Bost-Connes type systems associated with function fields

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Contents

Introduction

In 1995 Bost and Connes [**1**] constructed a quantum statistical mechanical system with properties related to the Riemann ζ-function and class field theory for $\mathbb{Q}$. Indeed, the system constructed answers Problem 1.1 of [**7**] (which we reproduce here) for the case $K = \mathbb{Q}$:

Problem 1. For a number field K, exhibit an explicit C*-dynamical system $(\mathcal{A}, \sigma)$ such that

(i) the partition function of the system is the Dedekind zeta function of K;

(ii) the quotient of the idèle class group C_K by the connected component D_K of the identity acts as symmetries of the system;

(iii) for each inverse temperature $0 < \beta \leq 1$ there is a unique KMS$_\beta$-state;

(iv) for each $\beta > 1$ the action of the symmetry group C_K/D_K on the extremal KMS$_\beta$-states is free and transitive;

(v) there is a K-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ such that the values of the extremal KMS$_\infty$-states on elements of $\mathcal{A}_0$ are algebraic numbers that generate the maximal abelian extension K^{ab} of K;

(vi) the Galois action of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ on these values is realized by the action of C_K/D_K on the extremal KMS$_\infty$-states via the class field theory isomorphism $s : C_K/D_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$.

This problem was solved in [**7**] for K an imaginary quadratic field. Furthermore, in [**15, 24**] a system satisfying (i)-(iv) has been constructed for arbitrary number fields. While the construction is not entirely explicit, it is shown in [**34**] that there exists an arithmetic subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ satisfying (v) and (vi).

It is natural to ask whether the analogue of Problem 1 can be solved for function fields. Since the basis for the full solutions in the number fields case [**1, 7**] has been the knowledge of explicit class field theory, the fact that one already has an explicit class-field theory for function fields through the work of Drinfeld [**10, 11**] and Hayes [**17**] is encouraging. This also indicates that the construction should involve Drinfeld modules in some fashion.

Steps in this direction, albeit down different paths, have been taken by Jacob [**18**] and Consani and Marcolli [**9**].

In his paper [**18**], Jacob considers the torsion points $\phi(\mathbb{C}_\infty)^{\text{tor}}$ of a Drinfeld module ϕ. He constructs an equivalence relation whose underlying space consists of characters on the set of torsion points of Drinfeld modules ϕ, where ϕ ranges through the set of sign-normalized rank one Drinfeld modules. This set of Drinfeld modules has been shown [**17**] to be closely related to the class field theory of a function field. The equivalence relation is induced by an ideal action on this object space, and the C*-algebra considered by Jacob is the C*-algebra of this equivalence relation with dynamics arising from the equivalence relation.

The dynamical system constructed satisfies (i)–(iv) of the problem given above, except that the symmetries of the system are given by $\text{Gal}(\mathbb{K}/K)$, where $\mathbb{K}$ is an abelian extension of the function field K which is somewhat smaller than the full maximal abelian extension. On the other hand, it is not possible to find an arithmetic subalgebra for this system. Indeed, the evaluation of elements of the algebra at extremal KMS_∞-states are complex numbers, while generators of K^{ab} over K live in a field of characteristic p.

Another approach to the problem is taken in [**9**], where Consani and Marcolli take as their starting point a notion of "pointed Drinfeld modules", which may equivalently be considered as pairs (Λ, ϕ), where Λ is a lattice in a certain field and ϕ is a homomorphism. The construction is reminicent of that carried out for number fields in for instance [**6**]. They then define an equivalence relation of commensurability on this space, similarly to the number field case, and consider the resulting groupoid up to scaling. With this groupoid as their basis they construct a quantum statistical mechanical system with values in a characteristic p field $\mathbb{C}_\infty$, and construct some KMS-states of the system.

The dynamical system satisfies appropriate analogues of (i), (ii) and (iv), but here the symmetries of the system are given by $\text{Gal}(K^{\text{ab},\infty}/K)$, where $K^{\text{ab},\infty}$ is the maximal abelian extension of K which is completely split at the distinguished place ∞.

In the current thesis we will proceed along both these paths.

The structure of the thesis is the following. In Chapter 1 we review briefly some background on characters from Tate's thesis [**30**]. We then give a very brief introduction to Drinfeld modules, before recalling the main results from explicit class field theory for function fields, following

[**17**]. We also state some results on KMS-states and type III factors which we will refer to in Chapter 2.

Chapter 2 mainly consists of results previously published in [**27**], and concerns complex-valued dynamical systems associated to function fields. In Section 2.1 we construct, for an abelian extension L of a function field K and a finite set of places S of K a dynamical system $(A_{L,S}, \sigma)$. We show that it satisfies (i)–(iv) of the problem given above, with the modification that the symmetries are given by $\mathrm{Gal}(L/K)$ as should be expected. In Section 2.2 we furthermore calculate the type of the unique KMS_β-state for $0 < \beta \leq 1$. This turns out to be $\mathrm{III}_{q^{-n\beta}}$ where q^n is the number of elements in the constant field of L, correcting a result of Jacob.

Section 2.3 shows how the system of Jacob fits into our framework, being isomorphic to our system in the case $L = \mathbb{K}$ and $S = \{\infty\}$. We also show that the groupoid of Consani and Marcolli is canonically isomorphic to the quotien of Jacob's groupoid by the action of $\mathrm{Gal}(\mathbb{K}/K^{\mathrm{ab},\infty})$, and that a small modification of the construction of Consani and Marcolli leads to a groupoid that is isomorphic to that of Jacob. Finally, in Section 2.4 we show that the system of Jacob can arise from a Hecke-algebra argument similar to that of [**25**].

In Chapter 3 we go down the other path, considering dynamical systems over fields of characteristic p. In Section 3.1 we recall the definition of dynamical systems over $\mathbb{C}_\infty$ given in [**9**], and construct such a system $(A_{L,S}, \sigma)$ associated to an abelian extension L of a function field K and a finite set S of places of K. The dynamical system considered in [**9**] corresponds to ours in the case $L = K^{\mathrm{ab},\infty}$ and $S = \{\infty\}$. In Section 3.2 we consider KMS-functionals on our dynamical system. In the complex-valued case there is a bijection between the set of KMS-functionals on $A_{L,S}$ and the set of probability measures on the object space of $Y_{L,S}$ of the equivalence relation satisfying a certain scaling condition. In the positive characteristic case a similar bijection exists, although we phrase our result in terms of probability type functionals in order to avoid having to introduce the language of characteristic p measure theory. This allows us to give a partial classification of the KMS-functionals of $(A_{L,S}, \sigma)$.

The rest of the chapter concerns the possibility of constructing an arithmetic subalgebra associated to the system $(A_{L,S}, \sigma)$. In Section 3.3 we show that the existence results for arithmetic subalgebras for number fields given in [**34**, Section 9] also apply in the function field case. This construction is, however, not entirely satisfying, since it is non-explicit. We are able to remedy this in two specific cases. First, in

Section 3.4 we explicitly construct an arithmetic subalgebra in the case $L = \mathbb{K}$, $S = \{\infty\}$ based on the explicit class field theory presented in [**17**]. Finally, in Section 3.5 we restrict to the case $K = \mathbb{F}_q(T)$, the rational function field, and construct an arithmetic subalgebra for the case $L = K^{\mathrm{ab}}$ and $S = \{\infty\}$ based on the work in [**16**].

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CHAPTER 1

Background

1.1. Function fields

Let p be a prime, and let K be a global function field of characteristic $p > 0$, that is, a finite algebraic extension of $\mathbb{F}_p(T)$. We let $q = p^k$ be such that $\mathbb{F}_q$ is the algebraic closure of $\mathbb{F}_p$ in K. We fix a distinguished place ∞ of K, and let $\mathcal{O}$ be the ring of functions in K which have no pole away from ∞.

Let us once and for all fix some notation. For a place v of K, let K_v be the completion of K at the place v, and let $\mathcal{O}_v = \{x \in K_v : |x|_v \leq 1\}$. Then $\mathcal{O}_v$ is the maximal compact subring of K_v. We write J for the group of divisors of K, so J is the free abelian group generated by the places of K. We also write $\mathbb{A}_K$ for the adele ring of K, that is the restricted product $\prod' K_v$ of the fields K_v with respect to $\mathcal{O}_v \subset K_v$. That is, an element of $\mathbb{A}_K$ is an element $(a_v)_v \in \prod K_v$ such that $a_v \in \mathcal{O}_v$ for all but finitely many places v of K. We also write $\mathcal{O}_{\mathbb{A}} = \prod_v \mathcal{O}_v$.

We will often want to ignore a finite set of places of K, and have use for corresponding notation. If S is a finite set of places of K, we will write $J_S \subset J$ for the subgroup of divisors with support in the complement S^c of S, so J_S is the free abelian group generated by places in S^c. Furthermore, we let $\mathbb{A}_{K,S} = \prod'_{v \in S^c} K_v$ and $\hat{\mathcal{O}}_S = \prod_{v \in S^c} \mathcal{O}_v$. In the case where $S = \{\infty\}$, we also write $\mathbb{A}_{K,f} = \mathbb{A}_{K,\{\infty\}}$, $\hat{\mathcal{O}} = \hat{\mathcal{O}}_{\{\infty\}}$ and $J_K = J_{\{\infty\}}$, which we identify with the fractional ideals of $\mathcal{O}$.

If K_∞ is the completion of K at ∞, the norm induced by ∞ extends uniquely to a norm on $\bar{K}_\infty$, the algebraic closure of K_∞. If we write $\mathbb{C}_\infty$ for the completion of $\bar{K}_\infty$ with respect to this norm, $\mathbb{C}_\infty$ is an extension of K which is both algebraically and topologically complete. It does in many ways correspond to the complex numbers.

In general, we will not be strict in distinguishing between a place $v \neq \infty$ of K and the corresponding prime ideal $\mathfrak{p}_v$ of $\mathcal{O}$. Hence we may occasionally let S consist of a finite set of primes, and the notation will still be as above. We will also (as long as $\infty \in S$) often consider J_S as the set of ideals of $\mathcal{O}$ which are relatively prime to the ideals in S.

1.1.1. Additive characters in characteristic p. We will need certain results about additive characters of various objects associated to function fields, mainly $\mathbb{A}_{K,f}$ and its quotient $\mathbb{A}_{K,f}/\hat{\mathcal{O}}$. This theory is developed in [**30**] in the characteristic zero case. The proofs go through to the positive characteristic case without modification, but can be somewhat simplified since any character of a torsion group of exponent p can only take values in the discrete set of pth roots of unity in $\mathbb{C}$. We give proofs for completeness.

Let us start by determining the character group $\hat{k}$ of a the additive group of local field k of characteristic p. In the following arguments, $|\cdot|$ is the norm on the local field k.

LEMMA 1.1.1. *If $x \mapsto \chi(x)$ is a nontrivial character of k then the map*

$$y \mapsto \chi(\cdot y)$$

is an isomorphism of k with $\hat{k}$.

PROOF. Since multiplication in k is continuous, the map $x \mapsto \chi(xy)$ is a character of k. Furthermore $\chi(\cdot(y+y')) = \chi(\cdot y)\chi(\cdot y')$, so the map respects the additive structure of k. To see that it is injective assume $\chi(\cdot y)$ is trivial. Since χ is nontrivial this is only the case if $k \cdot y \neq k$, which is only the case if $y = 0$.

It remains to show that the map is a homeomorphism. On the one hand, let $B \subset k$ be compact. Then there is an $M \geq 0$ such that $|x| \leq M$ for $x \in B$, and we may assume B consists of all such B. Since χ is continuous there is an $N \geq 0$ such that $\chi(x) = 1$ for all x with $|x| \leq N$. Let $V = \{y \in k : |y| \leq N/M\}$. Then for any $y \in V$ and $x \in B$ we have $|xy| \leq N$, so $\chi(xy) = 1$. Thus $y \mapsto \chi(\cdot y)$ is continuous.

On the other hand, let $x_0 \in k$ be such that $\chi(x_0) \neq 1$. Then if y is such that $\chi(By) = 1$ then $x_0 \notin By$, so we must have $|y| < |x_0|/M$. Thus the map is bicontinuous.

Finally note that the $\chi(\cdot y)$ separate points of k, since if $\chi(xy) = 0$ for all $y \in k$ then $x \cdot k \neq k$, so $x = 0$. Hence the characters $\chi(\cdot y)$ are dense in $\hat{k}$. Since the map is bicontinuous this implies that the characters of the form $\chi(\cdot y)$ form a closed and dense subgroup of $\hat{k}$, so our map is a homeomorphism. □

We need to fix a nontrivial character of k. To this end, let $\pi \in k$ be a uniformizer of k. Then every element of k can be written in the form

$$x = \sum_n a_n \pi^n,$$

where a_n are elements of the residue field of k. This residue field can be identified with $\mathbb{F}_q$ for some prime power $q = p^n$. Letting $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ be the trace map we can define a character of k by

$$\chi_1(x) = \lambda(\mathrm{Tr}(a_{-1})),$$

where $\lambda : \mathbb{F}_p \to \mathbb{C}^\times$ is an identification of the additive group of $\mathbb{F}_p$ with the multiplicative group of pth roots of unity in $\mathbb{C}$. This character is then nontrivial.

For a set $A \subset k$, let $A^\perp \subset \hat{k}$ consist of those $\chi \in \hat{k}$ such that $\chi|_A = 0$. Then under the identification of k with $\hat{k}$ given by χ_1 we have $\mathfrak{o}^\perp = \mathfrak{o}$.

We next want to analyze the characters of $\mathbb{A}_{K,f}$ and relate them to the result above. To this end, given a character $\chi \in \hat{\mathbb{A}}_{K,f}$ let χ_v be the character on K_v given by

$$\chi_v(x_v) = \chi((0, \ldots, 0, x_v, 0, \ldots)).$$

We will show that the characters χ_v determine χ uniquely.

Lemma 1.1.2. *The character χ_v is trivial on $\mathcal{O}_v$ for almost all v, and*

$$\chi(x) = \prod_v \chi_v(x_v).$$

Proof. Let N be a neighborhood of $0 \in \mathbb{A}_{K,f}$ such that $\chi(N) = 1$. We may assume that N is of the form $N = \prod_v N_v$. Let S be a finite set of places containing ∞ such that $v \in S$ if $N_v \neq \mathcal{O}_v$. Then $\mathbb{A}_{K,S} \subset N$ so $\chi(\mathbb{A}_{K,S}) = 1$. In particular $\chi_v(\mathcal{O}_v) = 1$ for $v \notin S$ as claimed.

If $x \in \mathbb{A}_{K,f}$, assume that S also contains all places v such that $x_v \notin \mathcal{O}_v$, and write $x = ((a_v)_{v \in S}, a_S)$ where $a_S = (a_v)_{v \notin S} \in \mathbb{A}_{K,S}$. Then

$$\chi(a) = \prod_{v \in S} \chi(a_v) \cdot \chi(a_S) = \prod_{v \in S} \chi_v(a_v) = \prod_v \chi(a_v).$$

□

Lemma 1.1.3. *Let $\chi_v \in \hat{K}_v$ be given for each v, and assume that $\mathcal{O}_v \subset \ker \chi_v$ for almost all v. Then*

$$\chi(x) = \prod_v \chi_v(x_v)$$

is a character of $\mathbb{A}_{K,f}$.

Proof. It is clear that χ is an algebraic character, and we only have to show continuity. To this end, let S be a finite set of places containing

∞ such that $v \in S$ if $\mathcal{O}_v \not\subset \ker \chi_v$. Now if N_v is a neighborhood $0 \in K_v$ such that $\chi_v(N_v) = 1$ for all $v \in S$ and $N_v = \mathcal{O}_v$ for $v \notin S$, we see that

$$\chi(\prod_v N_v) = 1,$$

so χ is continuous. □

THEOREM 1.1.4. *There is an isomorphism* $\hat{\mathbb{A}}_{K,f} \simeq \mathbb{A}_{K,f}$ *such that* $\hat{\mathcal{O}}^\perp = \hat{\mathcal{O}}$.

PROOF. Let $y \in \mathbb{A}_{K,f}$. If $\chi_{v,1}$ are the characters identifying $\hat{K}_v$ with K_v we want to map y to the character

$$\chi(\cdot y) = \prod_v \chi_{v,1}(\cdot y_v).$$

Since $y \in \mathbb{A}_{K,f}$ we have $y_v \in \mathcal{O}_v$ for almost all v, so $\chi_{v,1}(\mathcal{O}_v y_v) = 1$ for almost all v (since $\mathcal{O}_v^\perp = \mathcal{O}_v$). Thus by the preceding lemmas the map is an algebraic isomorphism. It remains to check that the topologies coincide.

To this end let B be a compact subset of $\mathbb{A}_{K,f}$. Since the $V(B) = \{\chi : \chi(B) = 1\}$ form a neighborhood basis of the trivial character in $\hat{\mathbb{A}}_{K,f}$, it suffices to show that this is carried to a neighborhood basis of $0 \in \mathbb{A}_{K,f}$ and vice versa.

Let $\chi \in V(B)$. We can write $B = \prod_v B_v$ where B_v is compact and $B_v = \mathcal{O}_v$ for almost all v. Let $M \geq 0$ be such that $|x_v|_v \leq M$ for $x_v \in B_v$, and let S be a finite set of places containing ∞ such that $v \in S$ if $B_v \neq \mathcal{O}_v$. Then for $v \in S$ we have $\chi_v(B_v) = 1$, so $\chi_v = \chi_{v,1}(\cdot y_v)$ where $|y_v|_v \leq 1/M$. For $v \notin S$ we have $\chi_v = \chi_{v,1}(\cdot y_v)$ for some $y_v \in \mathcal{O}_v^\perp = \mathcal{O}_v$, so

$$y = (y_v) \in \prod_{v \in S} \{x_v \in K_v : |x_v|_v \leq 1/M\} \times \hat{\mathcal{O}}_S.$$

Letting B run over an increasing net of compacts this is a neighborhood basis of $0 \in \mathbb{A}_{K,f}$.

Conversely, if y is in the set above then $\chi_1(\cdot y)$ maps the compact set

$$B_{M,S} = \prod_{v \in S} \{x_v \in K_v : |x_v|_v \leq M\} \times \hat{\mathcal{O}}_S$$

to $1 \in \mathbb{C}$, so the neighborhood basis of $0 \in \mathbb{A}_{K,f}$ maps to a neighborhood basis of the trivial character in $\hat{\mathbb{A}}_{K,f}$. □

1.1.2. Sign functions. Whereas the real numbers have a canonical sign function, in the function field case we must introduce one artificially, and it does not have all the properties that we would like. However, it is vital to several arguments that follow.

We will write $\mathbb{F}_\infty$ for the constant field of K_∞.

Definition 1.1.5. A sign function on $K_\infty^\times$ is a homomorphism

$$\mathrm{sgn} : K_\infty^\times \to \mathbb{F}_\infty^\times$$

which is the identity on $\mathbb{F}_\infty^\times$. We extend sgn to K_∞ by putting $\mathrm{sgn}(0) = 0$.

Say $x \in K_\infty$ is positive if $\mathrm{sgn}(x) = 1$, and write K_∞^+ for the set of positive elements of K_∞. We furthermore write $K^+ = K_\infty^+ \cap K$ for the set of positive elements of K.

It is obvious that any such function sgn is multiplicative, but (sadly) not additive, in that we cannot say anything about the sign of a sum of two positive elements.

Lemma 1.1.6. *Any sign function* $\mathrm{sgn} : K_\infty^\times \to \mathbb{F}_\infty^\times$ *is trivial on the one-units* $U_1 = 1 + \pi_\infty \mathcal{O}_\infty$.

Proof. By definition, every element of U_1 can be written in the form $1 + \sum_{v \geq 1} a_v \pi_\infty^v$ for some $a_v \in \mathbb{F}_\infty$. Let $U_1^{(n)}$ be the quotient of U_1 given by setting $\pi_\infty^n = 0$. Then $U_1 = \lim_{\leftarrow} U_1^{(n)}$ and the $U_1^{(n)}$ are finite groups in which every element has order a power of p. Hence in particular U_1 is a pro-p-group, in that any finite quotient of U_1 consists of elements of order a power of p.

Since $\mathbb{F}_\infty^\times$ has order prime to p this implies that the image of U_1 under sgn must be trivial. □

Sign functions can be constructed by choosing a uniformizer π_∞ and writing elements $x \in K_\infty$ as $x = \pi_\infty^a \cdot \zeta \cdot u$ where a is an integer, $\zeta \in \mathbb{F}_\infty$ and u is in U_1. Then $x \mapsto \zeta$ is a sign-function.

There is no canonical choice of a sign function. However, we will assume a sign function sgn to be fixed in any arguments given.

1.1.3. Exponentiation of ideals. One interesting feature which differentiates function fields from number fields is that there is a well-defined procedure of raising ideals to a non-integral exponent. While at first surprising, this mirrors the situation in $\mathbb{Q}$. Indeed, since $\mathbb{Z}$ is a principal ideal domain, ideals in $\mathbb{Z}$ (which are nothing but integers up to sign) can be raised to an arbitrary complex exponent. However, $\mathcal{O}$ is not a principal ideal domain, so staying at the level of elements of $\mathcal{O}$ is not sufficient.

In the characteristic zero case the domain of the exponent is the complex numbers $\mathbb{C}$, which indicates that we should define $\mathfrak{a}^x$ for all $x \in \mathbb{C}_\infty$. However, it turns out [**14**, Section 8.1] that the "most natural" choice is the group $\mathbb{C}_\infty^\times \times \mathbb{Z}_p$ which we denote by S_∞. We shall write this group additively, so $(x, y) + (x', y') = (xx', y + y')$.

We first define the exponentiation on K^+, the set of positive elements of K, which we identify with the set of positively generated principal ideals of $\mathcal{O}$, and then extend this exponentiation to all of J_K. Our exposition is based on [**14**, Sections 8.1–2].

Let π be a fixed positive uniformizer for K_∞ so $\operatorname{sgn}(\pi) = 1$, and let U_1 be the set of 1-units, that is the set of $u \in \mathcal{O}_\infty$ such that $u = 1 \mod \pi\mathcal{O}_\infty$. Then any $\alpha \in K_\infty^\times$ can be written (uniquely) as

$$\alpha = \operatorname{sgn}(\alpha)\pi^j\langle\alpha\rangle,$$

where $j = v_\infty(\alpha)$ and $\langle\alpha\rangle \in U_1$.

LEMMA 1.1.7. *If $u \in U_1$ then $\sum_{j=0}^{\infty} \binom{y}{j}(u-1)^j$ converges for all $y \in \mathbb{Z}_p$.*

REMARK 1.1.8. There are two things to note about the notation of this lemma. First, when we write $\binom{y}{j}$ for $y \in \mathbb{Z}_p$ and $j \in \mathbb{N}$, this should be read as

$$\binom{y}{j} = \frac{y(y-1)\cdots(y-j+1)}{j(j-1)\cdots 3 \cdot 2 \cdot 1}$$

which is a well-defined element of $\mathbb{Z}_p$. Secondly, the action of $a = \sum_{i \geq 0} a_i p^i \in \mathbb{Z}_p$ on $x \in K$ is given simply by $ax = a_0 x$, where $a_0 \in \mathbb{F}_p \subset K$. This extends the natural action of $\mathbb{N}$ on K.

PROOF OF LEMMA 1.1.7. Since $u \in U_1$ we get $u - 1 \in \pi\mathcal{O}_\infty$, so $|u-1|^j \leq p^{-j}$. Thus $|u-1|^j \to 0$, while $|\binom{y}{j}| = 1$, so the series is convergent. □

DEFINITION 1.1.9. For $\alpha \in K^+$ and $s = (x, y) \in S_\infty$, let $\deg \alpha = d_\infty v_\infty(\alpha)$, where d_∞ is the degree of the place ∞. This can for instance be defined as the integer such that $|\mathbb{F}_\infty| = q^{d_\infty}$, where $\mathbb{F}_\infty$ are the constants in K_∞. Furthermore, set

$$\alpha^s = x^{\deg(\alpha)}\langle\alpha\rangle^y = x^{d_\infty v_\infty(\alpha)}\langle\alpha\rangle^y.$$

Note that $\langle\alpha\rangle^y$ converges by the lemma. This exponential function works as expected. Indeed, we have the following.

LEMMA 1.1.10.

(i) *Let $\alpha, \beta \in K^+$ and let $s \in S_\infty$. Then*

$$(\alpha\beta)^s = \alpha^s\beta^s.$$

(ii) *Let $s, t \in S_\infty$ and $\alpha \in K^+$. Then*

$$\alpha^{s+t} = \alpha^s \alpha^t.$$

PROOF. Since $\langle \alpha\beta \rangle = \langle \alpha \rangle \langle \beta \rangle$ and $\deg(\alpha\beta) = \deg \alpha + \deg \beta$ we get

$$(\alpha\beta)^s = x^{\deg(\alpha\beta)} \langle \alpha\beta \rangle^y = x^{\deg \alpha} \langle \alpha \rangle^y x^{\deg \beta} \langle \beta \rangle^y = \alpha^s \beta^s.$$

For part (ii) let $s = (x, y)$ and $t = (z, w)$ to get

$$\alpha^{s+t} = (xz)^{\deg \alpha} \langle \alpha \rangle^{y+w} = x^{\deg \alpha} \langle \alpha \rangle^y z^{\deg \alpha} \langle \alpha \rangle^w = \alpha^s \alpha^t$$

as claimed. □

Next we want to extend the exponential to J_K. To start down this path, let $\hat{U}_1 \subset \mathbb{C}_\infty$ be the group of 1-units in $\mathbb{C}_\infty$. That is, $\hat{U}_1$ consists of the units of the ring $\{x \in \mathbb{C}_\infty : v_\infty(x) \geq 0\}$.

LEMMA 1.1.11. *The $\mathbb{Z}_p$-action on $\hat{U}_1$ given by exponentiation extends uniquely to an action of $\mathbb{Q}_p$.*

PROOF. Let $u = 1 + m$ with $|m| < 1$. If $y = \sum_{j > -\infty} c_j p^j$ with $0 \leq c_j < p$ we can set

$$u^y = \prod_{j > -\infty} (1 + m^{p^j})^{c_j}.$$

A slightly nasty computation shows that the two definitions of u^y coincide for $y \in \mathbb{Z}_p$ and $u \in U_1$. Since we have $u^{x+y} = u^x u^y$, the above definition then is the unique extension of the map defined on $\mathbb{Z}_p$ to $\mathbb{Q}_p$. □

Recall that an abelian group G is divisible if for every positive integer n and every $g \in G$ there is an element $h \in G$ such that $h^n = g$. This is equivalent to G being an injective object in the category of abelian groups. The group G is uniquely divisible if this h is unique.

COROLLARY 1.1.12. *The group $\hat{U}_1$ is uniquely divisible.*

PROOF. Let $g \in \hat{U}_1$ and n be a natural number. Then $1/n \in \mathbb{Q}_p$, so we can set $h = g^{1/n}$. This h is unique since $\hat{U}_1$ is torsion-free. □

Since thus $\hat{U}_1$ is injective, if G is an abelian group with a subgroup H any morphism $\phi : H \to \hat{U}_1$ extends to a morphism $\tilde{\phi} : G \to \hat{U}_1$. Furthermore, if G/H is finite then this extension is unique. Indeed, let $g \in G$. Then $g^n \in H$ for some natural number n and we must have $\phi(g^n) = \tilde{\phi}(g)^n$, which uniquely determines $\tilde{\phi}(g)$.

In applying the above to our case we want to identify K^+ with a subset of the fractional ideals J_K of $\mathcal{O}$. We say that a fractional ideal $\mathfrak{a}$ of $\mathcal{O}$ is positively generated if $\mathfrak{a} = (a)$ for some $a \in K^+$.

LEMMA 1.1.13. *The map $K^+ \to J_K$ given by $a \mapsto (a)$ is injective. That is, every positively generated principal ideal of $\mathcal{O}$ has a unique positive generator.*

PROOF. Let $a \in K^+$ be a generator of $\mathfrak{a} \in J_K$. Since $\mathcal{O}^* = \mathbb{F}_q^\times$ the set of generators of $\mathfrak{a}$ as a principal ideal is $\mathbb{F}_q^\times a$, and the only positive element here is a itself. □

If J_K is the set of fractional ideals of K the quotient J_K/K^+ is finite, where we identify K^+ with the set of positively generated fractional ideals by the lemma above. Indeed, we can write down an exact sequence

$$0 \to K^\times/K^+ \to J_K/K^+ \to J_K/K^\times \to 0.$$

Since both $K^\times/K^+$, which isomorphic to the image of $\operatorname{sgn} : K^\times \to F_q^\times$, and $J_K/K^\times$, which is nothing but the ideal class group of K, are finite, this implies that J_K/K^+ is finite.

Hence $\langle\cdot\rangle : K^+ \to \hat{U}_1$ extends uniquely to a map $\langle\cdot\rangle : J_K \to \hat{U}_1$.

DEFINITION 1.1.14. For a fractional ideal $\mathfrak{a}$ of K and an element $s = (x, y) \in S_\infty$ define

$$\mathfrak{a}^s = x^{\deg \mathfrak{a}} \langle \mathfrak{a} \rangle^y.$$

We then have $\mathfrak{a}^{s_1+s_2} = \mathfrak{a}^{s_1}\mathfrak{a}^{s_2}$ and $(\mathfrak{a}\mathfrak{b})^s = \mathfrak{a}^s\mathfrak{b}^s$. Furthermore, if $\mathfrak{a}$ is generated by a positive element $a \in K^+$ we have $\mathfrak{a}^s = a^s$.

LEMMA 1.1.15. *There is an isomorphism of groups between the group $\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*$ and the fractional ideals of $\mathcal{O}$ given by*

$$g \mapsto (g\hat{\mathcal{O}}) \cap K.$$

PROOF. By [**32**, Theorem 2, p. 84] there is a bijection between the set of fractional ideals of $\mathcal{O}$ and the set of $\hat{\mathcal{O}}$-modules $\Lambda \subset \mathbb{A}_{K,f}$ such that the v-component $\Lambda_v = \mathcal{O}_v$ for almost all finite places v of K (the theorem is stated for number fields, but the proof is identical in the function field case). Since $\hat{\mathcal{O}}$ is a principal ideal domain, such modules are given by $g\hat{\mathcal{O}}$ for some $g \in \mathbb{A}_{K,f}^*$. Hence the given map is an isomorphism. □

1.2. Drinfeld modules

Drinfeld modules were introduced in [**10**], generalizing a construction used by Carlitz [**3**] to almost construct the maximal abelian extension of $\mathbb{F}_q(t)$. This section summarizes the main properties of Drinfeld modules needed in this thesis. For a more complete account of Drinfeld modules see [**14, 17**], to which we also refer for some proofs.

1.2.1. Analytic construction. The most transparent way to discuss Drinfeld modules is probably by starting from the viewpoint of $\mathcal{O}$-lattices in $\mathbb{C}_\infty$. Let us work with the following definition:

Definition 1.2.1. An $\mathcal{O}$-lattice in $\mathbb{C}_\infty$ is a discrete finitely generated $\mathcal{O}$-submodule of $\mathbb{C}_\infty$.

Recall that a subgroup $\Lambda \subset \mathbb{C}_\infty$ is said to be discrete if there is a neighborhood U of $0 \in \mathbb{C}_\infty$ such that $U \cap \Lambda = \{0\}$. We will generally denote such an object simply by "lattice" unless this is likely to cause confusion.

Theorem 1.2.2. *The set of rank one lattices is parametrized by* $\mathbb{C}_\infty^\times \times_{K^\times} \mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*$ *via*

$$(\xi, r) \mapsto \xi(r\hat{\mathcal{O}} \cap K).$$

Proof. For each $(\xi, r) \in \mathbb{C}_\infty^\times \times_{K^\times} \mathbb{A}_{K,f}^*/\hat{\mathcal{O}}$, the set $\xi(r\hat{\mathcal{O}} \cap K)$ is clearly a lattice. We need to show that the representation is unique.

Assuming $\xi(r\hat{\mathcal{O}} \cap K) = \xi'(r'\hat{\mathcal{O}} \cap K)$, if we multiply with K we get $\xi K = \xi' K$, so $g = (\xi')^{-1}\xi \in K^\times$. Then $g(r\hat{\mathcal{O}} \cap K) = r'\hat{\mathcal{O}} \cap K$, and taking the completion in $\hat{\mathcal{O}}$ we get $gr\hat{\mathcal{O}} = r'\hat{\mathcal{O}}$. Thus $r' = gru$ for some $u \in \hat{\mathcal{O}}^*$. Thus $(\xi', r') = (\xi g^{-1}, gru)$, so they are representatives of the same equivalence class in $\mathbb{C}_\infty^\times \times_{K^\times} \mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*$. Hence the map is injective.

For surjectivity, let $\Lambda \subset \mathbb{C}_\infty$ be a lattice. Choose some $x \in K\Lambda$ and find an element $\xi \in \mathbb{C}_\infty$ such that $\xi^{-1}x \in K$. Then $\xi^{-1}\Lambda \subset K$ is an $\mathcal{O}$-submodule of K, so by Lemma 1.1.15 there is an $r \in \mathbb{A}_{K,f}^*$ such that $\xi^{-1}\Lambda = r\hat{\mathcal{O}} \cap K$. Thus $\Lambda = \xi(r\hat{\mathcal{O}} \cap K)$. □

We associate to a lattice $\Lambda \subset \mathbb{C}_\infty$ a function $e_\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty$ which we call the exponential function of Λ by

$$e_\Lambda(x) = x \prod_{\alpha \in \Lambda \setminus \{0\}} (1 - x/\alpha).$$

This product converges for all $x \in \mathbb{C}_\infty$. Indeed, since Λ is discrete, for all $r > 0$ there are only finitely many $\alpha \in \Lambda$ such that $|\alpha| < r$, so $1 - x/\alpha$ converges to 1 as α runs through Λ.

Lemma 1.2.3. *The exponential map e_λ is $\mathbb{F}_q$-linear, and e_Λ induces an isomorphism of Abelian groups $\mathbb{C}_\infty/\Lambda \to \mathbb{C}_\infty$.*

Proof. See [**14**] Proposition 4.2.5 and Corollary 4.2.6. □

Definition 1.2.4. For a lattice $\Lambda \subset \mathbb{C}_\infty$ and $a \in \mathcal{O}$, let

$$\phi_a^\Lambda(x) = x \prod_{\substack{\alpha \in a^{-1}\Lambda/\Lambda \\ \alpha \neq 0}} (1 - x/e_\Lambda(\alpha)).$$

Then $a \mapsto \phi_a^\Lambda$ is the Drinfeld module associated to Λ.

By [**14**, Corollary 1.2.2], the map $x \mapsto \phi_a^\Lambda(x)$ is $\mathbb{F}_q$-linear since its set of roots is an $\mathbb{F}_q$-vector space. This implies that ϕ_a^Λ can be written as a polynomial in $\tau = x^q$. (Note that $\tau^i(x) = x^{q^i}$ and in particular $\tau^0(x) = x$.) We will in the following consider ϕ_a^Λ to be an element of the ring $\mathbb{C}_\infty\{x^q\}$ of polynomials in x^q with coefficients in $\mathbb{C}_\infty$, where the multiplication is given by composition.

Remark 1.2.5. This is the basis for an alternative definition of a Drinfeld module as a homomorphism $\phi : \mathcal{O} \to \mathbb{C}_\infty\{x^q\}$ such that

(i) $D(\phi_a) = a$ for all $a \in \mathcal{O}$, where $D(\sum a_i x^{q^i}) = a_0$;

(ii) there is some $a \in \mathcal{O}$ such that $\phi_a \neq ax^0$.

It can be shown [**14**, Section 4.6][**17**, Section 8] that this definition is equivalent to the one given in this section in terms of lattices.

If one accepts that the exponential function of a lattice is an interesting object, then the motivation to study the Drinfeld module of the lattice is obvious from the following proposition. If one does not accept this, then the results summarized later in the current section should be convincing that at least Drinfeld modules themselves are interesting.

Proposition 1.2.6. *Given a lattice Λ and $a \in \mathcal{O}$ we have*

$$e_\Lambda(ax) = \phi_a^\Lambda(e_\Lambda(x)) \qquad \forall x \in \mathbb{C}_\infty.$$

Proof. See [**14**] Theorem 4.3.1 □

Note that this result can be interpreted as saying that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}_\infty/\Lambda & \xrightarrow{a} & \mathbb{C}_\infty/\Lambda \\ \downarrow{\scriptstyle e_\Lambda} & & \downarrow{\scriptstyle e_\Lambda} \\ \mathbb{C}_\infty & \xrightarrow{\phi_a^\Lambda} & \mathbb{C}_\infty. \end{array}$$

1.2.2. Rank.

LEMMA 1.2.7. *Let $\Lambda \subset \mathbb{C}_\infty$ be an $\mathcal{O}$-module. Then there are ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ such that as $\mathcal{O}$-modules*

$$\Lambda \simeq \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r.$$

PROOF. Since $\mathbb{C}_\infty$ is a field and the $\mathcal{O}$-module structure on $\mathbb{C}_\infty$ is given by the multiplication in the field, the $\mathcal{O}$-module $\mathbb{C}_\infty$ is torsion free. Hence Λ is also torsion free as an $\mathcal{O}$-module. Hence Λ is a torsion free finitely generated $\mathcal{O}$-module, and as $\mathcal{O}$ is a Dedekind ring Λ is projective. Hence there is an integer $r \geq 1$ and ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ of $\mathcal{O}$ such that

$$\Lambda \simeq \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r.$$

□

The integer r of the lemma is the rank of Λ. If ϕ^Λ is the Drinfeld module associated with Λ, we say that the rank of ϕ^Λ is r as well.

This rank can also be read out of the Drinfeld module itself. Indeed, let ϕ be a Drinfeld module and define $\nu_\phi : \mathcal{O} \to \mathbb{Z}$ by

$$\nu_\phi(a) = -\deg \phi_a(\tau)$$

(where we consider the degree in $\tau = x^q$). Then one can show, see for instance [**14**, Lemma 4.5.1, Proposition 4.5.3], that there is an integer r such that

$$\nu_\phi(a) = -r \deg(a).$$

One can furthermore show (see e.g. [**17**, Theorem 8.12]) that the rank defined in this fashion coincides with the rank as defined in terms of the underlying lattice.

If Λ_1 and Λ_2 are two lattices with the same rank, a morphism from Λ_1 to Λ_2 is an element $c \in \mathbb{C}_\infty$ with $c\Lambda_1 \subseteq \Lambda_2$.

PROPOSITION 1.2.8. *If $c : \Lambda_1 \to \Lambda_2$ is a morphism and if ϕ and ψ are the Drinfeld modules associated to Λ_1 and Λ_2 respectively, then there is a polynomial P in x^q with coefficients in $\mathbb{C}_\infty$ such that*

$$P\phi_a = \psi_a P$$

for all $a \in \mathcal{O}$.

PROOF. Consider the function $e_{\Lambda_2}(cx)$. By definition it is zero on $c^{-1}\Lambda_2 \supseteq \Lambda_1$. Since Λ_1 and $c^{-1}\Lambda_2$ have the same rank we see that $c^{-1}\Lambda_2/\Lambda_1$ is finite. Thus we can define

$$P(x) = cx \prod_{\substack{\alpha \in c^{-1}\Lambda_2/\Lambda_1 \\ \alpha \neq 0}} (1 - x/e_{\Lambda_1}(\alpha)).$$

Then $P(x)$ is $\mathbb{F}_q$-linear and $P(e_{\Lambda_1}(x))$ has a simple zero at each point of $c^{-1}\Lambda_2$ with derivative c. Hence

$$P(e_{\Lambda_1}(x)) = e_{\Lambda_2}(cx).$$

Now for $a \in \mathcal{O}$ we have

$$P\phi_a(e_{\Lambda_1}(x)) = P(e_{\Lambda_1}(ax)) = e_{\Lambda_2}(cax) = \psi_a(e_{\Lambda_2}(cx)) = \psi_a(P(e_{\Lambda_1}(x))),$$

so as e_{Λ_1} is surjective we get $P\phi_a = \psi_a P$ as claimed. □

By the proposition, we say a polynomial P in x^q with coefficients in $\mathbb{C}_\infty$ is a morphism from ϕ to ψ if $P\phi_a = \psi_a P$ for all $a \in \mathcal{O}$. Note that for P to be invertible we must have $P \in \mathbb{C}_\infty^\times$.

Remark 1.2.9. In the current thesis we will assume that all lattices and Drinfeld modules considered are of rank one.

1.2.3. Sign-normalization. Let ϕ be a (rank one) Drinfeld module and define $\mu_\phi(a) \in \mathbb{C}_\infty$ to be the leading coefficient of ϕ_a. Then

$$\mu_\phi(ab) = \mu_\phi(a)\mu_\phi(b),$$

so we can extend μ_ϕ to a map $\mu_\phi : K \to \mathbb{C}_\infty$.

Definition 1.2.10. Say that ϕ is normalized if $\mu_\phi(x) \in \mathbb{F}_\infty$ for all $x \in K$. If for some sign-function sgn there is an element σ in $\mathrm{Gal}(\mathbb{F}_\infty/\mathbb{F}_q)$ such that $\mu_\phi = \sigma \circ \mathrm{sgn}$ we say that ϕ is sgn-normalized.

Sign-normalized rank one Drinfeld modules are occasionally called Hayes modules [**14**, p. 199], but we will not use this terminology.

Theorem 1.2.11. *Let ϕ be a Drinfeld module and* sgn *be a sign function. Then ϕ is isomorphic over $\mathbb{C}_\infty$ to a* sgn*-normalized Drinfeld module.*

Proof. Consider K_∞, and let π_∞ be a uniformizer which is positive with respect to sgn. Then choose $\xi \in \mathbb{C}_\infty$ in such a way that we have $\xi^{q^{\deg\infty}-1} = 1/\mu_\phi(\pi_\infty^{-1})$. Then if $\psi = \xi\phi\xi^{-1}$ we get $\mu_\psi(\pi_\infty^{-1}) = 1$.

Write $x \in \mathcal{O}$ as $x = \zeta\pi_\infty^j u$ with $\zeta \in \mathbb{F}_\infty^\times$ and $u \in U_1$. Then

$$\mu_\psi(x) = \zeta = \mathrm{sgn}(x)$$

so ψ is sgn-normalized. □

1.2.4. Group actions on Drinfeld modules. There are two group actions on Drinfeld modules which will be relevant to our discussion: The action of the group of fractional ideals J_K and the action of certain Galois groups. Let us first consider the ideal action.

Lemma 1.2.12. *Every left ideal in $\mathbb{C}_\infty\{x^q\}$ is principal.*

PROOF. See [**17**] Proposition 4.1. □

Let $\mathfrak{a} \subset \mathcal{O}$ be an integral ideal and ϕ a Drinfeld module. Let $\mathbb{C}_\infty\{x^q\}\phi_\mathfrak{a}$ be the left ideal in $\mathbb{C}_\infty\{x^q\}$ generated by the ϕ_a for $a \in \mathfrak{a}$. By the lemma this is a principal left ideal. We write $\phi_\mathfrak{a}$ for its monic generator in $\mathbb{C}_\infty\{x^q\}$.

For $x \in \mathcal{O}$ we see that $\mathbb{C}_\infty\{x^q\}\phi_\mathfrak{a}\phi_x \subset \mathbb{C}_\infty\{x^q\}\phi_\mathfrak{a}$. Indeed, elements of $\mathbb{C}_\infty\{x^q\}\phi_\mathfrak{a}$ can be written as

$$\sum_{a\in\mathfrak{a}} \phi_{x_a}\phi_a,$$

and multiplying by ϕ_x on the right preserves this representation since $\mathfrak{a}$ is an ideal. Hence there is some $\phi'_x \in \mathbb{C}_\infty\{x^q\}$ such that

$$\phi_\mathfrak{a}\phi_x = \phi'_x\phi_\mathfrak{a}.$$

LEMMA 1.2.13. *The map $x \mapsto \phi'_x$ is a Drinfeld module.*

PROOF. See [**17**] Section 4. □

We write $\mathfrak{a} * \phi$ for the Drinfeld module ϕ'. We then have the following:

LEMMA 1.2.14. *The ideal action on Drinfeld modules has the following properties:*

(i) *If $\mathfrak{a} = a\mathcal{O}$ is a principal ideal and μ is the leading coefficient of ϕ_a, then $\phi_\mathfrak{a} = \mu^{-1}\phi_a$ and $(\mathfrak{a} * \phi)_x = \mu^{-1}\phi_x\mu$ for $x \in \mathcal{O}$;*

(ii) *If $\mathfrak{a}, \mathfrak{b}$ are ideals of $\mathcal{O}$ then*

$$\phi_{\mathfrak{a}\mathfrak{b}} = (\mathfrak{b} * \phi)_\mathfrak{a}\phi_\mathfrak{b}, \qquad \mathfrak{a} * (\mathfrak{b} * \phi) = (\mathfrak{a}\mathfrak{b}) * \phi.$$

PROOF. For (i), note that $\mathbb{C}_\infty\{x^q\}\phi_a$ is generated by $\mu^{-1}\phi_a$ which is a monic polynomial. For the second part we calculate

$$\mu^{-1} \cdot \phi_x \cdot \mu \cdot \phi_\mathfrak{a} = \mu^{-1} \cdot \phi_x \cdot \phi_a = \mu^{-1}\phi_{xa} = \mu^{-1}\phi_a\phi_x = \phi_\mathfrak{a} \cdot \phi_x,$$

proving the claim.

For (ii), since $\mathcal{O}$ is a Dedekind domain every integral ideal of $\mathcal{O}$ can be generated by two elements (see e.g. [**5**, Corollary 10.6.4]). Assume $\mathfrak{a}$ and $\mathfrak{b}$ have generating pairs (a, a') and (b, b') respectively. Then $\mathfrak{a}\mathfrak{b}$ is generated by $ab, ab', a'b, a'b'$. Consider the left ideal in $\mathbb{C}_\infty\{x^q\}$

generated by $(\mathfrak{b} * \phi)_{\mathfrak{a}}\phi_{\mathfrak{b}}$. We get

$$\begin{aligned}\mathbb{C}_\infty\{x^q\}(\mathfrak{b} * \phi)_{\mathfrak{a}}\phi_{\mathfrak{b}} &= \mathbb{C}_\infty\{x^q\}(\mathfrak{b} * \phi)_a\phi_{\mathfrak{b}} + \mathbb{C}_\infty\{x^q\}(\mathfrak{b} * \phi)_{a'}\phi_{\mathfrak{b}} \\ &= \mathbb{C}_\infty\{x^q\}\phi_{\mathfrak{b}}\phi_a + \mathbb{C}_\infty\{x^q\}\phi_{\mathfrak{b}}\phi_{a'} \\ &= \mathbb{C}_\infty\{x^q\}\phi_b\phi_a + \mathbb{C}_\infty\{x^q\}\phi_{b'}\phi_a \\ &\quad + \mathbb{C}_\infty\{x^q\}\phi_b\phi_{a'} + \mathbb{C}_\infty\{x^q\}\phi_{b'}\phi_{a'} \\ &= \mathbb{C}_\infty\{x^q\}\phi_{ab} + \mathbb{C}_\infty\{x^q\}\phi_{ab'} \\ &\quad + \mathbb{C}_\infty\{x^q\}\phi_{a'b} + \mathbb{C}_\infty\{x^q\}\phi_{a'b'} \\ &= \mathbb{C}_\infty\{x^q\}\phi_{\mathfrak{ab}}.\end{aligned}$$

Since $\phi_{\mathfrak{ab}}$ is the unique monic generator of this left ideal and $(\mathfrak{b} * \phi)_{\mathfrak{a}}\phi_{\mathfrak{b}}$ is monic we get $\phi_{\mathfrak{ab}} = (\mathfrak{b} * \phi)_{\mathfrak{a}}\phi_{\mathfrak{b}}$.

The second equality follows, since

$$\begin{aligned}(\mathfrak{a} * (\mathfrak{b} * \phi))_a(\mathfrak{b} * \phi)_{\mathfrak{a}}\phi_{\mathfrak{b}} &= (\mathfrak{b} * \phi)_{\mathfrak{a}}(\mathfrak{b} * \phi)_a\phi_{\mathfrak{b}} \\ &= (\mathfrak{b} * \phi)_{\mathfrak{a}}\phi_{\mathfrak{b}}\phi_a = \phi_{\mathfrak{ab}}\phi_a \\ &= (\mathfrak{ab} * \phi)_a\phi_{\mathfrak{ab}} \\ &= (\mathfrak{ab} * \phi)_a(\mathfrak{b} * \phi)_{\mathfrak{a}}\phi_{\mathfrak{b}}\end{aligned}$$

for all $a \in \mathcal{O}$. □

The action of integral ideals extends in the obvious way to an action of J_K on the set of Drinfeld modules.

In certain cases there is also a Galois action on Drinfeld modules. Indeed, assume the Drinfeld module ϕ has coefficients contained in some Galois extension L of K. Then there is an obvious action of $\mathrm{Gal}(L/K)$ on ϕ given by $(\sigma\phi)_a = \sigma(\phi_a)$, where σ acts on the coefficients. Since σ fixes K this $\sigma\phi$ is again a Drinfeld module with coefficients in L. Furthermore, this action commutes with the ideal action above since $\sigma(\phi_{\mathfrak{a}}) = (\sigma\phi)_{\mathfrak{a}}$.

1.3. Explicit class field theory for function fields

The problem of constructing explicit class fields originates in the statement of Hilbert's 12th problem asking for an extension of the Kronecker-Weber theorem on abelian extensions of $\mathbb{Q}$ to arbitrary number fields. For function fields the problem was solved in the case of $\mathbb{F}_q(t)$ by Carlitz [**3**], and in the general case by Drinfeld [**10, 11**]. A more explicit (and probably more readable) account is found in [**17**], which we follow here. We would like to point to that paper for proofs as well.

For this section let K be a function field with ∞ a distinguished place, sgn a sign function on K_∞, and let ϕ be a sgn-normalized rank one Drinfeld module.

For any $a \in \mathcal{O}$, write

$$\phi_a = a + \sum_{k=1}^{\deg a} c_k(\phi, a) x^{q^k}$$

where $c_k(\phi, a) \in \mathbb{C}_\infty$. Then for $\xi \in \mathbb{C}_\infty^\times$ we have

$$c_k(\xi \phi \xi^{-1}, a) = \xi^{1-r^k} c_k(\phi, a).$$

Now let $k_1, \ldots, k_s$ be the indices (depending on a) such that $c_{k_i}(\phi, a)$ is non-zero, and let g be the greatest common divisor of the numbers $r^{k_i} - 1$. We can then write

$$g = \sum_{i=1}^{s} e_i(r^{k_i} - 1)$$

for some integers e_i. Let

$$I_{k_i}(\phi, a) = c_{k_i}(\phi, a) \cdot \left(\prod_{j=1}^{s} c_{k_j}(\phi, a)^{e_j} \right)^{(1-r^{k_i})/g}.$$

Then a straightforward calculation shows that $I_{k_i}(\phi, a)$ depends only on a and the isomorphism class of ϕ, since $I_{k_i}(\phi, a) = I_{k_i}(\xi\phi\xi^{-1}, a)$ for all $\xi \in \mathbb{C}_\infty^\times$.

Definition 1.3.1. Let ϕ be a sgn-normalized rank one Drinfeld module and let $a \in K \setminus \mathbb{F}_q$. Then let H be the field extension of K generated by the $I_{k_i}(\phi, a)$ doe all k_i.

It can be shown [**17**, Theorem 15.4] that this extension is independent of the choices of ϕ and a. Furthermore, there is the following:

Theorem 1.3.2. *The extension H/K is completely split over ∞ and unramified at every finite place of K. We have*

$$\mathrm{Gal}(H/K) \simeq \mathrm{Pic}(\mathcal{O}),$$

and if ϕ is a Drinfeld module with coefficients in H then

$$\sigma_{\mathfrak{a}}\phi \simeq \mathfrak{a} * \phi$$

for every non-zero ideal $\mathfrak{a}$ of $\mathcal{O}$, where $\mathfrak{a} \mapsto \sigma_{\mathfrak{a}}$ is the Artin map.

The field H is known as the Hilbert class field [**28**] of $\mathcal{O}$.

Definition 1.3.3. Let ϕ be a sgn-normalized rank one Drinfeld module over $\mathcal{O}$ and let $a \in K \setminus \mathbb{F}_q$. Let H^+ be the field extension of K generated by the coefficients of ϕ_a.

Again this extension is independent of the choice of ϕ and a. We clearly have $H \subset H^+$.

Let $\mathcal{P}^+ = \{(a) \in J_K : \mathrm{sgn}(a) = 1\}$ be the set of positively generated principal fractional ideals of K. The quotient $\mathrm{Pic}^+(\mathcal{O}) = J_K/\mathcal{P}^+$ is called the narrow class group of $\mathcal{O}$ relative to sgn. We have the following result:

THEOREM 1.3.4. *The extension H^+/K is unramified at every finite place of K, and is Galois with Galois group isomorphic to* $\mathrm{Pic}^+(\mathcal{O})$*. For every non-zero ideal $\mathfrak{a}$ of $\mathcal{O}$ we have*

$$\sigma_{\mathfrak{a}}\phi = \mathfrak{a} * \phi$$

for every sgn*-normalized rank one Drinfeld module ϕ, where $\mathfrak{a} \mapsto \sigma_{\mathfrak{a}}$ is the Artin map.*

Given a non-zero element $a \in \mathcal{O}$ we consider ϕ_a as a map $\phi_a : \mathbb{C}_\infty \to \mathbb{C}_\infty$, and we write $\Lambda_\phi(a) = \{\zeta \in \mathbb{C}_\infty : \phi_a(\zeta) = 0\}$ for the kernel of this map. Similarily, if $\mathfrak{m}$ is a non-zero proper ideal of $\mathcal{O}$ we write

$$\Lambda_\phi(\mathfrak{m}) = \{\zeta \in \mathbb{C}_\infty : \phi_a(\zeta) = 0 \quad \forall a \in \mathfrak{m}\}$$

for the set of torsion points with respect to the action of $\mathfrak{m}$.

DEFINITION 1.3.5. Let $K_{\mathfrak{m}}$, the narrow ray class extension modulo $\mathfrak{m}$, be the extension of H^+ generated by $\Lambda_\phi(\mathfrak{m})$.

This extension is independent of the choice of ϕ. If we write $I_{\mathfrak{m}}$ for the semigroup of ideals of $\mathcal{O}$ which are relatively prime to $\mathfrak{m}$ and let

$$\mathcal{P}_{\mathfrak{m}} = \{(a) : a \in \mathcal{O}^\times, a \equiv 1 \mod \mathfrak{m}\},$$

we have the following:

THEOREM 1.3.6. *$K_{\mathfrak{m}}$ is a finite abelian extension of K, unramified away from ∞ and the prime ideals dividing $\mathfrak{m}$. It is independent of the choice of ϕ. The Artin map $\sigma : I_{\mathfrak{m}} \to \mathrm{Gal}(K_{\mathfrak{m}}/K)$ defines an isomorphism $\mathrm{Pic}^+_{\mathfrak{m}}(\mathcal{O}) \simeq \mathrm{Gal}(K_{\mathfrak{m}}/K)$, and for any $\mathfrak{a} \in I_{\mathfrak{m}}$ and $\lambda \in \phi[\mathfrak{m}]$ we have*

$$\sigma_{\mathfrak{a}}(\lambda) = \phi_{\mathfrak{a}}(\lambda).$$

The subfield $K^+_{\mathfrak{m}}$ of $K_{\mathfrak{m}}$ of elements fixed by the semigroup $\mathcal{P}_{\mathfrak{m}}$ is contained in K_∞, so the extension $K^+_{\mathfrak{m}}/K$ is completely split at infinity. If we continue to denote by ∞ the extension of the place ∞ of K to $K^+_{\mathfrak{m}}$ defined by the inclusion $K^+_{\mathfrak{m}} \subset K_\infty$, then $K_{\mathfrak{m}}/K^+_{\mathfrak{m}}$ is totally ramified at ∞.

Since $\mathrm{Gal}(K_{\mathfrak{m}}/K) \simeq \mathrm{Pic}^+_{\mathfrak{m}}(\mathcal{O})$ we have

$$\mathrm{Gal}(K^+_{\mathfrak{m}}/K) \simeq \mathrm{Pic}_{\mathfrak{m}}(\mathcal{O}) = I_{\mathfrak{m}}/\mathcal{P}_{\mathfrak{m}}.$$

Let $\mathbb{K} \subset \mathbb{C}_\infty$ be the union of the fields $K_{\mathfrak{m}}$ over non-zero proper ideals $\mathfrak{m}$ of $\mathcal{O}$ and let $K^{\mathrm{ab},\infty} \subset K_\infty$ be the union of the fields $K^+_{\mathfrak{m}}$. The field $K^{\mathrm{ab},\infty}$ is the maximal abelian etension of K in which ∞ splits completely.

Proposition 1.3.7. *The Artin map* $\mathbb{A}^*_{K,f} \to \mathrm{Gal}(K^{ab}/K)$ *induces isomorphisms* $K^+\backslash\mathbb{A}^*_{K,f} \simeq \mathrm{Gal}(\mathbb{K}/K)$ *and* $K^\times\backslash\mathbb{A}^*_{K,f} \simeq \mathrm{Gal}(K^{\mathrm{ab},\infty}/K)$.

Proof. Let us start with the first isomorphism. Recall that

$$\mathbb{K} = \bigcup_{\mathfrak{m}} K_{\mathfrak{m}},$$

so we start by computing the Artin map $r_{\mathfrak{m}} : \mathbb{A}^*_K \to \mathrm{Gal}(K_{\mathfrak{m}}/K)$.

Let $K^+_{\mathfrak{m},1} = \{x \in K^+ : x \equiv 1 \mod \mathfrak{m}\}$, where we write $x \equiv 1 \mod \mathfrak{m}$ if there are $a, b \in \mathcal{O}$ relatively prime to $\mathfrak{m}$ such that $x = a/b$ with $a \equiv b \mod \mathfrak{m}$. Then put $\mathcal{P}^+_{\mathfrak{m}} = \{(x) \in J_K : x \in K^+_{\mathfrak{m},1}\}$. We furthermore write $S(\mathfrak{m})$ for the set of places v such that $\mathfrak{p}_v|\mathfrak{m}$ together with ∞. Then $S(\mathfrak{m})$ is finite, so it makes sense to consider $\mathbb{A}_{S(\mathfrak{m})}$ and $\hat{\mathcal{O}}_{S(\mathfrak{m})}$.

We then have a commutative diagram

$$\begin{array}{ccccc}
\mathbb{A}^*_{S(\mathfrak{m})} & \longrightarrow & \mathbb{A}^*_K & \longrightarrow & \mathbb{A}^*_K/K^\times \\
\downarrow & & \downarrow r_{\mathfrak{m}} & \swarrow & \\
\mathbb{A}^*_{S(\mathfrak{m})}/\hat{\mathcal{O}}^*_{S(\mathfrak{m})}K^+_{\mathfrak{m},1} \xrightarrow{\simeq} \mathrm{Pic}^+_{\mathfrak{m}}(\mathcal{O}) & \xrightarrow{\simeq} & \mathrm{Gal}(K_{\mathfrak{m}}/K) & &
\end{array}$$

If $\mathfrak{m} = \prod_{r\in S(\mathfrak{m})} \mathfrak{p}_v^{n_v}$, put $U_{\mathfrak{m}} = \hat{\mathcal{O}}^*_{S(\mathfrak{m})} \times \prod_{v\in S(\mathfrak{m})}(1+\hat{\mathfrak{p}}_v^{n_v}) \subset \hat{\mathcal{O}}^*$, where $\hat{\mathfrak{p}}_v$ is the closure of $\mathfrak{p}_v$ in $\mathcal{O}_v$. Then by the diagram $\ker r_{\mathfrak{m}}$ contains $\hat{\mathcal{O}}^*_{S(\mathfrak{m})}(K^+_{\mathfrak{m},1})_{S(\mathfrak{m})}K^\times$, where

$$(K^+_{\mathfrak{m},1})_{S(\mathfrak{m})} = \{(x,1) : x \in K^+_{\mathfrak{m},1}\} \subset \mathbb{A}^*_{S(\mathfrak{m})} \times \left(\prod_{v\in S(\mathfrak{m})} K_v\right).$$

Now, by weak approximation, $K^+_{\mathfrak{m},1}$ is dense in $\prod_{v\in S(\mathfrak{m})}(1+\hat{\mathfrak{p}}_v^{n_v}) \times K^+_\infty$, whence

$$\ker r_{\mathfrak{m}} \supset (U_{\mathfrak{m}} \times K^+_\infty)K^\times.$$

Furthermore, the subgroup $(U_{\mathfrak{m}} \times K^+_\infty)K^\times$ is closed, and the map $\mathbb{A}^*_{S(\mathfrak{m})} \to \mathbb{A}^*_K$ induces an isomorphism

$$\mathbb{A}^*_{S(\mathfrak{m})}/\hat{\mathcal{O}}^*_{S(\mathfrak{m})}K^+_{\mathfrak{m},1} \simeq \mathbb{A}^*_K/(U_{\mathfrak{m}} \times K^+_\infty)K^\times,$$

whence $\ker r_{\mathfrak{m}} = (U_{\mathfrak{m}} \times K_\infty^+)K^\times$, so the Artin map induces an isomorphism

$$\mathbb{A}_K^*/(U_{\mathfrak{m}} \times K_\infty^+)K^\times \simeq \mathrm{Gal}(K_{\mathfrak{m}}/K).$$

Since $\bigcap_{\mathfrak{m}} \ker r_{\mathfrak{m}} = K_\infty^+ K^\times$ it follows that the Artin map

$$r_{\mathbb{K}/K} : \mathbb{A}_K^* \to \mathrm{Gal}(\mathbb{K}/K)$$

induces an isomorphism

$$\mathbb{A}_K^*/K_\infty^+ K^\times \simeq \mathrm{Gal}(\mathbb{K}/K),$$

or equivalently $\mathbb{A}_{K,f}^*/K^+ \simeq \mathrm{Gal}(\mathbb{K}/K)$.

The proof of the second isomorphism is identical except that we show that $\ker r_{\mathfrak{m},+} = (U_{\mathfrak{m}} \times K_\infty^\times)K^\times$, whence the Artin map induces an isomorphism

$$\mathbb{A}_K^*/K_\infty^\times K^\times \simeq \mathrm{Gal}(K^{\mathrm{ab},\infty}/K),$$

or equivalently the desired isomorphism $\mathbb{A}_{K,f}^*/K^\times \simeq \mathrm{Gal}(K^{\mathrm{ab},\infty}/K)$. □

LEMMA 1.3.8. *The Artin map induces an isomorphism*

$$\mathrm{Gal}(\mathbb{K}/H^+) \simeq \hat{\mathcal{O}}^*.$$

PROOF. Recall that the Artin map induces an isomorphism

$$\mathrm{Gal}(K_{\mathfrak{m}}/H^+) \simeq (I_{\mathfrak{m}} \cap \mathcal{P}^+)/\mathcal{P}_{\mathfrak{m}}^+.$$

By Lemma 1.1.13 there is a well-defined map $I_{\mathfrak{m}} \cap \mathcal{P}^+ \to (\mathcal{O}/\mathfrak{m})^*$ given by $(a) \mapsto a \mod \mathfrak{m}$ for positive a. The kernel of this map is exactly $\mathcal{P}^+$, so since $|\,\mathrm{Gal}(K_{\mathfrak{m}}/H^+)| = |(\mathcal{O}/\mathfrak{m})^*|$ this map is surjective. Hence the Artin map induces an isomorphism $\mathrm{Gal}(K_{\mathfrak{m}}/H^+) \simeq (\mathcal{O}/\mathfrak{m})^*$, and we get

$$\mathrm{Gal}(\mathbb{K}/H^+) \simeq \lim_{\mathfrak{m}} \mathrm{Gal}(K_{\mathfrak{m}}/H^+) = \lim_{\mathfrak{m}} (\mathcal{O}/\mathfrak{m})^* = \hat{\mathcal{O}}^*.$$

□

LEMMA 1.3.9. *The constant fields of $K_{\mathfrak{m}}$ and $K_{\mathfrak{m}}^+$ are both equal to $\mathbb{F}_{q^{d_\infty}}$.*

PROOF. Let us first consider $K_{\mathfrak{m}}^+$. On the one hand, the residue field of $K_{\mathfrak{m}}^+$ at infinity is $\mathbb{F}_{q^{d_\infty}}$, since $K_{\mathfrak{m}}^+$ is completely split at infinity. On the other hand, $K_{\mathfrak{m}}^+$ contains the Hilbert class field H. Since $\mathbb{F}_{q^{d_\infty}}K/K$ is unramified at every prime and completely split at infinity we have $\mathbb{F}_{q^{d_\infty}}K \subset H \subset K_{\mathfrak{m}}^+$. Hence $\mathbb{F}_{q^{d_\infty}}$ is both the constant field of $K_{\mathfrak{m}}^+$ and the residue field of $K_{\mathfrak{m}}^+$ at infinity. Since $K_{\mathfrak{m}}/K_{\mathfrak{m}}^+$ is totally ramified at infinity the residue field of $K_{\mathfrak{m}}$ at infinity is $\mathbb{F}_{q^{d_\infty}}$. Hence the constant field of $K_{\mathfrak{m}}$ is also $\mathbb{F}_{q^{d_\infty}}$. □

Corollary 1.3.10. *The algebraic closures of* $\mathbb{F}_q$ *in* $\mathbb{K}$ *and* $K^{\mathrm{ab},\infty}$ *are both equal to* $\mathbb{F}_{q^{d_\infty}}$.

1.4. KMS-states for dynamical systems arising from partial group actions

In [**24**] the authors present a framework for analysing the KMS states of C*-algebras associated to a certain class of groupoids. This class contains, amongst others, the groupoids we will consider in this thesis. We include the statement of the main result here for convenience.

Let X be a second countable locally compact Hausdorff space and Y be a clopen subset of X. Assume that G is a countable discrete group acting on X such that $GY = X$. This gives rise to the transformation groupoid $G \times X$ whose elements are (g, x) with source map $(g, x) \mapsto x$ and range map $(g, x) \mapsto gx$.

Consider the subgroupoid

$$G \boxtimes Y = \{(g, x) : x \in Y, gx \in Y\}$$

and let $C_r^*(G \boxtimes Y)$ be the reduced C*-algebra of this groupoid. Then $C_r^*(G \boxtimes Y) \simeq \mathbf{1}_Y(C_0(X) \rtimes_r G)\mathbf{1}_Y$.

Let $N : G \to (0, +\infty)$ be a homomorphism, and equip $C_r^*(G \boxtimes Y)$ with the dynamics σ given by

$$\sigma_t(f)(g, x) = N(g)^{it} f(g, x)$$

for $t \in \mathbb{R}$ and $f \in C_c(G \boxtimes Y) \subset C_r^*(G \boxtimes Y)$.

For a state ϕ on $C_r^*(G \boxtimes Y)$ the restriction of ϕ to $C_0(Y)$ gives rise to a Borel probability measure μ on Y. Conversely, if we write E for the conditional expectation from $C_0(X) \times_r G$ to $C_0(X)$, a Borel probability measure μ on Y gives rise to a state $\mu_* \circ E$ on $C_r^*(G \boxtimes Y)$ since the restriction of E to $C_r^*(G \boxtimes Y)$ has image $C_0(Y)$.

Recall that for $\beta \in \mathbb{R}$, a KMS state at inverse temperature β is a state ϕ such that $\phi(ab) = \phi(b\sigma_{i\beta}(a))$ for a and b in a set of σ-analytic elements with dense linear span.

Theorem 1.4.1. *Let* G, X, Y *and* N *be as above, and suppose there exists a sequence* $\{Y_n\}_{n=1}^\infty$ *of Borel subsets of* Y *and a sequence* $\{g_n\}_{n=1}^\infty$ *of elements of* G *such that*

(i) $\bigcup_{n=1}^\infty Y_n$ *contains the set of points in* Y *with nontrivial isotropy with respect to the action of* G *on* X;

(ii) $N(g_n) \neq 1$ *for all* n;

(iii) $g_n Y_n = Y_n$ *for all* n.

Then for each $\beta \neq 0$ *the map* $\mu \mapsto \phi = (\mu_* \circ E)|_{C_r^*(G \boxtimes Y)}$ *is an affine isomorphism between Radon measures* μ *on* X *satisfying* $\mu(Y) = 1$ *and the scaling condition* $\mu(gZ) = N(g)^{-\beta}\mu(Z)$ *for Borel sets* $Z \subset X$ *and* $g \in G$*, and KMS*$_\beta$*-states* ϕ *on* $C_r^*(G \boxtimes Y)$*.*

If furthermore S *is a subset of* G *and* $Y_0 \subset Y$ *is a nonempty Borel set such that*

(iv) $gY_0 \cap Y_0 = \emptyset$ *for* $g \in G \setminus \{e\}$*;*
(v) $SY_0 \subset Y$*;*
(vi) *if* $gY_0 \cap Y \neq \emptyset$ *then* $g \in S$*;*
(vii) $Y \setminus SU \subset \bigcup_n Y_n$ *for every open set* U *containing* Y_0*;*
(viii) $\zeta_S(\beta) := \sum_{s \in S} N(s)^{-\beta} < +\infty$*;*

then

(1) *the map* $\phi : \mu_* \circ E \mapsto \zeta_S(\beta)\mu|_{Y_0}$ *is an affine isomorphism between the KMS*$_\beta$*-states on* $C_r^*(G \boxtimes Y)$ *and the Borel probability measures on* Y_0*; the inverse map is given by* $\nu \mapsto \mu_* \circ E$*, where* μ *is the measure on* Y *defined by*

$$\mu(Z) = \zeta_S(\beta)^{-1} \sum_{s \in S} N(s)^{-\beta} \nu(s^{-1}Z \cap Y_0);$$

(2) *if* μ *is a measure on* Y *defined by a probability measure* ν *on* Y_0 *as above, and* H_S *is the subspace of* $L^2(Y, \mu)$ *consisting of functions* f *with* $f(sy) = f(y)$ *for* $y \in Y_0$ *and* $s \in S$*, then for* $f \in H_S$ *we have*

$$\|f\|_2^2 = \zeta_S(\beta) \int_{Y_0} |f(y)|^2 d\mu(y)$$

and the orthogonal projection $P : L^2(Y, \mu) \to H_S$ *is given by*

$$Pf(Sy) = \zeta_S(\beta)^{-1} \sum_{s \in S} N(s)^{-\beta} f(sy)$$

for $y \in Y_0$*.*

1.5. Type III factors

We will need some results on the classification of factors of type III, which we assemble here for convenience. The following results are proved in the Appendix of [**27**].

Let (X, μ) be a standard measure space with σ-finite positive measure μ, and let $\mathcal{R} \subset X \times X$ be a non-singular countable measurable equivalence relation on (X, μ). That is, for each $A \subset X$ of measure zero, the minimal $\mathcal{R}$-invariant subset of X containing A also has measure zero, and each equivalence class is countable.

Since X is nonsingular there exists a Radon-Nikodym cocycle for $\mathcal{R}$ defined as follows: Let $T : A \to B$ be a measurable bijective map with graph in $\mathcal{R}$. Then for $x \in A$ put

$$c_\mu(x, Tx) = \frac{dT^{-1}\mu}{d\mu}(x).$$

The map $c_\mu : \mathcal{R} \to \mathbb{R}_+^\times$ can be shown to be measurable (with respect to a measure class on $\mathcal{R}$ whose projection onto X coincides with the measure class of μ) and to satisfy $c_\mu(x, z) = c_\mu(x, y)c_\mu(y, z)$ for $(x, y), (y, z) \in \mathcal{R}$.

Definition 1.5.1. The ratio set $r(\mathcal{R}, \mu)$ is the intersection of the essential ranges of the restrictions of c_μ to $\mathcal{R} \cap (Z \times Z)$ for all measurable subsets $Z \subset X$ of poitive measure. That is, for $Z \subset X$ of positive measure and $\epsilon > 0$, let $r_{Z,\epsilon}(\mathcal{R}, \mu)$ be the set of $\lambda \geq 0$ such that there are measurable subsets $A, B \subset Z$ of positive measure and a measurable bijection $T : A \to B$ with graph in $\mathcal{R}$ such that $|c_\mu(x, Tx) - \lambda| < \epsilon$ for all $x \in A$. Then

$$r(\mathcal{R}, \mu) = \bigcap r_{Z,\epsilon}(\mathcal{R}, \mu),$$

with the intersection taken over all Z of positive measure and all $\epsilon > 0$.

It follows from the definition that $r(\mathcal{R}, \mu)$ is a subgroup of $\mathbb{R}_+^\times$.

Proposition 1.5.2. *Let $\mathcal{R}$ be a non-singular countable measurable equivalence relation on a standard measure space (X, μ). Assume (Y, ν) is another standard measure space, and define an equivalence relation $\mathcal{R} \times \mathrm{id}$ on $X \times Y$ such that $(x, y) \sim_{\mathcal{R}\times\mathrm{id}} (x', y')$ if and only if $x \sim_{\mathcal{R}} x'$ and $y = y'$. Then*

$$r(\mathcal{R} \times \mathrm{id}) \setminus \{0\} = r(\mathcal{R}, \mu) \setminus \{0\}.$$

Proposition 1.5.3. *Let $\mathcal{R}$ be the orbit equivalence relation defined by an action of a countable group Γ on a standard measure space (X, μ) by non-singular transformations. Assume $\{\xi_n\}_{n=1}^\infty$ is an increasing sequence of Γ-invariant measurable partitions such that $\vee_n \xi_n$ is the partition into points. Also assume that the measure μ_n induced by μ on $X_n = X/\xi_n$ is σ-finite and the Radon-Nikodym cocycle c_μ is ξ_n-measurable. Let $\mathcal{R}_n$ be the orbit equivalence relation defined by the action of Γ on X_n. Then*

$$r(\mathcal{R}, \mu) \setminus \{0\} = \bigcap r(\mathcal{R}_n, \mu_n) \setminus \{0\}.$$

Assume that we have an ergodic countable measurable equivalence relation $\mathcal{R}$ on (X, μ). Since $r(\mathcal{R}, \mu) \setminus \{0\}$ is a closed subgroup of $\mathbb{R}_+^\times$, the ratio set must be one of $\{1\}$, $\{0, 1\}$, $\{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$ for some

$\lambda \in (0,1)$, and $[0,+\infty)$. In the last three cases the relation is said to be of type III_0, III_λ and III_1, respectively.

Let $\mathcal{R}^0$ be the kernel of c_μ, that is the set of $(x,y) \in \mathcal{R}$ such that $c_\mu(x,y) = 1$. Then $\mathcal{R}^0$ is a measurable equivalence relation, and μ is $\mathcal{R}^0$-invariant.

PROPOSITION 1.5.4. *Let $\mathcal{R}$ be an ergodic non-singular countable measurable equivalence relation on a standard measure space (X,μ). Assume $\mathcal{R}^0$ is ergodic. Then $r(\mathcal{R},\mu)$ coincides with the essential range c_μ.*

Conversely, assume $r(\mathcal{R},\mu)$ coincides with the essential range of c_μ. Also assume that $r(\mathcal{R},\mu) \setminus \{0\}$ is discrete, so $\mathcal{R}$ is not of type III_1. Then $\mathcal{R}^0$ is ergodic.

Now consider a class of equivalence relations. Let (X_n,μ_n) for $n = 1,2,\ldots$ be a sequence of at most countable probability spaces. Put $(X,\mu) = \prod_n (X_n,\mu_n)$, and define an equivalence relation $\mathcal{R}$ on X by

$$x \sim y \text{ if } x_n = y_n \text{ for all sufficiently large } n.$$

This equivalence relation is non-singular and ergodic.

For a finite set $I \subset \mathbb{N}$ and $a \in \prod_{n\in I} X_n$, let

$$Z(a) = \{x \in X : x_n = a_n \quad \forall n \in I\}.$$

DEFINITION 1.5.5. The asymptotic ratio set $r_\infty(\mathcal{R},\mu)$ consists of all $\lambda \geq 0$ such that for $\epsilon > 0$ there exists a sequence I_n of mutually disjoint finite subsets of $\mathbb{N}$, disjoint subsets $K_n, L_n \subset \prod_{k\in I_n} X_k$ and bijections $\phi_n : K_n \to L_n$ such that

$$\left| \frac{\mu(Z(\phi_n(a)))}{\mu(Z(a))} - \lambda \right| < \epsilon$$

for all $a \in K_n$ and $n \geq 1$, and furthermore

$$\sum_{n=1}^{\infty} \sum_{a\in K_n} \mu(Z(a)) = +\infty.$$

It is known (see e.g. [**20**, Proposition 2.6]) that

$$r_\infty(\mathcal{R},\mu) \setminus \{0\} = r(\mathcal{R},\mu) \setminus \{0\},$$

so for calculating the type of an equivalence relation it very nearly suffices to calculate $r_\infty(\mathcal{R},\mu)$.

CHAPTER 2

Complex-valued Bost-Connes systems associated with function fields

An analogue of the Bost-Connes system for function fields was proposed by Jacob [**18**]. The system constructed there is associated to the extension $\mathbb{K}$ of a function field K. In the current chapter, we construct Bost-Connes systems for arbitrary abelian extensions L/K in the first section, while the second section analyzes their KMS states in the critical region $\beta \in [0, 1]$. The third section is devoted to showing that Jacob's system indeed is a special case of our construction, and to show the relation between that system and the one arising from the groupoid considered in [**9**] by Consani and Marcolli. In the fourth and final section we show that both these systems can arise from a Hecke-algebra construction similar to that of [**25**].

2.1. Systems associated to a function field

Let L/K be an abelian extension, finite or infinite, of the global function field K, and let S be a finite set of primes in K. Consider the space

$$X_{L,S} = \mathrm{Gal}(L/K) \times_{\hat{\mathcal{O}}_S^*} \mathbb{A}_{K,S}.$$

Here the action of $\hat{\mathcal{O}}_S^*$ on $\mathrm{Gal}(L/K)$ is defined using the Artin map $r_{L/K} \colon \mathbb{A}_K^* \to \mathrm{Gal}(L/K)$.

By Lemma 1.1.15 we can identify J_S with $\mathbb{A}_{K,S}^*/\hat{\mathcal{O}}_S^*$. Then the diagonal action of $\mathbb{A}_{K,S}^*$ on $\mathrm{Gal}(L/K) \times \mathbb{A}_{K,S}$ given by

$$g(x, y) = (x r_{L/K}(g)^{-1}, gy)$$

defines an action of J_S on $X_{L,S}$. Put $Y_{L,S} = \mathrm{Gal}(L/K) \times_{\hat{\mathcal{O}}_S^*} \hat{\mathcal{O}}_S \subset X_{K,S}$, and consider the C*-algebra

$$A_{L,S} = \mathbf{1}_{Y_{L,S}} (C_0(X_{L,S}) \rtimes J_S) \mathbf{1}_{Y_{L,S}}.$$

We can also write $A_{L,S}$ as the semigroup crossed product $C(Y_{L,S}) \rtimes J_S^+$, where $J_S^+ \subset J_S$ is the subsemigroup of effective divisors.

The action of $\mathrm{Gal}(L/K)$ by translations on itself defines an action of $\mathrm{Gal}(L/K)$ on $X_{L,S}$, which in turn defines an action on $A_{L,S}$.

We now want to define a dynamics σ on $A_{L,S}$. It is easier to explain its extension to the multiplier algebra of the whole crossed product $C_0(X_{L,S}) \rtimes J_S$, which we continue to denote by σ. We put $\sigma_t(f) = f$ for $f \in C_0(X_{L,S})$ and $\sigma_t(u_{\mathfrak{a}}) = N(\mathfrak{a})^{it} u_{\mathfrak{a}}$ for $\mathfrak{a} \in J_S$, where $N(\mathfrak{a}) = q^{\deg \mathfrak{a}}$ is the norm of $\mathfrak{a}$. The subalgebra $A_{L,S}$ is clearly preserved by this action, and so it defines a dynamics on $A_{L,S}$.

Recall that a KMS-state for σ at inverse temperature $\beta \in \mathbb{R}$, or a KMS$_\beta$-state, is a σ-invariant state φ such that $\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$ for a and b in a set of σ-analytic elements with dense linear span. A σ-invariant state φ is called a ground state if the holomorphic function $z \mapsto \varphi(a\sigma_z(b))$ is bounded on the upper half-plane for a and b in a set of σ-analytic elements spanning a dense subspace. If a state φ is a weak* limit point of a sequence of states $\{\varphi_n\}_n$ such that φ_n is a KMS$_{\beta_n}$-state and $\beta_n \to +\infty$ as $n \to \infty$, then φ is a ground state. Such ground states are called KMS$_\infty$-states.

THEOREM 2.1.1. *For the system $(A_{L,S}, \sigma)$ we have:*

(i) *for $\beta < 0$ there are no KMS$_\beta$-states;*

(ii) *for every $0 < \beta \le 1$ there is a unique KMS$_\beta$-state;*

(iii) *for every $1 < \beta < \infty$ the extremal KMS$_\beta$-states are indexed by the points of the subset*

$$Y^0_{L,S} = \mathrm{Gal}(L/K) \times_{\hat{\mathcal{O}}^*_S} \hat{\mathcal{O}}^*_S \cong \mathrm{Gal}(L/K)$$

of $Y_{L,S}$, with the state corresponding to $w \in Y^0_{L,S}$ given by

$$\varphi_{\beta,w}(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) = \frac{\delta_{\mathfrak{a},0}}{\zeta_{K,S}(\beta)} \sum_{\mathfrak{b} \in J^+_S} N(\mathfrak{b})^{-\beta} f(\mathfrak{b}w), \tag{2.1.2}$$

where $\zeta_{K,S}(\beta) = \sum_{\mathfrak{a} \in J^+_S} N(\mathfrak{a})^{-\beta}$, $\delta_{\mathfrak{a},0}$ is the Kronecker delta, and $\mathfrak{b}w$ is given by the ideal action on $X_{L,S}$; furthermore, every extremal KMS$_\beta$-state $\varphi_{\beta,w}$ is of type I_∞ and its partition function is $\zeta_{K,S}(\beta)$;

(iv) *the extremal ground states are indexed by $Y^0_{L,S}$, with the state corresponding to $w \in Y^0_{L,S}$ given by $\varphi_{\infty,w}(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) = \delta_{\mathfrak{a},0} f(w)$, and all ground states are KMS$_\infty$-states.*

PROOF. We would like to apply Theorem 1.4.1 to the case $X = X_{L,S}$, $Y = Y_{L,S}$ and $G = J_S$, with N as above. For the Y_n we let $\{v_n\}$ be some enumeration of the places v of S^c and let Y_n be the image in $X_{L,S}$ of the pairs $(r, \rho) \in \mathrm{Gal}(L/K) \times \hat{\mathcal{O}}_S$ with $\rho_v = 0$. For the g_n take the corresponding $\mathfrak{p}_{v_n}$ in our enumeration. Then the union of the Y_n contains every point of X of nontrivial isotropy, $N(\mathfrak{p}_{v_n}) \ne 1$ by definition, and $\mathfrak{p}_{v_n}$ clearly fixes Y_n. Hence by the proposition there is a bijection between KMS$_\beta$-states ϕ of $A_{L,S}$ and Radon measures μ on X with $\mu(Y) = 1$ satisfying the scaling condition $\mu(\mathfrak{a}Z) = N(\mathfrak{a})^{-\beta}\mu(Z)$.

For $\beta < 0$ there are no such measures. Indeed, let $v \in S^c$. Then $\mathfrak{p}_v Y_{L,S} \subset Y_{L,S}$ and by the scaling condition

$$\mu(Y_{L,S}) \geq \mu(\mathfrak{p}_v Y_{L,S}) = N(\mathfrak{p}_v)^{-\beta}\mu(Y_{L,S}),$$

while we know that $N(\mathfrak{p}_v) > 1$ which is a contradiction.

For $0 < \beta \leq 1$ define μ_β as the image of the measure

$$\lambda_{L/K} \times \prod_{v \in S^c} \mu_{\beta,v}$$

on $\mathrm{Gal}(L/K) \times \mathbb{A}_{K,S}$ under the quotient map $\mathrm{Gal}(L/K) \times \mathbb{A}_{K,S} \to X_{L,S}$, where $\lambda_{L/K}$ is the normalized Haar measure on $\mathrm{Gal}(L/K)$ and $\mu_{\beta,v}$ is the measure on K_v defined by letting $\mu_{1,v}$ being the Haar measure on K_v normalized by $\mu_{1,v}(\mathcal{O}_v) = 1$ and requiring $\mu_{\beta,v}$ to be absolutely continuous with respect to $\mu_{1,v}$ with

$$\frac{d\mu_{\beta,v}}{d\mu_{1,v}}(a) = \frac{1 - N(\mathfrak{p}_v)^{-\beta}}{1 - N(\mathfrak{p}_v)^{-1}} \|a\|_v^{\beta-1},$$

where $\|\cdot\|_v$ is the norm on K_v. Then one can see that μ_β is a measure on $X_{L,S}$ with $\mu_\beta(Y_{L,S}) = 1$ satisfying the scaling condition.

Assume μ is a measure defining a KMS$_\beta$-state for some $\beta \in (0, 1]$. We want to see that $\mu = \mu_\beta$.

If E is a finite extension of K contained in L we can identify $X_{E,S}$ with $X_{L,S}/\mathrm{Gal}(L/E)$. Since a continuous function with support in $Y_{L,S}$ can be approximated by a $\mathrm{Gal}(L/E)$-invariant function with support in $Y_{L,S}$ for sufficiently large E, it suffices to show that $\mu = \mu_\beta$ when restricted to $X_{E,S}$ for all finite extensions E/K.

Secondly, let S' be a finite set of primes of K with $S \subset S'$. The subset

$$X_{L,S,S'} = \mathrm{Gal}(L/K) \times_{\hat{\mathcal{O}}_S^*} \left(\prod_{v \in S' \setminus S} \mathcal{O}_v^* \times \mathbb{A}_{S'} \right)$$

of $X_{L,S}$ is $J_{S'}$-invariant by definition. Furthermore, if $J_{S,S'}$ is the group generated by the primes in $S' \setminus S$ then

$$X_{L,S} \setminus J_{S,S'} X_{L,S,S'} = \{(r, \rho) : \rho_v = 0 \text{ for some } v \in S' \setminus S\},$$

and this difference has measure zero with respect to μ_β since we have $\mu_{\beta,v}(\{0\}) = 0$. That is, $X_{L,S,S'}$ is a fundamental domain for the $J_{S,S'}$-action modulo a set of measure zero. Furthermore, the measure of

$$Y_{L,S,S'} = \mathrm{Gal}(L/K) \times_{\hat{\mathcal{O}}_S^*} \left(\prod_{v \in S' \setminus S} \mathcal{O}_v^* \times \mathbb{A}_{S'} \right)$$

is $\prod_{v \in S' \setminus S}(1 - N(\mathfrak{p}_v)^{-\beta})$, since clearly $Y_{L,S,S'} = Y_{L,S} \setminus \bigcup_{v \in S' \setminus S} \mathfrak{p}_v Y_{L,S}$. Hence classifying measures μ on $X_{L,S}$ defining KMS$_\beta$-states is the same as classifying measures ν on $X_{L,S,S'}$ satisfying

$$\nu(Y_{L,S,S'}) = \prod_{v \in S' \setminus S} (1 - N(\mathfrak{p}_v)^{-\beta})$$

and $\nu(\mathfrak{a}Z) = N(\mathfrak{a})^{-\beta}\nu(Z)$ for any Borel set $Z \subset X_{L,S,S'}$ and any $\mathfrak{a} \in J_{S'}$. However, we can identify $X_{L,S,S'}$ with $X_{L,S'}$, so if we have uniqueness of KMS$_\beta$-states for S' we also have it for S.

By these reductions we may assume that the extension L/K is finite, and that S is arbitrarily large. In particular we may assume that S contains all primes in K that ramify in L. In this case the kernel of the Artin map $\mathbb{A}_S^* \to \mathrm{Gal}(L/K)$ contains $\hat{\mathcal{O}}_S^*$, so this map factors through J_S, and

$$X_{L,S} = \mathrm{Gal}(L/K) \times \mathbb{A}_S / \hat{\mathcal{O}}_S^*$$

with the action of J_S on $X_{L,S}$ being diagonal.

To prove that $\mu = \mu_\beta$ we compute the projection P of $L^2(Y_{L,S}, d\mu)$ onto the subspace of J_S^+-invariant functions. It will turn out that this subspace consists only of constants, whence μ is ergodic with respect to the J_S^+-action. Since a nontrivial convex combination of measures is never ergodic this implies that μ is unique, so $\mu = \mu_\beta$.

To this end, let us calculate Pf for some functions f on $Y_{L,S}$. As noted above, we may assume that $Y_{L,S} = \mathrm{Gal}(L/K) \times \hat{\mathcal{O}}_S / \hat{\mathcal{O}}_S^*$. Suppose $A \subset S^c$ is some finite set of primes. We may assume f factors through $\mathrm{Gal}(L/K) \times \prod_{v \in A} \mathcal{O}_v / \mathcal{O}_v^*$, since such functions are dense in the set of functions on $Y_{L,S}$. Let $J_{S,A}^+$ be the subsemigroup of J_S^+ generated by the primes in A. Then

$$\left(\mathrm{Gal}(L/K) \times \prod_{v \in A} \mathcal{O}_v / \mathcal{O}_v^* \right) \setminus \left(\bigcup_{\mathfrak{a} \in J_{S,A}^+} \mathfrak{a}(\mathrm{Gal}(L/K) \times \{1\}) \right)$$
$$= \{(r, \rho) : \rho_v = 0 \text{ for some } v \in A\},$$

so the difference has measure zero. Hence, modulo a set of measure zero, f has support on $\bigcup_{\mathfrak{a}\in J^+_{S,A}} \mathfrak{a}(\mathrm{Gal}(L/K)\times\{1\})$. It follows that we may assume that f is of the form

$$f(a) = \begin{cases} \chi(\mathfrak{a}^{-1}a) & \text{if } a \in \mathfrak{a}(\mathrm{Gal}(L/K)\times\{1\}) \\ 0 & \text{otherwise,} \end{cases}$$

where χ is a character of $\mathrm{Gal}(L/K)$.

For B a finite subset of S^c let P_B be the projection onto the subspace H_B of $J^+_{S,B}$-invariant functions in $L^2(Y, d\mu)$. Then if we apply (2) of Theorem 1.4.1 with $G = J_{S,B} = (J^+_{S,B})^{-1}J^+_{S,B}$, $S = J^+_{S,B}$ and $Y_0 = Y_{L,S\cup B}$ we get that

$$P_B f|_{J^+_{S,B}y} = \zeta_{J^+_{S,B}}(\beta)^{-1} \sum_{\mathfrak{b}\in J^+_{S,B}} N(\mathfrak{b})^{-\beta} f(\mathfrak{b}y).$$

Now, for $\mathfrak{b} \in J^+_{S,B}$ we get that

$$\mathfrak{b}Y_{L,S\cup B} = \mathrm{Gal}(L/K) \times \mathfrak{b}\hat{\mathcal{O}}_{S\cup B}/\hat{\mathcal{O}}^*_{S\cup B},$$

so its intersection with $\mathrm{supp}(f) = \mathfrak{a}(\mathrm{Gal}(L/K)\times\{1\})$ is nonzero only if $\mathfrak{a}|\mathfrak{b}$, so $\mathfrak{a} \in J^+_{S,B}$ and furthermore $\mathfrak{b} \in \mathfrak{a}J^+_{S,B\setminus A}$. Since we may assume $A \subset B$ we get

$$\begin{aligned} P_B f|_{J^+_{S,B}y} &= \zeta_{J^+_{S,B}}(\beta)^{-1} \sum_{\mathfrak{c}\in J^+_{S,B\setminus A}} N(\mathfrak{a}\mathfrak{c})^{-\beta}\chi(\mathfrak{c}a) \\ &= \zeta_{J^+_{S,B}}(\beta)^{-1} N(\mathfrak{a})^{-\beta}\chi(a) \sum_{\mathfrak{c}\in J_{S,B\setminus A}} N(\mathfrak{c})^{-\beta}\chi(\mathfrak{c}) \\ &= N(\mathfrak{a})^{-\beta}\chi(a)\zeta_{J^+_{S,B}}(\beta)^{-1} \prod_{v\in B\setminus A} \frac{1}{1-\chi(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-\beta}} \end{aligned}$$

for any $a \in Y_{L,S\cup B}$.

Now if χ is trivial $P_B f$ is constant, so the same holds for Pf. For a nontrivial χ we see that

$$\begin{aligned} \|Pf\|_2 &= \lim_B \|P_B f\|_2 \\ &= N(\mathfrak{a})^{-\beta} \lim_B |\zeta_{J^+_{S,B}}(\beta)|^{-1} \prod_{v\in B\setminus A} \frac{1}{|1-\chi(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-\beta}|}. \end{aligned}$$

Divided by $N(\mathfrak{a})^{-\beta}$ the right-hand side is a nondecreasing function in β on $(0, +\infty)$, since $N(\mathfrak{p}_v) > 1$ for all but finitely many $\mathfrak{p}_v$. For $\beta > 1$

we have

$$\begin{aligned}\|Pf\|_2 &\le N(\mathfrak{a})^{-\beta}|\zeta_{S^c}(\beta)|^{-1}\cdot|L(\chi,\beta)|\cdot\prod_{v\in A\cup S}|1-\chi(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-\beta}|\\ &\le N(\mathfrak{a})^{-\beta}|\zeta_{S^c}(\beta)|^{-1}|L(\chi,\beta)|.\end{aligned}$$

Since $\zeta_{S^c}(\beta)$ has a pole at $\beta = 1$ while $L(\chi,\beta)$ does not, we see that $\|Pf\|_2 = 0$ if $\beta = 1$. Since the right-hand side (again ignoring $N(\mathfrak{a})^{-\beta}$) is increasing in β we see that $\|Pf\|_2 = 0$ for all $\beta \in (0,1]$. Hence $Pf = 0$, and in particular Pf is a constant. Thus the measure μ is ergodic, and hence $\mu = \mu_\beta$.

The statement for $\beta > 1$ is direct from Theorem 1.4.1 (1). □

The case $\beta = 0$ is special, as then there are KMS-states which do not factor through the conditional expectation $A_{L,S} \to C(Y_{L,S})$. In order to classify them, consider the subfield $L_S^{un} \subset L$ such that $\mathrm{Gal}(L/L_S^{un}) = r_{L/K}(\hat{\mathcal{O}}_S^*)$. This is the maximal subextension of L/K unramified at all primes in S^c. Let $J_{L,S}$ be the kernel of the Artin homomorphism $J_S \to \mathrm{Gal}(L_S^{un}/K)$, and $J_{L,S}^0 \subset J_{L,S}$ be the subgroup of degree zero divisors. Denote by μ_0 the unique $\mathrm{Gal}(L/K)$-invariant probability measure on $Y_{L,S}$ concentrated on the image of

$$\mathrm{Gal}(L/K)\times\{0\} \subset \mathrm{Gal}(L/K)\times\hat{\mathcal{O}}_S$$

in $Y_{L,S}$.

Proposition 2.1.3. *There is a one-to-one correspondence between the set of extremal KMS$_0$-states on $A_{L,S}$ and the set of characters of $J_{L,S}^0$, where the state τ_χ corresponding to a character χ is given by*

$$\tau_\chi(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) = \begin{cases}\chi(\mathfrak{a})\int f\,d\mu_0 & \text{if } \mathfrak{a}\in J_{L,S}^0\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Note that $A_{L,S}$ is the C*-algebra of the groupoid $J_S \boxtimes Y_{L,S}$. Then by [**26**, Theorem 1.3] there is a one-to-one correspondence between the set of KMS$_0$-states on $A_{L,S}$ and the set of pairs $(\mu,\{\tau_x\}_x)$, where x runs through Y, such that

(1) μ is a J_S-invariant measure on $X_{L,S}$ with $\mu(Y_{L,S})$;

(2) τ_x is a μ-measurable field of states τ_x on $C^*((J_S)_x)$, where $(J_S)_x$ is the stabilizer of x;

(3) $x \mapsto \tau_x$ satisfies $\tau_{\mathfrak{a}x} = \tau_x$ for μ-almost every $x \in X_{L,S}$;

(4) τ_x factors through the canonical conditional expectation

$$C^*((J_S)_x) \to C^*((J_S^0)_x),$$

where $(J_S^0)_x \subset (J_S)_x$ is the subgroup consisting of those ideals of $(J_S)_x$ which are of degree zero.

Assume $(\mu, \{\tau_x\}_x)$ is such a pair. Since the intersection of the sets $\mathfrak{a}Y_{L,S}$ for $\mathfrak{a} \in J_S$ coincides with the image Z of $\mathrm{Gal}(L/K) \times \{0\}$ in $Y_{L,S}$, the measure μ is concentrated on Z. The set Z can be identified with $\mathrm{Gal}(L/K)/r_{L/K}(\hat{\mathcal{O}}_S^*) = \mathrm{Gal}(L_S^{un}/K)$. Since the image of J_S in $\mathrm{Gal}(L_S^{un}/K)$ is dense, the Haar measure on $\mathrm{Gal}(L_S^{un}/K)$ is the unique J_S-invariant measure. Therefore $\mu = \mu_0$, and the action of J_S on $(X_{L,S}, \mu)$ is ergodic. It follows that the field $\{\tau_x\}_x$ is essentially constant. The stabilizer of every point in Z is $J_{L,S}$. Therefore we conclude that there is a one-to-one correspondence between KMS_0-states on $A_{L,S}$ and states on $C^*(J_{L,S}^0)$. Hence extremal KMS_0-states correspond to characters on $J_{L,S}^0$. □

2.2. The type of the KMS states in the critical region

We continue to use the notation of the previous section, so L is an abelian extension of a global function field K and S is a finite set of primes in K. Take $0 < \beta \leq 1$ and denote by ϕ_β the unique KSM_β-state on $A_{L,S}$. We want to prove the following result:

Theorem 2.2.1. *Let $\mathbb{F}_{q^n}$, $n \in \mathbb{N} \cup \{+\infty\}$, be the algebraic closure of the constant field $\mathbb{F}_q \subset K$ in L. Then the von Neumann algebra $\pi_{\sigma_\beta}(A_{L,S})''$ is an injective factor of type $III_{q^{-n\beta}}$.*

Since φ_β is extremal, the von Neumann algebra $\pi_{\varphi_\beta}(A_{L,S})''$ is a factor. It is the reduction of the von Neumann algebra

$$L^\infty(X_{L,S}, \mu_\beta) \rtimes J_S$$

by the projection $\mathbf{1}_{Y_{L,S}}$, where μ_β is the measure on $X_{L,S}$ defined in the previous section. Since $\mathbf{1}_{Y_{L,S}}$ is a full projection the von Neumann algebra $L^\infty(X_{L,S}, \mu_\beta) \rtimes J_S$ is also a factor. Hence the action of J_S on $(X_{L,S}, \mu_\beta)$ is ergodic. The equivalent formulation of the above theorem is therefore that this action (more precisely, the corresponding orbit equivalence relation) is of type $\mathrm{III}_{q^{-\beta n}}$, as defined in Section 1.5. Recall from for instance [**2**, Corollary IV.2.2.16] that the factor is injective since $L^\infty(X_{L,S}, \mu_\beta) \rtimes J_S$ is the crossed product of an injective von Neumann algebra with an amenable (in our case an abelian) group.

Our computation of the ratio set will rely on the following version of the Chebotarev density theorem for function fields. For a proof see [**12**, Proposition 6.4.8].

Theorem 2.2.2. *Let K be a function field with constant field $\mathbb{F}_q$, L a finite Galois extension of K with constant field $\mathbb{F}_{q^n}$, and $\mathcal{C}$ a conjugacy* *class in* $\mathrm{Gal}(L/K)$ *consisting of* c *elements. Let* $0 \leq a < n$ *be such that the restriction of every element in* $\mathcal{C}$ *to* $\mathbb{F}_{q^n}$ *is the* a*-th power of the Frobenius automorphism of* $\mathbb{F}_{q^n}/\mathbb{F}_q$*. Then every prime* $\mathfrak{p}$ *such that* $(\mathfrak{p}, L/K) = \mathcal{C}$ *has the property* $\deg \mathfrak{p} \equiv a \mod n$*, and*

$$\#\{\mathfrak{p} \mid (\mathfrak{p}, L/K) = \mathcal{C} \text{ and } \deg \mathfrak{p} = kn + a\} \\ = \frac{cn}{(kn+a)[L:K]} q^{kn+a} + O(q^{kn/2}) \quad \text{as} \quad k \to +\infty.$$

We are interested in the case when the extension L/K is abelian. Denote by $\mathrm{Spl}(L/K)$ the set of primes in K that are completely split in L.

Corollary 2.2.3. *Assume* L/K *is a finite abelian extension and* S *contains the set of primes in* K *that ramify in* L*. Then the degree of every element in the kernel of the Artin map* $J_S \to \mathrm{Gal}(L/K)$ *is divisible by* n*, and*

$$\#\{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spl}(L/K) \text{ and } \deg \mathfrak{p} = kn\} \\ = \frac{1}{k[L:K]} q^{kn} + O(q^{kn/2}) \quad \text{as} \quad k \to +\infty.$$

Proof. If $\mathfrak{p}$ is in the kernel of the Artin map, then the restriction of $r_{L/K}(\mathfrak{p})$ to $\mathbb{F}_{q^n}$ is the zeroth power of the Frobenius, so by the theorem $\deg \mathfrak{p} = 0 \mod n$.

As for the asymptotics, we are counting primes such that the restriction of $r_{L/K}(\mathfrak{p})$ to $\mathbb{F}_{q^n}$ is trivial. Hence every prime is completely split unless it is one of the finitely many primes that ramify, so the asymptotics do not change by restricting to the set of completely split primes. The formula given is then that of the theorem. □

Under the assumptions of the previous corollary consider the restricted product $X'_{L,S}$ of the spaces $K_v/\mathcal{O}_v^*$ with respect to $\mathcal{O}_v/\mathcal{O}_v^*$ over all $v \in S^c \cap \mathrm{Spl}(L/K)$, and consider the measure

$$\mu'_\beta = \prod_{v \in S^c \cap \mathrm{Spl}(L/K)} \mu_{\beta,v}$$

on $X'_{L,S}$, where $\mu_{\beta,v}$ is as defined in the proof of Theorem 1.4.1. Denote by J'_S the subgroup of J_S generated by places in $S^c \cap \mathrm{Spl}(L/K)$.

Lemma 2.2.4. *The action of* J'_S *on* $(X'_{L,S}, \mu'_\beta)$ *is of type* $III_{q^{-\beta n}}$*.*

Proof. The degree of every prime $\mathfrak{p}_v$ such that $v \in S^c \cap \mathrm{Spl}(L/K)$ is divisible by n. Since the measure μ'_β satisfies $\mu'_\beta(\mathfrak{a}\,\cdot) = N(\mathfrak{a})^{-\beta}\mu'_\beta$, the ratio set $r(\mathcal{R},\mu)$ is contained in the set $\{0\} \cup \{q^{-\beta nk} : k \in \mathbb{Z}\}$. Therefore it suffices to show that $q^{-\beta n}$ lies in the ratio set.

Instead of the orbit equivalence relation on $X'_{L,S}$ we may consider the relation induced on the subset $Y'_{L,S} = \prod_{v\in S^c\cap \mathrm{Spl}(L/K)} \mathcal{O}_v^\times/\mathcal{O}_v^*$, since this is a subset of positive measure.

But this is exactly the relation discussed in Section 1.5: Two points are equivalent if and only if their coordinates coincide outside a finite set of primes. The measure $\mu_{\beta,v}$ on $\mathcal{O}_v^\times/\mathcal{O}_v^*$ is given by

$$\mu_{\beta,v}(\pi_v^k\mathcal{O}_v^*) = N(\mathfrak{p}_v)^{-\beta k}(1-N(\mathfrak{p}_v)^{-\beta}),$$

where π_v is a uniformizer in $\mathcal{O}_v$.

Let m_k be the number of places in $S^c \cap \mathrm{Spl}(L/K)$ of degree kn. By Corollary 2.2.3 we have $m_k \sim [L:K]^{-1}q^{kn}/k$. In particular, for sufficiently large k we have $m_{2k} < m_{2k+1}$. Hence for every prime $\mathfrak{p}_v$ of degree $2kn$ we can choose a prime $\mathfrak{p}_{v'}$ of degree $(2k+1)n$ in such a way that the map $\mathfrak{p}_v \mapsto \mathfrak{p}_{v'}$ is injective.

For the sets I_j from the definition of the asymptotic ratio set we take all the pairs $\{\mathfrak{p}_v, \mathfrak{p}_{v'}\}$, and as the sets K_j and L_j we take the one-point sets $\{(\pi_v\mathcal{O}_v^*, \mathcal{O}_{v'}^*)\}$ and $\{(\mathcal{O}_v^*, \pi_{v'}\mathcal{O}_{v'}^*)\}$ respectively.

Then for $a \in K_n$, that is for $a = (\pi_v\mathcal{O}_v^*, \mathcal{O}_{v'}^*)$, we have

$$Z(a) = \prod_{w\neq v,v'} \mathcal{O}_w^\times/\mathcal{O}_w^* \times \{\pi_v\mathcal{O}_v^*\} \times \{\mathcal{O}_{v'}^*\},$$

while

$$Z(\phi(a)) = \prod_{w\neq v,v'} \mathcal{O}_w^\times/\mathcal{O}_w^* \times \{\mathcal{O}_v^*\} \times \{\pi_{v'}\mathcal{O}_{v'}\},$$

so we can see that $\mu(Z(a)) = N(\mathfrak{p}_v)^{-\beta}(1-N(\mathfrak{p}_v)^{-\beta})(1-N(\mathfrak{p}_{v'})^{-\beta})$ and $\mu(Z(\phi(a))) = N(\mathfrak{p}_{v'})^{-\beta}(1-N(\mathfrak{p}_v)^{-\beta})$. Hence

$$\frac{\mu(Z(\phi(a)))}{\mu(Z(a))} = \frac{N(\mathfrak{p}_{v'})^{-\beta}}{N(\mathfrak{p}_v)^{-\beta}} = \frac{q^{-(2n+1)k\beta}}{q^{-2nk\beta}} = q^{-k\beta}.$$

To conclude that $q^{-k\beta} \in r_\infty(\mathcal{R},\mu)$ it remains to show that

$$\sum_{j=1}^{\infty}\sum_{a\in K_j} \mu(Z(a)) = +\infty.$$

For each j such that $I_j = \{\mathfrak{p}_v, \mathfrak{p}_{v'}\}$ with $\deg \mathfrak{p}_v = nk$ we have

$$\mu(Z(a)) = q^{-2nk\beta}(1-q^{-2nk\beta})(1-q^{-(2n+1)k\beta}) > \frac{1}{2}q^{-2nk\beta},$$

so adding up the terms corresponding to primes of degree nk first we get that the sum is no smaller than

$$\sum_{k=1}^{\infty} q^{-2nk\beta} m_{2k}.$$

Since $m_{2k} \sim \frac{1}{k[L:K]} q^{2nk}$ this sum diverges to $+\infty$. Hence we see that $q^{-k\beta} \in r_\infty(\mathcal{R}, \mu)$, so $r_\infty(\mathcal{R}, \mu) = \{q^{-k\beta} : k \in \mathbb{Z}\}$. □

Proof of Theorem 2.2.1. Assume first that the extension L/K is finite. Since the action of J_S on $(X_{L,S}, \mu_\beta)$ is ergodic, in computing the ratio set of the orbit equivalence relation on $X_{L,S}$ we may in its place consider the relation induced on any subset of positive measure. In particular, for every $S' \supset S$ we may consider the subset $X_{L,S,S'}$ introduced in the proof of Theorem 2.1.1. As we discussed there, it can be identified with $X_{L,S'}$, and the equivalence relation we get on $X_{L,S'}$ is exactly the one defined by the action of the group $J_{S'}$. Therefore the type of the action of J_S on $(X_{L,S}, \mu_\beta)$ does not depend on S. Hence we may assume that S includes all primes that ramify in L. Then $X_{L,S} = \mathrm{Gal}(L/K) \times \mathbb{A}_{K,S}/\hat{\mathcal{O}}_S^*$.

Consider the subset $\{e\} \times \mathbb{A}_{K,S}/\hat{\mathcal{O}}_S^*$ of $X_{L,S}$. The equivalence relation induced on it is the one given by the action of the kernel G of the Artin map $J_S \to \mathrm{Gal}(L/K)$ on $\mathbb{A}_{K,S}/\hat{\mathcal{O}}_S^*$. By the first part of Corollary 2.2.3 the degree of every element in G is divisible by n. Hence the ratio set of the action of G on $(\mathbb{A}_{K,S}/\hat{\mathcal{O}}_S^*, \mu_\beta)$ is contained in $\{0\} \cup \{q^{-\beta nk} \mid k \in \mathbb{Z}\}$. On the other hand, by Lemma 2.2.4 the number $q^{-\beta n}$ is contained in the ratio set of the action of $J'_S \subset G$ on $(X'_{L,S}, \mu'_\beta)$. Hence, by Proposition 1.5.2, it is contained in the ratio set of the action of J'_S on $(\mathbb{A}_{K,S}/\hat{\mathcal{O}}_S^*, \mu_\beta)$. This proves the theorem when L/K is finite.

Now consider an arbitrary abelian extension L/K. It follows from Proposition 1.5.3 that the nonzero part of the ratio set of the action of J_S on $(X_{L,S}, \mu_\beta)$ is equal to the intersection of the nonzero parts of the ratio sets of the actions of J_S on the measure spaces $(X_{L,S}/\mathrm{Gal}(L/E), \mu_\beta)$ for all finite intermediate extensions E/K. Since $X_{L,S}/\mathrm{Gal}(L/E) = X_{E,S}$, from the first part of the proof we conclude that if $n < +\infty$ then the nonzero part of the ratio set of the action of J_S on $(X_{L,S}, \mu_\beta)$ equals $\{q^{-\beta nk} \mid k \in \mathbb{Z}\}$. Hence the action is of type $\mathrm{III}_{q^{-\beta n}}$.

In the case $n = +\infty$ we can only conclude that the ratio set is either $\{1\}$ or $\{0, 1\}$, so the factor $M_L = \pi_{\varphi_\beta}(A_{L,S})''$ is either semifinite or of type III_0. Since $M_K = M_L^{\mathrm{Gal}(L/K)}$ there exists a normal conditional expectation $M_L \to M_K$. As M_K is of type III, by [**29**, Proposition 10.21] it follows that M_L is also of type III, hence it is of type III_0. □

For every $\lambda \in (0,1]$ there exists a unique injective factor of type III_λ with separable predual. Therefore for $n < +\infty$ the above result completely describes the von Neumann algebra $\pi_{\varphi_\beta}(A_{L,S})''$. For type III_0 factors a complete invariant is the flow of weights [**21**, Theorem 8.4], which we will now briefly introduce.

Let Γ be a countable group acting ergodically by non-singular transformations on a standard measure space (X,μ). Letting λ be the Lebesgue measure on $\mathbb{R}$, define a new action of Γ on $(\mathbb{R}\times X, \lambda\times\mu)$ by

$$g(s,x) = \left(s - \log\frac{dg\mu}{d\mu}(gx), gx\right). \tag{2.2.5}$$

Let $\tilde{X}$ be the measure theoretic quotient of $(\mathbb{R}\times X, \lambda\times\mu)$ by this action. That is, $\tilde{X}$ is a standard Borel space with measure class $[\tilde{\mu}]$ such that $L^\infty(\tilde{X},\tilde{\mu}) = L^\infty(\mathbb{R}\times X, \lambda\times\mu)^\Gamma$. The $\mathbb{R}$-action on $\mathbb{R}\times X$ given by $t(s,x) = (s+t,x)$ induces a flow $\{F_t\}_{t\in\mathbb{R}}$ given on $(\tilde{X},\tilde{\mu})$. This flow depends up to isomorphism only on the measure class of μ and the orbit equivalence relation $\mathcal{R}$ on X defined by the Γ-action. This flow $\{F_t\}_{t\in\mathbb{R}}$ is the flow of weights of the factor $W^*(\mathcal{R})$ [**13**, Section 8].

Returning to the Bost-Connes systems, consider an abelian extension L/K. As before, let $\mathbb{F}_{q^n}$, $n\in\mathbb{N}\cup\{+\infty\}$, be the algebraic closure of $\mathbb{F}_q$ in L.

Define a continuous map $X_{L,S}\to \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ as the composition of the quotient map

$$X_{L,S} \to X_{L,S}/\operatorname{Gal}(L/\mathbb{F}_{q^n}K) = X_{\mathbb{F}_{q^n}K,S} = \mathrm{Gal}(\mathbb{F}_{q^n}K/K)\times \mathbb{A}_{K,S}/\hat{\mathcal{O}}_S^*,$$

(where we have used that any finite constant field extension of K is unramified at every prime) with the projection

$$\mathrm{Gal}(\mathbb{F}_{q^n}K/K)\times \mathbb{A}_{K,S}/\hat{\mathcal{O}}_S^* \to \mathrm{Gal}(\mathbb{F}_{q^n}K/K) = \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q).$$

Taking the direct product with the identity map on $\mathbb{R}$ we get a continuous map

$$\mathbb{R}\times X_{L,S} \to \mathbb{R}\times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q).$$

We equip $\mathbb{R}\times X_{L,S}$ with the action of J_S given in Equation 2.2.5 and $\mathbb{R}\times\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ with the action given by

$$\mathfrak{a}(s,g) = (s-\beta\log N(\mathfrak{a}), g\,\mathrm{res}_{\mathbb{F}_{q^n}}(r_{\mathbb{F}_{q^n}K/K}(\mathfrak{a}))^{-1}).$$

Note that $\mathrm{res}_{\mathbb{F}_{q^n}}(r_{\mathbb{F}_{q^n}K/K}(\mathfrak{a}))$ is simply the Frobenius automorphism raised to the power $\deg\mathfrak{a}$.

Denote by λ_n the normalized Haar measure on $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Then the map $\mathbb{R} \times X_{L,S} \to \mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ gives us a J_S-equivariant embedding

$$L^\infty(\mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda \times \lambda_n) \hookrightarrow L^\infty(\mathbb{R} \times X_{L,S}, \lambda \times \mu_\beta).$$

LEMMA 2.2.6. *We have*

$$L^\infty(\mathbb{R} \times X_{L,S}, \lambda \times \mu_\beta)^{J_S} = L^\infty(\mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda \times \lambda_n)^{J_S}.$$

PROOF. It suffices to check that the subalgebras of $\mathrm{Gal}(L/E)$-invariant elements on both sides coincide for all intermediate finite extensions E of K. Since $X_{L,S}/\mathrm{Gal}(L/E) = X_{E,S}$, this means that it is enough to prove the lemma for finite extensions.

Choose a finite set S' which contains S and all primes that ramify in L, and consider the set $X_{L,S,S'}$ introduced in the proof of Theorem 2.1.1. It can be identified with $X_{L,S'} = \mathrm{Gal}(L/K) \times \mathbb{A}_{S'}/\hat{\mathcal{O}}^*_{S'}$. Consider the subset $\{e\} \times \mathbb{A}_{S'}/\hat{\mathcal{O}}^*_{S'}$ of $X_{L,S'}$. We claim that if f is an element of $L^\infty(\mathbb{R} \times X_{L,S}, \lambda \times \mu_\beta)^{J_S}$ then the restriction of f to $\mathbb{R} \times \{e\} \times \mathbb{A}_{S'}/\hat{\mathcal{O}}^*_{S'}$ depends only on the first coordinate, and hence defines a function f_1 in $L^\infty(\mathbb{R}, \lambda)$, and that the function f_1 is $\beta n \log q$-periodic.

Indeed, let G be the kernel of the Artin map $J_{S'} \to \mathrm{Gal}(L/K)$. By the proof of Theorem 2.2.1 the nonzero part of the ratio set of the action of G on $\{e\} \times \mathbb{A}_{S'}/\hat{\mathcal{O}}^*_{S'}$ coincides with the set of values of the Radon-Nikodym derivatives. By Proposition 1.5.4 it follows that the subgroup $G_0 \subset G$ of divisors of degree zero acts ergodically on $\{e\} \times \mathbb{A}_{S'}/\hat{\mathcal{O}}^*_{S'}$. But the group G_0 acts trivially on $\mathbb{R}$, hence the G_0-invariant function f on $\mathbb{R} \times \{e\} \times \mathbb{A}_{S'}/\hat{\mathcal{O}}^*_{S'}$ depends only on the first coordinate, and therefore defines a function $f_1 \in L^\infty(\mathbb{R}, \lambda)$. Since f is G-invariant and the homomorphism $\deg\colon G \to n\mathbb{Z}$ is surjective, the function f_1 is $\beta n \log q$-periodic.

The function f_1 defines, in turn, a J_S-invariant function f_2 on

$$\mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{R} \times \mathbb{Z}/n\mathbb{Z}$$

such that $f_2(s, 0) = f_1(s)$. Namely, $f_2(s, m) = f_1(s - \beta m \log q)$. Since the J_S-orbit of almost every point in $\mathbb{R} \times X_{L,S}$ intersects the set

$$\mathbb{R} \times \{e\} \times \mathbb{A}_{S'}/\hat{\mathcal{O}}^*_{S'}$$

every J_S-invariant function is completely determined by its values on this set. Hence

$$f = f_2 \in L^\infty(\mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda \times \lambda_n)^{J_S}.$$

□

Assume now that $n = +\infty$, so that $\mathbb{F}_{q^n} = \bar{\mathbb{F}}_q$. We can identify the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ with the group $\lim\limits_{\leftarrow} \mathbb{Z}/k\mathbb{Z} = \hat{\mathbb{Z}}$. The action of J_S on $\mathbb{R} \times \hat{\mathbb{Z}}$ that we get is given by

$$\mathfrak{a}(s, a) = (s - \beta \deg \mathfrak{a} \log q, a - \deg \mathfrak{a}).$$

Rescaling the first coordinate, we can instead consider the action given by $\mathfrak{a}(s, a) = (s - \deg \mathfrak{a}, a - \deg \mathfrak{a})$. We can now formulate a refinement of Theorem 2.2.1 for $n = +\infty$.

THEOREM 2.2.7. *Assume the algebraic closure of $\mathbb{F}_q$ in L is infinite. Then $\pi_{\varphi_\beta}(A_{L,S})''$ is an ITPFI (infinite tensor product of finite type I factors) factor of type III_0. Its flow of weights is the flow on the compact group $(\mathbb{R} \times \hat{\mathbb{Z}})/\mathbb{Z}$ defined by*

$$F_t(s, a) = \left(s + \frac{t}{\beta \log q}, a \right).$$

PROOF. As follows from Lemma 2.2.6 and the subsequent discussion, the flow of weights of the factor $L^\infty(X_{L,S}, \mu_\beta) \rtimes J_S$ has the form given in the formulation of the theorem. We already know that this factor is of type III_0. The flow is approximately transitive [**8**], hence the factor is ITPFI. Since $\pi_{\varphi_\beta}(A_{L,S})''$ is a reduction of $L^\infty(X_{L,S}, \mu_\beta) \rtimes J_S$ and the flow of weights is preserved under reduction, the same assertions hold for $\pi_{\varphi_\beta}(A_{L,S})''$. Moreover, this in turn implies that the two factors are isomorphic. □

We next want to determine the center of the centralizer of the state φ_β on $\pi_{\varphi_\beta}(A_{L,S})''$. This centralizer can be expressed as the reduction of the von Neumann algebra $L^\infty(X_{L,S}, \mu_\beta) \rtimes J_S^0$ by the projection $\mathbf{1}_{Y_{L,S}}$. The center of $L^\infty(X_{L,S}, \mu_\beta) \rtimes J_S^0$ is $L^\infty(X_{L,S}, \mu_\beta)^{J_S^0}$. If the field $\mathbb{F}_q$ is not algebraically closed in L, then the ratio set of the action of J_S on $(X_{L,S}, \mu_\beta)$ is strictly smaller than the essential range of the Radon-Nikodym cocycle, so by Proposition 1.5.4 the action of J_S^0 on $(X_{L,S}, \mu_\beta)$ cannot be ergodic.

Consider the map $X_{L,S} \to \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ which was constructed before Lemma 2.2.6. It gives us an embedding of $L^\infty(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda_n)$ into $L^\infty(X_{L,S}, \mu_\beta)$.

PROPOSITION 2.2.8. *If L/K is an abelian extension and $\mathbb{F}_{q^n}$, $n \in \mathbb{N} \cup \{+\infty\}$, is the algebraic closure of $\mathbb{F}_q$ in L, then $L^\infty(X_{L,S}, \mu_\beta)^{J_S^0} = L^\infty(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda_n)$.*

Proof. As in the proof of Lemma 2.2.6, we may assume that L/K is finite. Consider the open subset

$$Z = \mathrm{Gal}(L/\mathbb{F}_{q^n}K) \times_{\hat{\mathcal{O}}_S^*} \mathbb{A}_{K,S} \subset X_{L,S}.$$

The equivalence relation on Z induced by the action of J_S on $X_{L,S}$ is the orbit equivalence relation defined by the action of the kernel H of the map $J_S \to \mathrm{Gal}(\mathbb{F}_{q^n}K/K) = \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. The group H is simply the subgroup of divisors in J_S of degree divisible by n. Since the action of J_S on $(X_{L,S}, \mu_\beta)$ is of type $\mathrm{III}_{q^{-\beta n}}$, by Proposition 1.5.4 we conclude that the action of $J_S^0 \subset H$ on (Z, μ_β) is ergodic. It follows that any J_S^0-invariant measurable subset of $X_{L,S}$ coincides, modulo a set of measure zero, with the union of translations of Z by elements of $\mathrm{Gal}(L/K)$. But the set Z is nothing else than the pre-image of the unit element $e \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ under the map $X_{L,S} \to \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Hence any J_S^0-invariant measurable subset of $X_{L,S}$ is the pre-image of a subset of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. □

2.3. Comparision with other systems

Inspired by the work of Drinfeld [**10, 11**] and Hayes [**17**] on explicit class field theory for function fields, Jacob [**18**] defined a dynamical system associated to a function field K. The main goal of this section is to show that Jacob's system fits into our framework.

Fix as before a distinguished prime ∞ of K. In this section we will consider the case $S = \{\infty\}$, $L = \mathbb{K}$, where $\mathbb{K}$ is as defined in Section 1.3.

Given a sgn-normalized rank-one Drinfeld module ϕ, we write

$$\phi(\mathbb{C}_\infty)^{\mathrm{tor}} = \cup_{\mathfrak{m}} \phi[\mathfrak{m}]$$

for the set of torsion points of the $\mathcal{O}$-action on $\mathbb{C}_\infty$ which is given by $a\xi = \phi_a(\xi)$. If $\phi = \phi^\Lambda$ then e_Λ defines an $\mathcal{O}$-module isomorphism $K\Lambda/\Lambda \cong \phi(\mathbb{C}_\infty)^{\mathrm{tor}}$. Let X_ϕ be the group of characters of $\phi(\mathbb{C}_\infty)^{\mathrm{tor}}$. Put $X = \bigsqcup_\phi X_\phi$, where the union is taken over the set of sgn-normalized rank-one Drinfeld modules. Since the set of such modules is finite and each X_ϕ is a profinite group, X is compact.

Define an action of J_K^+ on X by letting an ideal $\mathfrak{a}$ map the character $\chi \in X_\phi$ to the character $\chi^{\mathfrak{a}} = \chi \circ (\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}} \in X_{\mathfrak{a}^{-1} * \phi}$, where by $\mathfrak{a}^{-1} * \phi$ we mean the unique sgn-normalized Drinfeld module such that $\mathfrak{a} * (\mathfrak{a}^{-1} * \phi) = \phi$. This is a semigroup action. There is also an action of $\mathrm{Gal}(\mathbb{K}/K)$ on X given by $g\chi = \chi \circ g \in X_{g^{-1}(\phi)}$, where $g^{-1}(\phi)$ is the Drinfeld module defined by $g^{-1}(\phi)_a = g^{-1}(\phi_a)$. This action commutes with the ideal action.

The action of J_K on X defines a partially defined action of the group of fractional ideals of $\mathcal{O}$ on X, which gives rise to a transformation groupoid $\mathcal{G}_{\text{Jacob}}$. The C*-algebra $C_{K,\infty}$ underlying Jacob's system is the C*-algebra of this groupoid. Alternatively, one can say that $C_{K,\infty}$ is the semigroup crossed product $C(X) \rtimes J_K$ with respect to the action

$$(\mathfrak{a}f)(\chi) = \begin{cases} f(\chi') & \text{if } {\chi'}^{\mathfrak{a}} = \chi \\ 0 & \text{if no such } \chi' \text{ exists.} \end{cases}$$

The one-parameter family $\mathfrak{a} \mapsto N(\mathfrak{a})^{it}$ of characters of J_K defines a one-parameter group of automorphisms σ_t of $C_{K,\infty}$.

We want to show that the system $(C_{K,\infty}, \sigma)$ is isomorphic to the system $(A_{\mathbb{K},\{\infty\}}, \sigma)$ introduced in Section 2.1. The latter system was defined using the action of J_K^+ on $Y_{\mathbb{K},\{\infty\}} = \text{Gal}(\mathbb{K}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$. Define an action of $\text{Gal}(\mathbb{K}/K)$ on $Y_{\mathbb{K},\{\infty\}}$ by $g(x, y) = (g^{-1}x, y)$.

THEOREM 2.3.1. *There is a* $\text{Gal}(\mathbb{K}/K)$*- and* J_K*-equivariant homeomorphism*

$$\pi\colon X \to Y_{\mathbb{K},\{\infty\}}.$$

In particular, the C-dynamical systems* $(C_{K,\infty}, \sigma)$ *and* $(A_{\mathbb{K},\{\infty\}}, \sigma)$ *are isomorphic.*

PROOF. Fix a sgn-normalized rank-one Drinfeld module ϕ^0. Recall that the semigroup $\mathcal{P}^+$ of principal ideals with positive generators acts trivially on ϕ^0, so we have an action of $\mathcal{P}^+$ on X_{ϕ^0}. The semigroup $\mathcal{P}^+$ can be identified with $\mathcal{O}_+^\times$. The latter semigroup acts on $\hat{\mathcal{O}}$ by multiplication. Let us show first that there exists a $\mathcal{P}^+$-equivariant continuous isomorphism $\pi^0\colon X_{\phi^0} \to \hat{\mathcal{O}}$.

The $\mathcal{O}$-module $\phi^0(\mathbb{C}_\infty)^{\text{tor}}$ is isomorphic to $K/\mathcal{O}$. Indeed, if ϕ^0 is defined by a lattice Λ then $\phi^0(\mathbb{C}_\infty)^{\text{tor}} \cong K\Lambda/\Lambda$. The lattice Λ has the form $\xi\mathfrak{a}$ for some $\xi \in \mathbb{C}_\infty^\times$ and $\mathfrak{a} \subset \mathcal{O}$. Then $K\Lambda/\Lambda \cong K/\mathfrak{a}$. Next, the closure of $\mathfrak{a}$ in $\hat{\mathcal{O}}$ has the form $g\hat{\mathcal{O}}$ for some $g \in \mathbb{A}_{K,f}^* \cap \hat{\mathcal{O}}$. Therefore

$$K/\mathfrak{a} = \mathbb{A}_{K,f}/g\hat{\mathcal{O}} \cong \mathbb{A}_{K,f}/\hat{\mathcal{O}} = K/\mathcal{O},$$

and hence $\phi^0(\mathbb{C}_\infty)^{\text{tor}} \cong K/\mathcal{O}$.

As discussed in Section 1.1.1, the additive group is self-dual via a pairing $\mathbb{A}_{K,f} \times \mathbb{A}_{K,f} \to \mathbb{C}$ given by $(a, b) \mapsto \omega(ab)$ where ω is a character of $\mathbb{A}_{K,f}$. Furthermore, we can choose ω such that the annihilator of $\hat{\mathcal{O}} \subset \mathbb{A}_{K,f}$ is $\hat{\mathcal{O}}$. It follows that there exists an $\mathcal{O}$-module isomorphism

$$\widehat{K/\mathcal{O}} = \widehat{\mathbb{A}_{K,f}/\hat{\mathcal{O}}} \cong \hat{\mathcal{O}},$$

where the $\mathcal{O}$-module structure on $\widehat{K/\mathcal{O}}$ is defined by $a\chi = \chi(a\,\cdot)$.

We therefore get an $\mathcal{O}$-module isomorphism

$$X_{\phi^0} = \widehat{\phi^0(\mathbb{C}_\infty)^{\text{tor}}} \cong \hat{\mathcal{O}},$$

where the $\mathcal{O}$-module structure on $\widehat{\phi^0(\mathbb{C}_\infty)^{\text{tor}}}$ is defined by $a\chi = \chi \circ \phi^0_a$. Since $\phi^0_{(a)} = \phi^0_a$ for positive a this isomorphism is $\mathcal{P}^+$-equivariant.

Consider now the extension H^+/K defined in Section 1.3. As we observed there, the Artin map $\mathbb{A}^*_{K,f} \to \text{Gal}(\mathbb{K}/K)$ gives an isomorphism of $\hat{\mathcal{O}}^*$ onto $\text{Gal}(\mathbb{K}/H^+)$, so the open subset

$$Y^0 = \text{Gal}(\mathbb{K}/H^+) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}} \subset Y_{\mathbb{K},\{\infty\}}$$

can be identified with $\hat{\mathcal{O}}$. Hence the map π^0 can be considered to be a homeomorphism of X_{ϕ^0} onto Y^0.

We want to extend π^0 to a map $\pi\colon X \to Y_{\mathbb{K},\{\infty\}}$. Let ϕ be a sgn-normalized rank-one Drinfeld module and $\chi \in X_\phi$ be a character. There exists an ideal $\mathfrak{a}$ such that $\phi = \mathfrak{a} * \phi^0$. Then $\chi^{\mathfrak{a}} = \chi \circ \phi^0_{\mathfrak{a}} \in X_{\phi^0}$. By definition of $\phi^0_{\mathfrak{a}}$, the kernel of $\phi^0_{\mathfrak{a}}\colon \phi^0(\mathbb{C}_\infty)^{\text{tor}} \to \phi(\mathbb{C}_\infty)^{\text{tor}}$ is exactly $\phi^0[\mathfrak{a}]$. Therefore the kernel of $\chi^{\mathfrak{a}}$ contains $\phi^0[\mathfrak{a}]$. Hence, under our isomorphism of $\phi^0(\mathbb{C}_\infty)^{\text{tor}}$ with $K/\mathcal{O}$ the kernel of $\chi^{\mathfrak{a}}$ contains $\mathfrak{a}^{-1}\mathcal{O}/\mathcal{O}$. Since the annihilator of $\mathfrak{a}^{-1}\hat{\mathcal{O}}$ in $\mathbb{A}_{K,f}$ is $\mathfrak{a}\hat{\mathcal{O}}$ we conclude that $\pi^0(\chi^{\mathfrak{a}})$ is an element of $\mathfrak{a}\hat{\mathcal{O}}$. In particular $\pi^0(\chi^{\mathfrak{a}})$ is an element of $\mathfrak{a}Y_{\mathbb{K},\{\infty\}}$, so we can define

$$\pi(\chi) = \mathfrak{a}^{-1}\pi^0(\chi^{\mathfrak{a}}) \in Y_{\mathbb{K},\{\infty\}}.$$

Since π^0 is $\mathcal{P}^+$-equivariant this definition does not depend on the choice of $\mathfrak{a}$ (such that $\phi = \mathfrak{a} * \phi^0$).

We have therefore extended π^0 to a continuous map $\pi\colon X \to Y_{\mathbb{K},\{\infty\}}$. By construction this map is equivariant with respect to the ideal action.

Next let us show that π is $\text{Gal}(\mathbb{K}/K)$-equivariant. For every nonzero proper ideal $\mathfrak{m} \subset \mathcal{O}$ consider the finite sets

$$X_{\mathfrak{m}} = \bigsqcup_{\phi} \widehat{\phi[\mathfrak{m}]}$$

and

$$Y_{\mathfrak{m}} = \text{Gal}(K_{\mathfrak{m}}/K) \times_{\hat{\mathcal{O}}^*} (\hat{\mathcal{O}}/\mathfrak{m}\hat{\mathcal{O}}) = \text{Gal}(K_{\mathfrak{m}}/K) \times_{(\mathcal{O}/\mathfrak{m})^*} (\mathcal{O}/\mathfrak{m}).$$

Similarly to X and $Y_{\mathbb{K},\{\infty\}}$ these sets carry actions of $\text{Gal}(\mathbb{K}/K)$ and J_K; note, however, that if $\mathfrak{a}$ is not prime to $\mathfrak{m}$ then the actions by $\mathfrak{a}$ are defined by non-injective maps. Our isomorphism $\widehat{\phi^0(\mathbb{C}_\infty)^{\text{tor}}} \simeq \hat{\mathcal{O}}$ induces an isomorphism $\widehat{\phi^0[\mathfrak{m}]} \simeq \hat{\mathcal{O}}/\mathfrak{m}\hat{\mathcal{O}} = \mathcal{O}/\mathfrak{m}$. Since we know that $\text{Gal}(K_{\mathfrak{m}}/H^+) \simeq (\mathcal{O}/\mathfrak{m})^*$ we can identify

$$Y^0_{\mathfrak{m}} = \text{Gal}(K_{\mathfrak{m}}/H^+) \times_{(\mathcal{O}/\mathfrak{m})^*} (\mathcal{O}/\mathfrak{m}) \subset Y_{\mathfrak{m}}$$

with $\mathcal{O}/\mathfrak{m}$. Hence, similarly to the construction of π, by choosing for every ϕ an ideal $\mathfrak{a}_\phi \in I_\mathfrak{m}(\mathcal{O})$ such that $\phi = \mathfrak{a}_\phi * \phi^0$ we can extend the isomorphism $\widehat{\phi^0[\mathfrak{m}]} \cong \mathcal{O}/\mathfrak{m}$ to a map $\pi_\mathfrak{m} \colon X_\mathfrak{m} \to Y_\mathfrak{m}$.

Clearly the diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ \downarrow & & \downarrow \\ X_\mathfrak{m} & \xrightarrow[\pi_\mathfrak{m}]{} & Y_\mathfrak{m} \end{array}$$

is commutative, where the vertical arrows are the obvious quotient maps. It follows that $\pi_\mathfrak{m}$ is independent of the choice of $\mathfrak{a}_\phi$ and is I-equivariant.

Take a sgn-normalized rank-one Drinfeld module ϕ, a character $\chi \in \widehat{\phi[\mathfrak{m}]}$, and $\mathfrak{a} \in I_\mathfrak{m}(\mathcal{O})$. For $\lambda \in (\mathfrak{a}^{-1} * \phi)[\mathfrak{m}] = \sigma_\mathfrak{a}^{-1}(\phi)[\mathfrak{m}]$ we have

$$\sigma_\mathfrak{a}(\lambda) = (\mathfrak{a}^{-1} * \phi)_\mathfrak{a}(\lambda),$$

so $\sigma_\mathfrak{a}\chi = \chi \circ \sigma_\mathfrak{a} = \chi \circ (\mathfrak{a}^{-1} * \phi)_\mathfrak{a} = \chi^\mathfrak{a}$. Hence the Galois action of $\mathrm{Gal}(K_\mathfrak{m}/K)$ on $X_\mathfrak{m}$ corresponds to the ideal action of $I_\mathfrak{m}(\mathcal{O})$ via the Artin map.

Under the map $\pi_\mathfrak{m}$ above, this Galois action is then transported to $Y_\mathfrak{m}$ by

$$\pi_\mathfrak{m}(\sigma_\mathfrak{a}\chi) = \pi_\mathfrak{m}(\chi^\mathfrak{a}) = \mathfrak{a}\pi_\mathfrak{m}(\chi).$$

Recall now that the action of $\mathfrak{a}$ on $Y_\mathfrak{m}$ is defined using the action $\mathfrak{a}(x, y) = (x r_{K_\mathfrak{m}/K}(g)^{-1}, gy)$ on $\mathrm{Gal}(K_\mathfrak{m}/K) \times (\hat{\mathcal{O}}/\mathfrak{m}\hat{\mathcal{O}})$, where $g \in \mathbb{A}^*_{K,f} \cap \hat{\mathcal{O}}$ is any element such that $\mathfrak{a} = g\hat{\mathcal{O}} \cap \mathcal{O}$. Since $\mathfrak{a}$ is prime to $\mathfrak{m}$, we can take g such that $g_\mathfrak{p} = 1$ for all primes $\mathfrak{p}$ dividing $\mathfrak{m}$. Then the action of g on $\hat{\mathcal{O}}/\mathfrak{m}\hat{\mathcal{O}}$ is trivial and $r_{K_\mathfrak{m}/K}(g) = \sigma_\mathfrak{a}$. Therefore $\mathfrak{a}\pi_\mathfrak{m}(\chi) = \sigma_\mathfrak{a}\pi_\mathfrak{m}(\chi)$, so that

$$\pi_\mathfrak{m}(\sigma_\mathfrak{a}\chi) = \sigma_\mathfrak{a}\pi_\mathfrak{m}(\chi).$$

Since the Artin map $I_\mathfrak{m}(\mathcal{O}) \to \mathrm{Gal}(K_\mathfrak{m}/K)$ is surjective we conclude that $\pi_\mathfrak{m}$ is $\mathrm{Gal}(K_\mathfrak{m}/K)$-equivariant.

Now note that $X = \varprojlim X_\mathfrak{m}$ and $Y = \varprojlim Y_\mathfrak{m}$. Hence the $\mathrm{Gal}(\mathbb{K}/K)$-equivariance of $\pi \colon X \to Y$ follows from the $\mathrm{Gal}(K_\mathfrak{m}/K)$-equivariance of $\pi_\mathfrak{m}$.

It remains to show that π is a homeomorphism. The space X is the disjoint union of the open sets gX_{ϕ^0}, where g runs over representatives of

$$\mathrm{Gal}(\mathbb{K}/K)/\mathrm{Gal}(\mathbb{K}/H^+) \cong \mathrm{Gal}(H^+/K) \cong \mathrm{Pic}^+(\mathcal{O}).$$

Similarly, $Y_{\mathbb{K},\{\infty\}}$ is the disjoint union of the sets gY^0. Since π is $\mathrm{Gal}(\mathbb{K}/K)$-equivariant and defines a homeomorphism of X_{ϕ^0} onto Y^0, we conclude that π is a homeomorphism. $\square$

REMARK 2.3.2. The map π depends on the choice of a rank-one sgn-normalized Drinfeld module ϕ^0, the choice of an $\mathcal{O}$-module isomorphism $\phi^0(\mathbb{C}_\infty)^{\mathrm{tor}} \cong K/\mathcal{O}$, and the choice of a character ω of $\mathbb{A}_{K,f}$ defining a pairing on $\mathbb{A}_{K,f} \times \mathbb{A}_{K,f}$ such that $\hat{\mathcal{O}}^\perp = \hat{\mathcal{O}}$. It is not difficult to see that if $\pi'\colon X \to Y_{\mathbb{K},\{\infty\}}$ is another $\mathrm{Gal}(\mathbb{K}/K)$- and J_K-equivariant homeomorphism, e.g. one constructed using a different choice of the above three ingredients, then $\pi'(\chi) = g\pi(\chi)$ for a uniquely defined $g \in \mathrm{Gal}(\mathbb{K}/K)$.

Applied to the system $(A_{\mathbb{K},\{\infty\}}, \sigma)$, our Theorem 2.1.1 summarizes [**18**, Theorems 4.3.10, 4.4.15]. Furthermore, by Theorem 2.2.1 and Corollary 1.3.10 the type of the unique KMS_β-state of Jacob's system for $\beta \in (0,1]$ is $\mathrm{III}_{q^{-\beta d_\infty}}$. This corrects a mistake in [**18**, Theorem 4.5.8], which asserts that the type is $\mathrm{III}_{q^{-\beta}}$, a mistake partially caused by a wrong formulation of the Chebotarev density theorem.[1]

Another approach to defining a Bost-Connes system for function fields is that of Consani and Marcolli [**9**]. Their setting is different, as they develop a theory of dynamical systems for algebras of $\mathbb{C}_\infty$-valued functions. However, these algebras arise from groupoids, which makes it natural to consider the relationship between these groupoids and the ones we consider in this thesis.

Recall that a one-dimensional K-lattice in K_∞ is a pair (Λ, φ), where $\Lambda \subset K_\infty$ is a rank-one lattice and $\varphi\colon K/\mathcal{O} \to K\Lambda/\Lambda$ is an $\mathcal{O}$-module map. Two one-dimensional K-lattices (Λ_1, φ_1) and (Λ_2, φ_2) are called commensurable if Λ_1 and Λ_2 are commensurable (or equivalently, if $K\Lambda_1 = K\Lambda_2$) and the maps $K/\mathcal{O} \to K\Lambda_i/(\Lambda_1 + \Lambda_2)$ defined by φ_1 and φ_2 coincide.

By an argument identical to that of Theorem 1.2.2, the rank one K-lattices in K_∞ can be parametrized by the set

$$K_\infty^\times \times_{K^\times} \mathbb{A}_{K,f}^* \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}.$$

[1]The computation in [**18**] relies on this theorem, but in a way that is different from ours. Unfortunately, the strategy in [**18**] does not work even for $d_\infty = 1$, when the formulation of the Chebotarev density theorem becomes correct. The mistake is in the proof of crucial Lemma 4.5.5, the same lemma where the Chebotarev density theorem is used, which does not take into account that the elements $\mu_{\mathfrak{p}}$ and $\mu_{\mathfrak{q}}$ do not belong to $M[\mathfrak{d}]$. In fact, the assertion of that lemma is not correct, as it can be shown that already the center of $M[\mathfrak{d}]$ has elements that are not $\mathrm{Gal}(K_{\mathfrak{d}}/k)$-invariant.

There is a partial action of $\mathbb{A}^*_{K,f}$ on this space given by

$$g(\xi, x, y) = (\xi, xg^{-1}, gy)$$

as long as $gy \in \hat{\mathcal{O}}$, and which is undefined if this is not the case. This action descends to a partial action of fractional ideals of $\mathcal{O}$, and the corresponding orbit equivalence relation is exactly the relation of commensurability. The action defines a groupoid $\tilde{\mathcal{G}}$ with the set of one-dimensional K-lattices in K_∞ as its object space.

Let $\mathcal{G}_{CM}$ be the quotient groupoid obtained by identifying elements (ξ, x, y) and $(\zeta\xi, x, y)$ in the object space for $\zeta \in K_\infty^\times$. This is the main groupoid considered in [**9**].

It has the object space

$$\mathcal{G}^0_{CM} = K^\times \backslash \mathbb{A}^*_{K,f} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}} \simeq \mathrm{Gal}(K^{\mathrm{ab},\infty}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}} = Y_{K^{\mathrm{ab},\infty},\{\infty\}}.$$

This identification respects the actions of the semigroup J_K^+ of ideals, so the groupoid considered in [**9**] gives rise to the dynamical system $(A_{K^{\mathrm{ab},\infty},\{\infty\}}, \sigma)$. By the same reasoning as for Jacob's system above, the type of the unique KMS$_\beta$-state for this system for $\beta \in (0,1]$ is $\mathrm{III}_{q^{-\beta d_\infty}}$.

Theorem 2.3.1 clarifies the relation between the groupoids $\mathcal{G}_{CM}$ and $\mathcal{G}_{\mathrm{Jacob}}$ of Consani-Marcolli and Jacob: $\mathcal{G}_{CM}$ is isomorphic to the quotient of $\mathcal{G}_{\mathrm{Jacob}}$ by the action of $\mathrm{Gal}(\mathbb{K}/K^{\mathrm{ab},\infty}) \cong K^\times/K_+^\times \cong \mathbb{F}_\infty^\times$. Also note that in the case of the Bost-Connes system for $\mathbb{Q}$ we only divide out by scaling by positive reals. The natural analogue of the positive reals in our setting is the subgroup $K_\infty^+ \subset K_\infty^\times$ of positive elements. The groupoid $K_\infty^+ \backslash \tilde{\mathcal{G}}$ has the object space

$$K_\infty^+ \backslash K_\infty^\times \times_{K^\times} \mathbb{A}^*_{K,f} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}} \simeq \mathbb{F}_\infty^\times \times_{K^\times} \mathbb{A}^*_{K,f} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}} \simeq \mathrm{Gal}(\mathbb{K}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}},$$

and the ideal action is identical to the one on $Y_{\mathbb{K},\{\infty\}}$. Therefore, the modification of the construction of Consani-Marcolli obtained by considering K-lattices in K_∞ up to scaling by positive elements gives rise to a groupoid isomorphic to Jacob's groupoid $\mathcal{G}_{\mathrm{Jacob}}$.

2.4. Systems arising from Hecke algebras

Given a number field K, the authors of [**25**] consider the pair of groups

$$P^+_{\mathcal{O}} = \begin{pmatrix} 1 & \mathcal{O} \\ 0 & \mathcal{O}^+ \end{pmatrix} \subset P^+_K = \begin{pmatrix} 1 & K \\ 0 & K^+ \end{pmatrix}$$

where $\mathcal{O}^+$ and K^+ are the totally positive elements of $\mathcal{O}^*$ and K^* respectively, that is the elements which are positive in every real embedding of K. They show that the pair $(P^+_K, P^+_{\mathcal{O}})$ is a Hecke pair, and that the associated Hecke C*-algebra can be identified with a full corner of the Bost-Connes system of the number field K.

The situation in the function field case is very similar, although a bit simpler since every positively generated principal ideal has a unique positive generator. We will follow the presentation of [**25**] closely in this section.

In the function field case, the natural analogue of the totally positive elements are the positive elements K^+ with respect to our chosen sign-function sgn. Since $\mathcal{O}^* \subset \mathbb{F}_\infty^\times$ we see that $\mathcal{O}^* \cap K^+ = \{1\}$, so the groups we should consider are

$$P_{\mathcal{O}}^+ = \begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix} \subset P_K^+ = \begin{pmatrix} 1 & K \\ 0 & K^+ \end{pmatrix}.$$

LEMMA 2.4.1. *The pair $(P_K^+, P_{\mathcal{O}}^+)$ is a Hecke pair with*

$$\Delta \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} = \frac{|\mathcal{O}/(\mathcal{O} \cap x\mathcal{O})|}{|\mathcal{O}/(\mathcal{O} \cap x^{-1}\mathcal{O})|}.$$

To prove this, let us recall the following result, which is Proposition 1.3 of [**23**]:

PROPOSITION 2.4.2. *Let G be a group acting by automorphisms on a group V, and let G_0 be a subgroup of G leaving a subgroup V_0 of V invariant. Then $(V \rtimes G, V_0 \rtimes G_0)$ is a Hecke pair if an only if the following conditions are satisfied:*

(i) *(V, V_0) is a Hecke pair such that the action of G_0 on $V_0 \backslash V / V_0$ has finite orbits and*

(ii) *(G, G_0) is a Hecke pair such that V_0 and $g(V_0)$ are commensurable subgroups of V for every $g \in G$.*

Furthermore, if this is the case then

$$\Delta_{V_0 \rtimes G_0}(vg) = \Delta_{V_0}(v)\Delta_{G_0}(g)\frac{|V_0/(V_0 \cap g(V_0)|}{|V_0/(V_0 \cap g^{-1}(V_0))|}.$$

PROOF OF THE LEMMA. We apply the proposition to $(V, V_0) = (K, \mathcal{O})$ and $(G, G_0) = (K^+, 1)$, with the G-action given by multiplication. Then condition (i) is trivially satisfied since K is abelian and $G_0 = \{1\}$ is trivial.

We claim that for $b \in \mathcal{O}$ the ring $\mathcal{O}/b\mathcal{O}$ is finite. Indeed, let v be a prime of K such that $b \notin \mathcal{O}_v^*$. Then the map $\mathcal{O} \to \mathcal{O}_v/b\mathcal{O}_v$ has kernel $\mathcal{O} \cap b\mathcal{O}_v = b\mathcal{O}$, so it suffices to show that $\mathcal{O}_v/b\mathcal{O}_v$ is finite. But this ring has $|\mathbb{F}_v|^{v(b)}$ elements.

Now, to see that (ii) holds note that $(K^+, 1)$ is (trivially) a Hecke pair, and that for $a \in K^+$ with $a = b/c$ in lowest terms we have

$a\mathcal{O} \cap \mathcal{O} = b\mathcal{O}$. Hence $\mathcal{O}/b\mathcal{O}$ is finite by the claim above, while we have $a\mathcal{O}/b\mathcal{O} \simeq \mathcal{O}/c\mathcal{O}$ which again is finite.

Finally, the formula of the proposition gives

$$\Delta_{P^+_{\mathcal{O}}} \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} = \frac{|\mathcal{O}/(\mathcal{O} \cap x\mathcal{O})|}{|\mathcal{O}/(\mathcal{O} \cap x^{-1}\mathcal{O})|}$$

without any work. □

Recall that if (G, G_0) is a Hecke pair, we say that the pair is reduced if

$$\bigcap_{g \in G} g^{-1} G_0 g = \{e\}.$$

The following is Proposition 4.1 of [**31**].

Proposition 2.4.3. *(G, G_0) is a Hecke pair if and only if there is a reduced pair (G', G'_0) such that*

(i) *G' is a totally disconnected locally compact topological group;*

(ii) *G'_0 is a compact-open subgroup of G';*

(iii) *There exists a group homomorphism $\phi : G \to G'$ such that $\overline{\phi(G)} = G'$ and $\phi^{-1}(\phi(G) \cap G'_0) = G_0$.*

The pair (G', G'_0) is unique.

The pair (G', G'_0) of the proposition is called the Schlichting completion of (G, G_0). (See [**19**].)

Lemma 2.4.4. *The Schlichting completion of $(P^+_K, P^+_{\mathcal{O}})$ is given by $(\bar{P}^+_K, \bar{P}^+_{\mathcal{O}})$, where*

$$\bar{P}^+_{\mathcal{O}} = \begin{pmatrix} 1 & \hat{\mathcal{O}} \\ 0 & 1 \end{pmatrix} \subset \bar{P}^+_K = \begin{pmatrix} 1 & \mathbb{A}_{K,f} \\ 0 & K^+ \end{pmatrix}.$$

Proof. Note that for $g = \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}$ we have

$$g^{-1} \bar{P}^+_{\mathcal{O}} g = \begin{pmatrix} 1 & a\hat{\mathcal{O}} \\ 0 & 1 \end{pmatrix},$$

so since $\bigcap_{a \in K^+} a\hat{\mathcal{O}} = \{0\}$ we get $\bigcap_{g \in G} g^{-1} \bar{P}^+_{\mathcal{O}} g = \{e\}$. Thus $(\bar{P}^+_K, \bar{P}^+_{\mathcal{O}})$ is reduced.

The other conditions are immediate from the definitions. □

Proposition 2.4.5. *The C^*-algebra $C^*_r(P^+_K, P^+_{\mathcal{O}})$ is isomorphic to*

$$\mathbf{1}_{\hat{\mathcal{O}}}(C_0(\mathbb{A}_{K,f}) \rtimes_\alpha K^+)\mathbf{1}_{\hat{\mathcal{O}}},$$

where the action α is given by $\alpha_x(f) = f(x^{-1}\cdot)$.

PROOF. By Proposition 4.2 of [**31**]

$$C_r^*(P_K^+, P_{\mathcal{O}}^+) \simeq pC^*(\bar{P}_K^+)p,$$

where p is the characteristic function of $\bar{P}_{\mathcal{O}}^+ \subset \bar{P}_K^+$.

Since $\bar{P}_K^+ \simeq \mathbb{A}_{K,f} \rtimes K^+$ we have an isomorphism

$$C_r^*(\bar{P}_K^+) \simeq C_r^*(\mathbb{A}_{K,f}) \rtimes K^+.$$

Furthermore, since $\hat{\mathbb{A}}_{K,f} \simeq \mathbb{A}_{K,f}$ by the arguments of Section 1.1.1 we furthermore have an isomorphism $C_r^*(\mathbb{A}_{K,f}) \simeq C_0(\mathbb{A}_{K,f})$. Thus $C_r^*(\bar{P}_K^+) \simeq C_0(\mathbb{A}_{K,f}) \rtimes K^+$. The action of K^+ on $C_0(\mathbb{A}_{K,f})$ is here given by $\alpha_x(f) = f(x^{-1}\cdot)$. We claim that the projection p is carried to $\mathbf{1}_{\hat{\mathcal{O}}}$.

Indeed, under the isomorphism $C_r^*(\bar{P}_K^+) \simeq C_r^*(\mathbb{A}_{K,f}) \rtimes K^+$ the projection p is mapped to a projection $p' \in C_r^*(\mathbb{A}_{K,f})$ corresponding to $\hat{\mathcal{O}}$. Since our self-duality of $\mathbb{A}_{K,f}$ is chosen in such a way that $\hat{\mathcal{O}}^\perp = \hat{\mathcal{O}}$, the Fourier transform carries p' to $\mathbf{1}_{\hat{\mathcal{O}}}$ as claimed. □

Thus $C_r^*(P_K^+, P_{\mathcal{O}}^+)$ is a full corner in $C_0(\mathbb{A}_{K,f}) \rtimes K^+$. The K^+-action can be induced to an action of the group J_K of fractional ideals of $\mathcal{O}$ by considering the inclusion of the group of positively generated principal ideals $\mathcal{P}^+ \simeq K^+$ into J_K.

We will need the following result, which is Proposition 1.2 of [**25**]:

LEMMA 2.4.6. *Let G be a discrete group, and $H \subset G$ a subgroup. Let X be a locally compact space with an H-action. Let $i : X \to G \times_H X$ given by $i(x) = \overline{(e, x)}$. Then $i(X)$ is a clopen subset of $G \times_H X$, the corresponding projection in the multiplier algebra of $C_0(G \times_H X) \rtimes_r G$ is full, and*

$$C_0(X) \rtimes_r H \simeq \mathbf{1}_{i(X)}(C_0(G \times_H X) \rtimes_r G)\mathbf{1}_{i(X)}.$$

Let $X_K^+ = J_K \times_{K^+} \mathbb{A}_{K,f}$ and consider the subset $Y_K^+ \subset X_K^+$ given by

$$Y_K^+ = \{(g, \rho) \in X_K^+ : g\rho \in \hat{\mathcal{O}}/\hat{\mathcal{O}}^*\},$$

where we consider g as an element of $\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*$ when taking the product $g\rho$. Since $\hat{\mathcal{O}}$ is compact open in $\mathbb{A}_{K,f}$ and K^+ has finite index in J_K the set Y_K^+ is compact open in X_K^+. Let

$$A_K^+ = \mathbf{1}_{Y_K^+}(C_0(X_K^+) \rtimes J_K)\mathbf{1}_{Y_K^+}.$$

LEMMA 2.4.7. *The map $\phi : \mathbb{A}_{K,f}^* \times \mathbb{A}_{K,f} \to \mathbb{A}_{K,f}^* \times \mathbb{A}_{K,f}$ given by $\phi(x, y) = (x^{-1}, xy)$ induces a J_K-equivariant homeomorphism*

$$X_{\mathbb{K},\{\infty\}} \simeq X_K^+.$$

Under this homeomorphism $Y_{\mathbb{K},\{\infty\}}$ is mapped onto Y_K^+, and the set

$$Z_{H^+} = \mathrm{Gal}(\mathbb{K}/H^+) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}} \subset Y_K$$

is mapped onto $i(\hat{\mathcal{O}})$ where $i : \mathbb{A}_{K,f} \to X_K^+$ is the embedding.

Proof. Let $\mathbb{A}_{K,f}^* \times \mathbb{A}_{K,f}^*$ act on $\mathbb{A}_{K,f}^* \times \mathbb{A}_{K,f}$ by $(g,h)(x,y) = (gxh^{-1}, hy)$. Then $\phi((g,h)(x,y)) = (h,g)\phi(x,y)$. If we restrict the action to the subgroup $K^+ \times \hat{\mathcal{O}}^*$, the isomorphism ϕ thus induces a homeomorphism

$$(\mathbb{A}_{K,f}^* \times \mathbb{A}_{K,f})/(K^+ \times \hat{\mathcal{O}}^*) \simeq (\mathbb{A}_{K,f}^* \times \mathbb{A}_{K,f})/(\hat{\mathcal{O}}^* \times K^+).$$

The right-hand quotient is then $(\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*) \times_{K^+} \mathbb{A}_{K,f}$, which we recognize as X_K^+ after recalling the isomorphism $J_K \simeq \mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*$. The left-hand quotient is

$$K^+ \backslash \mathbb{A}_{K,f}^* \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f},$$

which is indeed $X_{\mathbb{K},\{\infty\}}$, since $K^+ \backslash \mathbb{A}_{K,f}^* \simeq \mathrm{Gal}(\mathbb{K}/K)$.

Now ϕ is $\mathbb{A}_{K,f}^*$-equivariant with respect to the actions given by $g(x,y) = (xg^{-1}, gy)$ on the domain and $g(x,y) = (gx, y)$ on the image. Hence the induced homeomorphism $X_{\mathbb{K},\{\infty\}} \to X_K^+$ is J_K-equivariant.

Furthermore, $Y_{\mathbb{K},\{\infty\}}$ is the image of $\mathbb{A}_{K,f}^* \times \hat{\mathcal{O}}$, which maps to the set $\{(x,y) \in \mathbb{A}_{K,f}^* \times \mathbb{A}_{K,f} : xy \in \hat{\mathcal{O}}\}$ under ϕ. The image of this set under the quotient map is nothing by the set Y_K^+, proving our second assertion.

Finally, recall from Lemma 1.3.8 that $\mathrm{Gal}(\mathbb{K}/H^+) \simeq \hat{\mathcal{O}}^*$ under the Artin map, so $\mathrm{Gal}(\mathbb{K}/H^+) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$ is the image of $\hat{\mathcal{O}}^* \times \hat{\mathcal{O}}$. This maps to $\hat{\mathcal{O}}^* \times \hat{\mathcal{O}}$ by ϕ, which is again maps to $\{\mathcal{O}\} \times \hat{\mathcal{O}} = i(\hat{\mathcal{O}}) \subset X_K^+$. □

Theorem 2.4.8. *The homeomorphism of the lemma gives rise to a canonical isomorphism of C^*-dynamical systems $(A_{\mathbb{K},\{\infty\}}, \sigma) \simeq (A_K^+, \sigma)$. This induces an isomorphism*

$$C_r^*(P_K^+, P_{\mathcal{O}}^+) \simeq p_K A_K p_K$$

of the Hecke algebra to the full corner of A_K defined by the projection p_K corresponding to the compact open subset $Z_{H^+} \subset Y_K$.

Proof. By the lemma, the homeomorphism of X_K to X_K^+ induces an isomorphism $(A_K, \sigma) \simeq (A_K^+, \sigma)$ mapping $p_K A_K p_K$ to

$$\mathbf{1}_{i(\hat{\mathcal{O}})} A_K^+ \mathbf{1}_{i(\hat{\mathcal{O}})} = \mathbf{1}_{i(\hat{\mathcal{O}})} (C_0(X_K^+) \rtimes J_K^+) \mathbf{1}_{i(\hat{\mathcal{O}})}.$$

This algebra in turn is isomorphic to $\mathbf{1}_{\hat{\mathcal{O}}}(C_0(\mathbb{A}_{K,f}) \rtimes K^+)\mathbf{1}_{\hat{\mathcal{O}}}$ by Lemma 2.4.6, which finally is isomorphic to $C_r^*(P_K^+, P_{\mathcal{O}}^+)$ by Proposition 2.4.5. □

Remark 2.4.9. The system we get here corresponds to the extension $\mathbb{K}/K$ which depends on the choices of both ∞ and sgn. If we instead consider the Hecke pair

$$P_{\mathcal{O}} = \begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix} \subset P_K = \begin{pmatrix} 1 & K \\ 0 & K^{\times} \end{pmatrix}$$

an identical argument would show that $C_r^*(P_K, P_{\mathcal{O}})$ is a full corner in $A_{K^{\mathrm{ab},\infty},\{\infty\}}$, which is the algebra arising from the considerations in [**9**]. In this case we have also removed the sgn-dependence from our construction.

CHAPTER 3

Function field-valued Bost-Connes systems associated with function fields

While Bost-Connes systems for function fields can be studied as in the previous chapter, one cannot hope to construct an arithmetic subalgebra as long as the algebras considered are complex-valued. For this one needs algebras with values in a field of positive characteristic.

Steps in this direction were taken by Consani and Marcolli [**9**], who defined the concept of a dynamical system over a field of positive characteristic and constructed a system associated to a function field K.

In the first section of this chapter we construct a dynamical system $(A_{L,S}, \sigma)$ associated with an arbitrary abelian extension L of K, while the second section is dedicated to a (partial) classification of the KMS states of this system. The final three sections are devoted to the problem of constructing an arithmetic subalgebra. In section three we show that such subalgebras exist in the general setting, although the construction is not explicit. The fourth and fifth sections include explicit constructions of such subalgebras for the cases $L = \mathbb{K}$, $S = \{\infty\}$ and $K = \mathbb{F}_q(T)$, $L = (\mathbb{F}_q(T))^{\mathrm{ab}}$, $S = \{\infty\}$, respectively.

3.1. Dynamical systems

As in the previous chapter consider the space

$$X_{L,S} = \mathrm{Gal}(L/K) \times_{\hat{\mathcal{O}}_S^*} \mathbb{A}_{K,S},$$

where the $\hat{\mathcal{O}}_S^*$-action on $\mathrm{Gal}(L/K)$ is defined using the Artin map

$$r_{L/K} : \mathbb{A}_K^* \to \mathrm{Gal}(L/K).$$

Consider the algebra $C_c(X_{L,S}, \mathbb{C}_\infty)$ of $\mathbb{C}_\infty$-valued functions on $X_{L,S}$ with compact support. The natural norm to consider on $C_c(X_{L,S}, \mathbb{C}_\infty)$ is the supremum norm $\|f\| = \sup_{x \in X_{L,S}} |f(x)|$. Let $C_0(X_{L,S})$ be the completion of $C_c(X_{L,S}, \mathbb{C}_\infty)$ with respect to this norm, where we suppress the $\mathbb{C}_\infty$ from the notation for simplicity.

The action of J_S on $X_{L,S}$ given by

$$\mathfrak{a}(x, y) = (x r_{L/K}(g)^{-1}, gy),$$

where $g \in \mathbb{A}^*_{K,S}$ is such that $\mathfrak{a} = g\hat{\mathcal{O}}_S \cap K$, induces an action of J_S on $C_0(X_{L,S})$ by $\alpha_\mathfrak{a}(f) = f(\mathfrak{a}^{-1}\cdot)$. This action is clearly norm-preserving.

DEFINITION 3.1.1. Let $C_0(X_{L,S}) \rtimes_{\mathrm{alg}} J_S$ be the $\mathbb{C}_\infty$-algebra whose underlying vector space is

$$\bigoplus_{\mathfrak{a}\in J_S} C_0(X_{L,S})u_\mathfrak{a}$$

with multiplication given by $f_1 u_{\mathfrak{a}_1} f_2 u_{\mathfrak{a}_2} = f_1 \alpha_{\mathfrak{a}_1}(f_2) u_{\mathfrak{a}_1\mathfrak{a}_2}$ and norm defined by $\| \sum_{\mathfrak{a}\in J_S} f_\mathfrak{a} u_\mathfrak{a}\| = \sup_\mathfrak{a} \|f_\mathfrak{a}\|$. Let $C_0(X_{L,S}) \rtimes J_S$ be the completion of $C_0(X_{L,S}) \rtimes_{\mathrm{alg}} J_S$ with respect to this norm.

It will be convenient to consider $C_0(X_{L,S})$ to be embedded into $C_0(X_{L,S}) \rtimes J_S$ via the map $f \mapsto f u_e$.

As in the complex-valued case we put

$$Y_{L,S} = \mathrm{Gal}(L/K) \times_{\hat{\mathcal{O}}^*_S} \hat{\mathcal{O}}_S \subset X_{L,S},$$

and write

$$A_{L,S} = \mathbf{1}_{Y_{L,S}}(C_0(X_{L,S}) \rtimes J_S)\mathbf{1}_{Y_{L,S}},$$

where $\mathbf{1}_{Y_{L,S}}$ is the characteristic function of $Y_{L,S}$. This is the $\mathbb{C}_\infty$-algebra of our dynamical system.

DEFINITION 3.1.2. Let $\mathcal{A}$ be a Banach algebra over $\mathbb{C}_\infty$. A (continuous) time evolution on $\mathcal{A}$ is a group homomorphism $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$ such that $y \mapsto \|\sigma_y(a)\|$ is continuous for all $a \in \mathcal{A}$. Say that σ extends to S_∞ if there is an extension $\tilde{\sigma} : S_\infty \to \mathrm{Aut}(\mathcal{A})$ of σ which still is continuous.

Recall from Section 1.1.3 that we have a map $J_S \times S_\infty \to \mathbb{C}_\infty^\times$, given for $z = (x, y) \in S_\infty$ by

$$\mathfrak{a}^z = x^{\deg \mathfrak{a}} \langle \mathfrak{a} \rangle^y.$$

PROPOSITION 3.1.3. *The map* $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(C_0(X_{L,S}) \rtimes J_S)$ *given on a dense subset by*

$$\sigma_y(f u_\mathfrak{a}) = \mathfrak{a}^{(1,-y)} f u_\mathfrak{a}$$

defines a continuous time evolution which extends to S_∞ *by*

$$\sigma_z(f u_\mathfrak{a}) = \mathfrak{a}^{-z} f u_\mathfrak{a}.$$

The σ_z *preserve the corner* $A_{L,S}$ *for each* $z \in S_\infty$, *so* σ *induces a continuous time evolution on* $A_{L,S}$.

Proof. It is clear that $z \mapsto \sigma_z$ is a homomorphism. To see continuity, let $fu_g \in C_0(X_{L,S}) \rtimes J_S$. Then if $z = (x, y) \in S_\infty$ we have

$$\|\sigma_z(fu_g)\| = \|x^{-\deg g}\langle g\rangle^{-y} fu_g\| = |x|^{-\deg g}\|fu_g\|,$$

so $z \mapsto \|\sigma_z(fu_g)\|$ is continuous.

Furthermore, σ_z preserves $\mathbf{1}_{Y_{L,S}}$ for all $z \in S_\infty$, so the corner $A_{L,S}$ is preserved, whence σ induces a continuous time evolution on $A_{L,S}$. □

Remark 3.1.4. The system constructed above generalises that of $\mathcal{A}_{\mathbb{C}_\infty}(\mathcal{L}_{K,n})$ in [**9**]. To see this, recall from Section 2.3 that the groupoid considered there is that of the equivalence relation with object space $\mathcal{G}^0_{CM} = K^\times \backslash \mathbb{A}^*_{K,f} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$ and equivalence given by $(r, \rho) \simeq (rg^{-1}, g\rho)$ for $g \in \mathbb{A}^*_{K,f}/\hat{\mathcal{O}}^*$ is such that $g\rho \in \hat{\mathcal{O}}$. The algebra considered is the the completion of the $\mathbb{C}_\infty$-valued groupoid algebra of this equivalence relation.

If we consider the case $L = K^{\mathrm{ab},\infty}$ and $S = \{\infty\}$, we clearly see that $\mathcal{G}_{CM} \simeq Y_{L,S}$ and that the equivalence relation arises from the J_S-action on $Y_{L,S}$. Hence the algebra considered in [**9**] is nothing else that $A_{L,S}$. Furthermore, in this case our time evolution mirrors exactly that defined there.

3.2. KMS functionals

Definition 3.2.1. Let $\mathcal{A}$ be a Banach algebra over $\mathbb{C}_\infty$ with a continuous time evolution σ which extends to S_∞. A continuous linear functional $\omega : \mathcal{A} \to \mathbb{C}_\infty$ is a KMS functional at inverse temperature x (or a KMS$_x$-functional) for $x \in \mathbb{C}^\times_\infty$ if $\|\omega\| \le 1$ and

$$\omega(f_1\sigma_x(f_2)) = \omega(f_2 f_1)$$

for all $f_1, f_2 \in \mathcal{A}$. If $\mathcal{A}$ is unital we also require $\omega(1) = 1$.

Let us consider our algebra $A_{L,S}$. The algebra $C(Y_{L,S})$ is embedded as a diagonal, and we have a natural linear map $\Phi : A_{L,S} \to C(Y_{L,S})$ given on an element $\mathbf{1}_{Y_{L,S}} fu_g \mathbf{1}_{Y_{L,S}}$ of the spanning subset by

$$\Phi(\mathbf{1}_{Y_{L,S}} fu_g \mathbf{1}_{Y_{L,S}}) = \begin{cases} f|_{Y_{L,S}} & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2.2. *The map Φ is a conditional expectation. That is, Φ is a continuous linear map such that*

(i) Φ *is idempotent;*

(ii) $\Phi(1) = 1$*;*

(iii) $\Phi(f_1 x f_2) = f_1\Phi(x) f_2$ *for* $x \in A_{L,S}$, $f_1, f_2 \in C(Y_{L,S})$.

PROOF. The only thing that is not immediate is part (iii). For this, let $\sum \mathbf{1}_{Y_{L,S}} f_g u_g \mathbf{1}_{Y_{L,S}}$ be a finite sum. Then

$$\Phi(f_1(\sum \mathbf{1}_{Y_{L,S}} f_g u_g \mathbf{1}_{Y_{L,S}})f_2) = f_1 f_e|_{Y_{L,S}} f_2 = f_1 \Phi(\sum \mathbf{1}_{Y_{L,S}} f_g u_g \mathbf{1}_{Y_{L,S}})f_2,$$

and by continuity this holds for all $x \in A_{L,S}$. □

In the complex-valued case all KMS states arise from probability measures on $Y_{L,S}$, as shown in Theorem 2.1.1. A similar result holds in our current setting if $|x| > 1$. The proof of the upcoming theorem is essentialy that of [**22**, Theorem 12].

If X is a topological space, we say that a functional $\phi : C(X) \to \mathbb{C}_\infty$ is probability type if $\phi(1) = 1$ and $|\phi(\mathbf{1}_E)| \leq 1$ for all $E \subset X$.

THEOREM 3.2.3. *Let $(A_{L,S}, \sigma)$ be as above and let $x \in \mathbb{C}_\infty^\times$ with $|x| > 1$. Then $\phi \mapsto \phi \circ \Phi$ is an affine isomorphism between the set of probability type functionals ϕ on $C(Y_{L,S})$ satisfying the scaling condition*

$$\phi \circ \alpha_{\mathfrak{a}} = \mathfrak{a}^{-x} \phi \tag{3.2.4}$$

for all $\mathfrak{a} \in J_S$, and the set of KMS$_x$-functionals on $A_{L,S}$.

PROOF. Since Φ is linear it suffices to prove that $\phi \mapsto \phi \circ \Phi$ is a bijection.

If ϕ is a probability type functional on $C(Y_{L,S})$ satisfying 3.2.4 then clearly $\phi \circ \Phi$ is a KMS$_x$-functional on $A_{L,S}$.

Conversely, assume $\omega : A_{L,S} \to \mathbb{C}_\infty$ is a KMS$_x$-functional. It suffices to show that $\omega(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) = 0$ whenever $\mathfrak{a} \neq \mathcal{O}$.

Let $\mathfrak{a} \in J_S \setminus \mathcal{O}$. Then we can write $\mathfrak{a} = \mathfrak{b}\mathfrak{c}^{-1}$ for $\mathfrak{b}, \mathfrak{c} \in J_S^+$ relatively prime. Note that we then have $\mathfrak{b}^{-1} Y_{L,S} \cap \mathfrak{c}^{-1} Y_{L,S} = Y_{L,S}$.

Assume first that $\mathfrak{b} \neq \mathcal{O}$. Then

$$\begin{aligned}
\omega(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) &= \omega(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{b}} u_{\mathfrak{c}^{-1}}) \\
&= \omega(\mathbf{1}_{Y_{L,S}} u_{\mathfrak{b}} \alpha_{\mathfrak{b}^{-1}}(f) \mathbf{1}_{\mathfrak{b}^{-1} Y_{L,S}} \mathbf{1}_{\mathfrak{c}^{-1} Y_{L,S}} u_{\mathfrak{c}^{-1}} \mathbf{1}_{Y_{L,S}}) \\
&= \omega\left((\mathbf{1}_{Y_{L,S}} u_{\mathfrak{b}} \mathbf{1}_{Y_{L,S}})(\mathbf{1}_{Y_{L,S}} \alpha_{\mathfrak{b}^{-1}}(f) u_{\mathfrak{c}^{-1}} \mathbf{1}_{Y_{L,S}})\right) \\
&= \mathfrak{b}^{-x} \omega\left((\mathbf{1}_{Y_{L,S}} \alpha_{\mathfrak{b}^{-1}}(f) u_{\mathfrak{c}^{-1}} \mathbf{1}_{Y_{L,S}})(\mathbf{1}_{Y_{L,S}} u_{\mathfrak{b}} \mathbf{1}_{Y_{L,S}})\right) \\
&= \mathfrak{b}^{-x} \omega(\mathbf{1}_{Y_{L,S}} \alpha_{\mathfrak{b}}^{-1})(f) u_{\mathfrak{c}^{-1}} u_{\mathfrak{b}} \mathbf{1}_{Y_{L,S}}) \\
&= \mathfrak{b}^{-x} \omega(\mathbf{1}_{Y_{L,S}} \alpha_{\mathfrak{b}^{-1}}(f) u_{\mathfrak{a}}) \mathbf{1}_{Y_{L,S}}),
\end{aligned}$$

and iterating this we get

$$\omega(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) = (\mathfrak{b}^{-x})^n \omega(\mathbf{1}_{Y_{L,S}} \alpha_{\mathfrak{b}^{-n}}(f) u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}).$$

Note that $\|\alpha_{\mathfrak{b}^{-n}}(f)\| = \|f\|$, so the norm of the right-hand side is equal to

$$|(\mathfrak{b}^{-x})^n| \|f\| = |x|^{-n \deg \mathfrak{b}} \|f\|.$$

As $\mathfrak{b} \neq \mathcal{O}$ we see that $\deg \mathfrak{b} > 0$, and since $|x| > 1$ this converges to zero as $n \to \infty$. Since the left-hand side is constant this implies that we must have $\omega(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) = 0$.

On the other hand, if $\mathfrak{b} = \mathcal{O}$ we have

$$\begin{aligned}\omega(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) &= \omega(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{c}^{-1}} \mathbf{1}_{Y_{L,S}}) \\ &= \omega\left((\mathbf{1}_{Y_{L,S}} f u_{\mathcal{O}} \mathbf{1}_{Y_{L,S}})(\mathbf{1}_{Y_{L,S}} u_{\mathfrak{c}^{-1}} \mathbf{1}_{Y_{L,S}})\right) \\ &= \mathfrak{c}^{-x} \omega(\mathbf{1}_{Y_{L,S}} u_{\mathfrak{c}}^{-1}) \mathbf{1}_{Y_{L,S}} \mathbf{1}_{Y_{L,S}} f u_{\mathcal{O}} \mathbf{1}_{Y_{L,S}}) \\ &= \mathfrak{c}^{-x} \omega(\mathbf{1}_{Y_{L,S}} \alpha_{\mathfrak{c}^{-1}}(f) u_{\mathfrak{c}^{-1}} \mathbf{1}_{Y_{L,S}}),\end{aligned}$$

and as above we can iterate to get

$$\omega(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) = (\mathfrak{c}^{-x})^n \omega(\mathbf{1}_{Y_{L,S}} \alpha_{\mathfrak{c}^{-n}}(f) u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}),$$

so by the same argument $\omega(\mathbf{1}_{Y_{L,S}} f u_{\mathfrak{a}} \mathbf{1}_{Y_{L,S}}) = 0$. Hence ω is concentrated on the diagonal, so $\omega = \omega \circ \Phi$. □

THEOREM 3.2.5. *Let $(A_{L,S}, \sigma)$ be as above. Then, for $x \in \mathbb{C}_\infty^\times$,*

(i) *for $|x| < 1$ there are no KMS$_x$-functional on $(A_{L,S}, \sigma)$;*

(ii) *for $|x| > 1$ all KMS$_x$-functionals are concentrated on the diagonal and correspond to functionals $\phi : C(Y_{L,S}) \to \mathbb{C}_\infty$ satisfying*

$$\phi(f(\mathfrak{a}^{-1}\cdot)) = \mathfrak{a}^{-x}\phi(f).$$

PROOF. The theorem proves part (ii). For part (i), assume that ω is a KMS$_x$-functional with $|x| < 1$ and let $\mathfrak{a} \in J_S^+$ be any element with $\deg \mathfrak{a} > 0$. Then by the scaling property we have

$$\omega(\mathbf{1}_{Y_{L,S}}) = \mathfrak{a}^{-x} \omega(\mathbf{1}_{\mathfrak{a} Y_{L,S}}),$$

but looking at norms this would require $|\omega(\mathbf{1}_{\mathfrak{a} Y_{L,S}})| > 1$ which contradicts the assumption that $\|\omega\| = 1$. Hence there can be no such ω. □

For $s \in S_\infty$, define the Goss zeta function relative to S by

$$Z_S(s) = \sum_{\mathfrak{a} \in J_S} \mathfrak{a}^{-s}.$$

LEMMA 3.2.6. *The Goss zeta function converges in the "half-plane"*

$$\{s = (x, y) \in S_\infty : |x| > 1\}.$$

PROOF. Since $|\mathfrak{a}^{-s}| = |x|^{-\deg \mathfrak{a}}$ converges to zero as $\deg \mathfrak{a} \to \infty$, the sum is convergent on the given set. □

PROPOSITION 3.2.7. *Let* $Y^0_{L,S} = \mathrm{Gal}(L/K) \times_{\hat{\mathcal{O}}^*_S} \hat{\mathcal{O}}^*_S$. *For* $x \in \mathbb{C}^\times_\infty$ *with* $|x| > 1$ *and* $z \in Y^0_{L,S}$ *the functional on* $A_{L,S}$ *given by*

$$\omega_{x,z}(f) = Z_S(x)^{-1} \sum_{\mathfrak{a} \in J_S^+} f(\mathfrak{a}z)\mathfrak{a}^{-x}$$

is a KMS_x*-state on* $A_{L,S}$.

PROOF. Let $\omega_0 = \omega_{x,z}|_{C(Y_{L,S})}$ be the restriction of $\omega_{x,z}$ to the subset $C(Y_{L,S})$ of $A_{L,S}$. By the above theorem it suffices to show that there is a functional ϕ on $X_{L,S}$ with $\phi|_{Y_{L,S}}$ satisfying 3.2.4 such that we have $\phi|_{C(Y_{L,S})} = \omega_0$.

We claim that ϕ given by $\phi(f) = Z_S(x)^{-1} \sum_{\mathfrak{a} \in J_S} f(\mathfrak{a}z)\mathfrak{a}^{-x}$ satisfies our conditions.

It is clear that ϕ satisfies $\phi(\mathbf{1}_{Y_{L,S}}) = 1$ and that $\phi|_{C(Y_{L,S})} = \omega_0$. Hence we have to show that ϕ satisfies the scaling condition. To this end, let $\mathfrak{b} \in J_S$ and calculate

$$\begin{aligned} Z_S(x)\phi(f(\mathfrak{b}^{-1}\cdot)) &= \sum_{\mathfrak{a} \in J_S} f(\mathfrak{b}^{-1}\mathfrak{a}z)\mathfrak{a}^{-x} \\ &= \mathfrak{b}^{-x} \sum_{\mathfrak{a} \in J_S} f(\mathfrak{b}^{-1}\mathfrak{a}z)\mathfrak{a}^{-x}\mathfrak{b}^{x} \\ &= \mathfrak{b}^{-x} \sum_{\mathfrak{b}^{-1}\mathfrak{a} \in \mathfrak{b}^{-1}J_S = J_S} f(\mathfrak{b}^{-1}\mathfrak{a}z)(\mathfrak{b}^{-1}\mathfrak{a})^{-x} \\ &= \mathfrak{b}^{-x} \sum_{\mathfrak{c} \in J_S} f(\mathfrak{c}z)\mathfrak{c}^{-x} \\ &= \mathfrak{b}^{-x} Z_S(x)\phi(f), \end{aligned}$$

so ϕ satisfies the scaling condition. Hence $\omega_{x,z}$ is a KMS$_x$-functional. □

DEFINITION 3.2.8. Let ω be a functional on $A_{L,S}$. Assume that there is a sequence ω_n of functionals on $A_{L,S}$ such that ω_n is a KMS$_{x_n}$-functional with $|x_n| \to \infty$ as $n \to \infty$. If $\omega_n(f) \to \omega(f)$ for all $f \in A_{L,S}$ as $n \to \infty$ we say that ω is a KMS$_\infty$-functional on $A_{L,S}$.

COROLLARY 3.2.9. *Let* $z \in Y^0_{L,S}$. *Then the functional* ω_z *on* $A_{L,S}$ *given by*

$$\omega_{\infty,z}(f) = f(z)$$

is a KMS_∞*-functional.*

Proof. Let x_n be any sequence in $\mathbb{C}_\infty^\times$ with $|x_n| \to \infty$ and consider the states $\omega_n = \omega_{x_n,z}$. Then for $f \in A_{L,S}$ we have

$$\omega_{\infty,z}(f) - \omega_{x_n,z}(f) = f(z) - Z_S(x_n)^{-1} \sum_{\mathfrak{a}\in J_S^+} f(\mathfrak{a}z)\mathfrak{a}^{-x_n}$$
$$= Z_S(x_n)^{-1} \sum_{\mathfrak{a}\in J_S^+} (f(\mathfrak{a}z) - f(z))\mathfrak{a}^{-x_n}.$$

Since the $\mathfrak{a} = \mathcal{O}$ term vanishes, $|\mathfrak{a}^{-x_n}| < 1$ for all nonzero terms. Hence $|\omega_{\infty,z}(f) - \omega_{x_n,z}(f)| \leq |x_n|^{-1}$, so as $|x_n| \to \infty$ we get that $\omega_{x_n,z}(f)$ converges to $\omega_{\infty,z}(f)$. Hence $\omega_{\infty,z}$ is a KMS_∞ functional on $A_{L,S}$. □

Remark 3.2.10.

(i) Our definition of KMS functionals differs from that of [**9**] in that we require $\|\omega\| \leq 1$ also in the unital case. In the complex-valued case this follows from $\omega(1) = 1$, but this is not the case here.

This assumption implies that the if we would consider the measure induced by ω on the diagonal subalgebra we would get a probability type measure, which is the case in the complex-valued case. It is also critical in proving the non-existence of KMS states of our system for $|x| < 1$.

(ii) The functionals constructed in the above proposition were constructed in [**9**, Theorem 4.10], and their parametrizing space $Y_{L,S}^0$ corresponds to the space parametrizing the extremal KMS_β-states in the number fields case. See for instance [**24**, Theorem 2.1].

(iii) We have not been able to show that the KMS functionals constructed in this section are all KMS functionals. In the complex-valued case this is done by considering extremal points of the simplex of KMS_x-functionals. In the present case it does not make sense to speak of extremal KMS_x-functionals, since there is no notion of positivity, and hence no way to define convex combinations.

3.3. Arithmetic subalgebras

The main (and rather elusive) goal in the theory of Bost-Connes systems is the construction of a dynamical system (A_K, σ_t) with an arithmetic subalgebra $\mathcal{A}_K$. Given a number field K with maximal abelian extension K^{ab} this would have the following properties:

(i) The group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ acts on A_K as symmetries compatible with σ_t.

(ii) The extremal KMS_∞-states evaluated on the arithmetic subalgebra $\mathcal{A}_K$ satisfy $\phi(a) \in \bar{K}$, and the elements $\phi(a)$, where a runs through

$\mathcal{A}_K$, generate K^{ab} over K.

(iii) The Galois actions on A_K and K^{ab} are compatible, in that for all elements $\alpha \in \mathrm{Gal}(K^{ab}/K)$, $a \in \mathcal{A}_K$ and all extremal KMS$_\infty$-states ϕ of (A_K, σ_t) we have

$$\alpha(\phi(a)) = \phi(\alpha(a)).$$

(iv) The $\mathbb{C}$-algebra generated by $\mathcal{A}_K$ is dense in A_K.

The explicit construction of such a system has been successful only in the case of the rational numbers $\mathbb{Q}$ [**1**] and in the imaginary quadratic case $K = \mathbb{Q}(\sqrt{-d})$ [**7**]. However, the existence of an arithmetic subalgebra for arbitrary number fields is proven in [**34**]. The results of [**34**, Section 9] can be mirrored in the function field setting, which we will do in this section.

Continuing with the notation of the previous sections, consider the algebra of locally constant L-valued functions on $X_{L,S}$. There is an action of $\mathrm{Gal}(L/K)$ on this algebra induced by that on $X_{L,S}$. Let A_0 be the algebra of $\mathrm{Gal}(L/K)$-equivariant functions on $X_{L,S}$ with values in L. Then let $\mathcal{A}_{L,S} = \mathbf{1}_{Y_{L,S}}(A_0 \rtimes J_S)\mathbf{1}_{Y_{L,S}}$.

THEOREM 3.3.1. *Let $(A_{L,S}, \sigma)$ and $\mathcal{A}_{L,S}$ be as above. Then*

(i) *the group* $\mathrm{Gal}(L/K)$ *acts on $A_{L,S}$ as symmetries compatible with the time evolution σ_t;*

(ii) *there is a set $\mathcal{E}_\infty$ of KMS$_\infty$ states parametrized by* $\mathrm{Gal}(L/K)$ *such that when $\phi \in \mathcal{E}_\infty$ is evaluated on $a \in \mathcal{A}_{L,S}$ we have $\phi(a) \in L$, and the elements $\phi(a)$, where a runs through $\mathcal{A}_{L,S}$, generates L over K;*

(iii) *the actions of* $\mathrm{Gal}(L/K)$ *on $A_{L,S}$ and L are compatible, in that for all $\alpha \in \mathrm{Gal}(L/K)$, $a \in \mathcal{A}_{L,S}$ and all $\phi \in \mathcal{E}_\infty$ we have*

$$\alpha(\phi(a)) = \phi(\alpha(a));$$

(iv) *the $\mathbb{C}_\infty$-algebra generated by $\mathcal{A}_{L,S}$ is dense in $A_{L,S}$.*

That is, $\mathcal{A}_{L,S}$ is an arithmetic subalgebra for the extension L/K.

PROOF. The action of $\mathrm{Gal}(L/K)$ on $A_{L,S}$ is induced by the action of $\mathrm{Gal}(L/K)$ on $X_{L,S}$. Since the time evolution is trivial on the subset $C(Y_{L,S})$ of $A_{L,S}$, the actions commute, and hence are compatible.

Let us now show density. It is clearly enough to show that the $\mathbb{C}_\infty$-algebra generated by A_0 is dense in $C(X_{L,S})$. By the Stone-Weierstrass theorem (for a proof of the postive characteristic case see for instance [**4**, Corollary p. 239]) it suffices to show that the $\mathbb{C}_\infty$-algebra generated by A_0 separates points and does not vanish identically at any point of $X_{L,S}$.

To show that A_0 separates points, recall that if E/K is an intermediate finite extension we can identify $X_{E,S}$ with $X_{L,S}/\operatorname{Gal}(L/E)$. Since any two points of $X_{L,S}$ are in a different $\operatorname{Gal}(L/E)$-orbit if E is large enough we may assume that L/K is finite. Next recall that $Y_{L,S}$ is a projective limit of finite $\operatorname{Gal}(L/K)$-sets, for instance as exhibited in Section 2.3. By these reductions it suffices to prove that if L/K is a finite Galois extension and Y is a finite $\operatorname{Gal}(L/K)$-set then the K-algebra of $\operatorname{Gal}(L/K)$-equivariant functions separates points of Y. But this is straightforward to prove.

Indeed, let $x, y \in Y$. If there is an $\alpha_0 \in \operatorname{Gal}(L/K)$ such that $\alpha_0(x) = y$ then we may find a $\xi \in L^{G_x}$ such that $\alpha_0(\xi) \neq \xi$, where G_x is the stabilizer of x. We may then define $f : Y \to L$ on $\operatorname{Gal}(L/K)x$ by $f(\alpha(x)) = \alpha(\xi)$, and $f(z) = 0$ for $z \notin \operatorname{Gal}(L/K)x$. On the other hand, if there is no such α_0 we may let f be the characteristic function of $\operatorname{Gal}(L/K)x$. Hence A_0 separates points of $X_{L,S}$. Since A_0 clearly is nowhere vanishing we have proved (iv).

For part (ii), let $\mathcal{E}_\infty = \{\phi_{\infty,z} : z \in Y^0_{L,S}\}$ where $\phi_{\infty,z}$ is as defined in Corollary 3.2.9. Then for $a = \sum_{\mathfrak{a} \in J_S} a_\mathfrak{a} \in \mathcal{A}_{L,S}$ and $z \in Y^0_{L,S}$ we have $\phi_{\infty,z}(a) = a_\mathcal{O}$. To see that these elements generate L, let $z \in Y^0_{L,S}$, and let E be the image of A_0 under the map $a \mapsto \phi_{\infty,z}(a)$. Then $E \subseteq L$. If $E \neq L$ there is a nontrivial element $\alpha \in \operatorname{Gal}(L/E) \subset \operatorname{Gal}(L/K)$. But then $\alpha z \neq z$ since $\operatorname{Gal}(L/K)$ acts freely on $Y^0_{L,S}$, while on the other hand for every $a \in A_0$ we have $a(\alpha z) = \alpha(a(z)) = a(z)$. But this contradicts A_0 separating points of $Y_{L,S}$. Hence we must have $E = L$, and the $\phi_{\infty,z}(a)$ generate L over K.

Finally, if $\alpha \in \operatorname{Gal}(L/K)$, $z \in Y^0_{L,S}$ and $a \in \mathcal{A}_{L,S}$, we have

$$\alpha(\phi_{\infty,z}(a)) = \alpha(\phi_{\infty,z}(a_\mathcal{O})) = \alpha(a_\mathcal{O}(z)) = a_\mathcal{O}(\alpha(z)) = \phi_{\infty,\alpha(z)}(a),$$

so the Galois actions are compatible. $\square$

The following useful characterisation of the algebra of locally constant functions $Y \to L$ which are equivariant with respect to the $\operatorname{Gal}(L/K)$-action is stated for number fields in [**34**, Section 9]. The proof is identical in the function field case, but we include it for convenience.

THEOREM 3.3.2. *Let E be a K-subalgebra of $C(Y_{L,S})$ satisfying*

(i) *every function in E is locally constant;*

(ii) *E separates points of $Y_{L,S}$;*

(iii) *E contains $\mathbf{1}_{\mathfrak{a}Y_{L,S}}$ for all ideals $\mathfrak{a} \subset \mathcal{O}$;*

(iv) *for every* $f \in E$ *we have* $f(Y_{L,S}) \subset L$*, and the map* $f : Y_{L,S} \to L$ *is* $\mathrm{Gal}(L/K)$*-equivariant.*
Then E *is the* K*-algebra of locally constant* L*-valued* $\mathrm{Gal}(L/K)$*-equivariant functions on* $Y_{L,S}$*.*

Proof. Let $f : Y_{L,S} \to \mathbb{K}$ be a locally constant $\mathrm{Gal}(L/K)$-equivariant function. We must show that $f \in E$.

Fix a point $y \in Y_{L,S}$ and let $L' \subset L$ be the subfield fixed by the stabilizer G_y of y in $\mathrm{Gal}(L/K)$. Then $f(y) \in L'$ by equivariance. We claim that the map $h \mapsto h(y)$ from E to L' is surjective.

Indeed, let L'' be the image of E under the map $h \mapsto h(y)$. Since E is a K-algebra, L'' is a subfield of L'. Assume $L'' \neq L'$. Then $\mathrm{Gal}(L''/L') \subset \mathrm{Gal}(L'/K) = \mathrm{Gal}(L/K)/G_y$ contains a nontrivial element, which can be lifted to an element $g \in \mathrm{Gal}(L/K)$. Then on the one hand we have $gy \neq y$, while on the other hand for every $h \in E$ we have $h(gy) = gh(y) = h(y)$. Since E separates points of $Y_{L,S}$ this is impossible, so we must have $L'' = L'$, which proves our claim.

By the claim, there exists $h \in E$ such that $h(y) = f(y)$. Since f and h are locally constant, there is some neighbourhood W of y such that f and h coincide on W. We may assume that W is the image in $Y_{L,S}$ of an open set in $\mathrm{Gal}(L/K) \times \hat{\mathcal{O}}$ of the form

$$W' \times \left(\prod_{v \in F} W_v \times \prod_{v \notin F} \mathcal{O}_v \right)$$

for some finite set F of places of K. Furthermore, we may split F into F' and F'' in such a way that F' contains those v with $0_v \notin W_v$. Then we may assume that F'' consists of those $v \in F$ such that $W_v = \mathfrak{p}_v^{n_v} \mathcal{O}_v$, while for $v \in F'$ we have an inclusion $W_v \subset \mathfrak{p}_v^{n_v} \mathcal{O}_v^\times$.

Since both f and h are $\mathrm{Gal}(L/K)$-equivariant, they coincide on the open set $U = \mathrm{Gal}(L/K)W$. However,

$$\mathrm{Gal}(L/K)W = \mathrm{Gal}(L/K) \times_{\hat{\mathcal{O}}^*} \left(\prod_{v \in F'} \mathfrak{p}_v^{n_v} \mathcal{O}_v^\times \times \prod_{v \in F''} \mathfrak{p}_v^{n_v} \mathcal{O}_v \times \prod_{v \notin F} \mathcal{O}_v \right),$$

so the characteristic function p of U is in E. Indeed, this characteristic function is the product of $\rho_{\mathfrak{p}_v^n} - \rho_{\mathfrak{p}_v^{n_v+1}}$ for $v \in F'$ and $\rho_{\mathfrak{p}_v^{n_v}}$ for $v \in F''$. Hence $fp = hp$ is an element of E.

Thus for every $y \in Y_{L,S}$ there is a neighbourhood U of y such that the characteristic function p of U belongs to E, and $fp \in E$. Since $Y_{L,S}$ is compact we can find a finite number of such neighborhoods covering $Y_{L,S}$, so as E is an algebra we have $f \in E$. □

REMARK 3.3.3. If we let $L = K^{\mathrm{ab}}$ and S be some set of primes the construction in Theorem 3.3.1 would give us a Bost-Connes system for the function field K. However, the above construction can hardly be called explicit, and hence is not entirely satisfactory.

It turns out, however, that in certain specific cases we can do better. This is the content of the following sections.

3.4. An arithmetic subalgebra for $L = \mathbb{K}$

In the previous section we showed that for any L and S there is an arithmetic subalgebra $\mathcal{A}_{L,S} \subset A_{L,S}$. The current section concentrates on the case $L = \mathbb{K}$ (where $\mathbb{K}$ is as given in Section 1.3) and $S = \{\infty\}$, and we will show that in this case the work of Hayes [**17**] gives us an explicit construction of generators of this subalgebra. It is worth recalling that the system $(A_{L,S}, \sigma)$ has as underlying groupoid that studied by Jacob [**18**].

For the purposes of this argument it will be convenient to use the point of view of Jacob given in Section 2.3 instead of the adelic picture. We furthermore write X and Y for $X_{\mathbb{K},\{\infty\}}$ and $Y_{\mathbb{K},\{\infty\}}$ respectively, and write

$$Y^0 = Y^0_{L,S} = \mathrm{Gal}(\mathbb{K}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}^* \subset Y.$$

Fix a sgn-normalized Drinfeld module ϕ^0, and recall that we in Section 2.3 defined for each integral ideal $\mathfrak{m}$ of $\mathcal{O}$ a Galois-equivariant isomorphism $\widehat{\phi^0[\mathfrak{m}]} \to \mathcal{O}/\mathfrak{m}$. Clearly $\phi^0[\mathfrak{m}]$ is equivariantly isomorphic to $\mathcal{O}/\mathfrak{m}$. Here the actions of $a \in (\mathcal{O}/\mathfrak{m})^* \simeq \mathrm{Gal}(K_\mathfrak{m}/H^+)$ are given by

$$\sigma_a(\chi) = \chi \circ \phi^0_a, \qquad \sigma_a(\lambda) = \phi^0_a(\lambda)$$

for $\chi \in \widehat{\phi^0[\mathfrak{m}]}$ and $\lambda \in \phi^0[\mathfrak{m}]$, and that on $\mathcal{O}/\mathfrak{m}$ is given by multiplication. Composing these maps we get a $\mathrm{Gal}(K_\mathfrak{m}/H^+)$-equivariant bijection $\Psi : \widehat{\phi^0[\mathfrak{m}]} \to \phi^0[\mathfrak{m}]$.

Let us now use this map to create generators of $\mathcal{A}_{\mathbb{K},\{\infty\}}$.

PROPOSITION 3.4.1. *For each proper ideal* $\mathfrak{m}$ *of* $\mathcal{O}$ *there is a map* $f_\mathfrak{m} : Y \to \mathbb{C}_\infty$ *such that*

(i) $f_\mathfrak{m}$ *takes values in* $K_\mathfrak{m}$*;*

(ii) *the values of* $f_\mathfrak{m}$ *generate* $K_\mathfrak{m}$ *over* K*;*

(iii) $f_\mathfrak{m}$ *is equivariant with respect to the* $\mathrm{Gal}(\mathbb{K}/K)$*-actions on* Y *and* $K_\mathfrak{m}$*, where the action of* $\mathrm{Gal}(\mathbb{K}/K)$ *on* $K_\mathfrak{m}$ *is defined through the quotient map* $\mathrm{Gal}(\mathbb{K}/K) \to \mathrm{Gal}(K_\mathfrak{m}/K)$*.*

PROOF. Fix $\mathfrak{m}$ and consider the map $f : \widehat{\phi^0[\mathfrak{m}]} \to \mathbb{C}_\infty$ given by $f(\chi) = \Psi(\chi)$. We want to extend f to a map defined on $Y_\mathfrak{m} = \bigsqcup_\phi \widehat{\phi[\mathfrak{m}]}$.

To this end, for $\chi \in \widehat{\phi[\mathfrak{m}]}$ find $\sigma \in \mathrm{Gal}(K_{\mathfrak{m}}/K)$ such that $\phi = \sigma\phi^0$. Then $\sigma\chi \in \widehat{\phi^0[\mathfrak{m}]}$, so we can define $f(\chi) = \sigma^{-1}\Psi(\sigma\chi)$. The choice of σ is unique up to an element of $\mathrm{Gal}(K_{\mathfrak{m}}/H^+)$, so this definition is independent of the choice of σ since Ψ is $\mathrm{Gal}(K_{\mathfrak{m}}/H^+)$-equivariant.

Then the range of $f : Y_{\mathfrak{m}} \to \mathbb{C}_\infty$ is exactly $\bigcup_\phi \phi[\mathfrak{m}]$ where ϕ runs through the sgn-normalized Drinfeld modules of K. Hence these values generate $K_{\mathfrak{m}}$ over K. Note that f is $\mathrm{Gal}(K_{\mathfrak{m}}/K)$-equivariant by construction.

Let $f_{\mathfrak{m}}$ be the lifting of f to Y through the quotient map $Y \to Y_{\mathfrak{m}}$. Then $f_{\mathfrak{m}}$ is $\mathrm{Gal}(\mathbb{K}/K)$-equivariant since both the quotient map and f are equivariant. Hence $f_{\mathfrak{m}}$ satisfies the conditions of the proposition. □

If we consider the K-algebra generated by the $f_{\mathfrak{m}}$, then it forms a subalgebra of $A_{\mathbb{K},\{\infty\}}$ such that when its functions are evaluated at the KMS_∞-functionals which correspond to elements of Y^0, the values we get generate $\mathbb{K}$ over K. However, the $\mathbb{C}_\infty$-algebra it generates is not dense in $A_{\mathbb{K},\{\infty\}}$. In fact it is not even dense in $C(Y) \subset A_{\mathbb{K},\{\infty\}}$. Indeed, if χ and χ' are the trivial characters on $\phi[\mathfrak{m}]$ and $\phi'[\mathfrak{m}]$ respectively, then $f_{\mathfrak{m}}(\chi) = 0 = f_{\mathfrak{m}}(\chi')$ for all $\mathfrak{m}$, and the algebra does not separate points.

To remedy this, we add additional functions. Let y be a fixed nonconstant element of $\mathcal{O}$, that is $y \in \mathcal{O} \setminus \mathbb{F}_q$. Recall from Section 1.3 that for any sgn-normalized rank one Drinfeld module ϕ the field H^+ is generated over K by the coefficients of ϕ_y, and has Galois group isomorphic to $\mathrm{Pic}^+(\mathcal{O})$ with an ideal $\mathfrak{a}$ acting on ϕ by $\sigma_{\mathfrak{a}}\phi = \mathfrak{a} * \phi$.

Define functions $g_{y,i} : X_{\mathbb{K},\{\infty\}} \to \mathbb{C}_\infty$ for $i = 0, 1, \ldots$ by letting $g_{y,i}(\chi)$ equal the coefficient of τ^i in $\phi_y(\tau)$, where ϕ is such that χ is a character of $\phi(\mathbb{C}_\infty)$. Since $\phi_y(\tau)$ is a polynomial of fixed degree there are only finitely many such functions $g_{y,i}$. Furthermore, each function $g_{y,i}$ is continuous and locally constant (since it is constant on the components of Y), takes values in $H^+ \subset \mathbb{K}$ by definition, and is $\mathrm{Gal}(\mathbb{K}/K)$-equivariant.

Let furthermore $\mathbf{1}_{\mathfrak{a}Y} : Y \to \mathbb{C}_\infty$ be the characteristic functions of $\mathfrak{a}Y \subset Y$, for $\mathfrak{a}$ running through the proper ideals of $\mathcal{O}$.

Let $\mathcal{E}_{\mathbb{K},\{\infty\}}$ be the algebra generated by the $f_{\mathfrak{m}}$, the $g_{y,i}$ and the $\mathbf{1}_{\mathfrak{a}Y}$.

PROPOSITION 3.4.2. *$\mathcal{E}_{\mathbb{K},\{\infty\}}$ is the algebra of locally constant $\mathbb{K}$-valued $\mathrm{Gal}(\mathbb{K}/K)$-equivariant functions on $Y_{\mathbb{K},\{\infty\}}$. Furthermore, the $\mathbb{C}_\infty$-algebra $\mathbf{1}_Y(\mathcal{E}_{\mathbb{K},\{\infty\}} \rtimes J_S)\mathbf{1}_Y$ is an arithmetic subalgebra for the extension $\mathbb{K}/K$.*

PROOF. The first part follows immediately from Theorem 3.3.2. The second then follows from Theorem 3.3.1. □

REMARK 3.4.3.

(i) The construction is not canonical, since the bijection Ψ is only determined up to a choice of an element of $\mathrm{Gal}(K_{\mathfrak{m}}/H^+) \simeq (\mathcal{O}/\mathfrak{m})^*$. However, the resulting arithmetic subalgebra is independent of this choice since it is characterised as in the previous section.

(ii) The system of the theorem is somewhat in between a Bost-Connes system proper and the partial Bost-Connes systems considered in [**33**] in that while it does not construct the full maximal abelian extension, the smaller field used is coupled by a reduced size of the symmetry group and algebra.

(iii) As in [**17**], if we would like to get the maximal abelian extension, we can take two different distinguished places ∞ and ∞', do the above construction, and compose the generated fields to get the maximal abelian extension K^{ab}. Indeed, by the calculation of the Galois group $\mathbb{K}$ is fixed by exactly the ideles $K_\infty^+ K^\times$, and the corresponding field for ∞' is fixed by $K_{\infty'}^+ K^\times$. Since the intersection of these two groups of ideles is exactly $K^\times$ and the Artin map has dense image, this implies that the composition of the fields is the maximal abelian extension.

3.5. A Bost-Connes system for the rational function field

In this section we specialize to the case $K = \mathbb{F}_q(T)$, and let ∞ be the place corresponding to $1/T \in K$. Then $\mathcal{O} = \mathbb{F}_q[T]$. The goal of this section is to show that the construction of Section 3.3 can be carried out explicitly in the case $L = K^{\mathrm{ab}}$, the maximal abelian extension of K, and $S = \{\infty\}$.

The maximal abelian extension K^{ab} is constructed in [**16**] as the composition of three pairwise linearly disjoint extensions E/K, F/K and L_∞/K. Let us consider them in turn:

First, let E_n be the extension of K generated by all roots of $u^{q^n} - u$ for $n = 1, 2, \ldots$, and let $E = \bigcup E_n$ (so $E = \bar{\mathbb{F}}_q$). Then $\mathrm{Gal}(E_n/K) \simeq \mathbb{Z}/n\mathbb{Z}$ so $\mathrm{Gal}(E/K) \simeq \hat{\mathbb{Z}}$, generated by the Frobenius, that is by the unique automorphism of E fixing K such that its restriction to $\bar{\mathbb{F}}_q \cap E$ is given by $u \mapsto u^q$.

Secondly, let $F_x = K(\phi[x])$ be the extension of K generated by the roots of ϕ_x for each $x \in \mathcal{O}$ and let $F = \bigcup_{x \in \mathcal{O}} F_x$. Then we know that $\mathrm{Gal}(F_x/K) \simeq (\mathcal{O}/x\mathcal{O})^*$, so recalling that $\mathcal{O}$ is a principal ideal domain we see that $\mathrm{Gal}(F/K) \simeq \hat{\mathcal{O}}^*$.

Finally we want to construct L_∞. To this end, write K as $\mathbb{F}_q(1/T)$, and let ∞' be the place corresponding to T. We can then consider the

Drinfeld module which is uniquely defined by

$$\phi'_{1/T}(x) = x^q + (1/T)x.$$

For $x \in \mathbb{F}_q[1/T]$, let $\phi'[x]$ be the roots of the polynomial ϕ'_x, and consider the field $\tilde{L}_v = K(\phi'[(1/T)^{v+1}])$. Then $\mathbb{F}_q^\times$ acts on $\tilde{L}_v$ by multiplication. We let $L_v = (\tilde{L}_v)^{\mathbb{F}_q^\times}$ be the fixed field of this action. Then

$$\mathrm{Gal}(L_v/K) \simeq (1 + (1/T)\mathbb{F}_q[1/T])/(1/T)^{v+1},$$

and when we set $L = \bigcup_v L_v$ we get

$$\mathrm{Gal}(L/K) \simeq (1 + (1/T)\mathbb{F}_q[[1/T]]).$$

Then $\mathrm{Gal}(L/K)$ is identified with a subgroup of $K_\infty^\times$ which we will denote by $K_\infty^{(1)}$.

THEOREM 3.5.1 (Hayes). *Let $K = \mathbb{F}_q(t)$. Then the maximal abelian extension K^{ab} of K decomposes into three linearly disjoing extensions $K^{\mathrm{ab}} = E \cdot F \cdot L_\infty$ where E, F and L_∞ are as above. Correspondingly, we have* $\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \mathrm{Gal}(E/K) \times \mathrm{Gal}(F/K) \times \mathrm{Gal}(L_\infty/K)$.

Next we want to consider the Artin map $r_K : \mathbb{A}_K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$. Since $\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \hat{\mathbb{Z}} \times \hat{\mathcal{O}}^* \times K_\infty^{(1)}$ we seek to decompose $\mathbb{A}_K^*$ in a similar manner.

Let v be a place of K different from ∞ and let $\mathfrak{p}_v$ be the corresponding prime ideal of $\mathcal{O}$. Since $\mathcal{O}$ is a principal ideal domain there is a unique monic irreducible $P_v \in \mathcal{O}$ which generates $\mathfrak{p}_v$. Hence we can make a canonical choice of uniformizer $\pi_v \in \mathcal{O}_v$ by for $v \neq \infty$ setting $\pi_v = P_v$ and setting $P_\infty = 1/T$. Then any element $x \in K_v^*$ can be written in the form $x = u\pi_v^k$ with $u \in \mathcal{O}_v^*$ and $k \in \mathbb{Z}$ uniquely determined. Write $\mathfrak{m}_v$ for the maximal idal of $\mathcal{O}_v$ and let $\mathrm{sgn}_v(x) = \bar{u}$, where $\bar{u}$ is the image of u in $\mathcal{O}_v/\mathfrak{m}_v$. We can identify $\mathcal{O}_v/\mathfrak{m}_v$ with $\mathbb{F}_v$, the constants of K_v, and consider sgn_v as a map $\mathrm{sgn}_v : K_v \to K_v$.

LEMMA 3.5.2. *Put $V_v = \ker(\mathrm{sgn}_v)$ and $K_v^{(1)} = V_v \cap \mathcal{O}_v^*$. Then $V_v \simeq K_v^{(1)} \times \mathbb{Z}$ as topological groups.*

PROOF. Let $x \in K_v^\times$. Then x can be written uniquely as $x = u\pi_v^k$ for some $u \in \mathcal{O}_v^\times$ and $k \in \mathbb{Z}$. If $x \in V_v$ then $u \in K_v^{(1)}$ since $\mathrm{sgn}_v(\pi_v) = 1$ by construction. Hence $V_v = K_v^{(1)} \times \mathbb{Z}$ as groups since the decomposition is unique.

Next note that $K_v^{(1)}$ is open in V_v since $\mathcal{O}_v^*$ is open in $\mathcal{O}_v$. Hence the map above is a map of topological groups. □

Note that the notation of the lemma coincides with the definition given before for $K_\infty^{(1)} \simeq \mathrm{Gal}(L_\infty/K)$.

Next, given an idèle $a = (a_v) \in \mathbb{A}_K^*$ define

$$d(a) = \mathrm{sgn}_\infty(a_\infty) \cdot \prod_{v \neq \infty} \pi_v^{\mathrm{ord}_v(a_v)}.$$

This is an element of $K^\times$, since $\pi_v \in \mathcal{O}$ for all v and ∞ is of degree one so $\mathbb{F}_\infty^\times = \mathbb{F}_q^\times \subset K^\times$. The map d is a surjection since any element of $K^\times$ can be written as a product of monic irreducibles times an element of $\mathbb{F}_q^\times$.

We will want to consider the following embeddings of groups into $\mathbb{A}_K^*$:

(i) $K^\times \to \mathbb{A}_K^*$ sitting on the diagonal;

(ii) $V_\infty = K_\infty^{(1)} \times \mathbb{Z} \subset K_\infty^\times \to K_\infty^\times \times \mathbb{A}_{K,f}^* \simeq \mathbb{A}_K^*$, which maps to idèles with value 1 at every coordinate except for the ∞-coordinate;

(iii) $\hat{\mathcal{O}}^* \subset \mathbb{A}_{K,f}^* \to K_\infty^\times \times \mathbb{A}_{K,f}^* \simeq \mathbb{A}_K^*$.

Since the intersection of any two of these embeddings is the unit element of $\mathbb{A}_K^*$, we see that $K^\times \times K_\infty^{(1)} \times \mathbb{Z} \times \hat{\mathcal{O}}^*$ sits as a subgroup inside $\mathbb{A}_K^*$.

Now any idèle $a = (a_v)$ can be written as $a = d(a) \cdot a'$ for some $a' \in \hat{\mathcal{O}}^* \times V_\infty$, where $d(a) \in K^\times$. Indeed, we by definition have that $a_v \in \pi_v^{\mathrm{ord}_v(a_v)} \mathcal{O}_v^*$ and $a_\infty \in \mathrm{sgn}_\infty(a_\infty) K_\infty^{(1)} \pi_\infty^{\mathbb{Z}}$. Hence $\mathbb{A}_K^* \simeq K^\times \times \hat{\mathcal{O}}^* \times V_\infty$ as a group, and as $\hat{\mathcal{O}}^* \times V_\infty$ is an open subgroup of $\mathbb{A}_K^*$ this decomposition is also a decomposition of $\mathbb{A}_K^*$ as a topological group. Hence, since we have $V_\infty = K_\infty^{(1)} \times \mathbb{Z}$, we get

$$\mathbb{A}_K^* \simeq K^\times \times \hat{\mathcal{O}}^* \times K_\infty^{(1)} \times \mathbb{Z}$$

as topological groups.

Theorem 3.5.3 (Hayes). *The map $\mathbb{A}_K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ given by mapping the components of $\mathbb{A}_K^*$ as above into the the respective Galois groups of the sub-extensions (with $K^\times$ as the kernel) is the Artin map r_K.*

Let us now construct several functions which will be the generators of the arithmetic subalgebra $\mathcal{A}_{K^{\mathrm{ab}},\{\infty\}} \subset A_{K^{\mathrm{ab}},\{\infty\}}$. For brevity we will write Y for $Y_{K^{\mathrm{ab}},\{\infty\}}$.

Let λ be a generator of E_n over K. Since Y is a direct product of topological spaces, the quotient map $Y \to \hat{\mathbb{Z}} \to Z/n\mathbb{Z}$ is equivariant with respect to the Galois action. Hence we can define a function $f_\lambda : Y \to \mathbb{C}_\infty^\times$ by

$$f_\lambda(a, b, c) = \psi_b^E(\lambda),$$

where $\bar{b} \mapsto \psi^E_{\bar{b}}$ is the isomorphism $\mathbb{Z}/n\mathbb{Z} \simeq \mathrm{Gal}(E_n/K)$. Then the values of f_λ on Y generate E_n as an extension of K, and f_λ is $\mathrm{Gal}(E_n/K)$-equivariant. Furthermore, f_λ is locally constant.

Next, let η be a generator of L_v over K. The quotient map

$$Y \mapsto K^{(1)}_\infty \to (1 + (1/T)\mathbb{F}_q[1/T])/(1/T)^{v+1}$$

is again equivariant, so we can define a function $g_\eta : Y \to \mathbb{C}^\times_\infty$ by

$$g_\eta(a, b, c) = \psi^L_{\bar{a}}(\eta)$$

where $\bar{a} \mapsto \psi^L_{\bar{a}}$ is the isomorphism

$$(1 + (1/T)\mathbb{F}_q[1/T])/(1/T)^{v+1} \simeq \mathrm{Gal}(L_v/K).$$

As above, g_η takes values which generate L_v, and g_η is $\mathrm{Gal}(L_v/K)$-equivariant and locally constant.

For the extension F of K, the quotient map $Y \to \hat{\mathcal{O}}$ is again equivariant, but the action of the Galois group $\hat{\mathcal{O}}^*$ is not transitive. Still, the $\hat{\mathcal{O}}^*$-action can be extended to one of $\hat{\mathcal{O}}$.

Indeed, recall that $\phi[x]$ is an $\mathcal{O}/x\mathcal{O}$-module via ϕ. We can use this to let the action of $\hat{\mathcal{O}}$ on $\phi[x]$ be given by $c \cdot \gamma = \phi_{\bar{c}}(\gamma)$ for $\gamma \in \phi[x]$, where $\bar{c}$ is the image of c in $\mathcal{O}/x$ under the map $\hat{\mathcal{O}} \to \hat{\mathcal{O}}/x\hat{\mathcal{O}} \simeq \mathcal{O}/x\mathcal{O}$. This is well defined, so we can define $h : Y \to \mathbb{C}^\times_\infty$ for $\gamma \in \phi[x]$ by

$$h_\gamma(a, b, c) = \psi^F_{\bar{c}}(\gamma)$$

where ψ^F is the module action described above. Then h_γ takes values which generate F_x over K, is $\mathrm{Gal}(F_x/K)$-equivariant and is locally constant.

Let $\mathcal{E}_{K^{\mathrm{ab}},\{\infty\}}$ be the K-algebra generated by the f_λ, g_η and h_γ as λ, η and γ run through the sets of generators for their respective field extensions together with the $\mathbf{1}_{\mathfrak{a}Y}$ for integral ideals $\mathfrak{a} \subset \mathcal{O}$. Let $\mathcal{A}_{K^{\mathrm{ab}},\{\infty\}} = \mathbf{1}_Y(\mathcal{E}_{K^{\mathrm{ab}},\{\infty\}} \rtimes J_S)\mathbf{1}_Y$.

Theorem 3.5.4. *The algebra $\mathcal{A}_{K^{\mathrm{ab}},\{\infty\}}$ is that of Theorem 3.3.1. That is, $\mathcal{A}_{K^{\mathrm{ab}},\{\infty\}}$ is an arithmetic subalgebra for the dynamical system $(A_{K^{\mathrm{ab}},\{\infty\}}, \sigma)$.*

Proof. By Theorems 3.3.1 and 3.3.2 it suffices to show that elements of $\mathcal{E}_{K^{\mathrm{ab}},\{\infty\}}$ separate points of Y and are nowhere vanishing. For the first, write Y as $K^{(1)}_\infty \times \hat{\mathbb{Z}} \times \hat{\mathcal{O}}^*$ and let $(a, b, c), (a', b', c') \in Y$. Then at least one coordinate differs, say $a \neq a'$. Since we can interprent $K^{(1)}_\infty$ as the Galois group of E/K, we can find a generator $\lambda \in E$ such that a and a' act differently on λ. Then

$$f_\lambda(a, b, c) = \phi^E_a(\lambda) \neq \phi^E_{a'}(\lambda) = f_\lambda(a', b', c').$$

An identical argument applies if $b \neq b'$ or $c \neq c'$. Hence $\mathcal{E}_{K^{\mathrm{ab}},\{\infty\}}$ separates points. For any $\lambda \in E$, $\lambda \neq 0$, we have that f_λ is nowhere zero, so $\mathcal{E}_{K^{\mathrm{ab}},\{\infty\}}$ is nowhere vanishing. $\square$

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