Generalized time-invariant overtaking*

Geir B. Asheim† Claude d'Aspremont‡ Kuntal Banerjee§

March 12, 2010

Abstract

We present a new version of the overtaking criterion, which we call *generalized time-invariant overtaking*. The generalized time-invariant overtaking criterion (on the space of infinite utility streams) is defined by extending proliferating sequences of complete and transitive binary relations defined on finite dimensional spaces. The paper presents a general approach that can be specialized to at least two, extensively researched examples, the utilitarian and the leximin orderings on a finite dimensional Euclidean space.

Keywords and Phrases: Intergenerational justice, Utilitarianism, Leximin.

JEL Classification Numbers: D63, D71.

*We thank an anonymous referee for many helpful suggestions, and Aanund Hylland, Mohamed Mabrouk, participants at the 9th Meeting of the Society for Social Choice and Welfare, the Economic Theory Conference in honor of Professor Tapan Mitra on his 60th birthday, the International Symposium on Choice, Rationality and Intergenerational Equity at Waseda University, and EEA-ESEM Barcelona'09 for comments. The paper is part of the research activities at the center of Equality, Social Organization, and Performance (ESOP) at the Department of Economics at the University of Oslo. ESOP is supported by the Research Council of Norway.

†Department of Economics, University of Oslo, P.O. Box 1095 Blindern, N-0317 Oslo, Norway (tel: 47-22855498; fax: 47-22855035; e-mail: `g.b.asheim@econ.uio.no`). *Corresponding author.*

‡CORE, Université catholique de Louvain, 34, Voie du Roman Pays, B-1348 Louvain-la-Neuve, Belgium (e-mail: `daspremont@core.ucl.ac.be`).

§Department of Economics, Florida Atlantic University, 777 Glades Road, Boca Raton, FL 33431 and Department of Economics, Cornell University, 426 Uris Hall, Ithaca, NY 14853, USA (e-mail: `kbanerje@fau.edu`).

1 Introduction

Recent contributions in welfare economics have suggested new social welfare relations for the purpose of evaluating infinite utility streams representing the welfare levels of an infinite and countable number of generations. In particular, Basu and Mitra (2007a) extend the utilitarian ordering on a finite dimensional Euclidian space to the infinite dimensional case. Also non-additive theories have been defended, and Bossert, Sprumont and Suzumura (2007) extend the leximin ordering. Both these social welfare relations are incomplete. Still, they may be effective in the sense of selecting a small set of optimal or maximal elements for a given class of feasible infinite utility streams. Suggestions have also come from the philosophical literature (e.g., Vallentyne and Kagan, 1997; Lauwers and Vallentyne, 2004), sticking to finitely additive moral value theories, but addressing the issue of ranking worlds with an infinite number of "locations of values". These may represent "times" and hence be naturally ordered, or "people" for which no natural ordering can be assigned.

It is easy to construct pairs of infinite utility streams incomparable according to the criteria of Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007), but where it is clear that the one infinite stream is socially preferred to the other both from a utilitarian and egalitarian point of view. To illustrate, consider the following two streams:

$$\begin{array}{lcccccccccc} \mathbf{u} & : & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots & \frac{1}{2^{n-1}} & \cdots \\ \mathbf{v} & : & -1 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots & \frac{1}{2^{n-2}} & \cdots \end{array}$$

It is intuitively clear that **u** is socially preferred to **v** from a utilitarian perspective since the sum of utility differences between **u** and **v** is unconditionally convergent and converges to 1. Likewise, it is intuitively clear that **u** is socially preferred to **v** from an egalitarian perspective since the minimal utility of **v** ($= -1$) is smaller than the greatest lower bound for the utility of **u** ($= 0$). Still, according to the criteria of Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007) these streams are incomparable since there is no cofinite set (a subset of all generations with finite complement) on which $\mathbf{u}$ equals or Pareto-dominates $\mathbf{v}$. *This motivates an investigation of social welfare relations for the evaluation of infinite utility streams which are more complete* than those proposed by Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007), while allowing for non-additive moral value theories and different interpretations for the locations of values.

Extensions of utilitarian and leximin orderings to the infinite-dimensional case are normally required to satisfy the axioms of Finite Anonymity (ensuring equal treatment of generations) and Strong Pareto (ensuring sensitivity for the interests for each generation). Recent work by Lauwers (2010) and Zame (2007) confirms the following conjecture, suggested by Fleurbaey and Michel (2003): no *definable* complete and transitive binary relation on the set of infinite utility streams can be proved to satisfy the axioms of Finite Anonymity and Strong Pareto. In this sense, no complete social welfare relation satisfying these axioms can be "explicitly described" (see Zame, 2007, Theorem 4).[1] We will here consider social welfare relations satisfying Finite Anonymity and Strong Pareto that can be "explicitly described", and hence completeness is an unreachable goal.

However, there might be reasons—other than issues of explicit description—why one should refrain from seeking excessive comparability. To make this argument, consider the following two infinite utility streams:

$$
\begin{array}{ccccccccccccc}
\mathbf{x} & : & \frac{3}{2} & 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 & \ldots \\
\mathbf{y} & : & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 0 & 1 & \ldots
\end{array}
$$

When overtaking (Atsumi, 1965; von Weizsäcker, 1965) is applied to the utilitarian or leximin ordering (see Asheim and Tungodden, 2004), then $\mathbf{x}$ is strictly preferred to $\mathbf{y}$ since the finite head of $\mathbf{x}$ is preferred to the finite head of $\mathbf{y}$ at all locations.

[1] By applying Szpilrajn's Lemma (whose proof uses the Axiom of Choice), Svensson (1980) has shown that complete social welfare relations satisfying Finite Anonymity and Strong Pareto exist.

This conclusion crucially depends on the sequencing of the locations, as permuting odd and even locations for both $\mathbf{x}$ and $\mathbf{y}$ makes the streams incomparable.

The strict ranking of $\mathbf{x}$ over $\mathbf{y}$ can be made robust to such re-sequencing by adding *Fixed-step Anonymity* (Lauwers, 1997; Mitra and Basu, 2007) to overtaking (as done by Kamaga and Kojima, 2009b). Then $\mathbf{y}$ becomes indifferent to

$$\begin{array}{lcccccccccc} \mathbf{z} & : & 1 & 0 & 1 & 0 & 1 & \dots & 0 & 1 & \dots \end{array}$$

and thus by Strong Pareto and transitivity strictly inferior to $\mathbf{x}$. However, imposing Fixed-step Anonymity comes at the cost of Koopmans' (1960) Stationarity axiom (in the sense that preference over future utilities should not depend on present utility if both streams have the same present utility). To see this, consider

$$\begin{array}{lccccccccccc} (0,\mathbf{y}) & : & 0 & 0 & 1 & 0 & 1 & 0 & \dots & 1 & 0 & \dots \\ (0,\mathbf{z}) & : & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 0 & 1 & \dots \end{array}$$

Fixed-step Anonymity, under which $\mathbf{y}$ and $\mathbf{z}$ are socially indifferent, combined with Strong Pareto forces us to conclude that $(0,\mathbf{z})$ is socially preferred to $(0,\mathbf{y})$, thereby contradicting Stationarity.

Furthermore, even in conjunction with Fixed-step Anonymity, overtaking is dependent on sequencing: By allowing for permutations that are not of the fixed-step kind, there exists an infinite permutation matrix P such that

$$\begin{array}{lcccccccccc} P\mathbf{x} & : & 0 & 0 & \frac{3}{2} & 0 & 1 & \dots & 0 & 1 & \dots \\ P\mathbf{y} & : & 1 & 1 & 0 & 1 & 0 & \dots & 1 & 0 & \dots \end{array}$$

implying that $P\mathbf{y}$ is socially preferred to $P\mathbf{x}$ by both the utilitarian and leximin overtaking criterion, thereby inverting the original ranking.[2]

[2]The concepts of a permutation and a permutation matrix are introduced in Section 2.2. The matrix P moves location 2 to location 1, all other even locations two periods backwards, and all odd locations two periods forwards.

These examples show that overtaking does not satisfy axioms of *Relative Anonymity*, in the sense the ranking of two streams should not change when the same permutation of locations is applied to both streams. In its traditional form, overtaking does not satisfy the axiom of *Fixed-step Relative Anonymity*, where 'fixed-step' reflects that only fixed-step permutations are considered. Even in conjunction with Fixed-step Anonymity, overtaking does not satisfy the axiom of *Strong Relative Anonymity*, where 'strong' reflects that all infinite permutations are considered.

In this paper we will insist on the axioms of Stationarity and Strong Relative Anonymity. An argument for Stationarity is that it is necessary for time-consistency if social preferences are assumed to be time-invariant.

An argument for Strong Relative Anonymity is, as discussed by Vallentyne and Kagan (1997), that there is no natural order; in this case the axiom coincides with Lauwers and Vallentyne's (2004, p. 317) *Isomorphism Invariance*. This argument may also apply in the intergenerational setting, where the generations follow each other in sequence. An interesting case is where the utilities of people within each generation are not aggregated into a single number,[3] but where the elements of the stream correspond to individual utilities. With an infinite number of individuals within each generation, the stream of individual utilities cannot have a natural order. With a finite population, there is no natural ordering of people within each generation. Even in the case where the elements of the stream represents generational utilities, one can argue that the order in which generations are counted should not matter for the ranking of streams if the generations are treated equally.

Relative Anonymity (in the sense the ranking of two streams does not change when the same permutation of locations is applied to *both* streams) is weaker than ordinary Anonymity (where a permutation is applied to only *one* stream). To illustrate: the incomplete social welfare relation generated by Strong Pareto alone

[3] See d'Aspremont (2007) for the assumptions required to reduce the welfare of each generation to a single number.

satisfies Strong Relative Anonymity, but fails to satisfy even the weakest form of Anonymity, Finite Anonymity, because Pareto-dominance can vanish when two elements of the one stream (only) are permuted.

The utilitarian and leximin social welfare relations proposed by Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007) respectively satisfy both Stationarity and Strong Relative Anonymity. It is the purpose of the present paper to expand the asymmetric parts of these binary relations without compromising Stationarity and Strong Relative Anonymity. In particular, we will present utilitarian and leximin social welfare relations that rank $\mathbf{u}$ strictly above $\mathbf{v}$, while deeming $\mathbf{x}$ and $\mathbf{y}$ (and $\mathbf{y}$ and $\mathbf{z}$, and $(0, \mathbf{y})$ and $(0, \mathbf{z})$, and $P\mathbf{x}$ and $P\mathbf{y}$) incomparable.

A simple but important fact is that, for comparing infinite utility streams, all welfare criteria, whether the utilitarian criterion of Basu and Mitra (2007b), the leximin criterion of Bossert, Sprumont and Suzumura (2007), as well as other utilitarian criteria such as overtaking and catching-up introduced by von Weizsäcker (1965) and Atsumi (1965), and the leximin criteria defined in Asheim and Tungodden (2004), use an infinite sequence of the standard finite version of either the utilitarian or the leximin social welfare ordering.

Using this fact, and a known property of these respective sequences, namely that of being "proliferating" (to impose the criterion for any finite number of individuals, it is sufficient to impose it in situations where only two individuals are involved), all these criteria can be given a "generalized" formulation. This generalized formulation is meaningful for any given proliferating sequence of social welfare relations defined on finite utility streams (and usually assumed to satisfy some Anonymity and Pareto conditions). The notion of a proliferating sequence was introduced for the analysis of generalized versions of infinite-dimensional SWRs by d'Aspremont (2007). It emphasizes the fact that value judgments made in the social evaluation of the welfare of the individuals within a generation, and in particular within the present generation, are binding in the evaluation of the welfare of all generations.

Here we suggest a version of the overtaking criterion within this general approach to the evaluation of infinite utility streams. We call this *generalized time-invariant overtaking*. The generalized time-invariant overtaking criterion (on the space of infinite utility streams) is defined by extending proliferating sequences of complete and transitive binary relations defined on finite dimensional spaces. Our general analysis specializes in a straightforward manner to the utilitarian and leximin cases. We establish as a general result (stated in Theorem 1) that generalized time-invariant overtaking satisfies Stationarity and Strong Relative Anonymity. We also note that the criterion ranks $\mathbf{u}$ strictly above $\mathbf{v}$. Moreover, we provide methods for determining the asymmetric and symmetric parts in the special cases of the utilitarian and leximin time-invariant overtaking criteria.

The paper is organized as follows: Section 2 contains preliminaries, Section 3 presents the concept of proliferating sequences, and Section 4 reviews different kinds of "generalized criteria". Section 5 defines and investigates the properties of generalized time-invariant overtaking, and Section 6 specializes this concept to the utilitarian and leximin cases. The concluding Section 7 contains a general analysis of the properties of pairs of utility streams that our criterion cannot compare, and a discussion of the close relationship between our analysis and the work Vallentyne and Kagan (1997) and Lauwers and Vallentyne (2004) in the utilitarian case.[4]

2 Preliminaries

2.1 Notation and Definitions

Let $\mathbb{N}$ denote the set of natural numbers $\{1, 2, 3, ...\}$ and $\mathbb{R}$ the set of real numbers. Let $\mathbf{X}$ denote the set $Y^{|\mathbb{N}|}$, where $Y \subseteq \mathbb{R}$ is an interval satisfying $[0, 1] \subseteq Y$. We let $\mathbf{X}$ be the domain of utility sequences (also referred to as "utility streams" or "utility profiles"). Thus, we write $\mathbf{x} \equiv (x_1, x_2, \ldots) \in \mathbf{X}$ iff $x_n \in Y$ for all $n \in \mathbb{N}$. Usually,

[4] We thank the referee for pointing out this close relationship.

x_n is interpreted as the utility of generation n, but more generally as the utility of individual n belonging to some generation. No natural order will be assumed. For $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ we will write $\mathbf{x} \geq \mathbf{y}$ iff $x_i \geq y_i$ for all $i \in \mathbb{N}$ and $\mathbf{x} > \mathbf{y}$ iff $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

Whenever we write about subsets M, N of $\mathbb{N}$, we will be dealing with subsets of finite cardinality, entailing that $\mathbb{N} \backslash M$, $\mathbb{N} \backslash N$ are cofinite sets (i.e., subsets of $\mathbb{N}$ which complements are finite). For all $\mathbf{x} \in \mathbf{X}$ and any $N \subset \mathbb{N}$, we will write $\mathbf{x}$ as $(\mathbf{x}_N, \mathbf{x}_{\mathbb{N} \backslash N})$. We will denote vectors (finite as well as infinite dimensional) by bold letters; example are $\mathbf{x}$**,** $\mathbf{y}$, etc. The components of a vector will be denoted by normal font. Negation of a statement is indicated by the logical quantifier $\neg$.

A *social welfare relation* (SWR) is a reflexive and transitive binary relation defined on $\mathbf{X}$ (and denoted $\succsim$) or $Y^{|M|}$ for some $M \subset \mathbb{N}$ (and denoted $\succsim_M$). A *social welfare order* (SWO) is a complete SWR.

An SWR $\succsim'$ is a *subrelation* to SWR $\succsim''$ if for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, (a) $\mathbf{x} \sim' \mathbf{y} \Rightarrow \mathbf{x} \sim'' \mathbf{y}$ and (b) $\mathbf{x} \succ' \mathbf{y} \Rightarrow \mathbf{x} \succ'' \mathbf{y}$.

2.2 Permutations

A *permutation* π is a one-to-one map from $\mathbb{N}$ onto $\mathbb{N}$. For any $\mathbf{x} \in \mathbf{X}$ and a permutation π, we write $\mathbf{x} \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \dots) \in \mathbf{X}$. Permutations can be *represented* by a permutation matrix, $P = (p_{ij})_{i,j \in \mathbb{N}}$, which is an infinite matrix satisfying:

(1) For each $i \in \mathbb{N}$, $p_{ij(i)} = 1$ for some $j(i) \in \mathbb{N}$ and $p_{ij} = 0$ for all $j \neq j(i)$.

(2) For each $j \in \mathbb{N}$, $p_{i(j)j} = 1$ for $i(j) \in \mathbb{N}$ and $p_{ij} = 0$ for all $i \neq i(j)$.

Given any permutation π, there is a permutation matrix P such that for $\mathbf{x} \in \mathbf{X}$, $\mathbf{x} \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \dots)$ can also be written as $P\mathbf{x}$ in the usual matrix multiplication. Conversely, given any permutation matrix P , there is a permutation π defined by $\pi = P\mathbf{a}$, where $\mathbf{a} = (1, 2, 3, \dots)$. The set of all permutations is denoted by $\mathcal{P}$.

A finite permutation π is a permutation such that there is some $N \subset \mathbb{N}$ with $\pi(i) = i$ for all $i \notin N$. Thus, a finite permutation matrix has $p_{ii} = 1$ for all $i \notin N$

for some $N \subset \mathbb{N}$. The set of all finite permutations is denoted by $\mathcal{F}$.

Given a permutation matrix $P \in \mathcal{P}$ and $n \in \mathbb{N}$, we denote the $n \times n$ matrix $(p_{ij})_{i,j \in \{1,\dots,n\}}$ by $P(n)$. Let

$$\begin{aligned}\mathcal{S} \;=\; \{P \in \mathcal{P} \mid & \text{ there is some } k \in \mathbb{N} \text{ such that, for each } n \in \mathbb{N}, \\ & P(nk) \text{ is a finite dimensional permutation matrix}\}\end{aligned}$$

denote the set of fixed-step permutations. It is easily checked that this is a group (with respect to matrix multiplication) of cyclic permutations.[5]

2.3 Axioms of Anonymity and Pareto

In this subsection we introduce the basic axioms that are repeatedly used in the rest of the paper. The first set of axioms pertains to SWRs defined on a finite-dimensional space, whereas the latter set is on the space of infinite utility streams.

Let $\succsim_M$ be an SWR defined on $Y^{|M|}$. Throughout we will as assume that $\succsim_M$ satisfies the following condition as a minimal requirement. It is an anonymity condition where the same permutation applies to the two utility vectors. Hence, we call it "relative anonymity". In the present intergenerational context it can be interpreted as a time invariance property, reflecting that no natural order is assumed.

Axiom m**-I** (m-Relative Anonymity) For all $\mathbf{x}_M$, $\mathbf{y}_M$, $\mathbf{u}_N$, $\mathbf{v}_N \in Y^m$ with $M = \{i_1, i_2, ..., i_m\} \subset \mathbb{N}$ and $N = \{j_1, j_2, ..., j_m\} \subset \mathbb{N}$ satisfying $|M| = |N| = m \geq 2$, if there exists a finite permutation $\pi : \{1, \dots, m\} \to \{1, \dots, m\}$ such that $x_{i_{\pi(k)}} = u_{j_k}$ and $y_{i_{\pi(k)}} = v_{j_k}$ for all $k \in \{1, \dots, m\}$, then $\mathbf{x}_M \succsim_M \mathbf{y}_M$ iff $\mathbf{u}_N \succsim_N \mathbf{v}_N$.

By satisfying m-**I**, $\succsim_M$ depends only on the dimension $|M|$. We will henceforth write $\succsim_m$ for an SWR on Y^m, thereby signifying that the SWR satisfies m-**I**.

[5]The permutation π is *cyclic* if for each $\mathbf{e}^i = (0, \dots, 0, 1, 0. \dots)$ (with 1 at the i^{th} place) there exists a $k \in \mathbb{N}$ such that $\pi^k(\mathbf{e}^i) = \mathbf{e}^i$. A class of cyclic permutations is not necessarily a group, while $\mathcal{P}$ is a group which does not contain only cyclic permutations.

It is useful to compare m-**I** to the usual anonymity condition where a permutation is applied to the one utility stream only.

Axiom m-**A** (m-Anonymity) For all $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $m \geq 2$, if $\mathbf{a}$ is a permutation of $\mathbf{b}$, then $\mathbf{a} \sim_m \mathbf{b}$.

Since $\succsim_m$ is transitive, m-**A** is equivalent to having $\mathbf{a} \sim_m \mathbf{b}$ whenever there exists i, $j \in \{1, \ldots, m\}$ such that $a_i = b_j$, $a_j = b_i$ and $a_k = b_k$ for all $k \neq i, j$.

The m-Pareto Principle ($\mathbf{a} \succsim_m^P \mathbf{b}$ if and only if $a \geq b$) illustrates that m-**I** does not imply m-**A**. However, as originally shown by d'Aspremont and Gevers (1977, Lemma 4), the two axioms are equivalent if $\succsim_m$ is complete.

Lemma 1 *If $\succsim_m$ with $m \geq 2$ is complete, then $\succsim_m$ satisfies m-**A**.*

Proof. Assume that $\succsim_m$ is complete (where the notation entails that the SWR satisfies m-**I**). Suppose by way of contradiction that there exists $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $a_i = b_j$, $a_j = b_i$ and $a_k = b_k$ for all $k \neq i, j$ such that $\neg(\mathbf{a} \sim_m \mathbf{b})$. Since $\succsim_m$ is complete, we may w.l.o.g. assume that $\mathbf{a} \succ_m \mathbf{b}$. However, by permuting the ith and jth element of both $\mathbf{a}$ and $\mathbf{b}$ and invoking m-**I**, we obtain $\mathbf{b} \succ_m \mathbf{a}$, which contradicts $\mathbf{a} \succ_m \mathbf{b}$. Hence, $\mathbf{a} \sim_m \mathbf{b}$ whenever there exists i, $j \in \{1, \ldots, m\}$ such that $a_i = b_j$, $a_j = b_i$ and $a_k = b_k$ for all $k \neq i, j$. ■

The other kind of basic axiom is the Pareto condition.

Axiom m-**P** (m-Pareto) For all $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $m \geq 2$, if $\mathbf{a} > \mathbf{b}$, then $\mathbf{a} \succ_m \mathbf{b}$.

Clearly, since $\succsim_m$ is transitive, m-**P** is equivalent to having $\mathbf{a} \succ_m \mathbf{b}$ whenever there exists $i \in \{1, \ldots, m\}$ such that $a_i > b_i$ and $a_k = b_k$ for all $k \neq i$. As a matter of notation, if it is clear from the context that an axiom on finite dimension is invoked, then we will drop the letter m from its abbreviation.

Let $\succsim$ be an SWR defined on $\mathbf{X}$. Consider the following versions of the anonymity and Pareto axioms on $\succsim$. Let $\mathcal{Q}$ be some fixed group of permutations equaling $\mathcal{F}$, $\mathcal{S}$

or $\mathcal{P}$, corresponding to the terms "Finite", "Fixed-step" and "Strong" respectively in the names of the axioms below.

Axiom $\mathcal{Q}$**I** (Finite/Fixed-step/Strong Relative Anonymity) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and all $P \in \mathcal{Q}$, $\mathbf{x} \succsim \mathbf{y}$ iff $P\mathbf{x} \succsim P\mathbf{y}$.

Axiom $\mathcal{Q}$**A** (Finite/Fixed-step/Strong Anonymity) For all $\mathbf{x} \in \mathbf{X}$ and all $P \in \mathcal{Q}$, $\mathbf{x} \sim P\mathbf{x}$.

Axiom FP (Finite Pareto) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with some subset $N \subset \mathbb{N}$ such that $x_i = y_i$ for all $i \in \mathbb{N} \backslash N$, if $\mathbf{x} > \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Axiom SP (Strong Pareto) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if $\mathbf{x} > \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Clearly, since $\succsim$ is transitive, $\mathcal{F}$**A** is equivalent to having $\mathbf{x} \sim \mathbf{y}$ whenever there exist i, $j \in \mathbb{N}$ such that $x_i = y_j$, $x_j = y_i$ and $x_k = y_k$ for all $k \neq i, j$. Likewise, **FP** is equivalent to having $\mathbf{x} \succ \mathbf{y}$ whenever there exists $i \in \mathbb{N}$ such that $x_i > y_i$ and $x_k = y_k$ for all $k \neq i$. This is what Basu and Mitra (2007b) refer to as Weak Dominance; hence, **FP** coincides with Weak Dominance. Note that for $\mathcal{Q} = \mathcal{F}$, $\mathcal{S}$ or $\mathcal{P}$, $\mathcal{Q}$**A** implies $\mathcal{Q}$**I**, while the converse is not true for incomplete infinite-dimensional SWRs. For an analysis of these issues and more generally on comparability of a social welfare evaluation in the intergenerational context we refer to Mabrouk (2008). It is also well-known that $\mathcal{P}$**A** cannot be combined with **SP**, while $\mathcal{S}$**A** can (since it is a group of cyclic permuations, cf. Mitra and Basu, 2007).

3 Proliferating sequences

Many well-known finite-dimensional SWRs form proliferating sequences. The structure imposed by this concept on a sequence of finite-dimensional SWR enables the extension to an infinite-dimensional SWR to be analyzed at a generalized level, without considering the specific nature of the finite-dimensional counterpart. Furthermore, it allows infinite-dimensional SWRs to be defined solely on the basis of the 2-dimensional version of the underlying finite-dimensional SWR.

An infinite-dimensional SWR $\succsim$ *extends* the finite-dimensional SWR $\succsim_m$ if, for all $M \subset \mathbb{N}$ with $|M| = m$ and all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i = y_i$ for every $i \in \mathbb{N} \backslash M$, $\mathbf{x}_M \succ_m \mathbf{y}_M$ implies $\mathbf{x} \succ \mathbf{y}$, and $\mathbf{x}_M \sim_m \mathbf{y}_M$ implies $\mathbf{x} \sim \mathbf{y}$.

Definition 1 A sequence of SWRs, $\{\succsim_m^*\}_{m=2}^{\infty}$, is *proliferating* if any SWR $\succsim$ that extends $\succsim_2^*$ also extends $\succsim_m^*$ for every $m \geq 2$.

The following result implies that the m-Grading Principle ($\mathbf{a} \succsim_m^S \mathbf{b}$ iff there exists a permutation $\mathbf{c}$ of $\mathbf{b}$ such that $\mathbf{a} \geq \mathbf{c}$) is proliferating.[6]

Lemma 2 *(i) If $\succsim_2$ is an SWR on Y^2 that satisfies $\boldsymbol{A}$, and $\succsim$ is an SWR on $\mathbf{X}$ that extends $\succsim_2$, then $\succsim$ satisfies $\mathcal{F}\boldsymbol{A}$.*
(ii) If $\succsim_2$ is an SWR on Y^2 that satisfies $\boldsymbol{P}$, and $\succsim$ is an SWR on $\mathbf{X}$ that extends $\succsim_2$, then $\succsim$ satisfies $\boldsymbol{FP}$.

Proof. (i) Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and for some i, $j \in \mathbb{N}$ $(i \neq j)$, $x_i = y_j$, $x_j = y_i$ and $x_k = y_k$ for all $k \neq i$, j. Set $M = \{i, j\}$. Since $\succsim_2$ satisfies $\mathbf{A}$, $\mathbf{x}_M \sim_2 \mathbf{y}_M$. By the fact that $x_k = y_k$ for all $k \in \mathbb{N} \backslash M$ and $\succsim$ extends $\succsim_2$, $\mathbf{x} \sim \mathbf{y}$.

(ii) Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for some $i \in \mathbb{N}$, $x_i > y_i$ and $x_k = y_k$ for all $k \neq i$. Set $M = \{i, k\}$ for some $k \neq i$. Since $\succsim_2$ satisfies $\mathbf{P}$, $\mathbf{x}_M \succ_2 \mathbf{y}_M$. By the fact that $x_j = y_j$ for all $j \in \mathbb{N} \backslash M$ and $\succsim$ extends $\succsim_2$, $\mathbf{x} \succ \mathbf{y}$. ■

The utilitarian and leximin SWOs, which will be defined and analyzed in Section 6, are other important examples of proliferating sequences. In the case of such

[6] The Grading Principle was introduced by Suppes (1966) and further analyzed by Sen (1970), Kolm (1972) and Hammond (1976, 1979). Its proliferating property is mentioned by Sen (1976, fn 26) as suggested by Hammond as a step to derive the same property for Leximin. For a proof, see Hammond (1979). The proof of d'Aspremont (1985, Lemma 3.1.1) can be immediately transposed to Y^m (in place of $\mathbb{R}^m$).

complete SWRs, the notion of proliferation yields added structure.[7]

Lemma 3 *A proliferating sequence* $\{\succsim^*_m\}_{m=2}^\infty$ *of SWOs satisfies:*

(i) For all $\mathbf{x}$*,* $\mathbf{y} \in \mathbf{X}$ *satisfying* $x_i = y_i$ *for some* $i \in \mathbb{N}\backslash M$*,* $\mathbf{x}_M \succsim^*_{|M|} \mathbf{y}_M$ *iff* $\mathbf{x}_{M\cup\{i\}} \succsim^*_{|M|+1} \mathbf{y}_{M\cup\{i\}}$.

(ii) Assume that $\succsim^*_m$ *satisfies* $\boldsymbol{P}$ *for each* $m \geq 2$*. For all* $\mathbf{x}$*,* $\mathbf{y} \in \mathbf{X}$ *satisfying that there exists* $M \subset \mathbb{N}$ *with* $|M| \geq 2$ *such that* $\mathbf{x}_N \sim^*_{|N|} \mathbf{y}_N$ *for all* $N \supseteq M$*,* $x_i = y_i$ *for all* $i \in \mathbb{N}\backslash M$*.*

Proof. *(i)* Let $\{\succsim^*_m\}_{m=2}^\infty$ be a proliferating sequence of SWOs, and let $\succsim$ extend $\succsim^*_2$, implying that $\succsim$ extends $\succsim^*_m$ for all $m \geq 2$. Assume that $\mathbf{x}_M \succsim^*_{|M|} \mathbf{y}_M$ and $x_i = y_i$ for some $i \in \mathbb{N}\backslash M$. Let $\mathbf{z} \in \mathbf{X}$ be an arbitrarily chosen utility stream. Since $\succsim$ extends $\succsim^*_{|M|}$, this implies $(\mathbf{x}_{M\cup\{i\}}, z_{\mathbb{N}\backslash(M\cup\{i\})}) \succsim (\mathbf{y}_{M\cup\{i\}}, z_{\mathbb{N}\backslash(M\cup\{i\})})$. Suppose $\mathbf{x}_{M\cup\{i\}} \prec^*_{|M|+1} \mathbf{y}_{M\cup\{i\}}$. Since $\succsim$ extends $\succsim^*_{|M|+1}$, this implies $(\mathbf{x}_{M\cup\{i\}}, z_{\mathbb{N}\backslash(M\cup\{i\})}) \prec (\mathbf{y}_{M\cup\{i\}}, z_{\mathbb{N}\backslash(M\cup\{i\})})$, leading to a contradiction. Hence, $\neg(\mathbf{x}_{M\cup\{i\}} \prec^*_{|M|+1} \mathbf{y}_{M\cup\{i\}})$, implying since the SWO $\succsim^*_{|M|+1}$ is complete that $\mathbf{x}_{M\cup\{i\}} \succsim^*_{|M|+1} \mathbf{y}_{M\cup\{i\}}$. Likewise, $\mathbf{x}_M \succ^*_{|M|} \mathbf{y}_M$ and $x_i = y_i$ for some $i \in \mathbb{N}\backslash M$ implies that $\mathbf{x}_{M\cup\{i\}} \succ^*_{|M|+1} \mathbf{y}_{M\cup\{i\}}$, thereby establishing the converse statement.

(ii) Let $\{\succsim^*_m\}_{m=2}^\infty$ be a proliferating sequence of SWOs with, for each $m \geq 2$, $\succsim^*_m$ satisfying $\mathbf{P}$. Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Suppose that $x_i \neq y_i$ for some $i \in \mathbb{N}\backslash M$; w.l.o.g. we can set $x_i > y_i$. Since $\succsim^*_{|M|+1}$ satisfies $\mathbf{P}$, it follows from part (i) that

$$\mathbf{x}_{M\cup\{i\}} \sim^*_{|M|+1} (\mathbf{y}_M, x_i) \succ^*_{|M|+1} \mathbf{y}_{M\cup\{i\}}\,,$$

contradicting that $\mathbf{x}_{M\cup\{i\}} \sim^*_{|M|+1} \mathbf{y}_{M\cup\{i\}}$. Hence, $x_i = y_i$ for all $i \in \mathbb{N}\backslash M$. ■

[7] Sakai (2010) refers to property (i) of the following lemma as "independence".

4 Generalized criteria

In this section we review "generalized criteria", namely infinite-dimensional SWRs that extend finite-dimensional SWRs that are both complete and proliferating. We first introduce two additional axioms on the space of infinite utility streams that will be used to differentiate these generalized criteria and in the rest of the paper.

Axiom ST (Stationarity) For all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v} \in \mathbf{X}$ with $x_1 = y_1$ and, for all $i \in \mathbb{N}$, $u_i = x_{i+1}$ and $v_i = y_{i+1}$, $\mathbf{x} \succsim \mathbf{y}$ iff $\mathbf{u} \succsim \mathbf{v}$.

Axiom IPC (Time-Invariant Preference Continuity) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if there exists $M \subset \mathbb{N}$ such that, for all $N \supseteq M$, $(\mathbf{x}_N, \mathbf{y}_{\mathbb{N}\backslash N}) \succ \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Let $\{\succsim^*_m\}_{m=2}^{\infty}$ be a proliferating sequence of SWOs with, for each $m \geq 2$, $\succsim^*_m$ satisfying axiom **P** (while, by Lemma 1, axiom **A** follows from the assumption that axiom **I** is satisfied). Let $\succsim$ extend $\succsim^*_2$, implying that $\succsim$ extends $\succsim^*_m$ for all $m \geq 2$.

For all $M \subset \mathbb{N}$ with $|M| = m \geq 2$ and all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i = y_i$ for every $i \in \mathbb{N}\backslash M$, $\mathbf{x}_M \succsim_m \mathbf{y}_M$ iff $\mathbf{x} \succsim \mathbf{y}$, since $\succsim^*_m$ is complete. Hence, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and $M \subset \mathbb{N}$ with $|M| \geq 2$, $(\mathbf{x}_M, \mathbf{x}_{\mathbb{N}\backslash M}) \succsim (\mathbf{y}_M, \mathbf{x}_{\mathbb{N}\backslash M})$ iff $(\mathbf{x}_M, \mathbf{y}_{\mathbb{N}\backslash M}) \succsim (\mathbf{y}_M, \mathbf{y}_{\mathbb{N}\backslash M})$. Therefore, axiom **IPC** does not depend on the specification of the common elements on $\mathbb{N}\backslash N$. Furthermore, axiom **IPC** is sufficient to ensure strict preference between $\mathbf{u}$ and $\mathbf{v}$ of the introduction. To see this, note that if $1 \in M$, then, for any $N \supseteq M$, $\mathbf{u}_N$ Pareto-dominates some permutation of $\mathbf{v}_N$.

The following generalized criteria extend $\succsim^*_m$ for every $m \geq 2$.

- **Equality on a cofinite set** (introduced here). $\succsim^*$ is the SWR defined by

$$\mathbf{x} \succsim^* \mathbf{y} \text{ iff there exists } N \subset \mathbb{N} \text{ such that } \mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N \text{ and } \mathbf{x}_{\mathbb{N}\backslash N} = \mathbf{y}_{\mathbb{N}\backslash N}\,.$$

- **Equality or Pareto-dominance on a cofinite set** (Basu and Mitra, 2007a; Bossert, Sprumont and Suzumura, 2007). $\succsim^*_{\mathcal{F}}$ is the SWR defined by

$$\mathbf{x} \succsim^*_{\mathcal{F}} \mathbf{y} \text{ iff there exists } N \subset \mathbb{N} \text{ such that } \mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N \text{ and } \mathbf{x}_{\mathbb{N}\backslash N} \geq \mathbf{y}_{\mathbb{N}\backslash N}\,.$$

- **Extended Anonymity** (Banerjee, 2006; Kamaga and Kojima, 2009a). $\succsim^*_{\mathcal{S}}$ is the SWR defined by

$$\mathbf{x} \succsim^*_{\mathcal{S}} \mathbf{y} \text{ iff there exists } P \in \mathcal{S} \text{ such that } \mathbf{x} \succsim^*_{\mathcal{F}} P\mathbf{y}\,.$$

- **Overtaking** (Atsumi, 1965; von Weizsäcker, 1965) $\succsim^*_{\mathcal{O}}$ is the SWR defined by

$$\mathbf{x} \succ^*_{\mathcal{O}} \mathbf{y} \text{ iff there exists } m \in \mathbb{N} \text{ such that } \mathbf{x}_{\{1,\dots,n\}} \succ^*_n \mathbf{y}_{\{1,\dots,n\}} \text{ for all } n \geq m\,,$$

$$\mathbf{x} \sim^*_{\mathcal{O}} \mathbf{y} \text{ iff there exists } m \in \mathbb{N} \text{ such that } \mathbf{x}_{\{1,\dots,n\}} \sim^*_n \mathbf{y}_{\{1,\dots,n\}} \text{ for all } n \geq m\,.$$

- **Fixed-step overtaking** (Lauwers, 1997; Fleurbaey and Michel, 2003; Kamaga and Kojima, 2009b). $\succsim^*_{\mathcal{SO}}$ is the SWR defined by

$$\mathbf{x} \succ^*_{\mathcal{SO}} \mathbf{y} \text{ iff there exists } k \in \mathbb{N} \text{ such that } \mathbf{x}_{\{1,\dots,nk\}} \succ^*_{nk} \mathbf{y}_{\{1,\dots,nk\}} \text{ for all } n \in \mathbb{N},$$

$$\mathbf{x} \sim^*_{\mathcal{SO}} \mathbf{y} \text{ iff there exists } k \in \mathbb{N} \text{ such that } \mathbf{x}_{\{1,\dots,nk\}} \sim^*_{nk} \mathbf{y}_{\{1,\dots,nk\}} \text{ for all } n \in \mathbb{N}.$$

The criteria, $\succsim^*$, $\succsim^*_{\mathcal{F}}$, $\succsim^*_{\mathcal{S}}$, $\succsim^*_{\mathcal{O}}$, and $\succsim^*_{\mathcal{SO}}$, are infinite-dimensional SWRs that illustrate the trade-offs between the axioms. By the definition of extension, $\succsim^*$ is a subrelation to any SWR extending $\succsim^*_m$ for every $m \geq 2$. Furthermore, $\succsim^*_{\mathcal{F}}$ is a subrelation to each of $\succsim^*_{\mathcal{S}}$ and $\succsim^*_{\mathcal{O}}$, and $\succsim^*_{\mathcal{S}}$ and $\succsim^*_{\mathcal{O}}$ are both subrelations to $\succsim^*_{\mathcal{SO}}$. All these SWRs satisfy $\mathcal{F}$**I** and $\mathcal{F}$**A**. Table 1 summarizes their properties in terms of the remaining axioms, where "violated by" means that, for a given SWR in the table, no alternative SWR to which this SWR is a subrelation satisfies the axiom. This leads to the following observations: Going from $\succsim^*_{\mathcal{F}}$ to $\succsim^*_{\mathcal{O}}$ we pick up **IPC**, but weaken $\mathcal{P}$**I** all the way to $\mathcal{F}$**I**. Going from $\succsim^*_{\mathcal{F}}$ to $\succsim^*_{\mathcal{SO}}$ we strengthen $\mathcal{F}$**A** to $\mathcal{S}$**A** and pick up **IPC**, but must weaken $\mathcal{P}$**I** to $\mathcal{S}$**I** and drop **ST**. This leads to the question: Is it possible to pick up **IPC** without weakening $\mathcal{P}$**I** and dropping **ST**?[8] We show that this is indeed possible by means of generalized time-invariant overtaking.

[8]The $(\mathbf{y}, \mathbf{z})$ example of Section 1 illustrates the problems of strengthening $\mathcal{F}$**A** to $\mathcal{S}$**A** while retaining **ST**. Mitra (2007) discusses the problem of combining **ST** with any kind of extended anonymity. Here we show how the asymmetric part of $\succsim^*_{\mathcal{F}}$ can be expanded, while retaining **ST**.

	$\mathcal{S}$**I**	$\mathcal{P}$**I**	$\mathcal{S}$**A**	$\mathcal{P}$**A**	**SP**	**ST**	**IPC**
$\succsim^*$	$+$	$+$	$+$	$+$		$+$	
$\succsim^*_{\mathcal{F}}$	$+$	$+$		$-$	$+$	$+$	
$\succsim^*_{\mathcal{S}}$	$+$	$-$	$+$	$-$	$+$	$-$	
$\succsim^*_{\mathcal{O}}$		$-$		$-$	$+$	$+$	$+$
$\succsim^*_{\mathcal{SO}}$	$+$	$-$	$+$	$-$	$+$	$-$	$+$

Table 1: Axioms satisfied ($+$) and violated ($-$) by various SWRs

5 A new criterion for infinite utility streams

We are now ready to state the definition of the generalized time-invariant overtaking criterion. Let $\{\succsim^*_m\}_{m=2}^{\infty}$ be a proliferating sequence of SWOs with $\succsim^*_m$ satisfying axiom **P** (while axiom **A** is implied by axiom **I**) for each $m \geq 2$.[9]

Definition 2 (Generalized time-invariant overtaking) The *generalized time-invariant overtaking criterion* $\succsim^*_{\mathcal{I}}$ generated by $\{\succsim^*_m\}_{m=2}^{\infty}$ satisfies, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y} \text{ iff there exists } M \subset \mathbb{N} \text{ with } |M| \geq 2 \text{ such that } \mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N \text{ for all } N \supseteq M.$$

We can now state our main result.

Theorem 1 *Let $\{\succsim^*_m\}_{m=2}^{\infty}$ be a proliferating sequence of SWOs with, for each $m \geq 2$, $\succsim^*_m$ satisfying axiom **P**. Then:*

*(i) $\succsim^*_{\mathcal{I}}$ is an SWR that satisfies $\mathcal{P}$**I**, $\mathcal{F}$**A**, **SP** and **ST**.*

*(ii) An SWR $\succsim$ extends $\succsim^*_2$ and satisfies **IPC** iff $\succsim^*_{\mathcal{I}}$ is a subrelation to $\succsim$.*

In the proof of Theorem 1, we make use of the following lemmas.

[9]Definition 2 is formulated as a "catching up" criterion. However, Lemma 4 shows that a formulation in terms of an "overtaking" criterion is equivalent, justify our terminology.

Lemma 4 *The SWR $\succsim^*_{\mathcal{I}}$ satisfies:*

(i) For all $\mathbf{x}$*,* $\mathbf{y} \in \mathbf{X}$*,* $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$ *iff there exist* $M \subset \mathbb{N}$ *with* $|M| \geq 2$ *such that* $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ *for all* $N \supseteq M$*.*

(ii) For all $\mathbf{x}$*,* $\mathbf{y} \in \mathbf{X}$*,* $\mathbf{x} \sim^*_{\mathcal{I}} \mathbf{y}$ *iff there exist* $M \subset \mathbb{N}$ *with* $|M| \geq 2$ *such that* $\mathbf{x}_N \sim^*_{|N|} \mathbf{y}_N$ *for all* $N \supseteq M$*.*

Proof. (Only-if part of (i): $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$ only if there exist $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.) Assume $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$ that is, (a) $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$ and (b) $\neg(\mathbf{y} \succsim^*_{\mathcal{I}} \mathbf{x})$. By (a), there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Note that $\neg(\mathbf{y} \succsim^*_{\mathcal{I}} \mathbf{x})$ implies that for any $M \subset \mathbb{N}$ there is some $M' \supset M$ such that $\mathbf{x}_{M'} \succ^*_{|M'|} \mathbf{y}_{M'}$. By way of contradiction, suppose that there does not exist $M'' \subset \mathbb{N}$ such $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M''$. In particular, since then $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$ does not hold, it follows from (a) that there exists $A \supseteq M$ such that $\mathbf{x}_A \sim^*_{|A|} \mathbf{y}_A$. We claim that there exists $B \subset \mathbb{N}$ with $A \cap B = \emptyset$ such that $\mathbf{x}_{A\cup B} \succ^*_{|A|+|B|} \mathbf{y}_{A\cup B}$. That is, the statement: for all $B \subset \mathbb{N}$ with $A \cap B = \emptyset$ we must have $\mathbf{y}_{A\cup B} \succsim^*_{|A|+|B|} \mathbf{x}_{A\cup B}$ is false. This possibility is ruled out since if it were correct, we would obtain $\mathbf{y} \succsim^*_{\mathcal{I}} \mathbf{x}$, which is contradicted by (b).

Since we suppose that there does not exist $M'' \subset \mathbb{N}$ such $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M''$, it does not hold that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq A \cup B$. Hence, by (a) there exists $C \subset \mathbb{N}$ with $(A \cup B) \cap C = \emptyset$ such that $\mathbf{x}_{A\cup B\cup C} \sim^*_{|A|+|B|+|C|} \mathbf{y}_{A\cup B\cup C}$. This leads to the first indifference in (1), while the second strict preference in (1) follows from Lemma 3(i):

$$\mathbf{y}_{A\cup B\cup C} \sim^*_{|A|+|B|+|C|} \mathbf{x}_{A\cup B\cup C} \succ^*_{|A|+|B|+|C|} (\mathbf{y}_{A\cup B}, \mathbf{x}_C)\,. \tag{1}$$

By transitivity we get $(\mathbf{y}_{A\cup B}, \mathbf{y}_C) \succ^*_{|A|+|B|+|C|} (\mathbf{y}_{A\cup B}, \mathbf{x}_C)$. So, $\mathbf{y}_C \succ^*_{|C|} \mathbf{x}_C$. [If $\neg(\mathbf{y}_C \succ^*_{|C|} \mathbf{x}_C)$, then $\mathbf{x}_C \succsim^*_{|C|} \mathbf{y}_C$. By Lemma 3(i), we obtain $(\mathbf{y}_{A\cup B}, \mathbf{x}_C) \succsim^*_{|A|+|B|+|C|} (\mathbf{y}_{A\cup B}, \mathbf{y}_C)$.] We now get:

$$\mathbf{y}_{A\cup C} \succ^*_{|A|+|C|} (\mathbf{y}_A, \mathbf{x}_C) \sim^*_{|A|+|C|} \mathbf{x}_{A\cup C} \succsim^*_{|A|+|C|} \mathbf{y}_{A\cup C}\,, \tag{2}$$

The first strict preference in (2) is a consequence of Lemma 3(i) and $\mathbf{y}_C \succ^*_{|C|} \mathbf{x}_C$. The second indifference in (2) is a consequence of Lemma 3(i) and $\mathbf{x}_A \sim^*_{|A|} \mathbf{y}_A$. The last weak preference in (2) follows from (a) and the fact that $A \cup C \supset M$. So (2) leads us to a contradiction.This completes the proof of the only-if part of (i).

(If part of (i): $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$ if there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Then $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$. By way of contradiction, suppose $\mathbf{y} \succsim^*_{\mathcal{I}} \mathbf{x}$. Then there exists $M' \subset \mathbb{N}$ with $|M'| \geq 2$ such that $\mathbf{y}_N \succsim^*_{|N|} \mathbf{x}_N$ for all $N \supseteq M'$. For $N \supseteq M' \cup M$ we must have $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ and $\mathbf{y}_N \succsim^*_{|N|} \mathbf{x}_N$. This leads to a contradiction. Hence, $\neg(\mathbf{y} \succsim^*_{\mathcal{I}} \mathbf{x})$ and, consequently, $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$.

(Only-if part of (ii): $\mathbf{x} \sim^*_{\mathcal{I}} \mathbf{y}$ only if there exist $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.) Let $\mathbf{x} \sim^*_{\mathcal{I}} \mathbf{y}$. Then there exists sets $M', M'' \subset \mathbb{N}$ such that $\mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M'$ and $\mathbf{y}_N \succsim^*_{|N|} \mathbf{x}_N$ for all $N \supseteq M''$. Then for all $N \supseteq M' \cup M''$ we must have $\mathbf{x}_N \sim^*_{|N|} \mathbf{y}_N$, as was required.

The if part of (ii) follows directly from the definition and we omit the details. ∎

Lemma 5 *The SWR $\succsim^*_{\mathcal{I}}$ satisfies $\mathcal{P}$**I**, **SP** and **ST**.*

Proof. ($\succsim^*_{\mathcal{I}}$ satisfies $\mathcal{P}$**I**.) Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $P \in \mathcal{P}$. Assume $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$. Let $\pi : \mathbb{N} \to \mathbb{N}$ be the equivalent representation of the infinite permutation matrix P. Clearly π is a one-to-one and onto function. Since $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$ there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Let the image of M under the function π be denoted by $\pi(M)$, that is $\pi(M) = \{i \in \mathbb{N} \mid \text{there exists } j \in M \text{ such that } \pi(j) = i\}$. Now for $N \supseteq \pi(M)$, we must have $\pi^{-1}(N) \supseteq M$, where $\pi^{-1} : \mathbb{N} \to \mathbb{N}$ is the inverse of π. Since $\succsim^*_m$ satisfies m-**I** for all $m \geq 2$, we must have for all $N \supseteq \pi(M)$, $(P\mathbf{x})_N \succsim^*_{\mathcal{I}} (P\mathbf{y})_N$. Hence, $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$ implies $P\mathbf{x} \succsim^*_{\mathcal{I}} P\mathbf{y}$ for any $P \in \mathcal{P}$. The converse is established in a similar manner.

($\succsim^*_{\mathcal{I}}$ satisfies **SP**.) Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ satisfy $\mathbf{x} > \mathbf{y}$. Pick $M \subset \mathbb{N}$ such that $\mathbf{x}_M \neq \mathbf{y}_M$. Since $\succsim^*_m$ satisfies **P** for all $m \geq 2$, we must have $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. By

Lemma 4 (i) we can conclude $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$.

($\succsim^*_{\mathcal{I}}$ satisfies **ST**.) Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v} \in \mathbf{X}$ satisfy $x_1 = y_1$, and for all $i \in \mathbb{N}$, $u_i = x_{i+1}$ and $v_i = y_{i+1}$. Assume $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$. Hence, there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such $\mathbf{x}_N \succsim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Construct M' as follows: $M' = \{i \in \mathbb{N} \mid i{+}1 \in M\}$, with an arbitrary element added in if the number of elements in M' would otherwise be 1. Consider any $N' \subseteq M'$, and construct N as follows: $N = \{i \in \mathbb{N} \mid i - 1 \in N'\} \cup \{1\}$. Since, by construction, $N \supseteq M$, $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$. By Lemma 3(i), $\mathbf{x}_{N\backslash\{1\}} \succ^*_{|N|-1} \mathbf{y}_{N\backslash\{1\}}$ since $x_1 = y_1$. Thus, $\mathbf{u}_{N'} \succsim^*_{|N|-1} \mathbf{v}_{N'}$ since $\succsim^*_m$ satisfies m-**I** for all m. Hence, $\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y}$ implies $\mathbf{u} \succsim^*_{\mathcal{I}} \mathbf{v}$. The converse is establish in a similar manner. ■

Proof of Theorem 1. (i) It can be easily checked that $\succsim^*_{\mathcal{I}}$ is reflexive and transitive provided that $\succsim^*_m$ is reflexive and transitive for each m; hence, $\succsim^*_{\mathcal{I}}$ is an SWR on $\mathbf{X}$. The rest of part (i) follows directly from Lemma 2(i) and Lemma 5.

(Only-if part of (ii): An SWR $\succsim$ extends $\succsim^*_2$ and satisfies **IPC** only if $\succsim^*_{\mathcal{I}}$ is a subrelation to $\succsim$.) Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$. If $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$, then using Lemma 4 (i) we must have that there exist $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. For all $N \supseteq M$, since $\succsim$ extends $\succsim^*_2$ and $\{\succsim^*_m\}_{m=2}^{\infty}$ is a proliferating sequence we obtain $(\mathbf{x}_N, \mathbf{y}_{\mathbb{N}\backslash N}) \succ \mathbf{y}$. Now, by **IPC**, we have $\mathbf{x} \succ \mathbf{y}$. If $\mathbf{x} \sim^*_{\mathcal{I}} \mathbf{y}$, then by Lemma 4 (ii) we must have that there exist $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. By Lemma 3 (ii), we have $x_i = y_i$ for all $i \in \mathbb{N}\backslash M$. Since $\succsim$ extends $\succsim^*_2$ and $\{\succsim^*_m\}_{m=2}^{\infty}$ is a proliferating sequence we get $\mathbf{x} \sim \mathbf{y}$.

(If part of (ii): An SWR $\succsim$ extends $\succsim^*_2$ and satisfies **IPC** if $\succsim^*_{\mathcal{I}}$ is a subrelation to $\succsim$.) We omit the straightforward proof of the result that $\succsim$ extends $\succsim^*_2$.

To show that $\succsim$ satisfies **IPC**, assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that, for all $N \supseteq M$, $(\mathbf{x}_N, \mathbf{y}_{\mathbb{N}\backslash N}) \succ \mathbf{y}$. Since $\succsim$ extends $\succsim^*_2$ and $\{\succsim^*_m\}_{m=2}^{\infty}$ is proliferating, it follows from the completeness of the SWO $\succsim^*_m$ for every m that $\mathbf{x}_N \succ^*_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Hence, $\mathbf{x} \succ^*_{\mathcal{I}} \mathbf{y}$ by Lemma 4(i), and $\mathbf{x} \succ \mathbf{y}$ since $\succsim^*_{\mathcal{I}}$ is a subrelation to $\succsim$. This shows that $\succsim$ satisfies condition **IPC**. ■

6 Applications

In this section we study specific criteria based on particular proliferating sequences. In particular, as the utilitarian SWO and the leximin SWO defined for pairs on any subset of the m-dimensional Euclidean space define two proliferating sequences, they lay the foundation for two specializations of the generalized time-invariant overtaking criterion: utilitarian and leximin time-invariant overtaking. Furthermore, we propose methods for determining the asymmetric and symmetric parts of the utilitarian and leximin time-invariant overtaking criteria.

6.1 The Utilitarian Case

To state the definition of the utilitarian SWO defined on Y^m we first introduce some additional notation. For each $N \subset \mathbb{N}$, where by our notational convention N is finite, the partial sum $\sum_{i \in N} x_i$ is written as $\sigma(\mathbf{x}_N)$. Let $\{\succsim_m^U\}_{m=2}^{\infty}$ denote the sequence of utilitarian SWOs, with each $\succsim_m^U$ defined on Y^m. Formally, for all $\mathbf{a}$, $\mathbf{b} \in Y^m$,

$$\mathbf{a} \succsim_m^U \mathbf{b} \text{ iff } \sigma(\mathbf{a}) \geq \sigma(\mathbf{b}) .$$

In order to rely on a standard characterization of utilitarianism, we first state the Translation Scale Invariance axiom for finite population social choice theory.

Axiom m**-TSI** (m-Translation Scale Invariance) For all $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $m \geq 2$, if $\mathbf{a} \succsim_m \mathbf{b}$ and $\boldsymbol{\alpha} \in \mathbb{R}^m$ satisfies $\mathbf{a} + \boldsymbol{\alpha} \in Y^m$ and $\mathbf{b} + \boldsymbol{\alpha} \in Y^m$, then $\mathbf{a} + \boldsymbol{\alpha} \succsim_m \mathbf{b} + \boldsymbol{\alpha}$.

This axiom says that utility differences can be compared interpersonally. A comprehensive treatment of the literature on social choice with interpersonal utility comparisons can be found in Bossert and Weymark (2004). The following characterization of finite-dimensional utilitarianism is well-known.[10]

[10]The argument is due to Milnor (1954) in the context of individual decision under risk. For a proof in the social choice context, see d'Aspremont and Gevers (2002).

Lemma 6 *For all $m \in \mathbb{N}$, the utilitarian SWO $\succsim_m^U$ is equal to $\succsim_m$ iff $\succsim_m$ satisfies **A**, **P** and **TSI**.*

Let $\succsim$ be an SWR defined on $\mathbf{X}$. Consider the following axiom on $\succsim$.

Axiom FTSI (Finite Translation Scale Invariance) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with some subset $N \subset \mathbb{N}$ such that $x_i = y_i$ for all $i \in \mathbb{N} \backslash N$, if $\mathbf{x} \succsim \mathbf{y}$ and $\boldsymbol{\alpha} \in \mathbb{R}^{\mathbb{N}}$ satisfies that $\mathbf{x} + \boldsymbol{\alpha} \in \mathbf{X}$ and $\mathbf{y} + \boldsymbol{\alpha} \in \mathbf{X}$ and $\alpha_i = 0$ for all $i \in \mathbb{N} \backslash N$, then $\mathbf{x} + \boldsymbol{\alpha} \succsim \mathbf{y} + \boldsymbol{\alpha}$.

By means of this axiom we can characterize the class of SWRs extending $\succsim_2^U$:

Proposition 1 *Let $\{\succsim_m^U\}_{m=2}^{\infty}$ be the utilitarian sequence of SWOs for each $m \geq 2$. Then:*

*(i) If $\succsim$ is an SWR on X that extends $\succsim_2^U$, then $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **FTSI**.*

*(ii) If $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **FTSI**, then $\succsim$ is an SWR on X that extends $\succsim_m^U$ for every $m \geq 2$.*

Proof of Proposition 1. (Proof of (i): $\succsim$ is an SWR on X that extends $\succsim_2^U$ only if $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **FTSI**.) Assume $\succsim$ is an SWR on X that extends $\succsim_2^U$. It follows from Lemma 2 that $\succsim$ satisfies $\mathcal{F}$**A** and **FP**. To show that $\succsim$ satisfies **FTSI**, consider $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ for which there exists some subset $N \subset \mathbb{N}$ such that $x_i = y_i$ for all $i \in \mathbb{N} \backslash N$, and $\boldsymbol{\alpha} \in \mathbb{R}^{\mathbb{N}}$ which satisfies $\mathbf{x} + \boldsymbol{\alpha} \in \mathbf{X}$ and $\mathbf{y} + \boldsymbol{\alpha} \in \mathbf{X}$ and $\alpha_k = 0$ for all $i \in \mathbb{N} \backslash N$. Since $\succsim$ extends $\succsim_2^U$ and satisfies **FP**, it follows from Lemma 8 of the appendix that $\mathbf{x} \succsim \mathbf{y}$ iff $\sigma(\mathbf{x}_N) \geq \sigma(\mathbf{y}_N)$ and $\mathbf{x} + \boldsymbol{\alpha} \succsim \mathbf{y} + \boldsymbol{\alpha}$ iff $\sigma(\mathbf{x}_N + \boldsymbol{\alpha}_N) \geq \sigma(\mathbf{y}_N + \boldsymbol{\alpha}_N)$. Clearly, $\sigma(\mathbf{x}_N) \geq \sigma(\mathbf{y}_N)$ implies $\sigma(\mathbf{x}_N + \boldsymbol{\alpha}_N) \geq \sigma(\mathbf{y}_N + \boldsymbol{\alpha}_N)$, thereby establishing that $\succsim$ satisfies **FTSI**.

(Proof (ii): $\succsim$ is an SWR on X that extends $\succsim_m^U$ if $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **FTSI**.) Assume that $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **FTSI**. Fix $\mathbf{z} \in \mathbf{X}$ and $M \in \mathbb{N}$ with $|M| = m \geq 2$. Construct $\succsim_m^{\mathbf{z}}$ as follows: $\mathbf{x}_M \succsim_m^{\mathbf{z}} \mathbf{y}_M$ iff $(\mathbf{x}_M, \mathbf{z}_{\mathbb{N} \backslash M}) \succsim (\mathbf{y}_M, \mathbf{z}_{\mathbb{N} \backslash M})$. Since $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **FTSI**, it follows that $\succsim_m^{\mathbf{z}}$ satisfies **A**, **P** and **TSI**.

Thus, by Lemma 6, $\succsim_m^U$ is equal to $\succsim_m^{\mathbf{z}}$. Since $\mathbf{z} \in \mathbf{X}$ and $M \in \mathbb{N}$ with $|M| = m$ are arbitrarily chosen, it follows that $\succsim$ extends $\succsim_m^U$. ∎

Proposition 1 implies the following result, which makes Theorem 1 applicable in the utilitarian case.

Proposition 2 *The sequence of utilitarian SWOs, $\{\succsim_m^U\}_{m=2}^{\infty}$, is proliferating.*

Proposition 2 is established by d'Aspremont (2007, Lemma 4) in the case where $Y = \mathbb{R}$. In the appendix we provide a direct proof of Proposition 2 in the present case where $Y \subseteq \mathbb{R}$ is an interval satisfying $[0, 1] \subseteq Y$.

Since, by Proposition 2, $\{\succsim_m^U\}_{m=2}^{\infty}$ is proliferating, we can now state the following specialization of generalized time-invariant overtaking.

Definition 3 (Utilitarian time-invariant overtaking) The *utilitarian time-invariant overtaking criterion* $\succsim_{\mathcal{I}}^U$ satisfies, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$\mathbf{x} \succsim_{\mathcal{I}}^U \mathbf{y}$ iff there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\sigma(\mathbf{x}_N) \geq \sigma(\mathbf{y}_N)$ for all $N \supseteq M$.

By Propositions 1 and 2, the following characterization of utilitarian time-invariant overtaking is a direct consequence of Theorem 1 and Lemma 4:

Corollary 1 *(i) $\succsim_{\mathcal{I}}^U$ is an SWR that satisfies $\mathcal{P}$**I**, **SP** and **ST**.*

*(ii) An SWR $\succsim$ satisfies $\mathcal{F}$**A**, **FP**, **FTSI** and **IPC** iff $\succsim_{\mathcal{I}}^U$ is a subrelation to $\succsim$.*

(iii) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succ_{\mathcal{I}}^U \mathbf{y}$ iff there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\sigma(\mathbf{x}_N) > \sigma(\mathbf{y}_N)$ for all $N \supseteq M$.

(iv) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \sim_{\mathcal{I}}^U \mathbf{y}$ iff there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\sigma(\mathbf{x}_N) = \sigma(\mathbf{y}_N)$ for all $N \supseteq M$.

To facilitate its use, we provide a characterization of the asymmetric and symmetric parts of the utilitarian generalized overtaking criterion.

Proposition 3 *Utilitarian time-invariant overtaking satisfies:*

(i) For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succ_{\mathcal{I}}^{U} \mathbf{y}$ *iff there exists* $M^+ \subseteq \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ *such that* $\sigma(\mathbf{x}_{M^+ \cup M^-}) > \sigma(\mathbf{y}_{M^+ \cup M^-})$ *for all* $M^- \subseteq \{i \in \mathbb{N} \mid x_i - y_i < 0\}$.

(ii) For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \sim_{\mathcal{I}}^{U} \mathbf{y}$ *iff* $M^+ := \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ *and* $M^- := \{i \in \mathbb{N} \mid x_i - y_i < 0\}$ *are finite sets satisfying* $\sigma(\mathbf{x}_{M^+ \cup M^-}) = \sigma(\mathbf{y}_{M^+ \cup M^-})$.

Proof. (If part of (i).) Assume that there exists $M^+ \subseteq \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ such that $\sigma(\mathbf{x}_{M^+ \cup M^-}) > \sigma(\mathbf{y}_{M^+ \cup M^-})$ for all $M^- \subseteq \{i \in \mathbb{N} \mid x_i - y_i < 0\}$. Let $M = M^+$ and choose $N \supseteq M$. We can partition N into $A := \{i \in N \mid x_i - y_i \geq 0\}$ and $M^- := \{i \in N \mid x_i - y_i < 0\}$, implying that $x_i - y_i \geq 0$ for all $A \backslash M^+$. Hence,

$$\sigma(\mathbf{x}_N) - \sigma(\mathbf{y}_N) = \sigma(\mathbf{x}_{A \cup M^-}) - \sigma(\mathbf{y}_{A \cup M^-}) \geq \sigma(\mathbf{x}_{M^+ \cup M^-}) - \sigma(\mathbf{y}_{M^+ \cup M^-}) > 0\,,$$

where the partitioning of N into A and M^- implies the first equality, $x_i - y_i \geq 0$ for all $A \backslash M^+$ implies the second weak inequality, and the premise implies the third strong inequality.

(Only-if part of (i).) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\sigma(\mathbf{x}_N) > \sigma(\mathbf{y}_N)$ for all $N \supseteq M$. Let $M^+ := M \cap \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ and choose $M^- \subseteq \{i \in \mathbb{N} \mid x_i - y_i < 0\}$. Note that $x_i \leq y_i$ for all $i \in M \backslash (M^+ \cap M^-)$. Hence,

$$\sigma(\mathbf{x}_{M^+ \cup M^-}) - \sigma(\mathbf{y}_{M^+ \cup M^-}) \geq \sigma(\mathbf{x}_{M \cup M^-}) - \sigma(\mathbf{y}_{M \cup M^-}) > 0$$

by the premise since $M \cup M^- \supseteq M$.

(If part of Part (ii).) Assume that $M^+ := \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ and $M^- := \{i \in \mathbb{N} \mid x_i - y_i < 0\}$ are finite sets satisfying $\sigma(\mathbf{x}_{M^+ \cup M^-}) = \sigma(\mathbf{y}_{M^+ \cup M^-})$. Let $M = M^+ \cup M^-$ and choose $N \supseteq M$. Since $x_i = y_i$ for all $i \in \mathbb{N} \backslash M$, it follows that

$$\sigma(\mathbf{x}_N) - \sigma(\mathbf{y}_N) = \sigma(\mathbf{x}_M) - \sigma(\mathbf{y}_M) = \sigma(\mathbf{x}_{M^+ \cup M^-}) - \sigma(\mathbf{y}_{M^+ \cup M^-}) = 0$$

by the premise.

(Only-if part of (ii).) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\sigma(\mathbf{x}_N) = \sigma(\mathbf{y}_N)$ for all $N \supseteq M$. By Lemma 3(ii) and the fact that $\{\succsim^U_m\}^\infty_{t=2}$ is proliferating, it follows that $x_i = y_i$ for all $i \in \mathbb{N}\backslash M$. Hence, $M^+ := \{i \in \mathbb{N} \mid x_i - y_i > 0\}$ and $M^- := \{i \in \mathbb{N} \mid x_i - y_i < 0\}$ are finite sets satisfying $\sigma(\mathbf{x}_{M^+ \cup M^-}) = \sigma(\mathbf{y}_{M^+ \cup M^-})$.

The if parts can easily be amended to ensure that $|M| \geq 2$. ■

This characterization can be illustrated by the $(\mathbf{u}, \mathbf{v})$ example of Section 1. In this example $\{i \in \mathbb{N} \mid u_i - v_i > 0\} = \{1\}$ and $\{i \in \mathbb{N} \mid u_i - v_i < 0\} = \mathbb{N}\backslash\{1\}$. By choosing $M^+ = \{1\}$ so that $\sigma(\mathbf{u}_{M^+}) - \sigma(\mathbf{v}_{M^+}) = 2$, and noting $\sigma(\mathbf{u}_{M^-}) - \sigma(\mathbf{v}_{M^-}) < 1$ for all $M^- \subset \mathbb{N}\backslash\{1\}$, it follows from Proposition 3(i) that $\mathbf{u} \succ^U_{\mathcal{I}} \mathbf{v}$.

The utilitarian criterion proposed by Basu and Mitra (2007a), which we discussed in Section 1 and denote $\succsim^U_{\mathcal{F}}$ (cf. the notation of Section 4), yields comparability only if there is equality or Pareto-dominance on a cofinite set:

$$\mathbf{x} \succsim^U_{\mathcal{F}} \mathbf{y} \text{ iff there exists } N \subset \mathbb{N} \text{ such that } \sigma(\mathbf{x}_N) \geq \sigma(\mathbf{y}_N) \text{ and } \mathbf{x}_{\mathbb{N}\backslash N} \geq \mathbf{y}_{\mathbb{N}\backslash N}\,.$$

It follows from Proposition 3 that $\succsim^U_{\mathcal{F}}$ is a subrelation to $\succsim^U_{\mathcal{I}}$, since the symmetric parts, $\sim^U_{\mathcal{I}}$ and $\sim^U_{\mathcal{F}}$, coincide, while $\succ^U_{\mathcal{I}}$ strictly expands $\succ^U_{\mathcal{F}}$, as illustrated by the $(\mathbf{u}, \mathbf{v})$ example of Section 1.

6.2 The Leximin Case

To state a precise definition of the leximin order we introduce additional notation. For any $\mathbf{x}_M$, $(x_{(1)}, \dots, x_{(|M|)})$ denotes the rank-ordered permutation of $\mathbf{x}_M$ such that $x_{(1)} \leq \cdots \leq x_{(|M|)}$, ties being broken arbitrarily. For all $\mathbf{x}_M$ and $\mathbf{y}_M$, $\mathbf{x}_M \succ^L_{|M|} \mathbf{y}_M$ iff there exists $m \in \{1, \dots, |M|\}$ such that $x_{(k)} = y_{(k)}$ for all $k \in \{1, \dots, m-1\}$ and $x_{(m)} > y_{(m)}$ and $\mathbf{x}_M \sim^L_{|M|} \mathbf{y}_M$ iff $x_{(k)} = y_{(k)}$ for all $k \in \{1, \dots, |M|\}$.

We first recall through Lemma 7 below a standard characterization of finite-dimensional leximin using the Hammond (1976) Equity axiom. This axiom states, in our intergenerational context, that if there is a conflict between two generations, with every other generation being as well off in the compared profiles, then society should weakly prefer the profile where the least favored generation is better off.

Axiom m**-HE** (m-Hammond Equity) For all $\mathbf{a}$, $\mathbf{b} \in Y^m$ with $m \geq 2$, if there exist i, $j \in \{1, \ldots, m\}$ such that $b_i > a_i > a_j > b_j$ and $a_k = b_k$ for all $k \neq i$, j, then $\mathbf{a} \succsim_m \mathbf{b}$.

Lemma 7 *For all $m \in \mathbb{N}$, the leximin SWO $\succsim_m^L$ is equal to $\succsim_m$ iff $\succsim_m$ satisfies **A**, **P** and **HE**.*

Let $\succsim$ be an SWR defined on $\mathbf{X}$. Consider also the **HE** axiom on $\succsim$.

Axiom HE (Hammond Equity) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if there exist i, $j \in \mathbb{N}$ such that $y_i > x_i > x_j > y_j$ and $x_k = y_k$ for all $k \neq i$, j, then $\mathbf{x} \succsim \mathbf{y}$.

By means of this axiom we can characterize the class of SWRs extending $\succsim_2^L$:

Proposition 4 *Let $\{\succsim_m^L\}_{m=2}^{\infty}$ be the leximin sequence of SWOs for each $m \geq 2$. Then:*

*(i) If $\succsim$ is an SWR on X that extends $\succsim_2^L$, then $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE.***

*(ii) If $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**, then $\succsim$ is an SWR on X that extends $\succsim_m^L$ for every $m \geq 2$.*

Proof. (Proof of (i): $\succsim$ is an SWR on X that extends $\succsim_2^L$ only if $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**.) Assume $\succsim$ is an SWR on X that extends $\succsim_2^L$. It follows from Lemma 2 that $\succsim$ satisfies $\mathcal{F}$**A** and **FP**. To show that $\succsim$ satisfies **HE**, let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ satisfy that there exist i, $j \in \mathbb{N}$ such that $y_i > x_i > x_j > y_j$ and $x_k = y_k$ for all $k \neq i$, j. Then $\mathbf{x}_{\{i,j\}} \succsim_2^L \mathbf{y}_{\{i,j\}}$ (since $\succsim_2^L$ satisfies 2-**HE**) and $\mathbf{x} \succsim \mathbf{y}$ (since $\succsim$ extends $\succsim_2^L$). This establishes that $\succsim$ satisfies **HE**.

(Proof of (ii): $\succsim$ is an SWR on X that extends $\succsim_m^L$ if $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**.) Assume that $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**. Fix $\mathbf{z} \in \mathbf{X}$ and $M \in \mathbb{N}$ with

$|M| = m \geq 2$. Construct $\succsim_m^{\mathbf{z}}$ as follows: $\mathbf{x}_M \succsim_m^{\mathbf{z}} \mathbf{y}_M$ iff $(\mathbf{x}_M, \mathbf{z}_{\mathbb{N}\setminus M}) \succsim (\mathbf{y}_M, \mathbf{z}_{\mathbb{N}\setminus M})$. Since $\succsim$ satisfies $\mathcal{F}$**A**, **FP** and **HE**, it follows that $\succsim_m^{\mathbf{z}}$ satisfies **A**, **P** and m-**HE**. Thus, by Lemma 7, $\succsim_m^L$ is equal to $\succsim_m^{\mathbf{z}}$. Since $\mathbf{z} \in \mathbf{X}$ and $M \in \mathbb{N}$ with $|M| = m$ are arbitrarily chosen, it follows that $\succsim$ extends $\succsim_m^L$. ■

Proposition 4 implies the following result, which makes Theorem 1 applicable in the utilitarian case.

Proposition 5 *The sequence of leximin SWOs, $\{\succsim_m^L\}_{m=2}^{\infty}$, is proliferating.*

d'Aspremont (2007, Lemma 5) proves Proposition 5 through a direct argument which is applicable also to the present case where $Y \subseteq \mathbb{R}$ is an interval satisfying $[0,1] \subseteq Y$.

Since, by Proposition 5, $\{\succsim_m^L\}_{m=2}^{\infty}$ is proliferating, we can now state the following specialization of generalized time-invariant overtaking.

Definition 4 (Leximin time-invariant overtaking) The *leximin time-invariant overtaking criterion* $\succsim_{\mathcal{I}}^L$ satisfies, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim_{\mathcal{I}}^L \mathbf{y} \text{ iff there exists } M \subset \mathbb{N} \text{ with } |M| \geq 2 \text{ such that } \mathbf{x}_N \succsim_{|N|}^L \mathbf{y}_N \text{ for all } N \supseteq M.$$

By Propositions 4 and 5, the following characterization of leximin time-invariant overtaking is a direct consequence of Theorem 1 and Lemma 4:

Corollary 2 *(i) $\succsim_{\mathcal{I}}^L$ is an SWR that satisfies $\mathcal{P}$**I**, **SP** and **ST**.*

*(ii) An SWR $\succsim$ satisfies $\mathcal{F}$**A**, **FP**, **HE** and **IPC** iff $\succsim_{\mathcal{I}}^L$ is a subrelation to $\succsim$.*

(iii) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succ_{\mathcal{I}}^L \mathbf{y}$ iff there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ_{|N|}^L \mathbf{y}_N$ for all $N \supseteq M$.

(iv) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \sim_{\mathcal{I}}^L \mathbf{y}$ iff there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim_{|N|}^L \mathbf{y}_N$ for all $N \supseteq M$.

We provide a characterization of the asymmetric and symmetric parts of the leximin generalized overtaking criterion. For this purpose, we need some additional notation. Let $\mathcal{N}$ be the class of all cofinite subsets of $\mathbb{N}$. We denote the set of all utility streams defined on some element of $\mathcal{N}$ and taking values in Y by $\mathbf{X}^c$. Since a utility stream can be viewed as a function from the domain of generations to the set Y, we can formally write $\mathbf{X}^c := \{\mathbf{x} : \mathbb{N}^{\mathbf{x}} \to Y \mid \mathbb{N}^{\mathbf{x}} \in \mathcal{N}\}$. Observe that for $\mathbf{x} \in \mathbf{X}^c$, we denote that cofinite subset of $\mathbb{N}$ which is the domain of $\mathbf{x}$ by $\mathbb{N}^{\mathbf{x}}$.

For any $\mathbf{x} \in \mathbf{X}^c$, write $\mathbb{N}^{\mathbf{x}}_{\min} := \{i \in \mathbb{N}^{\mathbf{x}} \mid x_i = \inf_{j \in \mathbb{N}^{\mathbf{x}}} x_j\}$. Say that $\mathbf{x} \in \mathbf{X}^c$ and $\mathbf{y} \in \mathbf{X}^c$ have *the same minimum and the same number of minimal elements* if $\inf_{j \in \mathbb{N}^{\mathbf{x}}} x_j = \inf_{j \in \mathbb{N}^{\mathbf{y}}} y_j$ and $0 < |\mathbb{N}^{\mathbf{x}}_{\min}| = |\mathbb{N}^{\mathbf{y}}_{\min}| < \infty$.

Define the operator $R : (\mathbf{X}^c)^2 \to (\mathbf{X}^c)^2$ as follows, where $\mathbf{x}'$ denotes the restriction of $\mathbf{x}$ to $\mathbb{N}^{\mathbf{x}} \backslash \mathbb{N}^{\mathbf{x}}_{\min}$ and $\mathbf{y}'$ is restriction of $\mathbf{y}$ to $\mathbb{N}^{\mathbf{y}} \backslash \mathbb{N}^{\mathbf{y}}_{\min}$ if $\mathbf{x} \in \mathbf{X}^c$ and $\mathbf{y} \in \mathbf{X}^c$ satisfy that $|\mathbb{N}^{\mathbf{x}}_{\min}|$ and $|\mathbb{N}^{\mathbf{y}}_{\min}|$ are positive and finite:

$$R(\mathbf{x}, \mathbf{y}) = \begin{cases} (\mathbf{x}', \mathbf{y}') & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ have the same minimum and} \\ & \text{the same number of minimal elements,} \\ (\mathbf{x}, \mathbf{y}) & \text{otherwise.} \end{cases}$$

Write $R^0(\mathbf{x}, \mathbf{y}) := (\mathbf{x}, \mathbf{y})$ and, for $n \in \mathbb{N}$, $R^n(\mathbf{x}, \mathbf{y}) := R(R^{n-1}(\mathbf{x}, \mathbf{y}))$.

Proposition 6 *Leximin time-invariant overtaking satisfies:*

(i) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succ^L_{\mathcal{I}} \mathbf{y}$ *iff*

(a) there is $P \in \mathcal{F}$ *such that* $P\mathbf{x} > \mathbf{y}$, *or*

(b) there exists m *such that* $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ *for all* $n \geq m$ *and one of following is true:*

$$\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j > \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$$
$$\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j \quad \textit{and} \quad 0 \leq |\mathbb{N}^{\mathbf{x}'}_{\min}| < |\mathbb{N}^{\mathbf{y}'}_{\min}| \leq \infty.$$

(ii) For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \sim^L_{\mathcal{I}} \mathbf{y}$ *iff there is* $P \in \mathcal{F}$ *such that* $P\mathbf{x} = \mathbf{y}$.

Proof. Write $(\mathbf{x}^n, \mathbf{y}^n) = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq 0$.

(If part of (i).) First assume that there is $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$. By the definition of $\succsim^L_{|M|}$, there exists $M \subset \mathbb{N}$ such that $\mathbf{x}_M \succ^L_{|M|} \mathbf{y}_M$ and $x_i \geq y_i$ for all $i \in \mathbb{N} \backslash M$. Hence, $\mathbf{x}_N \succ^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.

Then assume that there exists m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$. Let m be the smallest such integer. Then, for all $k \in \{0, \ldots, m-1\}$, $\mathbf{x}^k$ and $\mathbf{y}^k$ have the same minimum and the same number of minimal elements. Write

$$M^{\mathbf{y}} := \bigcup_{k \in \{0,\ldots,m-1\}} \mathbb{N}^{\mathbf{y}^k}_{\min} .$$

If $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j > \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$, choose $i' \in \mathbb{N}^{\mathbf{y}'}$ so that $y'_{i'} < \inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j$. Let $M = M^{\mathbf{y}} \cup \{i'\}$. Then $\mathbf{x}_N \succ^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. If $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$ and $0 \leq |\mathbb{N}^{\mathbf{x}'}_{\min}| < |\mathbb{N}^{\mathbf{y}'}_{\min}| \leq \infty$, let $N^{\mathbf{y}'}$ be a subset of $\mathbb{N}^{\mathbf{y}'}_{\min}$ with a larger number of elements than $\mathbb{N}^{\mathbf{x}'}_{\min}$. Let $M = M^{\mathbf{y}} \cup N^{\mathbf{y}'}$. Then $\mathbf{x}_N \succ^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.

(Only-if part of (i).) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \succ^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. Suppose that (a) and (b) are not true. We must show that, for all $M \subset \mathbb{N}$ with $|M| \geq 2$, there exists $N \supseteq M$ such that $\mathbf{x}_N \precsim^L_{|N|} \mathbf{y}_N$.

Suppose there is no $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$, and there exists no m such that $(\mathbf{x}', \mathbf{y}'^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$. Then, for all $n \geq 0$, $\mathbf{x}^n$ and $\mathbf{y}^n$ have the same minimum and the same number of minimal elements, and $\bigcup_{n \geq 0} \mathbb{N}^{\mathbf{y}^n}_{\min}$ is an infinite set. For any $M \subset \mathbb{N}$, one can choose $N \supseteq M$ such that N contains at least as many $\mathbb{N}^{\mathbf{x}^n}_{\min}$ elements as $\mathbb{N}^{\mathbf{y}^n}_{\min}$ elements for any $n \geq 0$, and more for some n'. Then $\mathbf{x}_N \prec^L_{|N|} \mathbf{y}_N$.

Suppose there is no $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$ and that, even though there exists m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$ and $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$, we have that $|\mathbb{N}^{\mathbf{x}'}_{\min}| = |\mathbb{N}^{\mathbf{y}'}_{\min}| = \infty$. Let m be the smallest such integer. Independently of how $M^{\mathbf{y}}$ is complemented to form $M \subset \mathbb{N}$, one can always choose $N \supseteq M$ such that N in addition to including $\bigcup_{k \in \{0,\ldots,m-1\}} \mathbb{N}^{\mathbf{x}^k}_{\min}$ contains more $\mathbb{N}^{\mathbf{x}'}_{\min}$ elements than $\mathbb{N}^{\mathbf{y}'}_{\min}$ elements. Then $\mathbf{x}_N \prec^L_{|N|} \mathbf{y}_N$.

Suppose there is no $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$ and that, even though there exists

m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$ and $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$, we have that $|\mathbb{N}^{\mathbf{x}'}_{\min}| = |\mathbb{N}^{\mathbf{y}'}_{\min}| = 0$. Let m be the smallest such integer. Independently of how $M^{\mathbf{y}}$ is complemented to form $M \subset \mathbb{N}$, one can always choose $N \supseteq M$ such that N in addition to including $\bigcup_{k \in \{0,\dots,m-1\}} \mathbb{N}^{\mathbf{x}^k}_{\min}$ contains $i' \in \mathbb{N}^{\mathbf{x}'}$ so that $x'_{i'} < \min_{j \in N \cap \mathbb{N}^{\mathbf{y}'}} y'_j$. Then $\mathbf{x}_N \prec^L_{|N|} \mathbf{y}_N$.

Suppose that, even though there exists m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$, we have that (1) $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j < \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$ or (2) $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$ and $\infty \geq |\mathbb{N}^{\mathbf{x}'}_{\min}| > |\mathbb{N}^{\mathbf{y}'}_{\min}| \geq 0$. Then there is no $P \in \mathcal{F}$ such that $P\mathbf{x} > \mathbf{y}$, and it follows from the if-part above that $\mathbf{x} \prec^L_{\mathcal{I}} \mathbf{y}$.

(If part of (ii).) Assume that there is $P \in \mathcal{F}$ such that $P\mathbf{x} = \mathbf{y}$. By the definition of $\succsim^L_{|M|}$, there exists $M \subset \mathbb{N}$ such that $\mathbf{x}_M \sim^L_{|M|} \mathbf{y}_M$ and $x_i = y_i$ for all $i \in \mathbb{N} \backslash M$. Hence, $\mathbf{x}_N \sim^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$.

(Only-if part (ii).) Assume that there exists $M \subset \mathbb{N}$ with $|M| \geq 2$ such that $\mathbf{x}_N \sim^L_{|N|} \mathbf{y}_N$ for all $N \supseteq M$. By Lemma 3(ii) and the fact that $\{\succsim^L_m\}^{\infty}_{t=2}$ is proliferating, it follows that $x_i = y_i$ for all $i \in \mathbb{N} \backslash M$. It now follows from the definition of $\succsim^L_{|M|}$ that there is $P \in \mathcal{F}$ such that $P\mathbf{x} = \mathbf{y}$.

The if parts can easily be amended to ensure that $|M| \geq 2$. ■

This characterization can be illustrated by the $(\mathbf{u}, \mathbf{v})$ example of Section 1. In this example $\mathbb{N}^{\mathbf{u}} = \mathbb{N}^{\mathbf{v}} = \mathbb{N}$ and $\inf_{j \in \mathbb{N}} u_j > \inf_{j \in \mathbb{N}} v_j$ so that $\mathbf{u}$ and $\mathbf{v}$ do not have the same minimum, implying that $(\mathbf{u}, \mathbf{v}) = R^n(\mathbf{u}, \mathbf{v})$ for all $n \geq 1$. By Proposition 6(i)(b) it follows that $\mathbf{u} \succ^L_{\mathcal{I}} \mathbf{v}$.

To illustrate part (i) of Proposition 6 further, we also consider the comparison of $\mathbf{v}$ of Section 1 to

$$\mathbf{w} \quad : \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \dots \quad 0 \quad 0 \quad \dots$$

Then $\mathbf{v}$ and $\mathbf{w}$ have the same minimum and the same number of minimal element, implying that $(\mathbf{v}', \mathbf{w}') = R(\mathbf{v}, \mathbf{w})$ with $\mathbf{v}'$ and $\mathbf{w}'$ being the restrictions of $\mathbf{v}$ and $\mathbf{w}$ to $\mathbb{N} \backslash \{1\}$. Furthermore, $\inf_{j \in \mathbb{N} \backslash \{1\}} v'_j = \inf_{j \in \mathbb{N} \backslash \{1\}} w'_j = 0$ and $0 = |\mathbb{N}^{\mathbf{v}'}_{\min}| < |\mathbb{N}^{\mathbf{w}'}_{\min}| = \infty$.

This entails that $(\mathbf{v}', \mathbf{w}') = R^n(\mathbf{v}, \mathbf{w})$ for all $n \geq 1$. By Proposition 6(i)(b) it follows that $\mathbf{v} \succ^L_{\mathcal{I}} \mathbf{w}$.

The leximin criterion proposed by Bossert, Sprumont and Suzumura (2007), which we discussed in Section 1 and denote $\succsim^L_{\mathcal{F}}$ (cf. the notation of Section 4), yields comparability only if there is equality or Pareto-dominance on a cofinite set:

$$\mathbf{x} \succsim^L_{\mathcal{F}} \mathbf{y} \text{ iff there exists } N \subset \mathbb{N} \text{ such that } \mathbf{x}_N \succsim^L_{|N|} \mathbf{y}_N \text{ and } \mathbf{x}_{\mathbb{N}\setminus N} \geq \mathbf{y}_{\mathbb{N}\setminus N}\,.$$

It follows from Proposition 6 that $\succsim^L_{\mathcal{F}}$ is a subrelation to $\succsim^L_{\mathcal{I}}$, since the symmetric parts, $\sim^L_{\mathcal{I}}$ and $\sim^L_{\mathcal{F}}$, coincide, while $\succ^L_{\mathcal{I}}$ strictly expands $\succ^L_{\mathcal{F}}$, as illustrated by the $(\mathbf{u}, \mathbf{v})$ example of Section 1.

7 Discussion

We have defined the generalized time-invariant overtaking criterion $\succsim^*_{\mathcal{I}}$. This criterion can be specialized in various cases, corresponding to different moral values theories, as long as these theories are specified by a proliferating sequence of Paretian SWOs. In the utilitarian and leximin cases it leads to $\succsim^U_{\mathcal{I}}$ and $\succsim^L_{\mathcal{I}}$. We have shown that through $\succsim^U_{\mathcal{I}}$ and $\succsim^L_{\mathcal{I}}$ we can expand the asymmetric parts of the utilitarian and leximin criteria suggested by Basu and Mitra (2007a) and Bossert, Sprumont and Suzumura (2007), $\succsim^U_{\mathcal{F}}$ and $\succsim^L_{\mathcal{F}}$ respectively, without compromising desirable properties like Stationarity (**ST**) and Strong Relative Anonymity ($\mathcal{P}$**I**).

When evaluating the merit of this exercise one should keep in mind that it is the expansion of the asymmetric part that matters if one seeks to reduce the set of maximal elements for a given class of feasible infinite utility streams. In this section we analyze whether further expansions of the asymmetric part are compatible with $\mathcal{P}$**I**, before discussing the earlier and related contributions by Vallentyne and Kagan (1997) and Lauwers and Vallentyne (2004) for finitely additive moral value theories.

Fix a proliferating sequence of Paretian SWOs, $\{\succsim^*_m\}^\infty_{m=2}$. Note that Lemmas 3(ii) and 4(ii) imply that the symmetric part of $\succsim^*_{\mathcal{I}}$ coincides with the symmetric parts of $\succsim^*_{\mathcal{F}}$ and $\succsim^*$. Since $\succsim^*$ is a subrelation to any SWR extending $\succsim^*_m$ for every $m \geq 2$, it follows that the asymmetric part of $\succsim^*_{\mathcal{I}}$ cannot be expanded at the expense of its symmetric part. The asymmetric part of $\succsim^*_{\mathcal{I}}$ can only be expanded by making comparable pairs of utility streams which $\succsim^*_{\mathcal{I}}$ does not rank. The following proposition characterizes the pairs of utility streams that $\succsim^*_{\mathcal{I}}$ does not rank.

Proposition 7 *Let $\{\succsim^*_m\}^\infty_{m=2}$ be a proliferating sequence of SWOs with, for each $m \geq 2$, $\succsim^*_m$ satisfying axiom **P**. Then, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, the following two statements are equivalent:*

(1) $\succsim^*_{\mathcal{I}}$ deems $\mathbf{x}$ and $\mathbf{y}$ as incomparable, i.e., $\neg(\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y})$ and $\neg(\mathbf{x} \precsim^*_{\mathcal{I}} \mathbf{y})$.

(2) (i) There exists $P^+ \in \mathcal{P}$ such that $\mathbf{x}^+ = P^+\mathbf{x}$ and $\mathbf{y}^+ = P^+\mathbf{y}$ satisfy that $\mathbf{x}^+_{\{1,\ldots,n\}} \succ^*_n \mathbf{y}^+_{\{1,\ldots,n\}}$ for all $n \in \mathbb{N}$.

(ii) There exists $P^- \in \mathcal{P}$ such that $\mathbf{x}^- = P^-\mathbf{x}$ and $\mathbf{y}^- = P^-\mathbf{y}$ satisfy that $\mathbf{x}^-_{\{1,\ldots,n\}} \prec^*_n \mathbf{y}^-_{\{1,\ldots,n\}}$ for all $n \in \mathbb{N}$.

Proof. *(1) implies (2).* Let $\{\succsim^*_m\}^\infty_{m=2}$ be a proliferating sequence of Paretian SWOs. Assume that $\neg(\mathbf{x} \succsim^*_{\mathcal{I}} \mathbf{y})$ and $\neg(\mathbf{x} \precsim^*_{\mathcal{I}} \mathbf{y})$. Since, for each $m \in \mathbb{N}$, $\succsim^*_m$ is complete, by Definition 2 it is a fact that, for all $M \subset \mathbb{N}$ with $|M| \geq 2$, there exist N', $N'' \supseteq M$ such that $\mathbf{x}_{N'} \succ^*_{|N'|} \mathbf{y}_{N'}$ and $\mathbf{x}_{N''} \prec^*_{|N''|} \mathbf{y}_{N''}$.

Part (i). By this fact, a sequence $\langle N_i \rangle_{i\in\mathbb{N}}$ can be constructed inductively as follows: Let $m_1 = 2$. For $i \in \mathbb{N}$, let $N_i \supseteq \{1, \ldots, m_i\}$ and $m_{i+1} \in \mathbb{N}$ satisfy that $\mathbf{x}_{N_i} \succ^*_{|N_i|} \mathbf{y}_{N_i}$ and $\{1, \ldots, m_{i+1}\} \supsetneq N_i$. Clearly, $\langle N_i \rangle_{i\in\mathbb{N}}$ satisfies, for all $i \in \mathbb{N}$, $\emptyset \neq \{1, \ldots, m_i\} \subseteq N_i \subsetneq \{1, \ldots, m_{i+1}\} \subseteq N_{i+1} \subset \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} N_i = \mathbb{N}$. Let $M_1 = N_1$ and, for $i \geq 2$, $M_i = N_i \setminus N_{i-1}$, implying that $\{M_1, M_2, \ldots, M_i, \ldots\}$ is a partition of $\mathbb{N}$. Write $n_0 = 0$ and, for all $i \in \mathbb{N}$, $n_i = |N_i|$. Since, for all $i \in \mathbb{N}$, $|M_i| = n_i - n_{i-1}$, and $\{\{n_0 + 1, \ldots, n_1\}, \{n_1 + 1, \ldots, n_2\}, \ldots, \{n_{i-1} + 1, \ldots, n_i\}, \ldots\}$ is a partition of $\mathbb{N}$, we can construct $P^+ \in \mathcal{P}$ as follows, writing $\mathbf{x}^+ = P^+\mathbf{x}$ and $\mathbf{y}^+ = P^+\mathbf{y}$: For all

$i \in \mathbb{N}$, elements in M_i are mapped onto $\{n_{i-1}+1, \ldots, n_i\}$ such that

$$x^+_{n_{i-1}+1} - y^+_{n_{i-1}+1} \geq x^+_{n_{i-1}+2} - y^+_{n_{i-1}+2} \geq \cdots \geq x^+_{n_i-1} - y^+_{n_i-1} \geq x^+_{n_i} - y^+_{n_i}\,.$$

We must establish that, for each $i \in \mathbb{N}$, $\mathbf{x}^+_{\{1,\ldots,m\}} \succ^*_m \mathbf{y}^+_{\{1,\ldots,m\}}$ for all $m \in \{n_{i-1}+1, \ldots, n_i\}$. For each $i \in \mathbb{N}$, this is shown by induction. Since $\mathbf{x}_{N_i} \succ^*_{n_i} \mathbf{y}_{N_i}$, it follows by axiom **I** and the properties of P^+ that $\mathbf{x}^+_{\{1,\ldots,n_i\}} \succ^*_{n_i} \mathbf{y}^+_{\{1,\ldots,n_i\}}$. Assume that $\mathbf{x}^+_{\{1,\ldots,m\}} \succ^*_m \mathbf{y}^+_{\{1,\ldots,m\}}$ for all $m \in \{\ell+1, \ldots n_i\}$, where $\ell \in \{n_{i-1}+1, \ldots n_i - 1\}$. The inductive proof is completed by showing that $\mathbf{x}^+_{\{1,\ldots,\ell\}} \succ^*_\ell \mathbf{y}^+_{\{1,\ldots,\ell\}}$.

If $x^+_{\ell+1} > y^+_{\ell+1}$, then $\mathbf{x}^+_{\{n_{i-1}+1,\ldots,\ell\}} > \mathbf{y}^+_{\{n_{i-1}+1,\ldots,\ell\}}$ by the properties of P^+. If $i = 1$, so that $n_{i-1}+1 = n_0 + 1 = 1$, then axiom **P** implies $\mathbf{x}^+_{\{1,\ldots,\ell\}} \succ^*_\ell \mathbf{y}^+_{\{1,\ldots,\ell\}}$. If $i \geq 2$, then $\mathbf{x}_{N_{i-1}} \succ^*_{n_{i-1}} \mathbf{y}_{N_{i-1}}$, and axiom **I** and the properties of P^+ imply that $\mathbf{x}^+_{\{1,\ldots,n_{i-1}\}} \succ^*_{n_{i-1}} \mathbf{y}^+_{\{1,\ldots,n_{i-1}\}}$. Hence, it follows from axiom **P** and Lemma 3(i) that

$$\mathbf{x}^+_{\{1,\ldots,\ell\}} \succ^*_\ell \left(\mathbf{x}^+_{\{1,\ldots,n_{i-1}\}}, \mathbf{y}^+_{\{n_{i-1}+1,\ldots,\ell\}}\right) \succ^*_\ell \mathbf{y}^+_{\{1,\ldots,\ell\}}\,.$$

If $x^+_{\ell+1} \leq y^+_{\ell+1}$, then axiom **P** implies that

$$\left(\mathbf{x}^+_{\{1,\ldots,\ell\}}, y^+_{\ell+1}\right) \succsim^*_{\ell+1} \mathbf{x}^+_{\{1,\ldots,\ell+1\}} \succ^*_{\ell+1} \mathbf{y}^+_{\{1,\ldots,\ell+1\}}\,.$$

It now follows from Lemma 3(i) that $\mathbf{x}^+_{\{1,\ldots,\ell\}} \succ^*_\ell \mathbf{y}^+_{\{1,\ldots,\ell\}}$.

Part (ii) follows by interchanging the roles of $\mathbf{x}$ and $\mathbf{y}$.

(2) implies (1). This follows directly from Definition 2 and the fact that $\succsim^*_{\mathcal{I}}$ satisfies $\mathcal{P}$**I** (cf. Theorem 1(i)). ■

Proposition 7 yields the following conclusion: If an SWR $\succsim$ to which $\succsim^*_{\mathcal{I}}$ is a subrelation strictly ranks a utility pair $\mathbf{x}$ and $\mathbf{y}$ deemed incomparable by $\succsim^*_{\mathcal{I}}$, then $\succsim$ cannot both satisfy axiom $\mathcal{P}$**I** and be determined from the sequence of finite-dimensional Paretian SWOs by means of an overtaking procedure.

By Proposition 3, in the utilitarian case an incomparable pair of utility streams, $\mathbf{x}$ and $\mathbf{y}$, satisfies that the sets of positive differences, $\{i \in \mathbb{N} \mid x_i - y_i > 0\}$, and negative differences, $\{i \in \mathbb{N} \mid x_i - y_i < 0\}$, are both infinite, and either (i) the sum of the positive differences and the sum of the negative differences both diverge, or (ii) the sum of the positive differences converges to $\sigma^+_{\mathbf{x}-\mathbf{y}} \in (0, \infty)$ and the sum of the negative differences converges to $\sigma^-_{\mathbf{x}-\mathbf{y}} \in (-\infty, 0)$, with $\sigma^+_{\mathbf{x}-\mathbf{y}} + \sigma^-_{\mathbf{x}-\mathbf{y}} = 0$.

By Proposition 6, in the leximin case an incomparable pair of utility streams, $\mathbf{x}$ and $\mathbf{y}$, satisfies that (a) there is no $P \in \mathcal{F}$ such that $P\mathbf{x} \geq \mathbf{y}$ or $P\mathbf{x} \leq \mathbf{y}$ and (b) there exists m such that $(\mathbf{x}', \mathbf{y}') = R^n(\mathbf{x}, \mathbf{y})$ for all $n \geq m$ and $\inf_{j \in \mathbb{N}^{\mathbf{x}'}} x'_j = \inf_{j \in \mathbb{N}^{\mathbf{y}'}} y'_j$ with the sets $\mathbb{N}^{\mathbf{x}'}_{\min}$ and $\mathbb{N}^{\mathbf{y}'}_{\min}$ both being empty or infinite.

Axiom **IPC** has earlier been proposed by Vallentyne and Kagan (1997, p. 10) under the name of RSBI ('rejected strengthened basic idea') and applied to the utilitarian case. The utilitarian SWR generated by RSBI coincides with the asymmetric part of the utilitarian time-invariant overtaking criterion $\succsim^U_{\mathcal{I}}$. Vallentyne and Kagan (1997, p. 10–11) reject RSBI in favor of SBI1 ('strengthened basic idea 1'), which is equivalent to Lauwers and Vallentyne's (2004) "Differential Betterness" principle.

As shown by Lauwers and Vallentyne (2004), the utilitarian SWR generated by SBI1 ranks $\mathbf{x}$ above $\mathbf{y}$ iff the sum of the positive differences converges to $\sigma^+_{\mathbf{x}-\mathbf{y}}$ and the sum of the negative differences converges to $\sigma^-_{\mathbf{x}-\mathbf{y}}$, with $\sigma^+_{\mathbf{x}-\mathbf{y}} + \sigma^-_{\mathbf{x}-\mathbf{y}} > 0$, or the sum of positive difference diverges and the sum of negative difference converges. In both cases, $\succsim^U_{\mathcal{I}}$ makes the same rankings.

It differs from $\succsim^U_{\mathcal{I}}$ in the case where the sum of the positive differences converges to $\sigma^+_{\mathbf{x}-\mathbf{y}}$ and the sum of the negative differences converges to $\sigma^-_{\mathbf{x}-\mathbf{y}}$, with $\sigma^+_{\mathbf{x}-\mathbf{y}} + \sigma^-_{\mathbf{x}-\mathbf{y}} = 0$. In this case, the utilitarian SWR generated by SBI1 yields no ranking. In contrast, it follows from Proposition 3 that

(1) $\mathbf{x} \succ^U_{\mathcal{I}} \mathbf{y}$ if $\{i \in \mathbb{N} \mid x_i - y_i > 0\}$ is finite and $\{i \in \mathbb{N} \mid x_i - y_i < 0\}$ is infinite,

(2) $\mathbf{x} \sim^U_{\mathcal{I}} \mathbf{y}$ if $\{i \in \mathbb{N} \mid x_i - y_i > 0\}$ and $\{i \in \mathbb{N} \mid x_i - y_i < 0\}$ are both finite,

(3) $\mathbf{x} \prec^U_{\mathcal{I}} \mathbf{y}$ if $\{i \in \mathbb{N} \mid x_i - y_i > 0\}$ is infinite and $\{i \in \mathbb{N} \mid x_i - y_i < 0\}$ is finite,

(4) $\mathbf{x}$ and $\mathbf{y}$ are incomparable by $\succsim^U_{\mathcal{I}}$ if $\{i \in \mathbb{N} \mid x_i - y_i > 0\}$ and $\{i \in \mathbb{N} \mid x_i - y_i < 0\}$ are both infinite; this follows from Proposition 7.

However, Lauwers and Vallentyne (2004) introduce a second principle, "Differential Indifference", which for each of these sub-cases deems x indifferent to y.

To illustrate this difference, compare stream $\mathbf{u}$ of the introduction with

$$(0,\mathbf{u}) \quad : \quad 0 \quad 1 \quad \tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{8} \quad \tfrac{1}{16} \quad \dots \quad \tfrac{1}{2^{n-2}} \quad \dots$$

By choosing $M^+ = \{1\}$, it follows from Proposition 3(i) that $\mathbf{u} \succ_{\mathcal{I}}^{U} (0,\mathbf{u})$. However, $\sigma^+_{\mathbf{u}-(0,\mathbf{u})} = 1$ and $\sigma^-_{\mathbf{u}-(0,\mathbf{u})} = -1$. Hence, $\sigma^+_{\mathbf{u}-(0,\mathbf{u})} + \sigma^-_{\mathbf{u}-(0,\mathbf{u})} = 0$, implying that an SWR satisfying Differential Indifference deems $\mathbf{u}$ and $(0,\mathbf{u})$ indifferent.

Hence, by deeming two streams indifferent when the sum of differences is unconditionally convergent and converges to zero, Differential Indifference reduces incomparability of a utilitarian SWR to the case where the sum of the positive differences and the sum of the negative differences both diverge. However, it also reduces the asymmetric part of the utilitarian time-invariant overtaking criterion and may thus increase the set of maximal elements. Furthermore, it is not clear how to adapt this principle to a generalized infinite-dimensional criterion obtained from some proliferating sequence of Paretian finite-dimensional SWOs, thus making it compatible with our purpose: to develop a generalized criterion that allows for non-additive moral value theories and different interpretations for the locations of values.

Appendix

Lemma 8 *Let the SWR* $\succsim$ *extends* $\succsim_2^U$*. If* $\mathbf{x}$, $\mathbf{u} \in \mathbf{X}$ *satisfy that there exists* $N \subset \mathbb{N}$ *such that* $u_i = \sigma(\mathbf{x}_N)/|N|$ *for* $i \in N$ *and* $u_i = x_i$ *for* $i \in \mathbb{N}\backslash N$*, then* $\mathbf{x} \sim \mathbf{u}$.

Proof. The result is shown by induction. Consider the statement that $\mathbf{x} \sim \mathbf{u}$ whenever $\mathbf{x}$, $\mathbf{u} \in \mathbf{X}$ satisfy that there exists $N \subset \mathbb{N}$ such that $u_i = \sigma(\mathbf{x}_N)/|N|$ for $i \in N$ and $u_i = x_i$ for $i \in \mathbb{N}\backslash N$.

This statement is true for all $N \subset \mathbb{N}$ with $|N| = 1$ by the reflexivity of $\succsim$.

Assume that the statement is true for all $M \subset \mathbb{N}$ with $|M| \leq m$. It remains to be shown that then the statement is true for all $N \subset \mathbb{N}$ with $|N| = m + 1$, provided that $\succsim$ extends $\succsim_2^U$. This is shown in the remainder of the proof.

Suppose $\mathbf{u} \in \mathbf{X}$ satisfy that there exists $N \subset \mathbb{N}$ such that $u_i = \sigma(\mathbf{x}_N)/|N|$ for $i \in N$ and $u_i = x_i$ for $i \in \mathbb{N}\backslash N$, where $|N| = m + 1$. W.l.o.g., $N = \{1, \ldots, m + 1\}$. Consider any $M \subset \mathbb{N}$ such that $M \subset \mathbb{N}$ and $|M| = m$. W.l.o.g., $M = \{1, \ldots, m\}$. Construct $\mathbf{v} \in \mathbf{X}$ by $v_i = \sigma(\mathbf{x}_M)/|M|$ for $i \in M$ and $v_i = x_i$ for $i \in \mathbb{N}\backslash M$.

Let the sequence $\{\mathbf{y}^k\}_{k=0}^m$, where $\mathbf{y}^k \in \mathbf{X}$ for each k, be constructed as follows:

$$\mathbf{y}_M^k = \begin{cases} \mathbf{v}_M & \text{for } k = 0 \\ (\mathbf{u}_{\{1,\ldots,k\}}, \mathbf{v}_{\{k+1,\ldots,m\}}) & \text{for } k = 1, \ldots, m-1 \\ \mathbf{u}_M & \text{for } k = m\,, \end{cases}$$

while for all k, $y_{m+1}^k = x_{m+1}^k + k(v_1 - u_1)$, and $y_i^k = u_i$ for $i \in \mathbb{N}\backslash N$. Then $\mathbf{y}^{k-1} \sim \mathbf{y}^k$ for $k \in \{1, \ldots, m\}$ by the property that $\succsim$ extends $\succsim_2^U$, since $y_k^{k-1} + y_{m+1}^{k-1} = y_k^k + y_{m+1}^k$ and $y_i^{k-1} = y_i^k$ for $i \in \mathbb{N}\backslash\{k, m+1\}$. By transitivity, $\mathbf{v} = \mathbf{y}^0 \sim \mathbf{y}^m = \mathbf{u}$. By assumption, $\mathbf{x} \sim \mathbf{v}$, leading by transitivity to the conclusion that $\mathbf{x} \sim \mathbf{u}$. ■

Direct proof of Proposition 2. Assume that the SWR $\succsim$ extends $\succsim_2^U$. We must show that $\succsim$ extends $\succsim_m^U$ for all $m \geq 2$. Consider $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ for which there exists some subset $M \subset \mathbb{N}$ such that $x_i = y_i$ for all $i \in \mathbb{N}\backslash M$.

If $\mathbf{x}_M \sim_{|M|}^U \mathbf{y}_M$, then $\sigma(\mathbf{x}_M) = \sigma(\mathbf{y}_M)$ and, by Lemma 8, $\mathbf{x} \sim \mathbf{u} \sim \mathbf{y}$, where $u_i = \sigma(\mathbf{x}_M)/|M|$ for $i \in M$ and $u_i = x_i$ for $i \in \mathbb{N}\backslash M$. By transitivity, $\mathbf{x} \sim \mathbf{y}$.

If $\mathbf{x}_M \succ_{|M|}^U \mathbf{y}_M$, then $\sigma(\mathbf{x}_M) > \sigma(\mathbf{y}_M)$ and, by Lemma 8 and **FP**, $\mathbf{x} \sim \mathbf{u} \succ \mathbf{v} \sim \mathbf{y}$, where $u_i = \sigma(\mathbf{x}_M)/|M|$ and $v_i = \sigma(\mathbf{y}_M)/|M|$ for $i \in M$ and $u_i = v_i = x_i = y_i$ for $i \in \mathbb{N}\backslash M$. By transitivity, $\mathbf{x} \succ \mathbf{y}$. ■

References

Asheim, G.B. and Tungodden, B. (2004), Resolving distributional conflicts between generations, Economic Theory **24**, 221–230.

d'Aspremont, C. (1985), Axioms for social welfare orderings. In: L. Hurwicz, D. Schmeidler and H. Sonnenschein (eds.), *Social Goals and Social Organizations: Essay in Memory of Elisha Pazner*, Cambridge University Press, Cambridge, pp. 19–76.

d'Aspremont, C. (2007), Formal welfarism and intergenerational equity. In: J. Roemer and K. Suzumura (eds.), *Intergenerational Equity and Sustainability*, Palgrave-Macmillan, Basingstoke, pp. 113–130.

d'Aspremont, C. and Gevers, L. (1977), Equity and the informational basis for social choice. Review of Economic Studies **44**, 199–208.

d'Aspremont, C. and Gevers, L. (2002), Social welfare functionals and interpersonal comparibility. In: K. Arrow, A. Sen and K. Suzumura (eds.), *Handbook of Social Choice and Welfare*, vol. 1, Elsevier, Amsterdam, pp. 459–541.

Atsumi, H. (1965), Neoclassical growth and the efficient program of capital accumulation. Review of Economic Studies **32**, 127–136.

Banerjee, K. (2006), On the extension of the utilitarian and Suppes-Sen social welfare relations to infinite utility streams. Social Choice and Welfare **27**, 327–339.

Basu, K. and Mitra, T. (2007a), Utilitarianism for infinite utility streams: A new welfare criterion and its axiomatic characterization. Journal of Economic Theory **133**, 350–373.

Basu, K. and Mitra, T. (2007b), Possibility Theorems for Aggregating Infinite Utility Streams Equitably. In: J. Roemer and K. Suzumura (eds.), Intergenerational Equity and Sustainability, Palgrave-Macmillan, Basingstoke, pp. 69–84.

Bossert, W. and Weymark, J.A. (2004), Utility in Social Choice. In: S. Barbara, P. Hammond and C. Seidl (eds.), Handbook of Utility Theory, Volume 2: Extensions, Kluwer, Boston, pp. 1099–1177.

Bossert, W., Sprumont, Y., and Suzumura, K. (2007), Ordering infinite utility streams. Journal of Economic Theory **135**, 579–589.

Fleurbaey, M. and Michel, P. (2003), Intertemporal equity and the extension of the Ramsey criterion. Journal of Mathematical Economics **39**, 777–802.

Hammond, P.J. (1976), Equity, Arrow's condition, and Rawls' difference principle, Econometrica **44**, 793–804.

Hammond, P.J. (1979), Equity in two person situations: Some consequences, Econometrica **47**, 1127–1135.

Kamaga, K. and Kojima, T. (2009a), $\mathcal{Q}$-anonymous social welfare relations on infinite utility streams, Social Choice and Welfare **33**, 405–413.

Kamaga, K. and T. Kojima (2009b), On the leximin and utilitarian overtaking criteria with extended anonymity. G-COE GLOPE II Working Paper No. 9, Waseda University (http://globalcoe-glope2.jp/modules/working/visit.php?cid=0&lid=9).

Kolm, S.C. (1972), *Justice et Equite*, CNRS, Paris.

Koopmans, T.C. (1960), Stationary ordinal utility and impatience. Econometrica **28**, 287–309.

Lauwers, L. (1997), Infinite utility: Insisting on strong monotonicity. Australasian Journal of Philosophy **75**, 222–233.

Lauwers, L. (2010), Ordering infinite utility streams comes at the cost of a non-Ramsey set. Journal of Mathematical Economics **46**, 32–37.

Lauwers, L. and Vallentyne, P. (2004), Infinite utilitarianism: More is always better. Economics and Philosophy **20**, 307–330.

Mabrouk, M. (2008), Comparing incomparable alternatives when certain axioms are to be satisfied. Mimeo, ESSAI, Tunis.

Milnor, J.W. (1954), Games against nature. In: R. Thrall, C. Coombs and R. Davis (eds), *Decision Processes*, John Wiley, London, pp. 49–60.

Mitra, T. (2007), Stationary Paretian social welfare quasi-orderings and intergenerational equity. Mimeo, Cornell University.

Mitra, T. and Basu, K. (2007), On the Existence of Paretian Social Welfare Relations for Infinite Utility Streams with Extended Anonymity. In: J. Roemer and K. Suzumura (eds.), Intergenerational Equity and Sustainability, Palgrave-Macmillan, Basingstoke, pp. 85–99.

Sakai, T. (2010), Intergenerational equity and an explicit construction of welfare criteria. Forthcoming in Social Choice and Welfare.

Sen, A. (1970), *Collective Choice and Social Welfare*, Holden-Day, San Francisco.

Sen, A. (1976), Welfare inequalities and Rawlsian axiomatics, Theory and Decision **7**, 243–262.

Suppes, P. (1966), Some formal models of the grading principle, Synthese **6**, 284–306.

Svensson, L.-G. (1980), Equity among generations. Econometrica **48**, 1251–1256.

Vallentyne, P. and Kegan, S. (1997), Infinite Value and Finitely Additive Value Theory. Journal of Philosophy **94**, 5–26.

von Weizsäcker, C. C. (1965), Existence of optimal program of accumulation for an infinite time horizon. Review of Economic Studies **32**, 85–104.

Zame, W. (2007), Can intergenerational equity be operationalized? Theoretical Economics **2**, 187–202.