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INTEGRAL MAJORIZATION POLYTOPES

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The majorization polytope $M(a)$ consists of all vectors dominated (or majorized, to be precise) by a given vector $a \in \mathbb{R}^n$; this is a polytope with extreme points being the permutations of a. For integral vector a, let $\nu(a)$ be the number of integral vectors contained in $M(a)$. We present several properties of the function ν and provide an algorithm for computing $\nu(a)$.

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1. Introduction

Let n be a positive integer. We may write (or split) n as sums of n nonincreasing nonnegative integers $p_1, p_2, \ldots, p_n$ in different ways (or partitions; see Section 2). For example, $3 = 3 + 0 + 0 = 2 + 1 + 0 = 1 + 1 + 1$. If we denote by $P(n)$ the number of different partitions of n, then $P(3) = 3$. One may check that $P(5) = 7$. As n gets large, $P(n)$ increases rapidly. It is astounding that $P(200)$ is about 4 trillion [1, p. 68]. The determination of $P(n)$ is an intriguing and difficult problem in number theory and combinatorics (see, e.g., [12] and [9, Chapter 15]). It has much to do with the theories of majorization and polytopes. In the language of majorization, $P(5) = 7$ means that there are 7 nonincreasing integral vectors in $\mathbb{R}^5$ that are majorized by the vector $(5, 0, 0, 0, 0)$. Equivalently, there are 7 nonincreasing integral vectors in $\mathbb{R}^5$ that are contained in the majorization polytope generated by $(5, 0, 0, 0, 0)$. In this paper we study majorization polytopes for more general integral vectors.

For vectors x and a in $\mathbb{R}^n$, we say that x is *majorized* by a, denoted by $x \preceq a$, provided that $\sum_{j=1}^{k} x_{[j]} \leq \sum_{j=1}^{k} a_{[j]}$ for $k = 1, 2, \ldots, n$, where there is an equality for $k = n$. Here $x_{[j]}$ is the jth largest component of x, $j = 1, 2, \ldots, n$. Roughly speaking, that x is majorized by a means that the components of x are dominated by or less "spread-out" than the components of a. For given $a \in \mathbb{R}^n$, the *majorization polytope* $M(a)$ is the collection of all vectors majorized by a, that is,

$$M(a) = \{x \in \mathbb{R}^n : x \preceq a\}.$$

We remark that $M(a)$ is known under different names in the literature, e.g., "permutohedron" (see [10]) or "permutation polytope"[a] (see [2], where facial properties of this polytope are presented). Our $M(a)$ may also be interpreted as the convex hull $\mathrm{conv}S_n(a)$ of the set $S_n(a)$ consisting of all permutations of vector a.

Majorization theory was first formally introduced in the Hardy-Littlewood-Pólya's well known book *Inequalities* [7, p. 45]. The monograph [8] contains a comprehensive study of majorization and its applications. Also, [3] treats majorization in connection with several combinatorial classes of matrices. In [14] majorization is discussed in detail in connection with matrix theory, particularly the spectral properties of matrices, etc.

Let $M_I(a)$ be the set of all integral vectors (i.e., all components are integers) contained in $M(a)$. We are interested in the cardinality of $M_I(a)$ and its dependence on a. As we shall see, the cardinality $\nu(a)$ of $M_I(a)$ is closely related to integer partitions. In Section 2 we show several properties of the function $a \to \nu(a)$, and in Section 3 we introduce an operation *splitting* and a recursive algorithm for computing $\nu(a)$ based on operations on Ferrers diagrams.

2. Properties of ν and ν^*

As usual, the jth unit vector of $\mathbb{R}^n$ is denoted by e_j, i.e., e_j has jth component 1 and 0 elsewhere. For a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, its jth component is x_j. Let $x_{[j]}$ denote its jth largest component: $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and write $x^{\downarrow} = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})$. We say that $x \in \mathbb{R}^n$ is *monotone* if $x_1 \geq x_2 \geq \cdots \geq x_n$. So for any $x \in \mathbb{R}^n$, $x^{\downarrow}$ is monotone.

We consider the sets $M(a)$ and $M_I(a)$ as defined in Section 1. Note that $M(a) = M(a')$ when a' is a permutation of a. For $a \in \mathbb{R}^n$, define

$$\nu(a) = |M_I(a)|.$$

So $\nu(a)$ is the number of *integral* vectors majorized by a. Let $M_I^*(a)$ denote the set of monotone vectors in $M_I(a)$, and define

$$\nu^*(a) = |M_I^*(a)|.$$

Our goal is to investigate the functions ν and ν^*. Apparently, $\nu(a)$ equals zero if the sum of the components of a is not an integer. For example, $a = (2, \frac{1}{2})$, $\nu(a) = 0$.

[a]Note: the term "permutation polytope" sometimes refers to a different object, namely, the convex hull of a group of permutation matrices.

We assume that the vectors in the study are integral. From the previous section, if $a = (5, 0, 0, 0, 0)$, we know that $\nu^*(a) = 7$.

Example 2.1. Let $a = (4, 2, 1)$. Then $\nu(a) = 12$ and $\nu^*(a) = 3$. In fact, $M_I(a)$ consists of: the six permutations of $(4, 2, 1)$, along with the three (different) permutations of $(3, 3, 1)$ and the three permutations of $(3, 2, 2)$. $M(a)$ is the convex hull of the permutations of $(4, 2, 1)$.

If a is a constant vector (i.e., all components are equal), then $x \preceq a$ implies $x = a$, so $M_I(a) = M_I^*(a) = \{a\}$ and thus $\nu(a) = \nu^*(a) = 1$. In general, for an integral $a \in \mathbb{R}^n$, since every vector in $M_I^*(a)$ generates through permutation at most $n!$ vectors in $M_I(a)$, we have $\nu(a) \leq \nu^*(a)n!$. In addition, $\nu(-a) = \nu(a)$ and $\nu^*(-a) = \nu^*(a)$. For integral $a = (\alpha, \beta) \in \mathbb{R}^2$ with $\alpha \geq \beta$, we observe that $\nu(a) = \alpha - \beta + 1$ and $\nu^*(a) = \lfloor \frac{\alpha-\beta}{2} \rfloor + 1$.

The following result from [6] will be useful to prove our Proposition 2.3.

Theorem 2.2. *([6]) Let $a, b \in \mathbb{R}^n$ be monotone vectors. Then $M(a+b) = M(a) + M(b)$. If, in addition, a and b are integral, then $M_I(a+b) = M_I(a) + M_I(b)$. (Here $S + T = \{s + t : s \in S, t \in T\}$.)*

Below are some basic observations about the function ν.

Proposition 2.3. *Let $a, b \in \mathbb{R}^n$ be integral vectors. Then the following hold:*

(i) If $a \preceq b$, then $\nu(a) \leq \nu(b)$.
(ii) If a is a constant vector, then $\nu(a + b) = \nu(b)$.
(iii) $\nu(a+b) \leq \nu(a)\nu(b)$. Equality holds if and only if a or b is a constant vector.
(iv) $\nu(ka) \leq \nu^k(a)$ for any positive integer k. Equality occurs if and only if $k = 1$ or a is a constant vector.

Proof. (i). The majorization order is transitive. So $a \preceq b$ implies that $M_I(a) \subseteq M_I(b)$. The cardinality inequality follows immediately.

(ii). If a is a constant vector, then $x \preceq b$ if and only if $a + x \preceq a + b$. There is a bijection between $M_I(b)$ and $M_I(a + b)$. So $\nu(a + b) = \nu(b)$.

(iii). Note that $a + b \preceq a^{\downarrow} + b^{\downarrow}$. It follows that $\nu(a + b) = |M_I(a + b)| \leq |M_I(a^{\downarrow} + b^{\downarrow})| = |M_I(a^{\downarrow}) + M_I(b^{\downarrow})| \leq \nu(a)\nu(b)$ (the second equality is by Theorem 2.2). Assuming that a and b are not constant vectors, we show that the strict inequality holds. To this end, it suffices to show that $M_I(a) + M_I(b)$ contains at least one duplicated element. Since a and b are non-constant integral vectors, there are permutations a' and b' of a and b, respectively, with $a' = (\alpha, \beta, \dots)$, $\alpha > \beta$, and $b' = (p, q, \dots)$, $p < q$. Set $\tilde{a} = (\alpha - 1, \beta + 1, \dots)$ and $\tilde{b} = (p + 1, q - 1, \dots)$, where $\tilde{a}$ and $\tilde{b}$ have the same remaining components as a' and b', respectively. Then $\tilde{a} \preceq a$ and $\tilde{b} \preceq b$. Apparently, $\tilde{a} \neq a'$ and $\tilde{b} \neq b'$. However, $a' + b' = \tilde{a} + \tilde{b}$.

(iv). This is a consequence (repeated use) of (iii) by setting $a = b$. □

Remark. Regarding (i), one may show that strict inequality holds if a is not a permutation of b. Furthermore, $\nu^*(a) \leq \nu^*(b)$, with equality if and only if a is a permutation of b. We also point out that these inequalities do not generalize to weak majorizations. Property (ii) reveals that the cardinality of $M_I(b)$ remains unchanged through "shifting". Thus the vectors may be assumed to be nonnegative. In addition, if a is a constant vector, then $\nu^*(a+b) = \nu^*(b)$. The analogous result of (iii) for ν^*, i.e., $\nu^*(a+b) \leq \nu^*(a)\nu^*(b)$, does not hold in general. For example, take $a = b = (1,0)$. Then $\nu^*(a) = \nu^*(b) = 1$, however, $\nu^*(a+b) = 2$.

Given $v \in M_I^*(a)$, let v have k distinct components $\tilde{v}_1 > \tilde{v}_2 > \cdots > \tilde{v}_k$, $1 \leq k \leq n$, and let $\tilde{v}_i$ occur n_i times in v. So $n_1 + \cdots + n_k = n$. Denote

$$\kappa(v) = \frac{n!}{n_1! \cdots n_k!}.$$

Proposition 2.4. *Let $a \in \mathbb{R}^n$. Then*

$$\nu(a) = \sum_{v \in M_I^*(a)} \kappa(v).$$

Proof. This is because each v in $M_I^*(a)$ generates $\kappa(v)$ vectors in $M_I(a)$. □

Example 2.5. Let $a = (4,2,1)$. Then $M_I^*(a) = \{a, u, v\}$, where $u = (3,3,1)$, $v = (3,2,2)$. Moreover, $\kappa(a) = 6$, $\kappa(u) = 6/2 = 3$, $\kappa(v) = 6/2 = 3$, so $\nu(a) = 6 + 3 + 3 = 12$ as we found in Example 2.1.

Corollary 2.6. *If $a = (s+t, \ldots, s+t, s, \ldots, s) \in \mathbb{R}^n$, where the first k $(1 \leq k < n)$ components are $s+t$, for some integer s and positive integer t, then*

$$\nu(a) \leq \binom{n}{k}^t.$$

Equality holds if and only if $t = 1$.

Proof. Write $a = se + t(1, \ldots, 1, 0, \ldots, 0)$, where e is the all-ones vector. By Proposition 2.3 (ii) and (iv), we have $\nu(a) = \nu\big(t(1, \ldots, 1, 0, \ldots, 0)\big) \leq \binom{n}{k}^t$. Equality holds if and only if $t = 1$ because a is not a constant vector. □

For the equality ($t = 1$) case, alternatively, the only vectors majorized by a are the permutations of a. Any such permutation of a corresponds to a selection of the k positions containing $s+1$, and the number of such selections is $\binom{n}{k}$. One can also see this from Proposition 2.4: $\nu(a) = \kappa(a) = \frac{n!}{k!(n-k)!} = \binom{n}{k}$.

The cardinality functions ν and ν^* are related to integer partitions. A *partition* of a positive integer n is a nonincreasing sequence $p_1, p_2, \ldots, p_k$ of positive integers whose sum is n. (We may add trailing zeros for convenience.) Clearly, such a partition may be represented by a monotone integral vector $(p_1, p_2, \ldots, p_k)$ (with the correct sum of its components). Each p_i is a *part* of the partition. Let $\mathcal{P}(n)$ be the set of all partitions of n (a subset of $\mathbb{R}^n$) and denote the number of partitions of an integer n by $P(n)$. So $P(n) = |\mathcal{P}(n)|$. It has been evident that determination of $P(n)$ is an intriguing and difficult problem in number theory and combinatorics; see [12] and [9, Chapter 15] for related results in this area. We observe that $\mathcal{P}(n)$ coincides with $M_I^*(a)$ when $a = (n, 0, \dots, 0) \in \mathbb{R}^n$. Thus integer partition may be described and studied by means of majorization.

Proposition 2.7. *Let* $a = (n, 0, \dots, 0) \in \mathbb{R}^n$. *Then*

$$\nu^*(a) = P(n), \;\; \nu(a) = \sum_{p \in \mathcal{P}(n)} \kappa(p) = \sum_{\substack{n \geq p_1 > \cdots > p_q \geq 1 \\ n_1 p_1 + \cdots + n_q p_q = n}} \frac{n!}{n_1! \cdots n_q! (n - \sum_{i=1}^q n_i)!}.$$

Proof. The first part is obvious because $M_I^*(a)$ contains exactly the partitions of n, that is, $M_I^*(a)$ coincides with $\mathcal{P}(n)$. The second part follows from Proposition 2.4. Note that in a partition $n = p_1 + \cdots + p_1 + \cdots + p_q + \cdots + p_q + 0 + \cdots + 0$, $p_1 > \cdots > p_q \geq 1$, each p_i appears n_i times, 0 appears $n - n_1 - \cdots - n_q$ times. □

A classical result of Euler (see, e.g., [9, p. 155] or [12, p. 7]) gives the generating function of $P(n)$ (in the summation below we define $P(0) = 1$):

$$\begin{aligned}\sum_{n=0}^{\infty} P(n) x^n &= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \\ &= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots) \cdots .\end{aligned}$$

It is possible to compute the numbers $P(n)$ recursively. Define $P_k(n)$ as the number of partitions of n into k parts. This is the same as the number of integral solutions of $x_1 + x_2 + \cdots + x_k = n$, $x_1 \geq x_2 \geq \cdots \geq x_k \geq 1$, which again equals the number of integral solutions of $z_1 + z_2 + \cdots + z_k = n - k$, $z_1 \geq z_2 \geq \cdots \geq z_k \geq 0$. Considering the number of these z_i's that are 1 reveals the recursion ([9, p. 152])

$$P_k(n) = \sum_{s=1}^{k} P_s(n-k) \qquad (1 \leq k \leq n-1),$$

with $P_1(n) = P_n(n) = 1$ for all n and $P_k(n) = 0$ when $k > n$. This makes it possible to compute the $P_k(n)$'s efficiently. Finally, one may compute $P(n)$ by $P(n) = \sum_{k=1}^n P_k(n)$. For instance, if we view the numbers $P_k(n)$ as the (k, n) entry of a matrix P, this matrix may be computed row by row, and its column sums are the numbers $P(1), P(2), P(3), \dots$. Although no explicit formula for $P(n)$ is known, several estimates are available; see, e.g., [12] and [9, Chapter 15].

Given a positive integer n, the number of ways that n is written as a sum of at most m parts can be described by the function ν and such function is bounded by m^n. To see this, let $(n, 0, \dots, 0) \in \mathbb{R}^m$ and write $a = n(1, 0, \dots, 0)$. By Proposition 2.3 (iv), we have $\nu(a) \leq \nu^n(1, 0, \dots, 0) = m^n$.

Proposition 2.8. *For positive integers m, n, let $a = (n, 0, \ldots, 0) \in \mathbb{R}^m$. Then*

$$\nu^*(a) = \sum_{k=1}^{\min\{m,n\}} P_k(n); \quad \nu(a) \le \sum_{k=1}^{\min\{m,n\}} \frac{m!}{(m-k)!} P_k(n).$$

Proof. The identity for $\nu^*(a)$ follows from the aforementioned discussions. For $\nu(a)$, it is sufficient to notice that each element $(p_1, \ldots, p_k, 0, \ldots, 0) \in \mathbb{R}^m$ counted in $P_k(n)$ can generate at most $\frac{m!}{(m-k)!}$ vectors in $M_I(a)$. □

Example 2.9. Let $a = (5, 0, 0) \in \mathbb{R}^3$, $n = 5, m = 3$. Then

$$\sum_{k=1}^{\min\{m,n\}} \frac{m!}{(m-k)!} P_k(n) = \sum_{k=1}^{3} \frac{3!}{(3-k)!} P_k(5) = 27.$$

Example 2.10. Let $a = (3, 0, 0, 0, 0) \in \mathbb{R}^5$, $n = 3, m = 5$. Then

$$\sum_{k=1}^{\min\{m,n\}} \frac{m!}{(m-k)!} P_k(n) = \sum_{k=1}^{3} \frac{5!}{(5-k)!} P_k(3) = 85.$$

The following result gives an upper bound for $\nu(a)$ in terms of m and n. This bound, not necessarily the best, but we believe, is better than m^n (that we discussed prior to Proposition 2.8). However, no proof is available yet.

Corollary 2.11. *For positive integers m, n, let $a = (n, 0, \ldots, 0) \in \mathbb{R}^m$. Then*

$$\nu(a) \le \sum_{k=1}^{\min\{m,n\}} \binom{m}{k} \binom{n + \frac{k(k-1)}{2} - 1}{k-1}.$$

Proof. It is known [9, p. 154] that $k! P_k(n) \le \binom{n + \frac{k(k-1)}{2} - 1}{k-1}$. So

$$\frac{m!}{(m-k)!} P_k(n) = \binom{m}{k} k! P_k(n) \le \binom{m}{k} \binom{n + \frac{k(k-1)}{2} - 1}{k-1}.$$

The upper bound is immediate from Proposition 2.8. □

Proposition 2.12. *Let $a = (a_1, a_2, \ldots, a_n)$ be an integral vector. Then*

$$\nu(a) \le \min\left\{ n^{\sum_{k=1}^n |a_k|},\ \prod_{k=1}^{n} \left(\sum_{t=1}^{\min\{n, |a_k|\}} \frac{n!}{(n-t)!} P_t(|a_k|) \right) \right\}.$$

Proof. Write $a = a_1e_1 + a_2e_2 + \cdots + a_ne_n$. By Proposition 2.3 (iii), we have

$$\nu(a) \le \prod_{k=1}^{n} \nu(a_ke_k) = \prod_{k=1}^{n} \nu(|a_k|e_k) \le \prod_{k=1}^{n} \left(\sum_{t=1}^{\min\{n,|a_k|\}} \frac{n!}{(n-t)!} P_t(|a_k|) \right).$$

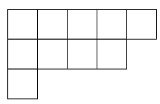
The last inequality is by Proposition 2.8. On the other hand, by Proposition 2.3 (iv), we have for each k,

$$\nu(a_ke_k) = \nu(|a_k|e_k) \le n^{|a_k|}.$$

Thus $\nu(a) \le n^{\sum_{k=1}^{n} |a_k|}$. Combining these reveals the desired inequality. □

3. The splitting operation

Let $a = (a_1, a_2, \ldots, a_n)$ be a nonnegative integral vector in $\mathbb{R}^n$ and let $N = \sum_{j=1}^{n} a_j$. Then each vector in $M_I^*(a)$ is a partition of N "controlled" by a. In this section, we study partitions using Ferrers diagrams (or Young diagrams). For instance, the partition $p = (5, 4, 1)$ of $N = 10$ corresponds to the Ferrers diagram

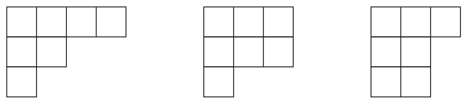


in which the number of boxes (squares) in the first row is the first part $p_1 = 5$, etc.

Let N and n be positive integers and let $\mathcal{P}_{N,n}$ be the set of all monotone integral vectors of length n whose sum of components equals N. This corresponds to partitions of N into at most n parts. Then $\mathcal{P}_{N,n}$ equipped with majorization ordering becomes a partially ordered set (poset) which has been studied in, e.g., [4]. This poset has a unique maximal element $(N, 0, \ldots, 0) \in \mathbb{R}^n$ and a unique minimal element $(v + 1, \ldots, v + 1, v, \ldots, v) \in \mathbb{R}^n$, where $v = \lfloor N/n \rfloor$ and the number of components being $v + 1$ is $N - nv$.

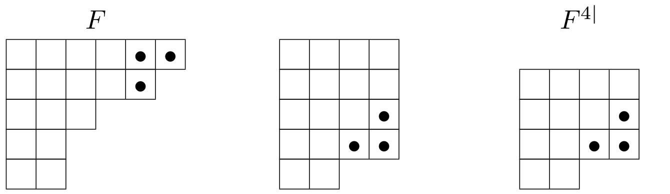
Given a monotone integral vector $a \in \mathbb{R}^n$, the set $M_I^*(a)$ is a subset of the poset $\mathcal{P}_{N,n}$, where $N = \sum_{j=1}^{n} a_j$, so $M_I^*(a)$ is a subposet. It is actually the principal ideal in $\mathcal{P}_{N,n}$ generated by a (see, e.g., [5] or [11, Chapter 3]). The set $M_I^*(a)$, or rather the corresponding Ferrers diagrams, may be constructed recursively as follows: Start with the Ferrers diagram of a and, repeatedly, choose a box at the end of a row and move it to the end of some row below, assuming monotonicity of the parts is preserved. Moving a box in this way corresponds to an integral transfer and it is known that any integral vector majorized by a may be produced by a sequence of such transfers (see [8, Chapter 5]). An enumeration like this is, of course, only practical for computing $\nu^*(a)$ when this number is reasonably small. A better (more efficient) approach is introduced in this section. That is, instead of moving one box each time, we move multiple boxes each time by so-called splitting vectors that are less spread-out and have lower dimensions.

Example 2.1 - continued. Let again $a = (4, 2, 1)$. Then $M_I^*(a)$ contains 3 vectors and their Ferrers diagrams are



Now we introduce an operation on monotone integral vectors which is convenient to explain using Ferrers diagrams. Let F be the Ferrers diagram of a monotone integral vector $a = (a_1, a_2, \ldots, a_n)$, define $N = \sum_{j=1}^n a_j$, and consider an integer j with $\lceil N/n \rceil \leq j \leq a_1$. Let $F^{j|}$ be the Ferrers diagram obtained from F by moving all boxes in columns $j+1, j+2, \ldots, a_1$ to other rows with preference to the uppermost rows, and then deleting the first row. (If $j = a_1$, no boxes are moved, but still delete the first row.) The corresponding integral vector, whose Ferrers diagram is $F^{j|}$, is denoted by $a^{j|}$ and this vector lies in $\mathbb{R}^{n-1}$. We call $a^{j|}$ a *splitting* of a.

Example 3.1. Let $a = (6, 5, 3, 2, 2)$ and $j = 4$. Then $a^{4|} = (4, 4, 4, 2)$. The Ferrers diagram of a and $a^{4|}$ are shown below



$a = (6, 5, 3, 2, 2)$ $\qquad$ $a^{4|} = (4, 4, 4, 2)$

Bullets indicate the boxes that were moved, and the intermediate Ferrers diagram (before the first row was deleted) is also shown.

For fixed j with $\lceil N/n \rceil \leq j \leq a_1$, we can give an explicit expression for $a^{j|}$. Define $\bar{a}_{1:s} = \frac{1}{s} \sum_{i=1}^s a_i \quad (1 \leq s \leq n)$ which is the (arithmetic) mean of the first s components of a. Since a is monotone,

$$a_1 = \bar{a}_{1:1} \geq \bar{a}_{1:2} \geq \cdots \geq \bar{a}_{1:n} = \frac{1}{n} \sum_{i=1}^n a_i = \frac{N}{n}.$$

Now let q $(1 \leq q \leq n)$ be the largest integer such that $\bar{a}_{1:q} \geq j$; such q exists and is unique (and depends on j). The ith component $a_i^{j|}$ of the vector $a^{j|} \in \mathbb{R}^{n-1}$ is given by

$$a_i^{j|} = \begin{cases} j & (1 \leq i \leq q-1) \\ \sum_{t=1}^{q+1} a_t - qj & (i = q) \\ a_{i+1} & (q+1 \leq i \leq n-1), \end{cases} \tag{3.1}$$

that is, written out explicitly,

$$a^{j|} = \Big(j, \ldots, j, \sum_{t=1}^{q+1} a_t - qj, \ a_{q+1}, \ldots, a_n\Big) \in \mathbb{R}^{n-1}.$$

One may verify that $a^{j|}$ is monotone and $\sum_{i=1}^{n-1} a_i^{j|} = N - j$.

The construction of $a^{j|}$ leads to the following proposition concerning splittings. It also gives a recursive expression for the counting function ν^*.

Proposition 3.2. *Let $x = (x_1, x_2, \ldots, x_n)$ and $a = (a_1, a_2, \ldots, a_n)$ be monotone integral vectors in $\mathbb{R}^n$. Then $x \preceq a$ if and only if $(x_2, x_3, \ldots, x_n) \preceq a^{x_1|}$. Moreover, with $N = \sum_{j=1}^n a_j$, we have*

$$\nu^*(a) = \sum_{j=\lceil N/n \rceil}^{a_1} \nu^*(a^{j|}). \tag{3.2}$$

Proof. By definition, $x \in M_I^*(a)$ means that x is (integral) monotone and majorized by a. Considering the vectors in $M_I^*(a)$ with the first component being j and from the construction of the Ferrers diagram of $a^{j|}$ (with moved boxes in the *topmost* rows), we see that the set $\{x \in M_I^*(a) : x_1 = j\}$ is the same as

$$\{x \in \mathbb{Z}^n : x \text{ is monotone}, x_1 = j, \ (x_2, x_3, \ldots, x_n) \preceq a^{j|}\}.$$

(Here $\mathbb{Z}^n$ for integral vectors in $\mathbb{R}^n$.) This proves the first statement of the theorem.

Next, note that every vector x in $M_I^*(a)$ satisfies $x_1 \geq \lceil N/n \rceil$ (due to monotonicity and $\sum_{j=1}^n x_j = N$). We count $M_I^*(a)$ by partitioning this set according to the value of the first component of its vectors. Thus, by the first part of this theorem, we have

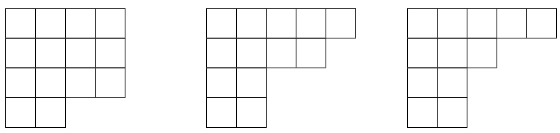
$$\begin{aligned}
\nu^*(a) &= \textstyle\sum_{j=\lceil N/n \rceil}^{a_1} |\{x \in M_I^*(a) : x_1 = j\}| \\
&= \textstyle\sum_{j=\lceil N/n \rceil}^{a_1} |\{(j,z) \in \mathbb{Z} \times \mathbb{Z}^{n-1} : (j,z) \text{ is monotone}, z \preceq a^{j|}\}| \\
&= \textstyle\sum_{j=\lceil N/n \rceil}^{a_1} |M_I^*(a^{j|})| \\
&= \textstyle\sum_{j=\lceil N/n \rceil}^{a_1} \nu^*(a^{j|}),
\end{aligned}$$

so (3.2) holds. □

Equation (3.2) in Proposition 9 clearly gives an algorithm for computing $\nu^*(a)$. Combined with the formula in Proposition 2.4, this algorithm may also be used to compute $\nu(a)$. Note that in (3.2) the computation of $\nu^*(a^{j|})$ can sometimes be simplified, especially for "small" j, by using the property $\nu^*(b) = \nu^*(b - b_n e)$ for a monotone vector b (where e is the all-ones vector and b_n is the smallest component of b), see Proposition 2.3. Proposition 3.2 leads to an algorithm for enumeration of the set $M_I^*(a)$: compute the $a^{j|}$'s, and then repeat this process for each of the constructed $a^{j|}$'s, etc.

Example 3.1 - continued. Consider again $a = (6, 5, 3, 2, 2)$. So $n = 5$, $N = 18$ and $\lceil N/n \rceil = 4$. The Ferrers diagram of $F^{j|}$ for $j = 4, 5, 6$ are

$F^{4|}$ $\quad$ $F^{5|}$ $\quad$ $F^{6|}$



$$a^{4|} = (4, 4, 4, 2) \quad a^{5|} = (5, 4, 2, 2) \quad a^{6|} = (5, 3, 2, 2)$$

Using formula (3.2) recursively, we compute

$$\nu^*(a^{4|}) = 2, \;\; \nu^*(a^{5|}) = 4, \;\; \nu^*(a^{6|}) = 4.$$

Therefore, by (3.2), $\nu^*(a) = 2 + 4 + 4 = 10$.

Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ be a nonnegative integral monotone vector, and define $m = a_1$. The *conjugate* of a is the vector $a^* = (a_1^*, a_2^*, \ldots, a_m^*) \in \mathbb{R}^m$, where

$$a_k^* = |\{i : a_i \geq k\}| \quad (1 \leq k \leq m).$$

If F is the Ferrers diagram corresponding to a, the row sums in F (viewing boxes as ones, and otherwise having zeros) are the components in a while the column sums are the components in a^*. In particular, the Ferrers diagram of a^* is the transpose of F (making rows into columns, as for matrices). For instance, if $a = (2, 2, 2, 1, 1)$, then $a^* = (5, 3)$.

Proposition 3.3. *Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ be a nonnegative monotone integral vector. Then*

$$\nu(a) \leq \prod_{k=1}^{a_1} \binom{n}{a_k^*}. \tag{3.3}$$

Equality holds if and only if $a = (s + 1, \ldots, s + 1, s, \ldots, s)$ for some s.

Proof. Let $\xi_k = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^n$ be the vector with k leading ones and otherwise zeros ($k \leq n$). We decompose $a = (a_1, a_2, \ldots, a_n)$ as

$$a = \sum_{k=1}^{m} \xi_{a_k^*} \qquad (m = a_1).$$

By Proposition 2.3 (iii) and Corollary 2.6, we have

$$\nu(a) = \nu\Big(\sum_{k=1}^{m} \xi_{a_k^*}\Big) \leq \prod_{k=1}^{m} \nu(\xi_{a_k^*}) = \prod_{k=1}^{m} \binom{n}{a_k^*}. \tag{3.4}$$

Equality in (3.3) occurs if and only if overall equality in (3.4) holds, which is true, by Proposition 2.3 (iii), if and only if one of $\xi_{a_k^*}$'s is non-constant. By Corollary 2.6, a is of the desired form. $\square$

Example 3.4. Let $a = (2, \ldots, 2, 1, \ldots 1) \in \mathbb{R}^n$, in which the number of 2's is k. Then $a^* = (n, k)$. So Proposition 3.3 gives

$$\nu(a) \leq \binom{n}{n} \cdot \binom{n}{k} = \binom{n}{k},$$

which we know (by Proposition 2.6) is tight, i.e., $\nu(a) = \binom{n}{k}$. If $a = (4, 2, 1)$ (Example 2.1), then $a^* = (3, 2, 1, 1)$ and $\nu(a) = 12$ while the bound in Proposition 3.3 is $1 \cdot 3 \cdot 3 \cdot 3 = 27$.

As this example shows, the quality of the bounds we have found is highly dependent on vector a itself. We believe that the bound in Proposition 3.3 may be acceptable when "the span" $a_1 - a_n$ is rather small.

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