

# A stochastic control approach to robust duality in utility maximization

Bernt Øksendal<sup>1,2</sup> Agnès Sulem<sup>3</sup>

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## Abstract

A celebrated financial application of convex duality theory gives an explicit relation between the following two quantities:

- (i) The optimal terminal wealth  $X^*(T) := X_{\varphi^*}(T)$  of the classical problem to maximize the expected  $U$ -utility of the terminal wealth  $X_{\varphi}(T)$  generated by admissible portfolios  $\varphi(t); 0 \leq t \leq T$  in a market with the risky asset price process modeled as a semimartingale
- (ii) The optimal scenario  $\frac{dQ^*}{dP}$  of the dual problem to minimize the expected  $V$ -value of  $\frac{dQ}{dP}$  over a family of equivalent local martingale measures  $Q$ . Here  $V$  is the convex dual function of the concave function  $U$ .

In this paper we consider markets modeled by Itô-Lévy processes, and in the first part we give a new proof of the above result in this setting, based on the maximum principle in stochastic control theory. An advantage with our approach is that it also gives an explicit relation between the optimal portfolio  $\varphi^*$  and the optimal measure  $Q^*$ , in terms of backward stochastic differential equations.

In the second part we present robust (model uncertainty) versions of the optimization problems in (i) and (ii), and we prove a relation between them. In particular, we show explicitly how to get from the solution of one of the problems to the solution of the other.

We illustrate the results with explicit examples.

**Keywords:** Utility maximization, Itô-Lévy market, duality method, stochastic control, maximum principles, backward stochastic differential equations, replicability, optimal scenario, optimal portfolio, Malliavin calculus

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<sup>1</sup>Center of Mathematics for Applications (CMA), Dept. of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo, Norway, email: [oksendal@math.uio.no](mailto:oksendal@math.uio.no). The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no [228087].

<sup>2</sup>Norwegian School of Economics and Business Administration, Helleveien 30, N-5045 Bergen, Norway.

<sup>3</sup>INRIA Paris-Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, Le Chesnay Cedex, 78153, France, and Université Paris-Est, email: [agnes.sulem@inria.fr](mailto:agnes.sulem@inria.fr)

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# 1 Introduction

The purpose of this paper is to use stochastic control theory to obtain new results and new proofs of important known results in mathematical finance, which have been proved using convex duality theory.

The advantage with this approach is that it gives an explicit relation between the optimal scenario in the dual formulation and the optimal portfolio in the primal formulation. We now explain this in more detail.

First, let us briefly recall the terminology and main results from the duality method in mathematical finance, as presented in e.g. [7]:

Let  $U : [0, \infty] \rightarrow \mathbb{R}$  be a given utility function, assumed to be strictly increasing, strictly concave, continuously differentiable ( $C^1$ ) and satisfying the Inada conditions:

$$\begin{aligned} U'(0) &= \lim_{x \rightarrow 0^+} U'(x) = \infty \\ U'(\infty) &= \lim_{x \rightarrow \infty} U'(x) = 0. \end{aligned}$$

Let  $S(t) = S(t, \omega) ; 0 \leq t \leq T, \omega \in \Omega$ , represent the discounted unit price of a risky asset at time  $t$  in a financial market. We assume that  $S(t)$  is a semimartingale on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ . Let  $\varphi(t)$  be an  $\mathcal{F}_t$ -predictable portfolio process, giving the number of units held of the risky asset at time  $t$ . If  $\varphi(t)$  is self-financing, the corresponding wealth process  $X(t) = X_\varphi^x(t)$  is given by

$$X(t) = x + \int_0^t \varphi(s) dS(s) ; 0 \leq t \leq T, \quad (1.1)$$

where  $T \geq 0$  is a fixed terminal time and  $x > 0$  is the initial value of the wealth. We say that  $\varphi$  is *admissible* and write  $\varphi \in \mathcal{A}$  if the integral in (1.1) converges and

$$X_\varphi(t) > 0 \text{ for all } t \in [0, T], \text{ a.s..} \quad (1.2)$$

The classical optimal portfolio problem in finance is to find  $\varphi^* \in \mathcal{A}$  (called an optimal portfolio) such that

$$u(x) := \sup_{\varphi \in \mathcal{A}} E[U(X_\varphi^x(T))] = E[U(X_{\varphi^*}^x(T))]. \quad (1.3)$$

The duality approach to this problem is as follows: Let

$$V(y) := \sup_{x > 0} \{U(x) - xy\} ; y > 0 \quad (1.4)$$

be the *convex dual* of  $U$ . Then it is well-known that  $V$  is strictly convex, decreasing,  $C^1$  and satisfies

$$V'(0) = -\infty, V'(\infty) = 0, V(0) = U(\infty) \text{ and } V(\infty) = U(0). \quad (1.5)$$

Moreover,

$$U(x) = \inf_{y>0} \{V(y) + xy\}; \quad x > 0, \quad (1.6)$$

and

$$U'(x) = y \Leftrightarrow x = -V'(y). \quad (1.7)$$

Let  $\mathcal{M}$  be the set of probability measures  $Q$  which are equivalent local martingale measures (ELMM), in the sense that  $Q$  is equivalent to  $P$  and  $S(t)$  is a local martingale with respect to  $Q$ . We assume that  $\mathcal{M} \neq \emptyset$ , which means absence of arbitrage opportunities on the financial market. The dual problem to (1.3) is for given  $y > 0$  to find  $Q^* \in \mathcal{M}$  (called an optimal scenario measure) such that

$$v(y) := \inf_{Q \in \mathcal{M}} E \left[ V \left( y \frac{dQ}{dP} \right) \right] = E \left[ V \left( y \frac{dQ^*}{dP} \right) \right]. \quad (1.8)$$

One of the main results in [7] is that, under some conditions,  $\varphi^*$  and  $Q^*$  both exist and they are related by

$$U'(X_{\varphi^*}^x(T)) = y \frac{dQ^*}{dP} \quad \text{with } y = u'(x) \quad (1.9)$$

i.e.

$$X_{\varphi^*}^x(T) = -V' \left( y \frac{dQ^*}{dP} \right) \quad \text{with } x = -v'(y). \quad (1.10)$$

In this paper we will give a new proof of a result of this type by using stochastic control theory. We will work in the slightly more special market setting with a risky asset price  $S(t)$  described by an Itô-Lévy process. This enables us to use the machinery of the maximum principle and backward stochastic differential equations (BSDE) driven by Brownian motion  $B(t)$  and a compensated Poisson random measure  $\tilde{N}(dt, d\zeta)$ ;  $t \geq 0$ ;  $\zeta \in \mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ . The advantage with this approach is that it gives explicit relation between the optimal scenario and the optimal portfolio. This is shown in Section 3 (see Theorem 3.1). As a step on the way, we prove in Section 2 a result of independent interest, namely that the existence of an optimal scenario is equivalent to the replicability of a related  $T$ -claim. In Section 4 we extend the discussion to robust (model uncertainty) optimal portfolio problems. More precisely, we formulate robust versions of the primal problem (1.3) and of the dual problem (1.8) and we show explicitly how to get from the solution of one to the solution of the other.

## 2 Optimal scenario and replicability

We now specialize the setting described in Section 1 as follows:

Suppose the financial market has a risk free asset with unit price  $S_0(t) = 1$  for all  $t$  and a risky asset with price  $S(t)$  given by

$$\begin{cases} dS(t) = S(t^-) \left( b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right); & 0 \leq t \leq T \\ S(0) > 0 \end{cases} \quad (2.1)$$

where  $b(t), \sigma(t)$  and  $\gamma(t, \zeta)$  are predictable processes satisfying  $\gamma > -1$  and

$$E \left[ \int_0^T \left\{ |b(t)| + \sigma^2(t) + \int_{\mathbb{R}} \gamma^2(t, \zeta) \nu(d\zeta) \right\} dt \right] < \infty. \quad (2.2)$$

Here  $B(t)$  and  $\tilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)dt$  is a Brownian motion and an independent compensated Poisson random measure, respectively, on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  satisfying the usual conditions,  $P$  is a reference probability measure and  $\nu$  is the Lévy measure of  $N$ . In the following we assume that

$$\sigma(t) \neq 0 \text{ for all } t \in [0, T]. \quad (2.3)$$

This is more because of convenience and notational simplicity than of necessity. See the remark after Theorem 2.2. Note that this assumption is not used before (2.19).

Let  $\varphi(t), \mathcal{A}$  be as in Section 1, with the condition

$$E \left[ \int_0^T \varphi(t)^2 S(t)^2 \left\{ b(t)^2 + \sigma^2(t) + \int_{\mathbb{R}} \gamma^2(t, \zeta) \nu(d\zeta) \right\} dt \right] < \infty.$$

Let  $X(t) = X_\varphi^x(t)$  be the corresponding wealth process given by

$$\begin{cases} dX(t) = \varphi(t)S(t^-) \left[ b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \\ X(0) = x > 0. \end{cases} \quad (2.4)$$

For  $\varphi$  to be admissible we require moreover that we have, for some  $\epsilon > 0$

$$E \left[ \int_0^T |X(t)|^{2+\epsilon} dt \right] < \infty \quad (2.5)$$

and

$$E[U'(X(T))^{2+\epsilon}] < \infty. \quad (2.6)$$

As in (1.3), for given  $x > 0$ , we want to find  $\varphi^* \in \mathcal{A}$  such that

$$u(x) := \sup_{\varphi \in \mathcal{A}} E[U(X_\varphi^x(T))] = E[U(X_{\varphi^*}^x(T))]. \quad (2.7)$$

In our model we represent  $\mathcal{M}$  by the family of positive measures  $Q = Q_\theta$  of the form

$$dQ_\theta(\omega) = G_\theta(T)dP(\omega) \text{ on } \mathcal{F}_T, \quad (2.8)$$

where

$$\begin{cases} dG_\theta(t) = G_\theta(t^-) \left[ \theta_0(t)dB(t) + \int_{\mathbb{R}} \theta_1(t, \zeta) \tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \\ G_\theta(0) = y > 0, \end{cases} \quad (2.9)$$

and  $\theta = (\theta_0, \theta_1)$  is a predictable process satisfying the conditions

$$E \left[ \int_0^T \left\{ \theta_0^2(t) + \int_{\mathbb{R}} \theta_1^2(t, \zeta) \nu(d\zeta) \right\} dt \right] < \infty, \quad \theta_1(t, \zeta) \geq -1 \quad \text{a.s.} \quad (2.10)$$

and

$$b(t) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) = 0; \quad t \in [0, T]. \quad (2.11)$$

If  $y = 1$  this condition characterises  $Q_\theta$  as an equivalent local martingale measure (ELMM) for this market. See e.g. [12, Chapter 1].

We let  $\Theta$  denote the set of all  $\mathcal{F}_t$ -predictable processes  $\theta = (\theta_0, \theta_1)$  satisfying the above conditions.

Thus the dual problem corresponding to (1.8) is for given  $y > 0$  to find  $\theta^* \in \Theta$  and  $v(y)$  such that

$$-v(y) := \sup_{\theta \in \Theta} E[-V(G_\theta^y(T))] = E[-V(G_{\theta^*}^y(T))]. \quad (2.12)$$

We will use the maximum principle for stochastic control to study the problem (2.12) and relate it to (2.7).

We first prove the following useful auxiliary result, which may be regarded as a special case of Proposition 4.4 in [4].

**Proposition 2.1** *Let  $\hat{\varphi}(t) \in \mathcal{A}$ . Then  $\hat{\varphi}(t)$  is optimal for the primal problem (2.7) if and only if the (unique) solution  $(\hat{p}, \hat{q}, \hat{r})$  of the BSDE*

$$\begin{cases} d\hat{p}(t) = \hat{q}(t)dB(t) + \int_{\mathbb{R}} \hat{r}(t, \zeta)\tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ \hat{p}(T) = U'(X_{\hat{\varphi}}^x(T)). \end{cases} \quad (2.13)$$

satisfies the equation

$$b(t)\hat{p}(t) + \sigma(t)\hat{q}(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\hat{r}(t, \zeta)\nu(d\zeta) = 0; \quad t \in [0, T]. \quad (2.14)$$

*Proof.* (i) First assume that  $\hat{\varphi} \in \mathcal{A}$  is optimal for the primal problem (2.7). Then by the necessary maximum principle (Theorem A.2) the corresponding Hamiltonian, given by

$$H(t, x, \varphi, p, q, r) = \varphi S(t^-)(b(t)p + \sigma q + \int_{\mathbb{R}} \gamma(t, \zeta)r(\zeta)\tilde{N}(dt, d\zeta)) \quad (2.15)$$

satisfies

$$\frac{\partial H}{\partial \varphi}(t, x, \varphi, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \Big|_{\varphi=\hat{\varphi}(t)} = 0,$$

where  $(\hat{p}, \hat{q}, \hat{r})$  satisfies (2.13). This implies (2.14).

(ii) Conversely, suppose the solution  $(\hat{p}, \hat{q}, \hat{r})$  of the BSDE (2.13) satisfies (2.14). Then  $\hat{\varphi}$ , with the associated  $(\hat{p}, \hat{q}, \hat{r})$  satisfies the conditions for the sufficient maximum principle (Theorem A.1) and hence  $\hat{\varphi}$  is optimal.  $\square$

We now turn to the dual problem (2.12). The Hamiltonian  $H$  associated to (2.12) is, by (2.9)

$$H(t, g, \theta_0, \theta_1, p, q, r) = g\theta_0q + g \int_{\mathbb{R}} \theta_1(\zeta)r(\zeta)\nu(d\zeta). \quad (2.16)$$

(We refer to e.g. [12] for more information about the maximum principle).

The adjoint equation for  $(p, q, r)$  is the following backward stochastic differential equation (BSDE):

$$\begin{cases} dp(t) = -\frac{\partial H}{\partial g}(t, G_\theta(t), \theta_0(t), \theta_1(t, \cdot), p(t), q(t), r(t, \cdot))dt \\ \quad + q(t)dB(t) + \int_{\mathbb{R}} r(t, \zeta)\tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \\ p(T) = -V'(G_\theta(T)). \end{cases} \quad (2.17)$$

In our setting this equation becomes

$$\begin{cases} dp(t) = - \left[ \theta_0(t)q(t) + \int_{\mathbb{R}} \theta_1(t, \zeta)r(t, \zeta)\nu(d\zeta) \right] dt \\ \quad + q(t)dB(t) + \int_{\mathbb{R}} r(t, \zeta)\tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \\ p(T) = -V'(G_\theta(T)). \end{cases} \quad (2.18)$$

By (2.3) the constraint (2.11) can be written

$$\theta_0(t) = \tilde{\theta}_0(t) = -\frac{1}{\sigma(t)} \left\{ b(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) \right\}; \quad t \in [0, T]. \quad (2.19)$$

Substituting this into (2.16) we get

$$\begin{aligned} H_1(t, g, \theta_1, p, q, r) &:= H(t, g, \tilde{\theta}_0, \theta_1, p, q, r) \\ &= g \left( -\frac{q}{\sigma(t)} \left\{ b(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(\zeta)\nu(d\zeta) \right\} + \int_{\mathbb{R}} \theta_1(\zeta)r(\zeta)\nu(d\zeta) \right), \end{aligned} \quad (2.20)$$

and this gives

$$\begin{cases} dp(t) = - \left[ -\frac{q(t)}{\sigma(t)} \left\{ b(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) \right\} \right. \\ \quad \left. + \int_{\mathbb{R}} r(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) \right] dt \\ \quad + q(t)dB(t) + \int_{\mathbb{R}} r(t, \zeta)\tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \\ p(T) = -V'(G_\theta(T)) \end{cases}$$

i.e.

$$\begin{cases} dp(t) = \left[ \frac{q(t)}{\sigma(t)} b(t) + \int_{\mathbb{R}} \theta_1(t, \zeta) \left( \frac{q(t)}{\sigma(t)} \gamma(t, \zeta) - r(t, \zeta) \right) \nu(d\zeta) \right] dt \\ + q(t) dB(t) + \int_{\mathbb{R}} r(t, \zeta) \tilde{N}(dt, d\zeta); 0 \leq t \leq T \\ p(T) = -V'(G_{\theta}(T)). \end{cases} \quad (2.21)$$

If there exists a maximiser  $\hat{\theta}_1$  for  $H_1$  then

$$(\nabla_{\theta_1} H_1)_{\theta_1 = \hat{\theta}_1} = 0, \quad (2.22)$$

i.e.

$$-\frac{q(t)}{\sigma(t)} \gamma(t, \zeta) + r(t, \zeta) = 0; 0 \leq t \leq T. \quad (2.23)$$

Substituting this into (2.21) we get

$$\begin{cases} dp(t) = \frac{q(t)}{\sigma(t)} \left[ b(t) dt + \sigma(t) dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right]; 0 \leq t \leq T \\ p(T) = -V'(G_{\hat{\theta}}(T)). \end{cases} \quad (2.24)$$

Equation (2.24) states that the contingent claim  $F := -V'(G_{\hat{\theta}}(T))$  is replicable, with replicating portfolio  $\varphi(t)$  given by

$$\varphi(t) := \frac{q(t)}{\sigma(t)S(t^-)}; t \in [0, T]. \quad (2.25)$$

and initial value  $x = p(0)$ . We have proved (i)  $\Rightarrow$  (ii) in the following theorem:

**Theorem 2.2** *The following are equivalent:*

(i) *For given  $y > 0$ , there exists  $\hat{\theta} \in \Theta$  such that*

$$\sup_{\theta \in \Theta} E[-V(G_{\theta}^y(T))] = E[-V(G_{\hat{\theta}}^y(T))] < \infty.$$

(ii) *For given  $y > 0$ , there exists  $\hat{\theta} \in \Theta$  such that the claim  $F := -V'(G_{\hat{\theta}}^y(T))$  is replicable, with initial value  $x = p(0)$ , where  $p$  solves (2.24).*

Moreover, if (i) or (ii) holds, then

$$\varphi(t) := \frac{\hat{q}(t)}{\sigma(t)S(t^-)} \quad (2.26)$$

is a replicating portfolio for  $F := -V'(G_{\hat{\theta}}^y(T))$ , where  $(\hat{p}(t), \hat{q}(t), \hat{r}(t, \zeta))$  is the solution of the linear BSDE

$$\begin{cases} d\hat{p}(t) = \frac{\hat{q}(t)}{\sigma(t)} b(t) dt + \hat{q}(t) dB(t) + \int_{\mathbb{R}} \hat{r}(t, \zeta) \tilde{N}(dt, d\zeta); 0 \leq t \leq T \\ \hat{p}(T) = -V'(G_{\hat{\theta}}^y(T)). \end{cases} \quad (2.27)$$

Proof. It remains to prove that (ii)  $\Rightarrow$  (i): Suppose that  $(\hat{\theta}_0, \hat{\theta}_1) \in \Theta$  is such that  $F := -V'(G_{\hat{\theta}}(T))$  is replicable with initial value  $x = p(0)$ , and let  $\varphi \in \mathcal{A}$  be a replicating portfolio. Then  $X(t) = X_{\varphi}^x(t)$  satisfies the equation

$$\begin{cases} dX(t) = \varphi(t)S(t^-) \left[ b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right]; & 0 \leq t \leq T \\ X(T) = -V'(G_{\hat{\theta}}(T)). \end{cases} \quad (2.28)$$

Define

$$\hat{p}(t) := X(t), \hat{q}(t) := \varphi(t)\sigma(t)S(t^-) \text{ and } \hat{r}(t, \zeta) := \varphi(t)\gamma(t, \zeta)S(t^-). \quad (2.29)$$

Then by (2.28),  $(\hat{p}, \hat{q}, \hat{r})$  satisfies the BSDE

$$\begin{cases} d\hat{p}(t) = \frac{\hat{q}(t)}{\sigma(t)}b(t)dt + \hat{q}(t)dB(t) + \int_{\mathbb{R}} \hat{r}(t, \zeta) \tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ \hat{p}(T) = -V'(G_{\hat{\theta}}(T)), \end{cases} \quad (2.30)$$

and  $\hat{p}(0) = X(0) = x = p(0)$ . Since  $(\hat{\theta}_0, \hat{\theta}_1) \in \Theta$  we get by (2.11) and (2.29) that (2.30) can be written

$$\begin{cases} d\hat{p}(t) = - \left[ \hat{\theta}_0(t)\hat{q}(t) + \int_{\mathbb{R}} \hat{\theta}_1(t, \zeta)\hat{r}(t, \zeta)\nu(d\zeta) \right] dt \\ \quad + \hat{q}(t)dB(t) + \int_{\mathbb{R}} \hat{r}(t, \zeta) \tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ \hat{p}(T) = -V'(G_{\hat{\theta}}(T)). \end{cases} \quad (2.31)$$

Comparing with (2.18) we see that this is the BSDE for the adjoint equation corresponding to the stochastic control problem (2.12). Therefore, since the functions  $g \rightarrow -V(g)$  and

$$g \rightarrow \sup_{\theta_1} H_1(t, g, \theta_1, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) = -g \frac{\hat{q}(t)}{\sigma(t)} b(t)$$

are concave, it follows from the sufficient maximum principle that  $(\hat{\theta}_0, \hat{\theta}_1)$  is optimal for the problem (2.12). Hence (i) holds.

The last statement follows from (2.24) and (2.25).  $\square$

*Remark 2.3* So far we have assumed that (2.3) holds. This is convenient, because it allows us to rewrite the constraint (2.11) in the form (2.19). If we do not assume (2.3), then we can use the Lagrange multiplier method in stead, as follows:

Let  $\lambda(t)$  be the Lagrange multiplier process and consider

$$\begin{aligned} H_1(\theta_0, \theta_1, \lambda) &:= g\theta_0q + g \int_{\mathbb{R}} \theta_1(\zeta)r(\zeta)\nu(d\zeta) \\ &\quad + \lambda(t) \left( b(t) + \sigma(t)\theta_0 + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(\zeta)\nu(d\zeta) \right). \end{aligned} \quad (2.32)$$



Maximising  $H_1$  over all  $\theta_0$  and  $\theta_1$  gives the following first order conditions

$$gq + \lambda(t)\sigma(t) = 0 \quad (2.33)$$

$$gr(\cdot) + \lambda(t)\gamma(t, \cdot) = 0. \quad (2.34)$$

Since  $g = G_\theta(t) \neq 0$ , we can write these as follows:

$$q(t) = -\frac{\lambda(t)}{G_\theta(t)}\sigma(t) \quad (2.35)$$

$$r(t, \zeta) = -\frac{\lambda(t)}{G_\theta(t)}\gamma(t, \zeta) \quad (2.36)$$

Substituting this into (2.18) we get

$$\begin{cases} dp(t) = -\frac{\lambda(t)}{G_\theta(t)} \left[ \left\{ -\theta_0(t)\sigma(t) - \int_{\mathbb{R}} \theta_1(t, \zeta)\gamma(t, \zeta)\nu(d\zeta) \right\} dt \right. \\ \quad \left. + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \\ p(T) = -V'(G_\theta(T)). \end{cases} \quad (2.37)$$

In view of (2.11) this can be written

$$\begin{cases} dp(t) = -\frac{\lambda(t)}{G_\theta(t)} \left[ b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \\ p(T) = -V'(G_\theta(T)) \end{cases} \quad (2.38)$$

Note that

$$\text{If } \sigma(t) \neq 0 \text{ then } -\frac{\lambda(t)}{G_\theta(t)} = \frac{q(t)}{\sigma(t)} \quad (2.39)$$

$$\text{If } \gamma(t, \zeta) \neq 0 \text{ then } -\frac{\lambda(t)}{G_\theta(t)} = \frac{r(t, \zeta)}{\gamma(t, \zeta)} \quad (2.40)$$

If  $\sigma(t) = \gamma(t, \zeta) = 0$ , then by (2.35) and (2.36) we have  $q(t) = r(t, \zeta) = 0$  and hence by (2.18) we have  $dp(t) = 0$ . Therefore we can summarize the above as follows:

Define

$$\varphi(t) = \frac{q(t)}{\sigma(t)S(t^-)}\chi_{\sigma(t) \neq 0} + \frac{r(t, \zeta)}{\gamma(t, \zeta)S(t^-)}\chi_{\sigma(t)=0, \gamma(t, \zeta) \neq 0}. \quad (2.41)$$

Then by (2.38)

$$\begin{cases} dp(t) = \varphi(t)S(t^-) \left[ b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \\ p(T) = -V'(G_\theta(T)). \end{cases} \quad (2.42)$$

Therefore  $-V'(G_\theta(T))$  is replicable, with replicating portfolio  $\varphi(t)$  given by (2.41).

Thus we see that Theorem 2.2 still holds without assumption (2.4), if we replace (2.26) by (2.41).

### 3 Optimal scenario and optimal portfolio

We proceed to show that the method above actually gives an explicit connection between an optimal  $\hat{\theta} \in \Theta$  for problem (2.12) and an optimal portfolio  $\hat{\varphi} \in \mathcal{A}$  for problem (2.7):

**Theorem 3.1 a)** *Suppose  $\hat{\varphi} \in \mathcal{A}$  is optimal for problem (2.7) with initial value  $x$ . Let  $(p_1(t), q_1(t), r_1(t, \zeta))$  be the solution of the BSDE*

$$\begin{cases} dp_1(t) = q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t, \zeta) \tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ p_1(T) = U'(X_{\hat{\varphi}}^x(T)). \end{cases} \quad (3.1)$$

Define

$$\hat{\theta}_0(t) = \frac{q_1(t)}{p_1(t)}, \quad \hat{\theta}_1(t, \zeta) = \frac{r_1(t, \zeta)}{p_1(t^-)}. \quad (3.2)$$

Suppose

$$E\left[\int_0^T \{\hat{\theta}_0^2(t) + \int_{\mathbb{R}} \hat{\theta}_1^2(t, \zeta) \nu(d\zeta)\} dt\right] < \infty; \quad \hat{\theta}_1 > -1. \quad (3.3)$$

Then  $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1) \in \Theta$  is optimal for problem (2.12) with initial value  $y = p_1(0)$  and

$$G_{\hat{\theta}}^y(T) = U'(X_{\hat{\varphi}}^x(T)). \quad (3.4)$$

**b)** *Conversely, suppose  $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1) \in \Theta$  is optimal for problem (2.12) with initial value  $y$ . Let  $(p(t), q(t), r(t, \zeta))$  be the solution of the BSDE*

$$\begin{cases} dp(t) = \frac{q(t)}{\sigma(t)}b(t)dt + q(t)dB(t) + \int_{\mathbb{R}} r(t, \zeta) \tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ p(T) = -V'(G_{\hat{\theta}}^y(T)). \end{cases} \quad (3.5)$$

Suppose the portfolio

$$\hat{\varphi}(t) := \frac{q(t)}{\sigma(t)S(t^-)}; \quad 0 \leq t \leq T \quad (3.6)$$

is admissible. Then  $\hat{\varphi}$  is an optimal portfolio for problem (2.7) with initial value  $x = p(0)$  and

$$X_{\hat{\varphi}}^x(T) = -V'(G_{\hat{\theta}}^y(T)). \quad (3.7)$$

□

Proof.

**a)** Suppose  $\hat{\varphi}$  is optimal for problem (2.7) with initial value  $x$ . Then the adjoint processes  $p_1(t), q_1(t), r_1(t, \zeta)$  for problem (2.7) satisfy both the BSDE

$$\begin{cases} dp_1(t) = q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t, \zeta) \tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ p_1(T) = U'(X_{\hat{\varphi}}^x(T)) \end{cases} \quad (3.8)$$

and the equation (see [4, Proposition 4.4])

$$b(t)p_1(t) + \sigma(t)q_1(t) + \int_{\mathbb{R}} \gamma(t, \zeta)r_1(t, \zeta)\nu(d\zeta) = 0. \quad (3.9)$$

Define

$$\tilde{\theta}_0(t) := \frac{q_1(t)}{p_1(t^-)}, \quad \tilde{\theta}_1(t, \zeta) := \frac{r_1(t, \zeta)}{p_1(t^-)}. \quad (3.10)$$

Then  $(\tilde{\theta}_0, \tilde{\theta}_1) \in \Theta$  and (3.1) can be written

$$\begin{cases} dp_1(t) = p_1(t^-) \left[ \tilde{\theta}_0(t)dB(t) + \int_{\mathbb{R}} \tilde{\theta}_1(t, \zeta)\tilde{N}(dt, d\zeta) \right] \\ p_1(T) = U'(X_{\tilde{\varphi}}^x(T)). \end{cases} \quad (3.11)$$

Therefore  $G_{\tilde{\theta}}(t) := p_1(t)$  satisfies the equation (2.9) with initial value  $y = p_1(0) > 0$  and we have that, by (1.7)

$$U'(X_{\tilde{\varphi}}^x(T)) = G_{\tilde{\theta}}^y(T), \text{ i.e. } X_{\tilde{\varphi}}^x(T) = -V'(G_{\tilde{\theta}}^y(T)). \quad (3.12)$$

Therefore  $-V'(G_{\tilde{\theta}}^y(T))$  is replicable and by Theorem 2.2 we conclude that  $\hat{\theta} := \tilde{\theta}$  is optimal for problem (2.12).

b) Suppose  $\hat{\theta} \in \Theta$  is optimal for problem (2.12) with initial value  $y$ . Let  $p(t), q(t), r(t, \cdot)$  be the associated adjoint processes, solution of the BSDE (3.5). Then by (2.24), they satisfy the equation

$$\begin{cases} dp(t) = \frac{q(t)}{\sigma(t)} \left[ b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right] \\ p(T) = -V'(G_{\hat{\theta}}(T)). \end{cases} \quad (3.13)$$

Define

$$\tilde{\varphi}(t) := \frac{q(t)}{\sigma(t)S(t^-)}. \quad (3.14)$$

Then

$$X_{\tilde{\varphi}}^x(T) = -V'(G_{\hat{\theta}}^y(T)) \text{ i.e. } G_{\hat{\theta}}^y(T) = U'(X_{\tilde{\varphi}}^x(T)), \quad (3.15)$$

with  $x = p(0)$ . Therefore  $G_{\hat{\theta}}^y(t) = G_{\hat{\theta}}(t)$  satisfies the equation

$$\begin{cases} dG_{\hat{\theta}}(t) = G_{\hat{\theta}}(t^-) \left[ \hat{\theta}_0(t)dB(t) + \int_{\mathbb{R}} \hat{\theta}_1(t, \zeta)\tilde{N}(dt, d\zeta) \right]; \quad 0 \leq t \leq T \\ G_{\hat{\theta}}(T) = U'(X_{\tilde{\varphi}}^x(T)). \end{cases} \quad (3.16)$$

Define

$$p_0(t) := G_{\hat{\theta}}(t), \quad q_0(t) := G_{\hat{\theta}}(t)\hat{\theta}_0(t), \quad r_0(t, \zeta) := G_{\hat{\theta}}(t)\hat{\theta}_1(t, \zeta). \quad (3.17)$$

Then by (3.16)  $(p_0, q_0, r_0)$  solves the BSDE

$$\begin{cases} dp_0(t) = q_0(t)dB(t) + \int_{\mathbb{R}} r_0(t, \zeta) \tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\ p_0(T) = U'(X_{\tilde{\varphi}}^x(T)). \end{cases} \quad (3.18)$$

Moreover, since  $\hat{\theta} \in \Theta$  we have, by (2.11)

$$b(t) + \sigma(t)\hat{\theta}_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\hat{\theta}_1(t, \zeta)\nu(d\zeta) = 0 ; 0 \leq t \leq T \quad (3.19)$$

i.e.,  $(p_0, q_0, r_0)$  satisfies the equation

$$b(t) + \sigma(t)\frac{q_0(t)}{p_0(t)} + \int_{\mathbb{R}} \gamma(t, \zeta)\frac{r_0(t, \zeta)}{p_0(t)}\nu(d\zeta) = 0 ; 0 \leq t \leq T. \quad (3.20)$$

It follows from Proposition 2.1 that  $\hat{\varphi} := \tilde{\varphi}$  is an optimal portfolio for problem (2.7) with initial value  $x = p(0)$ .  $\square$

**Example 3.1** As an illustration of Theorem 3.1 let us apply it to the situation when there are no jumps ( $N = 0$ ). Then  $\Theta$  has just one element  $\hat{\theta}$  given by

$$\hat{\theta}(t) = -\frac{b(t)}{\sigma(t)}.$$

So for any given  $y > 0$ ,  $\hat{\theta}$  is optimal for the problem (2.12), and

$$G_{\hat{\theta}}(T) = y \exp \left( - \int_0^T \frac{b(s)}{\sigma(s)} dB(s) - \frac{1}{2} \int_0^T \frac{b^2(s)}{\sigma^2(s)} ds \right). \quad (3.21)$$

Then, by Theorem 3.1b), if  $(p, q)$  is the solution of the BSDE

$$\begin{cases} dp(t) = \frac{q(t)}{\sigma(t)}b(t)dt + q(t)dB(t) ; 0 \leq t \leq T \\ p(T) = -V'(G_{\hat{\theta}}(T)), \end{cases} \quad (3.22)$$

then  $\hat{\varphi}(t) := \frac{q(t)}{\sigma(t)S(t^-)}$  is an optimal portfolio for the problem

$$\sup_{\varphi \in \mathcal{A}} E[U(X_{\varphi}(T))]$$

with initial value  $x = p(0)$ .

In particular, if  $U(x) = \ln x$ , then  $V(y) = -\ln y - 1$  and  $V'(y) = -\frac{1}{y}$ . So the BSDE (3.22) becomes

$$\begin{cases} dp(t) = \frac{q(t)}{\sigma(t)}b(t)dt + q(t)dB(t) ; 0 \leq t \leq T \\ p(T) = \frac{1}{y} \exp \left( \int_0^T \frac{b(s)}{\sigma(s)}dB(s) + \frac{1}{2} \int_0^T \frac{b^2(s)}{\sigma^2(s)}ds \right). \end{cases} \quad (3.23)$$

To solve this equation we try

$$q(t) = p(t) \frac{b(t)}{\sigma(t)}. \quad (3.24)$$

Then

$$dp(t) = p(t) \left[ \frac{b^2(t)}{\sigma^2(t)}dt + \frac{b(t)}{\sigma(t)}dB(t) \right], \quad (3.25)$$

which has the solution

$$p(t) = p(0) \exp \left( \int_0^t \frac{b(s)}{\sigma(s)}dB(s) + \frac{1}{2} \int_0^t \frac{b^2(s)}{\sigma^2(s)}ds \right) ; 0 \leq t \leq T. \quad (3.26)$$

Hence (3.23) holds and we conclude that the optimal portfolio is

$$\hat{\varphi}(t) = p(t) \frac{b(t)}{\sigma^2(t)S(t^-)}. \quad (3.27)$$

for the primal problem with initial value  $x = \frac{1}{y}$ . Note that with this portfolio we get

$$\begin{aligned} dX_{\hat{\varphi}}(t) &= p(t) \frac{b(t)}{\sigma^2(t)} [b(t)dt + \sigma(t)dB(t)] \\ &= p(t) \left[ \frac{b^2(t)}{\sigma^2(t)}dt + \frac{b(t)}{\sigma(t)}dB(t) \right] = dp(t). \end{aligned} \quad (3.28)$$

Therefore

$$\hat{\varphi}(t) = X_{\hat{\varphi}}(t) \frac{b(t)}{\sigma^2(t)S(t^-)} \quad (3.29)$$

which means that the optimal fraction of wealth to be placed in the risky asset is

$$\hat{\pi}(t) = \frac{\hat{\varphi}(t)S(t^-)}{X_{\hat{\varphi}}(t)} = \frac{b(t)}{\sigma^2(t)}, \quad (3.30)$$

which agrees with the classical result of Merton.

## 4 Robust duality

### 4.1 The robust primal problem

In this section we extend our study to a robust optimal portfolio problem and its dual. Thus we replace the price process  $S(t)$  in (2.1) by the perturbed process

$$\begin{cases} dS_\mu(t) = S_\mu(t^-)[(b(t) + \mu(t)\sigma(t))dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta)]; & 0 \leq t \leq T \\ S_\mu(0) > 0, \end{cases} \quad (4.1)$$

for some perturbation process  $\mu(t)$ , assumed to be predictable and satisfy

$$E \left[ \int_0^T |\mu(t)\sigma(t)| dt \right] < \infty.$$

We let  $\mathcal{M}$  denote this set of perturbation processes  $\mu$ .

Let  $\mathcal{A}$  denote the set of portfolios  $\varphi(t)$  such that

$$E \left[ \int_0^T \varphi(t)^2 S_\mu(t)^2 \left\{ (b(t) + \mu(t)\sigma(t))^2 + \sigma^2(t) + \int_{\mathbb{R}} \gamma^2(t, \zeta)\nu(d\zeta) \right\} dt \right] < \infty, \quad (4.2)$$

$$X_{\varphi, \mu}(t) > 0 \text{ for all } t \in [0, T] \text{ a.s.}, \quad (4.3)$$

(2.5), and (2.5), where  $X(t) = X_{\varphi, \mu}(t)$  is the wealth corresponding to  $\varphi$  and  $\mu$ , i.e.

$$\begin{cases} dX(t) = \varphi(t)S_\mu(t^-)[(b(t) + \mu(t)\sigma(t))dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta)]; & 0 \leq t \leq T \\ X(0) = x > 0. \end{cases} \quad (4.4)$$

Let  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  be a convex penalty function, assumed to be  $\mathcal{C}^1$ .

**Definition 4.1** *The robust primal problem is to find  $(\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}$  such that*

$$\inf_{\mu \in \mathcal{M}} \sup_{\varphi \in \mathcal{A}} I(\varphi, \mu) = I(\hat{\varphi}, \hat{\mu}) = \sup_{\varphi \in \mathcal{A}} \inf_{\mu \in \mathcal{M}} I(\varphi, \mu), \quad (4.5)$$

where

$$I(\varphi, \mu) = E \left[ U(X_{\varphi, \mu}(T)) + \int_0^T \rho(\mu(t)) dt \right], \quad (4.6)$$

where  $U$  is as in Section 1.

The problem (4.5) is a stochastic differential game. To handle this, we use an extension of the maximum principle to games, as presented in e.g. [13]. Define the Hamiltonian by

$$H_1(t, x, \varphi, \mu, p_1, q_1, r_1) = \rho(\mu) + \varphi S_\mu(t^-) \left[ (b(t) + \mu\sigma(t))p_1 + \sigma(t)q_1 + \int_{\mathbb{R}} \gamma(t, \zeta)r_1(\zeta)\nu(d\zeta) \right]. \quad (4.7)$$

The associated BSDE for the adjoint processes  $(p_1, q_1, r_1)$  is

$$\begin{cases} dp_1(t) = -\frac{\partial H_1}{\partial x}(t, X(t), \varphi(t), \mu(t), p_1(t), q_1(t), r_1(t))dt \\ \quad + q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t, \zeta) \tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\ p_1(T) = U'(X_{\varphi, \mu}(T)). \end{cases}$$

Since  $\frac{\partial H_1}{\partial x} = 0$ , this reduces to

$$\begin{cases} dp_1(t) = q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t, \zeta) \tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\ p_1(T) = U'(X_{\varphi, \mu}(T)). \end{cases} \quad (4.8)$$

The first order conditions for a maximum point  $\hat{\varphi}$  and a minimum point  $\hat{\mu}$ , respectively, for the Hamiltonian are

$$(b(t) + \hat{\mu}(t)\sigma(t))p_1(t) + \sigma(t)q_1(t) + \int_{\mathbb{R}} \gamma(t, \zeta)r_1(t, \zeta)\nu(d\zeta) = 0 ; t \in [0, T] \quad (4.9)$$

$$\rho'(\hat{\mu}(t)) + \hat{\varphi}(t)S_{\hat{\mu}}(t^-)\sigma(t)p_1(t) = 0 ; t \in [0, T]. \quad (4.10)$$

Since  $H_1$  is concave with respect to  $\varphi$  and convex with respect to  $\mu$ , these first order conditions are also *sufficient* for  $\hat{\varphi}$  and  $\hat{\mu}$  to be a maximum point and a minimum point, respectively. Therefore we obtain the following characterization of a solution (saddle point) of (4.5):

**Theorem 4.2** (*Robust primal problem*) *A pair  $(\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}$  is a solution of the robust primal problem (4.5) if and only if the solution  $(p_1, q_1, r_1)$  of the BSDE*

$$\begin{cases} dp_1(t) = q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t, \zeta) \tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\ p_1(T) = U'(X_{\hat{\varphi}, \hat{\mu}}(T)). \end{cases} \quad (4.11)$$

*satisfies (4.9), (4.10).*

*Alternatively, we can formulate this as follows:*

*$(\varphi, \mu) = (\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}$  is optimal for (4.5) if and only if the solution  $(p_1, q_1, r_1)$  of the FBSDE (4.4) and (4.8) satisfies (4.9)-(4.10).*

## 4.2 The robust dual problem

It is not a priori clear what should be a dual formulation of the robust primal problem in subsection 4.1. One formulation is studied in [5]. Here we will choose a different duality model, as follows:

**Definition 4.3** *The robust dual problem is to find  $\tilde{\theta} \in \Theta, \tilde{\mu} \in \mathcal{M}$  such that*

$$\sup_{\mu \in \mathcal{M}} \sup_{\theta \in \Theta} J(\theta, \mu) = J(\tilde{\theta}, \tilde{\mu}) = \sup_{\theta \in \Theta} \sup_{\mu \in \mathcal{M}} J(\theta, \mu) \quad (4.12)$$

where

$$J(\theta, \mu) = E \left[ -V(G_\theta(T)) - \int_0^T \rho(\mu(t)) dt \right], \quad (4.13)$$

and  $V$  is the convex dual of  $U$ , as in Section 1.

Here  $G_\theta(t) = G_{\theta, \mu}(t)$  is given by

$$\begin{cases} dG_\theta(t) = G_\theta(t^-) \left[ \theta_0(t) dB(t) + \int_{\mathbb{R}} \theta_1(t, \zeta) \tilde{N}(dt, d\zeta) \right]; & 0 \leq t \leq T \\ G_\theta(0) = y > 0 \end{cases} \quad (4.14)$$

with the constraint that if  $y = 1$ , then the measure  $Q_\theta$  defined by

$$dQ_\theta = G_\theta(T) dP \text{ on } \mathcal{F}_T$$

is an ELMM for the perturbed price process  $S_\mu(t)$  in (4.1). By the Girsanov theorem for Itô-Lévy processes [12] this is equivalent to requiring that  $(\theta_0, \theta_1)$  satisfies the equation

$$b(t) + \mu(t)\sigma(t) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) = 0; \quad t \in [0, T]. \quad (4.15)$$

Substituting

$$\theta_0(t) = -\frac{1}{\sigma(t)} \left[ b(t) + \mu(t)\sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d, \zeta) \right] \quad (4.16)$$

into (4.14) we get

$$\begin{cases} dG_\theta(t) = G_\theta(t^-) \left( -\frac{1}{\sigma(t)} \left[ b(t) + \mu(t)\sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) \right] dB(t) \right. \\ \quad \left. + \int_{\mathbb{R}} \theta_1(t, \zeta) \tilde{N}(dt, d\zeta) \right); & 0 \leq t \leq T \\ G_\theta(0) = y > 0. \end{cases} \quad (4.17)$$

The Hamiltonian for the problem (4.12) then becomes

$$\begin{aligned} H_2(t, g, \theta_1, \mu, p_2, q_2, r_2) &= -\rho(\mu) - \frac{gq_2}{\sigma(t)} \left[ b(t) + \mu\sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(\zeta)\nu(d\zeta) \right] \\ &\quad + g \int_{\mathbb{R}} \theta_1(\zeta)r_2(\zeta)\nu(d\zeta). \end{aligned} \quad (4.18)$$



The BSDE for the adjoint processes  $(p_2, q_2, r_2)$  is

$$\begin{cases} dp_2(t) = \left( \frac{q_2(t)}{\sigma(t)} \left[ b(t) + \mu(t)\sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) \right] \right. \\ \quad \left. - \int_{\mathbb{R}} \theta_1(t, \zeta)r_2(t, \zeta)\nu(d\zeta) \right) dt \\ \quad + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t, \zeta)\tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\ p_2(T) = -V'(G_{\tilde{\theta}}(T)). \end{cases} \quad (4.19)$$

The first order conditions for a maximum point  $(\tilde{\theta}, \tilde{\mu})$  for  $H_2$  are

$$(\nabla_{\theta_1} H_2 =) - \frac{q_2(t)}{\sigma(t)} \gamma(t, \zeta) + r_2(t, \zeta) = 0 \quad (4.20)$$

$$\left( \frac{\partial H_2}{\partial \mu} = \right) \rho'(\tilde{\mu}(t)) + G_{\tilde{\theta}}(t)q_2(t) = 0. \quad (4.21)$$

Substituting (4.20) into (4.19) we get

$$\begin{cases} dp_2(t) = \frac{q_2(t)}{\sigma(t)} [b(t) + \tilde{\mu}(t)\sigma(t)] dt \\ \quad + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t, \zeta)\tilde{N}(dt, d\zeta) ; t \in [0, T] \\ p_2(T) = -V'(G_{\tilde{\theta}}(T)). \end{cases} \quad (4.22)$$

Therefore we get the following.

**Theorem 4.4 (Robust dual problem)** *A pair  $(\tilde{\theta}, \tilde{\mu}) \in \Theta \times \mathcal{M}$  is a solution of the robust dual problem (4.12)-(4.13) if and only the solution  $(p_2, q_2, r_2)$  of the BSDE (4.22) also satisfies (4.20)-(4.21).*

*Alternatively, we can formulate this as follows:*

*$(\tilde{\theta}, \tilde{\mu}) \in \Theta \times \mathcal{M}$  is optimal for (4.12)-(4.13) if and only if the solution  $(p_2, q_2, r_2)$  of the FBSDE (4.17) & (4.22) satisfies (4.20)-(4.21).*

### 4.3 From robust primal to robust dual

We now use the characterizations above of the solutions  $(\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}$  and  $(\tilde{\theta}, \tilde{\mu}) \in \Theta \times \mathcal{M}$  of the robust primal and the robust dual problem, respectively, to find the relations between them.

First, assume that  $(\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}$  is a solution of the robust primal problem and let  $(p_1, q_1, r_1)$  be as in Theorem 4.2, i.e. assume that  $(p_1, q_1, r_1)$  solves the FBSDE (4.4) & (4.11) and satisfies (4.9)-(4.10).

We want to find the solution  $(\tilde{\theta}, \tilde{\mu}) \in \Theta \times \mathcal{M}$  of the robust dual problem. By Theorem 4.4 this means that we must find a solution  $(p_2, q_2, r_2)$  of the FBSDE (4.17) & (4.22) which satisfies (4.20)-(4.21).

To this end, choose

$$\tilde{\mu} := \hat{\mu} \quad (4.23)$$

and define

$$\tilde{\theta}_0(t) := \frac{q_1(t)}{p_1(t)} \text{ and } \tilde{\theta}_1(t, \zeta) := \frac{r_1(t, \zeta)}{p_1(t)}. \quad (4.24)$$

Then by (4.9) we have

$$b(t) + \tilde{\mu}(t)\sigma(t) + \sigma(t)\tilde{\theta}_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{\theta}_1(t, \zeta)\nu(d\zeta) = 0. \quad (4.25)$$

Assume that (2.10) holds. Then  $\tilde{\theta} \in \Theta$ .

Substituting (4.24) into (4.8), we obtain

$$\begin{cases} dp_1(t) = p_1(t^-) \left[ \tilde{\theta}_0(t)dB(t) + \int_{\mathbb{R}} \tilde{\theta}_1(t, \zeta)\tilde{N}(dt, d\zeta) \right] ; t \in [0, T] \\ p_1(T) = U'(X_{\hat{\varphi}, \hat{\mu}}(T)). \end{cases} \quad (4.26)$$

Comparing with (4.14) we see that

$$\frac{dG_{\tilde{\theta}}(t)}{G_{\tilde{\theta}}(t)} = \frac{dp_1(t)}{p_1(t)}$$

and hence, for  $y = G_{\tilde{\theta}}(0) = p_1(0) > 0$  we have

$$p_1(t) = G_{\tilde{\theta}}(t) ; t \in [0, T]. \quad (4.27)$$

In particular,

$$U'(X_{\hat{\varphi}, \hat{\mu}}(T)) = G_{\tilde{\theta}}(T). \quad (4.28)$$

Define

$$p_2(t) := X_{\hat{\varphi}, \hat{\mu}}(t), q_2(t) := \hat{\varphi}(t)\sigma(t)S_{\mu}(t^-), r_2(t, \zeta) := \hat{\varphi}(t)\gamma(t, \zeta)S_{\mu}(t^-). \quad (4.29)$$

Then by (4.4) and (4.28), combined with (1.7),

$$\begin{cases} dp_2(t) = \hat{\varphi}(t)S_{\mu}(t^-) \left[ (b(t) + \hat{\mu}(t)\sigma(t))dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right] \\ = \frac{q_2(t)}{\sigma(t)}[b(t) + \hat{\mu}(t)\sigma(t)]dt + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t, \zeta)\tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\ p_2(T) = X_{\hat{\varphi}, \hat{\mu}}(T) = -V'(G_{\tilde{\theta}}(T)). \end{cases} \quad (4.30)$$

Hence  $(p_2, q_2, r_2)$  solves the BSDE (4.22), as requested.

It remains to verify that (4.20) and (4.21) hold: By (4.29) we have

$$-\frac{q_2(t)}{\sigma(t)}\gamma(t, \zeta) + r_2(t, \zeta) = -\hat{\varphi}(t)S_\mu(t^-)\gamma(t, \zeta) + \hat{\varphi}(t)S_\mu(t^-)\gamma(t, \zeta) = 0,$$

which is (4.20).

By (4.23), (4.27), (4.29) and (4.10),

$$\rho'(\tilde{\mu}) + G_{\tilde{\theta}}(t)q_2(t) = \rho'(\hat{\mu}) + p_1(t)\hat{\varphi}(t)\sigma(t)S(t^-) = 0,$$

which is (4.21).

We have thus proved the theorem:

**Theorem 4.5** *Assume  $(\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}$  is a solution of the robust primal problem and let  $(p_1, q_1, r_1)$  be the associated adjoint processes satisfying (4.11). Define*

$$\tilde{\mu} := \hat{\mu} \tag{4.31}$$

$$\tilde{\theta}_0(t) := \frac{q_1(t)}{p_1(t)} ; \tilde{\theta}_1(t, \zeta) = \frac{r_1(t, \zeta)}{p_1(t)}. \tag{4.32}$$

*and suppose they satisfy (2.10). Then, they are optimal for the dual problem with initial value  $y = p_1(0)$ .*

#### 4.4 From robust dual to robust primal

Next, assume that  $(\tilde{\theta}, \tilde{\mu}) \in \Theta \times \mathcal{M}$  is optimal for the robust dual problem (4.12)-(4.13) and let  $(p_2, q_2, r_2)$  be as in Theorem 4.4.

We will find  $(\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}$  and  $(p_1, q_1, r_1)$  satisfying Theorem 4.2. Choose

$$\hat{\mu} := \tilde{\mu} \tag{4.33}$$

and define

$$\hat{\varphi}(t) := \frac{q_2(t)}{\sigma(t)S_\mu(t^-)} ; t \in [0, T]. \tag{4.34}$$

Assume that  $\hat{\varphi}$  is admissible. Then by (4.22) and (4.20)

$$\begin{cases} dp_2(t) = \hat{\varphi}(t)S_\mu(t^-) \left[ (b(t) + \hat{\mu}(t)\sigma(t))dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \\ p_2(T) = -V'(G_{\tilde{\theta}}(T)). \end{cases}$$

Hence

$$dp_2(t) = dX_{\hat{\varphi}, \hat{\mu}}(t)$$

and we obtain that, for  $x = p_2(0) > 0$ ,

$$p_2(T) = X_{\hat{\varphi}, \hat{\mu}}(T). \tag{4.35}$$

Therefore,

$$X_{\hat{\varphi}, \hat{\mu}}(T) = -V'(G_{\tilde{\theta}}(T)), \text{ i.e. } G_{\tilde{\theta}}(T) = U'(X_{\hat{\varphi}, \hat{\mu}}(T)). \quad (4.36)$$

We now verify that with  $\varphi = \hat{\varphi}$ ,  $\mu = \hat{\mu}$ , and  $p_1, q_1, r_1$  defined by

$$p_1(t) := G_{\tilde{\theta}}(t), q_1(t) := G_{\tilde{\theta}}(t)\tilde{\theta}_0(t), r_1(t, \zeta) := G_{\tilde{\theta}}(t)\tilde{\theta}_1(t, \zeta). \quad (4.37)$$

all the conditions of Theorem 4.2 hold: By (4.17) and (4.36),

$$\begin{cases} dp_1(t) = dG_{\tilde{\theta}}(t) = G_{\tilde{\theta}}(t^-) \left( -\frac{1}{\sigma(t)} \left[ b(t) + \hat{\mu}(t)\sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) \right] dB(t) \right. \\ \quad \left. + \int_{\mathbb{R}} \theta_1(t, \zeta)\tilde{N}(dt, d\zeta) \right); 0 \leq t \leq T \\ p_1(T) = G_{\tilde{\theta}}(T) = U'(X_{\hat{\varphi}, \hat{\mu}}(T)). \end{cases} \quad (4.38)$$

Hence (4.11) holds.

It remains to verify (4.9) and (4.10). By (4.37) and (4.15) for  $\theta = \tilde{\theta}$ , we get

$$\begin{aligned} & (b(t) + \hat{\mu}(t)\sigma(t))p_1(t) + \sigma(t)q_1(t) + \int_{\mathbb{R}} \gamma(t, \zeta)r_1(t, \zeta)\nu(d\zeta) \\ &= G_{\tilde{\theta}}(t) \left[ b(t) + \hat{\mu}(t)\sigma(t) + \sigma(t)\tilde{\theta}_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{\theta}_1(t, \zeta)\nu(d\zeta) \right] = 0, \end{aligned}$$

which is (4.9).

By (4.33), (4.47), (4.37) and (4.21) we get

$$\rho'(\hat{\mu}(t)) + \hat{\varphi}(t)S_{\mu}(t^-)\sigma(t)p_1(t) = \rho'(\tilde{\mu}(t)) + q_2(t)G_{\tilde{\theta}}(t) = 0,$$

which is (4.10).

We have thus proved the theorem

**Theorem 4.6** *Let  $(\tilde{\theta}, \tilde{\mu}) \in \Theta \times \mathcal{M}$  be optimal for the robust dual problem and let  $(p_2, q_2, r_2)$  be the associated adjoint processes satisfying (4.22). Define*

$$\hat{\mu} := \tilde{\mu} \quad (4.39)$$

$$\hat{\varphi}_t := \frac{q_2(t)}{\sigma_t S_{\tilde{\mu}}(t^-)}; t \in [0, T]. \quad (4.40)$$

*Assume that  $\hat{\varphi} \in \mathcal{A}$ . Then  $(\hat{\mu}, \hat{\varphi}_t)$  are optimal for primal problem with initial value  $x = p_2(0)$ .*

*Remark 4.7* Note that the optimal adjoint process  $p_1$  for the robust primal problem coincides with the optimal density process  $G_{\tilde{\theta}}$  for the robust dual problem.

Similarly, the optimal adjoint process  $p_2$  for the robust dual problem coincides with the optimal state process  $X_{\hat{\varphi}}$  for the robust primal problem.

**Example 4.1** We consider a robust version of Example 3.1. We want to study the robust primal problem

$$\inf_{\mu \in \mathcal{M}} \sup_{\varphi \in \mathcal{A}} E \left[ U(X_{\varphi, \mu}(T)) + \int_0^T \rho(\mu(t)) dt \right]. \quad (4.41)$$

in the case with no jumps ( $N = \gamma = 0$ ). In this case there is only one ELMM for the price process  $S_\mu(t)$  for each given  $\mu(t)$ . So the corresponding robust dual problem simplifies to a plain stochastic control problem

$$\sup_{\mu \in \mathcal{M}} E \left[ -V(G_\mu(T)) - \int_0^T \rho(\mu(t)) dt \right], \quad (4.42)$$

where

$$\begin{cases} dG_\mu(t) = -G_\mu(t^-) \left[ \frac{b_t}{\sigma_t} + \mu(t) \right] dB_t; & 0 \leq t \leq T \\ G_\mu(0) = y > 0. \end{cases} \quad (4.43)$$

The first order conditions for the Hamiltonian reduces to:

$$\tilde{\mu}(t) = (\rho')^{-1}(-G_{\tilde{\mu}}(t)q(t)) \quad (4.44)$$

which substituted into the adjoint BSDE equation gives:

$$\begin{cases} dp(t) = q(t) \left[ \frac{b_t}{\sigma_t} + (\rho')^{-1}(-G_{\tilde{\mu}}(t)q(t)) \right] dt + q(t) dB_t; & t \in [0, T] \\ p(T) = -V'(G_{\tilde{\mu}}(T)). \end{cases} \quad (4.45)$$

We get that  $\tilde{\mu}$  is optimal for the robust dual problem if and only if there is a solution  $(p, q, G_{\tilde{\mu}})$  of the FBSDE consisting of (4.45) and

$$\begin{cases} dG_{\tilde{\mu}}(t) = -G_{\tilde{\mu}}(t^-) \left[ \frac{b_t}{\sigma_t} + \tilde{\mu}(t) \right] dB_t; & 0 \leq t \leq T \\ G_{\tilde{\mu}}(0) = y > 0 \end{cases} \quad (4.46)$$

Hence, the optimal  $\hat{\mu}$  for the primal robust problem is given by  $\hat{\mu} := \tilde{\mu}$ , and the optimal portfolio is

$$\hat{\varphi}_t = \frac{q(t)}{\sigma_t S_{\tilde{\mu}}(t^-)}; \quad t \in [0, T]. \quad (4.47)$$

We have proved:

**Theorem 4.8** *The solution  $\hat{\mu}, \hat{\varphi}$  of the robust primal problem (4.41) is given by (4.44) and (4.47), respectively, where  $(G_{\tilde{\mu}}, p, q)$  is the solution of the FBSDE (4.46)-(4.45).*

# A Maximum principles for optimal control

Consider the following controlled stochastic differential equation

$$\begin{aligned} dX(t) &= b(t, X(t), u(t), \omega)dt + \sigma(t, X(t), u(t), \omega)dB(t) \\ &+ \int_{\mathbb{R}} \gamma(t, X(t), u(t), \omega, \zeta)\tilde{N}(dt, d\zeta); 0 \leq t \leq T \\ X(0) &= x \in \mathbb{R} \end{aligned} \tag{A.1}$$

The performance functional is given by

$$J(u) = E \left[ \int_0^T f(t, X(t), u(t), \omega)dt + \phi(X(T), \omega) \right] \tag{A.2}$$

where  $T > 0$  and  $u$  is in a given family  $\mathcal{A}$  of admissible  $\mathcal{F}$ -predictable controls. For  $u \in \mathcal{A}$  we let  $X^u(t)$  be the solution of (A.1). We assume this solution exists, is unique and satisfies, for some  $\epsilon > 0$ ,

$$E \left[ \int_0^T |X^u(t)|^{2+\epsilon} dt \right] < \infty. \tag{A.3}$$

We want to find  $u^* \in \mathcal{A}$  such that

$$\sup_{u \in \mathcal{A}} J(u) = J(u^*). \tag{A.4}$$

We make the following assumptions

$$f \in C^1 \text{ and } E \left[ \int_0^T |\nabla f|^2(t) dt \right] < \infty, \tag{A.5}$$

$$b, \sigma, \gamma \in C^1 \text{ and } E \left[ \int_0^T (|\nabla b|^2 + |\nabla \sigma|^2 + \|\nabla \gamma\|^2)(t) dt \right] < \infty, \tag{A.6}$$

$$\begin{aligned} &\text{where } \|\nabla \gamma(t, \cdot)\|^2 := \int_{\mathbb{R}} \gamma^2(t, \zeta) \nu(d\zeta) \\ &\phi \in C^1 \text{ and for all } u \in \mathcal{A}, \exists \epsilon \text{ s.t. } E[\phi'(X(T))^{2+\epsilon}] < \infty. \end{aligned} \tag{A.7}$$

Let  $\mathbb{U}$  be a convex closed set containing all possible control values  $u(t); t \in [0, T]$ .

The Hamiltonian associated to the problem (A.4) is defined by

$$H : [0, T] \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \mapsto \mathbb{R}$$

$$H(t, x, u, p, q, r, \omega) = f(t, x, u, \omega) + b(t, x, u, \omega)p + \sigma(t, x, u, \omega)q + \int_{\mathbb{R}} \gamma(t, x, u, \zeta, \omega)r(t, \zeta)\nu(d\zeta).$$

For simplicity of notation the dependence on  $\omega$  is suppressed in the following. We assume that  $H$  is Fréchet differentiable in the variables  $x, u$ . We let  $m$  denote the Lebesgue measure on  $[0, T]$ .

The associated BSDE for the adjoint processes  $(p, q, r)$  is

$$\begin{cases} dp(t) = -\frac{\partial H}{\partial x}(t) + q(t)dB(t) + \int_{\mathbb{R}} r(t, \zeta)\tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ p(T) = \phi'(X(T)). \end{cases} \quad (\text{A.8})$$

Here and in the following we are using the abbreviated notation

$$\frac{\partial H}{\partial x}(t) = \frac{\partial}{\partial x}(t, X(t), u(t)) \text{ etc}$$

We first formulate a sufficient maximum principle, with weaker conditions than in [13].

**Theorem A.1 (Sufficient maximum principle)** *Let  $\hat{u} \in \mathcal{A}$  with corresponding solutions  $\hat{X}, \hat{p}, \hat{q}, \hat{r}$  of equations (A.1)-(A.8). Assume the following:*

- *The function  $x \mapsto \phi(x)$  is concave*
- *(The Arrow condition) The function*

$$\mathcal{H}(x) := \sup_{v \in \mathbb{U}} H(t, x, v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \quad (\text{A.9})$$

*is concave for all  $t \in [0, T]$ .*

•

$$\sup_{v \in \mathbb{U}} H(t, \hat{X}(t), v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) = H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)); \quad t \in [0, T]. \quad (\text{A.10})$$

*Then  $\hat{u}$  is an optimal control for the problem (A.4).*

Next, we state a necessary maximum principle. For this, we need the following assumptions:

- For all  $t_0 \in [0, T]$  and all bounded  $\mathcal{F}_{t_0}$ -measurable random variables  $\alpha(\omega)$  the control

$$\beta(t) := \chi_{[t_0, T]}(t)\alpha(\omega)$$

belongs to  $\mathcal{A}$ .

- For all  $u, \beta \in \mathcal{A}$  with  $\beta$  bounded, there exists  $\delta > 0$  such that the control

$$\tilde{u}(t) := u(t) + a\beta(t); \quad t \in [0, T]$$

belongs to  $\mathcal{A}$  for all  $a \in (-\delta, \delta)$ .

- The derivative process

$$x(t) := \frac{d}{da} X^{u+a\beta}(t) |_{a=0},$$

exists and belongs to  $L^2(dm \times dP)$ , and

$$\begin{cases} dx(t) = \left\{ \frac{\partial b}{\partial x}(t)x(t) + \frac{\partial b}{\partial u}(t)\beta(t) \right\} dt + \left\{ \frac{\partial \sigma}{\partial x}(t)x(t) + \frac{\partial \sigma}{\partial u}(t)\beta(t) \right\} dB(t) \\ \quad + \int_{\mathbb{R}} \left\{ \frac{\partial \gamma}{\partial x}(t, \zeta)x(t) + \frac{\partial \gamma}{\partial u}(t, \zeta)\beta(t) \right\} \tilde{N}(dt, d\zeta) \\ x(0) = 0 \end{cases} \quad (\text{A.11})$$

**Theorem A.2 (Necessary maximum principle)** *The following are equivalent*

- 

$$\frac{d}{da} J(u + a\beta) |_{a=0} = 0 \text{ for all bounded } \beta \in \mathcal{A}$$

- 

$$\frac{\partial H}{\partial u}(t) = 0 \text{ for all } t \in [0, T].$$

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