Product differentiation and entry games

Simulating dynamic oligopoly games

Philippe Sørensen

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Preface

This thesis is an integral part of my master degree in Economics at the University of Oslo. It was written during the fall semester of 2008 and in the start of year 2009. This paper has permitted me to do an in-depth study in a subject that has truly captivated me. Industrial Organization Theory allows us to model and think about different issues in a unique manner. The study of market structure and oligopolies is central to this way of modelling.

My supervisor Alfonso Irarrázabal was the person who triggered my interest in Industrial Organization as he taught the first course I took in the subject. He also introduced me to a facinating set of models on entry and product differentiation and guided me in my work with them to form my thesis. My deepest thanks, hence goes to him for all his useful input and for motivating me.

I also wish to thank Fredrik Willumsen for always keeping his door open to students. I have asked for his help in several occasions and he has always taken time to listen to my questions.

Finally I wish to thank my family, my friends and my girlfriend for all the moral support, help and patience they have shown during this process.

I wish to point out that any errors or flaws in the thesis are entirely my responsibility.

Oslo, February 2009

Philippe Sørensen

Contents

Abstract

The aim for this thesis is to make an analysis of entry in industries with differentiated products. We will attempt to achieve this task by presenting a game. Studying thoroughly this game will make it possible at the same time to discover a framework. The framework will consist of a full derivation and presentation of a theoretical model, as well as a numerical method. We will concentrate on the case of oligopolies, and develop our framework to be able to solve dynamic problems. The model will rest on discrete choice theory, and in order to illustrate the dynamics we will in our derivations make use of the Bellman equation and Markov processes. The firms in the industries in question will, as stated, be able to differentiate their product from other firms' product. We will present two different types of industries.
First of all industries where firms are global players, and secondly industries where firms are local players. With global players we mean firms that have the opportunity of being active in different markets at the same time. In this setup the firm differentiates itself from the competitors by location. The choice of which market to enter into will decide its location, and thus how differentiated its product will be to competing products. We also mentioned local players. By them we mean players present exclusively in one local market. These players differentiate themselves from competitors with the price and quality of the product. In each case we will see that the equilibrium is found by solving a game. The firms will maximize their value function and build best response functions. The first part of our thesis will concentrate on a static set up. This is done in order to have a foundation on which to support the analysis of the dynamic models. The analysis of the dynamic models, takes place in our second part, when we introduce the event of entry. We will present a dynamic model, both for the case where firms are global players and for the case where firms are local players. Finally, in the third part, we implement both the static model and the global player model and use the global player implementation to perform a simulation with Matlab. The simulation will attempt to shed some light on different issues regarding entry in the Norwegian airline industry.

1. Static game

In this part we present a Static game. It will be based on the discussion in Anderson, de Palma, and Thisse (1992). We make use of a Logit model where price and quality are the decision variables of the firms. We will first set up the game by introducing the different variables and the framework. Then we work to find the solution to this game and finally we prove that this solution is unique.

1.1 Game setup

We start our setup by creating a market with a number of consumers and firms. There are S consumers, and n potential firms. This market can be seen for example as a local retail market. The number of consumers denotes the size of the market. In this market we introduce the fact that firms can have different qualities on their product, where w_i is the quality of firm i's product. We define **Ω** as the set of feasible product qualities. This opens up the market for product differentiation. Firms in this model hence sets price but quality as well, and the pair price-quality are the only factors that consumers care about in their choice of product. Proportional to the product demand, we find the market share of each product and thus of each firms. The market share of each product is equal to the probability of that product being purchased over the others. We assume here a Logit density function and hence a Logit probability.

$$(1.1) \quad P_i = \frac{e^{\frac{w_i - p_i}{\mu}}}{\sum_{j=1}^{n} e^{\frac{w_j - p_j}{\mu}}}, \qquad i = 1 \dots n$$

where μ is the consumer preference for diversity. Making use of expression (1.1) for the market share of product i, the market demand for this product can be expressed.

$$(1.2)\quad \widetilde{X_i} = SP_i, \qquad\qquad i = 1 \ldots n$$

(1.2) is called the Logit demand system. We introduce as well firm i's profit function.

$$(1.3)\quad \pi_i = \theta_v SP_i m(w_i, n) - K, \qquad\qquad i = 1 \ldots n$$

θ_v is a parameter and represents the degree of horizontal product differentiation, $m(w_i, n)$ is the price-cost margin and K is the sum of all the fixed costs.

Now that the basic expressions and variables are in place we can present the game at hand. We study a two-stage game. In the first stage, either the product quality is chosen by the potential entrants, or the potential entrant stays out. In the second stage, the entrants set their prices. By using backwards induction it is possible to find a subgame perfect Nash equilibrium to this game. The choices made in the first stage affect the decisions made in the second stage. In the case where two firms select the same quality, these two compete à la Bertrand. If K>0, two firms choosing the same quality will make a loss, and hence we will not see two similar quality choices in equilibrium. In the following section we use our setup to solve for equilibrium prices and quality.

1.2 Solving the game

Some derivations are needed in order to commence the analysis. We will as previously described, make use of backwards induction.

$$(1.4)\quad \frac{\partial P_i}{\partial p_j} = \frac{P_i P_j}{\mu}, \qquad\qquad i, j = 1 \ldots n, i \neq j$$

$$(1.5)\quad \frac{\partial P_i}{\partial p_i} = \frac{P_i(P_i - 1)}{\mu}, \qquad\qquad i = 1 \ldots n$$

Having in mind that $m(w_i, n)=p_i-c_i$, we try to solve for the second stage of our game. We thus want to find a solution for the equilibrium price. We derive our profit function so as to make use of the first order condition.

$$(1.6)\quad \frac{\partial \pi_i}{\partial p_i} = \theta_v \left[(p_i - c_i) S \frac{\partial P_i}{\partial p_i} + SP_i \right]$$

$$\Leftrightarrow \frac{\partial \pi_i}{\partial p_i} = \theta_v \left[(p_i - c_i) S \frac{P_i(P_i - 1)}{\mu} + SP_i \right]$$

We derive once again to find the double derivative, in order to make use of the second order condition.

$$(1.7)\quad \frac{\partial^2 \pi_i}{\partial p_i^2} = \theta_v \left\{ (p_i - c_i) \frac{S}{\mu} \left[\frac{P_i(P_i-1)^2 + P_i^2(P_i-1)}{\mu} \right] + 2S \frac{P_i(P_i-1)}{\mu} \right\}$$

$$\Leftrightarrow \frac{\partial^2 \pi_i}{\partial p_i^2} = \theta_v \left[(p_i - c_i) S \frac{P_i(P_i - 1)(2P_i - 1)}{\mu^2} + 2S \frac{P_i(P_i - 1)}{\mu} \right]$$

Setting (1.6) equal to zero, we find: $(p_i - c_i) S \frac{P_i(P_i-1)}{\mu} = -SP_i$

Hence, inserting this for (1.7), we can rewrite the expression for the second order derivative.

$$(1.8)\quad \frac{\partial^2 \pi_i}{\partial p_i^2} = \theta_v \left[-SP_i \frac{(2P_i-1)}{\mu} + 2S \frac{P_i(P_i-1)}{\mu} \right]$$

$$\Leftrightarrow \frac{\partial^2 \pi_i}{\partial p_i^2} = -\frac{\theta_v SP_i}{\mu} < 0$$

We can from (1.8) infer that, as our profit function is concave, the solution for the first order condition is a maximum. We now solve the first order condition to find the solution of the second stage game.

$$(1.9)\quad p_i^* = c_i + \frac{\mu}{1-P_i} \qquad\qquad i = 1 \dots n$$

We continue our analysis by attempting to compute the solution of the first stage game. Noticing from (1.9) that our expression for the equilibrium price depends on the quality through P_i, we can make the following deduction. We understand that in our case w^* and $p^*(w^*)$ define a subgame perfect Nash equilibrium for all w. We understand as well that the equilibrium of the quality game is found for w^*, where $\pi_i(w_i^*, w_{-i}^*) \geq \pi(w_i, w_{-i}^*)$ for all w_i and i=1…n. For computational reasons we redefine w_i: $w_i = \alpha q_i$, where α is the consumer's valuation of quality, and q_i is a measure of quality. α is a positive constant. Assume now that all firms choose quality w, except firm i who chooses quality w_i.

$$(1.10)\ p_i^* = c(q_i) + \frac{\mu}{1-\{exp[(\alpha q_i - p_i^*)/\mu]\}/D} \qquad i = 1 \dots n$$

$$(1.11)\ p^* = c(q) + \frac{\mu}{1-\{exp[(\alpha q - p^*)/\mu]\}/D},$$

where $D = exp[(\alpha q_i - p_i^*)/\mu] + (n-1)exp[(\alpha q - p^*)/\mu]$. After some algebra it can be found that:

$$(1.12)\ p^* - p_i^* = c(q) - c(q_i) + \frac{\mu}{n-2+\Delta} - \frac{\mu\Delta}{n-1},$$

where $\Delta = \frac{exp[(\alpha q_i - p_i^*)/\mu]}{exp[(\alpha q - p^*)/\mu]}$ is the ratio of firm i's output to the output of any other firm. The expression (1.12) will be usefull later on. Let's insert (1.10) in our expression for the profit.

$$(1.13)\ \pi_i = \theta_v \left[\frac{\mu}{1-\{exp[(\alpha q_i - p_i^*)/\mu]\}/D}\right] S \frac{exp[(\alpha q_i - p_i^*)/\mu]}{D} - K$$

$$\Leftrightarrow \pi_i = \theta_v \left[\frac{D}{(n-1)exp[(\alpha q - p^*)/\mu]}\right] S \frac{exp[(\alpha q_i - p_i^*)/\mu]}{D} - K$$

$$\Leftrightarrow \pi_i = \theta_v \frac{\mu S}{n-1} \Delta - K$$

We repeat the procedure used in the second stage. The derivative of the profit function with respect to our measure of quality q_i, will yield a solution given the first order condition. The sign of the second order derivative will prove whether or not this solution is in fact a maximum.

$$(1.14)\ \frac{\partial \pi_i}{\partial q_i} = \theta_v \frac{\mu S}{n-1} \frac{\partial \Delta}{\partial q_i}$$

$$\Leftrightarrow \frac{\partial \pi_i}{\partial q_i} = \theta_v \frac{\mu S}{n-1} \left[\alpha + (p^* - p_i^*) \frac{\partial}{\partial q_i}\right]$$

This is why the expression (1.12) is useful: It will ease the computation of the first derivative of the profit function. Before we attempt this derivation it is worthwhile to notice that the only way $\partial \pi_i / \partial q_i = 0$ is if $\partial \Delta / \partial q_i = 0$. Hence for any result of the first order condition we find the following.

$$(1.15)\ \frac{\partial (p^* - p_i^*)}{\partial q_i} = -c'(q_i) - \mu \left[\frac{1}{n-2+\Delta} + \frac{1}{n-1}\right] \frac{\partial \Delta}{\partial q_i}$$

$$\Leftrightarrow \frac{\partial (p^* - p_i^*)}{\partial q_i} = -c'(q_i)$$

Thus the answer of $\partial \pi_i / \partial q_i = 0$ is $\alpha = c'(q_i^*)$. The second order condition will now tell us if this is a maximum.

$$(1.16)\ \frac{\partial^2 \pi_i}{\partial q_i^2} = \frac{S\mu}{n-1} \frac{\partial^2 \Delta}{\partial q_i^2}$$

We wish to compute $\partial^2 \Delta / \partial q_i^2$.

$$(1.17)\ \frac{\partial \Delta}{\partial q_i} = \frac{\Delta}{\mu} \left[\alpha + \frac{\partial (p^* - p_i^*)}{\partial q_i}\right]$$

$$\Leftrightarrow \frac{\partial^2 \Delta}{\partial q_i^2} = \frac{\Delta}{\mu} \frac{\partial^2 (p^* - p_i^*)}{\partial q_i^2}$$

Again, we make use here of the fact that at a solution $\partial\Delta/\partial q_i = 0$. The expression (1.16) is rewritten as below.

$$(1.18)\ \frac{\partial^2\pi_i}{\partial q_i^2} = \frac{S\Delta}{n-1}\frac{\partial^2(p^*-p_i^*)}{\partial q_i^2}$$

It is straightforward to see that the sign of $\partial^2\pi_i/\partial q_i^2$ depends on the sign of $\partial^2(p^* - p_i^*)/\partial q_i^2$. Differentiating (1.15) once again by q_i we find the wanted result.

$$(1.19)\ \frac{\partial^2(p^*-p_i^*)}{\partial q_i^2} = -c''(q_i) - \mu\left[-\frac{2}{(n-2+\Delta)^3}\left(\frac{\partial\Delta}{\partial q_i}\right)^2 + \left(\frac{1}{(n-2+\Delta)^2} + \frac{1}{n-1}\right)\frac{\partial^2\Delta}{\partial q_i^2}\right]$$

$$\Leftrightarrow \frac{\partial^2(p^* - p_i^*)}{\partial q_i^2} = -c''(q_i) - \mu\left[\left(\frac{1}{(n-2+\Delta)^2} + \frac{1}{n-1}\right)\frac{\partial^2\Delta}{\partial q_i^2}\right]$$

We make use once more of $\partial\Delta/\partial q_i = 0$ since we are interested in the situation at the solution. $\partial^2\Delta/\partial q_i^2$ has already been found in (1.17), and is easily inserted.

$$\Leftrightarrow \frac{\partial^2(p^* - p_i^*)}{\partial q_i^2} = -c''(q_i) - \left(\frac{1}{(n-2+\Delta)^2} + \frac{1}{n-1}\right)\Delta\frac{\partial^2(p^* - p_i^*)}{\partial q_i^2}$$

$$\Leftrightarrow \frac{\partial^2(p^* - p_i^*)}{\partial q_i^2} = -\frac{c''(q_i)}{1 + \left(\frac{1}{(n-2+\Delta)^2} + \frac{1}{n-1}\right)\Delta} < 0$$

It can be concluded from this result that the solution $\alpha = c'(q_i^*)$ is a maximum. The analysis conducted here has brought forward an equilibrium solution for the quality and the price of the firms. This result is the solution of our static game.

$$p_i^* = c_i + \frac{\mu}{1 - P_i}$$

$$\alpha = c'(q_i^*)$$

1.3 Proof of uniqueness of solution

This section will have the aim of showing that the solution found in the preceding section is unique. We define two new variables.

(1.20) $x_i \equiv \frac{w_i - p_i}{\mu} \qquad i = 1 \dots n$

(1.21) $\gamma_i \equiv \frac{w_i - c_i}{\mu} \qquad i = 1 \dots n$

It is now possible to rewrite the solution for p_i^*.

(1.22) $p_i = c_i + \frac{\mu}{1 - P_i} \qquad \Leftrightarrow p_i - c_i = \frac{\mu}{1 - \frac{e^{x_i}}{\sum_{j=1}^{n} e^{x_j}}}$

$$\Leftrightarrow \gamma_i - x_i = \frac{1}{1 - \frac{e^{x_i}}{\sum_{j=1}^{n} e^{x_j}}} \qquad i = 1 \dots n$$

It is clear that the sum of exponential functions is positive. The expression (1.22) can be generalized, if we replace $\sum_{j=1}^{n} e^{x_j}$ by a positive parameter D. ($x_i < \ln D$)

(1.23) $G(x_i) = \gamma_i - x_i$

(1.24) $H(x_i) = \frac{1}{1 - \frac{e^{x_i}}{D}}$

We observe that the solution for the equilibrium price is found where the functions $G(x_i)$ and $H(x_i)$ are equal to each other. In other words, the equilibrium price is the one for which these two curves intersect. This is shown in Figure 1.1.

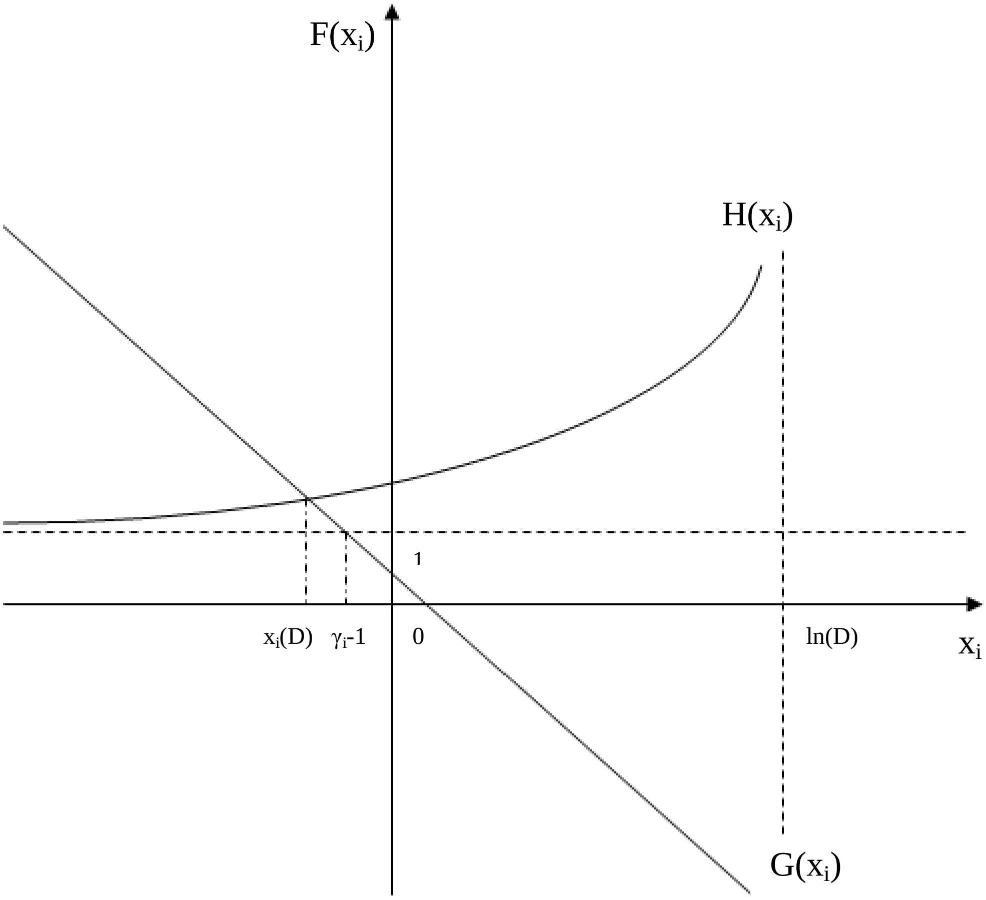

Figure 1.1: Graphical representation of the functions G(x_i) and H(x_i)
(Anderson, de Palma, and Thisse, 1992, p 265)

When taking a closer look at the expressions (1.23) and (1.24), there are several properties for these functions that can be outlined. First of all, G(x_i) is a decreasing linear function. It is also continuous and defined for all x_i. It is worth noticing as well that $\lim_{x_i \to -\infty} G(x_i) = \infty$ and $\lim_{x_i \to \infty} G(x_i) = -\infty$. For the function H($x_i$), it can be shown to be positive as $e^{x_i} < D$, and it is increasing and continuous in the defined interval $]-\infty, \ln D]$. We find that $\lim_{x_i \to -\infty} H(x_i) = \mu$, while $\lim_{x_i \to \ln D} H(x_i) = \infty$. Given the information found above, it is straightforward to see that in the interval $]-\infty, \ln D]$, there must be one and only one intersection point between the two functions. This proves that our equilibrium price is unique, and thus our quality equilibrium is unique as well.

2. Entry

The short description of a static game made in the first part, will allow us to start analysing a more advanced framework. Adding the possibility of entry for the firms, or potential firms, in the market at hand, moves our analysis from a static perspective to a dynamic one. The dynamics are introduced as the decision of entry is made through time. Our aim for this part is thus to present a dynamic model of product differentiation with entry. The firms in this setup are once again players in a game. As mentioned we will present two different cases. In the first case, the firms will be what are called global players. By that we mean players that have the choice of entering several markets. The choice of which markets to enter into, will affect the degree of differentiation. In the second case, the firms will be local players. Here the firms are only present in one market and make decisions on price and product quality. These decisions will again affect the level of differentiation of the firm's product compared to the competitor's products. We will define the equilibrium of the game for both cases, and present some examples of industries or businesses that fit each framework.

2.1 Global players

As we quickly described, a set up with global players will contain several local markets. The decision of the firms in this case will be to choose which markets to enter and which to stay out from, so as to maximize its aggregate value in the current period and in the future. We present the global player model as it is discussed in Aguirregabiria and Mira (2007). As the time dimension is introduced so is a time index t. There will be n potential firms, indexed with i. The number of local markets is given by J. We will call these local markets outlets. In addition we have what are called state variables. These describe the situation of the players and the outlets for a particular time period. We have both common knowledge variables and private information variables. The example we will use for the common knowledge variable is the size of the outlets. The size of the outlet is decided by the size of demand in that outlet. We define S_{jt} as our common knowledge variable in period t, for outlet j. S_t is a vector gathering each outlet size for the period t. This information is known by every player in the game. We will assume that the common knowledge variables are discrete and have a finite support.

We have also introduced a private information variable. This variable is information that is only available for the firm in question. An illustration of what the private information variable can be is the managerial ability of each firm. Competitors are not expected to detain that type of information, although it affects the player's wealth. We set the private information variable to be ε_{it}. Again a vector ε_t is created to collect all the firm's private information at period t. We will assume that the private values are independent and identically distributed, that they do not affect the outcome of the common knowledge variable, and that they appear additively in the profit function. Both state variables follow a Markov process. A Markov process maps a set of outcomes for a given variable, and assigns conditional probabilities on each outcome occurring given that a particular outcome occurred in the last period. These conditional probabilities are known as transition probabilities and we write them as such: $f(S_{t+1}, \varepsilon_{t+1}|S_t, \varepsilon_t)$.

As we mentioned, the firms make the decision of which outlet to be present in. We introduce a_{it} as being the decision of firm i, at period t. a_{it} is a vector that gathers for each outlet the choice of the firm at that period. Either the firm is present and $a = 1$, or the firm is not and $a = 0$. Each firm creates a decision rule or strategy function plotting the best strategy given the state variables. We set aside the time perspective for a moment and hence ignore the time index t. The set of decision rules are then given by $\sigma = \{\sigma_i(S, \varepsilon_i)\}$. To complete the set up of our game we now introduce some expressions that will be the building blocks for what we call the Bellman equation. The Bellman equation describes the net present value of all current and future cash flow of a given firm. This equation is central as it in fact illustrates the value of the firm, and is the one being maximized to find the optimal decision rule or choice probability.

Our first building block is the conditional choice probabilities. The conditional choice probability expresses the probability of the strategy a_i being chosen as the firm strategy given that S is common knowledge.

$$(2.1) \quad P_i^{\sigma}(a_i|S) \equiv Pr(\sigma_i(S,\varepsilon_i) = a_i|S) = \int I\{\sigma_i(S,\varepsilon_i) = a_i\} g_i(\varepsilon_i) d\varepsilon_i$$

$I\{\bullet\}$ is what we call an indicator function. It illustrates whether or not the element $\sigma_i(S,\varepsilon_i) = a_i$ is true for the subset in question. $g_i(\varepsilon_i)$ is the density function of our private information variable ε_i. As the common knowledge variable is given, the only other state variable to aggregate on is the private information variable. This means that the choice of a_i depends on the firm's private information, and thus the conditional choice probability depends on the probability density of ε_i.

The conditional choice probability permits us to build the current expected profit function, which is our second building block. This function provides us with information on the gains the firm can expect, when choosing a particular strategy at a given state.

$$(2.2) \quad \pi_i^{\sigma}(a_i,S) = \sum_{a_{-i}} \left(\prod_{j\neq i} P_j^{\sigma}(a_{-i}[j]|S)\right) \Pi_i(a_i,a_{-i},S)$$

We now take a moment to interpret the expression. We define the event t as being one combination of a's for all firms except the firm in question, i. The current expected profit is hence the profit in event t times the probability of event t summed over all events t. Let's take a look at the expression $\prod_{j\neq i} P_j^{\sigma}(a_{-i}[j]|S)$, and explain why that gives us the probability of one particular combination of a's. When we were discussing the conditional choice probability, we observed that the choice of "a" depended on nothing else than the private information variable. Hence we found that the firm's decision was independent of the competitor's decision. In that case the conditional choice probabilities are independent between players, and that means that the product of these probabilities gives us the probability that this particular combination of a's is chosen.

Our third building block is the transition probability of the common knowledge state variable S conditional on firm i choosing a_i and the other firms following the decision rule.

$$(2.3) \quad f_i^{\sigma}(S'|S,a_i) = \sum\nolimits_{a_{-i}} \left(\prod\nolimits_{j\neq i} P_j^{\sigma}(a_{-i}[j]|S)\right) f(S'|S,a_i,a_{-i})$$

This expression tells us the probability of S' occurring given that S occurred and the strategy a_i was played by firm i. To find the transition probability of the common knowledge variable given that firm i plays a_i, we make use of the following probability rule.

$$(2.4) \quad P(A) = P(A|B)P(B) + P(A|C)P(C) \qquad\qquad A \in B, C$$

This is called the law of total probability (Rice, 2007). Here the transition probability is conditioned on a_{-i} taking a particular value. Multiplying this conditional transition probability by the probability of that particular combination of a's, and summing over all possible combination of a's gives us the wanted transition probability.

With these expressions in place the Bellman equation is ready to be introduced. As described earlier, this will yield the value of the firm. The Bellman equation is actually the maximized value of the firm; hence it presents the value of the firm given that it behaves optimally at all points in time, while the other firms follow the decision rule.

(2.5)

$$V_i^{\sigma}(S,\varepsilon_i) = \max_{a_i}\left\{\pi_i^{\sigma}(a_i,S) + \varepsilon_i(a_i) + \beta\sum_{S'}\left[\int V_i^{\sigma}(S',\varepsilon_i{}')g_i(\varepsilon_i{}')d\varepsilon_i{}'\right] f_i^{\sigma}(S'|S,a_i)\right\}$$

We clearly see that the value of the firm is composed of the sum of the current expected profit and the net present value of future expected income. We have β as the discount factor and the function $\int V_i^\sigma(S', \varepsilon_i')g_i(\varepsilon_i')d\varepsilon_i'$ gives us the expectation of the next period Bellman equation over the private information. The expectation is taken over the common knowledge state variable as well, as we multiply it by the transition probability and sum it over all feasible states.

The setup of our game has been presented, and we now attempt to solve for the solution of the game. The solution in this case, will be found from the player's best response probability functions or Markov reaction functions. These are based entirely on pay-off relevant information, and supply an optimum strategy for each state of the world, given the strategy of the opponents. The solution will be what is called a Markov perfect equilibrium, and will consist of the strategy functions that maximize the Bellman equation. A Markov perfect equilibrium is defined as an equilibrium in the Markov reaction functions. In order to find the equilibrium, we need to work on the Bellman equation. We are interested in another form of the Bellman equation, the integrated Bellman equation.

$$(2.6) \quad V_i^\sigma(S) = \int V_i^\sigma(S, \varepsilon_i)g_i(\varepsilon_i)d\varepsilon_i$$

$$\Leftrightarrow V_i^\sigma(S) = \int \max_{a_i}\Bigg\{\pi_i^\sigma(a_i, S) + \varepsilon_i(a_i) + \beta\sum_{S'}\left[\int V_i^\sigma(S', \varepsilon_i')g_i(\varepsilon_i')d\varepsilon_i'\right] f_i^\sigma(S'|S, a_i)\Bigg\} g_i(\varepsilon_i)d\varepsilon_i$$

$$\Leftrightarrow V_i^\sigma(S) = \int \max_{a_i}\Bigg\{\pi_i^\sigma(a_i, S) + \varepsilon_i(a_i) + \beta\sum_{S'} V_i^\sigma(S')\ f_i^\sigma(S'|S, a_i)\Bigg\} g_i(\varepsilon_i)d\varepsilon_i$$

So as to simplify the analysis, we define the following expression.

$$(2.7) \quad v_i^\sigma(a_i, S) = \pi_i^\sigma(a_i, S) + \beta\textstyle\sum_{S'} V_i^\sigma(S')\ f_i^\sigma(S'|S, a_i)$$

We will now develop the integrated Bellman expression. To improve our understanding in the next steps we will study the case where there is only one market. This contradicts the idea of the global player set up, but it is necessary in order to fully grasp what is going on. Hence a_i is no longer a vector, but a number which can take two values: either zero or one.

$$(2.8)\quad V_i^{\sigma}(S) = \int max\{v_i^{\sigma}(0,S) + \varepsilon_i(0), v_i^{\sigma}(1,S) + \varepsilon_i(1)\} g_i(\varepsilon_i) d\varepsilon_i$$

Either a_i=0 or a_i=1 will yield the highest value, but it is unknown which one. Hence a best response is thus the response which maximizes the value by assigning a probability on each choice maximizing the expression. If we assume P^* is an equilibrium we can write the maximized Bellman as below.

(2.9)

$$V_i^{P^*}(S) = \int \left\{P_i^*(0|S)\left[v_i^{P^*}(0,S) + \varepsilon_i(0)\right] + P_i^*(1|S)\left[v_i^{P^*}(1,S) + \varepsilon_i(1)\right]\right\} g_i(\varepsilon_i) d\varepsilon_i$$

The "max" operator vanishes as we are located at maximum, when studying an equilibrium. We now go back to the general notation and rewrite (2.9).

$$(2.10)\ V_i^{P^*}(S) = \int \left\{\sum_{a_i} P_i^*(a_i|S) \left[v_i^{P^*}(a_i,S) + \varepsilon_i(a_i)\right]\right\} g_i(\varepsilon_i) d\varepsilon_i$$

At this point we can compute the integral.

$$(2.11)\ V_i^{P^*}(S) = \sum_{a_i} P_i^*(a_i|S)\, v_i^{P^*}(a_i,S) + \sum_{a_i} P_i^*(a_i|S) \int \varepsilon_i(a_i) g(\varepsilon_i) d\varepsilon_i$$

$$\Leftrightarrow V_i^{P^*}(S) = \sum_{a_i} P_i^*(a_i|S) \left[\pi_i^{P^*}(a_i,S) + e_i^{P^*}(a_i,S) + \beta \sum_{S'} V_i^{P^*}(S')\ f_i^{P^*}(S'|S,a_i)\right]$$

Here we make use of (2.7), and the fact that $e_i^{P^*}(a_i,S) = \int \varepsilon_i(a_i)\, g_i(\varepsilon_i) d\varepsilon_i$. We continue our computations on this expression, so as to be able to isolate the Bellman value function and thus get a solution. This will be possible since we are at equilibrium. The reason is that at equilibrium, the value function doesn't vary from one time period to another; it is at a fixed point. In other words we could say that the value function is at a steady state. We thus develop our expression.

(2.12)

$$V_i^{P^*}(S) = \sum_{a_i} P_i^*(a_i|S)\left[\pi_i^{P^*}(a_i,S) + e_i^{P^*}(a_i,S)\right] + \beta \sum_{S\prime} V_i^{P^*}(S') \sum_{a_i} P_i^*(a_i|S)\, f_i^{P^*}(S'|S,a_i)$$

We now insert for $f_i^{P^*}(S'|S,a_i)$ from (2.3).

(2.13)

$$V_i^{P^*}(S) = \sum_{a_i} P_i^*(a_i|S)\left[\pi_i^{P^*}(a_i,S) + e_i^{P^*}(a_i,S)\right] + \beta \sum_{S\prime} V_i^{P^*}(S') \sum_{a_i} P_i^*(a_i|S) \sum_{a_{-i}} \left(\prod_{j\neq i} P_j^*(a_{-i}[j]|S)\right) f(S'|S,a)$$

We define $\sum_a \left(\prod_j P_j^*\left(a_j\middle|S\right)\right) = \sum_{a_i} \sum_{a_{-i}} P_i^*(a_i|S)\left(\prod_{j\neq i} P_j^*(a_{-i}[j]|S)\right)$.

(2.14)

$$V_i^{P^*}(S) = \sum_{a_i} P_i^*(a_i|S)\left[\pi_i^{P^*}(a_i,S) + e_i^{P^*}(a_i,S)\right] + \beta \sum_{S\prime} V_i^{P^*}(S') \sum_{a} \left(\prod_{j} P_j^*\left(a_j\middle|S\right)\right) f(S'|S,a)$$

We define as well $f^{P^*}(S'|S) = \sum_a \left(\prod_j P_j^*\left(a_j\middle|S\right)\right) f(S'|S,a)$.

(2.15)

$$V_i^{P^*}(S) = \sum_{a_i} P_i^*(a_i|S)\left[\pi_i^{P^*}(a_i,S) + e_i^{P^*}(a_i,S)\right] + \beta \sum_{S\prime} V_i^{P^*}(S') f^{P^*}(S'|S)$$

In vector notation, it is possible to handle the expression more easily.

$$(2.16)\ V_i^{P^*} = \sum_{a_i} P_i^*(a_i) * \left[\pi_i^{P^*}(a_i) + e_i^{P^*}(a_i)\right] + \beta F^{P^*} V_i^{P^*}$$

$$\Leftrightarrow V_i^{P^*}\left(I - \beta F^{P^*}\right) = \sum_{a_i} P_i^*(a_i) * \left[\pi_i^{P^*}(a_i) + e_i^{P^*}(a_i)\right]$$

$$\Leftrightarrow \Gamma_i(P^*) \equiv V_i^{P^*} = \left(I - \beta F^{P^*}\right)^{-1} \left\{\sum_{a_i} P_i^*(a_i) * \left[\pi_i^{P^*}(a_i) + e_i^{P^*}(a_i)\right]\right\}$$

We define $\Gamma_i(P^*)$ as the solution of the system of equations, and hence the solution of the Bellman equation. $\Gamma_i(P^*)$ can hence be seen as being the expected maximized value of the firm, given the state variables and the strategies at hand. We observe that this result is not the Markov perfect equilibrium, but the solution of the system of Bellman equations. We will show that with this solution in hand, it is straight forward, to obtain the MPE from the best response probability functions. We therefore introduce the best response probability functions, built from our choice probability function in (2.1).

(2.17)

$$\Psi_i(a_i|S;P) = \int I\left(a_i = \underset{a_i}{\operatorname{argmax}}\left\{\pi_i(a_i,S) + \varepsilon_i(a_i) + \beta \sum_{S'} V_i(S')\ f_i(S'|S,a_i)\right\}\right) g_i(\varepsilon_i) d\varepsilon_i$$

Inserting for the solution found for the Bellman equation in this expression, gives us the wanted Markov reaction curves.

(2.18)

$$\Psi_i^*(a_i|S;P) = \int I\left(a_i = \underset{a_i}{\operatorname{argmax}}\left\{\pi_i(a_i,S) + \varepsilon_i(a_i) + \beta \sum_{S'} \Gamma_i(S';P^*)\ f_i(S'|S,a_i)\right\}\right) g_i(\varepsilon_i) d\varepsilon_i$$

The Markov perfect equilibrium is hence a fixed point originating from the Markov reaction curve mapping, and thus our solution is characterized. We observe that entry in the global player's case depends on the competitor's decision of entry, as equilibrium is found from the junction of the Markov reaction curves. As we will study more extensively in the part on local players, entry in this case is also affected by a cost perspective. The cost of entry, in addition to other costs, is included implicitly in the profit function in (2.2). We haven't discussed this perspective yet as the discussion of the Bellman equation was found to be intricate enough. We will also discover that the part on local player will bear more resemblance to our static analysis from part 1, than what the part on global players does.

Before moving on to the discussion of local players, we will go through some examples of industries with global players, show what defines the players of these industries as global players, and thus show why these industries fit our setup.

Let's take a look at the airline industry; it will be our first example. An airline company delivers services, such as transporting cargo or people, within or beyond a country's borders. As we have seen, an airline company will be a global player if it is present in several local markets. The key question in this case is what defines a market in this industry? Usually a market is seen as being a part of a city, a city, a region, or a country. Is this the case for the airline industry? An airline company offers transport from one city to another. But will the market be defined by the city of departure or the city of arrival? Actually, we can assume that neither is the case. As the demand is for transport between two cities, a city pair can be viewed as a local market in this case. The choice of which route to serve, determines which markets the airline company enters into. Whenever a company serves more than one route it is a global player. The cities vary in size and hence the demand for each route varies from one another. Adding for differences in demand between seasons, we can note that one route may experience changes in demand depending on time. Such changes in demand can also happen as a result of the economical climate. This fits well with our game set-up where the state variable changes from one time period to another through the Markov process. We can also assume that whenever an airline company changes its managers, the managerial abilities vary. It is clear as well that in the airline industry, which is very competitive, the choice of entry of one actor to a particular market, will affect the choice of entry of the competitors to the same market. This as well is in line with our model. The two main players in the Norwegian airline industry are Scandinavian Airlines (SAS) and Norwegian.

The supermarket industry is another example of an industry with global players. When visiting a supermarket, we often notice that the name or/and concept of the supermarket is identical to other supermarkets in the same city, region or country. The reason is that this supermarket belongs to a chain. The main examples of supermarket chains in Norway are Norgesgruppen, Coop Norge, ICA Norge, and Reitangruppen. They own several supermarket brands, which have stores all over the country. In this case, unlike the airline industry, markets are small local areas. These local areas can be neighborhoods or communities in a town or city, or small districts. Here as well the choice of where to enter is crucial for the division of the market share, and thus for the profit. The choice of entry will, as in our model, depend on the market sizes and the choice of the competitors. Similarly to the airline industry it is reasonable to assume that, due to management changes and changes in the economical climate, both the managerial ability and the market sizes can be modeled with a Markov process.

2.2 Local players

Since the model for local players is a dynamic one, some of the analysis in this part will resemble that of the global player part. The main resemblance will be the use of the Bellman equation. Taking a closer look we will see that the two models are very different. The local player set up studies, as we mentioned earlier, several firms in one local market. Hence the firms must make the decision of being present or not in each time period, for that market only. We use the analysis in Aguirregabiria, Mira and Roman (2007) as well as Aguirregabiria, Mira and Roman (2005) to model this case. We will introduce a cost structure, and hence the profit functions of the firms will differ whether they are inactive, entering, incumbent or exiting. In the static game part, the central issues were the decision of quality and price. This will also be a central issue in this part. As we move to a dynamic setup compared to the first part, investment costs will appear. These will able the firms to affect the quality level. We start the presentation of the model. We index time again by t, and firms by i. There are n_t potential firms and S_t consumers. The number of consumers determines the market size and evolves following a Markov process. We define $f(S_{t+1}|S_t)$ as the transition probability of the Markov process for the market size. Of the potential firms we have N_t active or incumbent firms. The remaining $E_t = n_t - N_t$ firms are potential entrants. We denote w_{it} to be the quality, and p_{it} to be the price variable of firm i at time t. We introduce the market share and variable profit functions. These are similar to the static game, but they vary each period.

$$(2.19)\ P_{it} = \frac{e^{\frac{w_{it}-p_{it}}{\mu}}}{\sum_{j=1}^{N_t} e^{\frac{w_{jt}-p_{jt}}{\mu}}}, \qquad i = 1 \dots n$$

$$(2.20)\ \pi_{it} = \theta_v S_t P_{it} m(w_{it}, N_t)$$

We remember that P_{it} is the probability of firm i being chosen by the consumers given that he achieves quality w_{it}, and sets price p_{it}. P_{it} is the Logit probability from discrete choice theory. θ_v is as before the horizontal product differentiation parameter. We recall as well $m(w_{it}, N_t)$ which we defined as the price-cost margin. In other words it's the mark-up firms take. The firms earn this variable profit, as long as they are active in the market. The quality follows as well a Markov process, but can be affected by investments. The transition probability is given by $F(w_{it+1}|w_{it}, i_{it})$, where i_{it} denotes the investment decision of firm i at period t. i_{it} takes either the value one or zero, where one indicates the firm invests, while zero signifies it doesn't. The transition probability function for the quality parameter has the following properties.

$$(2.21)\ f(w_{it+1}|w_{it}, i_{it} = 0) = 0, \qquad \forall\, w_{it+1} \geq w_{it}$$

$$(2.22)\ f(w_{it+1}|w_{it}, i_{it} = 1) = 0, \qquad \forall\, w_{it+1} < w_{it}$$

The first property states that without any investment, the quality cannot improve. The second property states that with a positive investment, the quality cannot depreciate. In relation to the investment decision, we introduce a parameter that is firm-specific. This parameter γ_i indicates how expensive the investment cost is for the firm in question. This parameter, as well as the initial quality, is drawn from a given probability distribution for each firm. The game allows the firms to exit from the market, but once a firm exits it isn't allowed to enter again. Each period the exiting firms disappear from the game and new players appear. The pool of potential entrants is thus updated. The new players must make their decision of whether to enter or not before they have information about the initial quality and the investment cost parameter. Once a firm enters the market, it competes in prices with the other active firms in a Bertrand manner.

We will now take a moment to describe the profit function. The form of the profit function depends on the firm's decision. There are four different profit functions. In the case where the firm decides to stay out of the market, the profit is zero. Next if an inactive firm decides to enter his profit is as follows.

$$(2.23)\ \Pi_{it} = -\theta_{EC} - \varepsilon_{E,it}$$

The entire profit is composed of the entry cost EC_{it}, where $EC_{it} = \theta_{EC} + \varepsilon_{E,it}$. First we have $\varepsilon_{E,it}$ which is the firm-specific part of the entry cost. It is also private information. On the other hand we have θ_{EC} which is the part of the entry cost common for all firms. We move over to the case where an already active firm decides to stay in the market. We present the profit function in this case.

$$(2.24)\ \Pi_{it} = \theta_v S_t P_{it} m(w_{it}, N_t) - FC_{it} - IC_{it}$$

As we have already mentioned, every period the firm i is active, it earns the variable profit. Each period it is present in the market it must also pay fixed costs FC_{it} and investment costs IC_{it}. The fixed cost has the form presented below.

(2.25) $FC_{it} = \theta_{FC,0} + \theta_{FC,1} w_{it}$

The interpretation is similar to the entry cost. We have a common component, and a firm-specific one. $\theta_{FC,0}$ is the fixed cost common for all active firms, while $\theta_{FC,1} w_{it}$ is the fixed cost specific to firm i. We assume as we can see that the fixed cost depends on the level of quality. The higher $\theta_{FC,1}$ is, the harder it is for a firm to sustain a high level of quality. We have an expression as well for the investment cost.

(2.26) $IC_{it} = \left(\theta_{IC} + \alpha_\gamma \gamma_i - \varepsilon_{I,it}\right) i_{it}$

We recall that $i_{it} \epsilon \{0,1\}$. Again, θ_{IC} is the common part of the investment cost. The firm-specific part of the investment is the expression $\alpha_\gamma \gamma_i + \varepsilon_{I,it}$, where α_γ is a parameter while γ_i is the type of the firm. The lower γ_i is, the more the firm has a propensity to invest. $\varepsilon_{I,it}$ is a private information variable. Lastly we present the profit function in the case where firm i decides to exit from the market.

(2.27) $\Pi_{it} = \theta_v S_t P_{it} m(w_{it}, N_t) - FC_{it} + EV_{it}$

Where EV_{it} is the exit value, meaning the income from terminating operations. This can be from selling the assets of the firm for example. The exit value or scrap value of the firm i is expressed accordingly.

(2.28) $EV_{it} = \theta_{EV,0} + \theta_{EV,1} w_{it} + \varepsilon_{X,it}$

Here again, we defined $\theta_{EV,0}$ as being the part of the exit value common for all firms, while $\theta_{EV,1} w_{it} + \varepsilon_{X,it}$ is specific for each firm. We observe that the quality is related to the exit value. We can assume that the more we spend on quality investment, the more the production processes become advanced and the higher we can expect the scrap value of being when these processes are sold. We have depicted the different forms the profit function can take depending on the choice of the firm in question.

We will in a moment work with the Bellman equation and the intertemporal value of the firm. We define first the state variables and decision variables of the firm. The state variables are the quality w_{it}, the market size S_t, the number of incumbent firms N_t, and the vector of firm-specific variables ε_{it}. The firm-specific variables are taken from the cost structure studied above.

$$(2.29)\ \varepsilon_{it} \equiv \left\{\varepsilon_{E,it}, \varepsilon_{I,it}, \varepsilon_{X,it}\right\}$$

We assume they are standard normally distributed. The decision variables of the firms are the price p_{it} and the decision i_{it} of investing in quality or not. The firms must also, as we have seen, make the decision of staying or exiting the market if they are incumbent, and make the decision of staying out or entering the market if they are potential entrants. We study the case of an incumbent firm.

$$(2.30)\ V(w_{it}, N_t, S_t) = \theta_v S_t P_{it} m(w_{it}, N_t) - FC_{it} + \int max \begin{Bmatrix} -IC_{it} + v^C(w_{it}, N_t, S_t, i_{it} = 1) \\ v^C(w_{it}, N_t, S_t, i_{it} = 0) \\ EV_{it} \end{Bmatrix} d\Phi(\varepsilon_I, \varepsilon_X)$$

The value or the income stream of firm i is given by the expression stated above. Whatever the choice taken by the incumbent, he will always earn the proceeds from being active and he will need to pay the fixed cost. As we can see, his choice will be the action that yields the highest expected flow of income. From the top, we find that he can either, stay in the market and invest in quality, stay in the market without investing or exit the market. If he decides to stay and invest, he must pay the investment cost, but earns the flow of income from subsequent periods given that he has invested in this period. If he decides to stay without investing, he needn’t pay the investment cost and he earns the flow of income from subsequent periods given that he has made no investment this period. If he exits the market, he only receives the exit value. The expected flow of income $v^C(w_{it}, N_t, S_t, i_{it})$ from subsequent periods is defined by the next expression.

(2.31)

$$v^C(w_{it}, N_t, S_t, i_{it}) \equiv \beta \sum_{w', N', S'} V(w', N', S') F(w' | w_{it}, i_{it}) f(S' | S_t)\, Q(N' | N_t, S_t)$$

We notice how we have aggregated over all relevant state variables. In equation (2.30) we integrate over the two private information variables. In equation (2.31) we sum over the occurrences of quality, number of active firms and market size. We introduce as well $Q(N' | N_t, S_t)$ which is the expected transition probability of the number of incumbents. This transition tells us how many incumbents we can expect the next period, given that the number of incumbent and market size this period are respectively N_t and S_t. β is our discount factor. We have presented and studied the value function of an incumbent firm and we are now interested in the value function of an entering firm. A potential entrant enters into the market as long as the value of entering is higher than the value of staying out. This is found to be the case when $v^E(N_t, S_t) - \theta_{EC} - \varepsilon_{E,it} > 0$. We define $v^E(N_t, S_t)$ as being the expected future flow of income for the entrant. It is described by the following expression.

(2.32)

$$v^E(N_t, S_t) \equiv \beta \sum_{w', N', S'} V(w', N', S') F_0(w') f(S' | S_t)\, Q(N' | N_t, S_t)$$

$F_0(w')$ refers to the first draw of quality. The entering firm earns, as we recall, nothing in the period at hand and thus the entrant’s income is discounted directly.

We are now ready to compute the choice probability functions. In other words we wish to find the probabilities that the firms choose either one of the strategies that we have presented. We will start by computing the probability that an incumbent firm i stays in the market without investing. Two conditions are required for this event to occur. First of all, the value of the firm in that case must be higher or equal to the value it achieves from exiting the market. Secondly, the value of the firm in that case must be higher or equal to the value it obtains from staying in the market and investing. Finding the probability of this event occurring, boils down to finding the probability that these two conditions are satisfied at the same time. As the shocks in cost ε_I and ε_X, have a standard normal distribution, we will be able to use the joint normal distribution to this end. Before we compute the probability we must express the two conditions.

$$(2.33)\quad v^C(w_{it}, N_t, S_t, i_{it} = 0) \geq EV_{it}$$
$$\Leftrightarrow v^C(w_{it}, N_t, S_t, i_{it} = 0) - \theta_{EV,0} - \theta_{EV,1}w_{it} \geq \varepsilon_{X,it}$$
$$\Leftrightarrow X \geq \varepsilon_{X,it}$$

$$(2.34)\quad v^C(w_{it}, N_t, S_t, i_{it} = 0) \geq v^C(w_{it}, N_t, S_t, i_{it} = 1) - IC_{it}$$
$$\Leftrightarrow v^C(w_{it}, N_t, S_t, i_{it} = 0) - v^C(w_{it}, N_t, S_t, i_{it} = 1) + \theta_{IC} + \sigma_\gamma \gamma_i \geq \varepsilon_{I,it}$$
$$\Leftrightarrow Y \geq \varepsilon_{I,it}$$

The inequality (2.33) expresses the case where staying in the market without investing is preferred to exiting the market. The inequality (2.34) expresses the case where staying in the market without investing is preferred to staying in the market and investing. With these two points found for $\varepsilon_{X,it}$ and $\varepsilon_{I,it}$, the joint standard normal distribution is now applied. The cumulative distribution function, at these values, and for correlation value ρ, produces the probability that an incumbent stays in the market without investing.

$$(2.35)\quad P_{NI} = \Phi\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_I}, \rho\right)$$

We repeat the same procedure for the probability that an incumbent firm i stays and invests. We find again two conditions for this to happen. The value of the firm in that case must be higher or equal to the value from exiting. At the same time, the value of the firm in that case must be higher or equal to the value it yields from staying without investing. We express these events in (2.36) and (2.37) respectively.

$$(2.36)\ v^C(w_{it}, N_t, S_t, i_{it} = 1) - IC_{it} \geq EV_{it}$$

$$\Leftrightarrow v^C(w_{it}, N_t, S_t, i_{it} = 1) - \theta_{IC} - \sigma_\gamma \gamma_i - \theta_{EV,0} - \theta_{EV,1} w_{it} \geq \varepsilon_{X,it} - \varepsilon_{I,it}$$

$$\Leftrightarrow Z \geq \varepsilon_{X,it} - \varepsilon_{I,it}$$

$$(2.37)\ v^C(w_{it}, N_t, S_t, i_{it} = 1) - IC_{it} \geq v^C(w_{it}, N_t, S_t, i_{it} = 0)$$

$$\Leftrightarrow v^C(w_{it}, N_t, S_t, i_{it} = 1) - v^C(w_{it}, N_t, S_t, i_{it} = 0) - \theta_{IC} - \sigma_\gamma \gamma_i \geq -\varepsilon_{I,it}$$

$$\Leftrightarrow W \geq -\varepsilon_{I,it}$$

The joint standard normal distribution can again be used. This is because the difference between two standard normally distributed variables is as well normally distributed. The mean of $\varepsilon_{X,it} - \varepsilon_{I,it}$ is found to be zero and the variance is computed below.

$$(2.38)\ Var(\varepsilon_{X,it} - \varepsilon_{I,it}) = Var(\varepsilon_{X,it}) + Var(\varepsilon_{I,it}) - 2Cov(\varepsilon_{X,it}, \varepsilon_{I,it})$$

$$\Leftrightarrow \sigma_{X-I} = \sqrt{\sigma_X^2 + \sigma_I^2 - 2\sigma_{X,I}}$$

We repeat the reasoning and find the probability that an incumbent stays in the market and invests.

$$(2.39)\ P_I = \Phi\left(\frac{Z}{\sigma_{X-I}}, \frac{W}{\sigma_I}, \rho\right)$$

With the two probabilities P_{NI} and P_I at hand, we can easily find the probability P_X that an incumbent firm i decides to exit. The reason is that the incumbent has three alternatives. If he doesn't choose the two first, he is bound to choose the last. We infer from this argument the probability that an incumbent exits.

$$(2.40)\ P_X = 1 - P_{NI} - P_I$$

We are interested in one last probability, and that is the probability P_E of a potential entrant entering the market. As we have already seen, the potential entrant enters only if one particular condition is satisfied. The value of entering must be higher than the value of the alternative of staying out.

$$(2.41)\ v^E(N_t, S_t) - \theta_{EC} - \varepsilon_{E,it} > 0$$
$$\Leftrightarrow v^E(N_t, S_t) - \theta_{EC} > \varepsilon_{E,it}$$

It is straight forward to apply the standard normal cumulative distribution function at this point to form the wanted probability.

$$(2.42)\ P_E = \Phi\left(\frac{v^E(N_t,S_t) - \theta_{EC}}{\sigma_E}\right)$$

Our last step is to find the equilibrium transition probability for the number of incumbents. In other words we wish to solve for $Q(N_{t+1}|N_t, S_t)$. The number of incumbents at a period of time is dependent on three elements. First of all, obviously it is dependent on the number of incumbents in the previous period. Furthermore it is dependent on one hand the number of entrants and on the other hand the number of exits in that period. We have taken some time to compute the probability P_X that an incumbent firm i exits, and the probability P_E that a potential entrant enters. This will be useful as we can use them to find first of all the conditional probability $B(e|E_t, P_E)$ that the number of entrants is e given the number of potential entrants and P_E. We will also be able to compute the conditional probability $B(x|N_t, P_X)$ that the number of exits is x given the number of incumbents and P_X. We will assume the probability density function $B(\cdot\,|N, P)$ is from the binomial distribution. In turn the conditional probabilities for the values of e and x will help build the following expression.

(2.43)

$$Q(N_{t+1}|N_t, S_t) = \sum_{e=0}^{E_t} \sum_{x=0}^{N_t} I\{N_{t+1} = N_t + e - x\} B(e|E_t, P_E) B(x|N_t, P_X)$$

Given that we know the value of N_t in period t and together with the expected values for e and x we will compute the expected value for N_{t+1}. We will rapidly go through the explanation of the binomial distribution as discussed in the literature (Rice, 2007) and study the way it is used here. We take the case of the number of entrants e.

(2.44) $B(e|E_t, P_E) = \binom{E_t}{e} {P_E}^e (1 - P_E)^{E_t - e}$

By ${P_E}^e$ we find the probability that a selection of players e choose to enter, while by $(1 - P_E)^{E_t - e}$ we find the probability that the remaining players choose to stay out. Therefore the element ${P_E}^e (1 - P_E)^{E_t - e}$ gives us the probability that a particular selection or combination of potential entrants actually enters. The number of combinations by which we get that e players enters from a pool of E_t potential entrants is given by $\binom{E_t}{e}$. We end up as we can see with $B(e|E_t, P_E)$, that pictures the probability that the number of entrants is e. The binomial distribution fits this setup perfectly as the potential entrant is standing before two alternatives, entering or staying out. The binomial distribution is used as well for determining $B(x|N_t, P_X)$ as the incumbent has the choice between staying and exiting.

Similarly to the analysis in the global player part, the collection of choice probability functions, the value functions and the transition probability for the number of incumbents creates a fixed point mapping $\Lambda(N_{t+1}|N_t, S_t)$. The fixed point mapping determines an equilibrium in the probability function $Q(N_{t+1}|N_t, S_t)$. The equilibrium is a Markov Perfect Equilibrium, as all the player's decisions are based only on pay-off relevant information. We define the equilibrium as Q^*. The solution of the model has been described, and we use the rest of this section to present several examples of industries that fit the local player setup.

Our first example is the restaurant business. Unless we study the case of a restaurant chain, a restaurant is located in one single local market. In large cities the competition is often very hard between restaurants and the market suffers many bankruptcies and exits. But simultaneously we often observe that there are many potential entrants, ready to take over some share of the same market. The turnover in those cases is high. When a restaurant is opened, many decisions must be made. One of the first choices is what market segment the restaurant wishes to serve. The choice of market segment imposes the restaurant to invest accordingly in quality. It needs to invest in the food, the staff and the interior among other things. With the market segment in place the restaurant is able to make the choice of the price level. It is reasonable to assume that the restaurant considers the market size and the number of incumbents to be state variables. Similarly the given managerial abilities determine the differences in cost structure between restaurants. Finally we can also assume that quality is a state of nature affected by investment. The reason is that quality is decided by the public. The consumer’s taste determines whether a restaurant is perceived as good or bad. As we can see, the interpretation of the restaurant sector fits nicely to the local player model presented earlier. Other examples of businesses that can be used in this framework are plumbers, electricians and local doctors.

3. Implementation

In the preceding sections, several models have been presented. In the first part, a static model was brought forward. In the second part, dynamic models were presented, as we introduced the event of entry of both global players and local players. In order to see how these models can be used, this part will focus on some applications, and present numerical methods which can be used to solve the models. We will concentrate on the global player model, although we display as well an application for the static game presented earlier. The application presented here for the static game could be extended as well to the local player framework. We will finally attempt towards the end to simulate the application of the global player model. This simulation will be inspired by the Norwegian airline industry.

3.1 An outside alternative

The discussion in this section will be built upon the analysis in Aguirregabiria, Mira and Roman (2005). The first setup will add a new assumption in the static game presented in the first part. In addition to be able to choose between the products of different firms, the consumer will now have a new option. This option is that of the outside alternative. The outside alternative can, for example, be interpreted as the choice of not purchasing any product at all. Our consumer will now hence be able to choose not to purchase anything. We assume that the consumer has a reservation utility, in other words that the consumer earns something even when she doesn't purchase anything. We introduce the consumer's indirect utility of purchasing from firm i.

$$(3.1) \quad U_i = w_i - p_i - v_i$$

Again, w_i is the quality of retailer i, p_i his price, and v_i is what is called the consumer's idiosyncratic taste. This variable is the consumer's preference for firm i, independent of the firm's price and quality. It describes the degree of horizontal product differentiation in the market. We assume that the v_i's are independently and identically distributed, and are extreme value distributed. The outside alternative is defined for i=0, and the indirect utility of this option is $U_0 = w_0 - v_0$.

We are now interested as in the first part, to compute the probability of a consumer purchasing the product of retailer i. A consumer will purchase from firm i, if the utility he derives from firm i is at least equal to the utility he derives from all the other options. The other options are either purchase from any other competitor, or not purchase anything. Hence we find that the probability of a consumer purchasing the product of retailer i is the probability that $w_i - p_i - v_i \geq w_j - p_j - v_j$ for all $j \in [0, n]$, where n is the number of retailers in the market. With this starting point, it is possible to find an expression for the probability.

$$(3.2) \quad P_i = \frac{e^{\frac{w_i - p_i}{\mu}}}{e^{\frac{w_0}{\mu}} + \sum_{j=1}^{n} e^{\frac{w_j - p_j}{\mu}}}$$

$$\Leftrightarrow P_i = \frac{e^{\frac{w_i - w_0 - p_i}{\mu}}}{1 + \sum_{j=1}^{n} e^{\frac{w_j - w_0 - p_j}{\mu}}}$$

Let the mark-up be defined as $m_i \equiv (p_i - c_i)/\mu$ and the cost-adjusted quality be defined as $\alpha_i \equiv (w_i - w_0 - c_i)/\mu$. From the first part, we know that the demand is given by $\widetilde{X}_i = SP_i$. With this information in place, an expression for the demand is computed easily.

$$(3.3) \quad \widetilde{X}_i = S \frac{e^{\alpha_i - m_i}}{1 + \sum_{j=1}^{n} e^{\alpha_j - m_j}} \qquad i = 1 \dots n$$

As the setup is static here, we set aside the matter of entry and are thus interested only in cases where the firms are present in the market for the period in question. This said, the profit function has the following form.

$$(3.4)\quad \pi_i = \theta_v m_i S \frac{e^{\alpha_i - m_i}}{1+\sum_{j=1}^{n} e^{\alpha_j - m_j}} - K \qquad\qquad i = 1 \ldots n$$

Each retailer, present in the market, will hence choose a mark-up which will maximize his profit.

A best response function exists for each firm, which expresses the optimal mark-up level for the firm in question, as a function of the other firm's mark-up. The system of best response function defines the Nash-Bertrand equilibrium. In order to find the best response functions, we first need to maximize the profit function. From the analysis in the first part, we know that the solution to the first order condition will yield a maximum. Hence we directly find the first order condition.

$$(3.5)\quad \frac{\partial \pi_i}{\partial m_i} = \theta_v S \left[(1 - m_i) \frac{e^{\alpha_i - m_i}}{1+\sum_{j=1}^{n} e^{\alpha_j - m_j}} + m_i \left(\frac{e^{\alpha_i - m_i}}{1+\sum_{j=1}^{n} e^{\alpha_j - m_j}} \right)^2 \right] = 0$$

$$\Leftrightarrow (1 - m_i) \frac{e^{\alpha_i - m_i}}{1+\sum_{j=1}^{n} e^{\alpha_j - m_j}} = m_i \left(\frac{e^{\alpha_i - m_i}}{1+\sum_{j=1}^{n} e^{\alpha_j - m_j}} \right)^2$$

$$\Leftrightarrow \frac{1 - m_i}{m_i} = -\frac{e^{\alpha_i - m_i}}{1+\sum_{j=1}^{n} e^{\alpha_j - m_j}}$$

$$\Leftrightarrow m_i = \left(1 - \frac{e^{\alpha_i - m_i}}{1+\sum_{j=1}^{n} e^{\alpha_j - m_j}} \right)^{-1}$$

The last equation is the expression for the best response of retailer i. From the system of equations consisting of all the best response functions, we have n equations, with n unknowns. It is thus possible to solve for the equilibrium price-cost margin. When the equilibrium price-cost margin is known, the equilibrium price is revealed as well. We can show how to solve for the mark-up numerically, simply by taking the case of a market with two retailers. By using the software Matlab, it is possible to use what is called the Newton method to find the root of a function. When finding the roots m_i to the functions $\partial\pi_i/\partial m_i$ (for i=1,2), we find at the same time the equilibrium price-cost margin from the best response functions.

3.2 Cournot competition

This second setup takes a closer look at the global player model from the second part. To demonstrate one implementation of the model, we will introduce the assumption that in each local market, firms compete à la Cournot. This is also treated in Aguirregabiria and Mira (2007). With Cournot competition firms compete in quantity rather than price. Hence for each market where more than one player is present, the firms will compete as such. As we will see in a moment this will give a particular form to the profit function. We will start by rapidly going through the Cournot analysis as discussed in the litterature (Tirole, 1982, p 218-221). We assume that there are N firms active in the market in question, they are indexed by i. They each choose a quantity q_{it} to produce of the good, which then determines the total quantity Q_t, of the good available in that market at period t. This supply, together with the market size, determines the actual price of the good. This is described by the following expression.

$$(3.6) \quad P_t = \alpha_0 - \frac{\alpha_1}{S_t} Q_t$$

We notice that the higher the supply, the lower we can expect the price to be. Equivalently, the higher the market size S_t, the higher will the price be. This is explained by the fact that demand increases with market size. We now move over to the study of the profit function. We define the marginal cost as c, which we assume is identical for all firms. We state firm i's variable profit, which we know from standard economical theory can be written $\Pi_i(q_{it}, q_{-it}) = (P_t - c)q_{it}$. It is clear that firm i makes his decision of which quantity to choose based on the maximization of this expression. To proceed we need to perform this task. We compute the first order condition.

(3.7) $\frac{\partial \Pi_i(q_{it},q_{-it})}{\partial q_{it}} = \alpha_0 - \frac{\alpha_1}{S_t} Q_t - \frac{\alpha_1}{S_t} q_{it} - c = 0$

We see directly here that the second order condition becomes $\frac{\partial^2 \Pi_i(q_{it},q_{-it})}{\partial q_{it}^2} = -2\frac{\alpha_1}{S_t}$, which is negative. Thus the solution for q_{it} is a maximum. For the case of a symmetric Cournot solution, all firms will produce the same quantity. We will have $q_{it} = q_{jt} = q_t$, and $Q_t = Nq_t$. We rewrite (3.7).

(3.8) $\alpha_0 - \frac{\alpha_1}{S_t} N q_t - \frac{\alpha_1}{S_t} q_t - c = 0$

$$\Leftrightarrow q_t^* = \frac{S_t}{\alpha_1} \frac{\alpha_0 - c}{1 + N}$$

We can use this solution to solve for the price and the profit function. Inserting for q_t^* in (3.6), we get the solution for the price.

(3.9) $P_t = \alpha_0 - \frac{\alpha_0 - c}{1+N} N$

$$\Leftrightarrow P_t = \frac{\alpha_0 + cN}{1 + N}$$

$$\Leftrightarrow P_t^* = c + \frac{\alpha_0 - c}{1 + N}$$

At this point we concentrate on the profit. As we recall the variable profit was given by $\Pi_i(q_{it}, q_{-it}) = (P_t - c)q_t$. We notice that both P_t and q_t have been found and can therefore be inserted directly to this expression.

(3.10) $\Pi_i = \frac{\alpha_0 - c}{1+N} \frac{S_t}{\alpha_1} \frac{\alpha_0 - c}{1+N}$

$$\Leftrightarrow \Pi_i = \frac{S_t \theta_R}{(1 + N)^2}$$

We define $\theta_R = \frac{(\alpha_0 - c)^2}{\alpha_1}$, which corresponds to θ_v in the static model. It represents the marginal transportation cost or in other words the degree of horizontal product differentiation. The Cournot analysis has been completed, and the variable profit found.

How can we implement this way of thinking into the model for global player? The profit function $\Pi_i(a_i, a_{-i}, S)$ from part two reveals that it is dependent on two things. Just as with the Cournot analysis, the profit is first of all dependent on the market size. But the profit is as well dependent on the choice of the firm’s a. At first glance this might appear different from the Cournot profit, but in fact it isn’t. For the market in question, the decision of being present or not that each firm face will actually give us the precise number of active firms when we aggregate this decision over all firms. This is because a_i takes only two values as we have already seen; either zero if firm i decides to be inactive, or one if it decides to be active in that particular market. From the discussion above, we can therefore state that $N = \sum_{j=0}^{n} a_{jt}$. Hence we note that there is a large difference between the number of potential firms (n), and the number of active firms (N). Assuming firm i is active, we can rewrite our variable profit as below.

$$(3.11)\ \Pi_{it}(a_{it}) = \frac{S_t\theta_R}{\left(1 + a_{it} + \sum_{j\neq i} a_{jt}\right)^2}$$

$$\Leftrightarrow \Pi_{it}(1) = \frac{S_t\theta_R}{\left(2 + \sum_{j\neq i} a_{jt}\right)^2}$$

For the case where firm i is inactive, the profit is assumed to be zero, thus we find $\Pi_{it}(0) = 0$. We now have our variable profit, and for the profit function to be complete, we need to incorporate the cost structure and the private information variable. This is our next step.

$$(3.12)\ \Pi_{it}(1) = \frac{S_t\theta_R}{\left(2 + \sum_{j\neq i} a_{jt}\right)^2} - \theta_{FC,i} + \varepsilon_{it}(1) - \left(1 - a_{i,t-1}\right)\theta_{EC}$$

$\theta_{FC,i}$ is firm i's fixed cost, $\varepsilon_{it}(1)$ is the private information variable introduced in part two, and $(1 - a_{i,t-1})\theta_{EC}$ is the entry cost for the case where firm i wasn't active in this market in the previous period. As we can see this element disappears if $a_{i,t-1}$ equals to one, hence if firm i actually was active in the previous period.

The method presented to solve the global player model in the second part, was a general method which can be replicated to solve more specific problems. This is exactly what we will attempt here using our newly found profit function from a Cournot analysis. Before we start, we make one new assumption. The private information variables are normally distributed with zero means. As we will see in a moment, our interest will be on the difference between these two private information variables. We place as well the following definition: $\sigma^2 \equiv var\left(\varepsilon_{it}(0) - \varepsilon_{it}(1)\right)$. We will now see why this difference is useful. A firm is active in a particular market if and only if, the value he gets from actually being active is higher then the value he gets from staying out. It's the decision which maximizes the expression (2.18) that is chosen. With this in mind we express our next condition.

(3.13)

$$\pi_i(0,S) + \varepsilon_{it}(0) + \beta \sum_{S'} \Gamma_i(S';P)\ f_i(S'|S,0) \leq \pi_i(1,S) + \varepsilon_{it}(1) + \beta \sum_{S'} \Gamma_i(S';P)\ f_i(S'|S,1)$$

$$\Leftrightarrow \varepsilon_{it}(0) - \varepsilon_{it}(1) \leq \pi_i(1,S) - \pi_i(0,S) + \beta \sum_{S'} \Gamma_i(S';P)\left[f_i(S'|S,1) - f_i(S'|S,0)\right] \equiv z$$

A firm is thus active if and only if (3.13) holds. We know from our initial assumption that $\{\varepsilon_{it}(0) - \varepsilon_{it}(1)\} \sim N(0,\sigma^2)$. Using the cumulative distribution function for the normal distribution we are able to find the probability that a firm actually will be active.

(3.14) $F(z) = \phi\left(\frac{z-\mu}{\sigma}\right)$

$$\Leftrightarrow \Psi_i(1|S;P) = \phi\left(\frac{1}{\sigma}\left[\pi_i(1,S) - \pi_i(0,S) + \beta\sum_{S'}\Gamma_i(S';P)\left[f_i(S'|S,1) - f_i(S'|S,0)\right]\right]\right)$$

In order to work with the model we need to introduce a few more building blocks. The next element we wish to present will be particularly central when we move over to the simulations using Matlab. This expression will give us what we can call a total transition probability. Given the market size from the previous period, the actions chosen by each firm in the previous period, and given that firm i plays a_i, this function will yield the probability that a particular combination of actions and market size occurs.

(3.15)

$$f_i^P(S', a_t|S, a_{t-1}, a_i) = f_S(S'|S)\prod_{j\neq i} P_j(0|S)^{1-a_{jt}} P_j(1|S)^{a_{jt}} I\{a_{it} = a_i\}$$

In this expression, we assume that the different actors in the market are independent, and that they do not collude. The probability that two independent events occur is defined as the product of the probability of each event: $P(A \cap B) = P(A) \cdot P(B)$. Hence we recognize from (3.15) the probability that the competitors of firm i each choose one particular action.

(3.16) $P(a_{-i}|S_t) = \prod_{j\neq i} P_j(0|S_t)^{1-a_{jt}} P_j(1|S_t)^{a_{jt}}$

Our last building block is the current expected profit function. This is, as we see below, constructed with our profit function from (3.12), and corresponds to the model from part two.

(3.17) $\pi_i(1, S_t) = \sum_{a_{-i}} \frac{S_t \theta_R P(a_{-i}|S_t)}{\left(2+\sum_{j\neq i} a_{jt}\right)^2} - \theta_{FC,i} - \left(1 - a_{i,t-1}\right)\theta_{EC}$

The building blocks are in place, and we will now work to simplify the notation using vector notation. From that point, a solution will be characterized. We define to ease the notation: $N_i(S_t) \equiv \sum_{a_{-i}} \frac{P(a_{-i}|S_t)}{\left(2+\sum_{j\neq i} a_{jt}\right)^2}$. We now introduce several vectors.

$$(3.18)\ \theta_\pi = \left(\theta_R, \theta_{FC,1}, \dots, \theta_{FC,n}, \theta_{EC}\right)'$$

$$(3.19)\ z_i(1, S_t) = \left(S_t N_i(S_t), -D_i, a_{i,t-1} - 1\right)$$

$$(3.20)\ z_i(0, S_t) = (0, \dots, 0)$$

We note that D_i is a $1 \times n$ row vector with values zero, except for column i, which has value one. With these tools we are now able to rewrite $\pi_i(1, S_t)$ and $\pi_i(0, S_t)$ in vector form.

$$(3.21)\ \pi_i(1, S_t) = \theta_\pi z_i(1, S_t)$$

$$(3.22)\ \pi_i(0, S_t) = \theta_\pi z_i(0, S_t)$$

Let's recall the expression (2.16) we had for the Bellman equation in vector notation from part two: $V_i^{P^*} = \sum_{a_i} P_i^*(a_i) * \left[\pi_i^{P^*}(a_i) + e_i^{P^*}(a_i)\right] + \beta F^{P^*} V_i^{P^*}$. Continuing our analysis we observe that we can make some modifications.

$$(3.23)\ Z_i^P = \begin{bmatrix} P_1(0|S_t) z_1(0, S_t) + P_1(1|S_t) z_1(1, S_t) \\ \vdots \\ P_n(0|S_t) z_n(0, S_t) + P_n(1|S_t) z_n(1, S_t) \end{bmatrix}$$

$$(3.24)\ \lambda_i^P \sigma = \begin{bmatrix} P_1(0|S_t) e_1^P(0, S_t) + P_1(1|S_t) e_1^P(1, S_t) \\ \vdots \\ P_n(0|S_t) e_n^P(0, S_t) + P_n(1|S_t) e_n^P(1, S_t) \end{bmatrix}$$

$$(3.25)\ \lambda_i^P = \begin{bmatrix} \phi\left(\Phi^{-1}\left(P_1(1|S_t)\right)\right) \\ \vdots \\ \phi\left(\Phi^{-1}\left(P_n(1|S_t)\right)\right) \end{bmatrix}$$

The last expression (3.25) will be proven in the appendix. From what is stated above, we can perform the following rewriting.

$$(3.26)\ \sum_{a_i} P_i^*(a_i) * \left[\pi_i^{P^*}(a_i) + e_i^{P^*}(a_i)\right] = Z_i^P \theta_\pi + \lambda_i^P \sigma$$

We are now ready to express the solution system to the value function of the Bellman equation. This is again based on expression (2.16), and the method presented in part two.

$$(3.27)\ \Gamma_i(S,P) \equiv (I - \beta F^P)^{-1}(Z_i^P \theta_\pi + \lambda_i^P \sigma)$$

$$\Leftrightarrow \Gamma_i(S,P) = \Gamma_i^Z(S,P)\theta_\pi + \Gamma_i^\lambda(S,P)\sigma$$

We deduce from (3.27) that $\Gamma_i^Z(P)$ and $\Gamma_i^\lambda(P)$ are defined by the following expressions.

$$(3.28)\ \Gamma_i^Z(S,P) = (I - \beta F^P)^{-1} Z_i^P$$

$$(3.29)\ \Gamma_i^\lambda(S,P) = (I - \beta F^P)^{-1} \lambda_i^P$$

Proceeding with the method from the global player model, we remember that there is one last step in characterizing the solution of the model. That is computing the best response probability functions or in other words the Markov reaction functions. From our previous analysis, we have in (3.14) an expression for this best response probability. We take a moment to study it.

(3.30)

$$\Psi_i(1|S;P) = \phi\left(\frac{1}{\sigma}\left[\pi_i(1,S) - \pi_i(0,S) + \beta\sum_{S'}\Gamma_i(S';P)\left[f_i(S'|S,1) - f_i(S'|S,0)\right]\right]\right)$$

As $\Gamma_i(S',P) = \Gamma_i^Z(S',P)\theta_\pi + \Gamma_i^\lambda(S',P)\sigma$, the next step is inferred.

$$\Leftrightarrow \Psi_i(1|S;P) = \phi\left(\frac{1}{\sigma}\left[\theta_\pi z_i(1,S) - \theta_\pi z_i(0,S) + \beta\sum_{S'}\left(\Gamma_i^Z(S',P)\theta_\pi + \Gamma_i^\lambda(S',P)\sigma\right)\left(f_i(S'|S,1) - f_i(S'|S,0)\right)\right]\right)$$

$$\Leftrightarrow \Psi_i(1|S;P) = \phi\left(\frac{\theta_\pi}{\sigma}\left[z_i(1,S) - z_i(0,S) + \beta\sum_{S'}\Gamma_i^Z(S',P)\left(f_i(S'|S,1) - f_i(S'|S,0)\right)\right] + \frac{1}{\sigma}\left[\beta\sum_{S'}\Gamma_i^\lambda(S',P)\sigma\left(f_i(S'|S,1) - f_i(S'|S,0)\right)\right]\right)$$

$$\Leftrightarrow \Psi_i(1|S;P) = \phi\left(\frac{\theta_\pi}{\sigma}\tilde{z}_i(S) + \tilde{\lambda}_i(S)\right)$$

Here our best response probability function has taken shape, and we see clearly from the last step the definition of $\tilde{z}_i(S)$ and $\tilde{\lambda}_i(S)$.

(3.31) $\tilde{z}_i(S) \equiv z_i(1,S) - z_i(0,S) + \beta\sum_{S'}\Gamma_i^Z(S',P)\left(f_i(S'|S,1) - f_i(S'|S,0)\right)$

(3.32) $\tilde{\lambda}_i(S) \equiv \beta\sum_{S'}\Gamma_i^\lambda(S',P)\left(f_i(S'|S,1) - f_i(S'|S,0)\right)$

Our application for the global player is complete. We have shown that it is possible to implement this model, so as to fit a set up with firms competing in each local market in a Cournot manner. This was done to give an example of a possible use for this model, and to demonstrate its scope. In the next section, our main interest will be the numerical way of solving this model. As we have already mentioned, the software program Matlab can help us with this task.

3.3 A numerical analysis

This section will consist of two elements. First of all, we will go through the procedure used to solve the global player model in Matlab. We will explain how the procedure brings us a solution. Then we will present some questions that are related to the theory, and use our program to attempt to answer them. The algorithm to the global player model can be found in the appendix.

Matlab enables us to create different loops. They make it possible to picture all the different events that are feasible, and permit us to work with them. By events we mean combinations of state variables and actions taken by the different players at different points in time. From an initial position, we are able with the program to determine and give form to unknown expressions. This will be the case for the Bellman value function as we will see in a moment. We will take some time and present the different elements needed to run the program. These are taken directly from the model. We will have different transition probabilities, the profit functions and the Bellman equation. We start by putting in place the different relevant transition probabilities in our program. Given $P(a_t|S_t)$, the probability of each firm choosing a particular action given a particular state of nature, we build $P(a_t|S_t, a_{t-1})$ and $P(S_{t+1}, a_t|S_t, a_{t-1}, a_i)$. The first expression yields the probability of a particular combination of actions occurring, conditioned on the fact that we are in a specific state and that a given combination of actions occurred in the previous period. The last expression is the expression we presented in (3.15). This expression yields the transition probability from one state and one set of actions given that the choice of firm i is a_i, to the next period's state and set of actions. In other words, with these probabilities in place we manage to map the set of outcomes and the transition between them from one time period to another. Furthermore we build both the profit function $\pi_i(a_i, S_t)$ and the value function $v_i(a_i, S_t)$ for each firm. The profit function will be a log-linear version of the current expected profit computed in the previous part. Hence it will be based on the expression (3.17). The value function is nothing else than the intertemporal income stream of a firm. It will have the following form.

(3.33)

$$v(a_{t-1}, S_t) = \max_a\{\pi(a_{t-1}, S_t) + \beta v(a_t, S_{t+1})P(S_{t+1}, a_t|S_t, a_{t-1}, a_i)\}$$

With the proper code, the program will take this value function and start to iterate it from an initial point. For each iteration, or time period, the profit and the transition probability will be updated. Each single iteration will, thus bring a suggestion for the solution to the value function. The iterations will stop once the solutions from one time period to the next become similar enough. The convergence criterion will decide what is meant by similar enough. The output produced is our solution for the model. We will try to explain why this is the case. We have seen in our analysis of the second part that it is possible at equilibrium to isolate the value function. The reason was that the equilibrium is a fixed point. The time aspect disappears and so does the variation in the value function. The value function become identical on each side of the equality and with some algebra we can solve for it. This was shown thoroughly in part two. What is done in Matlab is actually to work recursively. Since we know that at equilibrium the value functions are identical, we only need to get to the point where the value functions are close enough to know that we are approximately at equilibrium. We have presented the method and will now try to make use of it to answer several questions.

We will consider the airline industry in Norway, concentrating on domestic flights. We have mentioned earlier that there are two competitors, SAS and Norwegian. SAS is a large airline focusing on quality and diversity of destination, while Norwegian is a relatively new, low-price airline focusing on cheap flights on the largest markets. We focus on the routes that are considered as the main routes by TØI, the Norwegian Institute of Transport Economics. These are presented in Table 3.1 below. We retain hence these nineteen markets for our simulation. We see from this table that the market share is higher for SAS in all the city-pairs in question. To be able to express this difference in market shares in our simulation, we make the assumption that Norwegian has a higher fixed cost than SAS.

Route	Number of frequencies per week			Passengers	Seats	Norwegian's share of the:	
	Norwegian	SAS	Total	(in 1000)	(in 1000)	Frequencies	Traffic
Oslo-Bergen*	68	104	172	1632	2276	41	44
Oslo-Trondheim*	68	97	165	1624	2190	41	44
Oslo-Stavanger*	48	87	135	1241	1868	36	38
Bergen-Stavanger*	18	67	85	556	1097	27	37
Oslo-Tromsø*	19	52	71	775	1110	30	36
Oslo-Bodø*	18	44	62	635	867	41	35
Bergen-Trondheim*	10	44	54	380	632	40	43
Oslo-Ålesund	0	49	49	469	649	0	0
Oslo-Kristiansand	0	48	48	406	632	0	0
Oslo-Harstad/Narvik*	13	28	41	438	548	41	38
Bergen-Sandefjord	0	39	39	125	205	0	0
Kristiansand-Bergen	0	38	38	128	174	0	0
Tromsø-Bodø	0	36	36	168	392	0	0
Oslo-Haugesund	0	35	35	399	471	0	0
Oslo-Molde	0	32	32	268	408	0	0
Oslo-Kristiansund	0	30	30	158	259	0	0
Oslo-Bardufoss*	0	19	19	173	270	0	0
Oslo-Alta*	7	12	19	213	295	38	41
Oslo-Kirkeness	0	13	13	135	259	0	0

TØI report 974/2008
* Routes with competition in 2007

Table 3.1: Frequency and traffic on main routes by airline in 2007

We are now ready to study several issues. Our aim is to use our framework defined by the global player model and the program, to attempt answering those issues within the context of the Norwegian airline industry. Our first question is in what way product differentiation affects the choice of entry and the degree of turnover. We also wish to find out how the size of the entry cost affects the degree of entry and turnover. Our last goal is to pinpoint the different effects from the arrival of a new entrant. For the measure of entry cost we use the parameter θ_{EC}. The measure of product differentiation will be a version of the unit transportation cost θ_R, from the profit function in the Cournot analysis. When moving over to the log-linear form of the profit function, the parameters are rearranged, and θ_R is replaced by a parameter that we will call θ_{RN}. We will not proceed with the computations of the log linear profit as it does not enhance our understanding. θ_{RN} is similar to θ_R in that it represents the degree of horizontal product differentiation. But θ_{RN} differs from θ_R, in that higher θ_{RN} is translated by a lower degree of horizontal product differentiation, while by higher θ_R we mean more product differentiation. We present the results from the simulations in Table 3.2 and 3.3.

Descriptive Statistics	Exp.1 $\theta_{EC}=1$ $\theta_{RN}=0$	Exp.2 $\theta_{EC}=1$ $\theta_{RN}=1$	Exp.3 $\theta_{EC}=1$ $\theta_{RN}=2$	Exp.4 $\theta_{EC}=0$ $\theta_{RN}=1$	Exp.5 $\theta_{EC}=2$ $\theta_{RN}=1$	Exp.6 $\theta_{EC}=4$ $\theta_{RN}=1$
Number of firms						
Mean	1,4211	1,4737	1,0526	1,3684	1,3684	1,0000
Std. Dev.	0,8377	0,6967	0,7799	0,8307	0,7609	0,8165
Number of entrants						
Mean	0,2105	0,2632	0,4211	0,2105	0,2105	0,1053
Number of exits						
Mean	0,3158	0,1579	0,3684	0,4211	0,1053	0,1053
Excess Turnover	0,4210	0,3158	0,7368	0,4210	0,2106	0,2106
Correlation between entries and exits	0,2745	-0,2588	-0,2094	-0,1170	-0,1771	-0,1176
Probability of being active						
SAS	0,6842	0,7895	0,5789	0,7368	0,7895	0,6316
Norwegian	0,7368	0,6842	0,4737	0,6316	0,5789	0,3684

Table 3.2: Results for simulation with volatile entry cost and product differentiation

Descriptive Statistics	Exp.1 $\theta_{EC}=1$ $\theta_{RN}=1$	Exp.2 $\theta_{EC}=1$ $\theta_{RN}=1$	Exp.3 $\theta_{EC}=0$ $\theta_{RN}=1$	Exp.4 $\theta_{EC}=0$ $\theta_{RN}=1$	Exp.5 $\theta_{EC}=1$ $\theta_{RN}=0$	Exp.6 $\theta_{EC}=1$ $\theta_{RN}=0$
Number of players	2	3	2	3	2	3
Number of firms						
Mean	1,4737	1,8947	1,3684	1,9474	1,4211	2,0000
Std. Dev.	0,6967	0,9941	0,8307	0,9113	0,8377	1,1055
Number of entrants						
Mean	0,2632	0,3158	0,2105	0,6316	0,2105	0,0526
Number of exits						
Mean	0,1579	0,4211	0,4211	0,5263	0,3158	0,3684
Excess Turnover	0,3158	0,6316	0,4210	1,0526	0,4210	0,1052
Correlation between entries and exits	- 0,2588	0,0908	-0,1170	-0,2700	-0,2745	-0,1494
Probability of being active						
SAS	0,7895	0,6842	0,7368	0,8421	0,6842	0,7895
Norwegian	0,6842	0,6842	0,6316	0,4737	0,7368	0,5263
Entrant	N/A	0,5263	N/A	0,6316	N/A	0,6842

Table 3.3: Results for simulation with the arrival of a new entrant

Each experiment consists of 1,000 Monte Carlo simulations, and the turnover statistic is computed as follows. Turnover = Entry+Exit-abs(Entry-Exit). For the entry and exit values in this computation, we use the mean values from the simulation.

We will start interpreting the results from Table 3.2, which are linked to the two first issues raised. The first issue is, as we know, the effect on entry and turnover of altered product differentiation. Experiments 1, 2 and 3 concentrate on the effect of product differentiation, by holding the entry cost constant and steadily increasing θ_{RN}. In other words we see what happens when the degree of product differentiation decreases. We can observe several effects. Less product differentiation seems to lead to less active firms. This can be interpreted as follows. When a firm has less room to differentiate itself from other firms, it must resort to fiercer competition in order to attract demand. Fiercer competition leaves the firms with tighter profit, and thus they are only able to stay active in fewer outlets. We notice as well that the number of entrants and the number of exits increase as the degree of product differentiation decreases. As a result, we observe as well a higher level of turnover in the industry. This is also caused by more aggressive competition. As the level of product differentiation decreases, we know that the firms will be more similar as the quality converges, and that prices will be pushed down. As some inefficient outlets are dropped by the different firms, new ones may be found and entered into. With fiercer competition it is necessarily harder to deter entry in local markets with price cuts.

We move over to our interpretation of experiments 4, 5 and 6. These experiments illustrate the effect of variation in the entry cost. This time we keep the level of product differentiation constant. The first remark we can make is that higher entry cost seems to limit the number of active firms. The reason is that, an increase in entry cost gives a higher total fixed cost. When this is the case we can assume that fewer firms are efficient enough to be active. We observe that higher entry cost reduces entry in the different outlets. This relationship is straight forward. In the same way a higher price restrains consumption, a higher entry cost restrains entry. The level of exit is also reduced because of higher entry cost. The explanation is that, as it becomes more expensive to enter new local markets, the players limit expenditure by holding on to the outlets they already serve. With both fewer entrants and fewer exits, we necessarily find a lower degree of turnover. We finally notice that we have succeeded in picturing SAS as a larger player. By this we refer to the fact that Scandinavian airline's probability of being active is systematically higher for nearly all studied cases.

Let's study what happens when a new player enters the game. This is depicted in Table 3.3. For the simulation, we have chosen an entrant with similar cost structure than Norwegian. We have chosen approximately the same level of fixed cost. In other words, we can expect the new entrant to be in more direct competition to Norwegian than with SAS. This seems to be confirmed when we compare Norwegian's probability of being active before and after the new player's entry in the game. Norwegian seems to stand less of a chance of remaining active in the different outlets as the entrant appears. SAS on the other hand gets a higher probability of being active in two out of the three cases. With the arrival of a new player, the number of entrants for the different outlets rises as well in two out of the three cases, and the number of exits rises in all cases. This is attributed to the higher level of competition reached from more competitors. In the two cases where both entry and exit numbers are rising the turnover rate is rising as well. We realize that a few of the numbers are in conflict with the interpretations we have presented, and point to the fact that, for simplifying reasons we have chosen only nineteen outlets. Taking into account more city pairs in our analysis would enlarge the number of observations, and we can assume that the results would then be more consistent. This concludes our interpretation, and hence our simulation section.

4. Conclusion

We have gone through the study of a static game of product differentiation, as well as dynamic entry models with product differentiation. In our analysis we have distinguished between dynamic models with global and local players. Furthermore we have implemented the static model, as well as the global player model, to get an idea of how these models can be used. Finally we have attempted a simulation of the global player model, inspired by the Norwegian airline industry. The simulation produced some output, which was discussed in-depth. Based on the results we were able to analyze different issues regarding the relationship between entry, turnover, product differentiation and entry costs. We analyzed as well the arrival of a new airline into the market of domestic flights in Norway.

We have shown how this framework can be used to study a particular industry. In the same way we simulated the airline industry, we could have used the same framework to analyze the supermarket industry. But there are some limits to the framework presented here, and the limits indicate how this model can be extended.

One of these limits was the way we pictured the higher market share Scandinavian airlines enjoys in the industry. To underline that fact, we allowed SAS to have a lower level of fixed cost. But the model would be closer to reality if we introduced a measure of market power in our analysis. One way would be to use an average of the frequencies of offered flights on all routes in question. A second suggestion would be to compute market shares based on the airline's share of commuters.

We observe yet another limit, and that is how we model the different outlets. In our analysis and our simulation, all the nineteen local markets are assumed to have identical transition probability for the market sizes. They are also assumed to move between the same different occurrences of market sizes. But this is questionable as we can see from Table 3.1. The city pair Oslo-Bergen had approximately 1.6 million passengers in 2007, while the city pair Tromsø-Bodø had approximately 160 thousand passengers in the same year. We can't expect these two markets to move between the same states at the same probability. It is for example unlikely that the city pair Oslo-Bergen becomes ten times smaller relatively to the other city pairs in the near future. An extension to the model would be to allow the different markets to have their own set of transition probabilities. In that way, we could keep the same amount of state occurrences but alter the probability that one city pair reaches this state compared to another. This will be more realistic but much harder computationally. Only for the nineteen markets, it would mean nineteen different transition probability matrices instead of one. In order to create those matrices correctly, we would as well need to study the industry even more extensively, to discover annual variations in the sizes of the city pairs.

Lastly, it is possible to extent the model in order to include demand shocks. Seasonal effect in demand for air transport is a well known fact, and is poorly pictured here.

We believe the model performs well, despite the limitations we point out. The results produced by the simulations, seem to be in accordance with industrial organization theory. In that respect the framework is valuable and can thus be used to study other industries with global player firms. A similar framework can also be built for the local player case and may provide important insight on entry processes in oligopoly markets with product differentiation.

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Appendix

- **Appendix A**

We will take a moment and show how the expression (3.25) is reached with help from the literature (Aguirregabiria and Mira, 2007, p 12). We start from (3.24). The vector $\lambda_i^P \sigma$ consists of the rows $P_i(0|S_t)e_i^P(0,S_t) + P_i(1|S_t)e_i^P(1,S_t)$.

$$P_i(0|S_t)e_i^P(0,S_t) + P_i(1|S_t)e_i^P(1,S_t)$$
$$= \left\{\frac{1}{\sigma}\left[var\big(\varepsilon_i(0)\big) - cov\big(\varepsilon_i(0),\varepsilon_i(1)\big)\right]\phi\left(\Phi^{-1}\big(P_i(0|S_t)\big)\right) + \frac{1}{\sigma}\left[var\big(\varepsilon_i(1)\big) - cov\big(\varepsilon_i(0),\varepsilon_i(1)\big)\right]\phi\left(\Phi^{-1}\big(P_i(1|S_t)\big)\right)\right\}$$

From the properties of the normal distribution, we can assume from symmetry, that $\phi\left(\Phi^{-1}\big(P_i(0|S_t)\big)\right) = \phi\left(\Phi^{-1}\big(P_i(1|S_t)\big)\right)$.

$$P_i(0|S_t)e_i^P(0,S_t) + P_i(1|S_t)e_i^P(1,S_t)$$
$$= \left\{\frac{1}{\sigma}\left[var\big(\varepsilon_i(0)\big) - cov\big(\varepsilon_i(0),\varepsilon_i(1)\big) + var\big(\varepsilon_i(1)\big) - cov\big(\varepsilon_i(0),\varepsilon_i(1)\big)\right]\phi\left(\Phi^{-1}\big(P_i(1|S_t)\big)\right)\right\}$$

We observe that $var\big(\varepsilon_i(0)\big) - cov\big(\varepsilon_i(0),\varepsilon_i(1)\big) + var\big(\varepsilon_i(1)\big) - cov\big(\varepsilon_i(0),\varepsilon_i(1)\big) = \sigma^2$. Hence we can conclude the following.

$$P_i(0|S_t)e_i^P(0,S_t) + P_i(1|S_t)e_i^P(1,S_t) = \sigma\phi\left(\Phi^{-1}\big(P_i(1|S_t)\big)\right)$$

We can thus see that λ_i^P consists of the rows $\phi\left(\Phi^{-1}\big(P_i(1|S_t)\big)\right)$.

QED.

• Appendix B

Algorithm for global players

- Define the parameters n, θ_R, $\theta_{FC,i}$, θ_{EC}, σ and β
- Define the state space $\{S, \varepsilon,\}$, and build the transition probability

$f_i(S'|S, a_i \epsilon\{0,1\})$

- Initialize $P_i^{\sigma}(a_i|S)$
- Initiate a loop over $\Gamma_i(S, P)$

(2.16)

- Given $P_i^{\sigma}(a_i|S)$, construct the conditional choice probabilities

$P_i(a_t|S_t, a_{t-1})$ and $P_i(a_t|S_t, a_{t-1}, a_i = \{0,1\})$

- Construct the profit function

(The log-linear version of (3.17) is used here)

- Introduce the value function $V_i^{P^*}(S)$ of the n firms

(2.15)

- Update the choice probabilities
- Iterate until $|V_{t+1} - V_t| < \delta$, where δ is the convergence criterion.

- **Appendix C**

In the second part of our thesis we presented the global player as well as the local player model. We went through the numerical analysis of the global player model in the third part, but time didn't permit us to go through the same analysis for the local player model. We will nevertheless examine how we can simulate this model on Matlab by presenting the algorithm.

Algorithm for local players

- Define the parameters θ_v, μ, θ_{EC}, σ_E, $\theta_{FC,0}$, $\theta_{FC,1}$, θ_{IC}, α_γ, $g(\gamma)$, σ_I, $\theta_{EV,0}$, $\theta_{EV,1}$ and σ_X.
- Define the state space $\{S, w, N\}$, and build the transition probabilities $F(w'|w_{it}, i_{it}\epsilon\{0,1\})$ and $f(S'|S_t)$
- Construct the profit structure (2.20), (2.23), (2.24) and (2.27)
- Introduce the initial choice probabilities P_{NI}, P_I, P_X and P_E, respectively (2.35), (2.39), (2.40) and (2.42)
- Initialize $Q(N_0)$ and $V(w_{i0}, N_0, S_0)$
- Initiate a loop over $\Lambda(N_{t+1}|N_t, S_t)$
- Compute $V(w', N', S')$
- Compute the number of incumbents for the period at hand as well as the transition probability $Q(N_{t+1}|N_t, S_t)$
- Construct the value function of the firms $v^C(w_{it}, N_t, S_t, i_{it})$ and $v^E(N_t, S_t)$
- Update the choice probabilities
- Iterate until $|Q_{t+1} - Q_t| < \delta$, where δ is the convergence criterion.