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AMBIT FIELDS VIA FOURIER METHODS IN THE CONTEXT OF POWER MARKETS

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Abstract. In their paper Barndorff-Nielsen, Benth and Veraart [2] employ so called Ambit fields to model electricity spot-forward dynamics. We briefly introduce and discuss Ambit fields in the setting of modelling electricity forward markets, and introduce a novel method for approximating general ambit fields by a linear combination of ambit fields driven by exponential kernel functions (as has already been done in the null-spatial case of Lévy semistationary processes by Benth, Eyjolfsson and Veraart [12]) by representing the deterministic kernel function as an integral over its Fourier domain. Moreover we shall study examples of ambit fields with ill behaved kernel functions to illustrate the usefulness of our method for pricing purposes.

1. Introduction

Recently some effort has been put into studying, and applying so called ambit fields to model various tempo-spatial phenomena continuously in time and space. Roughly speaking, an ambit field is a tempo-spatial random field which is defined as an integral over a random measure (a Lévy basis) where the integrand is a deterministic function times a stochastic volatility/intermittency field. Initially, tempo-spatial ambit fields and their null-spatial analogues were suggested as tools for modelling turbulence in physics (see Barndorff-Nielsen and Schmiegel [5, 6]), but have also been successfully applied to model tumor growth [18].

In the current paper we shall however mostly be concerned with applications coming from mathematical finance, namely electricity markets. Indeed some very particular features of electricity market dynamics such as its non-storability (unless indirectly, e.g. in water-magazines) and (semi-) heavy tails of logarithmic returns of forward prices call for the application of non-standard models. Thus effectively traditional buy-and-hold hedging strategies break down and it is no longer necessary to stay within the semimartingale framework. In their paper Barndorff-Nielsen, Benth and Veraart [2] suggest using ambit fields as a general modelling framework for electricity forward contracts. They maintain that the general structure of ambit fields is very well suited for catching the various idiosyncratic features displayed by such markets and illustrate how previously suggested

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modelling frameworks (such as the HJM modelling framework [17]) relate to the ambit field framework. In particular they show that the very flexible structure of ambit fields can be employed to model the spot forward dynamics and to price derivatives by means of evaluating particular expected values over ambit fields.

Since these aforementioned expected values can rarely be evaluated explicitly, a matter of interest is to develop efficient ways of estimating them. Indeed, supposing that we are interested in pricing several derivatives depending on a parameter based on some underlying ambit field spot-forward relationship, it may not be entirely straightforward to do so since ambit fields can generally not be simulated be means of *incremental* simulation algorithms. The reason for this non-applicability of incremental algorithms is that the integrand of a general ambit field changes with the tempo-spatial parameter, so if one wants to evaluate a given ambit field on a specific grid one has to perform a complete integration at each grid point. This contrast the situation where the integrand is the same at each grid point, since for such integrands it is sufficient to evaluate an incremental integral and add it to the value at the previous close grid point to obtain the value at each point.

There are however some notable kernel functions that do accommodate incremental simulation schemes for their corresponding ambit fields despite being dependent on the tempo-spatial position. The most prominent ones being exponential functions. Indeed, in their paper Barndorff-Nielsen, Benth and Veraart [2] suggest some ambit field models with exponential kernel functions, and even outline how such a simulation algorithm can be obtained for a specific example. In the current paper we shall outline how one can approximate general ambit fields by means of a sum of finitely many ambit fields driven by complex-valued exponential functions. This in turn means that we can obtain an incremental simulation algorithm for general ambit fields.

More specifically, it is our goal is to extend the Fourier approximation methods first introduced in the setting of power markets by Benth and Eyjolfsson [11] and analysed further in Benth, Eyjolfsson and Veraart [12] in the setting of null-spatial ambit fields, called Lévy semistationary processes, to more general ambit fields. Thus we first give a general introduction to ambit fields and how they are defined, after which we introduce the Fourier approximation method in the setting of ambit fields. Analogously to the null-spatial case of Lévy semistationary processes the method consists of representing the deterministic kernel function as an integral over its Fourier transform and approximating the integral with a carefully selected finite sum. After which one may commute the sum with the stochastic integral to obtain an approximation of a general ambit field as a sum over finitely many ambit fields driven by ambit fields with exponential kernel functions.

As already mentioned, the novelty here is that for a given ambit field one obtains an approximation of it as a finite sum over ambit fields with exponential kernel functions. Thus enabling us to consider an iterative simulation scheme which would otherwise have been impossible due to the tempo-spatial dependency of general kernel functions. Furthermore we shall consider an example which illustrates the usefulness of our approach.

The paper is structured as follows. We begin with a preliminary section in which we recount definitions needed to define ambit fields and give appropriate references to more detailed accounts. This is followed by a section in which we define ambit fields and give some results on the L^2 continuity of ambit fields with respect to kernel functions and volatility fields, which are results that will be used throughout the rest of the paper. In section 4 we introduce and analyse the Fourier representation method for ambit fields. In section 5 we put our methods to work in the electricity forward setting, before drawing some conclusive remarks in section 6. Finally in the Appendix we give proofs to some auxiliary results.

2. Preliminaries

In the current section we shall briefly introduce the tools that are needed to define ambit fields and give references to relevant papers. We begin by listing the tools needed to define random measures and Lévy bases, followed by a brief analysis of Lévy bases and the integral we shall work with in this paper. The main reference for the survey in this section are Rajput and Rosinski [22] and Walsh [25], but in the setting of ambit processes this theory is also recounted in e.g Barndorff-Nielsen, Benth and Veraart [4].

Throughout the paper let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space. Moreover for a Borel set $S \subset \mathbb{R}^n$, where $n \geq 1$ let $(S, \mathcal{S}, \mathrm{Leb})$ denote the Lebesgue-Borel space where $\mathcal{S} = \mathcal{B}(S)$ and Leb denotes the Lebesgue measure. Now consider the subset of $\mathcal{S}$ which contains Borel sets of bounded Lebesgue measure, $\mathcal{B}_b(S) = \{A \in \mathcal{S} : \mathrm{Leb}(A) < \infty\}$. Clearly it holds that $\mathcal{B}_b(S)$ is a δ-ring, i.e. it is closed under finite union, relative complementation and countable intersection. Furthermore, there exists an increasing sequence $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_b$ such that $\cup_{n\in\mathbb{N}} S_n = S$. Now, following Rajput and Rosinski [22], by a *random measure* M on $(S, \mathcal{S})$ we mean a family $\{M(A)\}_{A\in\mathcal{B}_b(S)}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any sequence $\{A_n\}_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{B}_b(S)$ we have that $M(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} M(A_n)$ holds almost surely, where the series converges almost surely. If moreover, for any sequence $\{A_n\}_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{B}_b(S)$ the random variables $M(A_1), M(A_2), \ldots$ are independent then we say that M is an *independently scattered random measure*. Finally if for any $A \in \mathcal{B}_b(S)$ the law of $M(A)$ is infinitely divisible, i.e. if μ denotes the law of $M(A)$ then for each $n \in \mathbb{N}$ there exists a law μ_n such that $\mu_n^{*n} = \mu$, where μ_n^{*n} denotes the n-fold convolution of μ_n with itself, then we say that M is an *infinitely divisible random measure.*

We are now ready to define the objects which are used as integrators in general ambit fields. By a *Lévy basis* L on $(S, \mathcal{S})$ we mean an independently scattered, infinitely divisible random measure. We remark that the concept of a Lévy basis generalizes Lévy processes, since by taking $S = [0, \infty)$, letting $L(\{0\}) = 0$ and assuming stationarity of increments one obtains a Lévy process by considering $t \mapsto L([0, t])$.

2.1. **Rajput and Rosinski's integration theory.**

The following notation is useful in the setting of Rajput and Rosinski's integration theory. For a given random variable X we shall throughout the paper employ the notation

$$C\{\zeta \ddagger X\} := \log(\mathbb{E}[\exp(\mathrm{i}\zeta X)])$$

to denote the *cumulant* (i.e. log-characteristic) function of X, where $\zeta \in \mathbb{R}$. Here and in the sequel $C\{\zeta \ddagger X\}$ denotes the unique real solution of $\exp(C\{\zeta \ddagger X\}) = \mathbb{E}[\exp(\mathrm{i}\zeta X)]$.

Due to infinite divisibility a general Lévy basis has the Lévy-Khinchin representation

$$C\{\zeta \ddagger L(A)\} = \mathrm{i}\zeta a^*(A) - \frac{1}{2}\zeta^2 b^*(A) + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}\zeta x} - 1 - \mathrm{i}\zeta x 1_{[-1,1]}(x)\right) n(dx, A), \tag{2.1}$$

where a^* is a signed measure on $\mathcal{B}_b(S)$, b^* is a measure on $\mathcal{B}_b(S)$ and $n(dx, A)$ is a Lévy measure on $\mathbb{R}$ for fixed $A \in \mathcal{B}_b(S)$ and a measure on $\mathcal{B}_b(S)$ for fixed dx (see Rajput and Rosinski [22]). Another important object is the *control measure* associated to a given Lévy basis L with Lévy-Kinchin representation (2.1), which for a given $A \in \mathcal{B}_b(S)$ is defined by

$$c(A) = |a^*|(A) + b^*(A) + \int_{\mathbb{R}} \min(1, x^2) n(dx, A), \tag{2.2}$$

where $|\cdot|$ denotes total variation. Now again, by Rajput and Rosinski [22] it holds that the Lévy measure n factorises as $n(dx, d\boldsymbol{z}) = \nu(dx, \boldsymbol{z})c(d\boldsymbol{z})$ where $\nu(dx, A)$ is a Lévy measure on $\mathbb{R}$ for fixed $A \in \mathcal{B}_b(S)$ and a measure on $\mathcal{B}_b(S)$ for fixed dx and c is the control measure (2.2), and that a^* and b^* are absolutely continuous with respect to c, so that we may write $a^*(d\boldsymbol{z}) = a(\boldsymbol{z})c(d\boldsymbol{z})$ and $b^*(d\boldsymbol{z}) = b(\boldsymbol{z})c(d\boldsymbol{z})$. It is moreover possible to employ the Lévy-Kinchin formula in differential form to find that

$$\begin{aligned} C\{\zeta \ddagger L(d\boldsymbol{z})\} &= \mathrm{i}\zeta a^*(d\boldsymbol{z}) - \frac{1}{2}\zeta^2 b^*(d\boldsymbol{z}) + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}\zeta x} - 1 - \mathrm{i}\zeta x 1_{[-1,1]}(x)\right) n(dx, d\boldsymbol{z}) \\ &= \left(\mathrm{i}\zeta a(\boldsymbol{z}) - \frac{1}{2}\zeta^2 b(\boldsymbol{z}) + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}\zeta x} - 1 - \mathrm{i}\zeta x 1_{[-1,1]}(x)\right) \nu(dx, \boldsymbol{z})\right) c(d\boldsymbol{z}) \\ &= C\{\zeta \ddagger L'(\boldsymbol{z})\}c(d\boldsymbol{z}), \end{aligned}$$

where $\zeta \in \mathbb{R}$ and $L'(\boldsymbol{z})$ denotes the *Lévy seed* of L at $\boldsymbol{z}$, which is defined as the infinitely divisible random variable having Lévy-Kinchin representation

$$C\{\zeta \ddagger L'(\boldsymbol{z})\} = \mathrm{i}\zeta a(\boldsymbol{z}) - \frac{1}{2}\zeta^2 b(\boldsymbol{z}) + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}\zeta x} - 1 - \mathrm{i}\zeta x 1_{[-1,1]}(x)\right) \nu(dx, \boldsymbol{z}). \tag{2.3}$$

Having introduced the control measure, Rajput and Rosinski [22] proceed to introduce a general integration theory for deterministic integrands integrated with respect to random measures. For a given Lévy basis L on $(S, \mathcal{S})$ this is achieved by first defining the integral for simple integrands of the type $f = \sum_{j=1}^n x_j 1_{A_j}$, where $A_j \in \mathcal{B}_b(S)$ for $j = 1, \dots, n$ are disjoint as

$$\int_A f dL := \sum_{j=1}^n x_j L(A \cap A_j),$$

for any $A \in \mathcal{S}$. Next consider the class of L-*measurable* functions, which are defined as measurable functions $f : (S, \mathcal{S}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that there exists a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ c-a.e. (where c is the control measure of L) and for every $A \in \mathcal{S}$ the sequence $\{\int_A f_n dL\}$ converges in probability as $n \to \infty$. Now for the class of L-measurable functions, define

$$\int_A f dL := \mathbb{P} - \lim_{n\to\infty} \int_A f_n dL,$$

for any $A \in \mathcal{S}$. It can be shown that the integral is well defined in the sense that it does not depend on the approximating sequence $\{f_n\}$. Moreover, necessary and sufficient conditions for the existence of the integral for a given integrand $f : (S, \mathcal{S}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and Lévy basis L with control measure (2.2) and Lévy seed (2.3) are as follows (see Rajput and Rosinski [22], Theorem 2.7):

$$\int_S |V_1(f(\boldsymbol{z}), \boldsymbol{z})| c(d\boldsymbol{z}) < \infty, \quad \int_S |f(\boldsymbol{z})|^2 c(d\boldsymbol{z}) < \infty, \quad \int_S |V_2(f(\boldsymbol{z}), \boldsymbol{z})| c(d\boldsymbol{z}) < \infty, \tag{2.4}$$

where for $\varrho(x) := x 1_{[-1,1]}(x)$,

$$V_1(u, \boldsymbol{z}) := u a(\boldsymbol{z}) + \int_{\mathbb{R}} (\varrho(xu) - u\varrho(x)) \nu(dx, \boldsymbol{z}), \quad V_2(u, \boldsymbol{z}) := \int_{\mathbb{R}} \min(1, |xu|^2) \nu(dx, \boldsymbol{z}).$$

Finally (by Proposition 2.6 in Rajput and Rosinski [22]) if f is L-measurable and $A \in \mathcal{S}$ then

$$C\left\{\zeta \ddagger \int_A f dL\right\} = \int_A C\{\zeta f(\boldsymbol{z}) \ddagger L'(\boldsymbol{z})\} c(d\boldsymbol{z}).$$

From which it follows that

$$\mathbb{E}\left[\int_A f dL\right] = \int_A f(\boldsymbol{z}) \mathbb{E}[L'(\boldsymbol{z})] c(d\boldsymbol{z}) \tag{2.5}$$

and

$$\mathbb{E}\left[\left(\int_A f dL\right)^2\right] = \left(\int_A f(\boldsymbol{z}) \mathbb{E}[L'(\boldsymbol{z})] c(d\boldsymbol{z})\right)^2 + \int_A f^2(\boldsymbol{z}) \operatorname{Var}(L'(\boldsymbol{z})) c(d\boldsymbol{z}), \tag{2.6}$$

where $A \in \mathcal{S}$. Thus in the case when the Lévy seed has zero-mean, $\mathbb{E}[L'(\boldsymbol{z})] = 0$ for all $\boldsymbol{z} \in S$, the second moment (2.6) equation resembles an Itô isometry. Although we highlight that here we are only dealing with deterministic f.

2.2. **Walsh's integration theory.** Now let us summarize the integration theory of Walsh [25], which as opposed to the integration theory of Rajput and Rosinski [22] is more liberal in the sense that it allows stochastic integrands, but more restrictive in the sense that it imposes a square integrability condition on the Lévy basis.

First of all, time and space are treated separately and we restrict the domain of the Lévy basis to a bounded Borel set. Thus, suppose that L is a Lévy basis on $(S \times [0, T], \mathcal{B}(S \times [0, T]))$ where S is a bounded Borel subset in $\mathbb{R}^d$, where $d \geq 1$ denotes the space dimension, and $T > 0$ is the finite time horizon. Then, we introduce a natural ordering induced by time, by defining

$$L_t(A) = L(A, t) = L(A \times (0, t]),$$

for any $A \in \mathcal{B}_b(S)$ and $0 \leq t \leq T$. Now in order to employ the measure valued process $L_t(\cdot)$ as an integrator analogously to standard stochastic integrals we impose that $L_t(\cdot)$ is square integrable with zero mean. That is

Assumption 1. *$L_t(A) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}[L_t(A)] = 0$ for all $A \in \mathcal{B}_b(S)$.*

Note that the zero mean part of the assumption can easily be circumvented by considering the Lévy basis $\overline{L}_t(\cdot) := L_t(\cdot) - \mathbb{E}[L_t(\cdot)]$ in case $L_t(\cdot)$ has non-zero mean. Next, consider the right-continuous filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ which is given by

$$\mathcal{F}_t = \bigcap_{n=1}^{\infty} \mathcal{F}^0_{t+1/n}, \quad \text{where} \quad \mathcal{F}^0_t = \sigma\{L_s(A) : A \in \mathcal{B}_b(S), 0 \le s \le t\} \vee \mathcal{N}$$

and $\mathcal{N}$ denotes the $\mathbb{P}$-null sets of $\mathcal{F}$. Now by adding the assumption $L_0(A) = 0$ a.s. it holds that $\{L_t(A)\}_{t\ge 0, A\in\mathcal{B}_b(S)}$ is a so-called *martingale measure* with respect to the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ (see Walsh [25]). Furthermore for $t \in [0,T]$ and disjoint $A, B \in \mathcal{B}_b(S)$ the random variables $L_t(A)$ and $L_t(B)$ are independent. Thus in the setting of Walsh [25] $\{L_t(A)\}_{t\ge 0, A\in\mathcal{B}_b(S)}$ is an *orthogonal martingale measure*, and thus *worthy*, which is a property that makes them suitable as integrators. Under Assumption 1 this implies that for each $A \in \mathcal{B}_b(S)$, $t \mapsto L_t(A)$ is a square integrable martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ and that $L_t(A)$ and $L_t(B)$ are independent whenever $A \cap B = \emptyset$ for any $0 \le t \le T$. Finally we need to define the *covariance measure* Q which is given by the quadratic covariation

$$Q(A \times [0,T]) = \langle L(A) \rangle_t \,, \tag{2.7}$$

for $t \in [0,T]$ and $A \in \mathcal{B}_b(S)$.

Now Walsh [25] defines stochastic integration in the L^2 sense as follows. For an *elementary* random field

$$\zeta(\boldsymbol{\xi}, s) = X 1_{(a,b]}(s) 1_A(\boldsymbol{\xi}),$$

where $0 \le a < t$, $a \le b$, X is bounded and $\mathcal{F}_a$-measurable and $A \in \mathcal{B}_b(S)$ the stochastic integral with respect to L is defined as

$$\int_0^t \int_B \zeta(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds) := X(L_{t\wedge b}(A \cap B) - L_{t\wedge a}(A \cap B)),$$

for every $B \in \mathcal{B}_b(S)$. Letting $\mathcal{T}$ denote the set of simple random fields, i.e. finite linear combinations of elementary random fields, we extend the definition of the integral to $\mathcal{T}$ by defining the integral of elements in $\mathcal{T}$ as the corresponding linear combination of integrals of elementary random fields. Now denote by $\mathcal{P}$ the σ-algebra generated by $\mathcal{T}$, we say that a random field is *predictable* if it is $\mathcal{P}$-measurable. Then given the predictable random fields consider the Hilbert space $\mathcal{P}_L := L^2(\Omega \times [0,T] \times S, \mathcal{P}, Q)$ of predictable random fields which fulfill $||\zeta||_L^2 < \infty$, where

$$||\zeta||_L^2 := \mathbb{E}\left[\int_{[0,T]\times S} \zeta^2(\boldsymbol{\xi}, s) Q(d\boldsymbol{\xi}, ds)\right].$$

For simple integrands $\zeta \in \mathcal{T}$ the following Itô isometry holds:

$$\mathbb{E}\left[\left(\int_{[0,T]\times S} \zeta(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds)\right)^2\right] = ||\zeta||_L^2. \tag{2.8}$$

Walsh [25] shows that $\mathcal{T}$ is dense in $\mathcal{P}_L$. Thus for for $\zeta \in \mathcal{P}_L$ there exists a Cauchy sequence $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathcal{T}$ such that $||\zeta - \zeta_n||_L \to 0$ as $n \to \infty$. But due to the isometry (2.8) the sequence $\{\int_{[0,t]\times A} \zeta_n(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ for any $0 \leq t \leq T$ and $A \in \mathcal{B}_b(S)$. Thus we define the stochastic integral of ζ as the $L^2(\Omega, \mathcal{F}, \mathbb{P})$ limit of the corresponding approximating sequence of integrals of simple functions. Finally Walsh [25] proves that the Itô isometry (2.8) also holds for predictable integrands $\zeta \in \mathcal{P}_L$, and moreover it holds that the integral is a martingale measure.

Now consider weakening Assumption 1 to assume that $L_t(A) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is not necessarily zero-mean and denote by $\overline{L}_t(\cdot) := L_t(\cdot) - \mathbb{E}[L_t(\cdot)]$ the centered process which fulfills Assumption 1 and let Q be defined as before with $\overline{L}$. Then, since the integral with respect to $\overline{L}$ is a zero mean martingale for each $B \in \mathcal{B}_b(S)$ it holds that

$$\mathbb{E}\left[\int_{[0,T]\times S} \zeta(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds)\right] = \mathbb{E}\left[\int_{[0,T]\times S} \zeta(\boldsymbol{\xi}, s) \mathbb{E}[L(d\boldsymbol{\xi}, ds)]\right] \tag{2.9}$$

and

$$\begin{aligned}\mathbb{E}\left[\left(\int_{[0,T]\times S} \zeta(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds)\right)^2\right] &= \mathbb{E}\left[\left(\int_{[0,T]\times S} \zeta(\boldsymbol{\xi}, s) \mathbb{E}[L(d\boldsymbol{\xi}, ds)]\right)^2\right] + ||\zeta||_L^2 \\ &+ 2\mathbb{E}\left[\int_{[0,T]\times S} \zeta(\boldsymbol{\xi}, s) \mathbb{E}[L(d\boldsymbol{\xi}, ds)] \int_{[0,T]\times S} \zeta(\boldsymbol{\xi}, s) \overline{L}(d\boldsymbol{\xi}, ds)\right]. \end{aligned} \tag{2.10}$$

Notice in particular that when the integrand ζ is deterministic, the martingale condition of the stochastic integral implies that

$$\mathbb{E}\left[\left(\int_{[0,T]\times S} \zeta(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds)\right)^2\right] = \left(\int_{[0,T]\times S} \zeta(\boldsymbol{\xi}, s) \mathbb{E}[L(d\boldsymbol{\xi}, ds)]\right)^2 + ||\zeta||_L^2. \tag{2.11}$$

In the setting of deterministic integrands it is particularly interesting to compare the moments (2.9) and (2.11) to the moments (2.5) and (2.6) we obtained before in the setting of Rajput and Rosinski [22].

2.3. **Comparison of integration concepts.** Now let us briefly study the similarities and differences of the two proposed integration concepts, in particular the second moment structure of the stochastic integrals.

Recall that the first and second moments of the Rajput and Rosinkski's integral are given by (2.5) and (2.6). By the Lévy-Kinchin representation (2.3) of the Lévy seed it moreover holds that

$$\mathbb{E}[L'(\boldsymbol{z})] = a(\boldsymbol{z}) + \int_{|x|>1} x\nu(dx, \boldsymbol{z}) \quad \text{and} \quad \mathrm{Var}(L'(\boldsymbol{z})) = b(\boldsymbol{z}) + \int_{\mathbb{R}} x^2\nu(dx, \boldsymbol{z}).$$

So by separating the time and space variables and integrating $f = 1$ over $[0, T] \times S$ where S is a bounded Borel set it follows that

$$\mathbb{E}[L(d\boldsymbol{\xi}, ds)] = \left(a(\boldsymbol{\xi}, s) + \int_{|x|>1} x\nu(dx, \boldsymbol{\xi}, s)\right) c(d\boldsymbol{\xi}, ds) \tag{2.12}$$

and

$$Q(d\boldsymbol{\xi}, ds) = \left(b(\boldsymbol{\xi}, s) + \int_{\mathbb{R}} x^2 \nu(dx, \boldsymbol{\xi}, s) \right) c(d\boldsymbol{\xi}, ds) \tag{2.13}$$

on $[0, T] \times S$. Thus, we reiterate what has already been pointed out by Barndorff-Nielsen et al [1], namely that due to the above measure equivalence the weak integration concept of Rajput and Rosinski is a generalisation of that of Walsh as long as deterministic integrands are considered. On the other hand the integral due to Walsh is more suitable for studying dynamic properties, such as martingale properties, since it is derived in the spirit of the Itô integral as a tool to study stochastic partial differential equations (see Walsh [25]).

3. Ambit fields

In the current section we shall employ the integration concepts discussed in the previous section to define ambit fields. Here we follow the definition of Barndorff-Nielsen et al [1, 4]. As mentioned in Barndorff-Nielsen et al [4] ambit fields were initially defined using the integration concept of Rajput and Rosinski [22]. We review how this can be achieved before we discuss how one may employ Walsh's [25] integration concept to define ambit fields. Having done this, we shall give an overview of the similarities and differences of the two respective approaches. We shall see that the integration concept of Walsh has the advantage of allowing more general stochastic integrands as well as being more suitable for studying dynamic properties, such as martingale properties, as already mentioned. Whereas the integration concept due to Rajput and Rosinski is more flexible in the sense that it does not impose any conditions on the Lévy basis. Finally we finish the section with a result which is important in our setting, in which we employ the second order structure of the integrals to prove a result on the L^2-proximity of two different ambit processes driven by the same integrator. A result which is formulated generally, so that it may be employed for ambit fields defined by means of either Rajput and Rosinski's theory, using the control measure, or Walsh's theory, using the quadratic covariation (see Lemma 3.1). We shall use this result repeatedly throughout this paper.

Now let us state what we mean by an ambit field, before we discuss in what way we define an ambit fields rigorously. Fix a spatial dimension $d \geq 1$. An *ambit field* is a tempo-spatial random field on the form

$$Y(\boldsymbol{x}, t) = \int_{A(\boldsymbol{x},t)} g(\boldsymbol{x}, t; \boldsymbol{\xi}, s) \sigma(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds), \tag{3.1}$$

where $(\boldsymbol{x}, t) \in \mathbb{R}^d \times \mathbb{R}$, $A(\boldsymbol{x}, t) \subset \mathbb{R}^d \times \mathbb{R}$ is the so-called *ambit set* over which the integration is performed for each $(\boldsymbol{x}, t) \in \mathbb{R}^d \times \mathbb{R}$, $g : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a deterministic kernel function, σ is a non-negative stochastic space-time volatility field and L is a Lévy basis defined on a suitable Borel subset of $\mathbb{R}^d \times \mathbb{R}$. For a given curve $\varpi(\theta) = (\boldsymbol{x}(\theta), t(\theta)) \in \mathbb{R}^d \times \mathbb{R}$ we call

$$X_\theta = \int_{A(\theta)} g(\boldsymbol{x}(\theta), t(\theta); \boldsymbol{\xi}, s) \sigma(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds),$$

where $A(\theta) = A(\boldsymbol{x}(\theta), t(\theta))$, the corresponding *ambit process*.

In many applications it is natural to assume that the ambit process is stationary in time and nonanticipative and homogeneous in space. These effects can be achieved by considering ambit fields of the type

$$Y(\boldsymbol{x},t)=\int_{A(\boldsymbol{x},t)} g(\boldsymbol{x}-\boldsymbol{\xi},t-s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds), \tag{3.2}$$

where $(\boldsymbol{x},t)\in\mathbb{R}^d\times\mathbb{R}$, σ is a non-negative stochastic volatility field and L is a Lévy basis and the ambit set is on the form $A(\boldsymbol{x},t)=A+(\boldsymbol{x},t)$ where A only involves negative time coordinates.

For our analysis later it will moreover be convenient to extend the definition (3.1) of ambit processes to allow complex kernel functions. Thus for a complex kernel function g a Lévy basis L, volatility field σ and an ambit set $A(\boldsymbol{x},t)$ we simply define a complex ambit field $Y(\boldsymbol{x},t)$ as the sum

$$Y(\boldsymbol{x},t)=\int_{A(\boldsymbol{x},t)} \operatorname{Re} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)+\mathrm{i}\int_{A(\boldsymbol{x},t)} \operatorname{Im} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds), \tag{3.3}$$

as long as the real-part $(\boldsymbol{x},t)\mapsto\int_{A(\boldsymbol{x},t)}\operatorname{Re} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)$ and imaginary-part $(\boldsymbol{x},t)\mapsto\int_{A(\boldsymbol{x},t)}\operatorname{Im} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)$ ambit fields are well defined as real valued ambit fields. For such a complex ambit field, we shall employ the notation (3.1), as before.

3.1. **Ambit fields via Rajput and Rosinski.** Recall from the previous section that in the setting of Rajput and Rosinski the integrand is deterministic. Therefore we can not directly apply the theory of Rajput and Rosinski to give meaning to ambit fields. If however we assume that the stochastic volatility field and the Lévy basis are independent, we may employ conditioning to define the ambit field. We shall therefore need the following assumption.

Assumption 2. *The stochastic volatility field σ is independent of the Lévy basis L.*

Now consider the σ-algebra

$$\mathcal{F}_t^\sigma(\boldsymbol{x}):=\sigma\{\sigma(\boldsymbol{\xi},s):(\boldsymbol{\xi},s)\in A(\boldsymbol{x},t)\}$$

which is generated by the history of σ in the ambit set $A(\boldsymbol{x},t)$. Then, for a given a Lévy basis L on $(S,\mathcal{S})$ with control measure c, we may extend the integration theory of the previous section to measurable integrands of the type $f:(\Omega\times S,\mathcal{F}_t^\sigma(\boldsymbol{x})\otimes\mathcal{S})\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ by conditioning on the σ-algebra $\mathcal{F}_t^\sigma(\boldsymbol{x})$. Thus under Assumption 2 we give meaning to the ambit field concept for any fixed $(\boldsymbol{x},t)\in\mathbb{R}^d\times\mathbb{R}$.

3.2. **Ambit fields via Walsh.** In the setting of Walsh, by contrast, we have restricted ourselves to Lévy bases on bounded domains, i.e. $S\times[0,T]$ where $S\subset\mathbb{R}^d$ is a bounded Borel set. However, extensions to unbounded S and infinite time intervals follow by standard arguments (see Walsh [25], p. 289). Moreover, following Barndorff-Nielsen et al [4] we need the following assumption.

Assumption 3. *For a Lévy basis L on $(-\infty, T] \times S$, where $S \in \mathcal{B}(\mathbb{R}^d)$ consider the centered Lévy basis $\overline{L} := L - \mathbb{E}[L]$. We extend the definition of the covariation measure* (2.7) *of $\overline{L}$ to an unbounded domain and define a Hilbert space $\mathcal{P}_{\overline{L}}$ as before with norm $||\cdot||_{\overline{L}}$ extended to an unbounded domain and an Itô isometry of the type* (2.8)*. For fixed $(\boldsymbol{x}, t) \in \mathbb{R}^d \times \mathbb{R}$ it holds that*

$$\zeta(\boldsymbol{\xi}, s) = 1_{A(\boldsymbol{x},t)}(\boldsymbol{\xi}, s) g(\boldsymbol{x}, t; \boldsymbol{\xi}, s) \sigma(\boldsymbol{\xi}, s)$$

is integrable, $\zeta \in \mathcal{P}_{\overline{L}}$, where

$$||\zeta||_{\overline{L}}^2 = \mathbb{E}\left[\int_{\mathbb{R}^d \times \mathbb{R}} \zeta^2(\boldsymbol{\xi}, s) Q(d\boldsymbol{\xi}, ds)\right] < \infty \quad \textit{and} \quad \int_{\mathbb{R}^d \times \mathbb{R}} |\zeta(\boldsymbol{\xi}, s)| \mathbb{E}[L(d\boldsymbol{\xi}, ds)] < \infty.$$

Given the above assumption we may employ the integration theory of Walsh to define ambit fields (for more details see Barndorff-Nielsen et al [4]).

3.3. **Comparison and L^2-proximity.** At this point we have given two different definitions of the ambit field concept. In this subsection we shall investigate to what extent they are compatible and introduce L^2-approximation results. In order to ease the notation, let us first introduce the following function spaces.

For a given Lévy basis L, with Lévy seed L' and control measure c, and an ambit set $A(\boldsymbol{x}, t)$ let

$$\mathcal{L}^1 = \mathcal{L}^1(c, L', A(\boldsymbol{x}, t)) := L^1(\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}), 1_{A(\boldsymbol{x},t)}(\boldsymbol{\xi}, s) |\mathbb{E}[L'(\boldsymbol{\xi}, s)]| c(d\boldsymbol{\xi}, ds))$$

and

$$\mathcal{L}^2 = \mathcal{L}^2(c, L', A(\boldsymbol{x}, t)) := L^2(\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}), 1_{A(\boldsymbol{x},t)}(\boldsymbol{\xi}, s) \operatorname{Var}[L'(\boldsymbol{\xi}, s)] c(d\boldsymbol{\xi}, ds))$$

denote weighted L^1 and L^2 spaces in space-time and denote their norms by $||\cdot||_{\mathcal{L}^1}$ and $||\cdot||_{\mathcal{L}^2}$ respectively.

Now let us briefly study the ambit field concept, which we have just defined via two different approaches. First of all in the case of deterministic integrands and square integrable Lévy bases the integrals coincide, since L^2-convergence implies convergence in probability. Moreover, when dealing with deterministic integrands, the Rajput and Rosinski integral can be viewed as an extension of the Walsh integral, in the sense that it allows more general Lévy bases as integrators, i.e. Lévy bases which are not necessarily square integrable. For stochastic integrands which are independent to the Lévy basis, i.e. in the case when Assumption 2 is fulfilled, it moreover holds that the resulting integrals have the same first moment. To see this notice first that the first integral moment by means of Rajput and Rosinski obtained by using (2.5) to conclude that for given real-valued ambit field $Y(\boldsymbol{x}, t)$ given by (3.1) it holds that

$$\begin{aligned}\mathbb{E}\left[Y(\boldsymbol{x}, t)\right] &= \mathbb{E}\left[\mathbb{E}\left[Y(\boldsymbol{x}, t) | \mathcal{F}_t^\sigma\right]\right] \\ &= \mathbb{E}\left[||g(\boldsymbol{x}, t; \cdot, \cdot)\sigma||_{\mathcal{L}^1}\right],\end{aligned}$$

which due to the measure equivalence (2.12) is equal to the moment (2.9) obtained by means of Walsh. As for the second moment, in the setting of Rajput and Rosinski notice that we may employ conditioning and (2.6) to obtain

$$\begin{aligned}\mathbb{E}\left[Y^2(\boldsymbol{x},t)\right] &= \mathbb{E}\left[\mathbb{E}\left[Y^2(\boldsymbol{x},t)\right]|\mathcal{F}_t^\sigma\right] \\ &= \mathbb{E}\left[||g(\boldsymbol{x},t;\cdot,\cdot)\sigma||^2_{\mathcal{L}^1}\right] + \mathbb{E}\left[||g(\boldsymbol{x},t;\cdot,\cdot)\sigma||^2_{\mathcal{L}^2}\right]. \end{aligned} \tag{3.4}$$

However, this second moment may not be equal to the corresponding second moment obtained by means of Walsh. Indeed unless the cross term (2.10) of the second moment as obtained by Walsh is equal to zero, the second moments are not equal. The cross term in turn is equal to zero whenever the corresponding Lévy basis L is centered, i.e. as zero mean, or if the integrands are deterministic, as we have already mentioned. Finally note that the Walsh integral is more general in the sense that it is well defined regardless of the dependence structure between the stochastic volatility field and the Lévy basis.

Now let us state and prove a result on the L^2-continuity of ambit fields with respect to kernel functions and volatility fields. We shall need the following square integrability assumption on the volatility field of a given complex ambit field (3.3). There exist a constant $\kappa_2 > 0$ such that

$$\mathbb{E}[\sigma(\boldsymbol{\xi},s)\sigma(\tilde{\boldsymbol{\xi}},\tilde{s})] < \kappa_2 \tag{3.5}$$

holds for all $(\boldsymbol{\xi},s),(\tilde{\boldsymbol{\xi}},\tilde{s}) \in A(\boldsymbol{x},t)$. We have the following result.

Lemma 3.1. *Consider a complex ambit field $Y(\boldsymbol{x},t)$ given by (3.3) and let*

$$Z(\boldsymbol{x},t) = \int_{A(\boldsymbol{x},t)} h(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds) \quad \text{and} \quad \widetilde{Z}(\boldsymbol{x},t) = \int_{A(\boldsymbol{x},t)} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\rho(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)$$

denote complex ambit fields with a different kernel and volatility field respectively. Then

(1) *if the condition (3.5) is fulfilled it holds that*

$$\mathbb{E}[|Y(\boldsymbol{x},t) - Z(\boldsymbol{x},t)|^2] \leq 2\kappa_2(2||g(\boldsymbol{x},t;\cdot,\cdot) - h(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^1} + ||g(\boldsymbol{x},t;\cdot,\cdot) - h(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^2}),$$

(2)

$$\mathbb{E}\left[\left|Y(\boldsymbol{x},t) - \widetilde{Z}(\boldsymbol{x},t)\right|^2\right] \leq 2(2||g(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^1} + ||g(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^2}) \sup_{(\boldsymbol{\xi},s)\in A(\boldsymbol{x},t)} \mathbb{E}\left[|\sigma(\boldsymbol{\xi},s) - \rho(\boldsymbol{\xi},s)|^2\right].$$

Proof. Let us first prove (1). To that end first assume that the ambit fields $Y(\boldsymbol{x},t)$ and $Z(\boldsymbol{x},t)$ are real valued. Then in the case of Rajput and Rosinski we apply (3.4) and in the case of Walsh the elementary inequality $(x+y)^2 \leq 2(x^2+y^2)$, $x,y \in \mathbb{R}$ applied to

$$Y^2(\boldsymbol{x},t) = \left(\int_{A(\boldsymbol{x},t)} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)\mathbb{E}[L(d\boldsymbol{\xi},ds)] + \int_{A(\boldsymbol{x},t)} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)\overline{L}(d\boldsymbol{\xi},ds)\right)^2,$$

together with the isometry (2.8) and (3.5) to conclude that

$$\begin{aligned}&\mathbb{E}[|Y(\boldsymbol{x},t) - Z(\boldsymbol{x},t)|^2] \\ &\leq 2(\mathbb{E}\left[||(g(\boldsymbol{x},t;\cdot,\cdot) - h(\boldsymbol{x},t;\cdot,\cdot))\sigma||^2_{\mathcal{L}^1}\right] + \mathbb{E}\left[||(g(\boldsymbol{x},t;\cdot,\cdot) - h(\boldsymbol{x},t;\cdot,\cdot))\sigma||^2_{\mathcal{L}^2}\right]) \\ &\leq 2\kappa_2(||g(\boldsymbol{x},t;\cdot,\cdot) - h(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^1} + ||g(\boldsymbol{x},t;\cdot,\cdot) - h(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^2}).\end{aligned}$$

Now for a complex ambit fields, notice that it holds that

$$\mathbb{E}[|Y(\boldsymbol{x},t)-Z(\boldsymbol{x},t)|^2]=\mathbb{E}[|\operatorname{Re}Y(\boldsymbol{x},t)-\operatorname{Re}Z(\boldsymbol{x},t)|^2]+\mathbb{E}[|\operatorname{Im}Y(\boldsymbol{x},t)-\operatorname{Im}Z(\boldsymbol{x},t)|^2]$$

So (1) follows by observing that $|\operatorname{Re}f|,|\operatorname{Im}f|\leq|f|$ holds for any complex valued function f. Similarly (2) follows by observing that for real-valued ambit fields it holds that

$$\begin{aligned}&\mathbb{E}[|Y(\boldsymbol{x},t)-\widetilde{Z}(\boldsymbol{x},t)|^2]\\&\leq 2(\mathbb{E}\left[||g(\boldsymbol{x},t;\cdot,\cdot)(\sigma-\rho)||^2_{\mathcal{L}^1}\right]+\mathbb{E}\left[||g(\boldsymbol{x},t;\cdot,\cdot)(\sigma-\rho)||^2_{\mathcal{L}^2}\right])\\&\leq 2(||g(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^1}+||g(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^2})\sup_{(\boldsymbol{\xi},s)\in A(\boldsymbol{x},t)}\mathbb{E}\left[|\sigma(\boldsymbol{\xi},s)-\rho(\boldsymbol{\xi},s)|^2\right]\end{aligned}$$

and extending the argument to complex ambit fields. □

Now let us explore what the Lemma implies in the case of ambit fields (3.2) that are stationary in time and nonanticipative.

Corollary 3.2. *Suppose that the kernel functions g and h in Lemma 3.1 are on the form*

$$g(\boldsymbol{x},t;\boldsymbol{\xi},s)=g(\boldsymbol{x}-\boldsymbol{\xi},t-s)\ \text{ and }\ h(\boldsymbol{x},t;\boldsymbol{\xi},s)=h(\boldsymbol{x}-\boldsymbol{\xi},t-s),$$

and that the ambit set is on the form $A(\boldsymbol{x},t)=A+(\boldsymbol{x},t)$ where $A\subset\mathbb{R}^d\times\mathbb{R}$ is a bounded set, and furthermore that the control measure c is absolutely continuous with respect to the Lebesgue measure with Radon-Nikodym derivative f_c such that

$$\mathbb{E}[L'(\boldsymbol{\xi},s)]\vee\operatorname{Var}(L'(\boldsymbol{\xi},s))\vee f_c(\boldsymbol{\xi},s)<K$$

holds for all $(\boldsymbol{\xi},s)\in A(\boldsymbol{x},t)$ where $K>0$ is a constant. Then

(1) *if the condition* (3.5) *is furthermore fulfilled it holds that*

$$\mathbb{E}[|Y(\boldsymbol{x},t)-Z(\boldsymbol{x},t)|^2]\leq 2\kappa_2(2K^4\operatorname{Leb}(-A)+K^2)||g-h||^2_{L^2(-A)}$$

(2)

$$\mathbb{E}\left[\left|Y(\boldsymbol{x},t)-\widetilde{Z}(\boldsymbol{x},t)\right|^2\right]\leq 2(2K^4\operatorname{Leb}(-A)+K^2)||g||^2_{L^2(-A)}\sup_{(\boldsymbol{\xi},s)\in A(\boldsymbol{x},t)}\mathbb{E}\left[|\sigma_s(\boldsymbol{\xi})-\rho_s(\boldsymbol{\xi})|^2\right],$$

where the integration in $L^1(-A)$ and $L^2(-A)$ is with respect to the Lebesgue measure.

Proof. We prove the first part of the Corollary, the second part is similar. By applying the translational change of variables $\phi:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d\times\mathbb{R}$, $(\boldsymbol{y},u)\mapsto(\boldsymbol{x}-\boldsymbol{y},t-u)$, Lemma 3.1 and the Cauchy-Schwarz inequality:

$$\begin{aligned}\mathbb{E}[|Y(\boldsymbol{x},t)-Z(\boldsymbol{x},t)|^2]&\leq 2\kappa_2(2K^4||g-h||^2_{L^1(-A)}+K^2||g-h||^2_{L^2(-A)})\\&\leq 2\kappa_2(2K^4\operatorname{Leb}(-A)+K^2)||g-h||^2_{L^2(-A)}.\end{aligned}$$

□

We end this section by introducing a particular regularity condition on Lévy bases, under which it follows that the corresponding control measure becomes absolutely continuous with respect to the Lebesgue measure. Thus, obtaining a Corollary to our Lemma 3.1 under the regularity condition.

More specifically, for a given spatial dimension $d \geq 1$ assume that S is a bounded Borel set and that $T > 0$ is the finite time horizon such that the Hilbert space

$$H := L^2(S \times [0,T], \mathcal{B}(S \times [0,T]), \text{Leb}),$$

is separable. By separability the Hilbert space H has a countable orthonormal basis, which we denote by $\{e_n\}_{n=1}^{\infty}$. Now, as in the theory of Walsh, consider a Lévy basis L on the space-time product space $(S \times [0,T], \mathcal{B}(S \times [0,T]))$. Assume moreover that the Lévy basis L has *nuclear covariance*, meaning that

$$\sum_{n=1}^{\infty} \mathbb{E}\left[\left(\int_{S\times[0,T]} e_n(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds)\right)^2\right] < \infty. \tag{3.6}$$

Here the integrals over the basis functions are understood in the sense of Rajput and Rosinski. We have the following Proposition which restates and reformulates results from section 2.3.3 in Barndorff-Nielsen et al [1] for our setting, for the sake of completeness we also give the proof.

Proposition 3.3. *Suppose that L is a Lévy basis which has nuclear covariance* (3.6)*. Then for every $A \in \mathcal{B}(S)$ and $t \in [0,T]$ it holds that $L(A,t) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and that that L is absolutely continuous with respect to* Leb *with Radon-Nikodym derivative $\dot{L} \in H$.*

Proof. For $A \in \mathcal{B}(S)$ and $t \in [0,T]$ it holds that $1_{A\times[0,t]} \in H$, so $1_{A\times[0,t]}$ has the representation

$$1_{A\times[0,t]}(\boldsymbol{\xi}, s) = \sum_{n=1}^{\infty} (1_{A\times[0,t]}, e_n)_H e_n(\boldsymbol{\xi}, s),$$

from which it follows that

$$L(A,t) = \int_{S\times[0,T]} 1_{A\times[0,t]}(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds) = \sum_{n=1}^{\infty} (1_{A\times[0,t]}, e_n)_H \int_{S\times[0,T]} e_n(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds).$$

Now by the Cauchy-Schwarz inequality for sums and Parseval's identity, it holds that

$$\begin{aligned} \mathbb{E}\left[L^2(A,t)\right] &\leq \sum_{n=1}^{\infty} \left|(1_{A\times[0,t]}, e_n)_H\right|^2 \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\int_{S\times[0,T]} e_n(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds)\right)^2\right] \\ &= ||1_{A\times[0,t]}||_H^2 \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\int_{S\times[0,T]} e_n(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds)\right)^2\right] < \infty. \end{aligned}$$

Now consider the linear functional in the dual space, $I \in H^*$, which is defined by

$$\phi \mapsto \int_{S\times[0,T]} \phi(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds).$$

Then I is bounded, since for a $\phi \in H$ with representation $\phi = \sum_{n=1}^{\infty} (\phi, e_n)_H e_n$ it holds that

$$I(\phi) = \sum_{n=1}^{\infty} (\phi, e_n)_H \int_{S\times[0,T]} e_n(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds),$$

so again, by the Cauchy-Schwarz inequality for sums and Parseval's identity it follows that

$$\mathbb{E}\left[|I(\phi)|^2\right] \leq ||\phi||_H^2 \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\int_{S\times[0,T]} e_n(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)\right)^2\right] < \infty.$$

Thus the integral is finite a.s. and I is a bounded linear functional on H. By a property of Hilbert spaces it follows that there exists a unique function $\dot{L} \in H$ such that $I(\phi) = (\phi, \dot{L})_H$ holds for all $\phi \in H$. In particular, since $S \times [0,T]$ is bounded in $\mathbb{R}^d \times \mathbb{R}$ this implies that

$$L(A,t) = \int_{A\times[0,t]} \dot{L}(\boldsymbol{\xi},s)d\boldsymbol{\xi}ds,$$

for $A \in \mathcal{B}(S)$ and $t \in [0,T]$. □

Given the above Proposition and Lemma 3.1 we also have the following Corollary.

Corollary 3.4. *Suppose that* L *is a Lévy basis on a bounded domain* $(S\times[0,T], \mathcal{B}(S\times[0,T])$ *which has nuclear covariance* (3.6)*. Suppose that* $Y(\boldsymbol{x},t)$ *is a complex ambit field given by* (3.3) *and let*

$$Z(\boldsymbol{x},t) = \int_{A(\boldsymbol{x},t)} h(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds) \quad \text{and} \quad \widetilde{Z}(\boldsymbol{x},t) = \int_{A(\boldsymbol{x},t)} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\rho(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)$$

denote complex ambit fields with a different kernel and volatility field respectively. Then

$$\mathbb{E}\left[|Y(\boldsymbol{x},t) - Z(\boldsymbol{x},t)|^2\right] \leq \mathbb{E}\left[||1_{A(\boldsymbol{x},t)}\sigma.(\cdot)\dot{L}||_H^2\right] ||1_{A(\boldsymbol{x},t)}(g(\boldsymbol{x},t;\cdot,\cdot) - h(\boldsymbol{x},t;\cdot,\cdot))||_H^2$$

and

$$\mathbb{E}\left[\left|Y(\boldsymbol{x},t) - \widetilde{Z}(\boldsymbol{x},t)\right|^2\right] \leq ||1_{A(\boldsymbol{x},t)}g(\boldsymbol{x},t;\cdot,\cdot)||_H^2 \mathbb{E}\left[||1_{A(\boldsymbol{x},t)}(\sigma-\rho)\dot{L}||_H^2\right].$$

Proof. Follows from the Cauchy-Schwarz inequality. □

4. Fourier representation

In this section we shall put to work the results which we have derived on ambit fields so far. More specifically, we shall for a given ambit field approximate its kernel function by a finite sum which in turn allows us to approximate general ambit fields as a finite sum of complex ambit fields with exponential kernel functions. Thus representing the ambit field as a finite sum of ambit fields which demonstrate an incremental property (see Proposition 4.6).

4.1. **Series representation of kernel functions.** Let $Y(\boldsymbol{x},t)$ denote a general ambit field (3.1). In this subsection our goal is to identify conditions which allow us to represent the kernel function g as a convergent series and to analyse the proximity of the corresponding ambit fields as we truncate the series.

To that end consider the ambit field $(\boldsymbol{x},t) \mapsto Y(\boldsymbol{x},t)$ on some *bounded* domain $\mathcal{D} \subset \mathbb{R}^d \times \mathbb{R}$ in space time. Associated to the ambit field is the corresponding sequence of ambit sets $\{A(\boldsymbol{x},t)\}_{(\boldsymbol{x},t)\in\mathcal{D}}$ over which we integrate at each point $(\boldsymbol{x},t) \in \mathcal{D}$ in space time. In general there are no restrictions on the structure of the ambit sets $A(\boldsymbol{x},t)$ where $(\boldsymbol{x},t) \in \mathcal{D}$.

For our purposes we shall however require that the union $\cup_{(\boldsymbol{x},t)\in\mathcal{D}} A(\boldsymbol{x},t)$ is bounded in space time $\mathbb{R}^d \times \mathbb{R}$. If this is not true, then we may approximate $Y(\boldsymbol{x},t)$ arbitrarily well in each $(\boldsymbol{x},t) \in \mathcal{D}$ by ambit fields that have bounded ambit sets $\{A(\boldsymbol{x},t) \cap A\}_{(\boldsymbol{x},t)\in\mathcal{D}}$, where $A \subset \mathbb{R}^d \times \mathbb{R}$ is bounded, by employing the approximation Lemma 3.1. Thus we have reduced the domain of the kernel function $g : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ to a bounded set $\mathcal{D} \times \cup_{(\boldsymbol{x},t)\in\mathcal{D}} A(\boldsymbol{x},t) \subset \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$.

Moreover we need to impose some constraints on the kernel function g. Since it is our intention to represent the kernel function as an integral over a Fourier transform, the kernel function must comply with the conditions that are required for this to be possible. In particular we shall require that the kernel function is continuous, since indeed the Fourier inversion theorem states that for an integrable function f which has an integrable Fourier transform, the function f agrees almost everywhere with a continuous function given by inverting the Fourier transform.

Therefore we shall henceforth assume that g is a continuous function. If this is not the case then we assume that one may approximate g arbitrarily well in the $\mathcal{L}^1$ and $\mathcal{L}^2$ norms presented in the previous section by a function that is continuous. We remark that this assumption is quite reasonable in light of the fact that the space of continuous functions on a Euclidean space with compact support is dense in any $L^p(\mu)$ space where $1 \leq p < \infty$ and μ is a Radon measure (see Proposition 7.9 in Folland [15]), and thus in particular any Borel measure μ that is finite on compact sets.

As a final remark on the structure of general kernel functions g, consider the following. In the previous section we introduced a class ambit fields (3.2) having kernel functions of the type $(\boldsymbol{x},t,\boldsymbol{\xi},s) \mapsto g(\boldsymbol{x}-\boldsymbol{\xi},t-s)$. Thus effectively reducing the dimension of the kernel function domain from $2d+2$ to $d+1$. To represent this dimension reduction we shall therefore assume that a given kernel function g on a bounded domain can be represented as

$$g = h \circ p \tag{4.1}$$

where for some $1 \leq n \leq 2d+2$, $p : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^n$ is a linear map and $h : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that for a given $0 < \tau_0 < \tau$ where

$$p\left(\mathcal{D} \times \cup_{(\boldsymbol{x},t)\in\mathcal{D}} A(\boldsymbol{x},t)\right) \subset [0,\tau_0]^n, \tag{4.2}$$

$$h|_{\mathbb{R}^n \setminus (\tau,\tau)^n} = 0 \text{ and } h(u_1,\ldots,u_k,\ldots,u_n) = h(u_1,\ldots,-u_k,\ldots,u_n) \tag{4.3}$$

holds for all $1 \leq k \leq n$ where $\boldsymbol{u} = (u_1,\ldots,u_n) \in \mathbb{R}^n$. In particular when no dimension reduction occurs one can take $n = 2d+2$ and let p be the identity map, whereas in the case where the ambit field in question is given by (3.2) one could take $n = d+1$ and $p(\boldsymbol{x},t,\boldsymbol{\xi},s) = (\boldsymbol{x}-\boldsymbol{\xi},t-s)$.

We summarize our hypothesis on general ambit fields (3.1) as follows.

Assumption 4. *Let $Y(\boldsymbol{x},t)$ be a given ambit field on a bounded domain $\mathcal{D}$. We assume that the kernel function g has the form* (4.1) *where* (4.2) *and* (4.3) *hold. It moreover holds that the Lévy seed, L', in question has bounded first and second moments*

$$\mathbb{E}[L'(\boldsymbol{\xi},s)] \vee \operatorname{Var}(L'(\boldsymbol{\xi},s)) < K$$

for all $(\boldsymbol{\xi}, s) \in \cup_{(\boldsymbol{x},t)\in\mathcal{D}} A(\boldsymbol{x}, t)$ *where* $K > 0$ *is a constant and* (3.5) *holds.*

Now for a given $\boldsymbol{\lambda} \in \mathbb{R}^n$, under Assumption 4, such that $\boldsymbol{\lambda} > \mathbf{0}$ assume that

$$h_{\boldsymbol{\lambda}}(\boldsymbol{u}) := h(\boldsymbol{u})\mathrm{e}^{\boldsymbol{\lambda}\cdot(|u_1|,\ldots,|u_n|)} \in L^1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mathrm{Leb}),$$

where for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + \cdots + u_n v_n$ denotes the dot product in $\mathbb{R}^n$. Then by construction $h_{\boldsymbol{\lambda}}$ is also symmetric around 0 in each coordinate, i.e.

$$h_{\boldsymbol{\lambda}}(u_1, \ldots, u_k, \ldots, u_n) = h_{\boldsymbol{\lambda}}(u_1, \ldots, -u_k, \ldots, u_n)$$

holds for all $1 \leq k \leq n$ where $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$. Assume that

$$h_{\boldsymbol{\lambda}} \in L^1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mathrm{Leb}). \tag{4.4}$$

Then the Fourier transform of $h_{\boldsymbol{\lambda}}$ is well defined (see Folland [15]) and is given by

$$\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}) = \int_{\mathbb{R}^n} h_{\boldsymbol{\lambda}}(\boldsymbol{u})\mathrm{e}^{-\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{v}} d\boldsymbol{u}.$$

If we furthermore suppose that $\widehat{h}_{\boldsymbol{\lambda}} \in L^1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mathrm{Leb})$ then the inverse Fourier transform exists and we have that

$$h(\boldsymbol{u}) = \frac{\mathrm{e}^{-\boldsymbol{\lambda}\cdot(|u_1|,\ldots,|u_n|)}}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v})\mathrm{e}^{\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{v}} d\boldsymbol{v}.$$

Thus, under our conditions, we have obtained a representation of h as an integral in the Fourier transform domain.

Given the above integral representation of h let us consider approximating it. To that end we employ the integral approximation

$$h(\boldsymbol{u}) \approx \frac{\mathrm{e}^{-\boldsymbol{\lambda}\cdot(|u_1|,\ldots,|u_n|)}}{(2\pi)^n} \sum_{\alpha\in\mathcal{I}} \widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)\mathrm{e}^{\mathrm{i}\boldsymbol{v}_\alpha\cdot\boldsymbol{u}} \Delta v_1 \cdots \Delta v_n, \tag{4.5}$$

where $\{\boldsymbol{v}_\alpha\}_{\alpha\in\mathbb{Z}^n}$ is a grid in $\mathbb{R}^n$ which is equidistant in each coordinate with positive step sizes $\Delta v_1, \ldots, \Delta v_n$ and $\mathcal{I}$ is a finite subset in $\mathbb{Z}^n$. In our ambit field setting we aim to employ approximations of the type (4.5) to approximate ambit field kernel functions g in (3.1) by means of Lemma 3.1 and Assumption 4. By introducing $h_{\boldsymbol{\lambda}}$ notice that we obtain an approximation which is an exponential function times a finite sum over exponential functions. Thus enabling us to exploit a nice property of the exponential function, which is that $\mathrm{e}^{z+w} = \mathrm{e}^z\mathrm{e}^w$ holds for all $z, w \in \mathbb{C}$. In particular for $\boldsymbol{u} \in [0, \tau_0]^n$ the above approximation (4.5) implies that

$$h(\boldsymbol{u}) \approx \frac{1}{(2\pi)^n} \sum_{\alpha\in\mathcal{I}} \widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v}_\alpha)\cdot\boldsymbol{u}} \Delta v_1 \cdots \Delta v_n. \tag{4.6}$$

Now let us identify the parameters which minimize the distance of the approximation (4.6) for a given finite parameter set $\mathcal{I} \subset \mathbb{Z}^n$. To that end we need to define exactly what it is we want to minimize. Suppose that we denote the right hand side of (4.6) with $\widetilde{h}(\boldsymbol{u})$ and let

$$Y(\boldsymbol{x},t) = \int_{A(\boldsymbol{x},t)} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma_s(\boldsymbol{\xi})L(d\boldsymbol{\xi},ds) \quad \text{and} \quad \widetilde{Y}(\boldsymbol{x},t) = \int_{A(\boldsymbol{x},t)} \widetilde{g}(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma_s(\boldsymbol{\xi})L(d\boldsymbol{\xi},ds)$$

denote ambit fields driven by the different kernel functions $g = h \circ p$ and $\widetilde{g} := \widetilde{h} \circ p$ with the same Lévy bases, volatility fields and ambit sets. Then, it follows by Lemma 3.1 and the Cauchy-Schwarz inequality that

$$\begin{aligned}
\mathbb{E}\left[\left|Y(\boldsymbol{x},t) - \widetilde{Y}(\boldsymbol{x},t)\right|^2\right] &\leq 2\kappa_2(2||g(\boldsymbol{x},t;\cdot,\cdot) - \widetilde{g}(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^1} + ||g(\boldsymbol{x},t;\cdot,\cdot) - \widetilde{g}(\boldsymbol{x},t;\cdot,\cdot)||^2_{\mathcal{L}^2}) \\
&\leq 2\kappa_2(2K^2 c(A(\boldsymbol{x},t)) + K) \int_{A(\boldsymbol{x},t)} |g(\boldsymbol{x},t;\boldsymbol{\xi},s) - \widetilde{g}(\boldsymbol{x},t;\boldsymbol{\xi},s)|^2 c(d\boldsymbol{\xi},ds). \qquad (4.7)
\end{aligned}$$

So, for a given finite subset $\mathcal{I} \in \mathbb{Z}^n$, our task at this stage is to attempt to identify the parameters $\{\boldsymbol{v}_\alpha\}_{\alpha\in\mathcal{I}}$ and $\Delta v_1,\ldots,\Delta v_n$ that minimize the integral

$$Z(\boldsymbol{x},t) := \int_{A(\boldsymbol{x},t)} |g(\boldsymbol{x},t;\boldsymbol{\xi},s) - \widetilde{g}(\boldsymbol{x},t;\boldsymbol{\xi},s)|^2 c(d\boldsymbol{\xi},ds), \qquad (4.8)$$

for all $(\boldsymbol{x},t) \in \mathcal{D}$. To that end consider the following.

Let μ denote a Borel measure that is absolutely continuous with respect to the Lebesgue measure, with a Radon-Nikodym derivative f_μ which is symmetric around 0 in each coordinate. That is, we assume that

$$f_\mu(u_1,\ldots,u_k,\ldots,u_n) = f_\mu(u_1,\ldots,-u_k,\ldots,u_n)$$

holds for all $1 \leq k \leq n$ where $\boldsymbol{u} = (u_1,\ldots,u_n) \in \mathbb{R}^n$. It is essential to our approach that we restrict ourselves to finite sums over orthogonal families. By which we mean that the family $\{e^{i\boldsymbol{v}_\alpha\cdot\boldsymbol{u}}\}_{\alpha\in\mathcal{I}}$ is such that

$$\int_{[-\tau,\tau]^n} e^{i\boldsymbol{v}_\alpha\cdot\boldsymbol{u}}\overline{e^{i\boldsymbol{v}_\beta\cdot\boldsymbol{u}}}\mu(d\boldsymbol{u}) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 2^n\mu([0,\tau]^n) & \text{if } \alpha = \beta \end{cases} \qquad (4.9)$$

for all $\alpha,\beta \in \mathcal{I}$. Given $\tau > 0$ the orthogonality condition is satisfied if

$$\boldsymbol{v}_\alpha = \left(\frac{\alpha_1\pi}{\tau},\ldots,\frac{\alpha_n\pi}{\tau}\right) \text{ for all } \alpha \in \mathcal{I}. \qquad (4.10)$$

Under the orthogonality condition (4.9) consider choosing parameters $\{\boldsymbol{v}_\alpha\}_{\alpha\in\mathcal{I}}$ to minimize the integral

$$\int_{[-\tau,\tau]^n} \left(h_{\boldsymbol{\lambda}}(\boldsymbol{u}) - \sum_{\alpha\in\mathcal{I}} c_\alpha e^{i\boldsymbol{v}_\alpha\cdot\boldsymbol{u}}\right)^2 \mu(d\boldsymbol{u})$$

in the least squares sense. For a given $\beta \in \mathcal{I}$ differentiation with respect to c_β together with the orthogonality relation (4.9) implies that taking

$$c_\alpha = \frac{\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)}{2^n\mu([0,\tau]^n)}, \qquad (4.11)$$

where $\boldsymbol{v}_\alpha$ is given by (4.10) for all $\alpha \in \mathcal{I}$ is optimal in the least squares sense. Together with (4.5) this suggests that taking

$$\Delta v_k = \frac{\pi}{\mu([0,\tau]^n)^{1/n}} \text{ for } k = 1,\dots,n, \tag{4.12}$$

and letting $\{\boldsymbol{v}_\alpha\}_{\alpha \in Z^n}$ be given by (4.10) yields an optimal approximation in the mean square sense. So for a given finite set $\mathcal{I} \subset \mathbb{Z}^n$ and $\boldsymbol{u} \in [0,\tau_0]^n$ the mean square optimal approximation (4.6) is

$$h(\boldsymbol{u}) \approx \sum_{\alpha \in \mathcal{I}} c_\alpha \mathrm{e}^{(-\boldsymbol{\lambda} + \mathrm{i}\boldsymbol{v}_\alpha)\cdot \boldsymbol{u}}, \tag{4.13}$$

where the c_α coefficients are given by (4.11) and $\{\boldsymbol{v}_\alpha\}_{\alpha \in \mathcal{I}}$ is given by (4.10).

Now before applying the above observations in the ambit field setting, let us make a few additional observations. For a given Borel measure μ, consider the Hilbert space

$$\mathcal{H}(\mu) := L^2([-\tau,\tau]^n, \mathcal{B}([-\tau,\tau]^n), \mu). \tag{4.14}$$

By amending Theorem 8.20 from Folland [15] (and its proof) it follows that the family

$$\{e_\alpha(\boldsymbol{u})\}_{\alpha \in \mathbb{Z}^n} := \left\{ \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{v}_\alpha \cdot \boldsymbol{u}}}{2^n \mu([0,\tau]^n)} \right\}_{\alpha \in \mathbb{Z}^n} \tag{4.15}$$

where $\{\boldsymbol{v}_\alpha\}_{\alpha \in \mathbb{Z}^n}$ is given by (4.10) constitutes an orthonormal basis for $\mathcal{H}(\mu)$. Given a finite subset $\mathcal{I} \subset \mathbb{Z}^n$, denote by

$$\mathcal{M} = \mathrm{sp}\{e_\alpha(\boldsymbol{u}) : \alpha \in \mathcal{I}\} \tag{4.16}$$

the closed linear subspace of $\mathcal{H}(\mu)$ which contains all linear combinations of the finite family $\{e_\alpha(\boldsymbol{u})\}_{\alpha \in \mathcal{I}}$, where each e_α is given by (4.15). Then it holds that

$$\mathcal{H}(\mu) = \mathcal{M} \oplus \mathcal{M}^\perp,$$

i.e. that for given $f \in \mathcal{H}(\mu)$, f can be expressed uniquely as $f = \phi + \psi$, where $\phi \in \mathcal{M}$ and $\psi \in \mathcal{M}^\perp$. Moreover it holds that ϕ and ψ are the unique elements of $\mathcal{M}$ and $\mathcal{M}^\perp$ whose distance to h is minimal (see Folland [15], Theorem 5.24). For our purposes this implies that for a given $h_{\boldsymbol{\lambda}} \in \mathcal{H}$, the linear combination

$$\phi_{\boldsymbol{\lambda}}(\boldsymbol{u}) = \sum_{\alpha \in \mathcal{I}} \widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha) e_\alpha(\boldsymbol{u}), \tag{4.17}$$

where $\{e_\alpha(\boldsymbol{u})\}_{\alpha \in \mathcal{I}}$ is given by (4.15), is the unique element of $\mathcal{M}$ which best approximates $h_{\boldsymbol{\lambda}}$. In fact $\phi_{\boldsymbol{\lambda}}$ is a partial sum of the Fourier series which converges to $h_{\boldsymbol{\lambda}}$ in $\mathcal{H}$. Let us summarize our observations in the following Proposition.

Proposition 4.1. *Given a function $h_{\boldsymbol{\lambda}} \in \mathcal{H}(\mu)$, the function (4.17) is the unique element of $\mathcal{M}$ which best approximates $h_{\boldsymbol{\lambda}}$ in $\mathcal{H}(\mu)$. If moreover $\{\mathcal{I}_k\}_{k=1}^\infty$ is an increasing sequence of subsets of $\mathbb{Z}^n$ in the sense that $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots$ and $\cup_{k=1}^\infty \mathcal{I}_k = \mathbb{Z}^n$ and $\{\mathcal{M}_k\}_{k=1}^\infty$ is the corresponding increasing sequence of linear subspaces of $\mathcal{H}(\mu)$ defined by (4.16) then it holds that*

$$\lim_{k\to\infty} ||\phi_{\boldsymbol{\lambda}}^k - h_{\boldsymbol{\lambda}}||_{\mathcal{H}(\mu)} = 0,$$

where $\phi_{\boldsymbol{\lambda}}^k$ *is given by* (4.17) *with* $\mathcal{I} = \mathcal{I}_k$ *for all* $k \geq 1$.

Now let us employ these observations in the ambit field setting. Consider the control measure c (2.2) associated to the Lévy basis L. We shall henceforth work under the following assumption.

Assumption 5. *There exists a Borel measure* $\mu : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}_+$ *which is absolutely continuous with respect to the Lebesgue measure, with a Radon-Nikodym derivative* f_μ *which is symmetric around* 0 *in each coordinate, i.e. such that*

$$f_\mu(u_1, \dots, u_k, \dots, u_n) = f_\mu(u_1, \dots, -u_k, \dots, u_n)$$

holds for all $1 \leq k \leq n$.

The above Assumption is really an assumption on the structure of the control measure c. For example in the case when the ambit field in question is given by (3.2), Assumption 5 is fulfilled if $c << \text{Leb}$ with a Radon-Nikodym derivative which is symmetric in each coordinate, by taking $\mu = c$. While in the case when $n > d+1$, Assumption 5 is fulfilled if $c << \text{Leb}$ with a Radon-Nikodym derivative which is symmetric in each coordinate by taking $\mu = c \times \text{Leb}$, or $\mu = c \times \nu$ where ν is such that $c \times \nu$ fulfills Assumption 5.

Now let us reconsider the integral (4.8). Suppose that

$$\widetilde{g}(\boldsymbol{x}, t; \boldsymbol{\xi}, s) = \mathrm{e}^{-\boldsymbol{\lambda} \cdot p(\boldsymbol{x}, t; \boldsymbol{\xi}, s)} \phi_{\boldsymbol{\lambda}}(p(\boldsymbol{x}, t; \boldsymbol{\xi}, s)), \tag{4.18}$$

where $\phi_{\boldsymbol{\lambda}}$ is given by (4.17). Then for a given $(\boldsymbol{x}, t) \in \mathcal{D}$ it follows that

$$Z(\boldsymbol{x}, t) \leq \int_{[-\tau,\tau]^{d+1}} \mathrm{e}^{-2\boldsymbol{\lambda} \cdot p(\boldsymbol{x}, t; \boldsymbol{\xi}, s)} |h_{\boldsymbol{\lambda}}(\boldsymbol{x}, t; \boldsymbol{\xi}, s) - \phi_{\boldsymbol{\lambda}}(\boldsymbol{x}, t; \boldsymbol{\xi}, s)|^2 c(d\boldsymbol{\xi}, ds),$$

and moreover for the measure μ introduced in Assumption 5 it follows by the elementary inequality $|x+y|^2 \leq 2(|x|^2 + |y|^2)$, Proposition 4.1 and the inequality $|\sum_\alpha f_\alpha|^2 \leq 2(\sum_\alpha |f_\alpha|)^2$, where f is a complex valued function that

$$\begin{aligned}
&||\mathrm{e}^{-\boldsymbol{\lambda} \cdot p}(h_{\boldsymbol{\lambda}} - \phi_{\boldsymbol{\lambda}})||^2_{\mathcal{H}(\mu)} \\
&\leq 2 \left(||\mathrm{e}^{-\boldsymbol{\lambda} \cdot p}(h_{\boldsymbol{\lambda}} - \sum_{\alpha \in \mathbb{Z}^n} \widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha) e_\alpha)||^2_{\mathcal{H}(\mu)} + ||\mathrm{e}^{-\boldsymbol{\lambda} \cdot p} \sum_{\alpha \in \mathbb{Z}^n \setminus \mathcal{I}} \widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha) e_\alpha||^2_{\mathcal{H}(\mu)} \right) \\
&\leq \frac{||\mathrm{e}^{-\boldsymbol{\lambda} \cdot p}||^2_{\mathcal{H}(\mu)}}{2^{n-2}|\mu([0,\tau]^n)|} \left(\sum_{\alpha \in \mathbb{Z}^n \setminus \mathcal{I}} |\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)| \right)^2 .
\end{aligned} \tag{4.19}$$

We summarize the consequences of this in the case of general ambit fields of the type (3.1) as follows.

Proposition 4.2. *Suppose that* $Y(\boldsymbol{x}, t) = \int_{A(\boldsymbol{x}, t)} g(\boldsymbol{x}, t; \boldsymbol{\xi}, s) \sigma(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds)$ *is a general ambit field which fulfills Assumptions 4 and 5. Then if*

$$\widetilde{Y}(\boldsymbol{x}, t) = \int_{A(\boldsymbol{x}, t)} \widetilde{g}(\boldsymbol{x}, t; \boldsymbol{\xi}, s) \sigma(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds),$$

where $\widetilde{g}$ is given by (4.18) *and μ in Assumption 5 is given by $\mu = c \times \nu$, it holds that*

(1)
$$||Y - \widetilde{Y}||^2_{L^2(\mathbb{P}\times\nu)} \leq \frac{\kappa_2(2K^2 \sup_{(\boldsymbol{x},t)\in\mathcal{D}} c(A(\boldsymbol{x},t)) + K)||\mathrm{e}^{-\boldsymbol{\lambda}\cdot p}||^2_{\mathcal{H}(c\times\nu)}}{2^{n-3}|(c\times\nu)([0,\tau]^n)|} \left(\sum_{\alpha\in\mathbb{Z}^n\setminus\mathcal{I}} |\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)| \right)^2,$$
where the integration in $||\cdot||_{L^2(\mathbb{P}\times\nu)}$ is over $\Omega\times\mathcal{D}$.

(2) *If the ambit field has a kernel function on the form* (3.2) *it moreover holds that*
$$\mathbb{E}\left[\left| Y(\boldsymbol{x},t) - \widetilde{Y}(\boldsymbol{x},t) \right|^2 \right] \leq \frac{\kappa_2(2K^2 c(A(\boldsymbol{x},t)) + K)||\mathrm{e}^{-\boldsymbol{\lambda}\cdot p}||^2_{\mathcal{H}(c)}}{2^{n-3}|c([0,\tau]^n)|} \left(\sum_{\alpha\in\mathbb{Z}^n\setminus\mathcal{I}} |\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)| \right)^2.$$

Proof. Follows by the inequalities (4.7) and (4.19). □

Thus at this point it is of interest to study the convergence rate of the series

$$\sum_{\alpha\in\mathbb{Z}^n\setminus\mathcal{I}} |\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)|. \tag{4.20}$$

To that end consider the following. Suppose that h is C^2 in each coordinate. Then if $\partial_j h_{\boldsymbol{\lambda}}(\boldsymbol{u})|_{u_j=\tau} = 0$ for a particular $1 \leq j \leq n$ and $v_{\alpha_j} \neq 0$ we may employ integration by parts to obtain that

$$\begin{aligned} \int_{-\tau}^{\tau} h_{\boldsymbol{\lambda}}(\boldsymbol{u})\mathrm{e}^{-\mathrm{i}u_j v_{\alpha_j}} du_j &= 0 + \frac{\tau}{\mathrm{i}\alpha_j\pi} \int_{-\tau}^{\tau} \partial_j h_{\boldsymbol{\lambda}}(\boldsymbol{u})\mathrm{e}^{-\mathrm{i}u_j v_{\alpha_j}} du_j \\ &= \left(\frac{\tau}{\alpha_j\pi}\right)^2 \left(0 - \int_{-\tau}^{\tau} \partial_j^2 h_{\boldsymbol{\lambda}}(\boldsymbol{u})\mathrm{e}^{-\mathrm{i}u_j v_{\alpha_j}} du_j \right). \end{aligned}$$

This can be employed together with the following representation result to obtain more precise error estimates in particular cases.

Lemma 4.3. *Let $n \geq 1$ and consider a sequence $\{\gamma_\alpha\}_{\alpha\in\mathbb{Z}^n}$ which is such that $\gamma_\alpha \geq 0$ and*
$$\gamma_{(\alpha_1,\ldots,\alpha_k,\ldots,\alpha_n)} = \gamma_{(\alpha_1,\ldots,-\alpha_k,\ldots,\alpha_n)}$$
for all $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n$ and $k = 1,\ldots,n$, assume furthermore that
$$\sum_{\alpha\in\mathbb{Z}^n} \gamma_\alpha < \infty.$$
Then it holds that
$$\sum_{\alpha\in\mathbb{Z}^n} \gamma_\alpha = \gamma_{\boldsymbol{0}} + \sum_{m=1}^{n} 2^m \sum_{j=1}^{\binom{n}{m}} \sum_{\alpha\in\mathbb{N}^m} \gamma_{\rho_j^{n,m}(\alpha)},$$
where $\mathbb{N} = \{1,2,\ldots\}$ denotes the natural numbers and $\rho_j^{n,m} : \mathbb{N}^m \to \mathbb{N}^n$ is the function that embeds $\mathbb{N}^m$ to the coordinates determined by the jth m-combination of $\mathbb{N}^n$ and is 0 in the other coordinates.

Remark 4.4. *That is, for the $j = 1, \dots, \binom{n}{m}$ possible ways of selecting m elements from a set with n elements we associate the functions $\{\rho_j^{n,m}\}_j$ which embed $\mathbb{N}^m$ in the coordinates determined by the jth m-combination with 0 in the other coordinates. Thus for instance if $n = 2$, we can take $\rho_1^{2,1}(\alpha) = (\alpha_1, 0)$, $\rho_2^{2,1}(\alpha) = (0, \alpha_2)$ and $\rho_1^{2,2}(\alpha) = \alpha$, for $\alpha \in \mathbb{N}^2$.*

Example 4.5. *In the case when $n = 2$ we can for a given $N \geq 1$ set*

$$\gamma_{(k_1,k_2)} = \begin{cases} \mathbf{0} & \text{if } |k_1|, |k_2| \leq N \\ |\widehat{h}_{\boldsymbol{\lambda}}\left(\frac{\pi}{\tau}(k_1, k_2)\right)| & \text{otherwise} \end{cases}$$

and employ the above Lemma to conclude that

$$\begin{aligned}
&\sum_{|k_1|>N} \sum_{|k_2|>N} |\widehat{h}_{\boldsymbol{\lambda}}\left(\frac{\pi}{\tau}(k_1, k_2)\right)| \\
&= 2\sum_{k=N}^{\infty} (|\widehat{h}_{\boldsymbol{\lambda}}\left(\frac{\pi}{\tau}(k, 0)\right)| + |\widehat{h}_{\boldsymbol{\lambda}}\left(\frac{\pi}{\tau}(0, k)\right)|) + 4\sum_{k_1=N}^{\infty} \sum_{k_2=N}^{\infty} |\widehat{h}_{\boldsymbol{\lambda}}\left(\frac{\pi}{\tau}(k_1, k_2)\right)|.
\end{aligned}$$

If furthermore it holds that $h_{\boldsymbol{\lambda}}$ is C^{m_1,m_2} where $m_1, m_2 \geq 2$ and that $\partial_k^{l_k} h_{\boldsymbol{\lambda}}(\boldsymbol{u})|_{u_k=\tau} = 0$ for $l_k = 1, \dots, m_k - 1$ and $k = 1, 2$, then

$$\begin{aligned}
&\sum_{k=N}^{\infty} (|\widehat{h}_{\boldsymbol{\lambda}}\left(\frac{\pi}{\tau}(k, 0)\right)| + |\widehat{h}_{\boldsymbol{\lambda}}\left(\frac{\pi}{\tau}(0, k)\right)|) \\
&\leq \sum_{k=N}^{\infty} \left(\left(\frac{\tau}{\pi}\right)^{m_1} \frac{||\partial_1^{m_1} h_{\boldsymbol{\lambda}}||_{L^1([-\tau,\tau]^2)}}{k^{m_1}} + \left(\frac{\tau}{\pi}\right)^{m_2} \frac{||\partial_2^{m_2} h_{\boldsymbol{\lambda}}||_{L^1([-\tau,\tau]^2)}}{k^{m_2}}\right),
\end{aligned}$$

and

$$\sum_{k_1=N}^{\infty} \sum_{k_2=N}^{\infty} |\widehat{h}_{\boldsymbol{\lambda}}\left(\frac{\pi}{\tau}(k_1, k_2)\right)| \leq \sum_{k_1=N}^{\infty} \sum_{k_2=N}^{\infty} \left(\frac{\tau}{\pi}\right)^{m_1+m_2} \frac{||\partial_1^{m_1} \partial_2^{m_2} h_{\boldsymbol{\lambda}}||_{L^1([-\tau,\tau]^2)}}{k_1^{m_1} k_2^{m_2}}.$$

So

$$\begin{aligned}
&\sum_{|k_1|>N} \sum_{|k_2|>N} |\widehat{h}_{\boldsymbol{\lambda}}\left(\frac{\pi}{\tau}(k_1, k_2)\right)| \\
&\leq \left(\left(\frac{\tau}{\pi}\right)^{m_1} ||\partial_1^{m_1} h_{\boldsymbol{\lambda}}||_{L^1([-\tau,\tau]^2)} \sum_{k=N}^{\infty} \frac{1}{k^{m_1}} + \left(\frac{\tau}{\pi}\right)^{m_2} ||\partial_2^{m_2} h_{\boldsymbol{\lambda}}||_{L^1([-\tau,\tau]^2)} \sum_{k=N}^{\infty} \frac{1}{k^{m_2}}\right) \\
&+ \left(\frac{\tau}{\pi}\right)^{m_1+m_2} ||\partial_1^{m_1} \partial_2^{m_2} h_{\boldsymbol{\lambda}}||_{L^1([-\tau,\tau]^2)} \sum_{k=N}^{\infty} \frac{1}{k^{m_1}} \sum_{k=N}^{\infty} \frac{1}{k^{m_2}}.
\end{aligned}$$

4.2. **Incremental approximation.** By employing the grid (4.10) for a given $\mathcal{I}$ with the step sizes (4.12), we have the following integral representation and approximation of the kernel function g in (3.1)

$$g(\boldsymbol{x}, t; \boldsymbol{\xi}, s) = \frac{\mathrm{e}^{-\boldsymbol{\lambda} \cdot p(\boldsymbol{x},t;\boldsymbol{\xi},s)}}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}) \mathrm{e}^{\mathrm{i}\boldsymbol{v} \cdot p(\boldsymbol{x},t;\boldsymbol{\xi},s)} d\boldsymbol{v}$$

$$\approx \frac{1}{2^n\mu([0,\tau]^n)}\sum_{\alpha\in\mathcal{I}}\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v}_\alpha)\cdot p(\boldsymbol{x},t;\boldsymbol{\xi},s)}, \tag{4.21}$$

for $p(\boldsymbol{x},t;\boldsymbol{\xi},s)\in[0,\tau_0]^n$. Thus for a general ambit field of the type (3.1), by (4.21) it holds that

$$\begin{aligned} Y(\boldsymbol{x},t) &= \int_{A(\boldsymbol{x},t)}\frac{\mathrm{e}^{-\boldsymbol{\lambda}\cdot p(\boldsymbol{x},t;\boldsymbol{\xi},s)}}{(2\pi)^n}\left(\int_{\mathbb{R}^n}\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v})\mathrm{e}^{\mathrm{i}\boldsymbol{v}\cdot p(\boldsymbol{x},t;\boldsymbol{\xi},s)}d\boldsymbol{v}\right)\sigma_s(\boldsymbol{\xi})L(d\boldsymbol{\xi},ds) \\ &\approx \frac{1}{2^n\mu([0,\tau]^n]}\sum_{\alpha\in\mathcal{I}}\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x},t,\boldsymbol{v}_\alpha), \end{aligned} \tag{4.22}$$

where

$$\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x},t,\boldsymbol{v}) = \int_{A(\boldsymbol{x},t)}\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x},t;\boldsymbol{\xi},s)}\sigma_s(\boldsymbol{\xi})L(d\boldsymbol{\xi},ds). \tag{4.23}$$

The field (4.23) has an important incremental property.

Proposition 4.6. *Let $\Delta\boldsymbol{x}\geq\boldsymbol{0}$ and $\Delta t\geq 0$ denote increments in space and time respectively, then*

$$\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}+\Delta\boldsymbol{x},t+\Delta t,\boldsymbol{v}) = C_{\boldsymbol{\lambda}}(\Delta\boldsymbol{x},\Delta t,\boldsymbol{v})\left(\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x},t,\boldsymbol{v})+\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{x},\Delta\boldsymbol{x},t,\Delta t,\boldsymbol{v})\right)$$

holds where

$$C_{\boldsymbol{\lambda}}(\Delta\boldsymbol{x},\Delta t,\boldsymbol{v}) = \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\Delta\boldsymbol{x},\Delta t;\boldsymbol{0},0)}$$

and

$$\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{x},\Delta\boldsymbol{x},t,\Delta t,\boldsymbol{v}) = \int_{A(\boldsymbol{x}+\Delta\boldsymbol{x},t+\Delta t)\setminus A(\boldsymbol{x},t)}\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x},t;\boldsymbol{\xi},s)}\sigma_s(\boldsymbol{\xi})L(d\boldsymbol{\xi},ds).$$

Proof. This follows by

$$\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}+\Delta\boldsymbol{x},t+\Delta t;\boldsymbol{\xi},s)} = \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\Delta\boldsymbol{x},\Delta t;\boldsymbol{0},0)}\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x},t;\boldsymbol{\xi},s)}$$

and the linearity of the integral. □

Given the above Proposition it is of interest to consider the *increment fields*

$$\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{x},\Delta\boldsymbol{x},t,\Delta t,\boldsymbol{v})$$

on a given domain. Indeed given a space time grid $\{(\boldsymbol{x}_j,t_j)\}_{j=0}^J\subset\mathbb{R}^d\times\mathbb{R}$ let $(\Delta\boldsymbol{x}_j,\Delta t_j) := (\boldsymbol{x}_j-\boldsymbol{x}_{j-1},t_j-t_{j-1})$ where $j=1,\dots,J$ denote the increments in the space time domain, where we assume that $(\Delta\boldsymbol{x}_j,\Delta t_j)\geq(\boldsymbol{0},0)$ for all $j=1,\dots,J$. Suppose that $\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}_0,t_0,\boldsymbol{v})=0$ for all $\boldsymbol{v}$. Then we find by iteration that

$$\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}_J,t_J,\boldsymbol{v}) = \sum_{j=1}^J C_{\boldsymbol{\lambda}}(\boldsymbol{x}_J-\boldsymbol{x}_{J-j},t_J-t_{J-j},\boldsymbol{v})\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{x}_{J-j},\Delta\boldsymbol{x}_{J+1-j},t_{J-j},\Delta t_{J+1-j},\boldsymbol{v}).$$

Thus we have obtained an approximation of a general ambit field as a finite sum of the $\epsilon_{\boldsymbol{\lambda}}$ fields, which are in turn complex ambit fields driven by an exponential function. Consider further approximating the $\epsilon_{\boldsymbol{\lambda}}$ field by

$$\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{x}, \Delta\boldsymbol{x}, t, \Delta t, \boldsymbol{v}) \approx \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x},t;\boldsymbol{x}^*,t^*)}\sigma(\boldsymbol{x}^*, t^*)\Delta L(\boldsymbol{x}, t),$$

where $(\boldsymbol{x}^*, t^*) \in \overline{A(\boldsymbol{x}+\Delta\boldsymbol{x}, t+\Delta t) \setminus A(\boldsymbol{x}, t)}$ and $\Delta L(\boldsymbol{x}, t) = L(A(\boldsymbol{x}+\Delta\boldsymbol{x}, t+\Delta t) \setminus A(\boldsymbol{x}, t))$. In particular in the case when when $p(\boldsymbol{x}, t; \boldsymbol{\xi}, s) = (\boldsymbol{x}-\boldsymbol{\xi}, t-s)$ and $(\boldsymbol{x}, t) \in \overline{A(\boldsymbol{x}+\Delta\boldsymbol{x}, t+\Delta t) \setminus A(\boldsymbol{x}, t)}$ one can take $(\boldsymbol{x}^*, t^*) = (\boldsymbol{x}, t)$ which yields the approximation

$$\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{x}, \Delta\boldsymbol{x}, t, \Delta t, \boldsymbol{v}) \approx \sigma(x, t)\Delta L(\boldsymbol{x}, t),$$

which is independent of $\boldsymbol{v}$! Now given our space time grid and the corresponding $(\boldsymbol{x}_j^*, t_j^*) \in \overline{A(\boldsymbol{x}_{j+1}, t_{j+1}) \setminus A(\boldsymbol{x}_j, t_j)}$, to ease the notation we introduce

$$\psi_j(\boldsymbol{v}) := \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{x}^*_{J-j},t^*_{J-j})},$$

and

$$A_j := A(\boldsymbol{x}_{J-j+1}, t_{J-j+1}) \setminus A(\boldsymbol{x}_{J-j}, t_{J-j})$$

for $j = 1, \ldots, J$. Now let

$$\mathcal{A}_J := \bigcup_{j=1}^{J} A_j = A(\boldsymbol{x}_J, t_J) \setminus A(\boldsymbol{x}_0, t_0) \tag{4.24}$$

and let

$$\widehat{Y}_J(\boldsymbol{v}) := \int_{\mathcal{A}_J} \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{\xi},s)}\sigma_s(\boldsymbol{\xi})L(d\boldsymbol{\xi}, ds) \tag{4.25}$$

denote a truncated approximation of the field $\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}_J, t_J, \boldsymbol{v})$. Now observe that by employing our approximation we get that

$$\begin{aligned}
\widehat{Y}_J(\boldsymbol{v}) &\approx \int_{\mathcal{A}_J} \sum_{j=1}^{J} 1_{A_j}(\boldsymbol{\xi}, s)\psi_j(\boldsymbol{v})\sigma(\boldsymbol{x}^*_{J-j}, t^*_{J-j})L(d\boldsymbol{\xi}, ds) \\
&= \int_{\mathcal{A}_J} \sum_{j=1}^{J} 1_{A_j}(\boldsymbol{\xi}, s)\psi_j(\boldsymbol{v})\sigma(\boldsymbol{\xi}, s)L(d\boldsymbol{\xi}, ds) \\
&\quad + \int_{\mathcal{A}_J} \sum_{j=1}^{J} 1_{A_j}(\boldsymbol{\xi}, s)\psi_j(\boldsymbol{v})(\sigma(\boldsymbol{x}^*_{J-j}, t^*_{J-j}) - \sigma(\boldsymbol{\xi}, s))L(d\boldsymbol{\xi}, ds).
\end{aligned}$$

By Lemma 3.1 it follows that

$$\mathbb{E}\left[\left|\int_{\mathcal{A}_J}\left(\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{\xi},s)} - \sum_{j=1}^{J} 1_{A_j}(\boldsymbol{\xi}, s)\psi_j(\boldsymbol{v})\right)\sigma(\boldsymbol{\xi}, s)L(d\boldsymbol{\xi}, ds)\right|^2\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left|\int_{\mathcal{A}_J}\left(\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{\xi},s)} - \sum_{j=1}^{J} 1_{A_j}(\boldsymbol{\xi},s)\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}v)\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{x}^*_{J-j},t^*_{J-j})}\right)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)\right|^2\right] \\
&\leq 2\kappa_2\Bigg(2||\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}_J,t_J;\cdot,\cdot)} - \sum_{j=1}^{J} 1_{A_j}(\cdot,\cdot)\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}v)\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{x}^*_{J-j},t^*_{J-j})}||^2_{\mathcal{L}^1} \\
&\quad + ||\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}_J,t_J;\cdot,\cdot)} - \sum_{j=1}^{J} 1_{A_j}(\cdot,\cdot)\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}v)\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{x}^*_{J-j},t^*_{J-j})}||^2_{\mathcal{L}^2}\Bigg) \\
&\leq 4\kappa_2\mathbb{E}[L^2(\mathcal{A}_J)]\max_{1\leq j\leq J}|\mathrm{e}^{(-\boldsymbol{\lambda}+i\boldsymbol{v})\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{x}_{j-1},t_{j-1})} - \mathrm{e}^{(-\boldsymbol{\lambda}+i\boldsymbol{v})\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{x}_j,t_j)}|^2.
\end{aligned}
$$

At this point we apply the following result.

Lemma 4.7. *Suppose that $n \geq 1$ and consider the function $f : \mathbb{R}^n \to \mathbb{C}$, $\boldsymbol{x} \to \mathrm{e}^{\boldsymbol{z}\cdot\boldsymbol{x}}$ where $\boldsymbol{z} \in \mathbb{C}^n$ is such that $\operatorname{Re}\boldsymbol{z} < \boldsymbol{0}$. Then*

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})|^2 \leq 4||\boldsymbol{z}||^2||\boldsymbol{x}-\boldsymbol{y}||^2$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, where $||\boldsymbol{w}||^2 := \sum_{k=1}^n |w_k|^2$ denotes the Euclidean norm for all $\boldsymbol{w} \in \mathbb{C}^n$.

Thus we may conclude by the above Lemma and the preceding calculations that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_{\mathcal{A}_J}\left(\mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}_J,t_J;\boldsymbol{\xi},s)} - \sum_{j=1}^{J} 1_{A_j}(\boldsymbol{\xi},s)\psi_j(\boldsymbol{v})\right)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)\right|^2\right] \\
&\leq 16\kappa_2\mathbb{E}[L^2(\mathcal{A}_J)]||-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v}||^2\max_{1\leq j\leq J}||(\Delta\boldsymbol{x}_j,\Delta t_j)||^2.
\end{aligned}
$$

Furthermore by Lemma 3.1 it holds that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_{\mathcal{A}_J}\sum_{j=1}^{J} 1_{A_j}(\boldsymbol{\xi},s)\psi_j(\boldsymbol{v})(\sigma(\boldsymbol{x}^*_{J-j},t^*_{J-j}) - \sigma(\boldsymbol{\xi},s))L(d\boldsymbol{\xi},ds)\right|^2\right] \\
&\leq 4\mathbb{E}[L^2(\mathcal{A}_J)]\max_{1\leq j\leq J}\mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1},t_{j-1}) - \sigma(\boldsymbol{x}_j,t_j)|^2\right].
\end{aligned}
$$

Thus by employing the notation

$$\eta_J(\boldsymbol{v}) := \int_{\mathcal{A}_J}\sum_{j=1}^{J} 1_{A_j}(\boldsymbol{\xi},s)\psi_j(\boldsymbol{v})\sigma(\boldsymbol{x}^*_{J-j},t^*_{J-j})L(d\boldsymbol{\xi},ds), \tag{4.26}$$

we have the following results.

Lemma 4.8. *It holds that*

$$\mathbb{E}[|\widehat{Y}_J(\boldsymbol{v}) - \eta_J(\boldsymbol{v})|^2] \leq 4\mathbb{E}[L^2(\mathcal{A}_J)]\Bigg(8\kappa_2||-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v}||^2\max_{1\leq j\leq J}||(\Delta\boldsymbol{x}_j,\Delta t_j)||^2$$

$$+2 \max_{1\le j\le J} \mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1},t_{j-1})-\sigma(\boldsymbol{x}_j,t_j)|^2\right]\Bigg),$$

where $\widehat{Y}_J$ is given by (4.25) and η is given by (4.26).

Proposition 4.9. *Let $\{(\boldsymbol{x}_j,t_j)\}_{j=0}^J \subset \mathbb{R}^d\times\mathbb{R}$ be a space time grid and denote by $(\Delta\boldsymbol{x}_j,\Delta t_j) := (\boldsymbol{x}_j-\boldsymbol{x}_{j-1},t_j-t_{j-1})$ where $j=1,\dots,J$ the increments in the space time domain, where $(\Delta\boldsymbol{x}_j,\Delta t_j)\ge(\boldsymbol{0},0)$ holds for all $j=1,\dots,J$. For $\widetilde{g}_{\mathcal{I}}$ given by (4.18) for a given $\mathcal{I}\subset\mathbb{Z}^n$ and $\mathcal{A}_J$ given by (4.24), it holds that*

$$\mathbb{E}\left[\left|\int_{\mathcal{A}_J}\widetilde{g}_{\mathcal{I}}(\boldsymbol{x}_J,t_J;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)-\sum_{\alpha\in\mathcal{I}}c_\alpha\eta_J(\boldsymbol{v}_\alpha)\right|^2\right]$$
$$\le 24\mathbb{E}[L^2(\mathcal{A}_J)]\left\{4\kappa_2\left(||\boldsymbol{\lambda}||^2\left(\sum_{\alpha\in\mathcal{I}}|c_\alpha|\right)^2+\left(\frac{\pi}{\tau}\right)^2\left(\sum_{\alpha\in\mathcal{I}}|c_\alpha|||\alpha||\right)^2\right)\max_{1\le j\le J}||(\Delta\boldsymbol{x}_j,\Delta t_j)||^2\right.$$
$$\left.\left(\sum_{\alpha\in\mathcal{I}}|c_\alpha|\right)^2\max_{1\le j\le J}\mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1},t_{j-1})-\sigma(\boldsymbol{x}_j,t_j)|^2\right]\right\},$$

where the c_α coefficients are given by (4.11) and $||\cdot||$ is the Euclidian norm on $\mathbb{C}^n$.

Proof. By Minkowski's inequality and Lemma 4.8 it follows that

$$\left(\mathbb{E}\left[\left|\int_{\mathcal{A}_J}\widetilde{g}_{\mathcal{I}}(\boldsymbol{x}_J,t_J;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)-\sum_{\alpha\in\mathcal{I}}c_\alpha\eta_J(\boldsymbol{v}_\alpha)\right|^2\right]\right)^{1/2}$$
$$=\left(\mathbb{E}\left[\left|\sum_{\alpha\in\mathcal{I}}c_\alpha\left(\widehat{Y}_J(\boldsymbol{v}_\alpha)-\eta_J(\boldsymbol{v}_\alpha)\right)\right|^2\right]\right)^{1/2}$$
$$\le\sum_{\alpha\in\mathcal{I}}|c_\alpha|\left(\mathbb{E}\left[\left|\widehat{Y}_J(\boldsymbol{v}_\alpha)-\eta_J(\boldsymbol{v}_\alpha)\right|^2\right]\right)^{1/2}$$
$$\le 2\left(\mathbb{E}[L^2(\mathcal{A}_J)]\right)^{1/2}\sum_{\alpha\in\mathcal{I}}|c_\alpha|\Bigg(8\kappa_2||-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v}_\alpha||^2\max_{1\le j\le J}||(\Delta\boldsymbol{x}_j,\Delta t_j)||^2$$
$$+2\max_{1\le j\le J}\mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1},t_{j-1})-\sigma(\boldsymbol{x}_j,t_j)|^2\right]\Bigg)^{1/2}.$$

Furthermore using that $\sqrt{x^2+y^2}\le x+y$ holds for non-negative $x,y\ge 0$, and the triangle inequality it follows that

$$\left(8\kappa_2||-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v}_\alpha||^2\max_{1\le j\le J}||(\Delta\boldsymbol{x}_j,\Delta t_j)||^2+2\max_{1\le j\le J}\mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1},t_{j-1})-\sigma(\boldsymbol{x}_j,t_j)|^2\right]\right)^{1/2}$$

$$\leq 2||-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v}_\alpha||\left(2\kappa_2\max_{1\leq j\leq J}||(\Delta\boldsymbol{x}_j,\Delta t_j)||^2\right)^{1/2}+\sqrt{2}\left(\max_{1\leq j\leq J}\mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1},t_{j-1})-\sigma(\boldsymbol{x}_j,t_j)|^2\right]\right)^{1/2}$$

$$\leq 2(||\boldsymbol{\lambda}||+||\boldsymbol{v}_\alpha||)\left(2\kappa_2\max_{1\leq j\leq J}||(\Delta\boldsymbol{x}_j,\Delta t_j)||^2\right)^{1/2}+\sqrt{2}\left(\max_{1\leq j\leq J}\mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1},t_{j-1})-\sigma(\boldsymbol{x}_j,t_j)|^2\right]\right)^{1/2}.$$

From which by applying the elementary inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ that

$$\mathbb{E}\left[\left|\int_{\mathcal{A}_J}\widetilde{g}_{\mathcal{I}}(\boldsymbol{x}_J,t_J;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)-\sum_{\alpha\in\mathcal{I}}c_\alpha\eta_J(\boldsymbol{v}_\alpha)\right|^2\right]$$

$$\leq 4\mathbb{E}[L^2(\mathcal{A}_J)]\left\{2\sum_{\alpha\in\mathcal{I}}|c_\alpha|(||\boldsymbol{\lambda}||+||\boldsymbol{v}_\alpha||)\left(2\kappa_2\max_{1\leq j\leq J}||(\Delta\boldsymbol{x}_j,\Delta t_j)||^2\right)^{1/2}\right.$$

$$\left.+\sqrt{2}\sum_{\alpha\in\mathcal{I}}|c_\alpha|\left(\max_{1\leq j\leq J}\mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1},t_{j-1})-\sigma(\boldsymbol{x}_j,t_j)|^2\right]\right)^{1/2}\right\}^2$$

$$\leq 24\mathbb{E}[L^2(\mathcal{A}_J)]\left\{4\kappa_2\left(||\boldsymbol{\lambda}||^2\left(\sum_{\alpha\in\mathcal{I}}|c_\alpha|\right)^2+\left(\sum_{\alpha\in\mathcal{I}}|c_\alpha|||\boldsymbol{v}_\alpha||\right)^2\right)\max_{1\leq j\leq J}||(\Delta\boldsymbol{x}_j,\Delta t_j)||^2\right.$$

$$\left.+\left(\sum_{\alpha\in\mathcal{I}}|c_\alpha|\right)^2\max_{1\leq j\leq J}\mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1},t_{j-1})-\sigma(\boldsymbol{x}_j,t_j)|^2\right]\right\}.$$

□

5. Application to forward pricing

In the current section we shall demonstrate the usefulness of the approximation method described in this paper by means of constructing an example.

To that end consider the problem of simulating the ambit field

$$Y(x,t)=\int_{-\infty}^{t}\int_{0}^{\infty}g(t-s+x)\varphi(\xi)\sigma_s(\xi)L(d\xi,ds). \tag{5.1}$$

In the setting of a general ambit field (3.1) this translates into $d=1$ with an ambit set on the form

$$A(x,t)=\{(\xi,s)\in\mathbb{R}^2:\xi\geq 0,s\leq t\},$$

for $t\in\mathbb{R}$, $x\geq 0$. We remark that this particular choice of ambit set is the same Barndorff-Nielsen et al. [2] make for their electricity forward modelling framework. Moreover the above specification of the kernel function is motivated by their paper, since they specify two possible factorisations of kernel functions

$$g(x,t;\xi,s)=\psi(t-s)\varphi(\xi,x)\quad\text{and}\quad g(x,t;\xi,s)=\psi(t-s,x)\varphi(\xi),$$

for suitable functions ψ and φ, and analyse ambit fields having the respective factorisations.

Suppose that we are interested in simulating a field $\{Y(x,t)\}_{(x,t)\in\mathcal{D}}$ where $\mathcal{D}\subset\mathbb{R}^2$ is a bounded domain. We assume that we may discretise the domain $\mathcal{D}$ to the finite set $\{(x_j,t_k)\}_{j=0,k=0}^{J,K}$ where $0=x_0<x_1<\cdots<x_J$ and $t_0<t_1<\cdots<t_K$ for some constants $J,K\geq 1$. If $0<\tau_0<\tau$ are constants such that $t_K+x_J\leq\tau_0$ and h defined by (4.3), then for Y given by (5.1) and a given $\lambda>0$ and $N\geq 1$ we employ the approximation (4.22) with $\mathcal{I}=\{n\in\mathbb{Z}:|n|\leq N\}$ and employ that $\widehat{h}_\lambda$ is symmetric around 0 to write

$$Y(x,t)\approx\frac{c_0}{2}\widehat{Y}_\lambda(x,t,0)+\operatorname{Re}\sum_{n=1}^{N}c_n\widehat{Y}_\lambda(x,t,n\pi/\tau),$$

where $c_n=\widehat{h}_\lambda(n\pi/\tau)/\tau$ for $n=0,\dots,N$,

$$\widehat{Y}_\lambda(x,t,v)=\int_{-\infty}^{t}\int_0^\infty \mathrm{e}^{(-\lambda+\mathrm{i}v)(t-s+x)}\varphi(\xi)\sigma_s(\xi)L(d\xi,ds), \tag{5.2}$$

and $(x,t)=(x_j,t_k)$ for some $0\leq j\leq J$ and $0\leq k\leq K$. Thus we obtain the discrete field $\{Y(x_j,t_k)\}_{j=0,k=0}^{J,K}$ be means of a finite linear combination of the complex ambit fields (5.2) which can easily be simulated by means of an iterative algorithm based on Proposition 4.6. In the setting of Proposition 4.6 we regard $L_\varphi:=\int_\cdot\varphi(\xi)L(d\xi,ds)$ as the driving Lévy basis, and furthermore assume that the discretised domain is equidistant in time and space with step sizes $\Delta x=x_j-x_{j-1}>0$ and $\Delta t=t_k-t_{k-1}>0$ for $j=1,\dots,J$ and $k=1,\dots,J$. Thus obtaining the incremental identities

$$\widehat{Y}_\lambda(x_j,t_k,v)=\mathrm{e}^{(-\lambda+\mathrm{i}v)\Delta t}\left(\widehat{Y}_\lambda(x_j,t_{k-1},v)+\int_{t_{k-1}}^{t_k}\int_0^\infty \mathrm{e}^{(-\lambda+\mathrm{i}v)(t_{k-1}-s+x_j)}\sigma_s(\xi)L_\varphi(d\xi,ds)\right)$$

for all $j=0,\dots,J$ and $k=1,\dots,K$, and

$$\widehat{Y}_\lambda(x_j,t_k,v)=\mathrm{e}^{(-\lambda+\mathrm{i}v)\Delta x}\widehat{Y}_\lambda(x_{j-1},t_k,v) \tag{5.3}$$

for all $j=1,\dots,J$ and $k=0,\dots,K$. Using these identities and the approximation

$$\int_{t_{k-1}}^{t_k}\int_0^\infty \mathrm{e}^{(-\lambda+\mathrm{i}v)(t_{k-1}-s+x)}\sigma_s(\xi)L_\varphi(d\xi,ds)\approx\int_0^\infty \mathrm{e}^{(-\lambda+\mathrm{i}v)x_j}\sigma_{t_{k-1}}(\xi)L_\varphi(d\xi,t_k-t_{k-1})$$

for all $k=1,\dots,K$ we may for given $v_0,\dots,v_N$ obtain the fields $\{\widehat{Y}_\lambda(x_j,t_k,v_n)\}_{j=0,k=0,n=0}^{J,K,N}$ by means of the approximation recursion relation

$$\widehat{Y}_\lambda(0,t_k,v_n)\approx\mathrm{e}^{(-\lambda+\mathrm{i}v_n)\Delta t}\left(\widehat{Y}_\lambda(0,t_{k-1},v_n)+\int_0^\infty\sigma_{t_{k-1}}(\xi)L_\varphi(d\xi,t_k-t_{k-1})\right),$$

which holds for all $k=1,\dots,K$ and $n=0,\dots,N$ and employ (5.3) to obtain the increments in the space direction.

Now let us consider a specific example, suppose that

$$g(u)=u^{\nu-1}\mathrm{e}^{-\alpha u},\ \varphi(u)=\mathrm{e}^{-\beta u},\ \text{and }\sigma=1 \tag{5.4}$$

for constants $1/2<\nu<1$ and $\alpha,\beta>0$ and L is a Gaussian Lévy basis such that $L(A)\sim N(0,\mathrm{Leb}(A))$ for all $A\in\mathcal{B}(\mathbb{R}^2)$. Here the choice of the gamma kernel g in

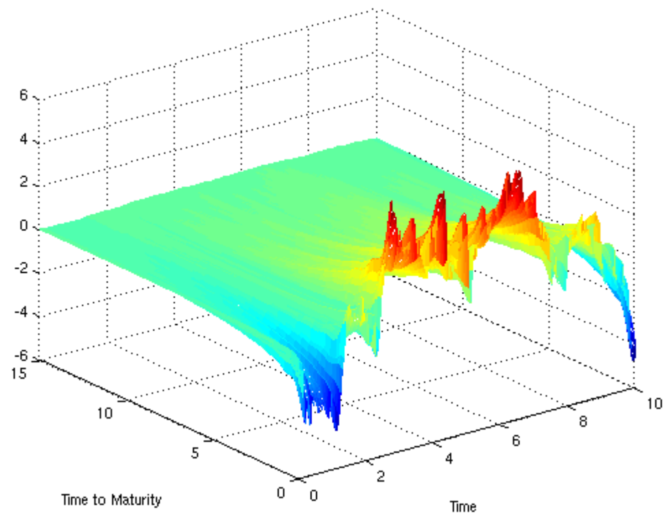

FIGURE 1. A simulated ambit field (5.1) with integrals truncated at $t = 0$ and $x = 15$, where (5.4) holds with $\nu = 0.75$, $\alpha = 0.2$ and $\beta = 0.1$, with a Gaussian ambit field $L(A) \sim N(0, \text{Leb}(A))$, for all $A \in \mathcal{B}(\mathbb{R}^2)$. Here $\lambda = 0.3$ and $N = 50$, with step sizes $\Delta t = \Delta x = 0.05$.

(5.4) is intended to demonstrate that our method can handle ill behaved kernel functions having a singularity at the origin. In Figure 1 we have simulated the ambit field (5.1) with integrals truncated at $t = 0$ and $x = 15$ by means of the method described in this section, where (5.4) holds with $\nu = 0.75$, $\alpha = 0.2$ and $\beta = 0.1$, with a Gaussian ambit field $L(A) \sim N(0, \text{Leb}(A))$, for all $A \in \mathcal{B}(\mathbb{R}^2)$. Here we have chosen $\lambda = 0.3$ and $N = 50$, and step sizes are given by $\Delta t = \Delta x = 0.05$. Moreover we have truncated the kernel function in its singularity point at the origin and in its tail by means of replacing it with the function

$$h(x) = \begin{cases} \phi_0(x) & \text{if } x \in [0, \epsilon] \\ g(x) & \text{if } x \in (\epsilon, \tau_0) \\ \phi_1(x) & \text{if } x \in [\tau_0, \tau] \end{cases}$$

where $\epsilon = 0.01$, $\tau_0 = 25$, $\tau = 26$ and ϕ_0, ϕ_1 are 5th degree interpolating polynomials with coefficients determined by $\phi_0^{(j)}(0) = g^{(j)}(\epsilon)$ and $\phi_0^{(j)}(\epsilon) = g^{(j)}(\epsilon)$ for $j = 0, 1, 2$ and $\phi_1^{(j)}(\tau_0) = g^{(j)}(\tau_0)$ and $\phi_1^{(j)}(\tau) = 0$ for $j = 0, 1, 2$ respectively. We remark that our method tends to be quite sensitive to the choice of λ, $\mathcal{I}$ and the function h. So one should always attempt to choose the parameters which makes the approximation (4.21) as accurate as possible before starting the simulation algorithm.

Figure 1 is obtained by simulating in Matlab, and in Figure 2 we compare the method described in the current section to obtain the field (5.1) to the more straightforward approach of employing numerical integration. By which we mean that for each (x_j, t_k) where

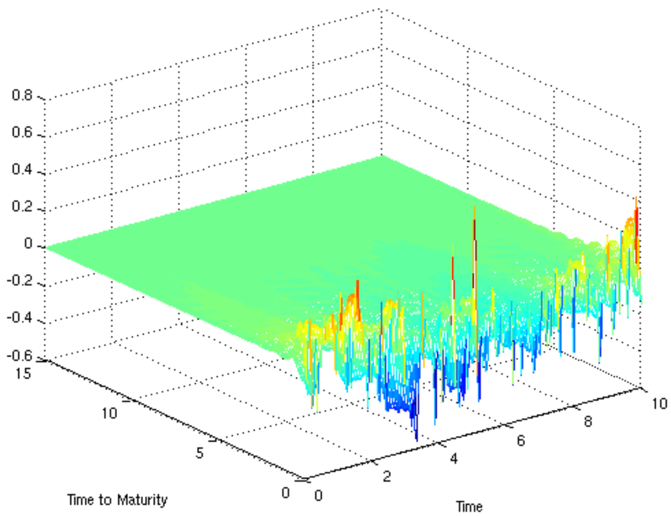

FIGURE 2. The difference $Y_{\text{Fourier}} - Y_{\text{NumInt}}$ between the simulated ambit field in Figure 1 as obtained by the described method versus numerical integration.

$j = 0, \ldots, J$ and $k = 0, \ldots, K$ we approximate $Y(x_j, t_k)$ by

$$Y(x_j, t_k) \approx \sum_{j'=0}^{j} \sum_{k'=0}^{k} g(t_k - t_{k'} + x_j)\varphi(x_{j'})\sigma_{t_{k'}}(x_{j'})L((x_{j'}, x_{j'+1}), (t_{k'}, t_{k'+1})).$$

In Figure 2 we can see that the two methods give reasonably similar fields, but that for values of x close to 0 the difference increases, which is caused by the singularity of the kernel function. Moreover we report that obtaining the field in Matlab by means of the Fourier method took 0.2587 seconds, whereas obtaining the field by means of the numerical integration approach took 17.2070 seconds. We remark that the computational time may be reduced even further, since in our calculations we used a for loop in the Fourier computation and a double for loop in the numerical integration computation. However by employing a 2 dimensional convolution in the Fourier case and a convolution for each x_j in the numerical integration case we could write a routine with no for loop in the Fourier case and one for loop (over the space dimension) in the numerical integration case. Even so it is apparent that computationally the Fourier method is more time efficient. Another advantage of the Fourier method is that at a given point in space time it is easy to compute an increment in a space and/or time direction by simply just computing the corresponding increments of $\widehat{Y}_\lambda$ and numerically integrate over the Fourier domain. Whereas a complete re-integration is required in the numerical integration case.

6. Conclusion

After giving a brief introduction to ambit fields, we have presented a general method for approximating ambit fields by means of a linear combination of ambit fields driven by complex exponential functions. Thus, by the factorisation property of the exponential function we can obtain a incremental simulation algorithm for general ambit fields by means of considering a Fourier transform of the kernel function of the original ambit field. Finally, as we demonstrated by an example, the existence of such a simulation algorithm makes the simulation of ambit fields more efficient, which in turn can be applied to get more efficient algorithms to simulate price of derivatives where the underlying is modelled using ambit fields.

Appendix A. Proofs of auxiliary results

Proof of Lemma 4.3. We proceed by induction on n. In the case when $n = 1$, the Lemma holds, since

$$\sum_{\alpha\in\mathbb{Z}}\gamma_\alpha = \gamma_0 + 2\sum_{\alpha\in\mathbb{N}}\gamma_\alpha.$$

Now suppose that the Lemma holds for a particular $n \geq 1$. Then

$$\begin{aligned}
\sum_{\alpha\in\mathbb{Z}^{n+1}}\gamma_\alpha &= \sum_{\alpha'\in\mathbb{Z}}\sum_{\alpha\in\mathbb{Z}^n}\gamma_{(\alpha,\alpha')} \\
&= \sum_{\alpha'\in\mathbb{Z}}\left(\gamma_{(\mathbf{0},\alpha')} + \sum_{m=1}^{n}2^m\sum_{j=1}^{\binom{n}{m}}\sum_{\alpha\in\mathbb{N}^m}\gamma_{(\rho_j^{n,m}(\alpha),\alpha')}\right) \\
&= \gamma_{\mathbf{0}} + 2\sum_{\alpha'\in\mathbb{N}}\gamma_{(\mathbf{0},\alpha')} + 2\sum_{j=1}^{n}\sum_{\alpha\in\mathbb{N}}\gamma_{(\rho_j^{n,1}(\alpha),0)} \\
&\quad + \sum_{m=2}^{n}2^m\sum_{j=1}^{\binom{n}{m}}\sum_{\alpha\in\mathbb{N}^m}\gamma_{(\rho_j^{n,m}(\alpha),0)} + \sum_{m=2}^{n+1}2^m\sum_{j=1}^{\binom{n}{m-1}}\sum_{\alpha\in\mathbb{N}^{m-1}}\sum_{\alpha'\in\mathbb{N}}\gamma_{(\rho_j^{n,m-1}(\alpha),\alpha')} \\
&= \gamma_{\mathbf{0}} + \sum_{m=1}^{n+1}2^m\sum_{j=1}^{\binom{n+1}{m}}\sum_{\alpha\in\mathbb{N}^m}\gamma_{\rho_j^{n+1,m}(\alpha)},
\end{aligned}$$

which completes the proof. □

Proof of Lemma 4.7. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ let $\phi : [0, 1] \to \mathbb{R}$, $t \mapsto \operatorname{Re} f(t\boldsymbol{x} + (1 - t)\boldsymbol{y})$ and $\psi : [0, 1] \to \mathbb{R}$, $t \mapsto \operatorname{Im} f(t\boldsymbol{x} + (1 - t)\boldsymbol{y})$. Then by applying the mean value theorem to the functions ϕ and ψ there exist constants $c_1, c_2 \in (0, 1)$ such that

$$f(\boldsymbol{x}) - f(\boldsymbol{y}) = (\nabla \operatorname{Re} f(c_1\boldsymbol{x} + (1 - c_1)\boldsymbol{y}) + \mathrm{i}\nabla \operatorname{Im} f(c_2\boldsymbol{x} + (1 - c_2)\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}).$$

Let $\boldsymbol{\gamma}_k := c_k\boldsymbol{x} + (1-c_k)\boldsymbol{y}$, for $k = 1, 2$. By applying the Cauchy-Schwarz inequality and noticing that $|\operatorname{Re} f|, |\operatorname{Im} f| < 1$ it follows that

$$\begin{aligned}
|f(\boldsymbol{x}) - f(\boldsymbol{y})|^2 &\leq ||\nabla \operatorname{Re} f(c_1\boldsymbol{x} + (1-c_1)\boldsymbol{y}) + \mathrm{i}\nabla \operatorname{Im} f(c_2\boldsymbol{x} + (1-c_2)\boldsymbol{y})||^2||\boldsymbol{x} - \boldsymbol{y}||^2 \\
&= \sum_{k=1}^{n} \big((\operatorname{Re} z_k \operatorname{Re} f(\boldsymbol{\gamma}_1) - \operatorname{Im} z_k \operatorname{Im} f(\boldsymbol{\gamma}_1))^2 \\
&\quad +(\operatorname{Re} z_k \operatorname{Im} f(\boldsymbol{\gamma}_2) + \operatorname{Im} z_k \operatorname{Re} f(\boldsymbol{\gamma}_2))^2\big) \, ||\boldsymbol{x} - \boldsymbol{y}||^2 \\
&\leq 4||\boldsymbol{z}||^2||\boldsymbol{x} - \boldsymbol{y}||^2.
\end{aligned}$$

□

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