

The integrand in this formula is less likely to be rounded off to zero, but the integral will likely converge slower.

Still there are challenges with certain combinations of parameters. If we let T grow we must increase R to maintain small error, but this will result in the integrand being rounded off to zero. It also results in the product $(T-t)\cdot c$ easily getting so big that $\frac{e^{(R^2Q_{t/R}-\gamma)\lambda R^2+2(T-t)c\sqrt{\pi\lambda}}}{\pi}$ is rounded off to inf.

Simulations

In figure 5.2 we can see the total emission and the price below it. In the price plot, the green plot shows the result from using the polynomial integrand, and the blue plot is from the exponential integrand. In this case, they seem to agree very well. We see that jumps in the total emission leads to jumps in the price. We also notice that, in accordance with proposition 3.3.4, since there are a lack of downwards jumps in the underlying, the price never jumps downwards. In some cases, not in this particular simulation, the downwards movement is so steep that it might be argued that it is equivalent to a jump, even though it is in fact continuous.

In the simulations we have used the quad function from the scipy library to do the numerical integration. The quad function provides an estimated upper bound for the error in the numerical integration. In figure 5.3 we can see this error multiplied by the constant $\frac{e^{(R^2Q_{t/R}-\gamma)\lambda R^2+2(T-t)c\sqrt{\pi\lambda}}}{\pi}$, which will be an estimation of the upper bound of the error in the price, due to the error in the numerical integration.

In figure 5.4 the total emission hits the total allowance, and the price becomes 1, as it should. We also notice that the exponential and the polynomial integrands disagree quite much as the price grows closer to 1. This is the numerical vulnerability of the polynomial integrand since its integral converges much slower. We can see that the error is huge by looking at figure 5.5. Strangely enough the error in 5.5 is not big enough to account for the difference between the prices, but at this point in the simulation python warned about round-off errors, and that it was not able to accurately calculate the error.

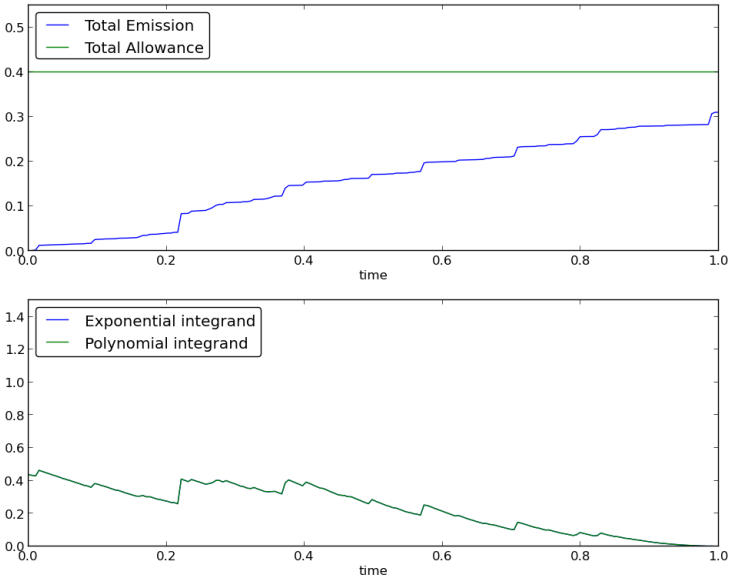

Figure 5.2: Above: Total emission, Below: Price. The parameters used were $T = 1, R = 1.\gamma = 0.4, c = 1, \lambda = 20$.

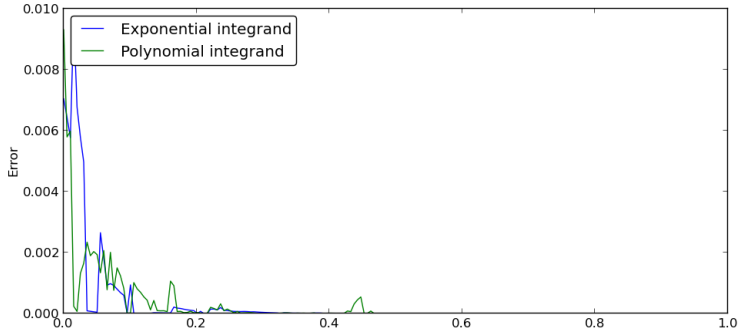

Figure 5.3: Estimated maximal error, due to numerical integration, of the price processes in figure 5.2.

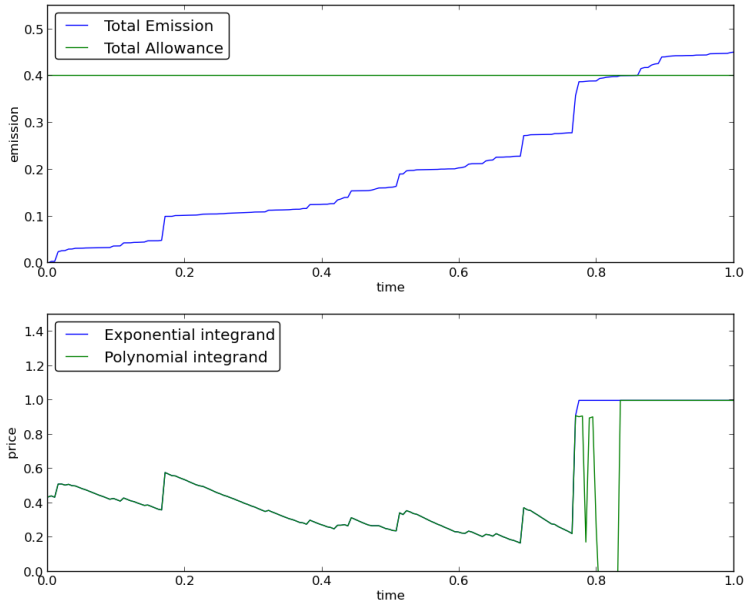

Figure 5.4: Above: Total emission, Below: Price. The parameters used were $T = 1, R = 1.\gamma = 0.4, c = 1, \lambda = 20$.

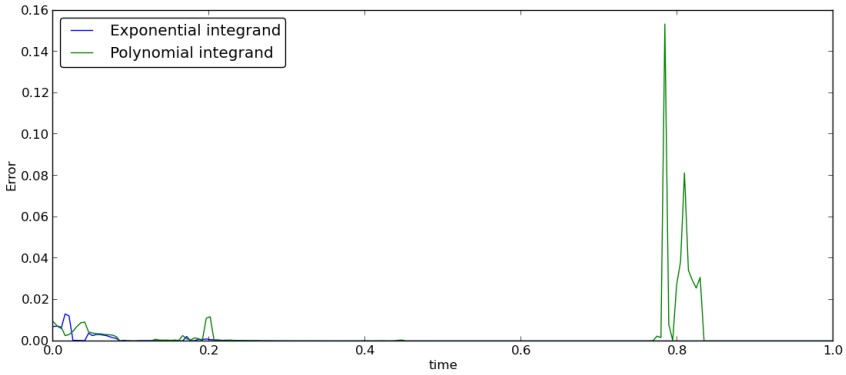

Figure 5.5: Estimated maximal error, due to numerical integration, of the price processes in figure 5.4.

5.3 Meixner

In this section there is some general information about Meixner processes which is given without proof, this information is from the report by Schoutens [21].

The Meixner distribution have enjoyed growing popularity in financial applications due to its flexibility and tractability. It as been shown to model the log returns of financial assets much better than for instance Normal distribution. It has also been shown that models based on the Meixner distribution shows great improvement over models where the underlying is Brownian motion.

Because of its flexibility and tractability we will try to use the Meixner process in our expression 4.2.1. The Meixner distribution is an infinitely divisible distribution with probability distribution function:

$$p(x;a,b,m,d) = \frac{2\cos^{2d}(b/2)}{2a\pi\Gamma(2d)} e^{\frac{b(x-m)}{a}} \left|\Gamma\left(d+i\frac{x-m}{a}\right)\right|^2 \tag{5.3.1}$$

where $a > 0, -\pi < b < \pi, m \in \mathbb{R}$ and $d > 0$. Since it is infinitely divisible, the Meixner distribution can be used to define a stochastic process called the Meixner process. The Meixner process is like the Inverse Gaussian process, a pure jump process, but contrary to the I.G it is not a subordinator. Now let Q_t be a Meixner process, then its characteristic function is given:

$$\Phi(z)_{Q_t} = \left(\frac{\cos(\frac{b}{2})}{\cosh(\frac{az-ib}{2})}\right)^{2dt} e^{imtz} \tag{5.3.2}$$

By theorem 4.2.1 we can express A_t as follows:

$$A_t = \frac{1}{2} + \text{p.v}\int_{-\infty}^{\infty} \frac{-ie^{(q+mh)i\omega}\cos^{2dh}(\frac{b}{2})}{2\pi\omega\cosh^{2dh}(\frac{a\omega-ib}{2})} d\omega \tag{5.3.3}$$

Because of the following proposition we will only need to consider the expression 5.3.3.

Proposition 5.3.1. *Let the total emission Q_t be modeled by a Meixner process $Q_t(a,b,m,d)$, then the risk neutral price $\bar{A}_t(a,b,m,d)$ is just:*

$$A_t(a,a\theta+b,m,d) = \frac{1}{2} + p.v\int_{-\infty}^{\infty} \frac{-ie^{qi\omega}}{2\pi i\omega}\left(\frac{\cos(\frac{(\theta a+b)}{2})}{\cosh(\frac{a\omega-i(\theta+b)}{2})}\right)^{2dh} d\omega$$

Proof. Using proposition 4.3.2 we see that

$$\bar{A}_t(a,b,m,d) = \frac{1}{2} + \text{p.v}\int_{-\infty}^{\infty} \frac{-ie^{qi\omega}\Phi(\omega - i\theta)_{Q_h}}{2\pi i\omega \Phi(-i\theta)_{Q_h}} d\omega \tag{5.3.4}$$

$$= \frac{1}{2} + \text{p.v}\int_{-\infty}^{\infty} \frac{-ie^{qi\omega}}{2\pi i\omega} \left(\frac{\cosh(\frac{-i(\theta a+b)}{2})}{\cosh(\frac{a\omega - i(\theta+b)}{2})} \right)^{2dh} d\omega \tag{5.3.5}$$

$$= \frac{1}{2} + \text{p.v}\int_{-\infty}^{\infty} \frac{-ie^{qi\omega}}{2\pi i\omega} \left(\frac{\cos(\frac{(\theta a+b)}{2})}{\cosh(\frac{a\omega - i(\theta+b)}{2})} \right)^{2dh} d\omega \tag{5.3.6}$$

□

We are now ready to investigate how the price acts when the underlying is a Meixner process. We will first look at the expression analytically, and then we will look at simulations.

5.3.1 Attempt at analytical solution

Let's see what we can say analytically about the price A_t when the underlying is a Meixner process. We will try to solve the integral in 5.3.3 using Cauchy's residue theorem. Therefore we need to identify the poles of the integrand. Let's denote the integrand by f

$$f = \frac{-ie^{(q+mh)i\omega}\cos^{2dh}(\frac{b}{2})}{2\pi\omega\cosh^{2dh}(\frac{a\omega - ib}{2})}$$

We will restrict ourselves to the case $2dh = n$ for $n \in \mathbb{N}$. In this case we see that f has simple pole at $\omega_0 = 0$. Also for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we see that f has a pole of order n at $w_k = \frac{i(b-(2k+1)\pi)}{a}$. That is, the integrand has an infinite amount of poles of order n spread out with regular intervals along the imaginary axis.

Now let C_{R^+} and C_{R^-} be the semi circles, with radius R, in the upper and lower half plane of $\mathbb{C}$ respectively. In order for us to use Jordan's lemma on C_{R^+} we need

$$\lim_{R\to\infty} \left| \frac{\cos^{2dh}(\frac{b}{2})}{Re^{i\theta}\cosh^{2dh}(\frac{aRe^{i\theta} - ib}{2})} \right| = 0 \tag{5.3.7}$$

for all $\theta \in [0,\pi]$, and similarly for C_{R^-}. This is not the case if we just let R pass continuously along the real line, since as we just concluded, f has a pole at $w_k = \frac{i(b-(2k+1)\pi)}{a}$ for $k \in \mathbb{Z}$. To avoid this difficulty we pick a sequence in

$\mathbb{R}\backslash\left\{\frac{|b-(2k+1)\pi|}{a} : k \in \mathbb{N}\right\}$ and let the radius R increase as the elements of that sequence. So, let the sequence R_j be defined as follows:

$$R_j = \frac{b + 2^j\pi}{a}$$

Then we have

$$\lim_{j\to\infty} \left| \frac{\cos^{2dh}(\frac{b}{2})}{R_j e^{i\theta} \cosh^{2dh}(\frac{aR_j e^{i\theta} - ib}{2})} \right| = 0 \tag{5.3.8}$$

$$\Rightarrow \lim_{j\to\infty} \text{p.v} \int_{C_{R_j^+}} f(\omega)d\omega = 0, \text{ for } q + hm > 0 \tag{5.3.9}$$

$$\text{and } \lim_{j\to\infty} \text{p.v} \int_{C_{R_j^-}} f(\omega)d\omega = 0, \text{ for } q + hm < 0 \tag{5.3.10}$$

by Jordan's lemma. This means that after accounting for the pole at 0, all we have to do, is to find and sum up all of the residues in the upper or lower half plane for $q + hm > 0$ or $q + hm < 0$ respectively. Let S^+ and S^- respectively denote the sum of the residues in the upper half and lower plane of $\mathbb{C}$. Then the price A_t is given:

$$A_t = \begin{cases} 1 - 2\pi i S^+ & \text{if } Q_t - \gamma + (T-t)m > 0 \\ -2\pi i S^- & \text{if } Q_t - \gamma + (T-t)m < 0 \end{cases} \tag{5.3.11}$$

This follows from computing the residue of f at 0 and using Cauchy's residue theorem.

The residue of a pole of order n is easily found, and the sums seems to converge, so that in principle there should be nothing stopping us from computing the price for arbitrary n. The only problem is that the size of the computation grows very quickly. Below is the computation for the simplest cases $dh = 1$ and $dh = 2$. For simpler notation, we introduce the following notation $A = q + hm$ and $B_k = (b + (2k+1)\pi)$, also the steps are kept to a minimum. In the following $F_1(.;.;.)$ denotes the Hypergeometric function.

- $2dh = 1$**:**

$$\text{Res } (f, \omega_k) = \lim_{\omega\to\omega_k} (\omega - \omega_k) f(\omega) \tag{5.3.12}$$

$$= \frac{2ie^{-\frac{AB_k}{a}}}{B_k} \tag{5.3.13}$$

For $A > 0$ we find:

$$S^+ = \sum_{k=0}^{\infty} \text{Res } (f, \omega_k) \tag{5.3.14}$$

$$= \sum_{k=0}^{\infty} \frac{2ie^{-\frac{A(b+(2k+1)\pi)}{a}}}{b+(2k+1)\pi} \tag{5.3.15}$$

$$= \frac{2iF_1\left(1, \frac{b+\pi}{2\pi}; \frac{b+3\pi}{2\pi}; e^{\frac{-2A\pi}{a}}\right)}{e^{\frac{A(b+\pi)}{a}}(b+\pi)} \tag{5.3.16}$$

For $A < 0$:

$$S^- = \sum_{k=1}^{-\infty} \text{Res } (f, \omega_k) \tag{5.3.17}$$

$$= \sum_{k=0}^{\infty} \frac{2ie^{-\frac{A(b-(2k+1)\pi)}{a}}}{b-(2k+1)\pi} \tag{5.3.18}$$

$$= \frac{2iF_1\left(1, \frac{\pi-b}{2\pi}; \frac{3\pi-b}{2\pi}; e^{\frac{2A\pi}{a}}\right)}{e^{\frac{A(b-\pi)}{a}}(b-\pi)} \tag{5.3.19}$$

- $2dh = 2$**:**

$$\text{Res } (f, \omega_k) = \lim_{\omega \to \omega_k} \frac{d}{dx}((\omega - \omega_k)^2 f(\omega)) \tag{5.3.20}$$

$$= \frac{4i\left(a + AB_k\right)e^{\frac{-AB_k}{a}}}{aB_k^2} \tag{5.3.21}$$

$$\tag{5.3.22}$$

For $A > 0$ we find:

$$S^+ = \sum_{k=0}^{\infty} \text{Res } (f, \omega_k) \tag{5.3.23}$$

$$= \sum_{k=0}^{\infty} \frac{4i\left(a + A(b+(2k+1)\pi)\right)e^{\frac{-A(b+(2k+1)\pi)}{a}}}{a(b+(2k+1)\pi)^2} \tag{5.3.24}$$

$$= \frac{4i(a + A(b+\pi))F_1\left(1, \frac{b+\pi}{2\pi}, \frac{b+\pi}{2\pi}, \frac{(b+3\pi)A+a}{2A\pi}; \frac{b+3\pi}{2\pi}, \frac{b+3\pi}{2\pi}, \frac{(b+\pi)A+a}{2A\pi}; e^{\frac{-2A\pi}{a}}\right)}{ae^{\frac{A(b+\pi)}{a}}(b+\pi)^2} \tag{5.3.25}$$

For $A < 0$:

$$S^- = \sum_{k=1}^{-\infty} \text{Res } (f, \omega_k) \tag{5.3.26}$$

$$= \sum_{k=0}^{\infty} \frac{4i\left(a + A(b-(2k+1)\pi)\right) e^{\frac{-A(b-(2k+1)\pi)}{a}}}{a(b-(2k+1)\pi)^2} \tag{5.3.27}$$

$$= \frac{4i(a + A(b-\pi))F_1\left(1, \frac{\pi-b}{2\pi}, \frac{\pi-b}{2\pi}, \frac{(3\pi-b)A-a}{2A\pi}; \frac{3\pi-b}{2\pi}, \frac{3\pi-b}{2\pi}, \frac{(\pi-b)A-a}{2A\pi}; e^{\frac{2A\pi}{a}}\right)}{ae^{\frac{A(b-\pi)}{a}}(b-\pi)^2} \tag{5.3.28}$$

We see that for these two cases the price A_t can be expressed in terms of the Hypergeometric function, which can be reasonably easily and well approximated. As we have already commented, there should be nothing in the way of computing the price for arbitrary n, except that the computation grows quickly. For large n it is likely more feasible to just numerically approximate the price. Another issue with the above solution is that it only holds when $2dh$ is an integer. This means that the number of times for which the price can be computed this way is dependent on the parameters T and d, and is potentially very limited for small d. In this case however, as we shall see, numerical approximation works well.

5.3.2 Simulation

We use the same simulation method as in the inverse Gaussian case. We choose a time window on which we want to model the price. Then we partition this window into n parts. We then draw n Meixner distributed samples, and model the total emission as the sum of these. It turns out that the Meixner process is not so easily sampled since the quantile function is not known in closed form [12]. In order to get around this problem, we use a method described in [1] to estimate a Meixner distributed sample. The method is as follows:

Meixner distributed sample

1. Use the probability distribution function (PDF) and numerical integration to estimate the cumulative distribution function(CDF).

2. Draw uniformly distributed random variables on the interval $[0, 1]$ using a pseudo random generator.

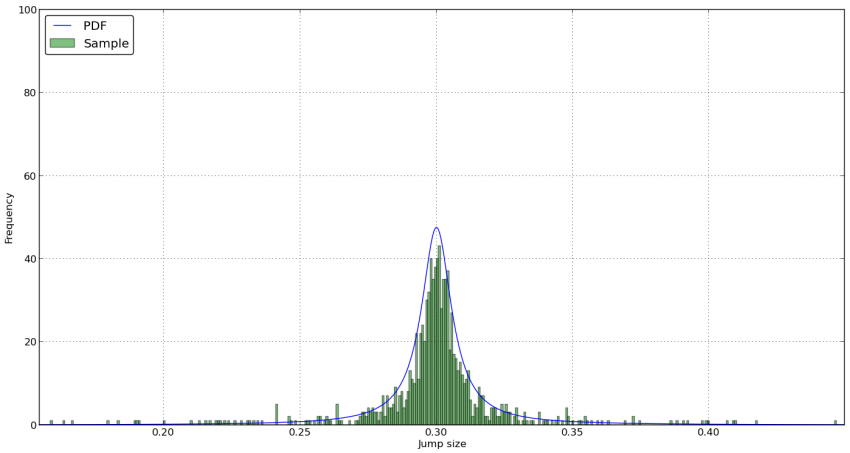

Figure 5.6: The PDF of the Meixner distribution and the distribution of the random samples.

3. Use the estimated CDF to inverse map the uniform random sample.

The computations were done using Python and the numpy library for numerical applications. There were some issues with the accuracy of the approximation to the CDF which led to the sampled variables disagreeing with the Meixner distribution. The accuracy depended on the parameters a, b, d and m of the Meixner distribution. The parameter that seemed to have the largest impact was d. For large d the CDF came completely off, even for very fine grids in the numerical integration in step 1. On the other hand, when d was small (0.01 or less), changing the other parameters only had minor effect on the accuracy. In figure 5.6 the PDF is plotted with parameters $a = 6.7, b = 0.6, d = 0.001$ and $m = 0.3$. Along with the PDF the approximated Meixner distributed samples with the same parameters is plotted. We see that the samples fit reasonably well with the PDF.

Price simulation Using the random samples the total emissions were modeled. Then the futures price was computed using 5.3.3. This shows that the expression 4.2.1 works well numerically when straight forward numerical integration is used.

Again there seemed to be a problem with roun-off errors for certain values of the parameters, and again d seemed to have the most impact. For large values of d Python report round off errors causing the numerical integration to become inaccurate. It should be mentioned that it was the product dT

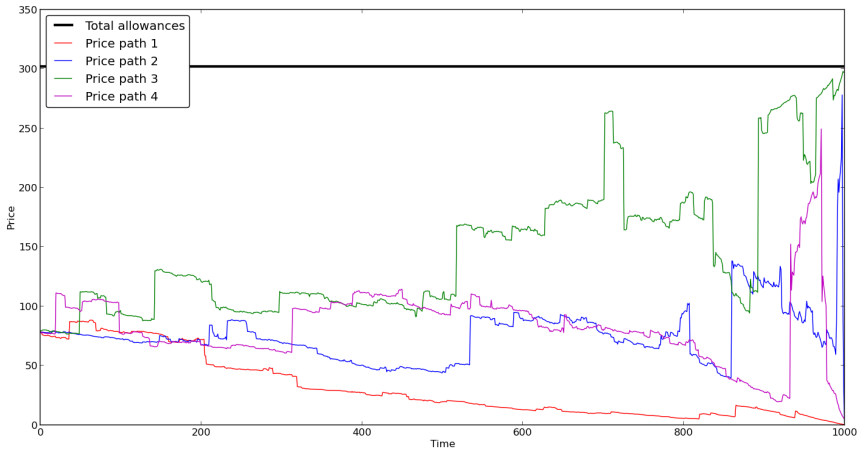

Figure 5.7: A_t when the underlying is Meixner process. The prices have been multiplied with the total emission cap which is set to be $E(A_t) + \mathrm{Var}(A_t)/8$.

that had to be kept from becoming to large. This is natural since part of the integrand in 5.3.3 is exponentiated by $2d(T-t)$. With $T = 1000$ it works well with $d = 0.001$. In figure 5.7 we see four different paths of the futures price: The parameters are the same as in the samples generated above. We see that the paths demonstrate both upwards and downwards jumps. We can also see that they seem to become more volatile as $t \to T$, just as in the I.G simulations.

In figure 5.8 we see the estimate for the error due to numerical integration provided by the numpy package. As in the I.G simulations, the error grows towards the end of the trading period. But it remains small throughout the whole trading period, and Python does not give any warnings.

5.4 Findings

We have now considered pricing of A_t using three examples of the total emission Q. We have not been able to express the price analytically in any of the examples, except semi analytically for some special times t in the Meixner case. Of the three examples we have looked at, the Meixner process seems most promising. It has great flexibility and is simulated numerically without much difficulty. Furthermore it allows for both positive and negative jumps in the price. The inverse Gaussian is less flexible and does not allow negative jumps. It might be argued that negative jumps are not so important since

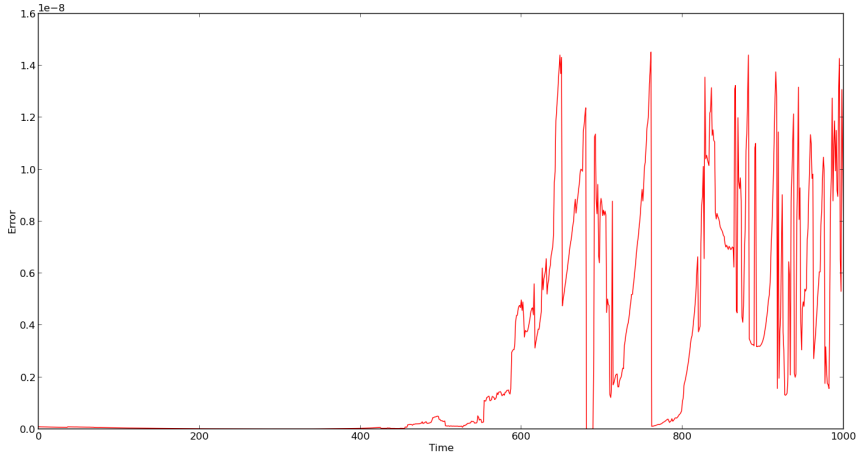

Figure 5.8: The figure shows estimated upper bound of the error due to numerical integration. The price path considered is price path 1 from figure 5.7.

they can be approximated by steep continuous decrease.[1]

An issue with the model of Grüll et al. was that it is hard to know how much the approximation differs from the "real" price. In the simulation method we used, we got a nice estimate of the error due to numerical integration. In the I.G case this should be the only source of error. Admittedly the estimate is completely off as t grows close to T, but we are warned about this by the integrator. In this way we have good control over the error. In the Meixner simulation another source of error is the estimation to Meixner distributed samples. In spite of this, the numerical error should be reasonably well under control if the error due to numerical integration is considered in combination the sample plot.

[1]The market data will always be jumpy since it is discretized. It is a matter of taste if one thinks about the data as the realization of some fundamentally jumpy, or continuous process

Chapter 6

Options on emission futures

In this chapter we will have a look at how to price standard call and put options on forward emission allowances. Using an SDE approach, such as in [4] and [6], the risk neutral option price can be expressed in terms of an SDE. The expression can be evaluated using the Monte Carlo method and the Markov property of the strong solution to the SDE [4].

We have not used the SDE approach, but the Fourier approach from Carr et al. [7]. Thus it would be more close at hand to try to express the option price in terms of an integral expression of the characteristic function of Q_t. Before we go on, let us quickly review some of the standard theory regarding options.

6.1 Options

We will only consider standard European call and put options. A European call option on an underlying asset is a contract allowing the holder to buy the underlying asset at an agreed upon price called the strike, at an agreed upon time, called the maturity of the option. The payoff function for a call option with strike price K and maturity t looks like $(\bar{A}_t - K)^+$ where $\bar{A}_t$ is the underlying asset at time t. In our case the underlying is forward contract on an emission allowance.

We know from the call-put parity relation that:

$$C_t(s, K) - P_t(s, K) = \bar{A}_s - e^{-r(t-s)}K \text{ [9]}$$

where C and P are the prices of call and put options respectively. The call-put parity relation can easily be deduced by looking at the general expressions for $P = E(\max(\bar{A}_t) - K, 0)$ and $Q = E(K - \max(\bar{A}_t), 0)$, and then using the

fact that $\max(x-K,0)-\max(K-x,0)=x-K$. Therefore, if we know one, we know the other. For this reason we will only consider call options.

In a complete market there would only be one price $C_t(s,K)$ of a call option, and it could be found by finding a replicating portfolio. Since our market is not complete, there is no replicating portfolio and from a modeling point of view the price will not be unique. Another way to find $C_t(s,K)$ is to find a risk neutral measure $\mathbb{Q}$ so that $C_t(s,K)=E_{\mathbb{Q}}(\left[\bar{A}_t-K\right]^+|\mathcal{F}_s)=E(\frac{d\mathbb{Q}}{d\mathbb{P}}\left[\bar{A}_t-K\right]^+|\mathcal{F}_s)$, and then calculate that expectation. This is in analogy with the way we found the market dynamics $\bar{A}_t$.

There are many ways of choosing the risk neutral measure [9], but we will not concern ourselves with this. We will just use $\mathbb{Q}$ to denote any risk neutral measure and do the computations in general.

6.2 Computation

We will try a direct approach using the technique we used to express $\bar{A}_t$ in terms of the characteristic function of Q_t. Before we start the computation, it is appropriate to make a remark regarding the payoff function for a call option. We have already used $\max(\bar{A}_t-K,0)$ and $[\bar{A}_t-K]^+$ to mean the same thing, which it is. Also note that that:

$$[\bar{A}_t-K]^+=\frac{\bar{A}_t-K+|\bar{A}_t-K|}{2} \tag{6.2.1}$$

$$=(\bar{A}_t-K)1_{\bar{A}_t-K>0} \tag{6.2.2}$$

At any time, we will pick the most convenient way of expressing the payoff function. Now recall that $\bar{A}_t=E(\frac{d\bar{P}}{dP}1_{Q_T>\gamma}|\mathcal{F}_t)$ we see that:

$$C_t(s,K)=E\left(\frac{d\mathbb{Q}}{dP}\left[E\left(\frac{d\bar{P}}{dP}1_{Q_T>\gamma}\Big|\mathcal{F}_t\right)-K\right]^+\Big|\mathcal{F}_s\right)$$

So C_t can be viewed as a function of Q_T. Let us use the Fourier technique. For practical reasons we will suppress the dependence on Q_T and just write $\bar{A}_t$. Recall that:

$$1_{x>b}=\frac{e^{-\beta b}}{2\pi}\int_{\mathbb{R}}\frac{e^{(x-b)iy}}{\beta+iy}dy$$

for some $\beta>0$. Therefore we can write $C_t(s,K)$ as follows:

$$C_t(s,K)=E_{\mathbb{Q}}\left(\frac{(\bar{A}_t-K)e^{-\beta K}}{2\pi}\int_{\mathbb{R}}\frac{e^{(\bar{A}_t-K)iy}}{\beta+iy}dy|\mathcal{F}_s\right)$$

using similar calculations as above we get the following:

$$C_t(s,K) = E_{\mathbb{Q}}\left(\frac{(\bar{A}_t - K)e^{-\beta K}}{2\pi}\int_{\mathbb{R}}\frac{e^{(\bar{A}_t-K)iy}}{\beta+iy}dy|\mathcal{F}_s\right) \tag{6.2.3}$$

$$= \int_{\mathbb{R}} E_{\mathbb{Q}}\left(\frac{(\bar{A}_t - K)e^{-\beta K}}{2\pi}\frac{e^{(\bar{A}_t-K)iy}}{\beta+iy}|\mathcal{F}_s\right)dy \tag{6.2.4}$$

$$= \frac{e^{-\beta K}}{2\pi}\int_{\mathbb{R}}\frac{e^{-Kiy}}{\beta+iy}E_{\mathbb{Q}}\left((\bar{A}_t - K)e^{\bar{A}_t iy}|\mathcal{F}_s\right)dy \tag{6.2.5}$$

$$\tag{6.2.6}$$

So now what we need to do is to calculate:

$$E_{\mathbb{Q}}\left((\bar{A}_t - K)e^{\bar{A}_t iy}|\mathcal{F}_s\right) \tag{6.2.7}$$

$$= E_{\mathbb{Q}}\left(\bar{A}_t e^{\bar{A}_t iy}|\mathcal{F}_s\right) - KE_{\mathbb{Q}}\left(e^{\bar{A}_t iy}|\mathcal{F}_s\right) \tag{6.2.8}$$

Unfortunately, by proposition 3.3.1 $\bar{A}$ is not a Levy process, so to find the expectation may be difficult. Therefore it seems doubtful whether this direct approach would be fruitful.

In the paper by Carr et al. the price of a European call option is expressed in terms of the characteristic function of the logarithm of the underlying. In our case that would translate to the characteristic function of $\ln(\bar{A}_t)$. Therefore, if we could express the characteristic function of $\ln(\bar{A}_t)$ in terms of the characteristic function of Q_t, we would arrive at a relatively explicit expression for C_t. Let us see what happens with $E(e^{iy\ln(\bar{A}_t)})$ if we write out the expression for $\bar{A}_t$:

$$E(e^{iy\ln(\bar{A}_t)}) = E(\bar{A}_t^{iy}) \tag{6.2.9}$$

$$= E\left(\left[E\left(\frac{d\bar{P}}{dP}1_{Q_T>\gamma}\Big|\mathcal{F}_t\right)\right]^{iy}\right) \tag{6.2.10}$$

$$= E\left(\left[\frac{1}{2} + p.v\int_{\mathbb{R}}\frac{e^{(Q_t-\gamma)i\omega+(T-t)(\psi_Q(\omega-i\theta)-\psi_Q(-i\theta))}}{2\pi i\omega}d\omega\right]^{iy}\right) \tag{6.2.11}$$

If we assume that Q_t has a known closed form probability density function q_t, we can write 6.2.11 as :

$$E\left(\left[\frac{1}{2} + \frac{e^{(t-T)\psi_Q(-i\theta)}}{2\pi}p.v\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{(T-t)\psi_Q+(x-\gamma)i\omega}}{i\omega}q_t(x)dxd\omega\right]^{iy}\right) \tag{6.2.12}$$

The integral inside the expectation can be numerically approximated, giving us an approximation of the characteristic function of $\ln(\bar{A}_t)$. This approximation can be used in the method of Carr et al. If Q_t has no known closed form probability density function, the Monte Carlo method may be used to estimate $E(\bar{A}^{iy})$. For the sake of clarity we summarize this idea in a proposition:

Proposition 6.2.1. *Let the risk-neutral price at time s of a call option with strike K and maturity t be denoted $C_s(t, K)$. If the total emission Q_t has a closed form probability density function, the price $C_s(t, K)$ can be approximated. This is done by solving 6.2.12 as an approximation of the characteristic function of $\ln(\bar{A}_t)$, and then using this approximation in the method described in Carr et al. [7].*

6.3 Analytical approximation

In this section we will try to gain more knowledge about the option price by finding upper and lower bounds in the risk neutral case. It is appropriate with a reminder that the futures price $\bar{A}_t$ is in time T money and normalized. Therefore, the strike price K and all expressions derived on $\bar{A}_t$ and K, will also be normalized. With this in mind, we continue:

In the case when the buyer is risk neutral, the price for call option $C(s, t, K)$ with maturity t and strike price K on an emission forward contract is given by the expected payoff $C_0(s, t, K)$:

$$C_0(s, t, K) = E(\max(\bar{A}_t - K, 0)|\mathcal{F}_s)$$

in time s. Thus lemma 6.3.1 can be interpreted as a lower bound for this price.

Lemma 6.3.1. *The expected value of an option on $\bar{A}_t$ is bounded below by:*

$$E(max(\bar{A}_t - K, 0)|\mathcal{F}_s) \geq (\bar{A}_s - K)1_{\bar{A}_> K}$$

Proof. Since $\bar{A}_t = E(\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}1_{Q_T>\gamma}|\mathcal{F}_t)$, it is a martingale with respect to the objective probability P. Also notice that the payoff function $\max(\bar{A}_t - K, 0)$ is convex. Thus we can use Jensen's inequality for conditional expectations as found in [22], so

$$\begin{aligned} E(\max(\bar{A}_t - K, 0)|\mathcal{F}_s) &\geq \max(E(\bar{A}_t|\mathcal{F}_s) - K, 0) && (6.3.1)\\ &= \max(A_s - K, 0) && (6.3.2)\\ &= (\bar{A}_s - K)1_{\bar{A}_> K} && (6.3.3) \end{aligned}$$

□

Now that we have a lower bound for the risk neutral option price, it would be interesting to see if we could find an upper bound. There has been much work done on finding converses to the Jensen inequality, such as the Lah-Ribarić inequality and the Jensen-Grüss inequality. The main focus in the literature seems to be on discrete versions of the inequality. Discrete inequalities is not without interest to us since any numerical approximation will be bounded above by such an inequality and also since many of these inequalities can no doubt be generalized to the integral case, as is indeed remarked in [10]. However, for us the main objective is to show that a decent upper bound can be found, and therefore we will not spend too much time trying to find the absolute best inequality, but rather use one that works in our case without too much customization. In the paper by Bakula and Pecarić [2] they prove an integral version of Lah-Pecarić for m-convex functions. In our case, with the max function, we have a 1-convex function (just convex) in which case it is just a special case of the inequality of Pecarić and Beesack in [17], which is the one we will use. Incidentally, by theorem 6.3.4, it turns out that this is the best maturity-independent upper bound we can hope for.

Lemma 6.3.2 (Converse Jensen [17])**.** *Let E be a non-empty set, and L be a linear class of functions $g : E \to \mathbb{R}$ satisfying $L1$ and $L2$. Also let T be a linear functional satisfying $T1, T2, T3$.*

L1: *If $g, h \in L$ then $ag + bh \in L$ for $a, b \in \mathbb{R}$*

L2: *We have $1 \in L$ where $1(t) = 1$ for all $t \in E$*

T1: *$A(ag + bh) = aA(g) + bA(h)$ for all $a, b \in \mathbb{R}$ and $g, h \in L$*

T2: *If $g \in L$ and $g(t) \geq 0 \ \forall t \in E$, then $A(g) \geq 0$*

T3: *$A(1) = 1$*

Now, if f is a convex function on an interval $I = [m, M] \subset \mathbb{R}$. Then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$ we have

$$T(f(g)) \leq \frac{M - T(g)}{M - m} f(m) + \frac{T(g) - m}{M - m} f(M)$$

Theorem 6.3.3 follows directly from lemma 6.3.2 and lemma 6.3.1

Theorem 6.3.3.

$$(\bar{A}_s - K)1_{\bar{A}_> K} \leq E(max(\bar{A}_t - K, 0)|\mathcal{F}_s) \leq \bar{A}_s(1 - K)$$

Furthermore, the upper bound is the best possible upper bound that is independent of maturity time t.

Proof. We see that the space $L^1(\Omega \times [0, \infty))$ of stochastic processes with finite expectation satisfies $L1$ and $L2$. Also, for all $s \in [0, \infty)$ we see that $E(\cdot|\mathcal{F}_s)$ satisfies $A1 - A3$. The forward price $\bar{A}_t$ is bounded $\bar{A}_t(\Omega) \subset [0, 1]$ and thus $\bar{A}_t \in L^1(\Omega \times [0, \infty))$. The function f:

$$f(\bar{A}_t) = \max(\bar{A}_t - K, 0) = \frac{(\bar{A}_t - K) + |\bar{A}_t - K|}{2}$$

is convex on $[0, 1]$ and $|f(\bar{A}_t)| \leq 1$, so $f(\bar{A}_t) \in L^1(\Omega \times [0, \infty))$. Therefore all conditions for using lemma 6.3.2 is satisfied, and the right side of the inequality follows. The left side of the inequality is from lemma 6.3.1. The fact that the upper bound is the best possible time-independent upper bound follows from theorem 6.3.4. □

Now we have both a lower and upper bound on the expected payoff of the call option $C(s, t, K)$. In the case $t = T$ we can do even better. To see this, notice that the expected payoff $C_0(s, t, K)$ is a bounded martingale for all t, $0 \leq C_0(s, t, K) \leq 1$, and $C_0(s, T, K)$ is binary at time $s = T$ s.t:

$$C_0(T, T, K) = 1 - K \text{ or } C_0(T, T, K) = 0 \text{ a.s.}$$

Hence we can use theorem 3.3.2 to see that

$$P(C_0(T, T, K) = 1 - K|\mathcal{F}_s) = \frac{C_0(s, T, K)}{1 - K} \tag{6.3.4}$$

Using lemma 3.3.2, we can also find an expression for $C_0(s, T, K)$, we summarize in a theorem:

Theorem 6.3.4. *For time $t = T$, the expected payoff $C_0(s, T, K)$ for the call option $C(s, t, K)$ can be computed:*

$$C_0(s, T, K) = E(max(\bar{A}_T - K, 0)|\mathcal{F}_s) = \bar{A}_t(1 - K) \tag{6.3.5}$$

Also the probability that the option $C(s, t, K)$ pays off can be computed:

$$P(C_0(T, T, K) = 1 - K|\mathcal{F}_s) = \bar{A}_t \tag{6.3.6}$$

Proof. In the special case when $t = T$ we know :

$$\max(\bar{A}_T - K, 0) = (1 - K)1_{\bar{A}_T > K}(\omega) = (1 - K)1_{\bar{A}_T = 1}(\omega)$$

And thus, using lemma 3.3.2 we find

$$C_0(s, T, K) = E(\max(\bar{A}_T - K, 0)|\mathcal{F}_s) = P(\bar{A}_T = 1|\mathcal{F}_s)(1 - K) = \bar{A}_t(1 - K)$$

which gives 6.3.5. Combining this with equation 6.3.4 we get 6.3.6. The expression 6.3.5 says that $C_0(s, t, K)$ attains its upper bound for $t = T$. This means that our upper bound is indeed the best we can get that is not time dependent. □

6.4 Findings

We have found that the expected payoff can be explicitly computed for maturity time T. For many market participants this will be the most interesting time. These options are written on actual emission allowances, and not futures. Therefore they provide a way of insuring power producers that they will have enough allowances at the end of the trading period. Another useful tool is equation 6.3.6. It can be used to assess the risk of the option $C(s, T, K)$.

In figure 6.1 the price of a call option with maturity T is plotted as a function of A_t for different strikes. The path of A_t is simulated using a Meixner process to model Q_t. We see that the paths of the option prices follow the path of the futures price. We also see that the options with higher strikes are more volatile.

We have also found an upper and lower bound on the expected payoff of a call option. If we subtract the lower bound from the upper bound, we get a band within which the $C_0(s, T, K)$ must lie. This band can be thought of as the size of the uncertainty about $C_0(s, T, K)$. In figure 6.2 we see the size of the uncertainty for for different values of A_t and K. The largest uncertainty is 0.25 when $\bar{A}_t = K = 0.5$. For other values the uncertainty falls drastically, such as for $\bar{A}_t = 0.3$ and $K = 0.2$, then the uncertainty is 0.02. It should be mentioned that this uncertainty must be seen in relation to the size of the price. However, when $\bar{A}_t$ and K have the right values, the bounds may be used as a very quick way of getting an estimate of $C_0(s, T, K)$.

For Call options with maturities $t \neq T$, the method described in proposition 6.2.1 may be used to evaluate the expected payoff. We have not implemented this method, but it would be an interesting task for future work.

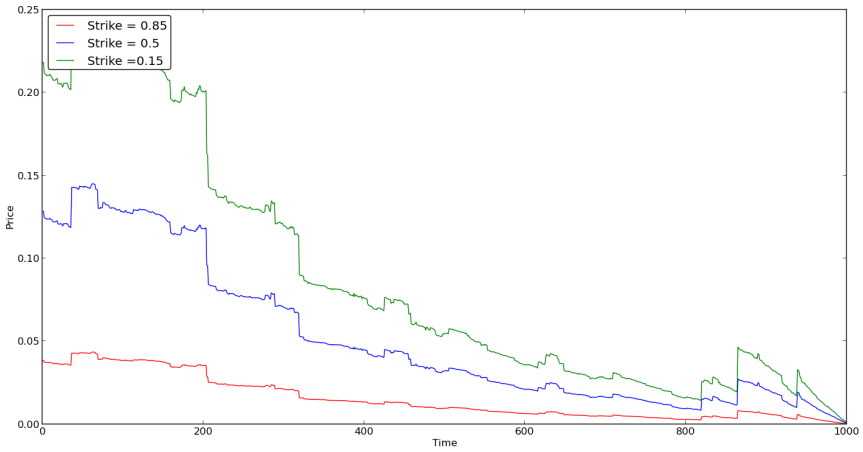

Figure 6.1: The underlying price path of A_t is price path 1 in fig 5.7.

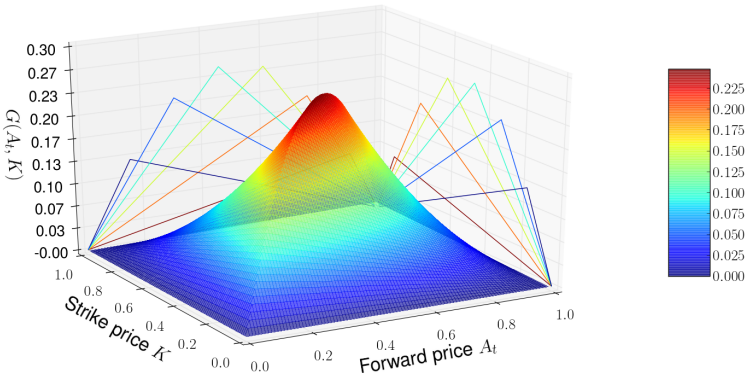

Figure 6.2: $G(A_t, K) = A_t(1 - K) - (A_t - K)1_{A_t>K}$. The plot shows the normalized uncertainty within which $C_0(s, t, K)$ must lie.

Chapter 7

Summary and ideas for further work.

In this thesis we have taken a reduced form approach based on the work of Carmona et al. [5]. By using their conditions for the existence of a market equilibrium, we have expressed the risk-neutral price of emission futures in terms of the characteristic function of Q_t in 4.1.3 and 4.2.1. We have done this by modeling the total emission Q_t directly as a Levy process. We have also found the expressions 4.3.2 and 4.3.3 for the market dynamics of emission futures. These expressions are reasonably explicit, and even though attempts at analytical solutions have not been successful, the expressions have been shown to allow numerical approximations for some examples of Q_t. It would be interesting to further investigate the price expressions for different ways of modeling the total emission, and see how well they can be calibrated to empirical data.

From the examples we have looked at, modeling the total emissions as a Meixner process seems most promising. The resulting price expression behaves well under numerical approximation, and even allows a semi analytical expression for some values of t. When the Mexiner process is used to model the underlying, the price simulation shows both upwards and downwards jumps. This is in accordance with market data. The flexibility of the Meixner process leaves hope that the model can be well approximated to empirical data.

In the last chapter we considered European call options on emission futures. We analytically expressed the risk-neutral price of a call with maturity T, and the probability of such an option being profitable, in theorem 6.3.4. For call options with general maturity time t we proposed a method for numerical approximation of the risk-neutral price in proposition 6.2.1. We also derived maturity independent bounds for the risk-neutral price and proposed

that these may be used as a way approximation for the right values of the futures price $\bar{A}_s$ and strike K. The analytical results on options in chapter 6 are based on the result 3.3.2. This result is new to the best of my knowledge.

It would be an interesting task for further work to test the method of proposition 6.2.1 and calibrate the results to empirical data. In adition, taking the option prices, computed either by equation 6.3.5 or by proposition 6.2.1, and deriving the implied volatilities of Black-76 would provide an interesting indication of our model's inherent ability to account for risk.

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