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Sobolev Differentiable Stochastic Flows for SDE's with Singular Coefficients: Applications to the Transport Equation

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Abstract

In this paper, we establish the existence of a stochastic flow of Sobolev diffeomorphisms

$$\mathbb{R}^d \ni x \longmapsto \phi_{s,t}(x) \in \mathbb{R}^d, \; s,t \in \mathbb{R},$$

for a stochastic differential equation (SDE) of the form

$$dX_t = b(t, X_t)\,dt + dB_t, \; s,t \in \mathbb{R}, \quad X_s = x \in \mathbb{R}^d.$$

The above SDE is driven by a *bounded measurable* drift coefficient $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and a d-dimensional Brownian motion B. More specifically, we show that the stochastic flow $\phi_{s,t}(\cdot)$ of the SDE lives in the space $L^2(\Omega; W^{1,p}(\mathbb{R}^d, w))$ for all s,t and all $p > 1$, where $W^{1,p}(\mathbb{R}^d, w)$ denotes a weighted Sobolev space with weight w possessing a p-th moment with respect to Lebesgue measure on $\mathbb{R}^d$. This result is counter-intuitive, since the dominant 'culture' in stochastic (and deterministic) dynamical systems is that the flow 'inherits' its spatial regularity from the driving vector fields.

The spatial regularity of the stochastic flow yields existence and uniqueness of a Sobolev differentiable weak solution of the (Stratonovich) stochastic transport equation

$$\begin{cases} d_t u(t,x) + & (b(t,x) \cdot Du(t,x))dt + \sum_{i=1}^d e_i \cdot Du(t,x) \circ dB_t^i = 0, \\ u(0,x) = & u_0(x), \end{cases}$$

where b is *bounded and measurable*, u_0 is C_b^1 and $\{e_i\}_{i=1}^d$ a basis for $\mathbb{R}^d$. It is well-known that the deterministic counter part of the above equation does not in general have a solution.

Using stochastic perturbations and our analysis of the above SDE, we establish a deterministic flow of Sobolev diffeomorphisms for classical one-dimensional (deterministic) ODE's

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driven by *discontinuous* vector fields. Furthermore, and as a corollary of the latter result, we construct a Sobolev stochastic flow of diffeomorphisms for one-dimensional SDE's driven by *discontinuous diffusion* coefficients.

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1 Introduction

In this article we analyze the spatial regularity in the initial condition $x \in \mathbb{R}^d$ for strong solutions $X_\cdot^x$ to the d-dimensional stochastic differential equation (SDE)

$$X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x})du + B_t - B_s, \;\; s,t \in \mathbb{R}. \tag{1}$$

In the above SDE, the drift coefficient $b : \mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ is only *Borel measurable and bounded*, and the equation is driven by standard Brownian motion $B_\cdot$ in $\mathbb{R}^d$.

More specifically, we construct a two-parameter pathwise Sobolev differentiable stochastic flow

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s,t,x) \longmapsto \phi_{s,t}(x) \in \mathbb{R}^d$$

for the SDE (1) such that each flow map

$$\mathbb{R}^d \ni x \longmapsto \phi_{s,t}(x) \in \mathbb{R}^d$$

is a Sobolev diffeomorphism in the sense that

$$\phi_{s,t}(\cdot) \text{ and } \phi_{s,t}^{-1}(\cdot) \;\in L^2(\Omega, W^{1,p}(\mathbb{R}^d; w)) \tag{2}$$

for all $s,t \in \mathbb{R}$, all $p > 1$. In (2) above, $W^{1,p}(\mathbb{R}^d, w)$ denotes a weighted Sobolev space of mappings $\mathbb{R}^d \to \mathbb{R}^d$ with any measurable weight function $w : \mathbb{R}^d \to [0,\infty)$ satisfying the integrability requirement

$$\int_{\mathbb{R}^d} (1 + |x|^p) w(x) dx < \infty. \tag{3}$$

In particular, $\phi_{s,t}(\cdot)$ is locally $\alpha-$Hölder continuous for all $\alpha < 1$. When the SDE (1) is autonomous, we show further that the stochastic flow corresponds to a Sobolev differentiable perfect cocycle on $\mathbb{R}^d$. For precise statements of the above results, see Theorem 3 and Corollary 5 in the next section.

In this article we offer a novel approach for constructing a Sobolev differentiable stochastic flow for the SDE (1). Our approach is based on Malliavin calculus ideas coupled with new probabilistic estimates on the spatial weak derivatives of solutions of the SDE. A unique (pleasantly surprising) feature of these estimates is that they do not depend on the spatial regularity of the drift coefficient b. Needless to say, the existence of differentiable flows for SDE's with measurable drifts is counter-intuitive: The dominant 'culture' in stochastic (and deterministic) dynamical systems is that the flow 'inherits' its spatial regularity from the driving vector fields. Furthermore, in the stochastic setting, the stochastic flow is in general even a little 'rougher' in the space variable than the driving vector fields. (cf. [22], [28]).

The existence of a Sobolev differentiable stochastic flow for the SDE (1) is exploited (Section 3) to obtain a unique weak solution $u(t,x)$ of the (Stratonovich) stochastic transport equation

$$\begin{cases} d_t u(t,x) + & (b(t,x)\cdot Du(t,x))dt + \sum_{i=1}^d e_i \cdot Du(t,x)\circ dB_t^i = 0 \\ u(0,x) = & u_0(x), \end{cases} \tag{4}$$

when b is just bounded and measurable, $u_0 \in C_b^1(\mathbb{R}^d)$, and $\{e_i\}_{i=1}^d$ a basis for $\mathbb{R}^d$. This result is surprising since the corresponding deterministic transport equation is in general ill-posed. Cf. [1], [8]. We also note that our result holds without the existence of the divergence of b; and furthermore, our solutions are spatially (and also Malliavin) Sobolev differentiable (cf. [14]).

In Section 4, we apply the ideas of Section 2 to show the existence of a family of solutions $\tilde{X}_t^x$ of the one-dimensional ODE

$$\frac{d\tilde{X}_t}{dt} = b(\tilde{X}_t),\ \ t\in\mathbb{R}, \quad \tilde{X}_0 = x\in\mathbb{R}, \tag{5}$$

which are locally of class $W^{1,2}$ in x (Theorem 26, Section 4). This result is obtained under the requirement that the coefficient b is monotone decreasing and is either bounded above or below. The proof of the result uses a stochastic perturbation argument via small Brownian noise coupled with local time techniques. As far as we know, it appears that the above result is new. Furthermore, solutions to the ODE (5) generate a one-parameter group of $W^{1,2}$ diffeomorphisms of $\mathbb{R}$ onto itself. As a consequence of the above result, we construct a $W^{1,2}$ perfect cocycle of diffeomorphisms for solutions of the one-dimensional Stratonovich SDE:

$$dX_t^x = b(X_t^x)\circ dB_t,\ \ t\in\mathbb{R}, \quad X_0^x = x\in\mathbb{R}. \tag{6}$$

It is surprising that such regularity of the flow is feasible despite the inherent discontinuities in the driving vector field of in the ODE (5) and the SDE (6). SDE's with discontinuous coefficients and driven by Brownian motion (or more general noise) have been an important area of study in stochastic analysis and other related branches of mathematics. Important applications of this class of SDE's pertain to the modeling of the dynamics of interacting particles in statistical mechanics and the description of a variety of other random phenomena in areas such as biology or engineering. See e.g. [33] or [23] and the references therein.

Using estimates of solutions of parabolic PDE's and the Yamada-Watanabe principle, the existence of a global unique strong solution to the SDE (1) was first established by A.K. Zvonkin [41] in the $1-$dimensional case, when b is bounded and measurable. The latter work is a significant development in the theory of SDE's. Subsequently, the result was generalized by A.Y. Veretennikov [39] to the multi-dimensional case. More recently, N.V. Krylov and M. Röckner employed local integrability criteria on the drift coefficient b to obtain unique strong solutions of (1) by using an argument of N. I. Portenko [33]. An alternative approach, which doesn't rely on a pathwise uniqueness argument and which also yields the Malliavin differentiability of solutions to (1) was recently developed in [27], [26]. We also refer to the recent article [5] for an extension of the previous results to a Hilbert space setting. In [5], the authors employ techniques based on solutions of infinite-dimensional Kolmogorov equations.

Another important issue in the study of SDE's with (bounded) measurable coefficients is the regularity of their solutions with respect to the initial data and the existence of stochastic flows. See [22], [28] for more information on the existence and regularity of stochastic flows for SDE's, and [29], [30] in the case of stochastic differential systems with memory. Using the method of stochastic characteristics, stochastic flows may be employed to prove uniqueness of solutions of stochastic transport equations under weak regularity hypotheses on the drift coefficient b. See for example [14], where the authors use estimates of solutions of backward Kolmogorov equations to show the existence of a stochastic flow of diffeomorphisms with $\alpha^{\prime}$-Hölder continuous derivatives for $\alpha^{\prime} < \alpha$, where $b \in C([0,1]; C_b^{\alpha}(\mathbb{R}^d))$, and $C_b^{\alpha}(\mathbb{R}^d)$ is the space of bounded $\alpha-$Hölder continuous functions. A similar result also holds true, when $b \in L^q([0,1]; L^p(\mathbb{R}^d))$ for p, q such that $p \geq 2, q > 2, \frac{d}{p} + \frac{2}{q} < 1$. See [12]. Here the authors construct, for any $\alpha \in (0,1)$, a stochastic flow of α-Hölder continuous homeomorphisms for the SDE (1). Furthermore, it is shown in [12] that the map

$$\mathbb{R}^d \ni x \longmapsto X_{\cdot}^x \in L^p([0,1] \times \Omega; \mathbb{R}^d)$$

is differentiable in the $L^p(\Omega)-$sense for every $p \geq 2$.

The approach used in [12] is based on a Zvonkin-type transformation [41] and estimates of solutions of an associated backward parabolic PDE. We also mention the recent related works [11], [10] and [2]. For an overview of this topic the reader may also consult the book [15].[5] In this connection, it should be noted that our method for constructing a stochastic flow for the SDE (1) is heavily dependent on Malliavin calculus ideas together with some difficult probabilistic estimates (cf. [26]).

Our paper is organized as follows: In Section, 2 we introduce basic definitions and notations and provide some auxiliary results that are needed to prove the existence of a Sobolev differentiable stochastic flow for the SDE (1). See Theorem 3 and Corollary 5 in Section 2. We also briefly discuss a specific extension of this result to SDE's with multiplicative noise. In Section 3 we give an application of our approach to the construction of a unique Sobolev differentiable solution to the (Stratonovich) stochastic transport equation (4). Ideas developed in Section 2 are used in Section 4 to show the existence and regularity of a deterministic flow for the one-dimensional ODE (5), and a perfect cocycle for the one-dimensional SDE (6).

2 Existence of a Sobolev Differentiable Stochastic Flow

Throughout this paper we denote by $B_t = (B_t^{(1)}, ..., B_t^{(d)})$, $t \in \mathbb{R}$, $d-$dimensional Brownian motion on the complete Wiener space $(\Omega, \mathcal{F}, \mu)$ where $\Omega := C(\mathbb{R}; \mathbb{R}^d)$ is given the compact open topology and $\mathcal{F}$ is its μ-completed Borel σ-field with respect to Wiener measure μ.

In order to describe the cocycle associated with the stochastic flow of our SDE, we define the μ-preserving (ergodic) Wiener shift $\theta(t, \cdot) : \Omega \to \Omega$ by

$$\theta(t, \omega)(s) := \omega(t+s) - \omega(t),\ \omega \in \Omega,\ t, s \in \mathbb{R}.$$

[5] After completing the preparation of this article, personal communication with F. Flandoli indicated work in preparation with E. Fedrizzi [13] on similar issues regarding the regularity of stochastic flows for SDE's, using a different approach.

The Brownian motion is then a *perfect helix* with respect to θ: That is

$$B_{t_1+t_2}(\omega) - B_{t_1}(\omega) = B_{t_2}(\theta(t_1,\omega))$$

for all $t_1, t_2 \in \mathbb{R}$ and all $\omega \in \Omega$. The above helix property is a convenient pathwise expression of the fact that Brownian motion B has stationary ergodic increments.

Our main focus of study in this section is the d-dimensional SDE

$$X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x})du + B_t - B_s, \quad s,t \in \mathbb{R},\, x \in \mathbb{R}^d, \tag{7}$$

where the drift coefficient $b : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a bounded Borel measurable function.

It is known that the above SDE has a unique strong global solution $X_\cdot^{s,x}$ for each $x \in \mathbb{R}^d$ ([39] or [26], [27]).

Here, we will establish the existence of a *Sobolev-differentiable* stochastic flow of diffeomorphisms for the SDE (7).

Definition 1 *A map $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s,t,x,\omega) \longmapsto \phi_{s,t}(x,\omega) \in \mathbb{R}^d$ is a* **stochastic flow of homeomorphisms** *for the SDE (7) if there exists a universal set $\Omega^* \in \mathcal{F}$ of full Wiener measure such that for all $\omega \in \Omega^*$, the following statements are true:*
(i) For any $x \in \mathbb{R}^d$, the process $\phi_{s,t}(x,\omega)$, $s,t \in \mathbb{R}$, is a strong global solution to the SDE (7).
(ii) $\phi_{s,t}(x,\omega)$ is continuous in $(s,t,x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$.
(iii) $\phi_{s,t}(\cdot,\omega) = \phi_{u,t}(\cdot,\omega) \circ \phi_{s,u}(\cdot,\omega)$ for all $s,u,t \in \mathbb{R}$.
(iv) $\phi_{s,s}(x,\omega) = x$ for all $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$.
(v) $\phi_{s,t}(\cdot,\omega) : \mathbb{R}^d \to \mathbb{R}^d$ are homeomorphisms for all $s,t \in \mathbb{R}$.

A stochastic flow $\phi_{s,t}(\cdot,\omega)$ of homeomorphisms is said to be *Sobolev-differentiable* if for all $s,t \in \mathbb{R}$, the maps $\phi_{s,t}(\cdot,\omega)$ and $\phi_{s,t}^{-1}(\cdot,\omega)$ are Sobolev-differentiable in the sense described below.

From now on we use $|\cdot|$ to denote the norm of a vector in $\mathbb{R}^d$ or a matrix in $\mathbb{R}^{d\times d}$.

In order to prove the existence of a Sobolev differentiable flow for the SDE (7), we need to introduce a suitable class of weighted Sobolev spaces. Fix $p \in (1,\infty)$ and let $w : \mathbb{R}^d \longrightarrow (0,\infty)$ be a Borel measurable function satisfying

$$\int_{\mathbb{R}^d} (1+|x|^p)w(x)dx < \infty. \tag{8}$$

Let $L^p(\mathbb{R}^d, w)$ denote the Banach space of all Borel measurable functions $u = (u_1, ..., u_d) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |u(x)|^p\, w(x)dx < \infty, \tag{9}$$

and equipped with the norm

$$\|u\|_{L^p(\mathbb{R}^d,w)} := \left[\int_{\mathbb{R}^d} |u(x)|^p\, w(x)dx \right]^{1/p}.$$

Furthermore, denote by $W^{1,p}(\mathbb{R}^d, w)$ the linear space of functions $u \in L^p(\mathbb{R}^d, w)$ with weak partial derivatives $D_j u \in L^p(\mathbb{R}^d, w)$ for $j = 1, ..., d$. We equip this space with the complete norm

$$\|u\|_{1,p,w} := \|u\|_{L^p(\mathbb{R}^d,w)} + \sum_{i,j=1}^{d} \|D_j u_i\|_{L^p(\mathbb{R}^d,w)} \,. \tag{10}$$

We will show that the strong solution $X_t^{s,\cdot}$ of the SDE (7) is in $L^2(\Omega, L^p(\mathbb{R}^d, w))$ when $p > 1$ (see Corollary 11). In fact, the SDE (7) implies the following estimate:

$$|X_t^{s,x}|^p \leq c_p(|x|^p + |t-s|^p \|b\|_\infty^p + |B_t - B_s|^p).$$

for all $s, t \in \mathbb{R}$, $x \in \mathbb{R}^d$.

On the other hand, it is easy to see that the solutions $X_t^{s,\cdot}$ of SDE (7) are in general not in $L^p(\mathbb{R}^d, dx)$ with respect to Lebesgue measure dx on $\mathbb{R}^d$: Just consider the special trivial case $b \equiv 0$. This implies that solutions of the SDE (7) (if they exist) may not belong to the Sobolev space $W^{1,p}(\mathbb{R}^d, dx), p > 1$. However, we will show that such solutions do indeed belong to the weighted Sobolev spaces $W^{1,p}(\mathbb{R}^d, w)$ for $p \geq 1$.

Remark 2 *(i) Let $w : \mathbb{R}^d \longrightarrow (0, \infty)$ be a weight function in Muckenhoupt's A_p-class $(1 < p < \infty)$, that is a locally (Lebesgue) integrable function on $\mathbb{R}^d$ such that*

$$\sup \left(\frac{1}{\lambda_d(B)} \int_B w(x)dx \right) \left(\frac{1}{\lambda_d(B)} \int_B (w(x))^{1/(1-p)} dx \right)^{p-1} =: c_{w,p} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^d$ and λ_d is Lebesgue measure on $\mathbb{R}^d$. For example the function $w(x) = |x|^\gamma$ is an A_p-weight iff $-d < \gamma < d(p-1)$. Other examples of weights are given by positive superharmonic functions. See e.g. [18] and [21] and the references therein. Denote by $H^{1,p}(\mathbb{R}^d, w)$ the completion of $C^\infty(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{1,p,w}$ in (10). If w is a A_p-weight, then we have

$$W^{1,p}(\mathbb{R}^d, w) = H^{1,p}(\mathbb{R}^d, w)$$

for all $1 < p < \infty$. See e.g. [18].
(ii) Let $p_0 = \inf\{q > 1 : w$ is a A_q-weight$\}$ and let $u \in W^{1,p}(\mathbb{R}^d, w)$. If $p_0 < p/d$, then u is locally Hölder continuous with any exponent α such that $0 < \alpha < 1 - dp_0/p$.

We now state our main result in this section which gives the existence of a Sobolev differentiable stochastic flow for the SDE (7).

Theorem 3 *In the SDDE (7), assume that the drift coefficient b is Borel-measurable and bounded. Then the SDE (7) has a Sobolev differentiable stochastic flow $\phi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$, $s, t \in \mathbb{R}$: That is*

$$\phi_{s,t}(\cdot) \text{ and } \phi_{s,t}^{-1}(\cdot) \ \in L^2(\Omega, W^{1,p}(\mathbb{R}^d, w))$$

for all $s, t \in \mathbb{R}$ and all $p > 1$.

Remark 4 *If w is a A_p-weight then it follows from Remark 2 (ii) that a version of $\phi_{s,t}(\cdot)$ is locally Hölder continuous for all $0 < \alpha < 1$ and all s, t.*

The following corollary is a consequence of Theorem 3 and the helix property of the Brownian motion.

Corollary 5 *Consider the autonomous SDE*

$$X_t^{s,x} = x + \int_s^t b(X_u^{s,x})du + B_t - B_s, \quad s,t \in \mathbb{R}, \tag{11}$$

with bounded Borel-measurable drift $b : \mathbb{R}^d \to \mathbb{R}^d$. Then the stochastic flow of the SDE (11) has a version which generates a perfect Sobolev-differentiable cocycle $(\phi_{0,t}, \theta(t,\cdot))$ where $\theta(t,\cdot) : \Omega \to \Omega$ is the μ-preserving Wiener shift. More specifically, the following perfect cocycle property holds ***for all*** *$\omega \in \Omega$ and all $t_1, t_2 \in \mathbb{R}$:*

$$\phi_{0,t_1+t_2}(\cdot,\omega) = \phi_{0,t_2}(\cdot,\theta(t_1,\omega)) \circ \phi_{0,t_1}(\cdot,\omega)$$

We will prove Theorem 3 through a sequence of lemmas and propositions. We begin by stating our main proposition:

Proposition 6 *Let $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be bounded and measurable. Let U be an open and bounded subset of $\mathbb{R}^d$. For each $t \in \mathbb{R}$ and $p > 1$ we have*

$$X_t^{\cdot} \in L^2(\Omega; W^{1,p}(U))$$

We will prove Proposition 6 using two steps. In the **first step** , we show that for a bounded smooth function $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ with compact support, it is possible to estimate the norm of $X_t^{\cdot}$ in $L^2(\Omega, W^{1,p}(U))$ independently of the size of b', with the estimate depending only on $\|b\|_\infty$. To do this we use the same technique as introduced in [26].

In the **second step**, we will approximate our bounded measurable coefficient b by a sequence $\{b_n\}_{n=1}^\infty$ of smooth compactly supported functions as in step 1. We then show that the corresponding sequence $X_t^{n,\cdot}$ of solutions is relatively compact in $L^2(\Omega)$ when integrated against a test function on $\mathbb{R}^d$. By step 1 we use weak compactness of the above sequence in $L^2(\Omega, W^{1,p}(U))$ to conclude that the limit point $X_t^{\cdot}$ of the above sequence must also lie in this space.

We now turn to the first step of our procedure. Note that if b is a compactly supported smooth function, the corresponding solution of the SDE (1) is (strongly) differentiable with respect to x, and the first order spatial Jacobian $\frac{\partial}{\partial x} X_t^x$ satisfies the linearized random ODE

$$\begin{cases} d\frac{\partial}{\partial x} X_t^x = & b'(t, X_t^x)\frac{\partial}{\partial x} X_t^x dt \\ \frac{\partial}{\partial x} X_0^x = & \mathcal{I}_d \end{cases}, \tag{12}$$

where $\mathcal{I}_d$ is the $d \times d$ identity matrix and $b'(t,x) = \left(\frac{\partial}{\partial x_i} b^{(j)}(t,x)\right)_{1\le i,j\le d}$ denotes the spatial Jacobian derivative of b.

A key estimate in the first step of the argument is provided by the following proposition:

Proposition 7 *Assume that b is a smooth function with compact support. Then for any $p \in [1,\infty)$ and $t \in \mathbb{R}$, we have the following estimate for the solution of the linearized equation (12):*

$$\sup_{x\in\mathbb{R}^d} E[|\frac{\partial}{\partial x} X_t^x|^p] \le C_{d,p}(\|b\|_\infty)$$

where $C_{d,p}$ is an increasing continuous function depending only on d and p.

The proof of Proposition 7 relies on the following of sequence lemmas which provide estimates on expressions depending on the Gaussian distribution and its derivatives. To this end we define for $P(t,z) := (2\pi t)^{d/2}e^{-|z|^2/2t}$, $t > 0$, where $|z|$ is the Euclidean norm of a vector $z \in \mathbb{R}^d$.

Lemma 8 *Let $\phi, h : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions such that $|\phi(s,z)| \le e^{-\|z\|^2/3s}$ and $\|h\|_\infty \le 1$. Also let $\alpha, \beta \in \{0,1\}^d$ be multiindices such that $|\alpha| = |\beta| = 1$. Then there exists a universal constant C (independent of ϕ, h, α and β) such that*

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)dydzdsdt \right| \le C\,.$$

Furthermore, there is a universal positive constant (also denoted by) C such that for measurable functions g and h bounded by 1, we have

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s,z)P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)dydzdsdt \right| \le C$$

and

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s,z)D^\gamma P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)dydzdsdt \right| \le C\,.$$

Proof.

We will only give a proof of the first estimate in the lemma. The proofs of the second and third estimates are left to the reader.

Denote the first integral in the lemma by I. Let $l, m \in \mathbb{Z}^d$ and define $[l, l+1) := [l^{(1)}, l^{(1)}+1) \times \cdots \times [l^{(d)}, l^{(d)}+1)$ and similarly for $[m, m+1)$. Truncate the functions ϕ, h by setting $\phi_l(s,z) := \phi(s,z)1_{[l,l+1)}(z)$ and $h_m(t,y) := h(t,y)1_{[m,m+1)}(y)$.

In the first integral, we replace ϕ, h by ϕ_l, h_m respectively, and thus define

$$I_{l,m} := \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_l(s,z)h_m(t,y)D^\alpha D^\beta P(t-s,y-z)dydzdsdt$$

Therefore we can write $I = \sum_{l,m\in\mathbb{Z}^d} I_{l,m}$. Below we let C be a generic constant that may vary from line to line.

Assume $\|l-m\|_\infty := \max_i |l^{(i)} - m^{(i)}| \ge 2$. For $z \in [l, l+1)$ and $y \in [m, m+1)$ we have $\|z-y\| \ge \|l-m\|_\infty - 1$. If $\alpha \ne \beta$ we have that

$$D^\alpha D^\beta P(t-s, z-y) = \frac{(z^{(i)} - y^{(i)})(z^{(j)} - y^{(j)})}{(t-s)^2} P(t-s, y-z)$$

for a suitable choice of i, j. Then we can find C such that

$$|D^\alpha D^\beta P(t-s,z-y)| \le Ce^{-(\|l-m\|_\infty - 2)^2/4}.$$

If $\alpha = \beta$, we have

$$(D^\alpha)^2 P(t-s,y-z) = \left(\frac{(y^{(i)} - z^{(i)})^2}{t-s} - 1 \right) \frac{P(t-s,y-z)}{t-s}$$

and similarly we find C such that

$$|(D^\alpha)^2 P(t-s,y-z)| \le Ce^{-(\|l-m\|_\infty-2)^2/4}.$$

In both cases we have $|I_{l,m}| \le Ce^{-\|l\|^2/8}e^{-(\|l-m\|_\infty-2)^2/4}$ and it follows that

$$\sum_{\|l-m\|_\infty\ge 2} |I_{l,m}| \le C.$$

Assume $\|l-m\|_\infty \le 1$ and let $\hat\phi_l(s,u)$ and $\hat h_m(t,u)$ be the Fourier transform in the second variable, defined by

$$\hat h_m(t,u) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} h(t,x)e^{-i(u,x)}dx$$

and similar for $\hat\phi_l(s,u)$. By the Plancherel theorem we have that

$$\int_{\mathbb{R}^d}\hat\phi_l(s,u)^2du = \int_{\mathbb{R}^d}\phi_l(s,z)^2dz \le Ce^{-\|l\|^2/6}$$

for all $s\in[0,1]$ and

$$\int_{\mathbb{R}^d}\hat h_m(t,u)^2du = \int_{\mathbb{R}^d} h_m(t,y)^2dy \le 1.$$

We can write

$$I_{l,m} = \int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat\phi_l(s,u)\hat h_m(t,-u)u^{(i)}u^{(j)}(t-s)e^{-(t-s)\|u\|^2/2}dudsdt. \tag{13}$$

To see this, start with the right hand side. Then we have by Fubini's theorem

$$\begin{aligned}
&\int_{\mathbb{R}^d}\hat h_m(t,-u)\hat\phi_l(s,u)u^iu^j(t-s)e^{-(t-s)\|u\|^2/2}du\\
&=(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_m(t,x)e^{i(u,x)}\phi_l(s,y)e^{-i(u,y)}u^iu^j(t-s)e^{-(t-s)\|u\|^2/2}dudxdy=\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_m(t,x)\phi_l(s,y)(t-s)\left[(2\pi)^{-d}\int_{\mathbb{R}^d}e^{i(u,x-y)}u^iu^je^{-(t-s)\|u\|^2/2}du\right]dxdy
\end{aligned}$$

.

Now look at the expression in the square brackets. Substitute $v=\sqrt{t-s}u$ to get

$$\begin{aligned}
&(2\pi)^{-d}\int_{\mathbb{R}^d}e^{i(u,x-y)}u^iu^je^{-(t-s)\|u\|^2/2}du\\
&=(2\pi)^{-d}(t-s)^{-d/2}\int_{\mathbb{R}^d}e^{i(\frac{v}{\sqrt{t-s}},x-y)}\frac{v^i}{\sqrt{t-s}}\frac{v^j}{\sqrt{t-s}}e^{-\|v\|^2/2}dv\\
&=(2\pi)^{-d}(t-s)^{-d/2}(t-s)^{-1}\int_{\mathbb{R}^d}e^{i(v,\frac{x-y}{\sqrt{t-s}})}v^iv^je^{-\|v\|^2/2}dv
\end{aligned}$$

Now put $f(v) = e^{-\|v\|^2/2}$ and $p(v) = v^{(i)}v^{(j)}$. From properties of the Fourier transform we know that $\widehat{pf} = D^\alpha D^\beta \hat{f}$ and $\hat{f} = f$. This gives that the above expression is equal to

$$(2\pi)^{-d/2}(t-s)^{-d/2}(t-s)^{-1}D^\alpha D^\beta f\left(\frac{x-y}{\sqrt{t-s}}\right) = (t-s)^{-1}D^\alpha D^\beta P(t-s, x-y)$$

This gives the equation (13).

Applying $ab \le \frac{1}{2}a^2c + \frac{1}{2}b^2c^{-1}$ with $a = \hat{\phi}_l(s,u)u^{(i)}$, $b = \hat{h}_m(t,-u)u^{(j)}$ and $c = e^{\|l\|^2/12}$ we get

$$\begin{aligned}|I_{l,m}| &\le \frac{1}{2}\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{\phi}_l(s,u)^2(u^{(i)})^2 e^{\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \\ &+ \frac{1}{2}\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{h}_m(t,-u)^2(u^{(j)})^2 e^{-\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \\ &\le \frac{1}{2}\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{\phi}_l(s,u)^2\|u\|^2 e^{\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \\ &+ \frac{1}{2}\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{h}_m(t,-u)^2\|u\|^2 e^{-\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt.\end{aligned}$$

For the first term, integrate first with respect to t in order to get

$$\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{\phi}_l(s,u)^2\|u\|^2 e^{\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \le Ce^{-\|l\|^2/12}$$

and for the second term, integrate with respect to s first to get

$$\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{h}_m(t,-u)^2\|u\|^2 e^{-\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \le Ce^{-\|l\|^2/12}$$

which gives $|I_{l,m}| \le Ce^{-\|l\|^2/12}$ and hence

$$\sum_{\|l-m\|_\infty \le 1} |I_{l,m}| \le C.$$

■

Using the previous lemma we can show the following:

Lemma 9 *There is a universal constant C such that for every Borel-measurable functions g and h bounded by 1, and $r \ge 0$*

$$\left|\int_{t_0}^t\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t_2,z)P(t_2-t_0,z)h(t_1,y)D^\alpha D^\beta P(t_1-t_2,y-z)(t-t_1)^r dydzdt_2dt_1\right| \le C(1+r)^{-1}(t-t_0)^{r+1}$$

and

$$\left|\int_{t_0}^t\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t_2,z)D^\gamma P(t_2-t_0,z)h(t_1,y)D^\alpha D^\beta P(t_1-t_2,y-z)(t-t_1)^r dydzdt_2dt_1\right| \le C(1+r)^{-1/2}(t-t_0)^{r+1/2}.$$

Proof.

We begin by proving the estimate for $t = 1$,$t_0 = 0$. From Lemma 9 we have that for each $k \geq 0$

$$\left| \int_{2^{-k-1}}^{2^{-k}} \int_{t/2}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s,z)P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)(1-t)^r dydzdsdt \right| \leq C(1-2^{-k-1})^r 2^{-k}.$$

To see this, make the substitutions $t' = 2^k t$ and $s' = 2^k s$. Use the easily verified fact that $P(at,z) = a^{-d/2}P(t,a^{-1/2}z)$ and substitute $z' = 2^{k/2}z$ and $y' = 2^{k/2}y$. Using $\tilde{h}(t,y) := \frac{(1-t)^r}{(1-2^{-k-1})^r}h(t,y)$ in Lemma (9), the result follows.

Summing this equation over k gives

$$\left| \int_0^1 \int_{t/2}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s,z)P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)(1-t)^r dydzdsdt \right| \leq C(1+r)^{-1}$$

Moreover from the bound (**??**)

$$\left| \int_0^1 \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s,z)P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)(1-t)^r dydzdsdt \right| \leq C \int_0^1 \int_0^{t/2} (t-s)^{-1}(1-t)^r dsdt \leq C(1+r)^{-1}$$

and combining these bounds gives the first assertion for $t = 1$,$t_0 = 0$. For general t and t_0 use the change of variables $t_1' = \frac{t_1-t_0}{t-t_0}$, $t_2 = \frac{t_2-t_0}{t-t_0}$, $y' = (t-t_0)^{-1/2}y$ and $z' = (t-t_0)^{-1/2}z$. The second assertion is proved similarly. ■

We now turn to the following key estimate (cf. [6, Proposition 2.2]):

Lemma 10 *Let B be a d-dimensional Brownian Motion starting from the origin and $b_1,\dots,b_n$ be compactly supported continuously differentiable functions $b_i : [0,1]\times\mathbb{R}^d \to \mathbb{R}$ for $i = 1,2,\dots n$. Let $\alpha_i \in \{0,1\}^d$ be a multiindex such that $|\alpha_i| = 1$ for $i = 1,2,\dots,n$. Then there exists a universal constant C (independent of $\{b_i\}_i$, n, and $\{\alpha_i\}_i$) such that*

$$\left| E\left[\int_{t_0<t_1<\cdots<t_n<t} \left(\prod_{i=1}^n D^{\alpha_i} b_i(t_i, x+B_{t_i}) \right) dt_1 \dots dt_n \right] \right| \leq \frac{C^n \prod_{i=1}^n \|b_i\|_\infty (t-t_0)^{n/2}}{\Gamma(\frac{n}{2}+1)} \tag{14}$$

where Γ is the Gamma-function and $x \in \mathbb{R}^d$. Here D^{α_i} denotes the partial derivative with respect to the j'th space variable, where j is the position of the 1 in α_i.

Proof. Without loss of generality, assume that $\|b_i\|_\infty \leq 1$ for $i = 1,2\dots,n$. Using the Gaussian density we write the left hand side of the estimate (14) in the form

$$\left| \int_{t_0<t_1<\cdots<t_n<t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n D^{\alpha_i} b_i(t_i, x+z_i) P(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n \right|.$$

Introduce the notation

$$J_n^\alpha(t_0,t,z_0) = \int_{t_0<t_1<\cdots<t_n<t}\int_{\mathbb{R}^{dn}}\prod_{i=1}^n D^{\alpha_i}b_i(t_i,x+z_i)P(t_i-t_{i-1},z_i-z_{i-1})dz_1\dots dz_n dt_1\dots dt_n$$

where $\alpha=(\alpha_1,\dots\alpha_n)\in\{0,1\}^{nd}$. We shall show that $|J_n^\alpha(t_0,t,z_0)|\le C^n(t-t_0)^{n/2}/\Gamma(n/2+1)$, thus proving the proposition.

To do this, we will use integration by parts to shift the derivatives onto the Gaussian kernel. This will be done by introducing the alphabet

$$\mathcal{A}(\alpha)=\{P, D^{\alpha_1}P,\dots,D^{\alpha_n}P, D^{\alpha_1}D^{\alpha_2}P,\dots D^{\alpha_{n-1}}D^{\alpha_n}P\}$$

where D^{α_i}, $D^{\alpha_i}D^{\alpha_{i+1}}$ denotes the derivatives in z of $P(t,z)$.

Take a string $S=S_1\cdots S_n$ in $\mathcal{A}(\alpha)$ and define

$$I_S^\alpha(t_0,t,z_0)=\int_{t_0<\cdots<t_n<t}\int_{\mathbb{R}^{dn}}\prod_{i=1}^n b_i(t_i,x+z_i)S_i(t_i-t_{i-1},z_i-z_{i-1})dz_1\dots dz_n dt_1\dots dt_n\,.$$

We will need only a special type of strings: Say that a string is *allowed* if, when all the $D^{\alpha_i}P$'s are removed from the string, a string of the form $P\cdot D^{\alpha_s}D^{\alpha_{s+1}}P\cdot P\cdot D^{\alpha_{s+1}}D^{\alpha_{s+2}}P\cdots P\cdot D^{\alpha_r}D^{\alpha_{r+1}}P$ for $s\ge 1$, $r\le n-1$ remains. Also, we will require that the first derivatives $D^{\alpha_i}P$ are written in an increasing order with respect to i.

We now claim that we can write

$$J_n^\alpha(t_0,t,z_0)=\sum_{j=1}^{2^{n-1}}\epsilon_j I_{S^j}^\alpha(t_0,t,z_0)$$

where each ϵ_j is either -1 or 1 and each S^j is an allowed string in $\mathcal{A}(\alpha)$. To see this, we proceed by induction:

The equation obviously holds for $n=1$. Assume the equation holds for $n\ge1$, and let b_0 be another function satisfying the requirements of the proposition. Likewise with α_0. Then

$$\begin{aligned} J_{n+1}^{(\alpha_0,\alpha)}(t_0,t,z_0) &= \int_{t_0}^t\int_{\mathbb{R}^d} D^{\alpha_0}b_0(t_1,x+z_1)P(t_1-t_0,z_1-z_0)J_n^\alpha(t_1,t,z_1)dz_1dt_1 \\ &= -\int_{t_0}^t\int_{\mathbb{R}^d} b_0(t_1,x+z_1)D^{\alpha_0}P(t_1-t_0,z_1-z_0)J_n^\alpha(t_1,t,z_1)dz_1dt_1 \\ &\quad -\int_{t_0}^t\int_{\mathbb{R}^d} b_0(t_1,x+z_1)P(t_1-t_0,z_1-z_0)D^{\alpha_0}J_n^\alpha(t_1,t,z_1)dz_1dt_1\,. \end{aligned}$$

Notice that

$$D^{\alpha_0}I_S^\alpha(t_1,t,z_1)=-I_{\tilde S}^{(\alpha_0,\alpha)}(t_1,t,z_1)$$

where

$$\tilde S=\begin{cases} D^{\alpha_0}P\cdot S_2\cdots S_n & \text{if } S=P\cdot S_2\cdots S_n \\ D^{\alpha_0}D^{\alpha_1}P\cdot S_2\cdots S_n & \text{if } S=D^{\alpha_1}P\cdot S_2\cdots S_n\,. \end{cases}$$

Here, $\tilde{S}$ is not an allowed string in $\mathcal{A}(\alpha)$. So from the induction hypothesis $D^{\alpha_0} J_n^{\alpha}(t_0, t, z_0) = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{\tilde{S}}^{(\alpha_0,\alpha)}(t_0, t, z_0)$ this gives

$$J_{n+1}^{(\alpha_0,\alpha)} = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{D^{\alpha_0} P\cdot S^j}^{(\alpha_0,\alpha)} + \sum_{j=1}^{2^{n-1}} \epsilon_j I_{P\cdot\tilde{S}^j}.$$

It is easily checked that when S^j is an allowed string in $\mathcal{A}(\alpha)$, both $D^{\alpha_0} P \cdot S^j$ and $P \cdot \tilde{S}^j$ are allowed strings in $\mathcal{A}(\alpha_0, \alpha)$.

This proves the claim.

For the rest of the proof of Lemma 10 we will bound I_S^{α} when S is an allowed string, i.e. we show that there is a positive constant M such that

$$I_S^{\alpha}(t_0, t, z_0) \leq \frac{M^n (t - t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

for all integers $n \geq 1$ and for each allowed string S in the alphabet $\mathcal{A}(\alpha)$.

We proceed by induction: The case $n = 0$ is immediate, so assume $n > 0$ and that this holds for all allowed strings of length less than n. There are three cases:

1. $S = D^{\alpha_1} P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_2, \ldots, \alpha_n)$
2. $S = P \cdot D^{\alpha_1} D^{\alpha_2} P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_3, \ldots, \alpha_n)$
3. $S = P \cdot D^{\alpha_1} P \cdots D^{\alpha_m} P \cdot D^{\alpha_{m+1}} D^{\alpha_{m+2}} P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_{m+3}, \ldots, \alpha_n)$.

In each case, S' is an allowed string in the given alphabet.

1. We use the inductive hypothesis to bound $I_{S'}^{\alpha'}(t_1, t, z_1)$ and the bound

$$\int_{\mathbb{R}^d} |D^{\alpha} P(t, z)| dz \leq C t^{-1/2} \tag{15}$$

to get

$$\begin{aligned} |I_S^{\alpha}(t_0, t, z_0)| &= \left| \int_{t_0}^{t} \int_{\mathbb{R}^d} b_1(t_1, z_1) D^{\alpha_1} P(t_1 - t_0, z_1 - z_0) I_{S'}^{\alpha'}(t_1, t, z_1) dz_1 dt_1 \right| \\ &\leq \frac{M^{n-1}}{\Gamma(\frac{n+1}{2})} \int_{t_0}^{t} (t - t_1)^{(n-1)/2} \int_{\mathbb{R}^d} |D^{\alpha_1} P(t_1 - t_0, z_1 - z_0)| dz_1 dt_1 \\ &\leq \frac{M^{n-1} C}{\Gamma(\frac{n+1}{2})} \int_{t_0}^{t} (t - t_1)^{(n-1)/2} (t_1 - t_0)^{-1/2} dt_1 \\ &= \frac{M^{n-1} C \sqrt{\pi} (t - t_0)^{k/2}}{\Gamma(\frac{n}{2} + 1)}. \end{aligned}$$

The result follows if M is large enough.

2. For this case we can write

$$\begin{aligned} I_S^{\alpha}(t_0, t, z_0) = &\int_{t_0}^{t} \int_{t_1}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b_1(t_1, z_1) b_2(t_2, z_2) \\ &\times P(t_1 - t_0, z_1 - z_0) D^{\alpha_1} D^{\alpha_2} P(t_2 - t_1, z_2 - z_1) I_{S'}^{\alpha'}(t_2, t, z_2) dz_1 dz_2 dt_2 dt_1. \end{aligned}$$

We set $h(t_2,z_2) := b_2(t_2,z_2) I_{S'}^{\alpha'}(t_2,z_2)(t-t_2)^{1-n/2}$ so that by the inductive hypothesis we have

$$\|h\|_\infty \le M^{n-2}/\Gamma(n/2)\,.$$

Use this in the first part of Lemma 9 with $g = b_1$ and integrate with respect to t_2 first, to get

$$|I_S^\alpha(t_0,t,z_0)| \le \frac{CM^{n-2}(t-t_0)^{n/2}}{n\Gamma(n/2)}\,,$$

and the result follows if M is large enough.

3. We have

$$\begin{aligned} I_S^\alpha(t_0,t,z_0) = \int_{t_0<\ldots t_{m+2}<t}\int_{\mathbb{R}^{(m+2)d}} &P(t_1-t_0,z_1-z_0)\prod_{j=1}^{m+2} b_j(t_j,z_j) \\ &\times \prod_{j=2}^{m} D^{\alpha_j} P(t_j-t_{j-1},z_j-z_{j-1}) D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2}-t_{m+1},z_{m+2}-z_{m+1}) \\ &\times I_{S'}^{\alpha'}(t_{m+2},t,z_{m+2}) dz_1\ldots dz_{m+2} dt_1\ldots dt_{m+2}\,. \end{aligned}$$

Let $h(t_{m+2},z_{m+2}) = b_{m+2}(t_{m+2},z_{m+2}) I_{S'}^{\alpha'}(t_{m+2},t,z)(t-t_{m+2})^{(2+m-n)/2}$, so that from the inductive hypothesis we have $\|h\|_\infty \le M^{n-m-2}/\Gamma((n-m)/2)$. Write

$$\begin{aligned} \Omega(t_m,z_m) := \int_{t_m}^t \int_{t_{m+1}}^t \int_{\mathbb{R}^{2d}} &b_{m+1}(t_{m+1},z_{m+1}) h(t_{m+2},z_{m+2}) \\ &\times (t-t_{m+2})^{(n-m-2)/2} D^{\alpha_m} P(t_{m+1}-t_m, z_{m+1}-z) \\ &\times D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2}-t_{m+1}, z_{m+2}-z_{m+1}) dz_{m+1} dz_{m+2} dt_{m+1} dt_{m+2}\,, \end{aligned}$$

so that from Lemma (9) we have that

$$|\Omega(t_m,z_m)| \le \frac{C(n-m)^{-1/2} M^{n-m-2}(t-t_m)^{(n-m-1)/2}}{\Gamma(\frac{n-m}{2})}\,.$$

Using this in

$$\begin{aligned} I_S^\alpha(t_0,t,z_0) = \int_{t_0<\ldots t_{m+2}<t}\int_{\mathbb{R}^{(m+2)d}} &P(t_1-t_0,z_1-z_0)\prod_{j=1}^{m} b_j(t_j,z_j) \\ &\times \prod_{j=1}^{m-1} D^{\alpha_j} P(t_j-t_{j-1},z_j-z_{j-1}) \Omega(t_m,z_m) dz_1\ldots dz_m dt_1\ldots dt_m\,, \end{aligned}$$

and using the bound (15) several times gives

$$\begin{aligned} |I_S^\alpha(t_0,t,z_0)| &\le C^{m+1}(n-m)^{-1/2}\frac{M^{n-m-2}}{\Gamma((n-m)/2)} \\ &\quad\times \int_{t_0<\ldots t_m<t} (t_2-t_1)^{-1/2}\ldots(t_m-t_{m-1})^{-1/2}(t-t_m)^{(n-m-1)/2} dt_1\ldots dt_m \\ &= C^{m+1}(n-m)^{-1/2}\frac{M^{n-m-2}\pi^{(m-1)/2}\Gamma(\frac{n-m+1}{2})}{\Gamma(\frac{n-m}{2})\Gamma(\frac{n}{2}+1)}(t-t_0)^{n/2}\,, \end{aligned}$$

and the result follows when M is large enough, thus proving the induction step.

■

We are now ready to complete the proof of Proposition 7.

Proof of Proposition 7. Iterating the linearized equation (12) we obtain

$$\frac{\partial}{\partial x}X_t^x = \mathcal{I}_d + \sum_{n=1}^{\infty}\int_{0<s_1<\dots s_n<t} b'(s_1, X_{s_1}^x) : \dots : b'(s_n, X_{s_n}^x)ds_1\dots ds_n.$$

Let $p\in[1,\infty)$ and choose $r,s\in[1,\infty)$ such that $sp=2^q$ for some integer q and $\frac{1}{r}+\frac{1}{s}=1$. Then by Girsanov's theorem and Hölder's inequality

$$\begin{aligned} E\left[|\frac{\partial}{\partial x}X_t^x|^p\right] =& E\left[|\mathcal{I}_d + \sum_{n=1}^{\infty}\int_{0<s_1<\dots s_n<t} b'(s_1, x+B_{s_1}) : \dots : b'(s_n, x+B_{s_n})ds_1\dots ds_n|^p \right.\\ & \left. \times\, \mathcal{E}(\int_0^1 b(u,x+B_u)dB_u)\right]\\ \le& C_1(\|b\|_\infty)\left\|\mathcal{I}_d + \sum_{n=1}^{\infty}\int_{0<s_1<\dots s_n<t} b'(s_1, x+B_{s_1}) : \dots : b'(s_n, x+B_{s_n})ds_1\dots ds_n\right\|_{L^{sp}(\mu,\mathbb{R}^{d\times d})}^p, \end{aligned}$$

where $\mathcal{E}(\int_0^1 b(u,x+B_u)dB_u)$ is the Doleans-Dade exponential of the martingale $\int_0^1 b(u,x+B_u)dB_u = \sum_{j=1}^d\int_0^1 b^{(j)}(u,x+B_u)dB_u^j$ and C_1 is a continuous increasing function.

Then we obtain

$$\begin{aligned} & E|\frac{\partial}{\partial x}X_t^x|^p\\ \le& C_1(\|b\|_\infty)\left\|\mathcal{I}_d + \sum_{n=1}^{\infty}\int_{0<s_1<\dots s_n<t} b'(s_1, x+B_{s_1}) : \dots : b'(s_n, x+B_{s_n})ds_1\dots ds_n\right\|_{L^{sp}(\mu,\mathbb{R}^{d\times d})}^p\\ \le& C_1(\|b\|_\infty)\left(1+\sum_{n=1}^{\infty}\sum_{i,j=1}^{d}\sum_{l_1,\dots l_{n-1}=1}^{d}\left\|\int_{t<s_1<\dots<s_n<s}\frac{\partial}{\partial x_{l_1}}b^{(i)}(s_1,x+B_{s_1})\frac{\partial}{\partial x_{l_2}}b^{(l_1)}(s_2,x+B_{s_2})\dots\right.\right.\\ & \left.\left.\dots\frac{\partial}{\partial x_j}b^{(l_{n-1})}(s_n,x+B_{s_n})ds_1\dots ds_n\right\|_{L^{ps}(\mu;\mathbb{R})}\right)^p. \end{aligned}$$

Now consider the expression

$$A := \int_{0<s_1<\dots<s_n<t}\frac{\partial}{\partial x_{l_1}}b^{(i)}(s_1,x+B_{s_1})\frac{\partial}{\partial x_{l_2}}b^{(l_1)}(s_2,x+B_{s_2})\dots\frac{\partial}{\partial x_{l_n}}b^{(l_n)}(s_n,x+B_{s_n})ds_1\dots ds_n.$$

Then, using (deterministic) integration by parts, repeatedly, it is easy to see that A^2 can be written as a sum of at most 2^{2n} terms of the form

$$\int_{0<s_1<\dots<s_{2n}<t} g_1(s_1)\dots g_{2n}(s_{2n})ds_1\dots ds_{2n}\,, \tag{16}$$

where $g_l\in\left\{\frac{\partial}{\partial x_j}b^{(i)}(\cdot,x+B.) : 1\le i,j\le d\right\}$, $l=1,2\dots 2n$. Similarly, by induction it follows that A^{2^q} is the sum of at most $2^{q2^q n}$ terms of the form

$$\int_{0<s_1<\dots<s_{2n}<t} g_1(s_1)\dots g_{2^q n}(s_{2^q n})ds_1\dots ds_{2^q n}\,, \tag{17}$$

Combining this with (10), we obtain the following estimate:

$$\left\| \int_{0<s_1<\cdots<s_n<t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x+B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, x+B_{s_2}) \ldots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, x+B_{s_n}) ds_1 \ldots ds_n \right\|_{L^{2^q}(\mu;\mathbb{R})}$$

$$\leq \left(\frac{2^{q2^q n} C^{2^q n} \|b\|_\infty^{2^q n} t^{2^{q-1} n}}{\Gamma(2^{q-1}n+1)} \right)^{2^{-q}} \leq \frac{2^{qn} C^n \|b\|_\infty^n}{((2^{q-1}n)!)^{2^{-q}}}.$$

Then it follows that

$$E\left[\| \frac{\partial}{\partial x} X_t^x \|^p \right]$$

$$\leq C_1(\|b\|_\infty) \left(1 + \sum_{n=1}^{\infty} \frac{d^{n+2} 2^{qn} C^n \|b\|_\infty^n}{((2^{q-1}n)!)^{2^{-q}}} \right)^p = C_{d,p}(\|b\|_\infty).$$

The right hand side of this inequality is independent of $x \in \mathbb{R}^d$, and the result follows. ■

As a consequence of Proposition 7 we obtain the following result:

Corollary 11 *Let $X_\cdot^{s,x}$ be the unique strong solution to the SDE (7) and $q > 1$ an integer. Then there exists a constant $C = C(d, \|b\|_\infty, q) < \infty$ such that*

$$E\left[\left| X_{t_1}^{s_1,x_1} - X_{t_2}^{s_2,x_2} \right|^q \right] \leq C(|s_1 - s_2|^{q/2} + |t_1 - t_2|^{q/2} + |x_1 - x_2|^q)$$

for all $s_1, s_2, t_1, t_2, x_1, x_2$.
In particular, there exists a continuous version of the random field $(s,t,x) \longmapsto X_t^{s,x}$ with Hölder continuous trajectories of Hölder constant $\alpha < \frac{1}{2}$ in s,t and $\alpha < 1$ in x, locally (see [22]).

Proof. Let $b_n : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, n \geq 1$ be a sequence of compactly supported smooth functions and let $b : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a bounded Borel measurable function. Suppose that $b_n(t,x) \longrightarrow b(t,x)$ $dt \times dx-$a.e. and such that $|b_n(t,x)| \leq M < \infty$ for all n, t, x for some constant M.

Denote by $X_\cdot^{n,s,x}$ the solution of (7) associated with the coefficient $b_n, n \geq 1$. Without loss of generality let $0 \le s_1 < s_2 < t_1 < t_2$. Then

$$
\begin{aligned}
& X_{t_1}^{n,s_1,x_1} - X_{t_2}^{n,s_2,x_2} \\
= \; & x_1 - x_2 + \int_{s_1}^{t_1} b_n(u, X_u^{n,s_1,x_1}) du - \int_{s_2}^{t_2} b_n(u, X_u^{n,s_2,x_2}) du \\
& + (B_{t_1} - B_{s_1}) - (B_{t_2} - B_{s_2}) \\
= \; & x_1 - x_2 + \int_{s_1}^{s_2} b_n(u, X_u^{n,s_1,x_1}) du + \int_{s_2}^{t_1} b_n(u, X_u^{n,s_1,x_1}) du \\
& - \int_{s_2}^{t_1} b_n(u, X_u^{n,s_2,x_2}) du - \int_{t_1}^{t_2} b_n(u, X_u^{n,s_2,x_2}) du \\
& + (B_{t_1} - B_{t_2}) + (B_{s_2} - B_{s_1}) \\
= \; & x_1 - x_2 + \int_{s_1}^{s_2} b_n(u, X_u^{n,s_1,x_1}) du - \int_{t_1}^{t_2} b_n(u, X_u^{n,s_2,x_2}) du \\
& + \int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_1}) - b_n(u, X_u^{n,s_1,x_2})) du \\
& + \int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_2}) - b_n(u, X_u^{n,s_2,x_2})) du \\
& + (B_{t_1} - B_{t_2}) + (B_{s_2} - B_{s_1}).
\end{aligned}
$$

So due to the uniform boundedness of $b_n, n \ge 1$ we get

$$
\begin{aligned}
& E[\left| X_{t_1}^{n,s_1,x_1} - X_{t_2}^{n,s_2,x_2} \right|^q] \\
\le \; & C_q(|x_1 - x_2|^q + |s_1 - s_2|^{q/2} + |t_1 - t_2|^{q/2} \\
& E[\left| \int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_1}) - b_n(u, X_u^{n,s_1,x_2})) du \right|^q] \\
& + E[\left| \int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_2}) - b_n(u, X_u^{n,s_2,x_2})) du \right|^q]).
\end{aligned} \tag{18}
$$

Using the fact that $X_t^{n,\cdot,s}$ is a stochastic flow of diffeomorphisms (see e.g. [22]), the mean value theorem and Proposition 7, we get

$$
\begin{aligned}
& E[\left| \int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_1}) - b_n(u, X_u^{n,s_1,x_2})) du \right|^q] \\
= \; & |x_1 - x_2|^q \, E[\left| \int_{s_2}^{t_1} \int_0^1 (b_n^{\shortmid}(u, X_u^{n,s_1,x_1+\tau(x_2-x_1)}) \frac{\partial}{\partial x} X_u^{n,s_1,x_1+\tau(x_2-x_1)}) d\tau du \right|^q] \\
\le \; & |x_1 - x_2|^q \int_0^1 E[\left| \int_{s_2}^{t_1} (b_n^{\shortmid}(u, X_u^{n,s_1,x_1+\tau(x_2-x_1)}) \frac{\partial}{\partial x} X_u^{n,s_1,x_1+\tau(x_2-x_1)}) du \right|^q] d\tau \\
= \; & |x_1 - x_2|^q \int_0^1 E[\left| \frac{\partial}{\partial x} X_{t_1}^{n,s_1,x_1+\tau(x_2-x_1)} - \frac{\partial}{\partial x} X_{s_2}^{n,s_1,x_1+\tau(x_2-x_1)} \right|^q] d\tau \\
\le \; & C_q \, |x_1 - x_2|^q \sup_{t \in [s_1,1], x \in \mathbb{R}^d} E[\left| \frac{\partial}{\partial x} X_{t_1}^{n,s_1,x} \right|^q] \\
\le \; & C_{d,q}(\|b\|_\infty) \, |x_1 - x_2|^q .
\end{aligned} \tag{19}
$$

Finally we observe that estimation of the last term of the right hand side of (18) can be reduced to the previous case (19) by applying the Markov property, since

$$\begin{aligned}
& E[\left|\int_{s_2}^{t_1}(b_n(u,X_u^{n,s_1,x_2})-b_n(u,X_u^{n,s_2,x_2}))du\right|^q] \\
\leq & \int_{s_2}^{t_1}E[|b_n(u,X_u^{n,s_1,x_2})-b_n(u,X_u^{n,s_2,x_2})|^q]du \\
= & \int_{s_2}^{t_1}E[E[|b_n(u,X_u^{n,s_2,y})-b_n(u,X_u^{n,s_2,x_2})|^q]|_{y=X_{s_2}^{n,s_1,x_2}}]du \\
\leq & CE[\left|X_{s_2}^{n,s_1,x_2}-x_2\right|^q]=CE[\left|X_{s_2}^{n,s_1,x_2}-X_{s_1}^{n,s_1,x_2}\right|^q] \\
\leq & M_q\left|s_2-s_1\right|^{q/2}
\end{aligned}$$

for a positive constant $M_q<\infty$.

Therefore, we have

$$E[\left|X_{t_1}^{n,s_1,x_1}-X_{t_2}^{n,s_2,x_2}\right|^q]\leq C_q(|s_1-s_2|^{q/2}+|t_1-t_2|^{q/2}+|x_1-x_2|^q)$$

for a constant C_q independent of n.

To complete the proof, we use the fact that $X_{t_1}^{n,s_1,x_1}\to X_{t_1}^{s_1,x_1}$ and $X_{t_2}^{n,s_2,x_2}\to X_{t_2}^{s_2,x_2}$ in $L^2(\mu)$ for $n\longrightarrow\infty$ (see [26]) together with Fatou's lemma applied to a.e. convergent subsequences of $\{X_{t_1}^{n,s_1,x_1}\}_{n=1}^{\infty}$ and $\{X_{t_2}^{n,s_2,x_2}\}_{n=1}^{\infty}$. ■

This concludes step one of our program.

We now proceed to Step 2.

As before, we continue to approximate the bounded Borel-measurable coefficient b by a uniformly bounded sequence $\{b_n\}_{n=1}^{\infty}$ of smooth functions with compact support. We then consider the corresponding sequence of solutions when b in (4) is replaced by b_n and denoted it by $\{X_t^{n,\cdot}\}_{n=1}^{\infty}$. The following lemma establishes relative compactness of the above sequence:

Lemma 12 *For any $\varphi\in C_0^{\infty}(\mathbb{R}^d;\mathbb{R}^d)$ and $t\in[0,1]$ the sequence*

$$\langle X_t^n,\varphi\rangle=\int_{\mathbb{R}^d}\langle X_t^{n,x},\varphi(x)\rangle_{\mathbb{R}^d}dx$$

is relatively compact in $L^2(\Omega,\mu)$. Moreover, there exists a subsequence converging to $\langle X_t,\varphi\rangle$ in $L^2(\Omega,\mu)$.

Proof. Denote by D_s the Malliavin derivative (see the Appendix) and by U the compact support of φ. By noting the inequalities

$$\begin{aligned}
E[|D_s\langle X_t^n,\varphi\rangle|^2] &= E[|\langle D_sX_t^n,\varphi\rangle|^2] \\
&\leq \|\varphi\|_{L^2(\mathbb{R}^d)}^2|U|\sup_{x\in U}E[|D_sX_t^{n,x}|^2]
\end{aligned}$$

and

$$\begin{aligned}
& E[|D_s\langle X_t^n,\varphi\rangle_{L^2(\mathbb{R}^d)}-D_{s'}\langle X_t^n,\varphi\rangle|^2] \\
&= E[|\langle D_sX_t^n-D_{s'}X_t^n,\varphi\rangle|^2]
\end{aligned}$$

$$\leq \|\varphi\|^2_{L^2(\mathbb{R}^d)}|U| \sup_{x\in U} E[|D_s X_t^{n,x} - D_{s'}X_t^{n,x}|^2]$$

we can invoke Corollary 32 together with Lemma 3.5 [26] to obtain a subsequence $\langle X_t^{n(k)}, \varphi\rangle_{L^2(\mathbb{R}^d)}$ converging in $L^2(\Omega,\mu)$ as $k\to\infty$. Denote the limit by $Y(\varphi)$.

Taking the S-transform (see [19] or [32]) for the definition; or alternatively just use the Wiener transform on the Wiener space) of $\langle X_t^n,\varphi\rangle$ and $\langle X_t,\varphi\rangle$ ($\langle X_t,\varphi\rangle$ is well-defined because of Corollary 11) we see that for any $\phi\in\mathcal{S}([0,1])$ (Schwartz test function space on $[0,1]$)

$$|S(\langle X_t^n,\varphi\rangle)(\phi) - S(\langle X_t,\varphi\rangle)(\phi)|^2 = |\langle S(X_t^n - X_t)(\phi),\varphi\rangle|^2$$

$$\leq \|\varphi\|^2_{L^2(\mathbb{R}^d)} \int_U |S(X_t^{n,x} - X_t^x)(\phi)|^2 dx$$

$$\leq \|\varphi\|^2_{L^2(\mathbb{R}^d)} \int_U CE[J_n(x)] \exp(68 \int_0^1 |\phi(s)|^2 ds) dx,$$

where C is a constant and

$$J_n(x) := \sum_{j=1}^d \left(2\int_0^1 \left(b_n^{(j)}(u,x+B_u) - b^{(j)}(u,x+B_u)\right)^2 du \right.$$

$$\left. + \left(\int_0^1 \left|(b_n^{(j)}(u,x+B_u))^2 - (b^{(j)}(u,x+B_u))^2\right| du\right)^2\right).$$

See [27] for a proof. Since $\{b_n\}$ is uniformly bounded, using dominated convergence, we get that

$$\langle X_t^n,\varphi\rangle \to \langle X_t,\varphi\rangle$$

in $(\mathcal{S})^*$ (Hida distribution space [19]), and in particular weakly in $L^2(\Omega,\mu)$. By uniqueness of the limits we can conclude that

$$Y(\varphi) = \langle X_t,\varphi\rangle.$$

■

We are now able to finalize the proof of Proposition 6.

Proof of Proposition 6. We know that $\{X_t^{n,x}\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(\Omega,\mu)$ for each fixed $t\in[0,1]$ and $x\in\mathbb{R}^d$. In particular, let $t\in[0,1]$ fixed and consider $x=0$. Then we can choose a subsequence such that

$$X_t^{n(k),0} \to X_t^0 \ \text{ in } L^2(\Omega,\mu)$$

We now claim that for any $\varphi\in C_0^\infty(U)$, the following convergence

$$\langle X_t^{n(k)},\varphi\rangle \to \langle X_t,\varphi\rangle \ \text{ in } L^2(\Omega,\mu)$$

holds for the same subsequence, $\{n(k)\}$. To see this, assume that there exists a $\varphi\in C_0^\infty(U)$, an $\epsilon>0$ and a subsequence $n(k(j))$ such that

$$\|\langle X_t^{n(k(j))},\varphi\rangle - \langle X_t,\varphi\rangle\|_{L^2(\Omega)} \geq \epsilon$$

By Lemma (12) we may extract a further subsequence $\langle X_t^{n(k(j))}, \varphi\rangle$ converging to $\langle X_t, \varphi\rangle$ giving the desired contradiction.

Using Proposition (7), we have

$$\sup_k E\left[|\frac{\partial}{\partial x} X_t^{n(k),x}|^p\right] < \infty.$$

Hence there exists a subsequence of $\frac{\partial}{\partial x} X_t^{n(k),x}$ (still denoted by $n(k)$ for simplicity) converging in the weak topology of $L^2(\Omega, L^p(U))$ to an element Y. Then we have for any $A \in \mathcal{F}$ and $\varphi \in C_0^\infty(U)$

$$E[1_A\langle X_t, \varphi'\rangle] = \lim_{k\to\infty} E[1_A\langle X_t^{n(k)}, \varphi'\rangle]$$

$$= \lim_{k\to\infty} -E[1_A\langle \frac{\partial}{\partial x} X_t^{n(k)}, \varphi\rangle] = -E[1_A\langle Y, \varphi\rangle].$$

Hence we have

$$\langle X_t, \varphi'\rangle = -\langle Y, \varphi\rangle$$

P-a.s. for every $\varphi \in C_0^\infty$ which gives the result. ■

Remark 13 *By a similar argument to the above proof we can show that there exists a subsequence* $\{n(k)\}$ *such that*

$$X_t^{n(k),x} \to X_t^x$$

in $L^2(\Omega)$ *for all* $(t, x) \in [0, 1] \times \mathbb{R}^d$*.*

We now return to the weighted Sobolev spaces. Using the same techniques as in the above lemma, we prove the following

Lemma 14 *For all* $p \in (1, \infty)$ *we have*

$$X_t^{\cdot} \in L^2(\Omega, W^{1,p}(\mathbb{R}^d, w))$$

Proof. For simplicity, we consider the case $d = 1$. It suffices to show that $E[(\int |\frac{\partial}{\partial x} X_t^x|^p w(x)dx)^{2/p}] < \infty$. To this end, let $X_t^{n,x}$ denote the sequence approximating X_t^x as in the previous lemma. Assume first that $p \geq 2$. Then by Hölder's inequality w.r.t. the Wiener measure μ we have

$$E[(\int |\frac{\partial}{\partial x} X_t^{n,x}|^p w(x)dx)^{2/p}]$$

$$\leq \left(E[\int |\frac{\partial}{\partial x} X_t^{n,x}|^p w(x)dx\right)^{2/p} \leq (\int w(x)dx)^{p/2} \sup_{x\in\mathbb{R}^d} E[|\frac{\partial}{\partial x} X_t^{n,x}|^p].$$

For $1 < p \leq 2$, by Hölder's inequality w.r.t. $w(x)dx$ we have

$$E[(\int |\frac{\partial}{\partial x} X_t^{n,x}|^p w(x)dx)^{2/p}] \leq (\int w(x)dx)^{(4-p)/2} \sup_{x\in\mathbb{R}^d} E[|\frac{\partial}{\partial x} X_t^{n,x}|^2].$$

In both cases we can find a subsequence converging to an element $Y \in L^2(\Omega, L^p(\mathbb{R}^d, w))$ in the weak topology, in particular for every $A \in \mathcal{F}$ and $f \in L^q(\mathbb{R}^d, w)$ (q is the Sobolev conjugate of p) we have

$$\lim_{k\to\infty} E[1_A \int \frac{\partial}{\partial x} X_t^{n(k),x} f(x)w(x)dx] = E[1_A \int Y(x)f(x)w(x)dx]$$

by choosing f such that $fw \in L^q(\mathbb{R}, dx)$ (e.g. put $f(x) = e^{-w(x)}\varphi(x)$ for $\varphi \in C_0^\infty(\mathbb{R})$), it follows that Y must coincide with the weak derivative of X_t^x. This proves the lemma. ■

We now complete the proof of our main theorem in this section (Theorem (3)) and its corollary:

Proof of Theorem 3. Denote by $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s,t,x) \longmapsto \phi_{s,t}(x) \in \mathbb{R}^d$ the continuous version of the solution map $(s,t,x) \longmapsto X_t^{s,x}$ provided by Corollary 11. Let Ω^* be the set of all $\omega \in \Omega$ such that the SDE (7) has a unique spatially Sobolev differentiable family of solutions. Then by completeness of the probability space $(\Omega, \mathcal{F}, \mu)$, it follows that $\Omega^* \in \mathcal{F}$ and $\mu(\Omega^*) = 1$. Furthermore, by uniqueness of solutions of the SDE (7), it is easy to check that the following two-parameter group property

$$\phi_{s,t}(\cdot,\omega) = \phi_{u,t}(\cdot,\omega) \circ \phi_{s,u}(\cdot,\omega), \qquad \phi_{s,s}(x,\omega) = x, \tag{20}$$

holds for all $s,u,t \in \mathbb{R}$, all $x \in \mathbb{R}^d$ and all $\omega \in \Omega^*$. Finally, we apply Lemma 14 and use the relation $\phi_{s,t}(\cdot,\omega) = \phi_{t,s}^{-1}(\cdot,\omega)$, to complete the proof of the theorem. ■

Proof of Corollary 5. Let Ω^* denote the set of full Wiener measure introduced in the above proof of Theorem 3. We claim that $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$. To see this, let $\omega \in \Omega^*$ and fix an arbitrary $t_1 \in \mathbb{R}$. Then from the autonomous SDE (11) it follows that

$$X_{t+t_1}^{t_1,x}(\omega) = x + \int_{t_1}^{t+t_1} b(X_u^{t_1,x}(\omega))du + B_{t+t_1}(\omega) - B_{t_1}(\omega), \;\; t_1, t \in \mathbb{R}, \tag{21}$$

By the helix property of B and a simple change of variable the above relation implies

$$X_{t+t_1}^{t_1,x}(\omega) = x + \int_0^t b(X_{u+t_1}^{t_1,x}(\omega))du + B_t(\theta(t_1(\omega))), \;\; t \in \mathbb{R}, \tag{22}$$

The above relation implies that the SDE (11) admits a Sobolev differentiable family of solutions when ω is replaced by $\theta(t_1,\omega)$. Hence $\theta(t_1,\omega) \in \Omega^*$. Thus $\theta(t_1,\cdot)(\Omega^*) \subseteq \Omega^*$, and since $t_1 \in \mathbb{R}$ is arbitrary, this proves our claim. Furthermore, using uniqueness in the integral equation (21) it follows that

$$X_{t_2+t_1}^{t_1,x}(\omega) = X_{t_2}^{0,x}(\theta(t_1,\omega)) \tag{23}$$

for all $t_1, t_2 \in \mathbb{R}$, all $x \in \mathbb{R}^d$ and $\omega \in \Omega^*$. To prove the following cocycle property for all $\omega \in \Omega^*$

$$\phi_{0,t_1+t_2}(\cdot,\omega) = \phi_{0,t_2}(\cdot,\theta(t_1,\omega)) \circ \phi_{0,t_1}(\cdot,\omega)$$

we rewrite the identity (23) in the form

$$\phi_{t_1,t_1+t_2}(x,\omega) = \phi_{0,t_2}(x,\theta(t_1,\omega)), \;\; t_1,t_2 \in \mathbb{R}, \, x \in \mathbb{R}^d, \omega \in \Omega^*, \tag{24}$$

replace x by $\phi_{0,t_1}(x,\omega)$ in the above identity and invoke the two-parameter flow property (20). This completes the proof of Corollary 5. ■

Finally, we give an extension of Theorem 3 to a class of non-degenerate $d-$dimensional Itô-diffusions.

Theorem 15 *Consider the time-homogeneous* $\mathbb{R}^d-$*valued SDE*

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dB_t,\ X_0^x = x \in \mathbb{R}^d,\ 0 \le t \le 1, \tag{25}$$

where the coefficients $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *and* $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$ *are Borel measurable. Suppose that* $\sigma(x)$ *has an inverse* $\sigma^{-1}(x)$ *for all* $x \in \mathbb{R}^d$. *Further assume that* $\sigma^{-1} : \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$ *is continuously differentiable such that*

$$\frac{\partial}{\partial x_k}\sigma_{lj}^{-1} = \frac{\partial}{\partial x_j}\sigma_{lk}^{-1}$$

for all $l, k, j = 1, ..., d$. *In addition, require that the function* $\Lambda : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *defined by*

$$\Lambda(x) := \int_0^1 \sigma^{-1}(tx) \cdot x dt$$

possesses a Lipschitz continuous inverse $\Lambda^{-1} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$. *Let* $D\Lambda : \mathbb{R}^d \longrightarrow L\left(\mathbb{R}^d, \mathbb{R}^d\right)$ *and* $D^2\Lambda : \mathbb{R}^d \longrightarrow L\left(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d\right)$ *be the existing corresponding derivatives of* Λ.
Assume that the function $b^* : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *given by*

$$\begin{aligned} b^*(x) &:= D\Lambda\left(\Lambda^{-1}(x)\right)\left[b(\Lambda^{-1}(x))\right] \\ &+\frac{1}{2}D^2\Lambda\left(\Lambda^{-1}(x)\right)\left[\sum_{i=1}^{d}\sigma(\Lambda^{-1}(x))[e_i], \sum_{i=1}^{d}\sigma(\Lambda^{-1}(x))[e_i]\right] \end{aligned}$$

is bounded and Borel measurable, where e_i, $i = 1, ..., d$, *is a basis of* $\mathbb{R}^d$.

Then there exists a stochastic flow $(s, t, x) \longmapsto \phi_{s,t}(x)$ *of the SDE (25) such that*

$$\phi_{s,t}(\cdot) \in L^2(\Omega, W^p(\mathbb{R}^d, w))$$

for all $0 \le s \le t \le 1$ *and all* $p > 1$.

Proof. Because of our assumptions we see that Λ^{-1} is twice continuously differentiable and that

$$D\Lambda(y)\sigma(y) = \mathcal{I}_d$$

for all $y \in \mathbb{R}^d$.

Then Itô's Lemma applied to (1) implies that

$$\begin{aligned} dY_t^x &= D\Lambda\left(\Lambda^{-1}(Y_t^x)\right)\left[b(\Lambda^{-1}(Y_t^x))\right] \\ &\quad +\frac{1}{2}D^2\Lambda\left(\Lambda^{-1}(Y_t^x)\right)\left[\sum_{i=1}^{d}\sigma(\Lambda^{-1}(Y_t^x))[e_i], \sum_{i=1}^{d}\sigma(\Lambda^{-1}(Y_t^x))[e_i]\right]dt + dB_t, \\ Y_0^x &= \Lambda(x),\ 0 \le t \le 1, \end{aligned}$$

where $Y_t^x = \Lambda(X_t^x)$. Because of Theorem 3 and a chain rule for functions in Sobolev spaces (see e.g. [40]) there exists a stochastic flow $(s, t, x) \longmapsto \phi_{s,t}(x)$ of the SDE (25) such that $\phi_{s,t}(\cdot) \in L^2(\Omega, W^p(\mathbb{R}^d, w))$ for all $0 \le s \le t \le 1$ and all $p > 1$. ■

3 Application to the Stochastic Transport Equation

In this section we will study the stochastic transport equation

$$\begin{cases} d_t u(t,x) + & (b(t,x)\cdot Du(t,x))dt + \sum_{i=1}^d e_i \cdot Du(t,x) \circ dB_t^i = 0 \\ u(0,x) = & u_0(x), \end{cases} \tag{26}$$

where $e_1, \dots e_d$ is the canonical basis of $\mathbb{R}^d$, $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ is a given bounded measurable vector field and $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a given initial data. The stochastic integration is understood in the Stratonovich sense.

In [22] it is proved that for smooth data and sufficiently regular vector field b, (26) has an explicit solution $u(t,x) = u_0(\phi_t^{-1}(x))$ where $\phi_t(x)$ is the flow map generated by the strong solutions $(X_t^x)_{t\geq 0}$ of the SDE (1). In fact this solution of the transport equation is strong in the sense that $u(t,\cdot)$ is differentiable everywhere in x almost surely for all t, and it satisfies the integral equation

$$u(t,x) + \int_0^t Du(s,x)\cdot b(s,x)ds + \sum_{i=1}^d \int_0^t e_i \cdot Du(s,x) \circ dB_s^i = u_0(x)$$

almost surely, for every t.

We shall use the following notion of weak solution (cf. Definition 12 in [14].

Definition 16 *Let b be bounded and measurable and $u_0 \in L^\infty(\mathbb{R}^d)$. A differentiable, weak L^∞-solution of the transport equation (26) is a stochastic process $u \in L^\infty(\Omega \times [0,1] \times \mathbb{R}^d)$ such that, for every t, the function $u(t,\cdot)$ is weakly differentiable a.s. with $\sup_{0\leq s\leq 1, x\in\mathbb{R}^d} E[|Du(s,x)|^4] < \infty$ and for every test function $\theta \in C_0^\infty(\mathbb{R}^d)$, the process $\int_{\mathbb{R}^d} \theta(x)u(t,x)dx$ has a continuous modification which is an $\mathcal{F}_t$-semi-martingale and*

$$\begin{aligned}\int_{\mathbb{R}^d} \theta(x)u(t,x)dx = & \int_{\mathbb{R}^d} \theta(x)u_0(x)dx \\ & - \int_0^t \int_{\mathbb{R}^d} Du(s,x)\cdot b(s,x)\theta(x)dxds \\ & + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s,x) D_i\theta(x)dx \right) \circ dB_s^i, \end{aligned} \tag{27}$$

where $Du(t,x)$ is the weak derivative of $u(t,x)$ in the space-variable.

Our definition of weak solution differs slightly from that in [14] due to the fact that we do not require any regularity on the coefficient b except Borel measurability and boundedness. To compensate for it, the expression depends on the weak derivative of $u(t,x)$.

It is easy to see that equation (26) can be written in the equivalent Itô form:

Lemma 17 *A process $u \in L^\infty(\Omega \times [0,1] \times \mathbb{R}^d)$ is a differentiable, weak L^∞ solution of the transport equation (26) if and only if, for every test function $\theta \in C_0^\infty(\mathbb{R}^d)$, the process*

$\int_{\mathbb{R}^d} \theta(x)u(t,x)dx$ *has a continuous $\mathcal{F}_t$-adapted modification and*

$$\begin{aligned}\int_{\mathbb{R}^d} \theta(x)u(t,x)dx = &\int_{\mathbb{R}^d} \theta(x)u_0(x)dx \\ &- \int_0^t \int_{\mathbb{R}^d} Du(s,x)\cdot b(s,x)\theta(x)dxds \\ &+ \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s,x)D_i\theta(x)dx\right) dB_s^i \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\theta(x)dxds.\end{aligned}$$

The main result of this section is the following existence and uniqueness theorem for solutions of the stochastic transport equation (26):

Theorem 18 *Let b be bounded and Borel measurable. Suppose $u_0 \in C_b^1(\mathbb{R}^d)$. Then there exists a unique $W^{1,\infty}$ weak solution $u(t,x)$ to the stochastic transport equation (26). (Moreover, for fixed t and x, this solution is Malliavin-differentiable.)*

Remark 19 *As noted in [14], the deterministic transport equation is generally ill-posed under the conditions of Theorem 18. It is remarkable that Brownian forcing on the transport equation induces uniqueness and regularity of the solution.*

We shall prove Theorem (18) using a sequence $b_n : [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ of uniformly bounded sequence of smooth functions with compact support converging almost everywhere to b. We then study the corresponding sequence of solutions of the transport equation (26) when b is replaced by b_n.

For the rest of this section we denote by ϕ_t the flow of the SDE (1) driven by the vector field b, and by $\phi_{n,t}$ the flow of the SDE (1) with b_n in place of b.

Following Remark (13), we will from now on assume that $\{\phi_{n,s}(x)\}$ is the subsequence chosen independently of s and x. In particular, reference to 'convergence of a sequence' indicates the 'convergence of a subsequence' of the sequence in question.

We begin with the following lemma:

Lemma 20 *Let $u_0 \in C_b^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$. Then the sequence*

$$\left(\int_{\mathbb{R}^d} u_0(\phi_{n,s}^{-1}(x))f(x)dx\right)_{n\geq 1}$$

converges to $\int_{\mathbb{R}^d} u_0(\phi_s^{-1}(x))f(x)dx$ in $L^2(\Omega)$ for every $s\in[0,1]$.

Proof. Consider

$$\|\int_{\mathbb{R}^d} u_0(\phi_{n,s}^{-1}(x))f(x)dx - \int_{\mathbb{R}^d} u_0(\phi_s^{-1}(x))f(x)dx\|_{L^2(\Omega)}$$

$$\leq \int_{\mathbb{R}^d} \|u_0(\phi_{n,s}^{-1}(x)) - u_0(\phi_s^{-1}(x))\|_{L^2(\Omega)}|f(x)|dx$$

We have $\|u_0(\phi_{n,s}^{-1}(x)) - u_0(\phi_s^{-1}(x))\|_{L^2(\Omega)} \leq \|Du_0\|_\infty \|\phi_{n,s}^{-1}(x) - \phi_s^{-1}(x)\|_{L^2(\Omega)}$ which goes to zero for every s and x by remark 13. Now

$$\|u_0(\phi_{n,s}^{-1}) - u_0(\phi_s^{-1})\|_{L^2(\Omega)}|f| \leq 2\|u_0\|_\infty |f| \in L^1(\mathbb{R}^d)$$

and the result follows by dominated convergence. ■

We also need the following result (see Theorem 2 in [17] and also [35], [36]):

Theorem 21 *Let $\mathcal{U}$ be open subset of $\mathbb{R}^d$ and $f \in W^{1,d}(\mathcal{U})$ be a homeomorphism. Then f satisfies the Lusin's condition, that is*

$$E \subset \mathcal{U},\ |E| = 0 \quad \Longrightarrow |f(E)| = 0.$$

Here $|A|$ stands for the Lebesgue measure of a set A.
Moreover, for every measurable function $g : \mathcal{U} \longrightarrow [0, \infty)$ and measurable set $E \subset \mathcal{U}$ the following change of variable formula is valid:

$$\int_E (g \circ f)\, |\det Jf|\, dx = \int_{f(E)} g(y) dy,$$

where $\det Jf$ is the determinant of the Jacobian of f.

Remark 22 *The random diffeomorphisms $\phi_t(\cdot), \phi_t^{-1}(\cdot) \in W_{loc}^{1,p}(\mathbb{R}^d)$ a.e. satisfy the conditions of Theorem 21 on each bounded and open subset $\mathcal{U}$ of $\mathbb{R}^d$.*

We are now ready to prove Theorem 18:

Proof of Theorem 18.

1. Existence of a weak solution:

We consider the approximation $\{b_n\}$ of b as described in Corollary 13. Then we know that there exists a unique strong solution to the transport equation (26) when b is replaced by b_n, which is uniquely given by $u_n(t, x) = u_0(\phi_{n,t}^{-1}(x))$, $n \geq 1$. In particular, u_n is a differentiable, weak L^∞-solution, such that for every $\theta \in C^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} \theta(x) u_n(t, x) dx = &\int_{\mathbb{R}^d} \theta(x) u_0(x) dx \\ &- \int_0^t \int_{\mathbb{R}^d} Du_n(s, x) \cdot b_n(s, x) \theta(x) dx ds \\ &+ \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u_n(s, x) D_i \theta(x) dx \right) dB_s^i \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u_n(s, x) \Delta\theta(x) dx ds.\end{aligned} \tag{28}$$

Let's now define $u(t,x) := u_0(\phi_t^{-1}(x))$ so that $u \in L^\infty(\Omega \times [0,1] \times \mathbb{R}^d)$, and $u(t,\cdot)$ is weakly differentiable, a.s. We now let n go to infinity to get that $u(t,x)$ is a solution of the transport equation.

The following two limits exist in $L^2(\Omega)$ by Lemma 20 and dominated convergence:

$$\int_{\mathbb{R}^d} \theta(x) u_n(t,x) dx \to \int_{\mathbb{R}^d} \theta(x) u(t,x) dx$$

$$\int_0^t \int_{\mathbb{R}^d} u_n(s,x) \Delta\theta(x) dx ds \to \int_0^t \int_{\mathbb{R}^d} u(s,x) \Delta\theta(x) dx ds$$

By the Itô isometry we have

$$\sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u_n(s,x) D_i\theta(x) dx \right) dB_s^i \to \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s,x) D_i\theta(x) dx \right) dB_s^i$$

in $L^2(\Omega)$. Finally, we claim that

$$\int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b_n(s,x) \theta(x) dx ds \to \int_0^t \int_{\mathbb{R}^d} Du(s,x) \cdot b(s,x) \theta(x) dx ds$$

in $L^2(\Omega)$. To see this observe that

$$\left(\int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b_n(s,x) \theta(x) dx ds \right)_n$$

is convergent in $L^2(\Omega)$ because of the convergence of the other terms in equality (28). Then the claim is proved once we show that $\int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b_n(s,x)\theta(x) dx ds$ converges **weakly** to $\int_0^t \int_{\mathbb{R}^d} Du(s,x) \cdot b(s,x)\theta(x) dx ds$. Then the strong and weak limit must coincide.

To prove weak convergence, we write the difference in three parts, namely:

$$\int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b_n(s,x) \theta(x) dx ds - \int_0^t \int_{\mathbb{R}^d} Du(s,x) \cdot b(s,x) \theta(x) dx ds$$

$$= \int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b_n(s,x) \theta(x) dx ds - \int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b(s,x) \theta(x) dx ds$$

$$+ \int_0^t \int_{\mathbb{R}^d} Du_0(\phi_{n,s}^{-1}(x)) D\phi_{n,s}^{-1}(x) \cdot b(s,x) \theta(x) dx ds - \int_0^t \int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x)) D\phi_{n,s}^{-1}(x) \cdot b(s,x) \theta(x) dx ds$$

$$+ \int_0^t \int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x)) D\phi_{n,s}^{-1}(x) \cdot b(s,x) \theta(x) dx ds - \int_0^t \int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x)) D\phi_s^{-1}(x) \cdot b(s,x) \theta(x) dx ds$$

$$= (i)_n + (ii)_n + (iii)_n$$

We shall deal with these terms separately.

(α): The first term $(i)_n$ converges to 0 strongly in $L^2(\Omega)$ as $n \to \infty$, since by Hölder's inequality and Fubini's theorem

$$E[(i)_n^2] = E\left[\left(\int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot (b_n(s,x) - b(s,x)) \theta(x) dx ds \right)^2 \right]$$

$$\leq \int_0^t \int_{\mathbb{R}^d} E[|Du_n(s,x)|^2]|b_n(s,x) - b(s,x)|^2|\theta(x)|dx\|\theta\|_{L^1(\mathbb{R})}$$

We have that

$$E[|Du_n(s,x)|^2] \leq \|Du_0\|_\infty^2 E[|D\phi_{n,s}^{-1}(x)|^2]$$

which is uniformly bounded in n, s and x by Proposition 7. Then, using dominated convergence, we obtain $\lim_{n\to\infty}(i)_n = 0$.

(β): The second term converges strongly to 0 in $L^2(\Omega)$, because of the following estimates:

$$E[(ii)_n^2] \leq \|b\|_\infty^2 E[\Big(\int_0^t \int_{\mathbb{R}^d} |Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))||D\phi_{n,s}^{-1}(x)||\theta(x)|dxds\Big)^2$$

$$\leq \|b\|_\infty^2 t\|\theta\|_{L^1(\mathbb{R}^d)} \int_0^t \int_{\mathbb{R}^d} E[|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))|^2|D\phi_{n,s}^{-1}(x)|^2]|\theta(x)|dxds$$

$$\leq \|b\|_\infty^2 t\|\theta\|_{L^1(\mathbb{R}^d)} \int_0^t \int_{\mathbb{R}^d} \left(E[|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))|^4]\right)^{1/2} \left(E[|D\phi_{n,s}^{-1}(x)|^4]\right)^{1/2} |\theta(x)|dxds$$

$$\leq \|b\|_\infty^2 t\|\theta\|_{L^1(\mathbb{R}^d)} \sup_{k,r,y} \left(E[|D\phi_{k,r}^{-1}(y)|^4]\right)^{1/2}$$

$$\times \int_0^t \int_{\mathbb{R}^d} \left(E[|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))|^4]\right)^{1/2} |\theta(x)|dxds.$$

The above estimates are consequences of Hölder's inequality. Since Du_0 is bounded and continuous, the right hand side of the above inequality converges to 0 by dominated convergence.

(γ): For the last term, let $X \in L^2(\Omega)$ and consider

$$E[(iii)_n X] = \int_0^t E[\int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x))(D\phi_{n,s}^{-1}(x) - D\phi_s^{-1}(x)) \cdot b(s,x)\theta(x)X\,dx]ds$$

Now, for each s, since Du_0, b and θ are bounded and $D\phi_s^{-1}$ is the weak limit of $D\phi_{n,s}^{-1}$, this expression tends to 0 as $n\to\infty$.

2. Uniqueness of weak solutions:

Let us assume that u is a weak solution to the stochastic transport equation (27) with $\sup_{0\leq s\leq 1, x\in\mathbb{R}^d} E[|Du(s,x)|^4] < \infty$. We will show that

$$u(t,x) = u_0(\phi_t^{-1}(x) \text{ a.e.}$$

This will guarantee uniqueness of the solution to the transport equation. So let V be a bounded and open subset of $\mathbb{R}^d$ and consider for the locally integrable function $u(t,\cdot)$ on $\mathbb{R}^d$ its mollification

$$u_\epsilon(t,x) = (u * \eta_\epsilon) = \int_{\mathbb{R}^d} u(t,y)\eta_\epsilon(x-y)dy,$$

with respect to the standard mollifier η.

We observe that u_ϵ satisfies the equation

$$u_\epsilon(t,x) = u_{0,\epsilon}(x) - \int_0^t (b\cdot Du)_\epsilon(s,x)ds - \int_0^t (Du)_\epsilon(s,x)\circ dB_s.$$

Then using the Itô-Ventzell formula applied to u_ϵ and $\phi_t(x)$ (see [22]) gives

$$u_\epsilon(t,\phi_t(x)) = u_{0,\epsilon}(x) + \int_0^t ((Du)_\epsilon(s,\phi_s(x))\cdot b(s,\phi_s(x)) - (b\cdot Du)_\epsilon(s,\phi_s(x)))ds. \tag{29}$$

Now let $\tau \in L^\infty(\Omega)$ and θ be a smooth function with compact support in V. Then it follows from (29) that

$$\begin{aligned} & E[\tau \int_V \theta(x) u_\epsilon(t,\phi_t(x))dx] \\ = \; & E[\tau \int_V \theta(x) u_{0,\epsilon}(x)dx] \quad (30)\\ & +E[\tau \int_0^t \int_V \theta(x)((Du)_\epsilon(s,\phi_s(x))\cdot b(s,\phi_s(x)) - (b\cdot Du)_\epsilon(s,\phi_s(x)))dxds]. \quad (31) \end{aligned}$$

Using Theorem 21 applied to $\phi_t^{-1}(\cdot)$ we obtain

$$\begin{aligned} & E[\tau \int_0^t \int_V \theta(x)((Du)_\epsilon(s,\phi_s(x))\cdot b(s,\phi_s(x)) - (b\cdot Du)_\epsilon(s,\phi_s(x)))dxds] \\ = \; & E[\tau \int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))((Du)_\epsilon(s,x)\cdot b(s,x) - (b\cdot Du)_\epsilon(s,x)) \left|\det(J\phi_s^{-1}(x))\right| dxds] \\ = \; & I_1 + I_2, \quad (32) \end{aligned}$$

where

$$I_1 := E[\tau \int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))((Du)_\epsilon(s,x)\cdot b(s,x)) \left|\det(J\phi_s^{-1}(x))\right| dxds] \tag{33}$$

and

$$I_2 := -E[\tau \int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))(b\cdot Du)_\epsilon(s,x) \left|\det(J\phi_s^{-1}(x))\right| dxds]. \tag{34}$$

Since V is bounded, there exists a $n\in\mathbb{N}$ such that $V \subset \overline{V} \subset W := (-n,n)^d$. Then we get

$$\begin{aligned} \|(Du)_\epsilon\|_{L^2(\phi_s(V))} &\leq \|Du\|_{L^2(\phi_s(W))}, \\ \|(b\cdot Du)_\epsilon\|_{L^2(\phi_s(V))} &\leq \|b\cdot Du\|_{L^2(\phi_s(W))} \\ &\leq \|b\|_\infty \|Du\|_{L^2(\phi_s(W))}. \end{aligned} \tag{35}$$

Using (35), Hölder's inequality, Fubini's theorem and Theorem 21, we obtain

$$\begin{aligned}
I_1 &\leq CE[\int_0^t (\int_{\mathbb{R}^d} (\chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))b(s,x)\left|\det(J\phi_s^{-1}(x))\right|)^2 dx)^{\frac{1}{2}} \\
&\quad \cdot(\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x)\left|Du(s,x)\right|^2 dx)^{\frac{1}{2}} ds] \\
&\leq C\int_0^t E[\int_{\mathbb{R}^d} (\chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))b(s,x)\left|\det(J\phi_s^{-1}(x))\right|)^2 dx]^{\frac{1}{2}} \\
&\quad \cdot E[\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x)\left|Du(s,x)\right|^2 dx]^{\frac{1}{2}} ds \\
&\leq C\int_0^t E[\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\left|\det(J\phi_s^{-1}(x))\right|^2 dx]^{\frac{1}{2}} \\
&\quad \cdot E[\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x)\left|Du(s,x)\right|^2 dx]^{\frac{1}{2}} ds \\
&\leq C\int_0^t (\int_{\mathbb{R}^d} E[\chi_{\phi_s(V)}(x)]^{\frac{1}{2}} E[\left|\det(J\phi_s^{-1}(x))\right|^4]^{\frac{1}{2}} dx)^{\frac{1}{2}} \\
&\quad \cdot(\int_{\mathbb{R}^d} E[\chi_{\phi_s(W)}(x)]^{\frac{1}{2}} E[\left|Du(s,x)\right|^4]^{\frac{1}{2}} dx)^{\frac{1}{2}} ds \\
&\leq C \sup_{0\leq s\leq 1, x\in\mathbb{R}^d} E[\left|\det(J\phi_s^{-1}(x))\right|^4]^{\frac{1}{2}} \sup_{0\leq s\leq 1, x\in\mathbb{R}^d} E[\left|Du(s,x)\right|^4]^{\frac{1}{2}} \\
&\quad \cdot\int_0^t (\int_{\mathbb{R}^d} E[\chi_{\phi_s(V)}(x)]^{\frac{1}{2}} dx) ds \\
&\leq C\int_0^t (\int_{\mathbb{R}^d} E[\chi_{\phi_s(V)}(x)]^{\frac{1}{2}} dx) ds
\end{aligned} \tag{36}$$

for a constant C depending on the sizes of V, θ and b, since

$$\sup_{0\leq s\leq 1, x\in\mathbb{R}^d} E[\left|\det(J\phi_s^{-1}(x))\right|^4] \leq M < \infty$$

because of Lemma 7 applied to $\phi_s^{-1}(x)$.

Further, it follows from Girsanov's theorem, Hölder's inequality and the symmetry of the distribution of the Brownian motion that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} E[\chi_{\phi_s(W)}(x)]^{\frac{1}{2}} dx ds \\
&= \int_0^t \int_{\mathbb{R}^d} (\mu(\phi_s^{-1}(x) \in W))^{\frac{1}{2}} dx ds \\
&\leq C\int_0^t \int_{\mathbb{R}^d} (\mu(B_s + x \in W))^{\frac{1}{4}} dx ds \\
&= C\int_0^t \int_{\mathbb{R}^d} (\mu(B_s + x \in (-n,n)^d)^{\frac{1}{4}} dx ds \\
&\leq C\int_0^t (2\int_0^\infty (1-\Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy)^d ds,
\end{aligned} \tag{37}$$

where Φ is the standard normal distribution function.

On the other hand we know that

$$1-\Phi(x) \leq \frac{1}{2\pi x}\exp(-x^2/2)$$

for all $x>0$ (see [3]).
So

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} E[\chi_{\phi_s(W)}(x)]^{\frac{1}{2}} dxds \\
\leq\ & C\int_0^t (2\int_0^n (1-\Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy + 2\int_n^\infty (1-\Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy)^d ds \\
\leq\ & K\int_0^t ((\int_0^n (1-\Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy)^d + (\int_n^\infty (1-\Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy)^d) ds \\
\leq\ & M(1+\int_0^t (\int_n^\infty (\frac{\sqrt{s}}{2\pi(y-n)}\exp(-(y-n)^2/2s))^{\frac{1}{4}} dy)^d) ds) \\
=\ & M(1+\int_0^t (\int_0^\infty (\frac{\sqrt{s}}{2\pi y}\exp(-y^2/2s))^{\frac{1}{4}} dy)^d) ds) \\
=\ & M(1+\int_0^t (\int_0^\infty \sqrt{s}(\frac{1}{2\pi y}\exp(-y^2/2))^{\frac{1}{4}} dy)^d) ds) \\
\leq\ & L<\infty. \qquad (38)
\end{aligned}$$

Furthermore, since

$$(Du)_\epsilon \longrightarrow Du \text{ in } L^p_{loc}(\mathbb{R}^d)$$

for all $p>1$ and since

$$\int_{\mathbb{R}^d} (\chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))b(s,x)\left|\det(J\phi_s^{-1}(x))\right|)^2 dx < \infty \text{ a.e.}$$

because of the above estimates, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))((Du)_\epsilon(s,x)\cdot b(s,x))\left|\det(J\phi_s^{-1}(x))\right| dx \\
\longrightarrow\ & \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))((Du)(s,x)\cdot b(s,x))\left|\det(J\phi_s^{-1}(x))\right| dx
\end{aligned}$$

for $\epsilon \searrow 0$ $\mu \times ds-$a.e.

On the other hand the latter expression w.r.t. ϵ is dominated by the integrable term

$$(\int_{\mathbb{R}^d} (\chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))b(s,x)\left|\det(J\phi_s^{-1}(x))\right|)^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x)\left|Du(s,x)\right|^2 dx)^{\frac{1}{2}}.$$

So using dominated convergence it follows from (36) and (38) that

$$\begin{aligned}
I_1 &= I_1(\epsilon) \longrightarrow \qquad (39)\\
& E[\tau\int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))((Du)(s,x)\cdot b(s,x))\left|\det(J\phi_s^{-1}(x))\right| dxds]
\end{aligned}$$

for $\epsilon \searrow 0$.

Similarly to I_1 we also get

$$\begin{aligned} I_2 &= I_2(\epsilon) \longrightarrow \\ &-E[\tau \int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))(b\cdot Du)(s,x)\left|\det(J\phi_s^{-1}(x))\right| dxds] \\ \text{for } \epsilon &\searrow 0 \end{aligned} \tag{40}$$

and

$$E[\tau \int_V \theta(x)u_\epsilon(t,\phi_t(x))dx] \longrightarrow E[\tau \int_V \theta(x)u(t,\phi_t(x))dx] \tag{41}$$

as $\epsilon \searrow 0$.

In addition, because of the assumptions on u_0 it is clear that

$$E[\tau \int_V \theta(x)u_{0,\epsilon}(x)dx] \longrightarrow E[\tau \int_V \theta(x)u_0(x)dx]$$

as $\epsilon \searrow 0$.

Altogether we can conclude that

$$E[\tau \int_{\mathbb{R}^d} \theta(x)u(t,\phi_t(x))dx = E[\tau \int_{\mathbb{R}^d} \theta(x)u_0(x)dx]$$

for all $\tau \in L^\infty(\Omega)$ and compactly supported smooth functions θ. Hence

$$u(t,\phi_t(x)) = u_0(x)$$

$\mu \times dx-$a.e.

Since $\phi_t^{-1}(\cdot)$ satisfies the Lusin condition in Theorem 21 on bounded open subsets we can find a Ω^* with $\mu(\Omega^*) = 1$ such that for all $\omega \in \Omega^*$

$$u(t,x) = u_0(\phi_t^{-1}(x)) \; dx - \text{a.e.}$$

Due to the continuity of u with respect to time the latter relation also holds uniformly in t.

Finally, the Malliavin differentiability of (a version) of $u(t,x)$ is a consequence of the fact that $\phi_t^{-1}(x)$ is Malliavin differentiable (see [26]) and of the chain rule for Malliavin derivatives (see [31]). ■

4 Application to ODE's

In this section we employ the approach developed in Section 2 to study the existence of absolutely continuous solutions $x \longmapsto X_t^x$ of the time-homogeneous (deterministic) ODE

$$dX_t^x = b(X_t^x)dt, X_0 = x \in \mathbb{R}, t \in \mathbb{R}, \tag{42}$$

where $b : \mathbb{R} \longrightarrow \mathbb{R}$ is a discontinuous function. More precisely, we show that the sequence of solutions $X_\cdot^{n,x}, n \geq 1$ to the perturbed equation

$$dX_t^{n,x} = b(X_t^{n,x})dt + \frac{1}{n}dB_t, X_0^{n,x} = x \in \mathbb{R}, t \in \mathbb{R},$$

converge to a solution $X_\cdot^{n,x}$ of the ODE (42). Furthermore, we show that this family of solutions to the ODE is absolutely continuous in $x \in \mathbb{R}$.

We begin with the following observation:

Proposition 23 *Let $b = (b_1, ..., b_d) : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a bounded Borel measurable function with at most countably many points of discontinuities in the space variable uniformly in time. Further, suppose that there exist constants $m_i > 0, i = 1, ..., d$ such that for each i*

$$\text{either } m_i \leq b_i(t,y) \text{ for all } t, y \text{ or } b_i(t,y) \leq -m_i \text{ for all } t, y. \tag{43}$$

Then there exists for all initial values $x \in \mathbb{R}^d$ a solution to the ODE

$$dX_t^x = b(t, X_t^x)dt, X_0 = x, 0 \leq t \leq 1. \tag{44}$$

Proof. By a result of A. Y. Veretennikov [39] we know that the perturbed equation

$$dX_t^{n,x} = b(t, X_t^{n,x})dt + \frac{1}{n}dB_t, X_0^{n,x} = x \in \mathbb{R}^d, 0 \leq t \leq 1$$

has a unique strong solution $X_\cdot^{n,x}$ with continuous paths for all $n \geq 1$.

On the other hand, we also know that the Brownian paths are $\alpha-$Hölder continuous a.e. for all $\alpha < \frac{1}{2}$ See e.g. [20]. Let us fix a ω in some Ω^* with $\mu(\Omega^*) = 1$ on which all those solutions and Brownian paths are concentrated. Then there exists a constant $C = C(\omega) < \infty$ such that for all $0 \leq t_1, t_2 \leq 1$ and $n \geq 1$

$$\begin{aligned} \left|X_{t_1}^{n,x} - X_{t_2}^{n,x}\right| &\leq M\left|t_1 - t_2\right| + \frac{C}{n}\left|t_1 - t_2\right|^\alpha \\ &\leq M\left|t_1 - t_2\right| + C\left|t_1 - t_2\right|^\alpha \end{aligned}$$

for a constant $M < \infty$. Clearly, we also have

$$\sup_{0 \leq t \leq 1} |X_t^{n,x}| \leq M < \infty$$

uniformly in n for some M. So it follows from the theorem of Arzela-Ascoli that

$$X_\cdot^{n_k,x} \underset{k \longrightarrow \infty}{\longrightarrow} X_\cdot^x = (X_\cdot^{x,(1)}, ..., X_\cdot^{x,(d)}) \text{ in } C([0,1]; \mathbb{R}^d)$$

for some subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$. Thus

$$X_t^{x,(j)} = x_j + \lim_{n \longrightarrow \infty} \int_0^t b_j(s, X_s^{n_k,x})ds$$

for all $t \in [0,1], j \in \{1, 2, \cdots, d\}$. So we obtain from (43) that for each $1 \leq j \leq d$ either

$$X_{t_1}^{x,(j)} - X_{t_2}^{x,(j)} = \lim_{n \longrightarrow \infty} \int_{t_1}^{t_2} b_j(s, X_s^{n_k,x})ds \geq m_j > 0, t_1 < t_2$$

or

$$X_{t_1}^{x,(j)} - X_{t_2}^{x,(j)} = \lim_{n \longrightarrow \infty} \int_{t_1}^{t_2} b_j(s, X_s^{n_k,x})ds \leq -m_j < 0, t_1 < t_2.$$

So any of the components $X_\cdot^{x,(j)}$ of $X_\cdot^x$ is a bijection on $[0,1]$. Hence $X_t^{x,(j)}$ can only hit the $j-$th projection of the points of discontinuities of b in the space variable at most countably many times for $t \in [0,1]$. Therefore X_t^x doesn't hit the discontinuity points of b $t-$a.e. Finally, using dominated convergence we get

$$X_t^x = x + \int_0^t b(s, X_s^x)ds, 0 \leq t \leq 1.$$

. ■

Remark 24 *In [38] it is shown that even if* $b : [0, \infty) \longrightarrow [a, b]$ *with* $a > 0$ *is Borel measurable, then the ODE (44) has a solution in* $[0, \infty)$.

In the sequel let us denote by

$$\int_0^t \int_{\mathbb{R}} f(s, y) L^{X^x}(dy, ds) \tag{45}$$

the integral of a bounded measurable function $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ with respect to the local time $L^{X^x}(dy, ds)$ of $X^x := X^{0,x}$ (in time and space). For more information about local time-space integration the reader is referred to [9] or [37].

We also need the following auxiliary result:

Lemma 25 *Let* $b : \mathbb{R} \longrightarrow \mathbb{R}$ *be a bounded Borel measurable function and let* $(b_n)_{n\geq 1} \subset C_0^\infty(\mathbb{R})$ *be a sequence of functions such that*

$$b_n(y) \underset{n\longrightarrow\infty}{\longrightarrow} b(y) \text{ a.e.},$$

and

$$|b_n(y)| \leq C$$

for all $n \geq 1, y \in \mathbb{R}$ *and some finite positive constant* C. *Denote by* $X_\cdot^{n,x}$ *the unique strong solution to*

$$dX_t^{n,x} = b_n(X_t^{n,x})dt + \delta dB_t, X_0^{n,x} = x \in \mathbb{R}, 0 \leq t \leq 1,$$

where $\delta > 0$ *is a constant. Let* $b_n^{'}$ *be the derivative of* b_n *for each* $n \geq 1$.
Then the following convergence

$$\exp(\int_0^t b_n'(X_s^{n,x})ds) \underset{n\longrightarrow\infty}{\longrightarrow} \exp(-\frac{1}{\delta^2} \int_0^t \int_{\mathbb{R}} b(y) L^{X^x}(dy, ds))$$

holds weakly in $L^2(\mathcal{U} \times [0, 1] \times \Omega, dx \times dt \times d\mu)$ *for all bounded open sets* $\mathcal{U} \subset \mathbb{R}$.

Proof.

We start by noting that the set of functions

$$\left(\varphi \otimes \exp\{\int_0^1 h(s)dB_s\}\right),$$

is total in $L^2(\mathcal{U} \times [0, 1] \times \Omega, dx \times dt \times d\mu)$ when φ ranges through $C_0^\infty(\mathcal{U} \times [0, 1])$ and h ranges through the step functions defined on $[0, 1]$.

By Girsanov's theorem we have

$$
\begin{aligned}
& \left| \left(\varphi \otimes \exp\{\int_0^1 h(s)dB_s\}, \exp(\int_0^t b_n'(X_s^{n,x})ds) - \exp(-\frac{1}{\delta^2}\int_0^t \int_{\mathbb{R}} b(y)L^{X^x}(dy,ds)) \right)_{L^2(\mathcal{U}\times[0,1]\times\Omega)} \right| \\
= & \left| \int_{\mathbb{R}}\int_0^1 \varphi(x,r)E[\exp\{\int_0^1 h(s)dX_s^{n,x}\}\exp\{-\int_0^t b_n'(x+B_s)ds\}\mathcal{E}(\int_0^t b_n(x+B_u)dB_u)]drdx \right. \\
- & \left. \int_{\mathbb{R}}\int_0^1 \varphi(x,r)E[\exp\{\int_0^t h(s)dX_s^x\}\exp\{-\frac{1}{\delta^2}\int_0^t\int_{\mathbb{R}} b(y)L^{\delta B_\cdot+x}(ds,dy)\}\mathcal{E}(\int_0^t b(x+B_u)dB_u)]drdx \right| \\
\leq & \left| \int_{\mathbb{R}}\int_0^1 \varphi(x,r)E[\left(\exp\{\int_0^1 h(s)dX_s^{n,x}\} - \exp\{\int_0^1 h(s)dX_s^x\}\right) \right. \\
& \left. \times \exp\{\int_0^t b_n'(x+B_s)ds\}\mathcal{E}(\int_0^1 b_n(x+B_u)dB_u)]drdx \right| \\
+ & \left| \int_{\mathbb{R}}\int_0^1 \varphi(x,r)E[\exp\{\int_0^1 h(s)dX_s^x\} \right. \\
& \left. \times \left(\exp\{\int_0^t b_n'(x+B_s)ds\} - \exp\{-\frac{1}{\delta^2}\int_0^t\int_{\mathbb{R}} b(y)L^{\delta B_\cdot+x}(ds,dy)\}\right)\mathcal{E}(\int_0^1 b_n(x+B_u)dB_u)]drdx \right| \\
+ & \left| \int_{\mathbb{R}}\int_0^1 \varphi(x,r)E[\exp\{\int_0^1 h(s)dX_s^x\}\exp\{-\frac{1}{\delta^2}\int_0^t\int_{\mathbb{R}} b(y)L^{\delta B_\cdot+x}(ds,dy)\} \right. \\
& \left. \times \left(\mathcal{E}(\int_0^1 b_n(x+B_u)dB_u) - \mathcal{E}(\int_0^1 b(x+B_u)dB_u)\right)]drdx \right| \\
& =: i)_n + ii)_n + iii)_n
\end{aligned}
$$

For the first term, since

$$
\exp\{\int_0^t b_n'(x+B_s)ds\} = 1 + \sum_{m\geq 1}\int_{0<s_1<\dots s_m<t}\prod_{j=1}^m b_n'(x+B_{s_j})ds_1\dots ds_m,
$$

we get that the sequence

$$
\left\{\exp\{\int_0^t b_n'(x+B_s)ds\}\mathcal{E}(\int_0^1 b_n(x+B_u)dB_u)\right\}_{n\geq 1}
$$

is bounded in $L^2(\Omega)$ and we have

$$i)_n \leq \int_0^1 \int_{\mathbb{R}} |\varphi(x,r)| \| \exp\{\int_0^1 h(s) dX_s^{n,x}\} - \exp\{\int_0^1 h(s) dX_s^x\} \|_{L^2(\Omega)} \times$$
$$\| \exp\{\int_0^t b_n'(x+B_s)ds\} \mathcal{E}(\int_0^1 b_n(x+B_u)dB_u) \|_{L^2(\Omega)} dx dr.$$

We know that $X_t^{n,x} \to X_t^x$ in $L^2(\Omega)$ and since h is a step function we get by dominated convergence that

$$\lim_{n\to\infty} i)_n = 0.$$

For the second term, by [9, Theorem 3.1] we have

$$\int_0^t b_n^{\shortmid}(\delta B_s + x) ds = - \int_0^t \int_{\mathbb{R}} \frac{1}{\delta^2} b_n(y) L^{\delta B + x}(ds, dy)$$

for all t, $\mu-$a.e.

On the other hand, we also know (see [37, p. 220]) that

$$\begin{aligned} & - \int_0^t \int_{\mathbb{R}} \frac{1}{\delta^2} b_n(y) L^{\delta B + x}(ds, dy) \\ = \;& 2(F_n(\delta B_t + x) - F_n(x) - \int_0^t \frac{1}{\delta^2} b_n(\delta B_s + x) \delta dB_s) \end{aligned}$$

where $F_n(y) := \int_0^y \frac{1}{\delta^2} b_n(u) du$. The last expressions holds is true when b_n is replaced by b. We see that the convergence

$$\int_0^t b_n^{\shortmid}(\delta B_s + x) ds \to - \int_0^t \int_{\mathbb{R}} \frac{1}{\delta^2} b(y) L^{\delta B + x}(ds, dy)$$

holds μ almost surely (possibly on a subequence). Similary as for $i)_n$ we may invoke dominated convergence to conclude

$$\lim_{n\to\infty} ii)_n = 0.$$

For the last term notice that $\mathcal{E}(\int_0^1 b_n(x+B_u)dB_u) \to \mathcal{E}(\int_0^1 b(x+B_u)dB_u)$ μ-almost surely (possibly on a subsequence). Since b_n is uniformly bounded we get that $\mathcal{E}(\int_0^1 b_n(x+B_u)dB_u) - \mathcal{E}(\int_0^1 b(x+B_u)dB_u)$ is bounded in, say, $L^4(\Omega)$, and thus the same sequence is uniformly integrable when squared. We then get that

$$\|\mathcal{E}(\int_0^1 b_n(x+B_u)dB_u) - \mathcal{E}(\int_0^1 b(x+B_u)dB_u)\|_{L^2(\Omega)} \to 0$$

by the Vitali Convergence theorem. By dominated convergence we get

$$\lim_{n\to\infty} iii)_n = 0.$$

■

Theorem 26 *Let $b:\mathbb{R}\longrightarrow\mathbb{R}$ be a bounded decreasing function such that either $m\leq b(y)$ for all y or $b(y)\leq -m$ for all y for some constant $m>0$. Then there exists a unique continuous function $(t,x)\longmapsto \widetilde{X}_t^x$ on $\mathbb{R}\times\mathbb{R}$ such that*

$$\widetilde{X}_t^x = x+\int_0^t b(\widetilde{X}_s^x)ds, \tag{46}$$

for all $t,x\in\mathbb{R}$. Moreover, the map

$$(t,x)\longmapsto \widetilde{X}_t^x$$

belongs to $L^2([0,1];W^{1,2}(\mathcal{U}))$ for any bounded open interval $\mathcal{U}$ in $\mathbb{R}$. The family $\mathbb{R}\ni x\longmapsto \widetilde{X}_t^x)\in\mathbb{R}$, $t\in\mathbb{R}$, is a group of $W^{1,2}$ Sobolev diffeomorphisms on $\mathbb{R}$.

Proof. Uniqueness is easy. Indeed, suppose Y_t^x is any solution of equation (46). Then

$$\frac{d}{dt}(\widetilde{X}_t^x - Y_t^x)^2 = 2(\frac{d}{dt}\widetilde{X}_t^x - \frac{d}{dt}Y_t^x)(\widetilde{X}_t^x - Y_t^x) = 2(b(\widetilde{X}_t^x)-b(Y_t^x))(\widetilde{X}_t^x - Y_t^x)\leq 0$$

since b is decreasing. Integrating the above inequality from 0 to t, we get

$$(\widetilde{X}_t^x - Y_t^x)^2 \leq (\widetilde{X}_0^x - Y_0^x)^2 = 0$$

This proves uniqueness.

We next prove existence of the flow for the ODE (46). It is sufficient to prove existence for Let $x\in\mathcal{U}$, an open bounded interval in $\mathbb{R}$ and for $t\in[0,1]$. Suppose $b_m, m\geq 1$ is a sequence of decreasing functions in $C_0^\infty(\mathbb{R})$ such that $b_m(y)\longrightarrow b(y)$ a.e. $m\longrightarrow\infty$ a.e. and $|b_m(y)|\leq C<\infty$ for all m,y and for some positive constant C.

Consider the solution $X_\cdot^{m,n,x}$ of the SDE

$$X_t^{m,n,x} = x+\int_0^t b_m(X_s^{m,n,x})ds+\frac{1}{n}B_t, 0\leq t\leq 1$$

for all $x\in\mathcal{U}, n,m\geq 1$ $\mu-$a.e.

We have (see [22])

$$\frac{\partial}{\partial x}X_t^{m,n,x} = 1+\int_0^t b_m^{\shortmid}(X_s^{m,n,x})\frac{\partial}{\partial x}X_s^{m,n,x}ds, 0\leq t\leq 1$$

for all $x\in\mathcal{U}, n,m\geq 1$ $\mu-$a.e.

Therefore,

$$\frac{\partial}{\partial x}X_t^{m,n,x} = \exp(\int_0^t b_m^{\shortmid}(X_s^{m,n,x})ds). \tag{47}$$

Let $\varphi\in C_0^\infty(\mathcal{U}), \xi\in L^\infty(\Omega), h\in L^\infty([0,1])$.

Now from the proof of Lemma 14 in Section 2, it follows that for each $n\geq 1$ and $p>1$

$$\sup_{0\leq t\leq 1}\sup_{x\in\mathcal{U}}\sup_{m\geq 1} E[\left|\frac{\partial}{\partial x}X_t^{m,n,x}\right|^p]\leq M<\infty,$$

where $M=M(n,p)$ is a positive constant.

The latter proof also shows that

$$
\begin{aligned}
&-\int_{\mathcal{U}}\int_0^1 E[X_t^{m,n,x}\xi]h(t)\frac{\partial}{\partial x}\varphi(x)dtdx\\
&\underset{m\longrightarrow\infty}{\longrightarrow} -\int_{\mathcal{U}}\int_0^1 E[X_t^{n,x}\xi]h(t)\frac{\partial}{\partial x}\varphi(x)dtdx.
\end{aligned}
$$

Since the map $(t,x)\longmapsto X_t^{x,n}$ belongs to $L^2([0,1]\times\Omega;W^{1,2}(\mathcal{U}))$, then it has a continuous version $(t,x)\longmapsto \widetilde{X}_t^{n,x}$ which is absolutely continuous in x.

Using (47) and Lemma 25 we find

$$
\begin{aligned}
&-\int_{\mathcal{U}}\int_0^1 E[X_t^{m,n,x}\xi]h(t)\frac{\partial}{\partial x}\varphi(x)dtdx\\
=&\int_{\mathcal{U}}\int_0^1 E[\frac{\partial}{\partial x}X_t^{m,n,x}\xi]h(t)\varphi(x)dtdx\\
\underset{m\longrightarrow\infty}{\longrightarrow}&\int_{\mathcal{U}}\int_0^1 E[\exp(-n^2\int_0^t\int_{\mathbb{R}} b(y)L^{X^{n,x}}(ds,dy))\xi]h(t)\varphi(x)dtdx.
\end{aligned}
$$

Hence

$$
\frac{\partial}{\partial x}\widetilde{X}_t^{n,x}=\exp(-n^2\int_0^t\int_{\mathbb{R}} b(y)L^{X^{n,x}}(ds,dy)) \tag{48}
$$

for all $n\geq 1, dt\times d\mu\times dx-$a.e. So we may identify $\frac{\partial}{\partial x}\widetilde{X}_t^{n,x}$ with the process on the right hand side of (48). Then $(t,x)\longmapsto \widetilde{X}_t^{n,x}$ is continuous $\mu-$a.s.

Furthermore, since $b_m^{\prime}(y)\leq 0, y\in\mathbb{R}$ in (47) we can argue by weak convergence that the right hand side of (48) is dominated by a constant $K\geq 0$ uniformly in $n,x,t,\mu-$a.e.

Thus

$$
\sup_{0\leq t\leq 1}\left|\widetilde{X}_t^{n,x_1}-\widetilde{X}_t^{n,x_2}\right|\leq K\left|x_1-x_2\right|
$$

for all $x_1,x_2\in\mathcal{U}, n\geq 1$ $\mu-$a.e.

On the other hand we may assume by Corollary 11 that $(t,x)\longmapsto X_t^{n,x}$ is continuous $\mu-$a.e. Hence we have

$$
\widetilde{X}_t^{n,x}=x+\int_0^t b(\widetilde{X}_s^{n,x})ds+\frac{1}{n}B_t
$$

for all $n\geq 1, t, x$, $\mu-$a.e.

So using the $\alpha-$Hölder continuity of Brownian paths, it follows that (for a fixed ω)

$$
\left|\widetilde{X}_{t_1}^{n,x}-\widetilde{X}_{t_2}^{n,x}\right|\leq M\left|t_1-t_2\right|+C(\omega)\left|t_1-t_2\right|^{\alpha} \tag{49}
$$

for all $0\leq t_1,t_2\leq 1, n\geq 1$ and all $x\in\mathcal{U}$, where $\alpha<\frac{1}{2}$, $C(\omega)=C(\omega,\alpha)<\infty$ and $M<\infty$.

Let $\mathcal{V}$ be a compact sub-interval of $\mathcal{U}$. Fix an appropriate $\omega\in\Omega$. Then by the Arzela-Ascoli theorem there is a subsequence (n_k) such that $(x\longmapsto \widetilde{X}_{\cdot}^{n_k,x}), k\geq 1$ converges in $C(\mathcal{V};C([0,1])$. Then repeated application of a weak compactness argument it follows that the limit, say $(t,x)\longmapsto \overline{X}_t^x$ belongs to $L^2([0,1];W^{1,2}(\mathring{\mathcal{V}}))$ ($\mathring{\mathcal{V}}$ the interior of $\mathcal{V}$). Finally, and as in the proof of Proposition 23, it follows that $\overline{X}_{\cdot}^x$ solves the ODE (46) for all x in $\mathring{\mathcal{V}}$. ■

Remark 27 *Using techniques of Malliavin calculus the authors in [25] prove that, for fixed $x \in \mathbb{R}$, the sequence $\{\widetilde{X}_t^{n,x}\}_{n=1}^{\infty}$ in the proof of Theorem 3 converges to $\widetilde{X}_t^x$ in $L^2(\mu)$ as $n \longrightarrow \infty$.*

Curiously enough, the next theorem is a consequence of the deterministic result in Theorem 26 above. It establishes the existence of a perfect cocycle of $W^{1,2}_{loc}$-Sobolev diffeomorphisms for solutions of the one-dimensional Stratonovich SDE:

$$dX_t^x = b(X_t^x) \circ dW(t), \;\; t \in \mathbb{R}, X_0 = x \in \mathbb{R},$$

driven by a bounded decreasing *diffusion* coefficient $b : \mathbb{R} \to \mathbb{R}$ satisfying the conditions of Theorem 26.

Theorem 28 *Let $b : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded decreasing function with a positive constant m such that either $m \leq b(y)$ for all $y \in \mathbb{R}$ or $b(y) \leq -m$ for all $y \in \mathbb{R}$. Suppose $W : \mathbb{R} \times \Omega \to \mathbb{R}$ is one-dimensional Brownian motion such that $W(0) = 0$, and $\theta : \mathbb{R} \times \Omega \to \Omega$ is the Wiener shift. Consider the Stratonovich SDE*

$$X_t^x = x + \int_0^t b(X_s^x) \circ dW(s), \;\; t, x \in \mathbb{R}. \tag{50}$$

Then the above SDE has a family of strong pathwise continuous solutions $\mathbb{R} \times \mathbb{R} \ni (t, x) \longmapsto X_t^x \in \mathbb{R}$ such that $(X_t^\cdot, \theta(t, \cdot))$ is a perfect cocycle of $W^{1,2}_{loc}$-Sobolev diffeomorphisms on $\mathbb{R}$.

Proof. The idea of the proof is simple: We rescale the deterministic flow of the ODE (46) using the one-dimensional Brownian motion W. This is feasible by an appropriate application of Itô's formula due to Fölmer, Protter and Shiryayev [16].

To simplify notation, we denote by $Y(t, x) := \widetilde{X}_t^x, t, x \in \mathbb{R}$, the deterministic flow of the ODE (46). Define the random field:

$$X_t^x := Y(W(t), x), \quad t, x \in \mathbb{R}. \tag{51}$$

We claim that the following equality

$$\int_0^{W(t)} b(Y(u, x))\, du = \int_0^t b(X_u^x) \circ dW(u), \;\; , X_0 = x \in \mathbb{R}, \tag{52}$$

holds for all $x \in \mathbb{R}$ a.s. for all $t \in \mathbb{R}$. To prove (52), we apply Itô's formula using the absolutely continuous change of variable $F(z) := \int_0^z b(Y(u, x))\, du, \;\; z \in \mathbb{R}$. Note that F is locally of class $W^{1,2}$ because $F'(z) = b(Y(z, x))$ for a.e. $z \in \mathbb{R}$. So by Itô's formula ([16]), it follows that

$$\begin{aligned} F(W(t)) &= \int_0^t F'(W(u))\, dW(u) + \frac{1}{2}[F'(W), W](t) \\ &= \int_0^t F'(W(u)) \circ dW(u) \\ &= \int_0^t b(X_u^x) \circ dW(u) \end{aligned} \tag{53}$$

a.s. for all $t \geq 0$. In the above relation, the bracket $[\cdot,\cdot]$ stands for the quadratic covariation. This proves our claim (52). Using (51), (46) and (52), we get

$$\begin{aligned} X_t^x &= Y(W(t),x) \\ &= x + \int_0^{W(t)} b(Y(u,x))\,du \\ &= x + \int_0^t b(X_u^x) \circ dW(u) \end{aligned} \tag{54}$$

a.s. for all $t \geq 0$. Hence X_t^x, $t,x \in \mathbb{R}$, is a family of solutions of the SDE (50). Since the map $[0,1] \ni t \mapsto Y(t,\cdot) \in W^{1,2}(U)$ belongs to $C([0,1],W^{1,2}(U))$, then so do the maps $[0,1] \ni t \mapsto X_t^{\cdot}(\omega) \in W^{1,2}(U)$ for a.a. $\omega \in \Omega$, where U is any bounded open interval in $\mathbb{R}$.

To prove the cocycle property for $(X_t^{\cdot},\theta(t,\cdot))$, we use the group property for the ODE (46):

$$Y(t_1,\cdot) \circ Y(t_2,\cdot) = Y(t_1+t_2,\cdot) \qquad t_1,t_2 \in \mathbb{R}. \tag{55}$$

Hence,

$$\begin{aligned} [X_{t_2}^{\cdot}(\theta(t_1,\omega)) \circ X_{t_1}^{\cdot}(\omega)](x) &= Y((\theta(t_1,\omega)(t_2), Y(\omega(t_1),x)) \\ &= Y(\omega(t_1+t_2),x) \\ &= X_{t_1+t_2}^x(\omega), \quad x,t_1,t_2 \in \mathbb{R},\ \omega \in \Omega. \end{aligned} \tag{56}$$

This completes the proof of the corollary. ■

Remark 29 *It is rather remarkable that the Stratonovich SDE (50) admits the existence of a perfect cocycle of $W_{loc}^{1,2}$-Sobolev diffeomorphisms with respect to a discontinuous diffusion coefficient. On the other hand, it is conceivable that the SDE (50) has more than one solution. In fact, we haven't even been able to find similar examples to (50) in the literature.*

5 Appendix

The following result which is due to [4] provides a compactness criterion for subsets of $L^2(\mu;\mathbb{R}^d)$ using Malliavin calculus. See e.g. [31], [24] or [7] for more information about Malliavin calculus.

Theorem 30 *Let $\{(\Omega,\mathcal{A},P);H\}$ be a Gaussian probability space, that is $(\Omega,\mathcal{A},P)$ is a probability space and H a separable closed subspace of Gaussian random variables of $L^2(\Omega)$, which generate the σ-field $\mathcal{A}$. Denote by $\mathbf{D}$ the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}(f(h_1,\ldots,h_n)) = \sum_{i=1}^n \partial_i f(h_1,\ldots,h_n)h_i,\ \ h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

Further let $\mathbf{D}_{1,2}$ be the closure of the family of elementary smooth random variables with respect to the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega;H)}\,.$$

Assume that C is a self-adjoint compact operator on H with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \left\|C^{-1}\mathbf{D}\, G\right\|_{L^2(\Omega;H)} \leq c \right\}$$

is relatively compact in $L^2(\Omega)$.

In order to formulate compactness criteria useful for our purposes, we need the following technical result which also can be found in [4].

Lemma 31 *Let $v_s, s \geq 0$ be the Haar basis of $L^2([0,1])$. For any $0 < \alpha < 1/2$ define the operator A_α on $L^2([0,1])$ by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \text{ if } s = 2^k + j$$

for $k \geq 0, 0 \leq j \leq 2^k$ and

$$A_\alpha 1 = 1.$$

Then for all β with $\alpha < \beta < (1/2)$, there exists a constant c_1 such that

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,1])} + \left(\int_0^1 \int_0^1 \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} dt\, dt' \right)^{1/2} \right\}.$$

A direct consequence of Theorem 30 and Lemma 31 is now the following compactness criterion which is essential for the proof of Lemma 12:

Corollary 32 *Let $X_n \in \mathbb{D}_{1,2}$, $n = 1, 2...$, be a sequence of $\mathcal{F}_1$-measurable random variables such that there are constants $\alpha > 0$ and $C > 0$ with*

$$\sup_n E[\|X_n\|^2] \leq C,$$

$$\sup_n E\left[\|D_t X_n - D_{t'} X_n\|^2\right] \leq C|t - t'|^\alpha$$

for $0 \leq t' \leq t \leq 1$ and

$$\sup_n \sup_{0 \leq t \leq 1} E\left[\|D_t X_n\|^2\right] \leq C\,.$$

where D_t denotes Malliavin differentiation. Then the sequence X_n, $n = 1, 2...$, is relatively compact in $L^2(\Omega)$ (D_t the Malliavin derivative).

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