Master thesis for the cand. oecon. degree

Sticky prices and the macroeconomy

A quantitative linear approximation analysis

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PREFACE

I was introduced to quantitative macroeconomics during my year as an exchange student at Humboldt University in Berlin 2002-2003. I had the pleasure to visit the classes *Quantitative Macroeconomics and Numerical Methods I* and *II* with Professor Harald Uhlig. The recent years I have become more and more aware of the importance of basing macroeconomic analysis on microeconomic foundations whenever possible. However, this implies developing highly non-linear dynamic, stochastic models that are hard to analyze and to solve.

Uhlig (1997) has developed methods to solve and analyze such models quantitatively in a manageable fashion based on linear approximation. I have been very impressed of the way Uhlig manages to develop an approach accessible for most economists – and the way he has provided MatLab computer programs to deal with the tedious, but rather straight forward components of the methods. In this way the analysis can kept at a level of time consumption acceptable for most researchers – and innumerable experiments can be carried out with the models.

Many economists still stick to old Keynesian-based models, due to their simplicity and since many of their implications still are appealing. These models can however be replaced by new-Keynesian models, based on sticky prices, that share many of the same features, but that are of the type mentioned above – and that are better equipped to analyze shocks to the economy, since they include an explicit utility function and allow an intertemporal approach through the intertemporal budget constraint. I therefore wanted to analyze such a model and to fit it into Uhlig's framework. Getting quantitative results, makes it possible to compare the model with real data.

The challenges have however been many in converting this idea into a master thesis. I wanted the paper to be complete – in the way Uhlig's methods are described and applied, so that the reader can verify everything that is done - and in the way the uncertainty regarding the methods is dealt with. Together with the wish to provide a thorough analysis of the model, showing the powerful possibilities of the methods, this has made the paper grow somewhat beyond the scope that was planned for the project.

I would like to thank my supervisor, Associate Prof. Harald Goldstein, for useful comments and guidance along the way and my study colleague from Berlin, Joachim Houeland, who kindly offered to read the final draft for the paper and contributed with helpful suggestions.

1. INTRODUCTION

Modern open economy macroeconomics is to a large extent characterized by dynamic models with explicit micro-foundations. The models are often highly non-linear, and hard or even impossible to solve explicitly. Rather than explicit solutions, analyses are often limited to determine signs of derivatives.

In order to find out whether a model corresponds to data, or in order to estimate the quantitative effect of a shock or of different kinds of fiscal and monetary policy, one needs a much wider analysis. It is not satisfactory only to be able to state the direction in which the variables of the model move, and not even know whether changes are significant or not.

This paper has two main objectives:

1. First it attempts to discuss methods to estimate how much macroeconomic variables such as production, consumption or the real interest rate change in the short- and the long run as a result of an exogenous shock. The methods presented here are based on methods in Uhlig (1997).
2. Secondly, an example of its application is shown, to discuss the effect of sticky prices and monopolistic competition in an open-economy framework. This is done by extending a model in Obstfeld and Rogoff (1995a and 1996) and fitting it into Uhlig's framework. It is shown what powerful results that can be found in the field of dynamic shock analysis. The main focus is on monetary shocks, and it is shown how shocks in the nominal money stock can lead to permanent real effects. Solving and simulating the model is done using MatLab™ source code.

In addition this paper provides long-run simulations and compares the results with historical data. Finally, the paper takes up important issues such as the quality of the linear approximations, and the sensitivity of the final model for mistakes in the calibration. Also, attempts are made to provide implications for fiscal and monetary policy of the impulse responses that are drawn.

The author has already, together with Houeland in Houeland and Lien (2003), shown how the baseline model in Obstfeld and Rogoff (1995a) can be analysed quantitatively using the framework in Uhlig (1997) – and thereby contributed somewhat to the second objective mentioned above. This paper builds to some extent on their work in reaching the second objective, but corrects some mistakes in addition to extending and generalizing the model.

Chapter 2 presents the general method of linearizing a dynamic stochastic model, to write it as a system of difference equation and how to solve this system. The chapter is not complete, as potential problems and more details follow in the subsequent chapters as an example of application is shown. Chapters 3 and 4 present an extended version of the open-economy model with sticky prices in Obstfeld and Rogoff (1995a) and show how to apply in practice the methods from chapter 2. Chapter 5 continues this by calibrating the model, and chapter 6 analyses impulse responses of shocks. Chapter 7 simulates first- and second-order moments of the model variables and compares with real data. Chapter 8 discusses possible weaknesses of the calibration, of the model setup and of the linear approximation methods. Chapter 9 concludes.

2. A GENERAL PROCEDURE FOR ANALYZING NON-LINEAR DYNAMIC STOCHASTIC MODELS

Uhlig (1997) explains a stepwise method of analyzing non-linear dynamic stochastic models that can be used for analyzing most models of this kind. The methods are based on using a linear approximation of the model and thereafter setting it up in a particular canonical form before solving it as a system of difference equations.

Before presenting the steps, we will start from the behind with the particular canonical form Uhlig uses. The purpose of all earlier steps is mainly to set up the model in the matrix form of (2.1)-(2.3):

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t \tag{2.1}$$

$$0 = E_t\left\{Fx_{t-1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right\} \tag{2.2}$$

$$z_{t+1} = Nz_t + \varepsilon_{t+1}; \qquad \varepsilon_{t+1} \sim N(0, \Sigma_\varepsilon) \tag{2.3}$$

x_t is a vector of endogenous state variables of length *m*, y_t is a vector of other endogenous variables of length *n* and z_t is a first-order exogenous stochastic process of dimension *k*. The distinction between the two types of endogenous variables will be explained later. Capital letters denote coefficient matrices.

(2.1) represents *l* equations in matrix form, i.e. *A, B, C* and *D* all have *l* rows. Furthermore it is assumed that $l \geq n$ and that *C* has rank *n*. Obviously, since (2.1) and (2.2) contains $m + n$ endogenous variables, the matrices *F, G, H, J, K, L* and *M* must all have $m + n - l$ rows to fully determine the system.

The stochastic processes described in (2.3) are assumed to be weakly stationary[1]. The matrix *N* has to be of size *k x k*. It must be assumed that the stochastic processes are stable, i.e. that the variables after a shock return to their steady state in a finite time horizon absent new shocks[2]. Consequently it is required that *N* has only stable eigenvalues (absolute value less than 1)[3]. The vector of error terms is of length *k* and it is assumed to be drawn from a normal distribution with mean zero and variance-covariance matrix Σ_ε. Typically (but not necessarily) Σ_ε is assumed to be a diagonal matrix which means that each element of z_t is i.i.d. (independently and identically distributed).

[1] Gujarati (1995), pg. 713: "Broadly speaking a stochastic process is said to be stationary if its mean and variance are constant over time and the value of covariance between two time periods depends only on the distance or lag between the two time periods and not on the actual time at which the covariance is computed."
[2] Strictly speaking we also open for permanent shocks. What is meant by stability here is a weak stability where the exogenous stochastic variables do not follow exponential paths. I.e. they either eventually return to steady state or follow random walks (if shocks are permanent).
[3] If allowing for unit root eigenvalues, the stochastic variables would move farther and farther away from steady state as time goes to infinity. The result of simulations would therefore depend strongly on the simulation length. Still, the case of permanent shocks, where *N* is a diagonal matrix with 1s along the diagonal will be treated as a reference case for interpreting impulse responses in chapter 7.

Later in this chapter possible "tricks" will be presented on how to manipulate models that apparently do not fit into this framework. Cf. in particular section 2.4 where it will be shown that most of the assumptions made above do not lead to any loss of generality. After finding the equations (2.1)-(2.3), the goal is to solve the model, which means to find the model's *recursive law of motion* (2.4)-(2.5):

$$x_t = Px_{t-1} + Qz_t \tag{2.4}$$

$$y_t = Rx_{t-1} + Sz_t \tag{2.5}$$

where P is of size m x m, Q of size m x k, R of size n x m and S of size n x k.

This implies to solve for the (unique) matrices P, Q, R and S which describe stable paths[4] for x_t and y_t. The solution process involves solving a system of difference equations. Uhlig (1997-2003) provides a MatLab™ source code[5] which numerically calculates P, Q, R and S given (2.1)-(2.3). The solution method involves solving (2.1)-(2.3) by the method of undetermined coefficients, of which the solution emerges implicitly as the solution of a matrix quadratic equation. Next the matrix quadratic equation is solved by using a QZ-decomposition[6] (also called generalized Schur-decomposition). Appendix I explains the solution method in detail, provides a formal proof and discusses the uniqueness and stability properties of the solution. Having solved for the recursive law of motion makes it easy to perform shock analysis by drawing impulse responses.

To get to (2.1)-(2.3), the following steps should be undertaken:

1. Find the equations that characterize the equilibrium of the original model
2. Find an explicit solution for a unique steady state if it exists, or choose one steady state if there are many

[4] Cf. subsection A1.4.2 in appendix I.
[5] The source code can be downloaded from the web page http://www.wiwi.hu-berlin.de/wpol/html/toolkit.htm. Too see what syntax to use and what variables to predefine, cf. the file *exampl0.m* or the source code *redux.m* that is listed in appendix IV.
[6] Cf. e.g. Sims (2000)

3. Log-linearize the equations found in step 1 (around the steady state from step 2)
4. If necessary, modify the equations in step 3 to fit into the canonical form (2.1)-(2.3)
5. Calibrate the model

These steps will be discussed in the following subsections.

2.1 FINDING EQUATIONS THAT CHARACTERIZE THE EQUILIBRIUM

Step 1 consists in finding the first-order conditions, constraints, definitions etc. that characterize the equilibrium. What is important at this point is to make sure that the dating convention is so that variables are dated in the period in which they are chosen or determined. (2.1)-(2.3) are based on variables being dated according to this principle. If agents e.g. have to choose in period $t - 1$ the capital stock in order to use it in period t, the variable should be dated $t - 1$.

2.2 FINDING A STEADY STATE

Step 2 is undertaken by dropping the time subscripts and attempting to find an explicit solution for each endogenous variable given the parameters and the exogenous variables of the model. The purpose is to have a baseline case to compare with when calculating percentage deviations from steady state when log-linearizing. This should be in mind if there are multiple steady states; the one should be chosen that one finds to be the most natural to compare with when analyzing shocks. In models with trends one might experience that there is no solution for the steady state. This can be the case e.g. in models with growth. A possible solution is to define new variables that are *detrended* versions of the variables that grow. Then one plugs in these definitions to substitute out the variables that grow, so that a steady state can be found. When replacing variables with transformed versions, one must just be careful with the interpretation of the variables in step 3, where the log-linearized versions of the detrended variables now are percentage deviations from the *trend* at any period in time.

Example: Consider a model with population growth, say where the log deviation of the population level (≈ population growth rate for small rates) follows an AR(1)-process, e.g.

$$\ln \frac{N_{t+1}}{N_t} = \rho_N \ln \frac{N_t}{N_{t-1}} + \varepsilon_{N,t+1} \tag{2.6}$$

where N_t denotes population *level* and (2.6) evolves according to the assumptions regarding (2.3). Say that total consumption, denoted C_t, has no steady state. When solving for steady state consumption, denoted $\overline{C}$, one might reach to an expression $\overline{C} = \overline{a} N_t$, where $\overline{a}$ denotes a constant that depends on the parameters of the model (N_t must remain with time subscript, since it has no steady state). Obviously population growth also leads to consumption growth. This problem is resolved simply be defining per capita consumption $c_t \equiv {C_t}/{N_t}$, which has the steady state $\overline{c} = \overline{a}$, and then substitute out C_t for c_t everywhere it appears in the model.

It should also be mentioned that it might also be that two or more variables have no individual steady state, but that a linear combination of them might have, i.e. the variables are co-integrated. In this case the variables should be replaced with the linear combination. Methods on how to perform such transformations are performed similarly as in the example above and will not be discussed further here.

2.3 LOG-LINEARIZATION

Step 3 implies to replace all the equations from step 1 with linear approximations, or more precisely with first order Taylor approximations around the steady state. Any function f of n variables x_1, x_2, …, x_n that have steady states $\overline{x}_1, \overline{x}_2, ..., \overline{x}_n$ can be approximated as in (2.7):

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) \approx f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) + f'_1(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)(x_1 - \overline{x}_1) + \\ f'_2(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)(x_2 - \overline{x}_2) + \ldots + f'_n(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)(x_n - \overline{x}_n) \end{aligned} \tag{2.7}$$

The quality of such an approximation will be discussed in chapter 8.

The log-deviation of a variable x_t, denoted $\hat{x}_t$, from its steady state value $\bar{x}$ is defined, where *ln* denotes the natural logaritm:

$$\hat{x}_t \equiv \ln x_t - \ln \bar{x} \approx \frac{x_t - \bar{x}}{\bar{x}} \quad \text{for small deviations} (x_t - \bar{x}) \tag{2.8}$$

For small deviations $\hat{x}_t$ can be interpreted as the percentage deviation from steady state[7].

To log-linearize an equation, i.e. to express it as an equation that is linear in log-deviations, the general recipe is to first take the natural logarithm of the equation and then apply the Taylor-approximation (2.7). Some standard manipulations yield an equation as wanted. However, there are some straightforward tricks that to a large extent simplify the process of log-linearization. Rewrite (2.8):

$$(2.8) \Rightarrow \hat{x}_t = \ln\left(\frac{x_t}{\bar{x}}\right) \Leftrightarrow \frac{x_t}{\bar{x}} = e^{\hat{x}_t} \Leftrightarrow$$

$$x_t = \bar{x} e^{\hat{x}_t} \tag{2.9}$$

Moreover, note that $e^{\hat{x}_t}$ for small[8] $\hat{x}_t$ can be approximated as:

$$e^{\hat{x}_t} \approx 1 + \hat{x}_t \tag{2.10}$$

Actually, by replacing all variables by the expression in (2.9), applying the approximation of (2.10) and exploiting the steady state relationships from step 2 to let

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[7] To define more precisely what is meant by *small*: A deviation that is actually 1%, 5%, 10% or 50%, leads to a log-deviation (times 100) of 1.00, 4.88, 9.53 and 40.55 respectively. As long as deviations are less than say 10%, no large errors are made.

[8] e^x for x equal to 0.01, 0.05, 0.10 and 0.50, equals 1.0101, 1.0513, 1.1052 and 1.6487 respectively. Also here we see that log-deviations/percentage deviations of up to 10% imply only small errors.

some constants drop out, one can log-linearize almost any equation with explicit functional forms in a far less tedious fashion than the standard method mentioned above. One obtains exactly the same results. As an example, consider the following equation:

$$ax_t = by_t z_t + cq_t \tag{2.11}$$

where x_t, y_t, z_t and q_t are variables and a, b and c parameters. Applying (2.9) and (2.10) to (2.11) yields:

$$\begin{aligned}&(2.11) \Rightarrow a\bar{x}e^{\hat{x}_t} = b\bar{y}\bar{z}e^{\hat{y}_t}e^{\hat{z}_t} + c\bar{q}e^{\hat{q}_t} = b\bar{y}\bar{z}e^{\hat{y}_t + \hat{z}_t} + c\bar{q}e^{\hat{q}_t}\\ &\Leftrightarrow a\bar{x}(1+\hat{x}_t) \approx b\bar{y}\bar{z}(1+\hat{y}_t+\hat{z}_t) + c\bar{q}(1+\hat{q}_t)\end{aligned}$$

Finally, use the steady state version of (2.11), namely $a\bar{x} = b\bar{y}\bar{z} + c\bar{q}$, so that the constant terms drop out of the equation:

$$\Leftrightarrow a\bar{x}\hat{x}_t \approx b\bar{y}\bar{z}\hat{y}_t + b\bar{y}\bar{z}\hat{z}_t + c\bar{q}\hat{q}_t \tag{2.12}$$

(2.12) is the log-linearized approximation of (2.11) and is linear in the log-deviations. Potential problems can appear in the case when the steady state value of a variable is exactly zero, since the logarithm of zero is not defined and accordingly the definition in (2.8) also not.

Assume that the variable z_t has steady state zero. Define a new variable $\tilde{\hat{z}}_t$ so that it expresses approximately the deviation of z_t from its steady state (zero) measured as a percentage of the steady state of some other variable, e.g. $\bar{y}$. In other words:

$$\tilde{\hat{z}}_t \equiv \frac{z_t - \bar{z}}{\bar{y}} = \frac{z_t}{\bar{y}}; \qquad \bar{y} \neq 0, \bar{z} = 0 \tag{2.13}$$

The first equality in (2.13) also holds if $\bar{z} \neq 0$. Actually assuming for a short while that $\bar{z} \neq 0$, makes it possible to relate (2.13) to (2.8) in the following way:

$$\tilde{\hat{z}}_t = \frac{z_t - \bar{z}}{\bar{y}} = \left(\frac{\bar{z}}{\bar{y}}\right)\left(\frac{z_t - \bar{z}}{\bar{y}}\right) \approx \left(\frac{\bar{z}}{\bar{y}}\right)\hat{z}_t; \quad \bar{z}, \bar{y} \neq 0 \tag{2.14}$$

However, if $\bar{z} = 0$, the relationship can be written as follows:

$$\tilde{\hat{z}}_t \approx \frac{z_t - \bar{z}}{\bar{y}} = \lim_{\bar{z} \to 0}\left[\left(\frac{\bar{z}}{\bar{y}}\right)\left(\frac{z_t - \bar{z}}{\bar{z}}\right)\right] \approx \lim_{\bar{z} \to 0}\left[\left(\frac{\bar{z}}{\bar{y}}\right)\hat{z}_t\right] \tag{2.15}$$

When log-linearizing an equation where at least one variable has steady state zero, one can still use the expression in (2.9) to rewrite all variables but to avoid using expressions that are not defined, one should at the same time take the limit of both sides of the expression as the variables with steady state zero go to zero.

Example: Consider once again equation (2.11), but now assume that $\bar{z} = 0$. Rewrite (2.11) in the same way as done above, but add limits for every step to avoid dividing by zero:

$$\begin{aligned}
&(2.11) \Rightarrow \lim_{\bar{z} \to 0}\left(a\bar{x}e^{\hat{x}_t}\right) = \lim_{\bar{z} \to 0}\left(b\bar{y}\bar{z}e^{\hat{y}_t}e^{\hat{z}_t} + c\bar{q}e^{\hat{q}_t}\right) = \lim_{\bar{z} \to 0}\left(b\bar{y}\bar{z}e^{\hat{y}_t + \hat{z}_t} + c\bar{q}e^{\hat{q}_t}\right) \\
&\Leftrightarrow \lim_{\bar{z} \to 0}\left[a\bar{x}(1 + \hat{x}_t)\right] \approx \lim_{\bar{z} \to 0}\left[b\bar{y}\bar{z}(1 + \hat{y}_t + \hat{z}_t + c\bar{q}(1 + \hat{q}_t))\right] \Leftrightarrow \\
&\lim_{\bar{z} \to 0}\left[a\bar{x}(1 + \hat{x}_t)\right] \approx \lim_{\bar{z} \to 0}\left[b\bar{y}\bar{z}(1 + \hat{y}_t + \hat{z}_t) + c\bar{q}(1 + \hat{q}_t)\right]
\end{aligned}$$

Also here use the steady state version of (2.11) so that constant terms drop out of the equation:

$$\Leftrightarrow \lim_{\bar{z} \to 0}\left[a\bar{x}\hat{x}_t\right] \approx \lim_{\bar{z} \to 0}\left[b\bar{y}\bar{z}\hat{y}_t + b\bar{y}\bar{z}\hat{z}_t + c\bar{q}\hat{q}_t\right] \tag{2.16}$$

Use the last (approximate) equality in (2.15) and plug it into (2.16):

$$(2.16) \Rightarrow \lim_{\bar{z}\to 0}\left[a\bar{x}\hat{x}_t\right] \approx \lim_{\bar{z}\to 0}\left[b\bar{y}\bar{z}\hat{y}_t + b\bar{y}\bar{z}\,\frac{\bar{y}}{\bar{z}}\,\tilde{\hat{z}}_t + c\bar{q}\hat{q}_t\right]$$

$$\Leftrightarrow a\bar{x}\hat{x}_t \approx b\bar{y}^2\hat{z}'_t + c\bar{q}\hat{q}_t \qquad (2.17)$$

When choosing which variable to be $\bar{y}$, one should choose the one that makes $\tilde{\hat{z}}_t$ as easy as possible to interpret. E.g., in the model that is presented in chapter 3, bond holdings has steady state zero. The log-deviation for bond holdings is therefore instead chosen relative to GDP. The interpretation of the variable is therefore the deviation of bond holdings from steady state (zero) measured as a percentage of GDP.

For variables that already in their original version denote percentages, such as the real interest rate, the interpretation of the definition in (2.8) might be confusing. If the real interest rate rises from 1% to 2%, this is a percentage deviation of 100. Often one wishes that deviations in such variables are denoted as absolute deviation instead of relative. Consider a variable r_t, assumed to denote such a variable. The definition in (2.8) would give approximately the percentage deviation in r_t. Obviously, multiplying with $\bar{r}$ gives approximately the absolute deviation in r_t. Thus define the following new variable $\tilde{\hat{r}}_t$:

$$\tilde{\hat{r}}_t \equiv \bar{r}\hat{r}_t = \bar{r}(\ln r_t - \ln \bar{r}) \approx r_t - \bar{r} \qquad (2.18)$$

When log-linearizing, r_t should everywhere that it appears, be replaced with the following expression:

$$r_t = \bar{r}e^{\frac{\tilde{\hat{r}}_t}{\bar{r}}} \qquad (2.19)$$

Chapter 3 shows the log-linearization of an entire model using these methods.

2.4 MODIFYING EQUATIONS IF NECESSARY

After log-linearizing the task is to set up the system of log-linearized equations in the canonical form (2.1)-(2.3).

The first to notice is that (2.1)-(2.3) describe a system of *second-order* difference equations. The system should only contain variables that are dated $t-1$, t or $t+1$. Usually this restriction imposes no problems. However, if the system contains additional leads or lags, this can be handled by using the same techniques as when reducing difference equations of higher order to second-order difference equations. E.g. if the system contains a variable $\hat{q}_{t-2}$, this is solved by defining a new variable $\tilde{\hat{q}}_t \equiv \hat{q}_{t-1}$. Then one adds this definition as a new equation and replaces $\hat{q}_{t-2}$ with $\tilde{\hat{q}}_{t-1}$ everywhere that it appears. Further lags can be handled by introducing additional definitions in the same fashion. The same goes if the system contains a variable such as $E_t\{\hat{w}_{t+2}\}$. Define $\tilde{\hat{w}}_t \equiv \hat{w}_{t+1}$. Also here, add the definition as a new equation and replace $E_t\{\hat{w}_{t+2}\}$ everywhere that it appears with $E_t\{\tilde{\hat{w}}_{t+1}\}$.

How does one define the vectors x_t and y_t? Let x_t and y_t be vectors of size m and n respectively. Consider the system of linear difference equations found in step 3. Endogenous variable that appear with time subscript $t-1$ are given at date t, i.e. cannot be changed. These variables are called *endogenous state variables*. This means that at any given point in time t, the solution for the endogenous variables depends on the value of the *endogenous state variables* from the last period. A typical example of a state variable in many models is the capital stock, i.e. models where e.g. production depends on the capital stock from last period. The vector y_t contains the *other endogenous variables*. As Uhlig (1997) points out, one makes no mistake by defining all endogenous variables as state variables. The solution process will simply confirm that some of the variables not are true state variables. This appears in the sense that the columns in the matrices P and R that correspond to those variables will be a column of zeros. This implies that it has been confirmed that none of the endogenous variables depend on the value of these falsely declared state variables from the last period. Uhlig (1997) calls this method *brute force*.

The vector z_t contains the exogenous variables of the model that follow first-order stochastic processes, specified in the matrix N. Often the stochastic processes are independent of each other, i.e. N is diagonal. Then stability simply requires that all the elements along the diagonal are less than or equal to one. Higher-order stochastic processes, such as AR(p)-processes, can also be included by *stacking* equation (2.3) as a first-order vector stochastic difference equations. Example: Assume that the log-linearized model contains the AR(2)-process (2.20) that determines the variable $\hat{w}_t$ and where $\rho_{w,1}$ and $\rho_{w,2}$ are coefficients of autocovariance and ε_t a normally distributed error term with properties as discussed earlier.

$$\hat{w}_{t+1} = \rho_{w,1}\hat{w}_t + \rho_{w,2}\hat{w}_{t-1} + \varepsilon_{t+1} \tag{2.20}$$

(2.20) can be rewritten as a matrix equation:

$$\begin{bmatrix} \hat{w}_{t+1} \\ \hat{w}_t \end{bmatrix} = \begin{bmatrix} \rho_{w,1} & \rho_{w,2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{w}_t \\ \hat{w}_{t-1} \end{bmatrix} \tag{2.21}$$

Define a new variable $\hat{v}_t$, to rewrite (2.21) as a 2-dimensional first-order difference equation:

$$\hat{v}_t \equiv \begin{bmatrix} \hat{w}_t \\ \hat{w}_{t-1} \end{bmatrix}$$

$$(2.21) \Rightarrow \hat{v}_{t+1} = \begin{bmatrix} \rho_{w,1} & \rho_{w,2} \\ 1 & 0 \end{bmatrix} \hat{v}_t \tag{2.22}$$

(2.22) is now in the form of (2.3). Higher-order systems can be stacked in the same fashion. Cf. e.g. Uhlig (2003b) for a formal discussion.

Denote the number of equations that do not contain variables dated $t + 1$, i.e. the equations in (2.1), as l. Since there are $m + n$ endogenous variables, there should necessarily be $m + n - l$ equations that contain variables dated $t + 1$, i.e. the equations in (2.2). Moreover, the solution method requires that there are at least as many equations in (2.1) than there are other endogenous variables, i.e. $l \geq n$, and that the matrix C which is of size $l \times n$, has full rank, i.e. rank n. Cf. the solution of (2.1)-(2.3) in appendix I to see why this restriction is required. This requirement might seem as a strict restriction. Remember, however, the argument made about defining the vectors x_t and y_t - that any *endogenous variable* in principle can be declared as a state variable. In the extreme case one can let n drop all the way to zero, in which $l \geq n$ always holds, and the rank of C always will be n.

2.5 CALIBRATING THE MODEL

By finding values for the parameters in the model, one can calculate the steady state values of the model variables quantitatively. The next step is to calculate the coefficient matrices in (2.1)-(2.3). Please cf. chapter 5 for an applied example of calibration.

3. A TWO-COUNTRY MODEL WITH MONOPOLISTIC COMPETITION AND STICKY PRICES

3.1 MAIN ASSUMPTIONS

The model consists of two countries, let us call them *Home* and *Foreign*. There is a continuum of agents; total world population is normalized to one, with a fraction n in Home and a fraction $1 - n$ in Foreign. All agents have the same preferences, and each of them produces a single differentiated good with labour as the only factor of production under monopolistic competition. Prices are sticky in the sense that producers have to set prices one period in advance[9]. Goods are freely traded, and there is also one internationally traded financial asset, a real bond. Agents can hold money or real bonds. Assume for simplicity that agents only hold the currency of their own country. They maximize their discounted future utility under an infinite time horizon. The government prints money and collects taxes. This model does not focus on the effect of distortionary taxes and it is therefore assumed that the government has access to lump sum taxes.

Some additional assumptions to rule out unreasonable solutions with speculative bubbles in the price level or with infinite borrowing will be introduced later.

Uncertainty will be introduced as exogenous variables assumed to follow AR(1)-processes, just as has been done in Houeland and Lien (2003). The approach here is however extended, as not only domestic government spending and money stock are stochastic, but also technology and the foreign versions of all three variables[10]. Most questions regarding how to justify the assumptions are gathered in section 8.2. Please confer this section whenever such questions arise.

3.2 SOLUTION METHOD

With the particular way of modelling sticky prices, it is clear that the long-run flex-price and sticky-price equilibria will be quite similar. When an unanticipated shock occurs, producers are unable to change prices until next period. One period after the initial shock (absent new shocks) prices will again be set optimally, and all equations characterizing the flex-price equilibrium will also hold in the sticky price case. This insight turns out to be particularly convenient for drawing impulse responses to one-time shocks.

Obstfeld and Rogoff (1995a) and (1996) solve the model by first setting it up with flexible prices. After solving for the log-linearized equilibrium, sticky prices are

[9] Another frequently used approach that is more complex is the so called Calvo sticky prices, where only a fraction α ($0<\alpha<1$) of the producers are allowed to change prices every period. The producers are randomly drawn each period. Cf. e.g. Woodford (1996) for a model similar to the one presented here.

[10] For analyzing impulse responses there is no loss of generality in allowing for shocks in domestic exogenous variables only since one always can interchange the two countries. However, for calculating simulations, results are more general when there are shocks in both countries at the same time.

introduced by considering how the sticky price equilibrium differs from the flexible price equilibrium in the short and the long run after a shock, respectively. However, they consider only shocks that are purely temporary or shocks with full persistence – and besides the analysis is purely qualitative. Another disadvantage is that the particular way of setting up the model makes it possible only to analyse one-time shocks. The latter drawback also applies to the framework in Houeland and Lien (2003).

When running simulations, the economy is hit by multiple shocks drawn from some random distributions every period, and the similarities between the flex-price and the sticky-price equilibria can no longer be exploited in the same way. This paper therefore modifies the model in Obstfeld and Rogoff to introduce the assumption of sticky prices from the very beginning. It is thereby possible to capture the full dynamics of the model outside steady state.

Equilibrium equations that are equal for the two countries, except for the notation, are derived for Home only. *Foreign* variables are denoted with a *. All real variables are measured as per capita sizes, i.e. for the representative agent.

3.3 THE STICKY-PRICE EQUILIBRIUM[11]

3.3.1 Finding equilibrium equations

Define a real domestic private consumption index, denoted C_t, of the Dixit-Stiglitz[12] type. There is a continuum of differentiated goods indexed from 0 to 1 (includes domestic as well as foreign goods).

$$C_t \equiv \left(\int_0^1 c_t(z)^{\frac{\theta-1}{\theta}} dz \right)^{\frac{\theta}{\theta-1}}; \qquad \theta > 1 \tag{3.1}$$

$c_t(z)$ denotes the representative domestic agent's consumption of a single good, θ denotes the elasticity of substitution between any pair of goods. It will be shown later why it has

[11] Some straight forward intermediate calculations regarding this section are added in appendix II.
[12] Cf. Dixit and Stiglitz (1977)

to be that $\theta > 1$. Let the term *domestic consumption* from now on refer to domestic composite private consumption C_t.

By minimizing the expenditure of buying one unit of the composite good, one gets the following domestic price index, denoted P_t;

$$P_t = \left(\int_0^1 p_{t-1}(z)^{1-\theta} dz \right)^{\frac{1}{1-\theta}} \tag{3.2}$$

$p_{t-1}(z)$ denotes the domestic currency price of a single good; the time subscript is $t - 1$ since the price charged in period t has been set in period $t - 1$[13]. The same calculation also yields the representative domestic agent's demand function for a single differentiated good;

$$c_t(z) = \left(\frac{p_{t-1}(z)}{P_t} \right)^{-\theta} C_t \tag{3.3}$$

We assume for simplicity that government spending is allocated in exactly the same way as private consumption. This means that we get the following domestic government consumption index, denoted G_t, and demand function, where $g_t(z)$ denotes domestic public consumption of a single differentiated good;

$$G_t \equiv \left(\int_0^1 g_t(z)^{\frac{\theta-1}{\theta}} dz \right)^{\frac{\theta}{\theta-1}} \tag{3.4}$$

$$g_t(z) = \left(\frac{p_{t-1}(z)}{P_t} \right)^{-\theta} G_t \tag{3.5}$$

[13] Cf. the dating convention described in section 2.1.

Equation (3.1)-(3.5) correspond to equivalent equations for *Foreign*, just with a superscript * for all the variables. An overview of all the equations characterizing the equilibrium will be provided later.

No trading costs in the economy implies that the law of one price (LOOP) holds;

$$p_{t-1}(z) = \varepsilon_t p^*_{t-1}(z) \tag{3.6}$$
$$P_t = \varepsilon_t P^*_t \tag{3.7}$$

ε_t denotes the exchange rate; the price of foreign money in terms of domestic money.

Remember that Home has a fraction *n* of total population and Foreign accordingly a fraction *1-n*. Define C_t^w and G_t^w as the population-weighted averages of private consumption and government spending, respectively (from now on called *world consumption* and *world government spending*):

$$C_t^w \equiv nC_t + (1-n)C_t^* \tag{3.8}$$
$$G_t^w \equiv nG_t + (1-n)G_t^* \tag{3.9}$$

The total demand that a single producer faces (domestic or foreign producer), denoted by $y_t(z)$, will consist of total demand from each of the countries, weighted with their relative population size:

$$y_t(z) = n\left[c_t(z) + g_t(z)\right] + (1-n)\left[c_t^*(z) + g_t^*(z)\right] \tag{3.10}$$

Plug (3.3) and (3.5)-(3.9) into (3.10), to get an expression that depends only on relative prices and total world demand (cf. appendix II for details):

$$y_t(z) = \left(\frac{p_{t-1}(z)}{P_t} \right)^{-\theta} \left(C_t^w + G_t^w \right) \tag{3.11}$$

The agents hold money and real bonds, sell a differentiated good, consume and pay taxes. The budget constraint for the representative agent is given by:

$$F_t + \frac{M_t}{P_t} = (1 + r_{t-1}) F_{t-1} + \frac{M_{t-1}}{P_t} + \frac{p_{t-1}(z)}{P_t} y_t(z) - C_t - T_t \tag{3.12}$$

F_t denotes real bond holdings at the end of period t, r_t the real interest rate between period t and $t+1$, M_t money holdings and T_t real taxes. The budget constraint says that end-of-period wealth must be equal to initial wealth plus income less consumption and taxes.

The utility function U_t depends positively on consumption and money holdings and negatively on work effort (measured as output);

$$U_t = E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[\ln C_s + \chi \ln \left(\frac{M_s}{P_s} \right) - \frac{\kappa_s}{2} y_t(z)^2 \right] \right\} \tag{3.13}$$

$\beta \in <0,1>$ is the subjective discount factor (the agent's "patience"), $\chi > 0$ is a parameter that can be interpreted as the magnitude of transaction costs[14], and $\kappa_s > 0$ can be interpreted as a technology variable[15]. There is no independent production function as work effort enters directly into the utility function. A low κ_s means that it takes little effort to produce some given amount of production and vice versa. κ_s can therefore be interpreted as the inverse of productivity. From now on κ_s will be called *technology*.

The consumer's problem below, (3.14), is the same every period and consists in maximizing utility (3.13) with respect to consumption, product price, money and bond

[14] Feenstra (1986) shows how money in the utility function is equivalent to models with transaction costs (where the transaction costs of consuming depend negatively on money holdings).
[15] κ_s also captures the preference for working. But since it is assumed that all agents have the same preferences, a change in κ_s must be due to a change in productivity.

holdings subject to the budget constraint (3.12) and given the total demand function (3.11). Since all agents within a country are identical, they will all choose the same product price and end up with the same level of work effort.

Simplify the notation for work effort by dropping the index *z*, i.e. domestic and foreign work effort will be denoted y_t and y_t^* respectively. Operate equivalently only with country specific prices. Let the indices *h* and *f* denote a domestic and a foreign product, respectively. Let a * denote that the price is measured in foreign currency, and vice versa without a *. I.e. the domestic currency product price of a domestic product will be $p_t(h)$ and the foreign currency price of a foreign product $p_t{}^*(f)$ [16].

$$
\begin{aligned}
&\max U_t = E_t\left\{\sum_{s=t}^{\infty}\beta^{s-t}\left[\ln C_s + \chi \ln\left(\frac{M_s}{P_s}\right) - \frac{\kappa_s}{2}y_s^2\right]\right\} \\
&\text{w.r.t. } \{C_s\},\{M_s\},\{p_s(h)\},\{F_s\} \qquad (3.14) \\
&\text{s.t.} \qquad F_s + \frac{M_s}{P_s} = (1+r_{s-1})F_{s-1} + \frac{M_{s-1}}{P_s} + \frac{p_{s-1}(h)}{P_s}y_s - C_s - T_s \\
&\text{and} \qquad y_t = \left(\frac{p_{t-1}(h)}{P_t}\right)^{-\theta}\left(C_t^w + G_t^w\right)
\end{aligned}
$$

Note that in the flexible price case, it would have been irrelevant whether the agent had chosen work effort y_s or product price $p_s(z)$, since every level of production directly corresponds to a unique price level and vice versa[17]. However, in the sticky-price case, the agent chooses the price one period ahead – and thereby indirectly chooses expected work effort next period. If there are no new unanticipated shocks, the expectation will confirm and there is no difference to the flex-price case. But if there are shocks, demand might be higher or lower than expected when the price was set. Since there is monopolistic competition, the price is above marginal cost (price equals marginal revenue). If demand is higher than expected, even though the price cannot be changed, it

[16] Even though all agents set the same product price, they see themselves as small relative to the whole population. Certainly they therefore take the price indices as given.
[17] Cf. the total demand function (3.11).

will be profitable to meet the unexpected extra demand. And if demand is somewhat lower than expected, it will be profitable to meet that demand rather than closing down[18].

(3.14) yields the following first-order conditions, where $\beta^{s-t}\lambda_s$ denotes the marginal period-t utility of real wealth in period s:

$$\text{FOC1)}\quad \frac{\partial L}{\partial C_t}: \quad \frac{1}{C_t} = \lambda_t \tag{3.15}$$

$$\text{FOC2)}\quad \frac{\partial L}{\partial M_t}: \quad \frac{\chi}{M_t} = \frac{\lambda_t}{P_t} - \beta E_t\left\{\frac{\lambda_{t+1}}{P_{t+1}}\right\} \tag{3.16}$$

$$\text{FOC3)}\quad \frac{\partial L}{\partial p_t(h)}: E_t\left\{\kappa_{t+1} y_{t+1} \frac{\partial y_{t+1}}{\partial p_t(h)}\right\} = E_t\left\{\lambda_{t+1}\frac{y_{t+1}}{P_{t+1}} + \frac{p_t(h)}{P_{t+1}}\frac{\partial y_{t+1}}{\partial p_t(h)}\right\}$$

$$\Leftrightarrow p_t(h) = \frac{\theta}{\theta - 1} E_t\left\{\frac{\kappa_{t+1} y_{t+1} P_{t+1}}{\lambda_{t+1}}\right\} \tag{3.17}$$

$$\text{FOC4)}\quad \frac{\partial L}{\partial F_t}: \quad \lambda_t = \beta(1 + r_t) E_t\{\lambda_{t+1}\} \tag{3.18}$$

(3.15) states that consumption should be chosen so that the marginal utility of consumption equals the marginal utility of wealth. I.e. for the last unit of consumption, the agent should be indifferent between consuming and saving. (3.16) states that the marginal utility of real money (left hand side) should be equal to the cost of holding money in terms of utility (right hand side). The utility cost of holding money depends positively on inflation, since inflation increases the seignorage, and negatively on the change in marginal utility of wealth, since the utility loss is higher the higher future consumption is valued. (3.17) states that the price should be set so that the expected marginal disutility of work effort equals the marginal benefit of production – i.e. the marginal revenue valued with the marginal utility of wealth. We also see that θ has to be larger than 1, since the optimal price otherwise will be negative. This is because the

[18] This is a similar argument to a one made in Obstfeld and Rogoff (1995a). We implicitly assume here that shocks are small enough to avoid situations of rationing, i.e. that producers always meet demand at current prices.

marginal revenue in this case will be negative for any positive level of production. (3.18) is a kind of Lucas asset pricing equation. The equation can be rewritten as

$$1 = E_t\left\{(1+r_t)D_{t+1}\right\}; \qquad D_{t+1} \equiv \beta \frac{\dfrac{\partial U_t}{\partial C_{t+1}}}{\dfrac{\partial U_t}{\partial C_t}} = \beta \frac{C_t}{C_{t+1}} \tag{3.19}$$

where D_{t+1} is the stochastic discount factor between period t and $t+1$. In a situation where future consumption is expected to be high compared with current consumption, which means that the marginal utility of consumption will be lower in the future than now, it will be unattractive to save. The stochastic discount factor D_{t+1} will be low, and the equilibrium interest rate will be high. An equivalent argument goes for the opposite case.

For later use, rewrite the first order conditions (3.15)-(3.18) slightly (cf. appendix II for details):

$$\frac{M_t}{P_t} = \chi C_t E_t\left\{\frac{(1+r_t)\dfrac{P_{t+1}}{P_t}}{(1+r_t)\dfrac{P_{t+1}}{P_t}-1}\right\} \tag{3.20}$$

$$p_t(h) = \frac{\theta}{\theta-1}E_t\left\{\kappa_{t+1}y_{t+1}P_{t+1}C_{t+1}\right\} \tag{3.21}$$

$$E_t\left\{C_{t+1}\right\} = \beta(1+r_t)C_t \tag{3.22}$$

Apply to equation (3.2) the fact that all product prices within a country must be the same. The domestic and foreign price index can then be written as (cf. appendix II for details):

$$P_t = \left[np_{t-1}(h)^{1-\theta} + (1-n)\left(p^*_{t-1}(f)\mathcal{E}_t\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{3.23}$$

$$P^*_t = \left[n\left(\frac{p_{t-1}(h)}{\mathcal{E}_t}\right)^{1-\theta} + (1-n)p^*_{t-1}(f)^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{3.24}$$

Note this purchasing power parity (PPP) holds in this setup (combine (3.23) and (3.24)):

$$P_t = \varepsilon_t P_t^* \tag{3.25}$$

The government collects taxes, spends and prints money. Woodford (1996) shows that Ricardian equivalence holds in models with nominal rigidities with a setup as in this model. Government spending and the money stock are exogenously given as stochastic processes. The government could in theory determine taxes. However, the agents know that any tax-cut today leading to a government deficit must be financed by higher taxes in the future (the deficit cannot be financed by lower government spending in the future since government spending evolves exogenously according to an exogenous process). Therefore they save already today to finance these future taxes and smooth consumption. These private savings exactly equate government dissaving – and there are no real changes. The structure of the model can therefore be simplified by assuming that the government budget balances each period. The same simplification has been done in Obstfeld and Rogoff (1995a). The government budget constraint will therefore be given by;

$$G_t = T_t + \frac{M_t - M_{t-1}}{P_t} \tag{3.26}$$

I.e. government spending equals taxes plus seignorage. Plug (3.26) into the agent's intertemporal budget constraint (3.12) to substitute out taxes and money[19]:

$$F_t = (1 + r_{t-1})F_{t-1} + \frac{p_{t-1}(h)}{P_t} y_t - C_t - G_t \tag{3.27}$$

Note that the sum of real bond holdings in the two countries must be zero. The foreign bond holdings can therefore be expressed in terms of domestic bond holdings:

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[19] It is natural to assume that each country has many inhabitants, and that each single individual sees himself as small relative to the society. In other words, he takes taxes as given – which means that plugging

$$nF_t + (1-n)F_t^* = 0 \Leftrightarrow F_t^* = -\frac{n}{1-n}F_t \tag{3.28}$$

Note that bond holdings must be weighted with relative population size, since F_t and F_t* express real bond holdings per capita and since the countries may be of different size.

Finally, for welfare analysis purposes later, it would be convenient to measure how welfare develops over time. (3.13) expresses the discounted sum of utility in all future periods. From (3.13) it is clear that per capita welfare in a single period, from now on called only welfare and denoted W_t, should be defined as:

$$W_t \equiv \ln C_t + \chi \ln\left(\frac{M_t}{P_t}\right) - \frac{\kappa_t}{2} y_t^2 \tag{3.29}$$

We are now ready to provide a list of equations characterizing the sticky-price equilibrium, numbered as (E1)-(E17). This list includes the foreign version of equations (3.11), (3.20)-(3.22), (3.27) and (3.29), where (3.28) has been plugged into the foreign version of the agent's budget constraint (3.27).

There are 15 endogenous variables: C_t, C_t*, C_t^w, ε_t, F_t, G_t^w, r_t, P_t, P_t*, $p_{t-1}(h)$, p_{t-1}*(f), y_t, y_t*, W_t, W_t*. But there are 17 listed equations. Since they all hold, it means that 2 of the equations are abundant for solving the system. E.g. (E7) follows directly from combining (E8) and (E9). After log-linearizing the system we will build down the system to the correct number of equations and endogenous variables. (E1)-(E17) is anyway not a sufficient description of the equilibrium, since assumptions to rule out infinite-borrowing solutions and of speculative bubbles not yet has been introduced. This will be done in section 3.3.3.

---

the government budget constraint into the agent's budget constraint should be done *after* solving for the first order conditions, as here.

Equilibrium equations

(E1) $$E_t\left\{C_{t+1}\right\} = \beta(1+r_t)C_t$$

(E2) $$E_t\left\{C^*_{t+1}\right\} = \beta(1+r_t)C^*_t$$

(E3) $$\frac{M_t}{P_t} = \chi C_t E_t\left\{\frac{(1+r_t)\frac{P_{t+1}}{P_t}}{(1+r_t)\frac{P_{t+1}}{P_t}-1}\right\}$$

(E4) $$\frac{M^*_t}{P^*_t} = \chi C^*_t E_t\left\{\frac{(1+r_t)\frac{P^*_{t+1}}{P^*_t}}{(1+r_t)\frac{P^*_{t+1}}{P^*_t}-1}\right\}$$

(E5) $$C^w_t = nC_t + (1-n)C^*_t$$

(E6) $$G^w_t = nG_t + (1-n)G^*_t$$

(E7) $$P_t = \mathcal{E}_t P^*_t$$

(E8) $$P_t = \left[np_t(h)^{1-\theta} + (1-n)\left(p^*_t(f)\mathcal{E}_t\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

(E9) $$P^*_t = \left[n\left(\frac{p_t(h)}{\mathcal{E}_t}\right)^{1-\theta} + (1-n)\,p^*_t(f)^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

(E10) $$F_t = (1+r_{t-1})F_{t-1} + \frac{p_t(h)}{P_t}y_t - C_t - G_t$$

(E11) $$-\frac{n}{1-n}F_t = -\frac{n}{1-n}(1+r_{t-1})F_{t-1} + \frac{p^*_t(f)}{P^*_t}y^*_t - C^*_t - G^*_t$$

(E12) $$y_t = \left(\frac{p_t(h)}{P_t}\right)^{-\theta}\left(C^w_t + G^w_t\right)$$

(E13) $$y^*_t = \left(\frac{p^*_t(f)}{P^*_t}\right)^{-\theta}\left(C^w_t + G^w_t\right)$$

(E14) $$p_t(h) = \frac{\theta}{\theta-1}E_t\left\{\kappa_{t+1}y_{t+1}P_{t+1}C_{t+1}\right\}$$

(E15) $$p^*_t(f) = \frac{\theta}{\theta-1}E_t\left\{\kappa^*_{t+1}y^*_{t+1}P^*_{t+1}C^*_{t+1}\right\}$$

(E16) $$W_t = \ln C_t + \chi\ln\left(\frac{M_t}{P_t}\right) - \frac{\kappa_t}{2}y_t^2$$

(E17) $$W^*_t = \ln C^*_t + \chi\ln\left(\frac{M^*_t}{P^*_t}\right) - \frac{\kappa^*_t}{2}y_t^{*2}$$

3.3.2 Finding a steady state

The next step is to find a steady state for the system (E1)-(E17), i.e. a situation where all variables take the same value every period absent shocks. For a variable x_t, denote the steady state of that variable as $\bar{x}$. Finding the steady state consists in replacing all variables in (E1)-(E17) with their steady states and remove all expectations signs since there is no uncertainty in this problem. Finding the steady states then consists in solving the new system of equations that appears for the endogenous variables given the exogenous variables and the parameters.

Though straight forward, a list of the steady state versions of equations (E1)-(E17), denoted (S1)-(S17) is here included for completeness:

(S1) $$\bar{C} = \beta(1+\bar{r})\bar{C}$$

(S2) $$\bar{C}^* = \beta(1+\bar{r})\bar{C}^*$$

(S3) $$\frac{\bar{M}}{\bar{P}} = \chi\bar{C}\left[\frac{(1+\bar{r})\frac{\bar{P}}{\bar{P}}}{(1+\bar{r})\frac{\bar{P}}{\bar{P}}-1}\right]$$

(S4) $$\frac{\bar{M}^*}{\bar{P}^*} = \chi\bar{C}^*\left[\frac{(1+\bar{r})\frac{\bar{P}^*}{\bar{P}^*}}{(1+\bar{r})\frac{\bar{P}^*}{\bar{P}^*}-1}\right]$$

(S5) $$\bar{C}^w = n\bar{C} + (1-n)\bar{C}^*$$

(S6) $$\bar{G}^w = n\bar{G} + (1-n)\bar{G}^*$$

(S7) $$\bar{P} = \bar{\mathcal{E}}\bar{P}^*$$

(S8) $$\bar{P} = \left[n\bar{p}(h)^{1-\theta} + (1-n)\left(\bar{p}^*(f)\bar{\mathcal{E}}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

(S9) $$\bar{P}^* = \left[n\left(\frac{\bar{p}(h)}{\bar{\mathcal{E}}}\right)^{1-\theta} + (1-n)\bar{p}^*(f)^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

(S10) $$\bar{F} = (1+\bar{r})\bar{F} + \frac{\bar{p}(h)}{\bar{P}}\bar{y} - \bar{C} - \bar{G}$$

(S11) $$-\frac{n}{1-n}\bar{F} = -\frac{n}{1-n}(1+\bar{r})\bar{F} + \frac{\bar{p}^*(f)}{\bar{P}^*}\bar{y}^* - \bar{C}^* - \bar{G}^*$$

(S12) $\bar{y} = \left(\frac{\bar{p}(h)}{\bar{P}}\right)^{-\theta}\left(\bar{C}^{w} + \bar{G}^{w}\right)$

(S13) $\bar{y}^{*} = \left(\frac{\bar{p}^{*}(f)}{\bar{P}^{*}}\right)^{-\theta}\left(\bar{C}^{w} + \bar{G}^{w}\right)$

(S14) $\bar{p}(h) = \frac{\theta}{\theta - 1}\bar{\kappa}\bar{y}\bar{P}\bar{C}$

(S15) $\bar{p}^{*}(f) = \frac{\theta}{\theta - 1}\bar{\kappa}^{*}\bar{y}^{*}\bar{P}^{*}\bar{C}^{*}$

(S16) $\bar{W} = \ln\bar{C} + \chi\ln\left(\frac{\bar{M}}{\bar{P}}\right) - \frac{\bar{\kappa}}{2}\bar{y}^{2}$

(S17) $\bar{W}^{*} = \ln\bar{C}^{*} + \chi\ln\left(\frac{\bar{M}^{*}}{\bar{P}^{*}}\right) - \frac{\bar{\kappa}^{*}}{2}\bar{y}^{*2}$

(S1) (or [S2]) yields:

$$\bar{C} = \beta(1+\bar{r})\bar{C} \Leftrightarrow \bar{r} = \frac{1-\beta}{\beta} \qquad (3.30)$$

In this model there are multiple steady states, but we will choose to focus on a symmetric steady state. By symmetry here is meant *per capita symmetry*, since the parameter *n* that controls relative country size still is allowed to differ from ½. More precisely, the symmetric steady state is a steady state where both countries have the same technology, per capita consumption, output, government spending and bond holdings[20]. I.e. $\bar{C} = \bar{C}^{*}, \bar{y} = \bar{y}^{*}, \bar{F} = \bar{F}^{*}, \bar{G} = \bar{G}^{*}$ and $\bar{\kappa} = \bar{\kappa}^{*}$.

For bond holdings this has to imply:

$$\bar{F} = \bar{F}* = 0 \qquad (3.31)$$

[20] Other choices of the initial steady state would have been possible, and the model results would have been affected. However, this model is not meant to study the effects of an uneven initial wealth distribution - which would have characterized any other steady state than the symmetric one. Also, it turns out that the impulse reponse analysis in chapter 6 still gives valuable insights in what would have been the effects of such an uneven initial distribution. It namely turns out that an unanticipated money shock will lead to such a redistribution of wealth among countries.

Combining (S3) and (S4) and using that consumption is the same in both countries yields that the real money stock must be the same in both countries:

$$\frac{\overline{M}}{\overline{P}} = \frac{\overline{M}^*}{\overline{P}^*} = \chi\overline{C}\left(\frac{1+\overline{r}}{\overline{r}}\right) \tag{3.32}$$

Furthermore, in this symmetric steady state the price of a domestic and a foreign product measured in the same currency has to be the same, and equal to the two price indices. Use $\overline{P}$ as the numeraire:

$$\overline{p}(h) = \overline{\mathcal{E}}\overline{p}^*(f) = \overline{P} = \overline{\mathcal{E}}\overline{P}^* = 1 \tag{3.33}$$

By plugging in (3.33) into (3.32) we see that the choice of $\overline{\mathcal{E}}$ is also arbitrary as it has no real effects, and that we can set $\overline{\mathcal{E}} = 1$ for simplicity, to get $\overline{M} = \overline{M}^*$ and $\overline{P} = \overline{P}^*$, i.e. that also the two nominal steady state money stocks are the same.

(S10) and (S11) yields:

$$\overline{y} = \overline{y}^* = \overline{C} + \overline{G} \tag{3.34}$$

(S5) and (S6) yields that due to the symmetry, population weighted averages of per capita consumption and government spending are equal to the per capita level in each country:

$$\overline{C}^w = \overline{C} = \overline{C}^* \tag{3.35}$$
$$\overline{G}^w = \overline{G} = \overline{G}^* \tag{3.36}$$

Obstfeld and Rogoff (1995a) assume that government spending is equal to zero in both countries as it simplifies calculations to a great extent. Also, qualitative conclusions do not change. However, since we want to calibrate the model later and compare with real data, it would be an advantage to extend the model and open for a situation where government spending differs from zero in steady state and instead find an estimate for the relative size between private and public spending. Plug (3.34)-(3.36) into (SS14) and solve for C_t:

$$
\begin{aligned}
&(\overline{C}+\overline{G})^{\frac{\theta+1}{\theta}} = \overline{C}^{-1}\left(\frac{\theta-1}{\theta\overline{\kappa}}\right)\left(\overline{C}+\overline{G}\right)^{\frac{1}{\theta}} \\
&\Leftrightarrow \overline{C}+\overline{G} = \overline{C}^{-1}\left(\frac{\theta-1}{\theta\overline{\kappa}}\right) \\
&\Leftrightarrow \overline{C}^2 + \overline{G}\,\overline{C} - \frac{\theta-1}{\theta\overline{\kappa}} = 0 \\
&\Leftrightarrow \overline{C} = -\frac{\overline{G}}{2} \pm \sqrt{\left(\frac{\overline{G}}{2}\right)^2 + \frac{\theta-1}{\theta\overline{\kappa}}}
\end{aligned}
\tag{3.37}
$$

It has earlier been shown that $\theta > 1$. It is then clear that the argument of the square root exceeds $\overline{G}/2$. Thus it can be guaranteed that there is a unique positive solution:

$$
\Leftrightarrow \overline{C} = -\frac{\overline{G}}{2} + \sqrt{\left(\frac{\overline{G}}{2}\right)^2 + \frac{\theta-1}{\theta\overline{\kappa}}} \tag{3.38}
$$

After calculating (3.38), all the other variables follow recursively from the equations above. $\overline{W}$ and $\overline{W}^*$ follow directly from (S16) and (S17) – note that $\overline{W} = \overline{W}^*$ in the model. However, the steady state variables $\overline{W}$ and $\overline{W}^*$ have no real importance, since they are only scale parameters in the welfare functions ([E16] and [E17]). The consumer's maximization problem (3.14) is of course unchanged if utility in every single period is scaled up by some positive parameter. We therefore set $\overline{W} = \overline{W}^* = 1$ for simplicity.

Finally, notice in particular that combining (3.37) and (3.34) yields

$$
\overline{y} = \overline{C} + \overline{G} = \left(\frac{\theta-1}{\theta\overline{\kappa}\overline{C}}\right) \tag{3.39}
$$

3.3.3 Log-linearizing the equilibrium around the steady state

This section log-linearizes the equations (E1)-(E17) using the methods from section 2.3. The notation used will also be the same. The objective is to transform the (E1)-(E17) into linear equations in the log-deviations, or approximately percentage deviations from steady state. The approximate equality symbol ≈ is in the following dropped to avoid tedious notation. Note that the real variables that denote per capita sizes, also can be interpreted as aggregate sizes after log-linearizing[21].

The real interest rate r_t is already measured as a percentage. To ease the interpretation define $\tilde{\hat{r}}_t \equiv \bar{r}\hat{r}_t$, i.e. approximately the *absolute* deviation from its steady state, as suggested in section 2.3, equation (2.18). Domestic bond holdings, F_t, has steady state zero. The percentage deviation from steady state is not defined, let therefore $\tilde{\hat{F}}_t$ be defined as $\lim_{\bar{F}\to 0}\left(\frac{\bar{F}}{\bar{y}}\hat{F}_t\right)$, in accordance with the discussion in section 2.3, equation (2.15). $\tilde{\hat{F}}_t$ must therefore be interpreted as domestic bond holdings measured as a percentage of domestic production (this model's GNP).

Remember from (3.33) that the prices, price indices and the exchange rate all have a steady state of 1 and can be dropped whenever appearing multiplicatively. Define for convenience the following shortcuts:

$$\bar{c} \equiv \bar{C} / \bar{y} \qquad (3.40)$$

$$\bar{g} \equiv \bar{G} / \bar{y} \qquad (3.41)$$

Apply (3.34) to see that (3.40)-(3.41) satisfy $\bar{c} + \bar{g} = 1$ and can be interpreted as the private and public share of GNP, respectively. Due to the Ricardian equivalence of the model (cf. section 3.3.1), $\bar{g}$ will also be the net tax rate.

[21] Example: C_t denotes per capita domestic consumption. nC_t denotes aggregate domestic consumption. Since $\hat{C}_t = \ln C_t - \ln \bar{C} = \ln(nC_t) - \ln(n\bar{C})$, it is clear that $\hat{C}_t$ can be interpreted as both a per capita and an aggregate measure.

It is time to introduce formally the stochastic processes that determine the exogenous variables nominal money stock, government spending and technology. It is assumed that the log deviations of all six variables follow AR(1)-processes[22] with i.i.d. (identically and independently distributed) error terms $\varepsilon_{w,t}$ – and assume for simplicity that the error terms are normally distributed:

$$\hat{W}_{t+1} = \rho_w \hat{W}_t + \varepsilon_{w,t+1}; \quad \begin{aligned} &\varepsilon_{w,t+1} \sim N(0,\sigma_w^2) \\ &E_t\{\varepsilon_{w,t+1}\varepsilon_{w,t+s}\} = 0 \,\forall s \neq 1 \\ &E_t\{\varepsilon_{v,t+1}\varepsilon_{w,t+1}\} = 0 \\ &v \neq w;\;\; V,W = G,G^*,\kappa,\kappa^*,M,M^* \end{aligned} \tag{3.42}$$

ρ_w denotes the coefficient of autocorrelation. This process is chosen since it opens for a high degree of generality[23]. However, it is assumed for simplicity that there is no correlation between government spending, technology and the money stock. Also assume that $\rho_G = \rho_{G^*}, \rho_\kappa = \rho_{\kappa^*}$ and $\rho_M = \rho_{M^*}$. In other words, it is assumed in line with the per capita symmetry introduced in section 3.3.2 that there is no asymmetry between the two countries in how the three exogenous variables behave.

To rule out unreasonable solutions with speculative bubbles in the price level or with infinite borrowing, it is necessary to impose the following two assumptions on the system:

$$\lim_{T\to\infty}\left[\left(\frac{1}{1+\bar{r}}\right)^T E_t\{\hat{P}_T\}\right] = \lim_{T\to\infty}\left[\left(\frac{1}{1+\bar{r}}\right)^T E_t\{\hat{P}_T^*\}\right] = 0 \tag{3.43}$$

$$\lim_{T\to\infty}\left[\left(\frac{1}{1+\bar{r}}\right)^T \tilde{\hat{F}}_T\right] = 0 \tag{3.44}$$

[22] Note that since $\hat{W}_{t+1} \approx (W_t - \overline{W})/\overline{W}$, multiplying (3.42) with $\overline{W}$ yields approximately $(W_{t+1} - \overline{W}) = \rho_w (W_{t+1} - \overline{W}) + \overline{W}\varepsilon_{W,t+1}$. I.e. (3.42) is equivalent to an AR(1)-process for *absolute* deviation from steady state.

(3.43) states that it is unreasonable that agents should expect prices to grow exponentially in the future[24]. This bizarre situation could only occur as a self-inforcing *bubble* and is here ruled out. Since the nominal money stock is expected to remain constant on average, it is reasonable also to expect no inflation in the long run. Note that subtracting the foreign version of (3.43) from the domestic, leads to the result that there should be no speculative bubbles in the exchange rate.

(3.44) is a combination of the no-Ponzi-game-condition (NPG) and the transversality condition (TVC)[25]. The no-Ponzi-game condition rules out situations where agents borrow infinitely and consume infinitely, always take up new loans to repay the old ones and never pay back. Credit restrictions are not explicitly modelled here, but we instead assume that the discounted value of bond holdings have to be positive in an infinite time horizon. The transversality condition states that since the marginal utility of consumption is always positive, it cannot be optimal to leave something behind in an infinite time horizon – utility would always be higher if spending it. Therefore the discounted value of bond holdings in an infinite time horizon cannot be positive. The transversality condition is not an assumption - it is more a result of our choice of utility function, with positive marginal utility for all C. The only possibility when combining the no-Ponzi-game condition and the transversality condition is (3.44), that the discounted value of bond holding in an infinite time horizon is zero. This is quite reasonable, it just states that all debt has to be repaid someday, and that all wealth someday will be spent.

The complete list of the log-linearized equations of the model, denoted (L1)-(L17), and the six AR(1)-processes, denoted (A1)-(A6), follow below, numbered in the same order as the equilibrium equations and the steady state equations. The derivations follow directly from the methods presented in section 2.3 and (A1)-(A6) directly from (3.42). Cf. appendix III for details.

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[23] Obstfeld and Rogoff (1995a) analyze only purely temporary and fully permanent shocks. (3.42) opens for the intermediate case, and the possibility of calibrating ρ to real data.
[24] Note that the first term within the brackets of (3.43) goes to zero exponentially. Prices would have to grow exponentially and even faster to prevent the expression from going to zero. Cf. also Obstfeld and Rogoff (1996), pp. 518-519.
[25] Cf. Obstfeld and Rogoff (1996), pp. 64-66 for a detailed discussion on these constraints.

Log-linearized equations

(L1) $E_t\left\{\hat{C}_{t+1}\right\} = \hat{C}_t + \frac{1}{1+\bar{r}}\tilde{\hat{r}}_t$

(L2) $E_t\left\{\hat{C}^*_{t+1}\right\} = \hat{C}^*_t + \frac{1}{1+\bar{r}}\tilde{\hat{r}}_t$

(L3) $\hat{M}_t - \hat{P}_t = \hat{C}_t - \frac{1}{\bar{r}(1+\bar{r})}\tilde{\hat{r}}_t - \frac{1}{\bar{r}}\left(E_t\{\hat{P}_{t+1}\} - \hat{P}_t\right)$

(L4) $-\hat{P}^*_t = \hat{C}^*_t - \frac{1}{\bar{r}(1+\bar{r})}\tilde{\hat{r}}_t - \frac{1}{\bar{r}}\left(E_t\{\hat{P}^*_{t+1}\} - \hat{P}^*_t\right)$

(L5) $\hat{C}^w_t = n\hat{C}_t + (1-n)\hat{C}^*_t$

(L6) $\hat{G}^w_t = n\hat{G}_t$

(L7) $\hat{P}_t = \hat{\mathcal{E}}_t + \hat{P}^*_t$

(L8) $\hat{P}_t = n\hat{p}_{t-1}(h) + (1-n)\,\hat{p}^*_{t-1}(f) + (1-n)\hat{\mathcal{E}}_t$

(L9) $\hat{P}^*_t = n\hat{p}_{t-1}(h) + (1-n)\,\hat{p}^*_{t-1}(f) - n\hat{\mathcal{E}}_t$

(L10) $\tilde{\hat{F}}_t = (1+\bar{r})\tilde{\hat{F}}_{t-1} + \hat{p}_t(h) - \hat{P}_t + \hat{y}_t - \bar{c}\hat{C}_t - \bar{g}\hat{G}_t$

(L11) $-\frac{n}{1-n}\tilde{\hat{F}}_t = -\frac{n}{1-n}(1+\bar{r})\tilde{\hat{F}}_{t-1} + \hat{p}^*_t(f) - \hat{P}^*_t + \hat{y}^*_t - \bar{c}\hat{C}^*_t$

(L12) $\hat{y}_t = \theta\left(\hat{P}_t - \hat{p}_{t-1}(h)\right) + \bar{c}\hat{C}^w_t + \bar{g}\hat{G}^w_t$

(L13) $\hat{y}^*_t = \theta\left(\hat{P}^*_t - \hat{p}^*_{t-1}(f)\right) + \bar{c}\hat{C}^w_t + \bar{g}\hat{G}^w_t$

(L14) $\hat{p}_t(h) = E_t\left\{\hat{\kappa}_{t+1} + \hat{y}_{t+1} + \hat{P}_{t+1} + \hat{C}_{t+1}\right\}$

(L15) $\hat{p}^*_t(f) = E_t\left\{\hat{y}^*_{t+1} + \hat{P}^*_{t+1} + \hat{C}^*_{t+1}\right\}$

(L16) $\hat{W}_t = \hat{C}_t + \chi(\hat{M}_t - \hat{P}_t) - \left(\frac{\theta-1}{\theta\bar{c}}\right)\left(\frac{\hat{\kappa}_t}{2} + \hat{y}_t\right)$

(L17) $\hat{W}^*_t = \hat{C}^*_t + \chi(\hat{M}^*_t - \hat{P}^*_t) - \left(\frac{\theta-1}{\theta\bar{c}}\right)\left(\frac{\hat{\kappa}^*_t}{2} + \hat{y}^*_t\right)$

(A1) $\hat{G}_{t+1} = \rho_G\hat{G}_t + \varepsilon_{G,t+1}$

(A2) $\hat{G}^*_{t+1} = \rho_G\hat{G}^*_t + \varepsilon_{G^*,t+1}$

(A3) $\hat{\kappa}_{t+1} = \rho_\kappa\hat{\kappa}_t + \varepsilon_{\kappa,t+1}$

(A4) $\hat{\kappa}^*_{t+1} = \rho_\kappa\hat{\kappa}^*_t + \varepsilon_{\kappa^*,t+1}$

(A5) $\hat{M}_{t+1} = \rho_M\hat{M}_t + \varepsilon_{M,t+1}$

(A6) $\hat{M}^*_{t+1} = \rho_M\hat{M}^*_t + \varepsilon_{M,t+1}$

4. MODIFYING THE MODEL TO FIT INTO UHLIG'S CANONICAL FORM

The equations (L1)-(L17) and (A1)-(A6) seem to already be of the canonical form from chapter 2, cf. equations (2.1)-(2.3). However, these equations alone do not describe the model, since the solution also is constrained by the assumptions (3.43)-(3.45). The next step will be to modify the log-linearized equations (L1)-(L15) in such a way that (3.43)-(3.44) can be imposed, and so that the resulting equations still fit into the form (2.1)-(2.3).

The NPG- and TVC-conditions will e.g. be imposed by moving the intertemporal budget constraint repeatedly forward, plugging into itself, and finally taking the limit as time goes to infinity.

4.1 SOME INITIAL CALCULATIONS

At first we will introduce some new notation - for simplicity. Let superscript Δ denote the difference between the domestic and foreign version of a variable as in (4.1):

$$\begin{aligned} &\hat{C}_t^{\Delta} \equiv \hat{C}_t - \hat{C}_t^*; \qquad && \hat{y}_t^{\Delta} \equiv \hat{y}_t - \hat{y}_t^*; \qquad && \hat{p}_t^{\Delta} \equiv \hat{p}_t(h) - \hat{p}_t^*(f) \\ &\hat{G}_t^{\Delta} \equiv \hat{G}_t - \hat{G}_t^*; && \hat{\kappa}_t^{\Delta} \equiv \hat{\kappa}_t - \hat{\kappa}_t^*; && \hat{M}_t^{\Delta} \equiv \hat{M}_t - \hat{M}_t^* \end{aligned} \tag{4.1}$$

Call the three variables on top *consumption difference*, *output difference* and *price difference*[26] from now on. For the difference in price indices, we already have notation from (L7), namely $\hat{\mathcal{E}}_t = \hat{P}_t - \hat{P}_t^*$.

[26] The *price difference* has no interesting interpretation since the prices are measured in different currencies. However, the definition will be very useful for computational purposes.

Let superscript *w* as before denote the population-weighted average of the domestic and foreign version of a variable and introduce the following three new definitions:

$$\hat{y}_t^w \equiv n\hat{y}^t + (1-n)\hat{y}_t^*; \quad \hat{\kappa}_t^w \equiv n\hat{\kappa}_t + (1-n)\hat{\kappa}_t^*; \quad \hat{M}_t^w \equiv n\hat{M}_t + (1-n)\hat{M}_t^* \tag{4.2}$$

Call the first variable *world output*.

Now we will make use of the fact that most of the equations (L1)-(L15) appear in pairs of one domestic and one foreign equation. Subtract the foreign equation from the domestic equation in the pairs (L1)-(L2), (L3)-(L4), (L10)-(L11), (L12-L13) and (L14-L15):

$$\text{(L1)} \div \text{(L2)}: \qquad E_t\left\{\hat{C}_{t+1}^{\Delta}\right\} = \hat{C}_t^{\Delta} \tag{4.3}$$

$$\text{(L3)} \div \text{(L4)}: \qquad \hat{M}_t - \hat{\mathcal{E}}_t = \hat{C}_t^{\Delta} - \frac{1}{\bar{r}}\left(E_t\{\hat{\mathcal{E}}_{t+1}\} - \hat{\mathcal{E}}_t\right) \tag{4.4}$$

$$\text{(L10)} \div \text{(L11)}: \qquad \frac{\tilde{\hat{F}}_t}{1-n} = \frac{1+\bar{r}}{1-n}\tilde{\hat{F}}_{t-1} + \hat{p}_{t-1}^{\Delta} - \hat{\mathcal{E}}_t + \hat{y}_t^{\Delta} - \bar{c}\hat{C}_t^{\Delta} - \bar{g}\hat{G}_t^{\Delta} \tag{4.5}$$

$$\text{(L12)} \div \text{(L13)}: \qquad \hat{y}_t^{\Delta} = \theta(\hat{\mathcal{E}}_t - \hat{p}_{t-1}^{\Delta}) \tag{4.6}$$

$$\text{(L14)} \div \text{(L15)}: \qquad \hat{p}_t^{\Delta} = E_t\left\{\hat{\kappa}_{t+1}^{\Delta} + \hat{y}_{t+1}^{\Delta} + \hat{\mathcal{E}}_{t+1} + \hat{C}_{t+1}^{\Delta}\right\}$$

$$\text{apply (4.3)}: \quad \Leftrightarrow \hat{p}_t^{\Delta} = E_t\left\{\hat{\kappa}_{t+1}^{\Delta} + \hat{y}_{t+1}^{\Delta} + \hat{\mathcal{E}}_{t+1} + \hat{C}_t^{\Delta}\right\} \tag{4.7}$$

(4.3) is an interesting result - it states that the consumption difference is expected to remain the same as it is. This means that if there occurs an unanticipated shock in period *t*, the consumption difference will typically change from period *t – 1* to period *t*, but it will remain the same from period t and on absent new unanticipated shocks. In other words, shocks can have permanent effects on the difference between domestic and foreign per capita consumption. This does not mean that agents necessarily have flat consumption profiles after a shock. This will only be the case if the real interest rate is at its steady state level. But it follows from the consumption Euler equations (L1) and (L2), that if the real interest rate deviates from steady state, this will have the same effect on domestic and foreign consumption growth – keeping a stable difference. This actually turns out to be a fundamental result of the model. But these effects will be discussed more in detail when analysing impulse responses in chapter 6.

Secondly, compute a population weighted average of the pairs of equations (L1)-(L2), (L3)-(L4), (L12-L13) and (L14-L15), i.e. multiply the domestic equation with *n* and the foreign equation with *1 – n* and then add them together:

$$n(\text{L1}) + (1\text{-}n)(\text{L2}): \qquad E_t\left\{\hat{C}^w_{t+1}\right\} = \hat{C}^w_t + \frac{1}{1+\bar{r}}\tilde{\hat{r}}_t \tag{4.8}$$

$$n(\text{L3}) + (1\text{-}n)(\text{L4}): \qquad n\hat{M}_t =$$

$$\hat{C}^w_t - \frac{1}{\bar{r}(1+\bar{r})}\tilde{\hat{r}}_t - \frac{1}{\bar{r}}E_t\left\{n\hat{P}_{t+1} + (1-n)\hat{P}^*_{t+1}\right\} + \frac{1+\bar{r}}{\bar{r}}\left[n\hat{P}_t + (1-n)\hat{P}^*_t\right] \tag{4.9}$$

$$n(\text{L8}) + (1\text{-}n)(\text{L9}): \qquad n\hat{P}_t + (1-n)\hat{P}^*_t = \hat{p}^w_{t-1} \tag{4.10}$$

$$n(\text{L12}) + (1\text{-}n)(\text{L13}): \quad \hat{y}^w_t = \theta(n\hat{P}_t + (1-n)\hat{P}^*_t - n\hat{p}_{t-1}(h) - (1-n)\hat{p}^*_{t-1}(f)) + \bar{c}\hat{C}^w_t + \bar{g}\hat{G}^w_t$$

plug in (L8) - (L9) :

$$\Leftrightarrow \hat{y}^w_t = \theta\Big[n\left(n\hat{p}_{t-1}(h) + (1-n)\hat{p}^*_{t-1}(f) + (1-n)\hat{\varepsilon}_t\right) + (1-n)\left(n\hat{p}_{t-1}(h) + (1-n)\hat{p}^*_{t-1}(f) + -n\hat{\varepsilon}_t\right)$$

$$- n\hat{p}_{t-1}(h) - (1-n)\hat{p}^*_{t-1}(f)\Big] + \bar{c}\hat{C}^w_t + \bar{g}\hat{G}^w_t$$

$$\Leftrightarrow \hat{y}^w_t = \bar{c}\hat{C}^w_t + \bar{g}\hat{G}^\Delta_t \tag{4.11}$$

$$n(\text{L14}) + (1\text{-}n)(\text{L15}): \quad \hat{p}^w_t = E_t\left\{\hat{\kappa}^\Delta_{t+1} + \hat{y}^w_{t+1} + n\hat{P}_{t+1} + (1-n)\hat{P}^*_{t+1} + \hat{C}^w_{t+1}\right\}$$

$$\Leftrightarrow 0 = E_t\left\{\hat{\kappa}^\Delta_{t+1} + \hat{y}^w_{t+1} + \hat{C}^w_{t+1}\right\} \tag{4.12}$$

4.2 CALCULATIONS TO INCLUDE NPG, TVC AND NO-BUBBLES ASSUMPTIONS

Note that in the following the law of iterated expectations and formulas for the infinite sum of the elements in a geometric series will be used repeatedly. It will also be used several times from (A1)-(A6) that the AR(1)-processes satisfy the following properties:

$$E_t\{\hat{V}_s\} = \rho_V^{s-t}\hat{V}_t; \qquad\qquad s > t; \quad V = G, G^*, \kappa, \kappa^*, M, M^*$$

$$E_t\{\hat{W}^\Delta_s\} = \rho_W^{s-t}\hat{W}^\Delta_t; \qquad E_t\{\hat{W}^w_s\} = \rho_W^{s-t}\hat{W}^w_t \qquad s > t; \quad W = G, \kappa, M \tag{4.13}$$

where the last two equalities follow directly using the definitions of $\hat{W}_s^{\Delta}$ and $\hat{W}_s^{w}$.

Solve (4.4) for $\hat{\varepsilon}_t$:

$$(4.4) \Rightarrow \hat{\varepsilon}_t = \frac{\bar{r}}{1+\bar{r}}(\hat{M}_t^{\Delta} - \hat{C}_t^{\Delta}) + \frac{1}{1+\bar{r}} E_t\{\hat{\varepsilon}_{t+1}\} \tag{4.14}$$

Substitute out $\hat{\varepsilon}_{t+1}$ in (4.14) by plugging in the equation forwarded one period:

$$\hat{\varepsilon}_t = \frac{\bar{r}}{1+\bar{r}}(\hat{M}_t^{\Delta} - \hat{C}_t^{\Delta}) + \frac{1}{1+\bar{r}} E_t\left\{\frac{\bar{r}}{1+\bar{r}}(\hat{M}_{t+1}^{\Delta} - \hat{C}_{t+1}^{\Delta}) + \frac{1}{1+\bar{r}} E_{t+1}\{\hat{\varepsilon}_{t+2}\}\right\}$$

$$\Leftrightarrow \hat{\varepsilon}_t = \frac{\bar{r}}{1+\bar{r}}\left(\hat{M}_t^{\Delta} - \hat{C}_t^{\Delta} + \frac{1}{1+\bar{r}} E_t\{\hat{M}_{t+1}^{\Delta} - \hat{C}_{t+1}^{\Delta}\}\right) + \left(\frac{1}{1+\bar{r}}\right)^2 E_t\left\{\hat{\varepsilon}_{t+2}\right\} \tag{4.15}$$

Repeated substitution yields:

$$\hat{\varepsilon}_t = \frac{\bar{r}}{1+\bar{r}} E_t \sum_{s=t}^{T}\left[\left(\frac{1}{1+\bar{r}}\right)^{s-t}(\hat{M}_s^{\Delta} - \hat{C}_s^{\Delta})\right] + \left(\frac{1}{1+\bar{r}}\right)^{T} E_t\left\{\hat{\varepsilon}_{t+T}\right\} \tag{4.16}$$

Apply from (4.3) that $E_t\{\hat{C}_s^{\Delta}\} = \hat{C}_t^{\Delta}$ and take the limit of (4.16) as $T \to \infty$:

$$\lim_{T\to\infty} \hat{\varepsilon}_t = \frac{\bar{r}}{1+\bar{r}} \lim_{T\to\infty} E_t \sum_{s=t}^{T}\left[\left(\frac{1}{1+\bar{r}}\right)^{s-t}(\rho_M^{s-t}\hat{M}_t^{\Delta} - \hat{C}_t^{\Delta})\right] + \lim_{T\to\infty}\left[\left(\frac{1}{1+\bar{r}}\right)^{T} E_t\{\hat{\varepsilon}_{t+T}\}\right]$$

$$\Leftrightarrow \hat{\varepsilon}_t = \frac{\bar{r}}{1+\bar{r}}\left[\lim_{T\to\infty}\sum_{s=t}^{T}\left(\frac{\rho_M}{1+\bar{r}}\right)^{s-t}\hat{M}_t^{\Delta} - \lim_{T\to\infty}\sum_{s=t}^{T}\left(\frac{1}{1+\bar{r}}\right)^{s-t}\hat{C}_t^{\Delta}\right] + \lim_{T\to\infty}\left[\left(\frac{1}{1+\bar{r}}\right)^{T} E_t\{\hat{\varepsilon}_{t+T}\}\right]$$

$$\Leftrightarrow \hat{\mathcal{E}}_t = \frac{\bar{r}}{1+\bar{r}}\left(\frac{1+\bar{r}}{1+\bar{r}-\rho_M}\hat{M}_t^{\Delta} - \frac{1+\bar{r}}{\bar{r}}\hat{C}_t^{\Delta}\right) + \lim_{T\to\infty}\left[\left(\frac{1}{1+\bar{r}}\right)^T E_t\{\hat{\mathcal{E}}_{t+T}\}\right]$$

$$\Leftrightarrow \hat{\mathcal{E}}_t = \bar{r}\tilde{\rho}_M \hat{M}_t^{\Delta} - \hat{C}_t^{\Delta} \tag{4.17}$$

$$\text{where}\,\tilde{\rho}_M \equiv \frac{1}{1+\bar{r}-\rho_M}$$

The standard formulas for the sum of the elements in infinite geometric series have been applied and also in the last step, the assumption of no speculative bubbles in the exchange rate (3.43).

Plug (4.6) into (4.7) and solve for $E_t\{\hat{y}_{t+1}^{\Delta}\}$:

$$(4.9) \Rightarrow 0 = E_t\left\{\hat{\kappa}_{t+1}^{\Delta} + \hat{y}_{t+1}^{\Delta} + \hat{\mathcal{E}}_{t+1} - \hat{p}_t^{\Delta} + \hat{C}_{t+1}^{\Delta}\right\}$$

$$\Leftrightarrow 0 = E_t\left\{\hat{\kappa}_{t+1}^{\Delta} + \hat{y}_{t+1}^{\Delta}\left(\frac{\theta+1}{\theta}\right) + \hat{C}_{t+1}^{\Delta}\right\}$$

$$\Leftrightarrow E_t\{\hat{y}_{t+1}^{\Delta}\} = -\frac{\theta}{\theta+1}E_t\left\{\hat{\kappa}_{t+1}^{\Delta} + \hat{C}_{t+1}^{\Delta}\right\} \tag{4.18}$$

Forward (4.5) one period, solve for $\tilde{\hat{F}}_t$ and take the expectation conditional on information known in period *t* on both sides of the equation:

$$(4.5) \Rightarrow \tilde{\hat{F}}_t = E_t\left\{\frac{1}{1+\bar{r}}\tilde{\hat{F}}_{t+1} - \frac{1-n}{1+\bar{r}}\left[\hat{p}_t^{\Delta} - \hat{\mathcal{E}}_{t+1} + \hat{y}_{t+1}^{\Delta} - \bar{c}\hat{C}_{t+1}^{\Delta} - \bar{g}\hat{G}_{t+1}^{\Delta}\right]\right\} \tag{4.19}$$

Plug (4.6) and (4.18) into (4.19) to substitute out $\hat{p}_t - \hat{\mathcal{E}}_{t+1}$ and $\hat{y}_{t+1}^{\Delta}$:

$$(4.18) \Rightarrow \tilde{\hat{F}}_t = E_t\left\{\frac{1}{1+\bar{r}}\tilde{\hat{F}}_{t+1} - \frac{1-n}{1+\bar{r}}\left[\frac{\theta-1}{\theta}\hat{y}_{t+1}^{\Delta} - \bar{c}\hat{C}_{t+1}^{\Delta} - \bar{g}\hat{G}_{t+1}^{\Delta}\right]\right\}$$

$$\Leftrightarrow \tilde{\hat{F}}_t = E_t\left\{\frac{1}{1+\bar{r}}\tilde{\hat{F}}_{t+1} - \frac{1-n}{1+\bar{r}}\left[-\left(\frac{\theta-1}{\theta+1}\right)(\hat{C}_{t+1}^{\Delta} + \hat{\kappa}_{t+1}^{\Delta}) - \bar{c}\hat{C}_{t+1}^{\Delta} - \bar{g}\hat{G}_{t+1}^{\Delta}\right]\right\}$$

$$\Leftrightarrow \tilde{\hat{F}}_t = E_t\left\{\frac{1}{1+\bar{r}}\tilde{\hat{F}}_{t+1} + \frac{1-n}{1+\bar{r}}\left[\frac{\theta-1}{\theta+1}\hat{\kappa}_{t+1}^{\Delta} + \bar{g}\hat{G}_{t+1}^{\Delta} + \left(\bar{c} + \frac{\theta-1}{\theta+1}\right)\hat{C}_{t+1}^{\Delta}\right]\right\} \tag{4.20}$$

Plug (4.20) into equation (4.5):

$$E_t\left\{\frac{1}{1+\bar{r}}\tilde{\hat{F}}_{t+1}+\frac{1-n}{1+\bar{r}}\left[\frac{\theta-1}{\theta+1}\hat{\kappa}^{\Delta}_{t+1}+\bar{g}\hat{G}^{\Delta}_{t+1}+\left(\bar{c}+\frac{\theta-1}{\theta+1}\right)\hat{C}^{\Delta}_{t+1}\right]\right\}=$$
$$(1+\bar{r})\tilde{\hat{F}}_{t-1}+(1-n)\left[(\theta-1)\hat{\mathcal{E}}_t-(\theta-1)\hat{p}^{\Delta}_{t-1}-\bar{c}\hat{C}^{\Delta}_t-\bar{g}\hat{G}^{\Delta}_t\right]$$
$$\Leftrightarrow\frac{1-n}{1+\bar{r}}\left[\frac{\theta-1}{\theta+1}\rho_\kappa\hat{\kappa}^{\Delta}_t+\bar{g}\rho_G\hat{G}^{\Delta}_t+\left(\bar{c}+\frac{\theta-1}{\theta+1}\right)\hat{C}^{\Delta}_t\right]+E_t\left\{\frac{1}{1+\bar{r}}\tilde{\hat{F}}_{t+1}\right\}=$$
$$(1+\bar{r})\tilde{\hat{F}}_{t-1}+(1-n)\left[(\theta-1)\hat{\mathcal{E}}_t-(\theta-1)\hat{p}^{\Delta}_{t-1}-\bar{c}\hat{C}^{\Delta}_t-\bar{g}\hat{G}^{\Delta}_t\right] \qquad (4.21)$$

Forward (4.20) one period and plug into (4.21):

$$\frac{1-n}{1+\bar{r}}\left[\frac{\theta-1}{\theta+1}\rho_\kappa\left(1+\frac{\rho_\kappa}{1+\bar{r}}\right)\hat{\kappa}^{\Delta}_t+\bar{g}\rho_G\left(1+\frac{\rho_G}{1+\bar{r}}\right)\hat{G}^{\Delta}_t+\left(\bar{c}+\frac{\theta-1}{\theta+1}\right)\left(1+\frac{1}{1+\bar{r}}\right)\hat{C}^{\Delta}_t\right]+E_t\left\{\left(\frac{1}{1+\bar{r}}\right)^2\tilde{\hat{F}}_{t+2}\right\}=$$
$$(1+\bar{r})\tilde{\hat{F}}_{t-1}+(1-n)\left[(\theta-1)\hat{\mathcal{E}}_t-(\theta-1)\hat{p}^{\Delta}_{t-1}-\bar{c}\hat{C}^{\Delta}_t-\bar{g}\hat{G}^{\Delta}_t\right] \qquad (4.22)$$

Repeated plugging in yields:

$$\frac{1-n}{1+\bar{r}}\left[\frac{\theta-1}{\theta+1}\rho_\kappa\sum_{s=0}^{T}\left(\frac{\rho_\kappa}{1+\bar{r}}\right)^s\hat{\kappa}^{\Delta}_t+\bar{g}\rho_G\sum_{s=0}^{T}\left(\frac{\rho_G}{1+\bar{r}}\right)^s\hat{G}^{\Delta}_t+\left(\bar{c}+\frac{\theta-1}{\theta+1}\right)\sum_{s=0}^{T}\left(\frac{1}{1+\bar{r}}\right)^s\hat{C}^{\Delta}_t\right]+E_t\left\{\left(\frac{1}{1+\bar{r}}\right)^T\tilde{\hat{F}}_{t+T}\right\}=$$
$$(1+\bar{r})\tilde{\hat{F}}_{t-1}+(1-n)\left[(\theta-1)\hat{\mathcal{E}}_t-(\theta-1)\hat{p}^{\Delta}_{t-1}-\bar{c}\hat{C}^{\Delta}_t-\bar{g}\hat{G}^{\Delta}_t\right] \qquad (4.23)$$

Take the limit of (4.23) as $T\to\infty$ and apply the NPG+TVC-assumption (3.42):

$$1-n\left[\frac{\theta-1}{\theta+1}\left(\frac{\rho_\kappa}{1+\bar{r}-\rho_\kappa}\right)\hat{\kappa}^{\Delta}_t+\bar{g}\left(\frac{\rho_G}{1+\bar{r}-\rho_G}\right)\hat{G}^{\Delta}_t+\left(\bar{c}+\frac{\theta-1}{\theta+1}\right)\frac{1}{\bar{r}}\hat{C}^{\Delta}_t\right]=$$
$$(1+\bar{r})\tilde{\hat{F}}_{t-1}+(1-n)\left[(\theta-1)\hat{\mathcal{E}}_t-(\theta-1)\hat{p}^{\Delta}_{t-1}-\bar{c}\hat{C}^{\Delta}_t-\bar{g}\hat{G}^{\Delta}_t\right]$$
$$\Leftrightarrow\frac{\theta-1}{\theta+1}\tilde{\rho}_\kappa\hat{\kappa}^{\Delta}_t+\bar{g}\left(\tilde{\rho}_G+1\right)\hat{G}^{\Delta}_t+\left[\frac{\theta-1}{(\theta+1)\bar{r}}+\bar{c}\,\frac{1+\bar{r}}{\bar{r}}\right]\hat{C}^{\Delta}_t=\frac{1+\bar{r}}{1-n}\tilde{\hat{F}}_{t-1}+(\theta-1)\hat{\mathcal{E}}_t-(\theta-1)\hat{p}^{\Delta}_{t-1} \qquad (4.24)$$

where $\tilde{\rho}_\kappa\equiv\dfrac{\rho_\kappa}{1+\bar{r}-\rho_\kappa}$ and $\tilde{\rho}_G\equiv\dfrac{\rho_G}{1+\bar{r}-\rho_G}$

Plug (4.11) into (4.12):

$$(4.11) \Rightarrow 0 = E_t\left\{\hat{\kappa}^w_{t+1} + (\bar{c}+1)\hat{C}^w_{t+1} + \bar{g}\hat{G}^w_{t+1}\right\}$$
$$\Leftrightarrow E_t\{\hat{C}^w_{t+1}\} = E_t\left\{\frac{-\bar{g}}{\bar{c}+1}\hat{G}^w_{t+1} - \frac{1}{\bar{c}+1}\hat{\kappa}^w_{t+1}\right\} \tag{4.25}$$

Plug (4.25) into the left hand side of (4.8):

$$E_t\left\{\frac{-\bar{g}}{\bar{c}+1}\hat{G}^w_{t+1} - \frac{1}{\bar{c}+1}\hat{\kappa}^w_{t+1}\right\} = \hat{C}^w_t + \frac{1}{1+\bar{r}}\tilde{\hat{r}}_t$$
$$\Leftrightarrow \frac{-\bar{g}\rho_G}{\bar{c}+1}\hat{G}^w_t - \frac{\rho_\kappa}{\bar{c}+1}\hat{\kappa}^w_t = \hat{C}^w_t + \frac{1}{1+\bar{r}}\tilde{\hat{r}}_t \tag{4.26}$$

Solve (4.9) for $n\hat{P}_t + (1-n)\hat{P}^*_t$:

$$n\hat{P}_t + (1-n)\hat{P}^*_t = \frac{\bar{r}}{1+\bar{r}}\hat{M}^w_t - \frac{\bar{r}}{1+\bar{r}}\hat{C}^w_t + \frac{1}{(1+\bar{r})^2}\tilde{\hat{r}}_t + \frac{1}{1+\bar{r}}E_t\left\{n\hat{P}_{t+1} + (1-n)\hat{P}^*_{t+1}\right\} \tag{4.27}$$

Solve (4.26) for $\tilde{\hat{r}}_t$:

$$\tilde{\hat{r}}_t = -(1+\bar{r})\left(\frac{\bar{g}\rho_G}{\bar{c}+1}\hat{G}^w_t + \frac{\rho_\kappa}{\bar{c}+1}\hat{\kappa}^w_t + \hat{C}^w_t\right) \tag{4.28}$$

Plug (4.28) into (4.27):

$$n\hat{P}_t + (1-n)\hat{P}^*_t = \frac{\bar{r}}{1+\bar{r}}\hat{M}^w_t - \frac{\bar{r}}{1+\bar{r}}\hat{C}^w_t - \frac{1}{1+\bar{r}}\left(\frac{\bar{g}\rho_G}{\bar{c}+1}\hat{G}^w_t + \frac{\rho_\kappa}{\bar{c}+1}\hat{\kappa}^w_t + \hat{C}^w_t\right) + \frac{1}{1+\bar{r}}E_t\left\{n\hat{P}_{t+1} + (1-n)\hat{P}^*_{t+1}\right\}$$
$$\Leftrightarrow n\hat{P}_t + (1-n)\hat{P}^*_t = \frac{\bar{r}}{1+\bar{r}}\hat{M}^w_t - \hat{C}^w_t - \frac{1}{1+\bar{r}}\left(\frac{\bar{g}\rho_G}{\bar{c}+1}\hat{G}^w_t + \frac{\rho_\kappa}{\bar{c}+1}\hat{\kappa}^w_t\right) + \frac{1}{1+\bar{r}}E_t\left\{n\hat{P}_{t+1} + (1-n)\hat{P}^*_{t+1}\right\} \tag{4.29}$$

Forward (4.29) one period:

$$n\hat{P}_{t+1} + (1-n)\hat{P}^*_{t+1} = \frac{\bar{r}}{1+\bar{r}}\hat{M}^w_{t+1} - \hat{C}^w_{t+1} - \frac{1}{1+\bar{r}}\left(\frac{\bar{g}\rho_G}{\bar{c}+1}\hat{G}^w_{t+1} + \frac{\rho_\kappa}{\bar{c}+1}\hat{\kappa}^w_{t+1}\right) + \frac{1}{1+\bar{r}}E_{t+1}\left\{n\hat{P}_{t+2} + (1-n)\hat{P}^*_{t+2}\right\} \tag{4.30}$$

Plug (4.30) into (4.29):

$$\begin{aligned} n\hat{P}_t + (1-n)\hat{P}^*_t = {} & \frac{\bar{r}}{1+\bar{r}}\left(1+\frac{\rho_M}{1+\bar{r}}\right)\hat{M}^w_t - \left(\hat{C}^w_t + \frac{1}{1+\bar{r}}E_t\{\hat{C}^w_{t+1}\}\right) \\ & - \frac{1}{1+\bar{r}}\left[\frac{\bar{g}\rho_G}{\bar{c}+1}\left(1+\frac{\rho_G}{1+\bar{r}}\right)\hat{G}^w_t + \frac{\rho_\kappa}{\bar{c}+1}\left(1+\frac{\rho_\kappa}{1+\bar{r}}\right)\hat{\kappa}^w_t\right] + \left(\frac{1}{1+\bar{r}}\right)^2 E_t\left\{n\hat{P}_{t+2} + (1-n)\hat{P}^*_{t+2}\right\} \end{aligned} \tag{4.31}$$

Repeated plugging in yields:

$$\begin{aligned} n\hat{P}_t + (1-n)\hat{P}^*_t = {} & \frac{\bar{r}}{1+\bar{r}}\sum_{s=t}^{T}\left(\frac{\rho_M}{1+\bar{r}}\right)^{s-t}\hat{M}^w_t - \hat{C}^w_t - \sum_{s=t+1}^{T}\left[\left(\frac{1}{1+\bar{r}}\right)^{s-t}E_t\{\hat{C}^w_s\}\right] \\ & - \frac{1}{1+\bar{r}}\left[\frac{\bar{g}\rho_G}{\bar{c}+1}\sum_{s=t}^{T}\left(\frac{\rho_G}{1+\bar{r}}\right)^{s-t}\hat{G}^w_t + \frac{\rho_\kappa}{\bar{c}+1}\sum_{s=t}^{T}\left(\frac{\rho_\kappa}{1+\bar{r}}\right)^{s-t}\hat{\kappa}^w_t\right] + \left(\frac{1}{1+\bar{r}}\right)^T E_t\left\{n\hat{P}_{t+T} + (1-n)\hat{P}^*_{t+T}\right\} \end{aligned} \tag{4.32}$$

Substitute out $E_t\{\hat{C}^w_s\}$ by using (4.25):

$$\begin{aligned} n\hat{P}_t + (1-n)\hat{P}^*_t = {} & \frac{\bar{r}}{1+\bar{r}}\sum_{s=t}^{T}\left(\frac{\rho_M}{1+\bar{r}}\right)^{s-t}\hat{M}^w_t - \hat{C}^w_t + \sum_{s=t+1}^{T}\left(\frac{\rho_G}{1+\bar{r}}\right)^{s-t}\frac{\bar{g}}{\bar{c}+1}\hat{G}^w_t - \sum_{s=t+1}^{T}\left(\frac{\rho_\kappa}{1+\bar{r}}\right)^{s-t}\frac{\rho_\kappa}{\bar{c}+1}\hat{\kappa}^w_t \\ & - \frac{1}{1+\bar{r}}\left[\frac{\bar{g}\rho_G}{\bar{c}+1}\sum_{s=t}^{T}\left(\frac{\rho_G}{1+\bar{r}}\right)^{s-t}\hat{G}^w_t + \frac{\rho_\kappa}{\bar{c}+1}\sum_{s=t}^{T}\left(\frac{\rho_\kappa}{1+\bar{r}}\right)^{s-t}\hat{\kappa}^w_t\right] + \left(\frac{1}{1+\bar{r}}\right)^T E_t\left\{n\hat{P}_{t+T} + (1-n)\hat{P}^*_{t+T}\right\} \end{aligned} \tag{4.33}$$

Take the limit as $T \to \infty$ and apply the assumption of no speculative bubbles in prices, (3.43):

$$\begin{aligned} n\hat{P}_t + (1-n)\hat{P}^*_t = {} & \frac{\bar{r}}{1+\bar{r}-\rho_M}\hat{M}^w_t - \hat{C}^w_t + \left(\frac{\rho_G}{1+\bar{r}-\rho_G}\right)\left(\frac{\bar{g}}{\bar{c}+1}\right)\hat{G}^w_t + \left(\frac{\rho_\kappa}{1+\bar{r}-\rho_\kappa}\right)\left(\frac{\rho_\kappa}{\bar{c}+1}\right)\hat{\kappa}^w_t \\ & - \frac{1}{1+\bar{r}}\left[\left(\frac{\bar{g}\rho_G}{\bar{c}+1}\right)\left(\frac{1+\bar{r}}{1+\bar{r}-\rho_G}\right)\hat{G}^w_t + \left(\frac{\rho_\kappa}{\bar{c}+1}\right)\left(\frac{1+\bar{r}}{1+\bar{r}-\rho_\kappa}\right)\hat{\kappa}^w_t\right] \end{aligned}$$

$$\Leftrightarrow n\hat{P}_t + (1-n)\hat{P}_t^* = \bar{r}\tilde{\rho}_M \hat{M}_t^w - \hat{C}_t^w$$
$$\Leftrightarrow p_{t-1}^w = \bar{r}\tilde{\rho}_M \hat{M}_t^w - \hat{C}_t^w \tag{4.34}$$

Collecting the equations (4.5), (4.6), (4.7), (4.11), (4.12), (4.17), (4.24), (4.26) and (4.34) together with (A1)-(A6) yields a system of the form (2.1)-(2.3) - with 9 equations, 9 endogenous variables and the six exogenous processes (one then also has to *undo* the definitions of differences and population weighted of the exogenous variables, i.e. use [4.1] and [4.2]). However, some variables that might be interesting to analyze have been substituted out. E.g. domestic and foreign consumption no longer appear in the equations – only *consumption difference* and *world consumption*. Use the definitions of *consumption difference* and *world consumption* to verify equations (4.35) and (4.36):

$$\hat{C}_t = \hat{C}_t^w + (1-n)\hat{C}_t^\Delta \tag{4.35}$$
$$\text{since} \quad \hat{C}_t^w + (1-n)\hat{C}_t^\Delta = n\hat{C}_t + (1-n)\hat{C}_t^* + (1-n)\hat{C}_t - (1-n)\hat{C}_t^* = \hat{C}_t$$

$$\hat{C}_t^* = \hat{C}_t^w - n\hat{C}_t^\Delta \tag{4.36}$$
$$\text{since} \quad \hat{C}_t^w - n\hat{C}_t^\Delta = n\hat{C}_t + (1-n)\hat{C}_t^* - n\hat{C}_t + n\hat{C}_t^* = \hat{C}_t^*$$

Do the same operation to find expressions for domestic and foreign output:

$$\hat{y}_t = \hat{y}_t^w + (1-n)\hat{y}_t^\Delta \tag{4.37}$$
$$\hat{y}_t^* = \hat{y}_t^w - n\hat{y}_t^\Delta \tag{4.38}$$

For the price level, the most relevant variables are the two price indices. Rewrite (L8) and (L9) using (4.2):

$$\hat{P}_t = \hat{p}_{t-1}^w + (1-n)\hat{\mathcal{E}}_t \tag{4.39}$$
$$\hat{P}_t^* = \hat{p}_{t-1}^w - n\hat{\mathcal{E}}_t \tag{4.40}$$

4.3 THE FINAL MODEL

Collecting the equations mentioned and in addition (4.35)-(4.40) and (L16)-(L17), the final model consists of the equations below, where regular equations are denoted (M1)-(M17) and the AR(1)-processes (A1)-(A6) as before.

The vector of endogenous state variables (denoted x_t in chapter 2), of other endogenous variables (denoted y_t in chapter 2) and of exogenous stochastic processes (denoted z_t in chapter 2) will be respectively:

$$x_t = \left[\tilde{\hat{F}}_t, \hat{p}_t^{\Delta}, \hat{p}_t^{w} \right]^T .$$

$$y_t = \left[\hat{\mathcal{E}}_t, \hat{C}_t^{\Delta}, \hat{C}_t^{w}, \hat{y}_t^{\Delta}, \hat{y}_t^{w}, \tilde{\hat{r}}_t, \hat{C}_t, \hat{C}_t^{*}, \hat{y}_t, \hat{y}_t^{*}, \hat{P}_t, \hat{P}_t^{*}, \hat{W}_t, \hat{W}_t^{*} \right]^T$$

$$z_t = \left[\hat{G}_t, \hat{G}_t^{*}, \hat{\kappa}_t, \hat{\kappa}_t^{*}, \hat{M}_t, \hat{M}_t^{*} \right]^T$$

where the distinction of categorizing endogenous variables into the vector x_t and y_t respectively has been done in accordance with the discussion in section 2.4.

The matrix equations (2.1), (2.2) and (2.3) correspond to (M1)-(M15), (M16)-(M17) and (A1)-(A6) respectively.

(M1) $\quad 0 = -\frac{\tilde{\hat{F}}_t}{1-n} + \frac{1+\bar{r}}{1-n}\tilde{\hat{F}}_{t-1} + \hat{p}^{\Delta}_{t-1} - \hat{\mathcal{E}}_t + \hat{y}^{\Delta}_t - \bar{c}\hat{C}^{\Delta}_t - \bar{g}(\hat{G}_t - \hat{G}^*_t)$

(M2) $\quad 0 = -\hat{y}^{\Delta}_t + \theta(\hat{\mathcal{E}}_t - \hat{p}^{\Delta}_{t-1})$

(M3) $\quad 0 = -\hat{y}^w_t + \bar{c}\hat{C}^w_t + \bar{g}\left[n\hat{G}_t + (1-n)G^*_t\right]$

(M4) $\quad 0 = -\hat{\mathcal{E}}_t + \bar{r}\tilde{\rho}_M(\hat{M}_t - \hat{M}^*_t) - \hat{C}^{\Delta}_t$

(M5) $\quad 0 = -\frac{\theta-1}{\theta+1}\tilde{\rho}_{\kappa}(\hat{\kappa}_t - \hat{\kappa}^*_t) - \bar{g}(\tilde{\rho}_G + 1)(\hat{G}_t - \hat{G}^*_t) - \left[\frac{\theta-1}{(\theta+1)\bar{r}} + \bar{c}\,\frac{1+\bar{r}}{\bar{r}}\right]\hat{C}^{\Delta}_t + \frac{1+\bar{r}}{1-n}\tilde{\hat{F}}_{t-1} + (\theta-1)\hat{\mathcal{E}}_t - (\theta-1)\hat{p}^{\Delta}_{t-1}$

(M6) $\quad 0 = \frac{\bar{g}\rho_G}{\bar{c}+1}\left[n\hat{G}_t + (1-n)\hat{G}^*_t\right] + \frac{\rho_{\kappa}}{\bar{c}+1}\left[n\hat{\kappa}_t + (1-n)\hat{\kappa}^*_t\right] + \hat{C}^w_t + \frac{1}{1+\bar{r}}\tilde{\hat{r}}_t$

(M7) $\quad 0 = -\hat{p}^w_{t-1} + \bar{r}\tilde{\rho}_M\left[n\hat{M}_t + (1-n)\hat{M}^*_t\right] - \hat{C}^w_t$

(M8) $\quad 0 = -\hat{C}_t + \hat{C}^w_t + (1-n)\hat{C}^{\Delta}_t$

(M9) $\quad 0 = -\hat{C}^*_t + \hat{C}^w_t - n\hat{C}^{\Delta}_t$

(M10) $\quad 0 = -\hat{y}_t + \hat{y}^w_t + (1-n)\hat{y}^{\Delta}_t$

(M11) $\quad 0 = -\hat{y}^*_t + \hat{y}^w_t - n\hat{y}^{\Delta}_t$

(M12) $\quad 0 = -\hat{P}_t + \hat{p}^w_{t-1} + (1-n)\hat{\mathcal{E}}_t$

(M13) $\quad 0 = -\hat{P}^*_t(f) + \hat{p}^w_{t-1} - n\hat{\mathcal{E}}_t$

(M14) $\quad \hat{W}_t = \hat{C}_t + \chi(\hat{M}_t - \hat{P}_t) - \left(\frac{\theta-1}{\theta\bar{c}}\right)\left(\frac{\hat{\kappa}_t}{2} + \hat{y}_t\right)$

(M15) $\quad \hat{W}^*_t = \hat{C}^*_t + \chi(\hat{M}^*_t - \hat{P}^*_t) - \left(\frac{\theta-1}{\theta\bar{c}}\right)\left(\frac{\hat{\kappa}^*_t}{2} + \hat{y}^*_t\right)$

(M16) $\quad 0 = E_t\left\{-\hat{p}^{\Delta}_t + (\hat{\kappa}_{t+1} - \hat{\kappa}^*_{t+1}) + \hat{y}^{\Delta}_{t+1} + \hat{\mathcal{E}}_{t+1} + \hat{C}^{\Delta}_t\right\}$

(M17) $\quad 0 = E_t\left\{n\hat{\kappa}_{t+1} + (1-n)\hat{\kappa}^*_{t+1} + \hat{y}^w_{t+1} + \hat{C}^w_{t+1}\right\}$

(A1) $\quad \hat{G}_{t+1} = \rho_G\hat{G}_t + \varepsilon_{G,t+1}$

(A2) $\quad \hat{G}^*_{t+1} = \rho_G\hat{G}^*_t + \varepsilon_{G^*,t+1}$

(A3) $\quad \hat{\kappa}_{t+1|} = \rho_{\kappa}\hat{\kappa}_t + \varepsilon_{\kappa,t+1}$

(A4) $\quad \hat{\kappa}^*_{t+1} = \rho_{\kappa}\hat{\kappa}^*_t + \varepsilon_{\kappa^*,t+1}$

(A5) $\quad \hat{M}_{t+1} = \rho_M\hat{M}_t + \varepsilon_{M,t+1}$

(A6) $\quad \hat{M}^*_{t+1} = \rho_M\hat{M}^*_t + \varepsilon_{M^*,t+1}$

5. CALIBRATION

The purpose of the model can be said to be to focus on the effect of sticky prices in an environment of monopolistic competition and free trade. Notice that the model by assuming that a steady state can be found, abstracts away from many trends found in actual data, such as growth in population, government spending and total factor productivity. One can say that the model does this in order to study price stickiness isolated without other disturbing processes going on at the same time. Other simplifications are e.g. that labour is the only one factor of production in the model (and accordingly there are no investments), that it is assumed that all goods are tradable and that the government has access to lump-sum taxes.

Obviously, in such a stylized model, calibrating the model based on real data is difficult. Some of the steady state variables to be estimated do not exist in the same way in reality. Consider e.g. the parameters ρ_G and σ_G that describe the AR(1)-processes that determine government spending. It is assumed that government spending fluctuates around a steady state. However, real data for Norway and the U.S. show that government spending has a clear increasing trend in the entire post-war period[27]. Calibrating ρ_G and σ_G has to be based on some approximation.

The importance of finding the *"correct"* calibration should not be overstated. The purpose of calibrating in a setting as here is rather to be able to quantify some results found in the model, to investigate what results are considerable and which are negligible and to give an approximation of the magnitude of variable movements. Also one cannot expect simulations of the model to come very close to real data. In section 8.1 some sensitivity analysis will be performed and it will be discussed whether small changes in the calibration chosen might alter the results found severely. This paper is not meant to be an econometric paper and except for the sensitivity analysis, questions regarding uncertainty and validity of calibrations used will not be discussed thoroughly. The

[27] Sources: Bureau of Economic Analysis (BEA) and Statistisk Sentralbyrå (Statistics Norway).

analysis in chapter 6 will use four calibration sets, three of which country Home is calibrated to be the U.S. and one of which country Home is calibrated to be Norway. The two countries are chosen to illustrate the case of a large, open economy contra a typical small, open economy. In both cases it is natural to think of country Foreign as *the rest of the world* (or alternatively as the most important trading partners of the U.S. and Norway, respectively – this interpretation will be discussed later in this chapter).

The time period has been set to one year. This means that prices are set one *year* in advance. Cf. section 8.2 for a justification. Table 5.1 shows the chosen calibration sets. Only the benchmark case is shown for the U.S. The other two sets will be used for impulse response analysis only and are the special cases where shocks are fully persistent ($\rho_G = \rho_\kappa = \rho_M = 1$) or fully temporary $(\rho_G = \rho_\kappa = \rho_M = 0)$. To avoid confusion, it must be mentioned that all the log-linearized variables by convention should be interpreted so that a value of 1 means 1 % deviation from steady state. Since the model is linear, it does not matter whether one chooses 0.01 or 1 to denote 1% as long as one remembers to be consequent. The chosen convention includes the interpretations of the standard deviations of the error terms, σ_G, σ_κ and σ_M - e.g. σ_G = 1.37 means *1.37%*.

Calibrations are based on U.S. and Norwegian time series for the period 1970-2002 unless otherwise stated. These time series are included, together with exact descriptions and sources, in appendix VI.

Calibration set / Parameter	the U.S.	Norway
$\bar{c}$	0.77	0.71
$\bar{g}$	0.23	0.29
$\bar{r}$	0.046	0.026
n	0.32	0.006
θ	6	6
χ	0.046	0.027
ρ_G	0.9	0.9
σ_G	1.01	1.37
ρ_κ	0.95	0.95
σ_κ	1.4	1.4
ρ_M	0.49	0.15
σ_M	1.39	2.53

Table 5.1 Calibration sets

Remember the assumption from section 3.3.2 of per capita symmetry in consumption and government spending. This leads to a loss of generality when calibrating $\bar{c}$ and $\bar{g}$ if the *rest of world* ideally should have been calibrated differently than the country considered. Also this model assumes that goods and bonds are traded freely. Due to this simplification, it might make more sense to compare the two countries with their most important trading partners, rather than the entire rest of the world.

For calibrating $\bar{c}$ and $\bar{g}$, actual data for per capita private consumption and government spending as fractions of total consumption has been used. Since there are no investments in the model, investment spending has not been considered. Figures for Norway in the period 1970-2002 lead to $\bar{c} = 0.71$ and $\bar{g} = 0.29$. Figures for the U.S. in the same period lead to $\bar{c} = 0.77$ and $\bar{g} = 0.23$. It is simply assumed that Norway's most important trading partners have a similar composition of private and public consumption. The same is assumed for the U.S[28].

There is only one asset in the model, a real risk-free bond. In a stylized model as this it is natural to see this asset as an approximated aggregate of different kinds of risk-free deposits, loans and government bonds. Using e.g. the federal funds rate might lead to a too low estimate, since it is private agents that hold bonds in this model, and since the existence of banks and other intermediate financial institutions make the actual interest rate higher than the federal funds rate. Furthermore, since bonds are freely traded internationally, we are really talking about the equilibrium world interest rate.

For the U.S., Prescott (1986) states that the annual real interest rate is about 4%. Cooley and Prescott (1995) use a β of 0.947 which corresponds to a real interest rate of 5.60%. Woodford (1996) uses a real interest rate of 5%. For Norway, the *interest rate indicator* measures the average nominal interest rate on deposits, loans and bonds. The corresponding real interest rate can be calculated using CPI figures. Figures for the period 1970-2002 yield an average real interest rate of 2.6%. U.S. figures for the same period yield an average real interest rate of 4.6% (both calculations based on the geometric average). The estimates based on real data are used since some of the purpose of the model is to explain how the interest rate affects the trade off between consumption and saving and since we will compare simulations with real data later.

When setting *n*, one possibility would be to measure the fractions of U.S. and Norwegian GDP to world GDP. As mentioned above, it might be more reasonable to compare the two countries with their most important trading partners. The two *n*'s have therefore been set as the country's nominal GDP relative to the total GDP of itself and its 20 most

[28] This might be not such an unrealistic assumption since 69% of Norway's trade volume of 2002 was trade with the E.U., known to have similar high levels of government spending. And for the U.S. trade with Canada, Mexico, China and Japan constituted about 50% of 2002 trade volumes. (Source: Statistics Norway and Bureau of Economic Analysis)

important trading partners. Using data from 2002 leads to *n = 0.32* for the U.S. and *n = 0.006* for Norway[29].

For estimating θ, consider first the expected marginal disutility of production in period *t + 1* (take the expected derivative of (3.13) wrt. y_{t+1}):

$$E_t\left\{\frac{\partial U_t}{\partial y_{t+1}}\right\} = -\beta E_t\{\kappa_{t+1} y_{t+1}\} \tag{5.1}$$

$\beta E_t\{\lambda_{t+1}\}$ is the expected marginal utility of real wealth in the next period and $\beta E_t\left\{\frac{\lambda_{t+1}}{P_{t+1}}\right\}$ is thereby the expected marginal utility of nominal wealth. Dividing the marginal disutility with the marginal utility of nominal wealth yields the marginal cost of production in terms of nominal money, denoted MC_t. Reverse the sign since we are talking about a cost and wants to measure it as a positive number. The expected marginal cost in period *t + 1* conditional on information known in period *t* will then be:

$$E_t\{MC_{t+1}\} = -E_t\left\{\frac{\frac{\partial U_t}{\partial y_{t+1}}}{\frac{\lambda_{t+1}}{P_{t+1}}}\right\} = E_t\left\{\frac{-\kappa_{t+1} y_{t+1} P_{t+1}}{\lambda_{t+1}}\right\} \tag{5.2}$$

Comparing with (3.17), one can see that $\left(\frac{\theta}{\theta - 1}\right)$ will be the expected factor by which the price exceeds marginal costs, i.e. a mark-up factor. For U.S. data, it seems to be a kind of consensus in the literature that a mark-up of about 1.2 should be used, which corresponds to a θ of 6. Morrison (1990) calculates the average annual mark-up for the U.S. manufacturing industry for the period 1973-1986 to be 1.211. Schmitt-Grohé and Uribe (2001) also use 1.2 as the mark-up factor. Rotemberg and Woodford (1995), pp. 260-262,

[29] Source: Norwegian trading partners from Statistics Norway. U.S. trading partners from BEA. GDP levels in U.S. dollars from OECD and Nationmaster.com.

provide an overview on the literature on this field and on how to determine the mark up factor – and they also end up with the same mark-up factor. Due to lack of reliable data sources for Norway, it is simply assumed that the U.S. value also applies to Norway.

χ is an unobservable parameter, but by rewriting the steady state relationship (3.32) we see that the parameter can be expressed in terms of observable variables:

$$(3.32) \Rightarrow \chi = \frac{\overline{M}}{\overline{C}\,\overline{P}}\left(\frac{\bar{r}}{1+\bar{r}}\right) \tag{5.3}$$

$\frac{\overline{M}}{\overline{P}}$ is the real money stock per capita and $\overline{C}$ private consumption per capita. Using the same data as when calculating $\bar{c}$ and $\bar{g}$ and using $\bar{r}$ as in table 5.1 yields a χ of 0.046 and 0.027, using U.S. and Norwegian data, respectively.

As mentioned, when setting ρ_G and σ_G, a problem is that historical post-war data for both the U.S. and Norway show that real government spending increases over time. Rotemberg and Woodford (1995) correct detrended government spending by correcting for population growth and technological change. The remaining component is assumed to follow an AR(1)-process just as in our model. Based on US-data they set $\rho_G = 0.9$. Unfortunately they provide no standard-deviation since they only analyze impulse responses. Schmitt-Grohé and Uribe (2001) use the same process and the same ρ_G and set $\sigma_G = 3.02$ without very much justification for their calibration.

When calculating the calibrated values used in table 5.1, it is at first assumed that $\rho_G = 0.9$ is a good estimation, in line with the preceding discussion. Then time series for U.S. and Norwegian real government consumption spending for the period 1970-2002 have been HP-filtered (cf. appendix V) – and the percentage deviations from the trend have been calculated. Note that equation (A1) in the model can be rewritten in the following way:

$$(A1) \Rightarrow \qquad \hat{G}_{t+1} - \rho_G \hat{G}_t = \varepsilon_{G,t+1} \tag{5.5}$$

Then the HP-filtered data have been plugged into the left hand side of (5.5) also using the assumption that $\rho_G = 0.9$. Thereafter σ_G has been estimated by calculating the standard deviation of the estimated error terms. This yields σ_G *=1.01* and σ_G *=1.37* for U.S. and Norwegian data, respectively[30].

When estimating ρ_κ and σ_κ we should first discuss in a much more exact way what κ_t actually is. Denote work effort, measured e.g. as hours worked, by ζ_t and assume that the disutility of work effort is linear in the effort and given by:

$$\frac{\partial U_t}{\partial \zeta_t} = -\phi \zeta_t \tag{5.6}$$

where ϕ is some positive parameter. Furthermore, assume that the production function is given by:

---

[30] It is assumed that the remaining residuals $\varepsilon_{G,t}$ are normally distributed. Using the model $\varepsilon_{G,t} = \rho_\varepsilon \varepsilon_{G,t-1} + v_t$, where ρ_ε is the coefficient of autocorrelation and v_t a normally distributed error term, one can test for autocorrelation. This regression yields ρ_ε =-0.04 and ρ_ε =-0.15, for U.S. and Norwegian data, respectively. However, none of these coefficients are significantly different from zero, on a 5% level of significance. Rather than testing formally if the error terms are normally distributed, histograms for the residuals are provided below, with U.S. data to the left and Norwegian data to the right. The U.S. data series seems to be approximately normally distributed; the Norwegian data series does not fit that good to the normality assumption.

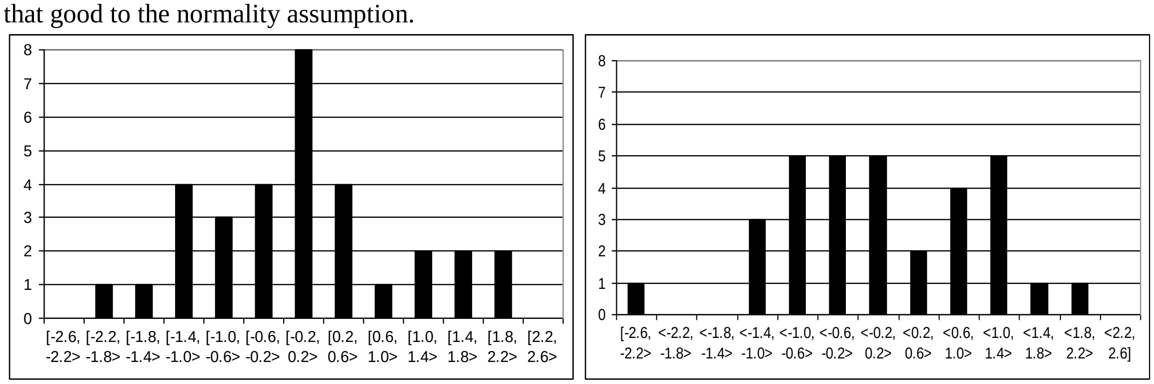


Figure 5.1 Histograms of residuals in equation (5.5). (y=no. of residuals x= std. dev)

$$y_t = A_t \zeta_t^{\frac{1}{2}} \tag{5.7}$$

where A_t denotes total factor productivity (TFP). Solve for ζ_t to get:

$$\zeta_t = \left(\frac{y_t}{A_t} \right)^2 \tag{5.8}$$

Then define the exogenous variable κ_t to be:

$$\kappa_t \equiv \frac{2\phi}{A_t^2} \Leftrightarrow A_t = \left(\frac{2\phi}{\kappa_t} \right)^{\frac{1}{2}} \tag{5.9}$$

The disutility from effort will then be – plug (5.8)-(5.9) into (5.6):

$$\frac{\partial U_t}{\partial \zeta_t} = -\phi \zeta_t = -\phi \left(\frac{y_t}{A_t} \right)^2 = -\phi \left(\frac{y_t}{\sqrt{\frac{2\phi}{\kappa_t}}} \right)^2 = -\phi \frac{y_t^2}{2\phi / \kappa_t} = -\frac{\kappa_t}{2} y_t^2 \tag{5.10}$$

(5.10) shows that the special cases of disutility of work effort (5.6) and production function (5.7) leads exactly to the specification of the utility function used in our model. κ_t will then be related to TFP via (5.9).

Assume that the log-deviation of A_t follows the following process:

$$\hat{A}_{t+1} = \rho_A \hat{A}_t + \varepsilon_{A,t+1} \tag{5.11}$$

where notation and error term properties are as in section 3.3.3. Plug (5.9) into (5.11):

$$(5.9) \Rightarrow \ln A_t - \ln \overline{A} = \frac{1}{2}\ln\left(\frac{\frac{2\phi}{\kappa_t}}{\frac{2\phi}{\overline{\kappa}}}\right) = -\frac{1}{2}(\ln\kappa_t - \ln\overline{\kappa}) = -\frac{1}{2}\hat{\kappa}_t$$

$$(5.11) \Rightarrow -\frac{1}{2}\hat{\kappa}_{t+1} = -\rho_A\frac{1}{2}\hat{\kappa}_t + \varepsilon_{A,t}$$

$$\Leftrightarrow \hat{\kappa}_{t+1} = \rho_\kappa\hat{\kappa}_{t+1} + \varepsilon_{\kappa,t}; \qquad \rho_\kappa \equiv \rho_A; \qquad \varepsilon_{\kappa,t} \equiv -2\varepsilon_{A,t} \Leftrightarrow \sigma_\kappa = 2\sigma_A \qquad (5.12)$$

Cooley and Prescott (1995), have estimated the coefficient of auto-covariance and the standard deviation in equation (5.10), denoted ρ_A and σ_A respectively, and found $\rho_A = 0.95$ and $\sigma_A = 0.7$. Their estimation is based on U.S. data series for TFP. Schmitt-Grohé and Uribe (2001) use $\rho_A = 0.82$ and $\sigma_A = 2.29$ in a similar model to the one here, but their calibration is based on a survey by Chari et. al. (1995) who use data series for labour productivity.

Labour is the only factor of production in this model, but since the purpose is to compare with real data, e.g. for output, it might make more sense to use a more general productivity measure, such as TFP. The figures from Cooley and Prescott (1995) have been used and they yield $\rho_\kappa = 0.95$ and $\sigma_\kappa = 1.4$. Remember however, that there is uncertainty on how to measure productitivity, and that this calibration should not be blindly trusted. As for the other coefficients of autocorrelation and standard deviations, one should analyze the consequences of changing these parameters. As earlier mentioned, this will be done in chapter 8.1. Also notice that no reliable data sources for Norway have been available, and that the same calibration has been chosen for Norway as for the U.S.

To calculate ρ_M data for the broad monetary aggregate M3 for the U.S. and Norway for the period 1970-2002 has been used. Thereafter the data has been HP-filtered and the percentage deviations from trend have been calculated. By assuming that these data series follow the equation (A5), one can run an OLS-regression. This yields ρ_M *=0.49* and σ_M *=1.39* for U.S. data and $\rho_M = 0.15$ and σ_M *=2.53* for Norwegian data[31].

[31] 95% confidence intervals for ρ_M are [0.19, 0.79] and [-0.21, 0.52], respectively – in other words the calibrations are quite uncertain, and for Norwegian data the coefficient is not even significantly different from zero. One can do as for government spending and apply a few tests on the remaining residuals $\varepsilon_{M,t}$. By testing for autocorrelation using the equation $\varepsilon_{M,t} = \rho_\varepsilon \varepsilon_{M,t-1} + v_t$ where ρ_ε is the coefficient of autocorrelation and v_t a normally distributed error term, one must for both data sets conclude that ρ_ε is not significantly different from zero on a 5% level of significance. Histograms of the residuals are provided below, U.S. data to the left and Norwegian data to the right. The Norwegian data series seems to be approximately normally distributed; the U.S. data series does not fit that good to the normality assumption.

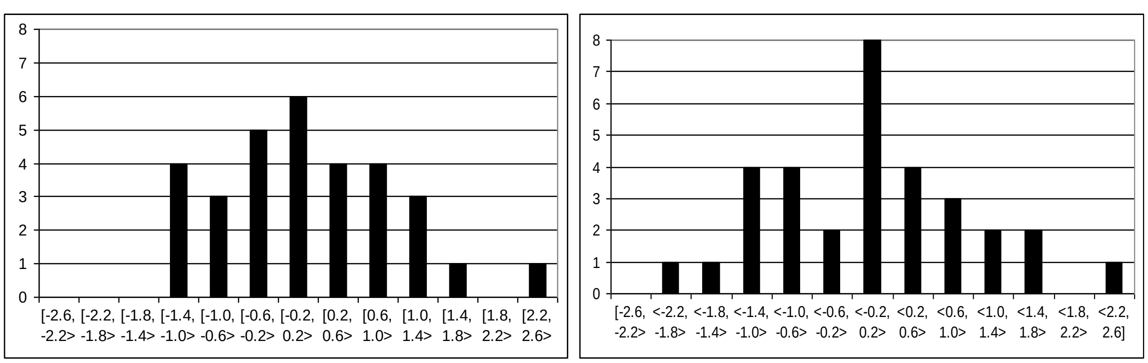


Figure 5.2 Histograms of residuals in equation (A5). (y=no. of residuals x= std. dev)

6. ANALYSIS

Chapter 6 calculates impulse responses to shocks using the MatLab source code *redux.m* in combination with Uhlig's MatLab-programs "*Toolkit*" (cf. appendix IV). Section 6.1 and 6.2 discuss the effect of entirely permanent and temporary shocks, respectively. The sections are meant as two reference cases for the later discussion and to explain the main mechanisms of the model. Sections 6.3-6.7 use the calibration sets in table 5.1 and attempt to address some policy issues related to the results found. All figures will show impulse reponses to a 1% shock in each of the exogenous variables[32]. Only shocks in domestic exogenous variables are analyzed. Domestic effects of shocks in foreign variables follow similarly by interchanging the two countries in the model. The main focus of the discussion will be the monetary shocks.

6.1 PERMANENT SHOCKS

All shocks are calculated using the U.S. calibration set. The shocks are permanent, i.e. $(\rho_G = \rho_\kappa = \rho_M = 1)$.

Money shocks:

Consider figure 6.1. A permanent increase in the domestic money stock leads to a temporary domestic income increase. In the short run prices cannot change; thus the real money stock also increases with 1%. In the long run, however, prices can again be set optimally. The optimal response to the wealth increase is to spend the increase over an infinite time horizon. In practice this implies to save over the current account and then spend only the interest income every period in the future. This follows directly from the NPG and TVC conditions.

[32] The value of the initial shock of 1% is chosen rather arbitrarily. Another possibility would be to e.g. let the shocks be of the size of one standard deviation, in accordance with table 5.1. Notice, however, that since the model is linear, the responses of all other variables will follow proportionally, i.e. the impulse responses to a 0.1% shock will be exactly one tenth of the impulse responses to a 1% shock. Nonetheless, the error of linearization will be larger, the larger the deviations from steady state. Cf. section 8.3 for a further discussion on the error made when linearizing.

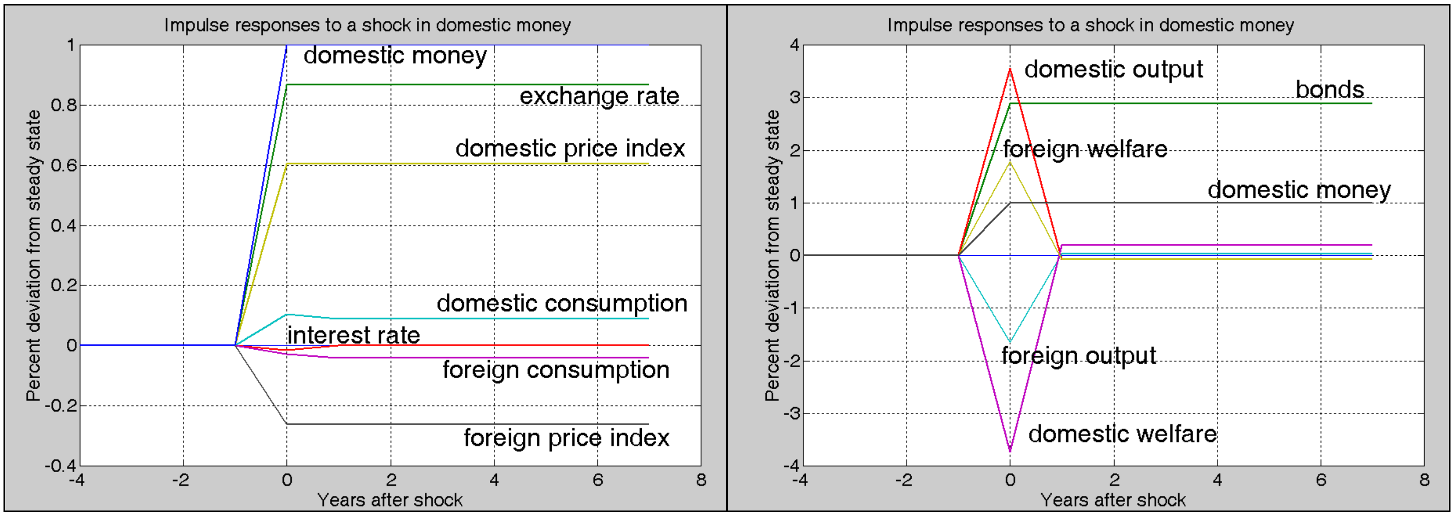


Figure 6.1 Impulse responses to a 1% permanent shock in money

A short run domestic current account surplus can only take place if foreign agents are willing to borrow. The equilibrium real interest rate therefore falls temporarily. After borrowing over the current account in period 0, the optimal response for foreign agents will be to pay back the debt over an infinite time horizon, i.e. pay only the interest payments. Since neither foreign nor domestic agents have an incentive to save or borrow in the long run, the equilibrium real interest rate returns to steady state and remains there. In the long run the *objective* real interest rate equals the *subjective* interest rate $\frac{1-\beta}{\beta}$. It is therefore optimal to choose the same level of consumption every period in the future, absent new shocks.

When the domestic money stock increases permanently with 1% one would normally expect a 1% depreciation of the exchange rate – for the real money stock to remain unchanged and to restore the money market equilibrium. This is what that would have happened in the case of flexible prices. With sticky prices and the calibration set here the exchange rate depreciates with only about 0.85%, however. This is because the increase in domestic consumption relative to foreign consumption raises domestic money demand relative to foreign. Thus the effect is partially offset.

There are two effects affecting foreign consumption in the short run. On the first hand, the reduced real interest rate affects consumption positively, since consumption gets more attractive relative to saving. On the other hand, the depreciated exchange rate, increases the demand for domestic output and lowers the demand for foreign output, i.e. a terms-of-trade effect. As argued in section 3.3.1, since prices are above marginal costs, domestic producers find it profitable to still meet the higher demand and oppositely abroad. This lowers foreign income, which affects foreign consumption negatively. The latter effect turns out to dominate.

In the long run prices adjust, and there is no longer any terms-of-trade effect - remember that purchasing power parity (PPP) holds in this model. But foreign output actually exceeds domestic output. This is because the domestic wealth alters the trade-off between work and consumption and vice versa abroad.

To conclude, the fundamental lesson to be learnt from figure 6.1 is how an unexpected shock in the money stock can affect real variables permanently. The shock leads to a permanent rise in domestic consumption and a permanent fall in foreign consumption due to current account movements.

Government spending shocks:

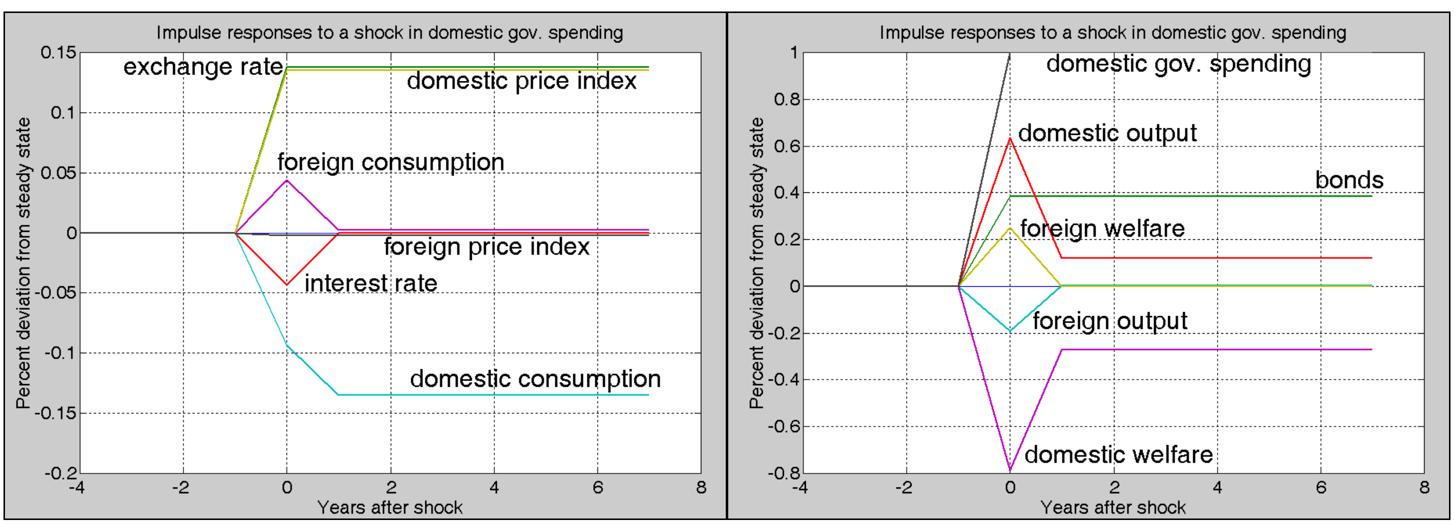


Figure 6.2 Impulse responses to a 1% permanent shock in government spending

Figure 6.2 shows a permanent shock in government spending. Remember that the government allocates its spending in the same way as private agents – and some of the increase will be spent at home and some of it abroad. I.e., demand increases in both countries ceteris paribus. However, the increased tax burden has to be paid by domestic agents only.

First of all domestic agents respond to the shock by lowering domestic consumption due to the higher tax burden. The lower domestic consumption relative to foreign consumption leads to exchange rate depreciation, since the consumption difference lowers domestic money demand relative to foreign money demand. The depreciation increases demand for domestic products due to the terms-of-trade effect mentioned. Output is demand driven in the short run and therefore rises.

The higher output leads to higher income and there is therefore no need for domestic consumption to decrease as much as the rise in taxes. Domestic consumption falls only with about 0.13%. Note that some of the increase in output lasts only temporarily. In the short run producers are unable to change prices. In the long run, equivalently to in the money shock case, prices are again set optimally, i.e. domestic prices increase to meet the higher demand. This means that the increase in output is partially offset.

Temporary increased domestic output means that domestic residents experience a temporary income increase. Agents prefer a smooth path of consumption and therefore decide to consume this temporary increase over an infinite time horizon. They save over the current account and spend only the interest income every period in the future. Since domestic residents save in the short run, foreign residents must borrow, and for them to be willing to do that, the interest rate must fall temporarily. The temporary low real interest rate makes it attractive for foreign agents to consume more and save less, and accordingly foreign consumption increases temporarily.

The positive difference between the foreign and domestic level of consumption leads to an increase in foreign money demand relative to domestic. Equilibrium is restored by a deprecation of the exchange rate. This permanent depreciation leads again to a terms-of-trade effect - higher demand for domestic products vis-à-vis foreign. That is the demand-side explanation for why domestic output increases permanently relative to foreign output. A more intuitive explanation is the supply-side explanation – that the higher domestic tax burden alters the trade-off between work and consumption, and that domestic agents substitute out of leisure and into work to compensate for some of the tax increase.

This leads to the fundamental result that a government spending shock unambiguously increases world output. The other most interesting feature to notice is how the shock makes domestic agents run a short run current account surplus – as explained above.

Technology shocks:

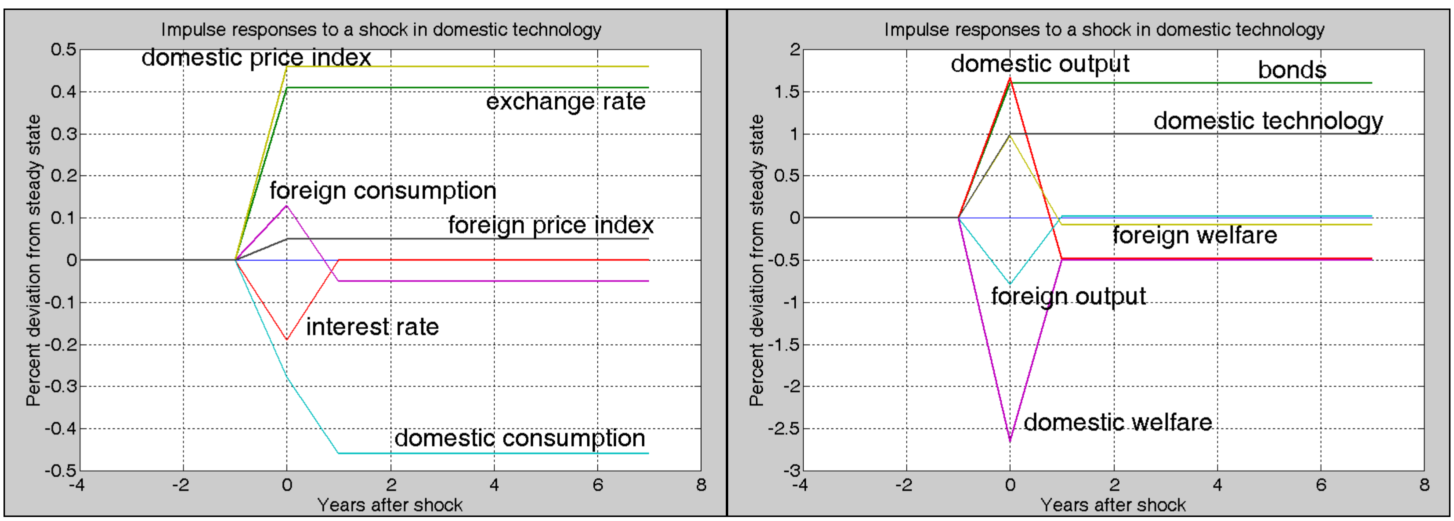


Figure 6.3 Impulse responses to a 1% permanent shock in technology

Cf. figure 6.3 that shows the impulse responses of a permanent shock in technology. Remember that the technology shock must be interpreted as a permanent *reduction* in domestic productivity. Certainly lower domestic productivity leads to lower domestic consumption. That is because agents immediately realize that the productivity decrease will worsen their situation. The difference between foreign and domestic consumption leads to lower domestic money demand relative to foreign. This leads to a depreciated exchange rate.

This depreciation increases demand for domestic products via the terms-of-trade effect. Therefore, surprisingly, domestic output actually increases in the short run, with about 1.7% (even though only to maintain the old level of production would require more work effort to be put in). In the long run prices are again set optimally, and the total effect on output is negative. Oppositely, foreign producers suffer from the terms-of-trade effect in the short run as they are unable to lower prices. In the long run they end up with production slightly above the old steady state level.

Just as in the government shock case, the temporary high domestic income makes it optimal for domestic agents to save much of the income, and spend it over an infinite time horizon. They run a current account surplus in the short run and spend only the interest income in every future period. This makes the long-run consumption decrease less severe than it otherwise would have been. In equilibrium this must correspond to a temporary reduction in the real interest rate.

A fundamental result here, which could not have been found in any flexible price model, is the short run output *increase* following a productivity *decrease*. The fall in productivity leads to exchange rate depreciation. And price stickiness then implies a short-run terms-of-trade effect. This offsets the fall in output that one would expect – and one might in fact experience that short-run output increases, as with the calibration here. The magnitude with which domestic output responds to exchange rate depreciation – depends of course on the price elasticity of demand – i.e. on θ. And as seen in chapter 5, it seems to be a consensus in the literature that θ should be about 6 (corresponding to a price mark-up of 20%). This certainty supports the hypothesis that the particular result found in figure 6.3 could hold in general.

6.2 TEMPORARY SHOCKS

All shocks are calculated using the U.S. calibration set. The shocks are temporary, i.e. $(\rho_G = \rho_\kappa = \rho_M = 0)$. Impulse responses are shown in figures 6.4-6.6. Many of the variable movements are similar as in the permanent shock case – often the movements only differ in magnitude. The results are discussed only where remarkable differences can be found.

Money shocks:

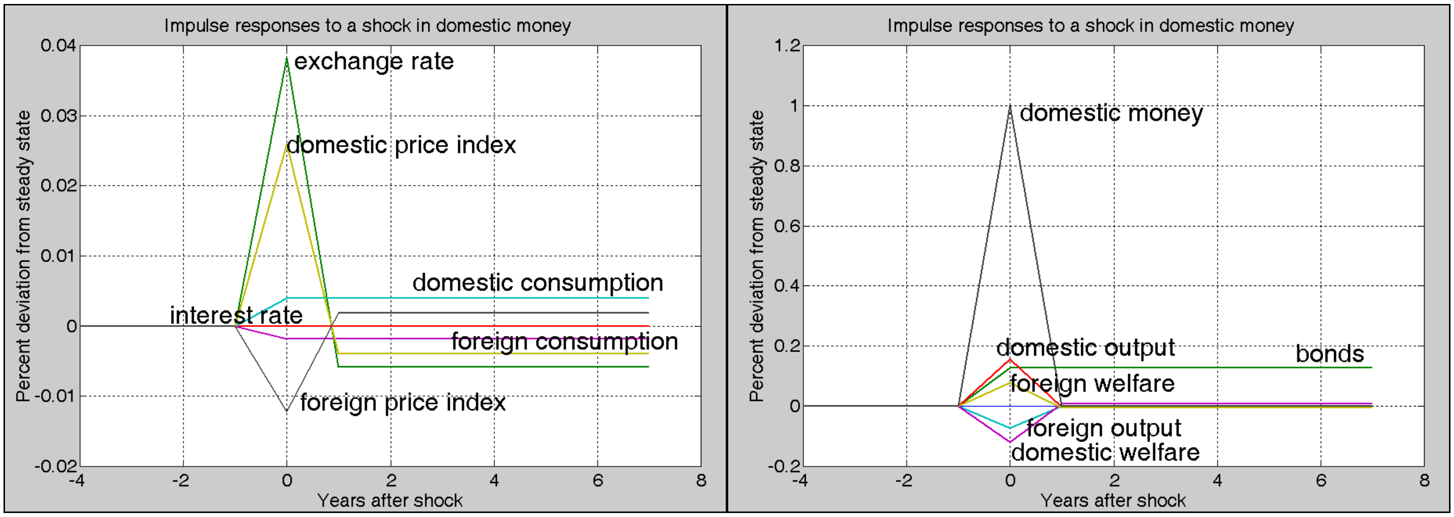


Figure 6.4 Impulse responses to a 1% temporary shock in money

In the temporary money shock case, cf. figure 6.4, the shock leads to a temporary domestic wealth increase, leading to movements in consumption and output similar to the permanent case. The exchange rate immediately depreciates in the short run, but only with about 0.04%. The small increase in the short run exchange rate comes from the temporary increase in demand for domestic money as consumption increases. And since the exchange rate movement is small, output and consumption movements are equivalently small.

Surprisingly, the exchange rate actually *appreciates* in the long run. In the first place, the domestic money stock has returned to steady state and can no longer contribute to depreciation. Secondly, since domestic consumption is permanently higher than foreign consumption, demand for domestic money rises relative to foreign money demand. Therefore, there must be an appreciation in the long run.

Also notice the surprising fact that the real interest rate is unchanged in this case. Foreign agents reduce production in the short run due to a terms-of-trade effect. Knowing that their output will go up again already next period (since the exchange rate appreciates), they borrow over the current account. Foreign long run consumption is therefore about 0.002% below the old steady state. The difference compared with the permanent shock case is that foreign agents have incentives to borrow without any reduction in the real interest rate.

Government spending shocks:

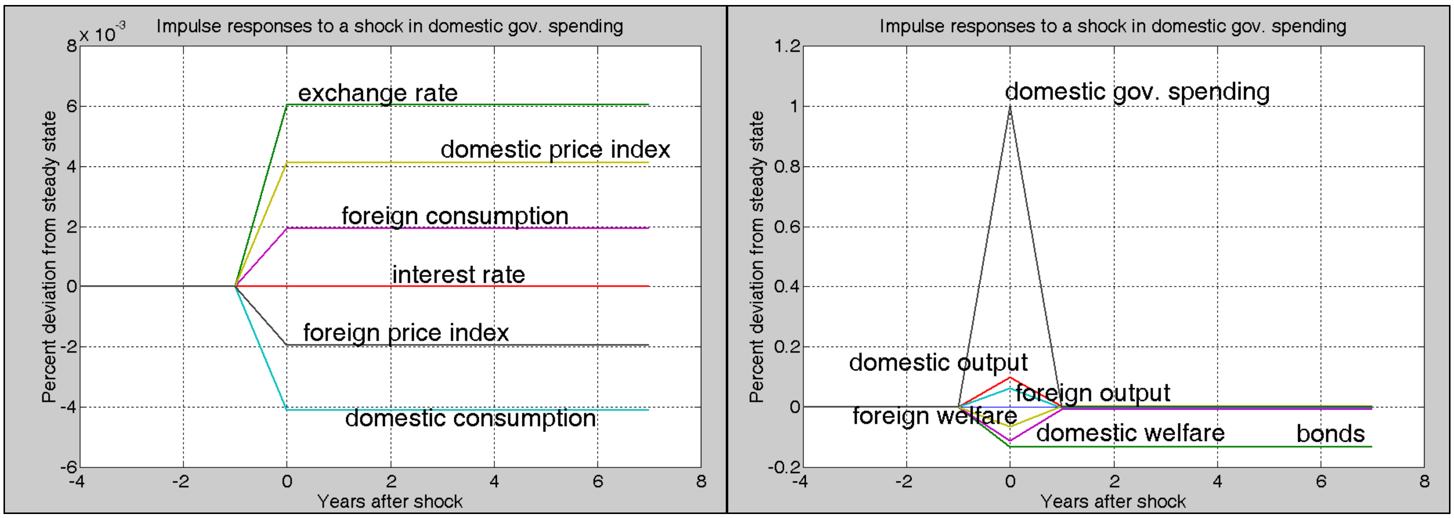


Figure 6.5 Impulse responses to a 1% temporary shock in government spending

Figure 6.5 shows the case of a temporary government spending shock. It is optimal for domestic agents to finance the temporary shock (the increased tax burden) by cutting consumption slightly all future periods, since agents prefer a smooth path of consumption. When comparing with the permanent shock, the redistribution of wealth is of opposite sign.

Foreign agents experience a temporary income increase since the higher total domestic consumption (public and private) leads to higher demand for foreign products. They spend the increase over an infinite time horizon by saving over the current account. Again, foreign agents are willing to save without a change in the interest rate. The fact that foreign consumption increases relative to domestic consumption every period in the future leads to immediate depreciation of the exchange rate, via relative money demand.

In the short run, both domestic and foreign output increase due to the higher total world demand. Domestic output increases more, however, due to the terms-of-trade effect of the depreciation. In the long run, world demand returns to steady state and prices adjust. The only effect left to affect output is the altered trade-off between work and consumption in the two countries. The foreign wealth makes foreign agents choose slightly more leisure (and less output) than domestic agents. This effect is almost negligible though.

Technology shocks:

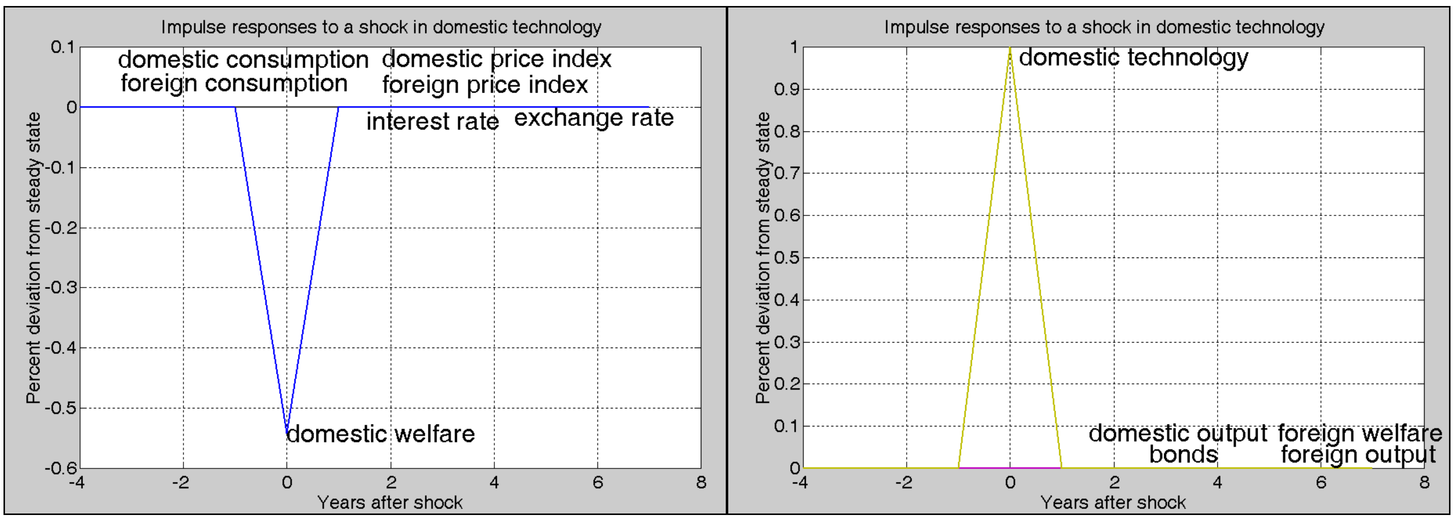


Figure 6.6 Impulse responses to a 1% temporary shock in technology

Cf. figure 6.6 that shows impulse responses of a 1% temporary shock in technology (fall in domestic productivity). Notice the surprising result that practically all variables remain unchanged. The technology shock affects supply only and output is demand driven in the short run. Demand is unchanged in the short run, and agents meet demand by working more, just enough to produce steady state output as before. The only variable affected is welfare, directly via the utility function. Since the short run equilibrium implies no changes in the exchange rate and no changes in savings, the economy returns to the old steady state in the first period after the shock.

6.3 GENERAL SHOCKS

We will now calculate the same impulse responses as above, but use the U.S. data calibration set from table 5.1. These responses will in many ways be a kind of intermediate case between fully permanent and temporary shocks.

Money shocks:

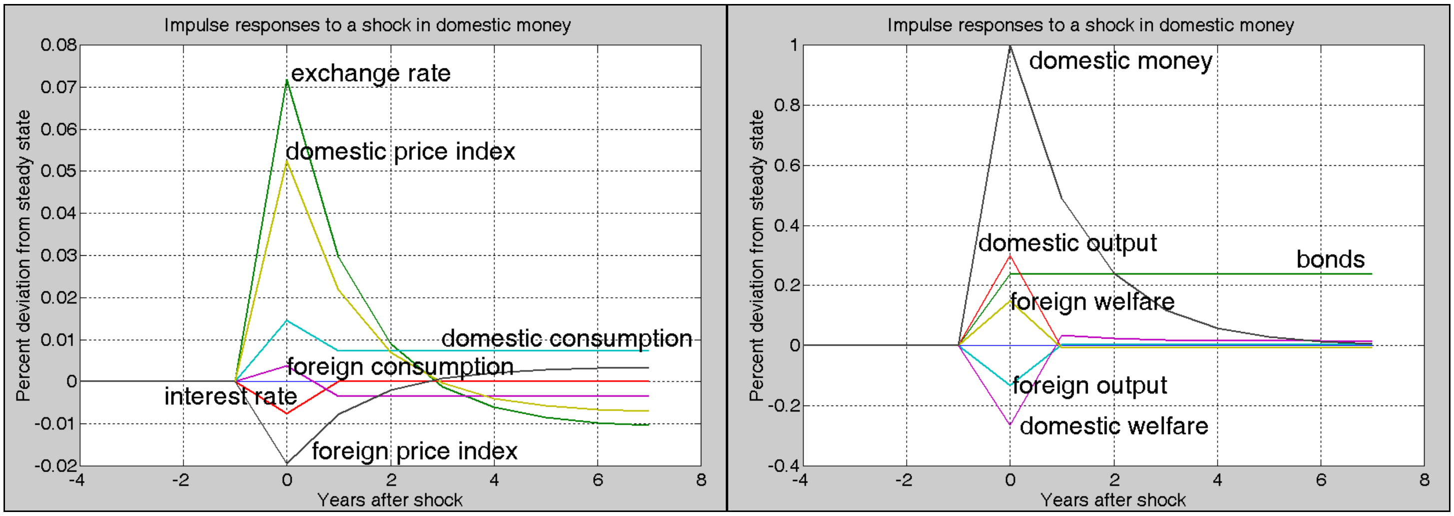


Figure 6.7 Impulse responses to a 1% general shock in money, calibrated to U.S. data

The main lesson to learn from figure 6.7 is how an unanticipated money shock can lead to a permanent redistribution of world wealth. The mechanisms are the same as mentioned in sections 6.1 and 6.2. Even though the shock is not permanent, the redistribution last permanently since agents have an infinite time horizon. Domestic bond holdings constitute about 0.25% of GDP in the long run. Accordingly, domestic consumption increases with about 0.008%, foreign consumption decreases with about the half.

Government spending shocks:

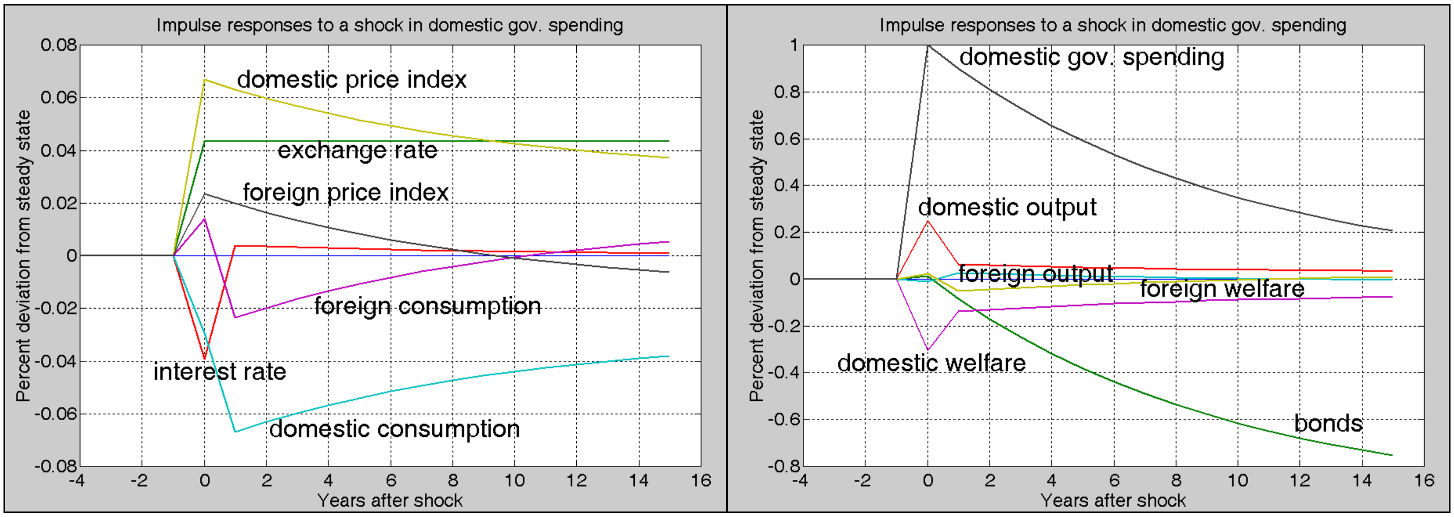


Figure 6.8 Impulse responses to a 1% general shock in government spending, calibrated to U.S. data

For the government spending shock case, the short run effects are fully equivalent to the effects mentioned in section 6.1. Notice how domestic agents actually run a current account surplus in the short run, but then a deficit all future periods.

Just as in the money shock case, the government spending shock leads to permanent redistribution of wealth in favour of foreign agents. A government spending shock is a real shock – so this result would also occur in the flexible price case.

The tax burden induced by the government spending shock falls with time. Domestic agents' optimal response from period 1 and on will be to run a current account deficit every period, but to run a large deficit when government spending is high, and gradually reduce the deficit as government spending closes the gap to steady state. The current account deficit goes to zero as time goes to infinity. As usual the debt is finally repaid in an infinite time horizon, i.e. only the interest payment is paid every period.

It is clear that domestic agents substitute future consumption with current consumption, to smooth out the consumption path. For foreign agents to be willing to save over the current account, the equilibrium real interest rate rises in period 1 – and then slowly falls as domestic agents slowly reduce their borrowing. The fact that the *objective* real interest rate r_t for some time exceeds domestic agents' *subjective* real interest rate , $\dfrac{1-\beta}{\beta}(=\bar{r})$, explains the upward sloping path of domestic consumption. Extending the time horizon shows that domestic and foreign consumption reach new long run steady states 0.03% below and 0.01% above their old steady states, respectively. This permanent difference in consumption levels corresponds entirely to the redistribution of wealth – bond holdings reach a new long run steady state about 0.8% (of domestic GDP) below the initial steady state.

Technology shocks:

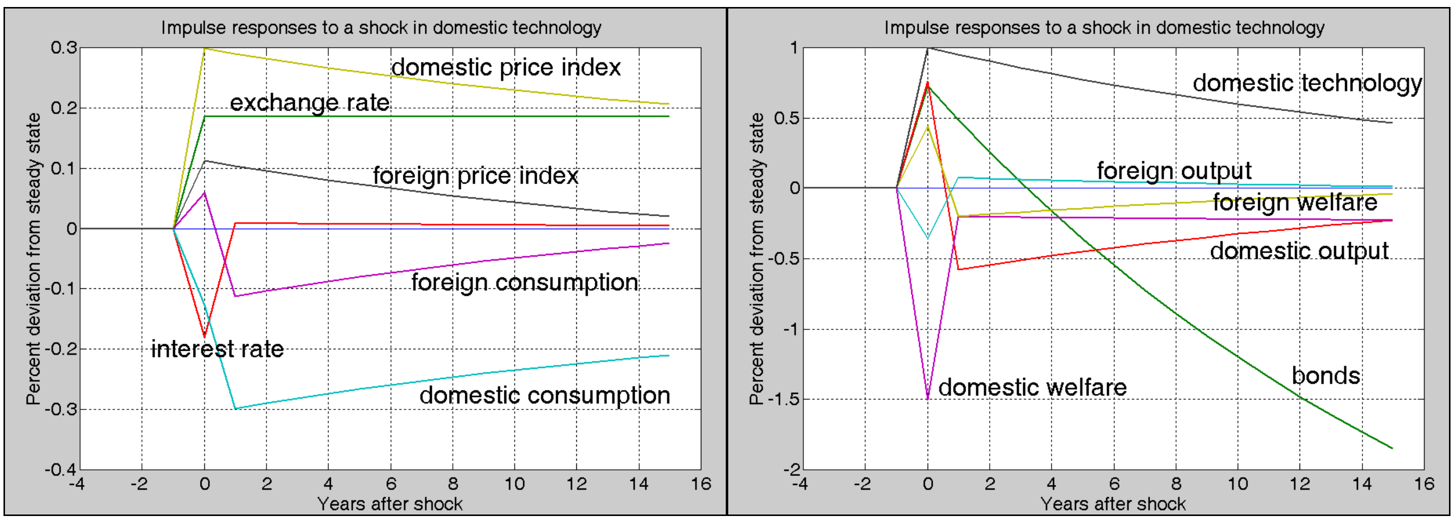


Figure 6.9 Impulse responses to a 1% general shock in technology, calibrated to U.S. data

In the technology shock case, the most surprising result is, as discussed in section 6.1, the short run rise in domestic output. Since output is demand driven in the short run, the depreciation leads to increased demand for domestic products.

The long run effects are quite similar to the government spending shock case above. The variables return more slowly to the new steady state, since the technology shock has been calibrated to be more persistent than the government spending shock. Knowing that domestic productivity eventually will return to steady state, domestic agents cut output with about 0.6% in period 1 and increase it slowly as productivity improves. At the same time agents substitute future consumption for current consumption by borrowing over the current account – thereby avoiding a drastic fall in consumption the periods shortly after the shock. The movement of the real interest rate from period 1 and on can be explained equivalently to as in the government spending shock case above.

By extending the time horizon, one finds that in the new long run steady state the changes in domestic and foreign consumption are about -0.13% and 0.06%, respectively. For domestic and foreign output the changes are about 0.10% and -0.05%, respectively. The latter effect follows from the altered trade-off between work and consumption following the redistribution of wealth – bond holdings reach a long run steady state that differs about -4.0% of GDP from the initial steady state.

6.4 THE IMPORTANCE OF COUNTRY SIZE

So far all impulse responses have been calculated using the calibration set based on U.S. data. Whereas the U.S. in this model is a typical example of a large, open economy (with the calibration of n=0.32, the U.S. constitutes about one third of the world economy), Norway is a typical example of a small, open economy (Norway constitutes only 6/1000th of the world economy). Figure 6.10-6.12 show impulse responses to shocks in money, government spending and technology with the Norwegian calibration set. Obviously an important difference will be that Norwegian agents can save or borrow over the current account with only negligible changes to the real interest rate. This means that they have better possibilities to carry out intertemporal substitution of consumption.

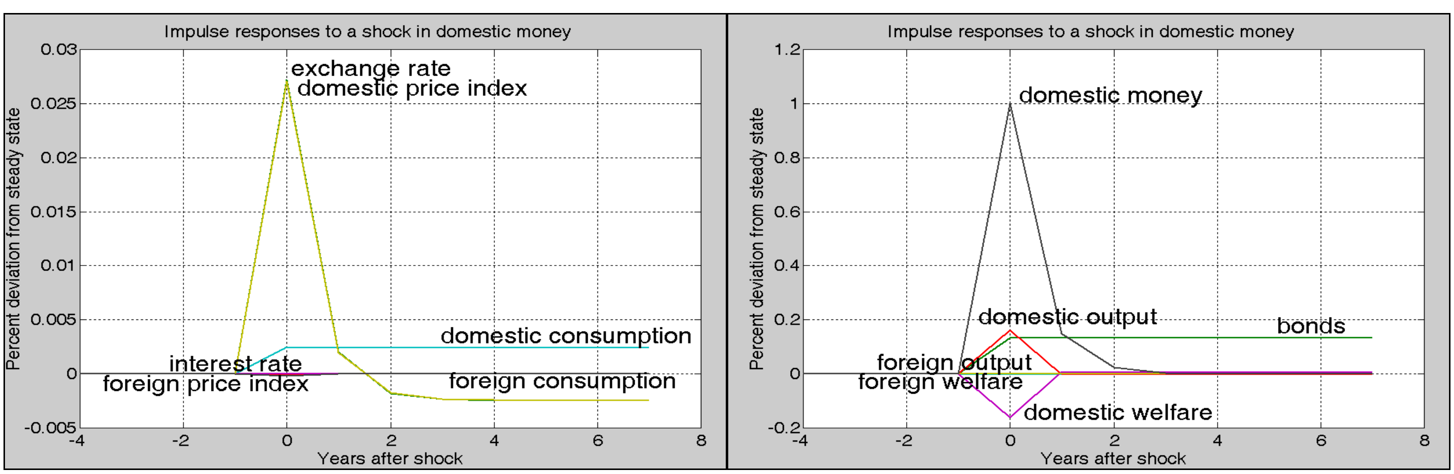


Figure 6.10 Impulse responses to a 1% general shock in money, calibrated to Norwegian data

On the other hand, figures 6.1, 6.4 and 6.7 show that the wealth redistribution of a money shock depends positively on the persistence of the shock. Cf. section 8.1 for a further discussion on this matter. Nonetheless, there are two effects of opposite direction that distinguish the Norwegian case from the U.S. Comparing figure 6.10 with 6.7, we see that Norwegian agents save about 0.15% of GDP in period 0, whereas U.S. agents save about 0.25%. Due to the different country sizes relative to the world economy, the interest rate does not respond at all in the Norwegian case, and with about 0.008 percentage points in the U.S. case. Norwegian agents are able to increase consumption every period after the shock with about 0.002%. Due to their impact on the real interest rate, U.S. agents instead consume about 0.015% above steady state in the short run and then 0.008% above steady state every period in the future.

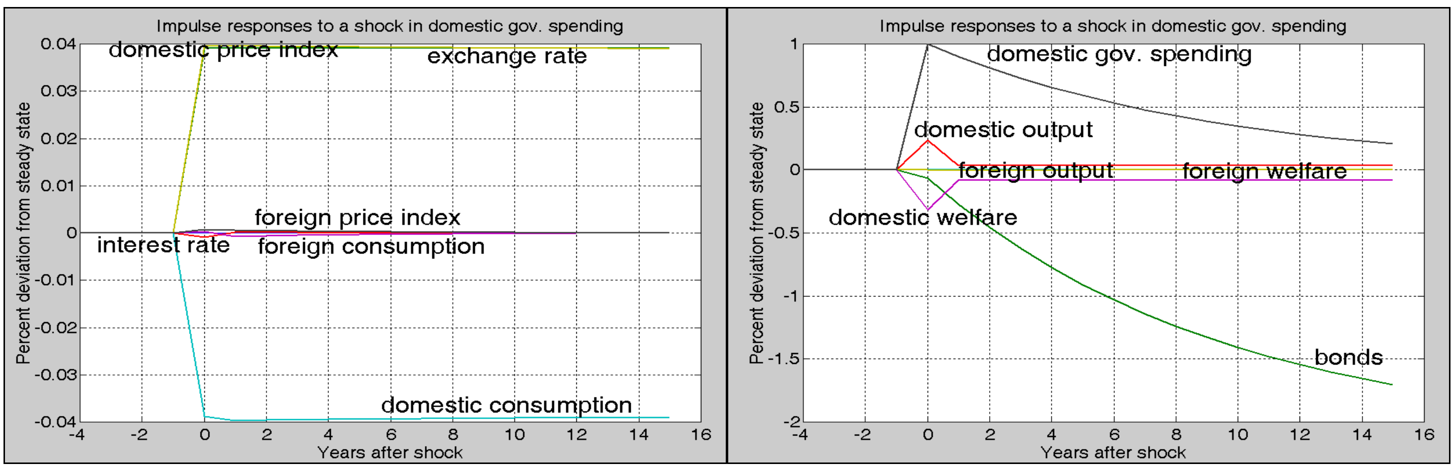


Figure 6.11 Impulse responses to a 1% general shock in government spending, calibrated to Norwegian data

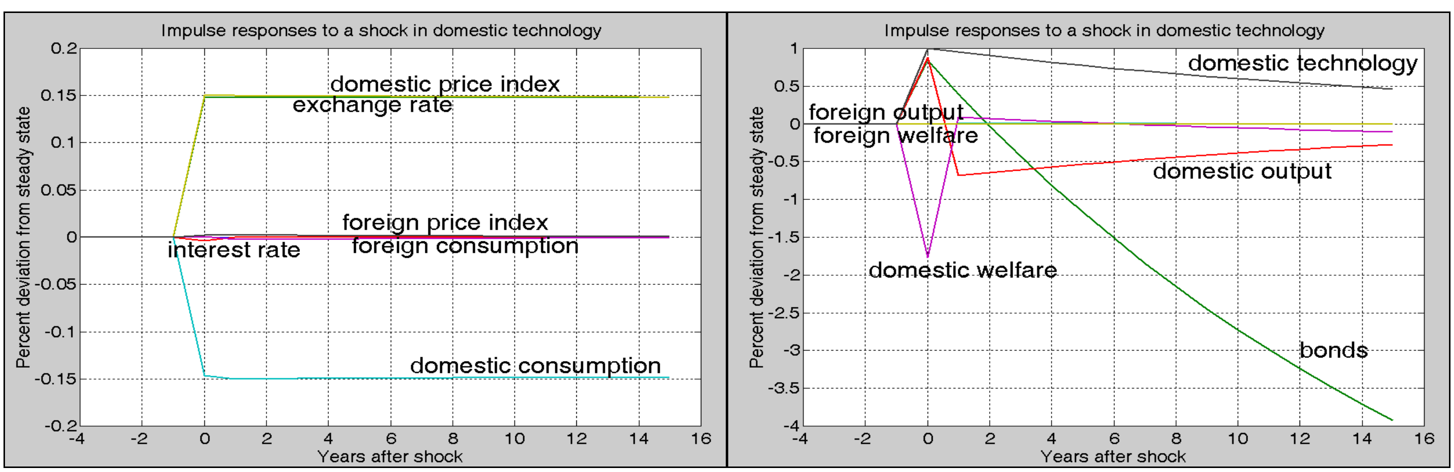


Figure 6.12 Impulse responses to a 1% general shock in technology, calibrated to Norwegian data

Comparing the figures 6.11 and 6.12 with the figures 6.8 and 6.9 yields equivalent insights as for the money shock case. Notice how Norwegian agents are able to maintain a smooth path of consumption every period after a shock since their impact on the real interest rate is negligible. A small, open economy has the advantage that effects of shocks to a higher degree can be smoothed out over time by saving and borrowing the current account to avoid fluctuations in real variables.

6.5 WELFARE ANALYSIS

It is now time to compute welfare effects of the shocks in figures 6.7-6.12. Notice that the calibrations of $\bar{r}=0.046$ and $\bar{r}=0.026$ from table 5.1 imply discount factors β of 0.956 and 0.975, respectively (cf. equation [3.30]). An often used objective of policy makers is to maximize the discounted sum of expected welfare in all future periods. In our model there is no explicitly modelled policy – only the exogenous processes. Still it would be interesting to see what effects the shocks in 6.7-6.12 have on the discounted sum of welfare. Combining the utility function (3.13) and the expression for single period welfare (3.29) yields:

$$U_t = E_t\left\{\sum_{s=t}^{\infty}\beta^{s-t}W_t\right\} \tag{6.1}$$

Then log-linearize equation (6.1):

$$\hat{U}_t = E_t\left\{\sum_{s=t}^{\infty}\beta^{s-t}\hat{W}_t\right\} \approx E_t\left\{\sum_{s=t}^{t+100}\beta^{s-t}\hat{W}_t\right\} \tag{6.2}$$

The approximation on the right hand side of (6.2) has been used so that the expression can be calculated numerically[33]. This approximation and its foreign counterpart have been used to calculate table 6.1 below[34].

[33] For simplicity it has been chosen to calculate welfare for the first hundred years following a shock only, since the remaining terms are so small that they can be neglected. Remember that $\hat{W}_t$ as *s* grows, either returns to steady state or reaches a new steady state (random walk), whereas β^{s-t} goes to zero as *s* grows.

Data set:	**U.S. data**		**Norwegian data**	
Shock in:	*U*	*U**	*U*	*U**
M	*0.0868*	*0.0012*	*0.0187*	*0.0001*
G	*-2.0977*	*0.0670*	*-3.1324*	*0.0008*
K	*-6.3521*	*-0.3503*	*-7.2088*	*-0.0129*

Table 6.1 Total welfare effect (in percentage deviation from steady state) of a 1% shock in exogenous variables

It seems that a domestic money shock raises welfare in both countries. How can this be? It turns out that the money shock partly offsets some of the inefficiency caused by monopolistic competition. In this case sticky prices actually help to reduce the production inefficiency. Remember that the domestic money shock raises short run domestic wealth since prices must remain the same. This again leads to an increase in demand of both countries' products. Since output is demand driven in the short run, world output in fact increases. The mark-up factor of price over marginal cost falls temporarily, thereby increasing output efficiency. In the long run, output is again determined by the product market equilibrium, and there is no such gain – there is only a redistribution of wealth in favour of domestic agents – which is neutral from a world welfare perspective. Nonetheless, it turns out that the short run welfare increase for foreign agents is large enough to dominate over the welfare decrease that they experience in the long run. A negative shock in the money stock would of course yield a welfare decrease for both countries.

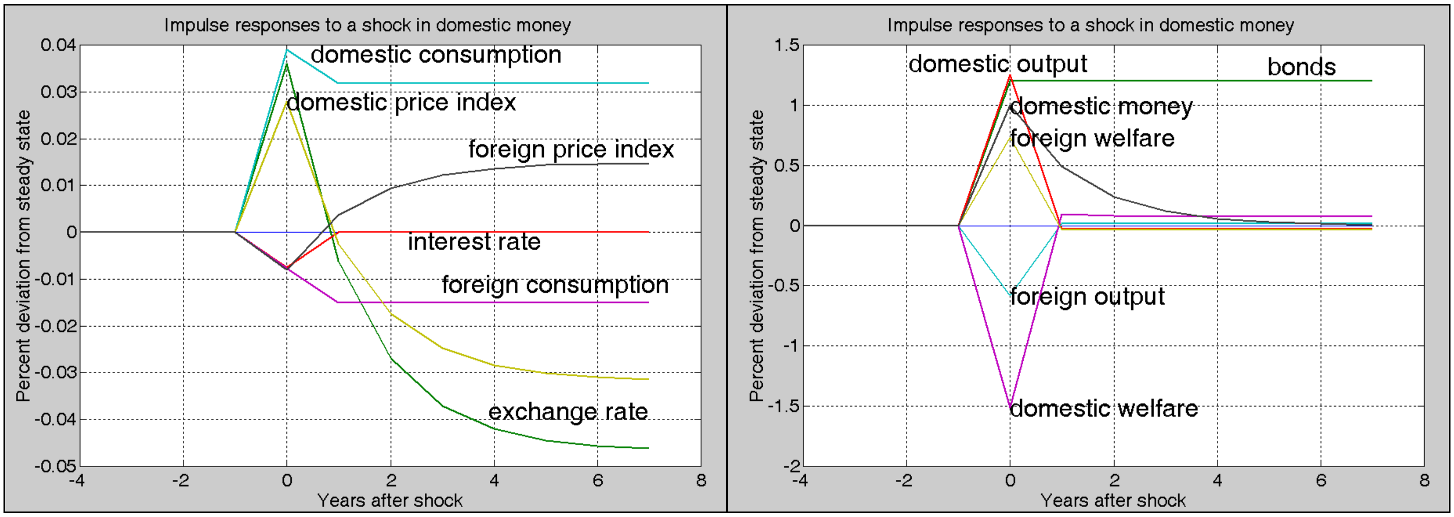
The other results found are less surprising. A government spending shock reduces domestic welfare – obviously because government spending is assumed not to enter the

With our calibration, one gets $\beta^{100}\approx 0.011$ *and* $\beta^{100}\approx 0.077$, respectively. Accordingly the expression $\beta^{s-t}\hat{W}_t$ also goes to zero for a large *s*.

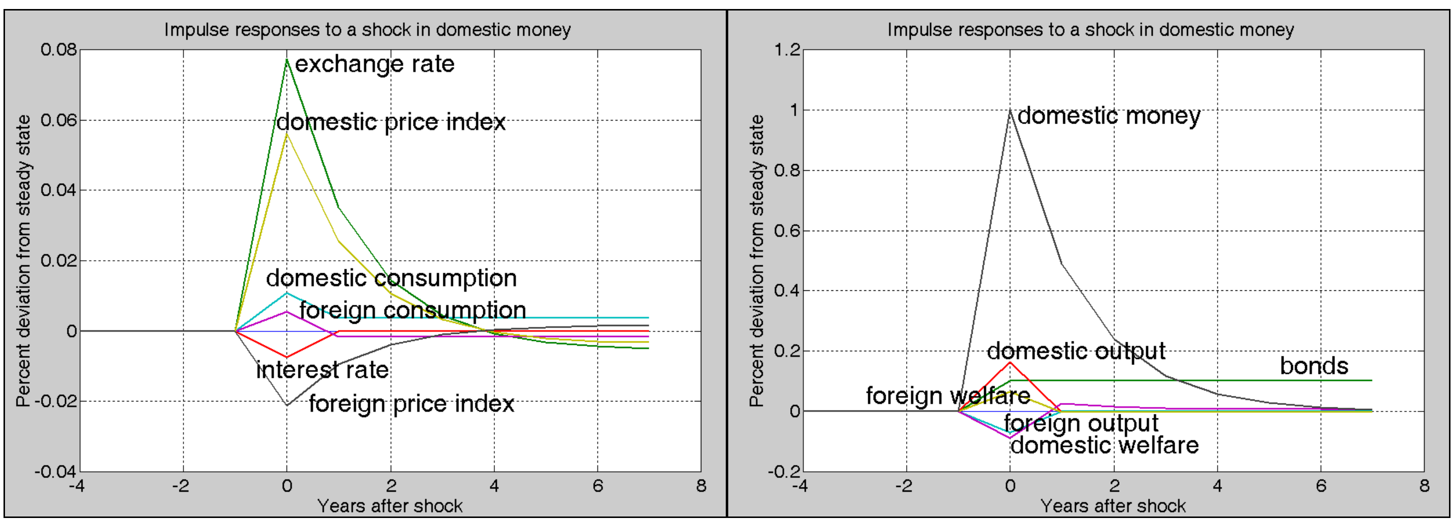
[34] Keep in mind that a monotone transformation on an intertemporal utility function does not lead to any real changes – therefore one should not put any real meaning into the magnitude of the welfare effects – only their relative ordering. Also one should not compare the relative magnitudes for a Norwegian and a U.S. agent directly, since the steady state scale parameter $\overline{W}$ was not calibrated, but set to 1 for simplicity.

utility function. However, the shock increases world demand and therefore leads unambiguously to a rise in welfare abroad – since foreign agents do not have to bear any of the tax burden associated with the shock. We also find that a fall in domestic productivity results in a fall in welfare in both countries.

6.6 THE IMPORTANCE OF THE LEVEL OF COMPETITION



6.13 Impulse responses to a 1% general shock in money, calibrated to U.S. data, but with $\theta = 51$



6.14 Impulse responses to a 1% general shock in money, calibrated to U.S. data, but with $\theta = 3$

Figure 6.13 and 6.14 show two experiments where the elasticity of substitution has been changed. The analysis here is limited to the money shock case only. Until now, $\theta = 6$ has been used, which corresponds to a price mark-up of 20%. 6.13 and 6.14 use $\theta = 51$ and $\theta = 3$ respectively, which correspond to a price mark-up of 2% and 50%, respectively. The two cases could be though of as cases with very high and very low level of competition, respectively.

In a competitive economy, where θ is high, producers face a flat demand curve – i.e. demand changes a lot for small changes in the price. For preset prices, even a small appreciation of the exchange rate will induce large changes in demand for domestic products via the terms-of-trade effect. Since output is demand driven, this directly affects output. This means that the temporary rise in income is much higher in the competitive economy than in the non-competitive economy. Accordingly, in the first case agents save about 1.2% of GDP over the current account, whereas agents in the latter case save only about 0.1% of GDP. It seems that one can conclude that competitive economies are much more heavily affected by shocks.

6.7 CONCLUDING REMARKS SO FAR…

All three shocks lead to permanent changes in consumption, output and bond holdings – and the degree of persistence of a shock mainly affects the magnitude of the random walk behaviour.

The most essential result found is how the money shock, which is not a real shock, in fact can lead to permanent real effects – and a permanent redistribution of wealth. The fact that monetary policy has real effects makes the choice of currency regime and of monetary policy in general more important. It also shows how the countries can affect each other with their monetary policies. This might call for countries that trade at a lot with each other in goods and bonds to coordinate monetary policy. The assumptions of sticky prices and of monopolistic competition, which make output demand driven in the short run, lead to these conclusions. Any thorough analysis of monetary policy should take the effects of these assumptions into account.

Besides from these general remarks, the model in this paper is not very suitable to discuss monetary policy. This is because monetary policy is not explicitly included in the model. The money shock in the model has to be random – so that it cannot be anticipated by private agents. If the money shock was used systematically, say to achieve some objective, e.g. maximize welfare, agents would eventually anticipate the particular monetary policy chosen (in accordance with the rational expectations approach). In such a case there would be no difference between the sticky price model and the same model with flexible prices – i.e. monetary policy would have no real effects at all. Only components of monetary policy that cannot be anticipated could make agents set prices that deviate from optimal prices ex-post. This will be discussed further in section 8.2.

Exchange rate movements between USD, EUR and NOK are heavily discussed in Norway, and movements between EUR and USD in the U.S. and in the E.U. – and in particular the effects for industries with a high level of international competition. As seen in chapter 5 empirical evidence supporting monopolistic competition is quite strong – and as will be seen in section 8.2, the same goes for sticky prices, not only in general, but also of the particular way of modelling this phenomenon that has been chosen in this paper. In the model the exchange rate changes immediately after an unanticipated shock, whereas product prices not until next period. Such findings are supported by data – and should be assumed to address monetary policy properly. A common main objective of many central banks, hereunder the European and the Norwegian, is to keep a stable level of inflation. However, the model predicts that an (unanticipated) change in monetary policy immediately affects the exchange rate – and domestic product prices only after some delay.

In particular short run effects of shocks yield in some cases almost the opposite results of what is found in flex-price models or Keynesian models – e.g. the model predicts that a country actually increases short run production following an unanticapted fall in productivity. Important assumptions that produce these results are the infinite time horizon, sticky prices and monopolistic competition. Without diving deeper into these surprising short run results, it will only be stated that since empirical evidence support these assumptions – their implications should be closer examined.

7. SIMULATIONS

This chapter will simulate data from the model and compare with annual real data time series for the U.S. and Norway for the period 1970-2002. Simulations are performed using the MatLab source code *redux.m* in combination with Uhlig's MatLab-programs "*Toolkit*" (cf. appendix IV).

At first, remember that the variable $\hat{y}_t$ is the model's domestic GDP per capita. But since one of the purposes of the model is to explain current account movements, it could be interesting when evaluating simulations also to consider GNP per capita – the sum of final output and net international factor payments. Only domestic GNP will be considered. In our model this will be the sum of domestic output and net interest from domestic bond holdings:

$$GNP_t = y_t + r_{t-1}F_{t-1} \tag{7.1}$$

Log-linearizing (7.1) yields:

$$G\hat{N}P_t = \hat{y}_t + \tilde{\hat{r}}_{t-1} + \bar{r}\tilde{\hat{F}}_{t-1} \tag{7.2}$$

Figure 7.1 shows plots of the most interesting variables for one simulated time series of length 33 years - starting from the symmetric steady state. In other words the figure shows one particular realization of data, and the figure is meant to give an impression of how some variables seem to be correlated and of their volatility.

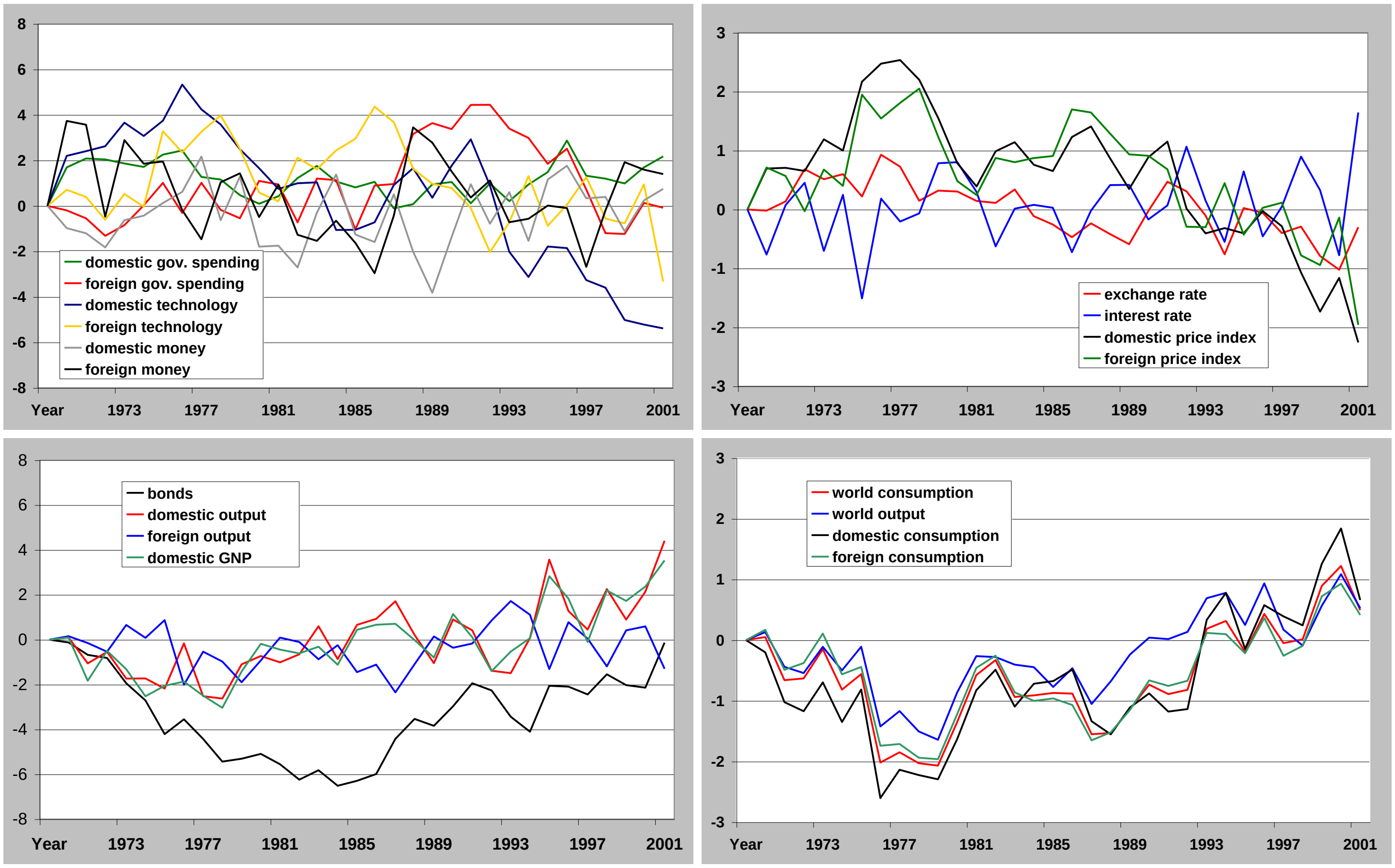


Figure 7.1 Simulated data with a 33 years time horizon using the U.S. data calibration set, starting from the symmetric steady state

Notice from the bottom figures how domestic and foreign output seem to be more volatile than domestic and foreign consumption – and how consumption co-move with world output rather than with output of the individual country. Even though there is only one risk-free asset in the model, it seems that each country is able to insure itself against idiosyncratic risk – volatility in own output independent of the other country. Certainly, by saving and borrowing over the current account both countries gain by the possibility to smooth consumption. Overall risk, however, in terms of the volatility of world output, cannot be prevented.

This risk sharing would also have been the case in the flex-price version of the model. But remember from chapter 6, how sticky prices in combination with monopolistic competition made unanticipated shocks lead to wealth redistributions. This is illustrated in the bottom right of figure 7.1 in the difference between domestic and foreign consumption. Since some of the variables follow random walks, one should be a bit careful in trusting long-run simulations. Cf. section 8.2.

To examine the statistical properties of the model variables in a more formal fashion, 100 time series of length 33 have been drawn. Then cross-correlations between GNP and the most relevant variables with up to 2 leads and lags have been computed together with standard deviations. The results are shown in table 7.1 and 7.2, for the U.S. and Norwegian calibration sets, respectively[35].

Variable v_t	Std. dev.		corr(v_{t+j}, GNP_t)									
dom. bond holdings	8.29	(5.55)	0.16	(0.42)	0.21	(0.41)	0.34	(0.39)	0.38	(0.39)	0.41	(0.39)
exchange rate	1.13	(0.62)	**-0.57**	**(0.28)**	**-0.61**	**(0.26)**	-0.43	(0.34)	-0.42	(0.35)	-0.40	(0.36)
world output	1.08	(0.47)	0.14	(0.44)	0.14	(0.43)	0.29	(0.41)	0.30	(0.41)	0.28	(0.42)
real interest rate	0.59	(0.07)	0.03	(0.14)	**0.35**	**(0.15)**	0.04	(0.14)	-0.03	(0.16)	-0.03	(0.16)
dom. consumption	1.42	(0.64)	0.42	(0.35)	0.45	(0.34)	0.53	(0.31)	0.53	(0.32)	0.50	(0.34)
fgn. consumption	1.52	(0.64)	-0.02	(0.46)	-0.02	(0.45)	0.15	(0.44)	0.17	(0.44)	0.16	(0.45)
dom. output	1.93	(0.63)	**0.53**	**(0.22)**	**0.61**	**(0.19)**	**0.94**	**(0.04)**	**0.58**	**(0.20)**	**0.52**	**(0.22)**
fgn. output	1.57	(0.58)	-0.13	(0.43)	-0.18	(0.42)	-0.21	(0.41)	0.00	(0.42)	0.01	(0.42)
dom. price index	1.44	(0.64)	-0.43	(0.35)	**-0.61**	**(0.29)**	-0.53	(0.31)	-0.51	(0.33)	-0.48	(0.35)
fgn. price index	1.56	(0.65)	0.01	(0.46)	-0.12	(0.45)	-0.16	(0.44)	-0.15	(0.44)	-0.14	(0.43)
dom. GNP	2.16	(0.76)	**0.53**	**(0.21)**	**0.59**	**(0.20)**	**1**	**(0.00)**	**0.59**	**(0.20)**	**0.53**	**(0.21)**
dom. gov. spending	2.06	(0.67)	0.00	(0.41)	0.00	(0.40)	0.03	(0.41)	0.02	(0.41)	0.01	(0.41)
fgn. gov. spending	2.16	(0.72)	0.07	(0.40)	0.05	(0.39)	0.05	(0.39)	0.04	(0.40)	0.03	(0.40)
dom. technology	3.51	(1.54)	**-0.72**	**(0.18)**	**-0.86**	**(0.10)**	**-0.65**	**(0.20)**	**-0.58**	**(0.23)**	**-0.52**	**(0.25)**
fgn. technology	3.35	(1.37)	0.17	(0.45)	0.06	(0.44)	-0.05	(0.43)	-0.06	(0.43)	-0.06	(0.43)
dom. money	1.56	(0.24)	0.00	(0.28)	-0.02	(0.28)	0.08	(0.27)	0.01	(0.26)	0.01	(0.26)
fgn. money	1.54	(0.23)	-0.03	(0.25)	-0.05	(0.24)	-0.12	(0.22)	-0.06	(0.22)	-0.03	(0.24)
Lead/lag j			-2		-1		0		1		2	

Table 7.1 Table of cross-correlations calibrated to U.S. data (simulation-based calculations). Small sample standard errors in brackets

[35] Cross-correlations printed in bold denote that the value deviates more than two small sample standard errors from zero. This means that the corresponding cross-correlations in real data (a particular realization of data) should be expected to be of the same sign.

Variable v_t	**Std. dev.**		**corr(v_{t+j} , GNP_t)**									
dom. bond holdings	17.74	(11.23)	0.30	(0.42)	0.36	(0.42)	0.45	(0.40)	0.47	(0.40)	0.49	(0.40)
exchange rate	0.93	(0.49)	**-0.69**	**(0.26)**	**-0.69**	**(0.26)**	-0.51	(0.35)	-0.50	(0.35)	-0.47	(0.35)
world output	1.43	(0.50)	-0.18	(0.49)	-0.22	(0.47)	-0.09	(0.47)	0.00	(0.47)	-0.01	(0.46)
real interest rate	0.81	(0.11)	-0.08	(0.16)	0.25	(0.15)	0.22	(0.13)	0.01	(0.17)	0.00	(0.14)
dom. consumption	1.57	(0.65)	0.02	(0.48)	-0.02	(0.48)	0.06	(0.47)	0.17	(0.44)	0.17	(0.44)
fgn. consumption	1.98	(0.82)	-0.27	(0.50)	-0.30	(0.49)	-0.18	(0.49)	-0.08	(0.49)	-0.07	(0.48)
dom. output	2.29	(0.64)	**0.57**	**(0.24)**	**0.70**	**(0.18)**	**0.92**	**(0.05)**	**0.56**	**(0.23)**	**0.51**	**(0.23)**
fgn. output	1.44	(0.51)	-0.18	(0.49)	-0.23	(0.47)	-0.10	(0.47)	-0.01	(0.47)	-0.01	(0.46)
dom. price index	1.58	(0.69)	0.03	(0.50)	-0.12	(0.47)	-0.18	(0.45)	-0.18	(0.45)	-0.17	(0.45)
fgn. price index	2.03	(0.84)	0.31	(0.49)	0.18	(0.49)	0.08	(0.49)	0.07	(0.48)	0.07	(0.48)
dom. GNP	2.54	(0.74)	**0.55**	**(0.25)**	**0.67**	**(0.20)**	**1**	**(0.00)**	**0.67**	**(0.20)**	**0.55**	**(0.25)**
dom. gov. spending	2.97	(0.93)	0.00	(0.45)	0.00	(0.44)	0.05	(0.43)	0.05	(0.43)	0.04	(0.43)
fgn. gov. spending	2.65	(0.86)	0.16	(0.39)	0.13	(0.40)	0.16	(0.38)	0.13	(0.38)	0.12	(0.38)
dom. technology	3.48	(1.43)	**-0.74**	**(0.20)**	**-0.83**	**(0.14)**	**-0.62**	**(0.23)**	**-0.56**	**(0.24)**	-0.50	(0.27)
fgn. technology	3.54	(1.40)	0.29	(0.50)	0.17	(0.49)	0.05	(0.49)	0.05	(0.48)	0.05	(0.48)
dom. money	2.47	(0.30)	-0.03	(0.21)	-0.03	(0.19)	0.07	(0.18)	0.00	(0.20)	-0.01	(0.21)
fgn. money	2.48	(0.34)	0.02	(0.20)	0.01	(0.21)	-0.06	(0.19)	0.00	(0.17)	0.00	(0.20)
Lead/lag j			**-2**		**-1**		**0**		**1**		**2**	

Table 7.2 Table of cross-correlations calibrated to Norwegian data (simulation-based calculations). Small sample standard errors in brackets

Both in table 7.1 and table 7.2 one can see that domestic technology has the highest correlation (negative) with GNP when lagged one period. Remember that output in the model is demand driven in the short run – and that a productivity decrease affected output fully one period after the initial shock.

Notice that several of the calculated figures have quite large small sample standard errors. That means that in one particular realization of data (of length 33), the cross-correlations and standard deviations might differ quite considerably from the results in table 7.1 and 7.2. The idea of this chapter is to compare statistical properties of the model with comparable figures for real data. However, if real data behave similarly to in our model, real data must be seen as one particular realization of data. Obviously, similarities or dissimilarities found are to some extent uncertain.

It would have been desirable to use a longer sample than the period 1970-2002, but it has turned out to be difficult to obtain reliable data for earlier periods – and it could also be that extending the sample longer back in time would introduce new problems as data for separate subperiods might not be comparable with each other due to structural changes.

Nonetheless, when comparing the simulated data with real data, it is important to use comparable data. Our model abstracts away from investment and growth, and except for some random walk behaviour, the data pivot around a stable steady state. Real data for consumption, output and prices all show a growing trend and this trend should be removed from the data before comparing with the simulations. If one calculates cross-correlations on unfiltered data, it would e.g. be natural to expect a very high correlation between private consumption and output. However, much of this correlation would occur since both variables follow a growing trend in historical data. By using the Hodrick-Prescott-filter (HP-filter) this trend will be filtered away and one is left with what is thought to be the business cycle component of the data. The idea is to see whether the exogenous processes included in our model can explain some features of business cycles, i.e. how real variables move relative to their trends.

Cf. appendix V for details on the HP-filter and how it is applied in the calculations in this chapter. Unfortunately, reliable real data has not been found for all variables included in table 7.1 and 7.2. All variables have been calculated using time series for the period 1970-2002 – included in appendix VI together with exact descriptions and sources.

	U.S. data						Norwegian data					
Variable v_t	**Std.dev.**	**corr(v_{t+j}, GNP_t)**					**Std.dev.**	**corr(v_{t+j}, GNP_t)**				
dom. bond holdings	10,12*	0,11*	0,02*	-0,06*	-0,18*	-0,15*	17,06*	0,00*	-0,03*	0,02*	0,27*	0,14*
real interest rate	2,69	-0,25*	-0,15	0,16*	0,11*	-0,03*	4,05	0,10*	0,11*	-0,06	-0,07*	0,02*
dom. consumption	1,10*	-0,21*	0,53*	0,89*	0,23*	-0,38	1,73*	-0,40	-0,11*	0,35*	0,35*	0,06*
dom. output	1,46*	-0,33	0,40*	0,97*	0,29*	-0,36	2,10	-0,67	0,09*	0,99*	-0,01*	-0,72
dom. price index	0,51*	0,06*	0,15	-0,35*	-0,41*	0,32	1,28*	0,08*	-0,34*	-0,25*	0,10*	0,20
dom. GNP	1,58*	-0,38	0,35*	1*	0,35*	-0,38	2,14	-0,67	0,05*	1*	0,05*	-0,67
dom. gov. spending	0,92*	0,04*	-0,08*	-0,02*	0,34*	0,38*	1,11*	-0,11*	-0,35*	-0,13*	0,27*	0,47*
dom. money stock	1,66*	-0,10*	0,44*	0,70	0,31*	-0,22*	2,52*	-0,33*	0,24*	0,48*	-0,10*	-0,24*
Lead/lag j		**-2**	**-1**	**0**	**1**	**2**		**-2**	**-1**	**0**	**1**	**2**

Table 7.3 Table of cross-correlations in U.S. and Norwegian HP-filtered-data (based on real data for the period 1970-2002[36]) A * denotes that the value lies within an interval of the simulated equivalent ± 2 small sample std. errors.

Before calculating the cross-correlations and standard deviations in table 7.3, all variables have been calculated as percentage deviations from their HP-filtered trend, except for bond holdings and the real interest rate which have been calculated as a percentage of the GDP-trend and as the absolute deviation from the average real interest rate respectively. This is to make the calculations comparable with our definitions of $\tilde{\hat{F}}_t$ and $\tilde{\hat{r}}_t$ from subsection 3.3.3. Values are marked with a * when they deviate less than 2 small sample standard errors from the equivalent simulated values in tables 7.1 and 7.2[37]. This seems to indicate that the correspondence is quite good. Remember however, that the length of the time series is only 33 periods – leading to large small sample standard errors in tables 7.1 and 7.2 (close to 0.5 for many variables). This means that one cannot rule out that many of the stars in table 7.3 occur due to pure coincidence rather than a remarkable model performance.

[36] All correlations are based on real, per capita variables as in the model. Domestic consumption, domestic output (GDP), domestic price index, domestic GNP and domestic money stock are all HP-filtered and transformed as percentage deviations from their respective trends before calculating correlations. Remember that the model assumes that bond holdings has an initial steady state of zero. To compare with real data the initial debt of Norway and the U.S. in 1970 is set as the initial steady state. In line with the definitions of $\tilde{\hat{F}}_t$ of $\tilde{\hat{r}}_t$ from subsection 3.3.3, bond holdings are transformed as the change in bond holdings after 1970 measured as a percentage of the domestic output (GDP)-trend. This is equivalent to the cumulated current account after 1970 relative to the GDP-trend. Using the average real interest rate to estimate the steady state - the real interest rate is transformed as the absolute deviations from the average.

[37] This is meant as an indication of whether the predications of the model correspond to the real data. However, no confidence level can be given since this requires information of what distributions the

Nonetheless, the model predicts that bond holdings and GNP should be more volatile in Norway than in the U.S., which is confirmed in the data[38]. Furthermore, the model predicts that GNP and domestic consumption should be more correlated in the U.S. than in Norway since Norway has better opportunities to save and borrow over the current account. This can also be confirmed in the data.

From tables 7.1 and 7.2 it seems that there is almost no correlation between the domestic and foreign money stock and domestic GNP. It is clear from the comparisons of permanent, temporary and general money shocks in chapter 6, that if money shocks have low persistence, the real effects are almost negligible. The calibrations of $\rho_M = 0.49$ and $\rho_M = 0.15$ in the U.S. and Norwegian case, respectively, lead to these results. On the contrary table 7.3 seems to indicate some positive correlation between domestic money stock and GNP in the same period.

From chapter 6 it is clear that the technology shock is the shock that implies the most significant real effects. This explains why domestic technology is the only exogenous variable that shows significant correlation with domestic GNP. In other words, technology seems to dominate over the other two shocks in creating volatility of real variables in the model. Unfortunately, this cannot be compared with real data, since productivity cannot be directly observed. However, also real data seem to indicate negligible correlation between domestic government spending and domestic GNP.

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standard deviations and the correlations are drawn from. These indications are weak though, since the small sample standard errors in many cases are quite large.

[38] Norway ran large current account deficits in the 1970s, among other things to finance large investments in the oil industry. The debt was to a large extent repaid in the 1980s. Thereafter Norway has since the second half of the 1990s ran current account surpluses to finance future pension entitlements. In other words, political and demographical conditions could explain the high volatility in Norwegian bond holdings. Nonetheless, the fact that Norway is a small, open economy has made it easier for the country to smooth out such intertemporal unevennesses in consumption – since the current account surpluses and deficits have been modest relative to the size of international capital markets.

8. WEAKNESSES OF THE MODEL – AND WHERE TO GO FROM NOW?

8.1 SENSITIVITY ANALYSIS

The purpose of this section is to examine whether the conclusions from the model are sensitive to the particular calibration used – i.e. whether choosing the wrong calibration might affect the qualitative and quantitative results severely. This will be examined by performing a few experiments rather than attempting to provide a complete analysis.

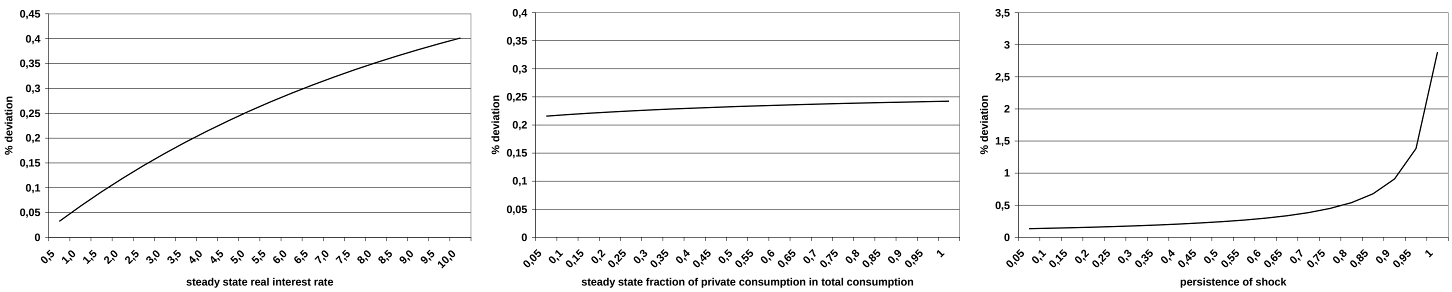

Figure 8.1 Long run bond holdings after a 1% money shock by letting $\bar{r}$, $\bar{c}$ and ρ_M deviate from the U.S. calibration set in table 5.1

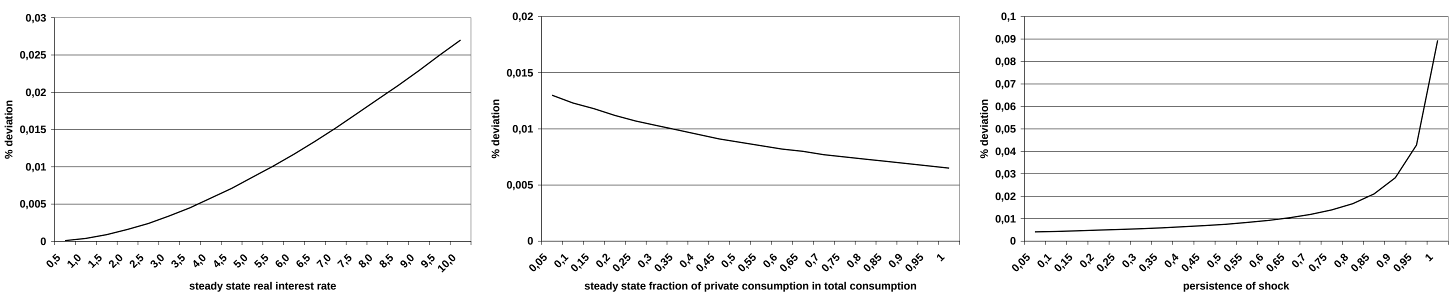

Figure 8.2 Long run domestic consumption after a 1% money shock by letting $\bar{r}$, $\bar{c}$ and ρ_M deviate from the U.S. calibration set in table 5.1

Figure 8.1 and 8.2 start from the 1% money shock using the U.S. calibration set (cf. figure 6.7). Remember that the money shock leads to a redistribution of wealth – and that bond holdings move immediately to the new long run steady state and domestic consumption one period after the shock. Figure 8.1 and 8.2 examine how this new long run steady state after a money shock, in terms of long run bond holdings and domestic consumption, is affected by changing $\bar{r}$, $\bar{c}$ *and* ρ_M .

When changing $\bar{r}$, notice that wealth redistribution depends positively on the steady state real interest rate. The quantitative consequences of some uncertainty when determining $\bar{r}$ do not seem dramatic, though. $\bar{c}$ denotes the steady state fraction of private consumption in total consumption. Since taxes are not distortionary we know that changing $\bar{c}$ does not affect efficiency and the output decision. As one should expect, the effect of changing $\bar{c}$ seems to have negligible effects on the wealth redistribution following a money shock. Bond holdings are almost unaffected. Remember that the higher $\bar{c}$ is, the higher steady state domestic consumption, $\overline{C}$, will be. Since $\hat{C}_t$ measures percentage deviation from steady state, this explains why middle curve in figure 8.2 has a negative slope.

When changing ρ_M we see that uncertainty about ρ_M has larger consequences the higher ρ_M is. In table 5.1 ρ_M was chosen to be 0.53 and 0.15 in the U.S. and Norwegian case, respectively. In particular the Norwegian calibration was very uncertain. But notice though, that with such low persistence, the consequence of choosing the wrong calibration seems quite moderate. Obviously, the results are similar for government spending shocks and technology shocks, where ρ_G =0.9 and ρ_κ =0.95 have been chosen. A small change in ρ_G and ρ_κ might affect the quantitative effects of a shock more severely.

When it comes to n and θ, changing these parameters have already been illustrated in sections 6.4 and 6.6.

8.2 CRITICAL DISCUSSION OF SOME OF THE ASSUMPTIONS IN THE MODEL

It is neither desirable nor possible to design a model that describes every feature of the economy. Undoubtedly, the model in this paper, an extended version of Obstfeld and Rogoff (1995 and 1996), has many limitations. Usually it is more interesting and more useful though, to discuss the topics a model tries to enlighten rather than topics the model does not cover. This section therefore attempts to avoid discussing the most obvious limitations of the model.

The typical justification for the price stickiness assumption is the menu cost approach. When prices initially are set optimally, a small change in demand leads to a small forgone profit if prices are not adjusted. And certainly, if there is a positive menu cost of price adjustment, the optimal response might be to keep prices unchanged until shocks are large enough. This model has used maybe the simplest way possible to model sticky prices, by assuming that prices must be set one year in advance. Other approaches have been common in the litterature. Rotemberg (1982) assumed that changing prices leads to a price adjustment cost which is proportional to the square of the percentage price increase per period. This means that it is more costly to change prices rapidly than slowly. Another famous approach was introduced by Calvo (1983). Calvo assumed that only a fraction $0 \leq \alpha \leq 1$ of randomly drawn firms are allowed to change prices every period – this means that firms must take into account the risk of not getting to change prices in the future whenever they set prices. Nonetheless, all these approaches have in common the implication that output to some extent may be demand driven in the short run – which is what we need for the main conclusions of the model.

When it comes to evidence on sticky prices, Obstfeld and Rogoff (1996), pg. 676, refer to several surveys that support the particular assumption used in this paper. Blinder (1991) studies prices in American manufacturing industry and finds that the average time span between two price changes is about one year. He actually estimates that 55% of GNP represents goods and services that are repriced once a year or more seldom. Kashyap (1995) studies the price adjustment of selected goods for three mail-order companies over a more than 30 year period. He finds that prices on average are changed every 12-18 months. It seems that our assumption that prices can change only once a year is a quite good approximation.

Obstfeld and Rogoff (1996), pg. 607, study the relationship between German and American consumer price indices and the DM/dollar exchange rate in the period 1970-1994. They find that the exchange rate is much more volatile than the relative price indices. Remember from chapter 6 how unanticipated shocks immediately affect the exchange rate whereas price indices are affected much less since product prices are sticky in the short run. It seems that our model is well equipped to explain such findings.

As seen in chapter 7, simulations were quite uncertain when comparing with real data (a particular realization of data) – much due to the short simulation length. As argued there, complete data sets from before 1970 are hard to obtain. A possible solution is therefore to instead change from annual data to quarterly data. This however imposes additional problems when it comes to sticky prices. One cannot simply apply quarterly data to the theoretical model presented here. The assumption that products are repriced once a year should be kept – i.e. one has to modify the model, by e.g. assuming that prices must be set four periods ahead. Considering the uncertain conclusions in chapter 7, though, this would have been a very interesting extension.

Remember that the random walk properties of the model followed directly from the assumption of an infinite time horizon. This assumption can be justified if agents also care about future generations. Agents will also behave almost as if they had an infinite time horizon if they are uncertain about how many periods they will live. Certainly an alternative approach would be to introduce overlapping generations – this would however complicate the model severely. It is nonetheless important to remember that the infinite horizon assumption is an approximation – and that the time horizon in reality may be long, but not infinite. This means that variables such as bond holdings, consumption and the nominal exchange rate will be *close to* random walks, but not exact random walks. For a relatively short simulation length, such as the 33 years simulations from chapter 7, the difference must be expected to be of relatively moderate magnitude.

Another critical assumption is the assumption that there is only one asset – a real riskfree bond. The other extremity would be the availability of Arrow-Debreu assets – i.e. that agents can insure themselves against any type of risk. This would mean that agents also could insure themselves against the risk of not being able to set prices optimally. Implications could be that sticky prices to a large degree lose their effect. However, Obstfeld and Rogoff (1995a) refer to another paper, Obstfeld and Rogoff (1995b), where it is shown that no insurance possibilities (only riskfree bonds) is an assumption much closer to data than the other extremity. Yet another possibility would be that only nominally risk free bonds were available. This would imply that Ricardian equivalence no longer would hold. And it would be possible for the government to use inflation as a lump sum tax. Cf. e.g. Schmitt-Grohé and Uribe (2001) for a model similar to the one in this paper, but with the assumption mentioned. One could argue that the model is this paper should be extended allowing for some insurance. Nonetheless, the qualitative results would not change – it would only yield that the effects of shocks might be somewhat less severe.

As mentioned in section 6.7 the lack of explicitly modelled fiscal and monetary policy, makes policy analysis in our framework difficult. This is maybe the main drawback of the approach. Except for the autocorrelation, policy cannot be anticipated at all. It is more natural to imagine that the authorities attempt to fulfil some objective. It could be either be in terms of some feedback rules for policy – e.g. a monetary Taylor-rule, or simply that the objective is to choose fiscal and monetary policy so that welfare is maximized. On the other hand, it seems reasonable that policy as perceived by private agents at least to some extent is unpredictable – i.e. has some uncertain components. This means that the approach in this paper simplifies by exaggerating this uncertainty. An even if one included policy explicitly as mentioned, the implications of the uncertain components of policy would still be similar to in the framework here. The conclusion must be that the model in this paper does not yield much insight in what policy to choose to maximize welfare. However, it yields important insight in what the consequences are of choosing a policy that cannot be fully predicted by private agents. Schmitt-Grohé and Uribe (2001) present a model somewhat similar to the model here, but where the authorities choose monetary and fiscal policy to maximize welfare (given the optimal response of private agents). Due to rational expectations, agents can predict what the authorities will do – and the only uncertainty left in their model is technology. In that sense their model shows the other extremity when it comes to the predictability of policy. Their model requires that the authorities are able to commit to rules that are optimal in the long run, but not necessarily in the short run (the classical time inconsistency problem). An assumption somewhat between the model in this paper and the model in Schmitt-Grohé and Uribe would have been the most realistic assumption[39].

It should be mentioned that the particular statistical properties chosen for the error terms of the stochastic processes, i.e. that the error terms are uncorrelated across countries and across the three types of uncertainty, is just a simplification chosen to focus on the effect of uncertainty in general. The way I see it, the model could not be significantly improved by introducing correlation between the error terms – only by introducing policy explicitly and preferably with microfoundations. One could e.g. assume that fiscal and monetary policy is chosen so that welfare is maximized – but assume that the authorities sometimes make mistakes, or that conflicts of interest make them deviate from optimum, and that this manifests itself as randomly distributed error terms in the choice variables.

The simplification that the economy initially is in a symmetric steady state, should be considered as a reference case. Even though there is initial symmetry, all three kinds of shocks lead to permanent asymmetry (due to the infinite horizon assumption). And from the discussion in chapter 6, it is evident what would have been the effect of initial asymmetry. In the asymmetric steady state, one country would have initial wealth, the other initial debt. The interest the first country would receive every period would alter the trade-off between work and consumption – and agents would choose more leisure. In the other country the opposite would be the case. Except for that, one would gain no new qualitative insight. The initial symmetry is chosen to abstract away from effects of initial asymmetry and focus on the shocks only.

[39] The ability to commit to rules in Schmitt-Grohé and Uribe should maybe not be interpreted as an assumption, but rather as a recommendation. And the "Growth and stability pact", providing rules for fiscal policy in the Euro zone, could be seen as a practical example of how to formalize such recommendations.

Finally, note that the utility function does not contain government spending, meaning that a government shock that leads to higher total consumption in fact lowers welfare. This limitation is important to remember in particular when analyzing government spending shocks. In our model it is assumed that government spending is allocated in exactly the same way as private consumption. Some people would claim that $ 1 in government spending is less worth than $ 1 in private consumption, measured in utility, since the government has less information on how to allocate resources optimally. Others would claim that government spending can be used to even out an uneven distribution of consumption across agents in the country. In relation to this the possibility of introducing distortionary taxes also rises. All of this, however, goes into the account for obvious limitations – and should be discussed in models specially designed for these purposes.

8.3 ACCURACY OF THE SOLUTION BASED ON LINEAR APPROXIMATION

Solving the model numerically was made possible using linear approximation techniques. Finding a general analytical solution that holds for each date and each state of the world goes far beyond the scope of this paper and seems to be an almost impossible task. It is important to realize that since the solution of the model includes some unit roots – i.e. some variables follow random walks, one should not trust long-run simulations. This is because the variables that follow random walks, are bound to eventually wander far away from the initial steady state. This means that the longer the length of a simulation, the more inaccurate one must expect the linear approximation to be. Notice from tables 7.1 and 7.2, which are based on simulations of length 33 years, that except for bond holdings, all variables have standard deviations less than 4%. Bond holdings have standard deviations of 8.29% and 17.74% of GDP in the U.S. and Norwegian case, respectively. And it is mostly in the latter case the value might give some concerns about how this affects the accuracy of the solution. Notice also that the volatility of the solution of course depends on the volatility of the stochastic processes. I.e. the more volatile one calibrates government spending, the money stock and technology, the more inaccurate one should expect the log-linear solution to be. Due to the reasons stated above, rather than attempting to solve the model exactly, this problem will be addressed by referring to literature where the log-linearized solution and the exact solution somewhat simpler models have been compared.

As mentioned in section 8.2, Schmitt-Grohé and Uribe (2001) discuss a model that has several similarities to the model in this paper. Their model is also a stochastic production economy without capital and with monopolistic competition, but it is a closed economy. And there is also price stickiness, but of the Rotemberg-type. There is only one type of bonds, but it is nominally risk free. But the main difference is that policy is explicitly modeled. Monetary policy involves choosing the nominal interest rate. Government spending is given as an exogenous stochastic process, but Ricardian equivalence does not hold in their setup and the government can therefore choose taxes, which are distortionary. In addition to government spending, technology is the only stochastic process. And it is assumed that fiscal and monetary policy is coordinated to maximize welfare of the representative agent. They show that if one simplifies the model by assuming that prices are flexible, finding an exact solution reduces to a single intertemporal time-separable problem. They compare the exact solution with the log-linear solution by calculating the mean and the standard deviation of the labour tax rate, the nominal interest rate and the rate of inflation. For the first two variables the differences are negligible. For the rate of inflation, the difference turns out to be somewhat larger, but the rate of inflation in their model is also the most volatile variable of the three. Its standard deviation is computed to be 7.92 and 6.8 percentage points, using the exact and the log-linear solution, respectively. Schmitt-Grohé and Uribe (2001) conclude the comparison by stating that *the quantitative results obtained using the exact numerical solution and a log-linear approximation are remarkably close*.

9. CONCLUSION

The objective of this paper was to examine the implications of sticky prices and monopolistic competition in open economies. Unlike most other approaches in this field, the focus has been to find quantitative results - and to develop a framework, using the methods in Uhlig (1997), in which many experiments easily can be conducted out in terms of changing the calibration or underlying assumptions.

One of the main results is how unanticipated monetary shocks lead to a redistribution of wealth – this might be a result that contributes to justify coordination of monetary policy for countries that trade a lot in goods and financial assets – which has been put into practice e.g. in the Euro zone. Despite the wealth redistribution, positive unanticipated money shocks are shown to undo some of the inefficiencies caused by monopolistic competition. This cannot be applied systematically though – as only unanticipated shocks have this effect.

Another important result is that technology shocks must have some persistence in order to yield real results. The model predicts that purely temporary shocks in technology have no effects on consumption, output and the current account. This results follows from the fact that sticky prices and monopolistic competition make output demand driven in the short run.

Furthermore, some first and second other moments when simulating the model was compared to HP-filtered real data for the U.S. and Norway in the period 1970-2002 – to examine whether the model is well equipped to explain cyclical movements in real data. The fit seems quite good. However, this conclusion is very uncertain. Firstly because the model predicts that the volatility among particular realizations of data for such short periods of time is quite high. Secondly, because one has to choose some way to detrend real data in order to compare with the model – and this imposes additional uncertainty.

Regardless of this, there is strong empirical evidence that supports the assumptions of sticky prices and monopolistic competition. And the approach provided here suggests methods on how to extend existing theoretical models in the field – so that they more easily can be compared with real data. Though limited as applied here the approach opens for numerous extensions and experiments to be undertaken.

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APPENDIX I - Formal solution of the system of second-order difference equations

This appendix gives a formal solution of the special kind of system of difference equations that appears in the main sections of the paper. The appendix is based on the approach found in Uhlig (1997), but is extended and explained much more thoroughly with the hope to make it accessible for a wider audience – in particular for people with only basic skills in linear algebra.

A1.1 OBJECTIVE

Consider the system of second-order stochastic difference equations from chapter 2, (2.1)-(2.3), and the recursive law of motion (2.4)-(2.5). The goal was to find matrices P, Q, R and S so that the equilibrium is stable.

A1.2 THE PSEUDOINVERSE FOR A MATRIX WITH FULL RANK[40]

Consider a system of linear equations $Cx = b$, where C is of size $l \times n$ with rank n, x is of size $n \times 1$ and $l \geq n$ as before and where b is of size $n \times 1$. This system is typically inconsistent in the case where $l \geq n$, since the system contains n variables and l equations. This means that there might be no x that perfectly solves the system. In order words, b is not a linear combination of the columns of C, i.e. not in the column space of C. A "second best solution" is to choose an $\bar{x}$ that minimizes the error $\|C\bar{x} - b\|$. This implies to find the point $p = C\bar{x}$ that is closer to b that any other point in the column space of C.

This problem is most easily solved by considering the special case where $l = 3$ and $n = 2$. Then the column space of C will be a plane, and b will be a three-dimensional vector. The shortest distance from the column space to b will a line that is perpendicular to the column space, in other word the projection of b onto the column space of C. See fig. A1.1. This property holds in general, i.e. for any $l \geq n$.

[40] This subsection follows the approach in Strang (1980), pp. 112-114 and 137-138.

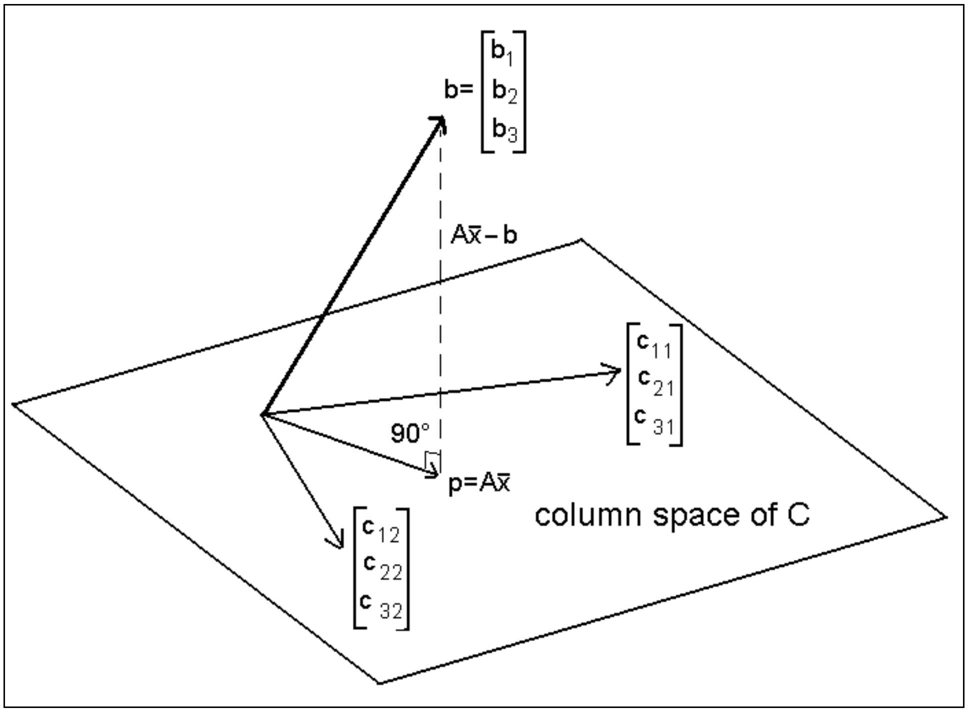

Figure A1.1 Projection of b onto the column space of C. Source: Strang (1980), pg. 113

All linear combinations of *C* are perpendicular to the error vector $C\overline{x}-b$. For any vector *y* of length *l*, we can therefore write

$$(Cy)^T(C\overline{x}-b)=0$$

since *Cy* and $C\overline{x}-b$ are orthogonal vectors. *T* denotes the transposed of a matrix. This can be rewritten

$$\begin{aligned} & y^T C^T (C\overline{x}-b)=0 \\ \Leftrightarrow\ & y^T(C^T C\overline{x}-C^T b)=0 \qquad (A1.1) \end{aligned}$$

Since (A1.1) must hold for any *y*, it is obvious that the brackets must be equal to zero:

$$C^T C\bar{x} - C^T b = 0$$
$$\Leftrightarrow \bar{x} = (C^T C)^{-1} C^T b \qquad (A1.2)$$

For the last equality we need $C^T C$ to be invertible. This follows from the fact that C has rank n, i.e. the columns of C are linearly independent. Then $C^T C$ is a symmetric matrix, also of rank n and is invertible (cf. Strang (1980), pg. 109).

In the special case where the rank of C is n, the coefficient matrix on b in (A1.2) is exactly the definition of the so called *pseudoinverse* of C (also called the *Moore-Penrose inverse*), denoted C^+[41]:

$$C^+ \equiv (C^T C)^{-1} C^T \qquad (A1.3)$$

A1.3 FORMAL SOLUTION

Plug the recursive law of motion (2.4)-(2.5) into equation (2.1). This yields

$$A(Px_{t-1} + Qz_t) + Bx_{t-1} + C(Rx_{t-1} + Sz_t) + Dz_t = 0$$
$$\Leftrightarrow (AP + CR + B)x_{t-1} + (AQ + CS + D)z_t = 0 \qquad (A1.4)$$

(A1.4) must hold for any x_{t-1} and any z_t. This means that the coefficient matrices on x_{t-1} and z_t must be zero.

$$AP + CR + B = 0 \qquad (A1.5)$$
$$AQ + CS + D = 0 \qquad (A1.6)$$

Plug the recursive law of motion (2.4)-(2.5) into equation (2.2) so that only the variables x_{t-1} and z_t are left. This yields

[41] Note that in the special case where $l = n$, C is quadratic and invertible since it has rank n. Then the pseudoinverse reduces to the normal inverse, C^{-1}.

$$
\begin{aligned}
0 = E_t\{&F(Px_t + Qz_{t+1}) + G(Px_{t-1} + Qz_t) + Hx_{t-1} + J(Rx_t + Sz_{t+1}) + \\
&K(Rx_{t-1} + Sz_t) + Lz_{t+1} + Mz_t\} \\
\Leftrightarrow 0 = E_t\{&F[P(Px_{t-1} + Qz_t) + Qz_{t+1}] + G(Px_{t-1} + Qz_t) + Hx_{t-1} + \\
&J[R(Px_{t-1} + Qz_t) + Sz_{t+1}] + K(Rx_{t-1} + Sz_t) + Lz_{t+1} + Mz_t\}
\end{aligned}
$$

use from $(A2.3)$ that $E_t\{z_{t+1}\} = Nz_t$

$$
\begin{aligned}
\Leftrightarrow 0 = &\left[(FP + JR + G)P + KR + H\right]x_{t-1} + \\
&\left[(FQ + JS + L)N + (FP + JR + G)Q + KS + M\right]z_t \qquad (A1.7)
\end{aligned}
$$

(A1.7) must also hold for all x_{t-1} and z_t. Thus

$$(FP + JR + G)P + KR + H = 0 \qquad (A1.8)$$
$$(FQ + JS + L)N + (FP + JR + G)Q + KS + M = 0 \qquad (A1.9)$$

(A1.5) can be rewritten as

$$CR = -(AP + B)$$

Premultiply with the pseudoinverse C^+ on both sides:

$$
\begin{aligned}
&(C^TC)^{-1}(C^TC)R = -C^+(AP + B) \\
&\Leftrightarrow R = -C^+(AP + B) \qquad (A1.10)
\end{aligned}
$$

Plug (A1.10) into (A1.8):

$$
\begin{aligned}
&0 = \left(FP + J\left[-C^+(AP + B)\right] + G\right)P + K\left[-C^+(AP + B)\right] + H \\
&\Leftrightarrow 0 = (F - JC^+A)P^2 - (JC^+B - G + KC^+A)P - KC^+B + H \qquad (A1.11)
\end{aligned}
$$

Now, consider the matrix C^T. It is an $n \times l$ matrix and the system of equations $C^Ty = b$ where y and b are vectors of length l, has $l - n$ degrees of freedom, since C^T has n

equations and l unknowns, and since C^T is of rank n. This means that a basis of the nullspace[42] of C^T must be $(l-n)$-dimensional. Define the matrix C^0 to be a matrix whose rows constitute a basis of the nullspace of C^T. Then the following must hold:

$$\begin{aligned} & C^T(C^0)^T = 0 \\ & \Leftrightarrow (C^0C)^T = 0 \Leftrightarrow C^0C = 0 \end{aligned} \qquad (A1.12)$$

Premultiply (A1.5) with C^0 and use (A1.12):

$$\begin{aligned} & C^0CR = -C^0(AP+B) \\ & \Leftrightarrow 0 = C^0AP + C^0B \end{aligned} \qquad (A1.13)$$

P is found by solving the matrix quadratic equation that consists of (A1.11) and (A1.13). The formal method for this is included in an own section, section A1.4. Given P, (A1.10) gives the solution for R.

To solve for Q and S given the solution for P and R, consider the coefficient matrices on z_t, i.e. equations (A1.6) and (A1.9). Take the columnwise vectorization[43] of (A1.6):

$$vec(AQ+CS) = vec(-D)$$

This can be rewritten as

$$(I_k \otimes A)vec(Q) + (I_k \otimes C)vec(S) = -vec(D) \qquad (A1.14)$$

where I_k denotes the identity matrix of dimension k and $\otimes$ denotes the Kronecker product[44]. Equivalently for (A1.9):

[42] The nullspace of a matrix A is the space of all vectors x that that solve the equation $Ax = 0$.
[43] Cf. e.g. Sydsæter et al. (1998) sect. 23.15.

$$vec\left[(FQ + JS)N + (FP + JR + G)Q + KS\right] = vec(-LN + M)$$
$$\Leftrightarrow vec\left[FQN + JSN + (FP + JR + G)Q + KS\right] = vec(-LN + M)$$

It turns out that *vec(FQN)* can be rewritten as $[N^T \otimes F]$ *vec(Q)*. Equivalently *vec(JSN)* becomes $[N^T \otimes J]$ *vec(S)*. Then the whole expression can be rewritten as

$$\left[N^T \otimes F + I_k \otimes (FP + JR + G)\right] vec(Q) + \left[N^T \otimes J + I_k \otimes K\right] vec(S) = -vec(LN + M) \qquad (A1.15)$$

Rewrite (A1.14)-(A1.15) in matrix form:

$$\mathbf{V}\begin{bmatrix}\mathbf{vec(Q)}\\ \mathbf{vec(S)}\end{bmatrix} = -\begin{bmatrix}\mathbf{vec(D)}\\ \mathbf{vec(LN+M)}\end{bmatrix} \qquad (A1.16)$$

where V is defined as

$$\mathbf{V} \equiv \begin{bmatrix}\mathbf{I_k} \otimes \mathbf{A}, & \mathbf{I_k} \otimes \mathbf{C}\\ \mathbf{N^T} \otimes \mathbf{F} + \mathbf{I_k} \otimes \mathbf{(FP + JR + G)}, & \mathbf{N^T} \otimes \mathbf{J} + \mathbf{I_k} \otimes \mathbf{K}\end{bmatrix} \qquad (A1.17)$$

If V is invertible, the unique solution for Q and S will be

$$\begin{bmatrix}\mathbf{vec(Q)}\\ \mathbf{vec(S)}\end{bmatrix} = -\mathbf{V^{-1}}\begin{bmatrix}\mathbf{vec(D)}\\ \mathbf{vec(LN+M)}\end{bmatrix} \qquad (A1.18)$$

[44] Cf. e.g. Sydsæter et al. (1998) sect. 23.1.

A1.4 SOLVING A MATRIX QUADRATIC EQUATION USING THE QZ-DECOMPOSITION[45]

Our goal is to solve the matrix quadratic equation

$$\Psi P^2 - \Gamma P - \Theta = 0 \qquad (A1.19)$$

for P which is of size $m \times m$. The same goes for the coefficient matrices Ψ, Γ and Θ.

(A1.11) and (A1.13) fits into this format by choosing

$$\begin{aligned}
\Psi &= \begin{bmatrix} \mathbf{0}_{(\mathbf{l-n}),\mathbf{m}} \\ \mathbf{F} - \mathbf{J}\mathbf{C}^{+}\mathbf{A} \end{bmatrix} \\
\Gamma &= \begin{bmatrix} \mathbf{C_0 A} \\ \mathbf{J}\mathbf{C}^{+}\mathbf{B} - \mathbf{G} + \mathbf{K}\mathbf{C}^{+}\mathbf{A} \end{bmatrix} \\
\Theta &= \begin{bmatrix} \mathbf{C_0 B} \\ \mathbf{K}\mathbf{C}^{+}\mathbf{B} - \mathbf{H} \end{bmatrix}
\end{aligned} \qquad (A1.20)$$

$0_{(l-n),m}$ denotes a matrix of zeros of size $(l - n) \times m$. Note that (A1.13) constitutes the first $l - n$ rows of the matrices, whereas (A1.16) constitutes the last $m + n - l$ rows. The solution as such follows in subsection A1.4.1. Subsection A1.4.2 discusses requirements needed to find a (unique) stable solution.

A1.4.1 Solution

Introduce the following notation; for a $2m \times 2m$ matrix X its partition will be written as

---

[45] Uhlig (1997) solves the matrix quadratic equations by turning the problem into a *generalized eigenvalue and eigenvector problem*. This approach, however, is not fully general as it has trouble to deal with repeated eigenvalues. Section A1.4 instead uses the more general method of *QZ-decomposition* (also called *generalized Schur decomposition*) to solve the equation. The section is based on the approach in Uhlig (2003).

$$\mathbf{X} \equiv \begin{bmatrix} \mathbf{X_{11}} & \mathbf{X_{12}} \\ \mathbf{X_{21}} & \mathbf{X_{22}} \end{bmatrix}$$

where X_{ij} denotes a $m \times m$ submatrix.

With this in mind, define the following $2m \times 2m$ matrices:

$$\Xi \equiv \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Theta} \\ \mathbf{I_m} & \mathbf{0_{m,m}} \end{bmatrix}$$
$$\Delta \equiv \begin{bmatrix} \mathbf{\Psi} & \mathbf{0_{m,m}} \\ \mathbf{0_{m,m}} & \mathbf{I_m} \end{bmatrix} \qquad (A1.21)$$

The next step is to find the so called QZ-decomposition[46] of Ξ and Δ. This implies to find unitary matrices[47] Y and Z and upper triangular matrices Σ and Φ, all of which may be complex, that satisfy

$$Y^T \Sigma Z = \Delta$$
$$Y^T \Phi Z = \Xi \qquad (A1.22)$$

This QZ-decomposition always exists even though it may not be unique. Let σ_{ij} and ϕ_{ij} denote element (i,j) of the matrices Σ and Φ, respectively. The absolute value of the ratios of the diagonal elements of Σ and Φ, $|\phi_{ii} / \sigma_{ii}|$, are called the *generalized eigenvalues* of Δ and Ξ. As stated by Sims (2000), the set of generalized eigenvalues is unique unless Δ and Ξ have one of more eigenvalues of zero that correspond to the same eigenvector[48]. This problem will not occur as long as (2.1)-(2.2) contains no redundant equations.

[46] Cf. Sims (2000), pg. 9

[47] A unitary matrix A satisfies A^TA=I, but unlike an orthogonal matrix, A may be complex. Cf. Strang (1980), pg. 223

[48] i.e. unless there exists a vector c so that $\Delta c= \Xi c=0$.

Furthermore, let the QZ decomposition be chosen so that the generalized eigenvalues appear in ascending order along the diagonals of Σ and Φ. Sims (2002) shows how this can be done[49]. Assume that the upper left *m* generalized eigenvalues are stable, i.e. that $|\phi_{mm} / \sigma_{mm}| < 1$. It turns out that this is exactly the requirement that makes *P* stable. A proof of this is provided in the end of the section. Furthermore, assume that Z_{21} and Y_{21} are invertible which is required to be able to find a solution for *P*.

Claim:

P is given by:

$$P = -Z_{21}^{-1} Z_{22} \qquad (A1.23)$$

Proof:

Solving (A1.19) is equivalent to showing that *P* satisfies (A1.24) for any vector $x \in \Re^m$.

$$(\Psi P^2 - \Gamma P - \Theta)x = 0 \qquad (A1.24)$$

Define the vector function *v(x)* of length *2m*:

$$v(x) \equiv \begin{bmatrix} \mathbf{Px} \\ \mathbf{x} \end{bmatrix} \qquad (A1.25)$$

Rewrite (A1.24) as

[49] The standard MatLab routine QZ(·) computes a QZ decomposition as described above, but does not sort the generalized eigenvalues. Sims has written an extra routine that performs the sorting, *qzdiv.m* which is available at the web page `http://www.princeton.edu/~sims/#gensys`.

$$\begin{aligned}
&\Psi P^2 X = \Gamma P x + \Theta x \\
&\Leftrightarrow \begin{bmatrix} \mathbf{\Psi P^2 X} \\ \mathbf{Px} \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma Px + \Theta x} \\ \mathbf{Px} \end{bmatrix} \\
&\Leftrightarrow \begin{bmatrix} \mathbf{\Psi} & \mathbf{0_{m,m}} \\ \mathbf{0_{m,m}} & \mathbf{I_m} \end{bmatrix} \begin{bmatrix} \mathbf{P^2 X} \\ \mathbf{Px} \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Theta} \\ \mathbf{I_m} & \mathbf{0_{m,m}} \end{bmatrix} \begin{bmatrix} \mathbf{Px} \\ \mathbf{x} \end{bmatrix} \\
&\Leftrightarrow \Delta v(Px) = \Xi v(x)
\end{aligned} \qquad (A1.26)$$

Define the vector function *w(x)* of length *2m* as *w(x) ≡ Z v(x)*. Plug (A1.25) into the definition:

$$w(x) = \begin{bmatrix} \mathbf{Z_{11}Px + Z_{12}x} \\ \mathbf{Z_{21}Px + Z_{22}x} \end{bmatrix} = \begin{bmatrix} \mathbf{Z_{11}Px + Z_{12}x} \\ 0_{m,1} \end{bmatrix} \qquad (A1.27)$$

where the last equality follows from plugging in (A1.23) for *P*.

Using the QZ-decomposition (A1.22) and the definition (A1.27), (A1.26) can be rewritten as

$$\begin{aligned}
&Y^T \Sigma Z v(Px) = Y^T \Phi Z v(x) \\
&\Leftrightarrow Y^T \Sigma w(Px) = Y^T \Phi w(x) \\
&\Leftrightarrow \begin{bmatrix} \mathbf{Y_{11}} & \mathbf{Y_{21}} \\ \mathbf{Y_{12}} & \mathbf{Y_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma_{11}} & \mathbf{\Sigma_{12}} \\ \mathbf{\Sigma_{21}} & \mathbf{\Sigma_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{Z_{11}P^2x + Z_{12}Px} \\ \mathbf{0_{m,1}} \end{bmatrix} = \begin{bmatrix} \mathbf{Y_{11}} & \mathbf{Y_{21}} \\ \mathbf{Y_{12}} & \mathbf{Y_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi_{11}} & \mathbf{\Phi_{12}} \\ \mathbf{\Phi_{21}} & \mathbf{\Phi_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{Z_{11}Px + Z_{12}x} \\ \mathbf{0_{m,1}} \end{bmatrix}
\end{aligned}$$

$$\Leftrightarrow \begin{bmatrix} \mathbf{Y_{11}\Sigma_{11}(Z_{11}P^2x + Z_{12}Px)} \\ \mathbf{Y_{21}\Sigma_{11}(Z_{11}P^2x + Z_{12}Px)} \end{bmatrix} = \begin{bmatrix} \mathbf{Y_{11}\Phi_{11}(Z_{11}Px + Z_{12}x)} \\ \mathbf{Y_{21}\Phi_{11}(Z_{11}Px + Z_{12}x)} \end{bmatrix} \qquad (A1.28)$$

Note that it has been used from (A1.22) that $\mathbf{\Sigma_{21}} = \mathbf{\Phi_{21}} = 0$ which follows from the fact that $\mathbf{\Sigma}$ and $\mathbf{\Phi}$ are upper triangular.

We see from both sides of the equation (A1.26) that the last *m* rows are equal ($Px = Px$). All that has been done in computing (A1.28) is making a claim about *P* and thereafter plugged in some definitions and the QZ-decomposition. Consequently, the last *m* rows must still be equal, no matter the claim about *P*. Thus

$$\begin{aligned} & Y_{21}\Sigma_{11}(Z_{11}P^2x + Z_{12}Px) = Y_{21}\Phi_{11}(Z_{11}Px + Z_{12}x) \\ & \Leftrightarrow \Sigma_{11}(Z_{11}P^2x + Z_{12}Px) = \Phi_{11}(Z_{11}Px + Z_{12}x) \end{aligned} \qquad (A1.29)$$

where the last manipulation uses that Y_{21} is invertible. Plug (A1.29) into the first *m* rows of (A1.28) to establish that the equality holds also for the first *m* rows. Thus the claim (A1.23) has been proved:

$$\begin{aligned} & P = -Z_{21}{}^{-1}Z_{22} \quad \Rightarrow \\ & \Psi P^2 x = Y_{11}\Sigma_{11}(Z_{11}P^2x + Z_{12}Px) = \\ & \qquad Y_{21}\Phi_{11}(Z_{11}Px + Z_{12}x) = \Gamma Px + \Theta x \end{aligned} \qquad (A1.30)$$

The equalities are obtained using (A1.21) and (A1.22)

A1.4.2 Stability and uniqueness of the solution

Finally, we will check what is required for *P* to be stable and unique. With stability it is meant that the vector of state variables *x* returns to steady state on an infinite time horizon absent new shocks. I.e.:

$$\lim_{n\to\infty} P^n x = 0 \qquad \forall x \qquad (A1.31)$$

With uniqueness it is meant that there only can be found one *P* that satisfies (A1.31). Apply (A1.25):

$$(A1.31) \Leftrightarrow \lim_{n\to\infty} v(P^n x) = \lim_{n\to\infty}\begin{bmatrix} P^{n+1}x \\ P^n x \end{bmatrix} = 0 \qquad \forall x \qquad (A1.32)$$

Obviously (A1.31) holds if and only if (A1.32) holds. Then, apply (A1.27):

$$(A1.32) \Leftrightarrow \lim_{n\to\infty} w(P^n x) = \lim_{n\to\infty} Zv(P^n x)$$
$$= \lim_{n\to\infty}\begin{bmatrix} Z_{11}P^{n+1}x + Z_{12}P^n x \\ 0_{m,1} \end{bmatrix} = 0 \;\; \forall x \qquad (A1.33)$$

Obviously (A1.32) holds if and only if (A1.33) holds[50]. It is clear that finding requirements such that P is stable is equivalent to finding requirements such that $w(P^n x)$ is stable.

Combine (A1.19) and (A1.25) to calculate the following two expressions:

$$\Delta v(P^n x) = \begin{bmatrix} \boldsymbol{\Psi} & \mathbf{0_{m,m}} \\ \mathbf{0_{m,m}} & \mathbf{I_m} \end{bmatrix}\begin{bmatrix} P^{n+1}x \\ P^n x \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}P^{n+1}x \\ P^n x \end{bmatrix} \qquad (A1.34)$$

$$\Xi v(P^{n-1}x) = \begin{bmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Theta} \\ \mathbf{I_m} & \mathbf{0_{m,m}} \end{bmatrix}\begin{bmatrix} P^n x \\ P^{n-1}x \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Gamma}P^n x + \boldsymbol{\Theta}P^{n-1}x \\ P^n x \end{bmatrix} \qquad (A1.35)$$

By postmultiplying (A1.19) with $P^{n-1}x$, it is clear that (A1.34) and (A1.35) are equal:

$$\Delta v(P^n x) = \Xi v(P^{n-1}x) \qquad (A1.36)$$

Plug (A1.22) into (A1.36):

[50] Since Z is unitary, it cannot be that $Z_{11} = Z_{12} = 0$.

$$
\begin{aligned}
& Y^T \Sigma Z v(P^n x) = Y^T \Phi Z v(P^{n-1} x) \\
& \Leftrightarrow Y^T \Sigma w(P^n x) = Y^T \Phi w(P^{n-1} x) \\
& \Leftrightarrow \Sigma w(P^n x) = \Phi w(P^{n-1} x)
\end{aligned}
\qquad (A1.37)
$$

where the last step follows from premultiplying both sides with Y and using that Y is unitary. Writing out (A1.37) using (A1.27) yields:

$$
(A1.37) \Leftrightarrow \begin{bmatrix} \Sigma_{11}\left(Z_{11}P^{n+1}x + Z_{12}P^n x\right) \\ \Sigma_{21}\left(Z_{11}P^{n+1}x + Z_{12}P^n x\right) \end{bmatrix} = \begin{bmatrix} \Phi_{11}\left(Z_{11}P^n x + Z_{12}P^{n-1}x\right) \\ \Phi_{21}\left(Z_{11}P^n x + Z_{12}P^{n-1}x\right) \end{bmatrix} \qquad (A1.38)
$$

Assume that Σ_{11} is invertible. The first m rows of (A1.38) then imply that:

$$
(A1.38) \Rightarrow Z_{11}P^{n+1}x + Z_{12}P^n x = \Sigma_{11}{}^{-1}\Phi_{11}\left(Z_{11}P^n x + Z_{12}P^{n-1}x\right) \qquad (A1.39)
$$

Rewrite (A1.39) in vector form by using (A1.27):

$$
\begin{aligned}
& (A1.39) \Rightarrow \begin{bmatrix} Z_{11}P^{n+1}x + Z_{12}P^n x \\ Z_{21}P^{n+1}x + Z_{22}P^n x \end{bmatrix} = \begin{bmatrix} \Sigma_{11}{}^{-1}\Phi_{11}\left(Z_{11}P^n x + Z_{12}P^{n-1}x\right) \\ 0_{m,1} \end{bmatrix} \\
& \Leftrightarrow w(P^n x) = \Sigma_{11}{}^{-1}\Phi_{11} w(P^{n-1}x)
\end{aligned}
\qquad (A1.40)
$$

By moving (A1.40) repeatedly backwards and plugging in, (A1.40) can be rewritten as:

$$
(A1.40) \Leftrightarrow w(P^n x) = \left(\Sigma_{11}{}^{-1}\Phi_{11}\right)^n w(x) \qquad (A1.41)
$$

Finally, take the limit of (A1.41) as $n \to \infty$:

$$
\lim_{n\to\infty} w(P^n x) = \lim_{n\to\infty}\left(\Sigma_{11}{}^{-1}\Phi_{11}\right)^n w(x) \qquad (A1.42)
$$

The eigenvalues of a triangular matrix are exactly the elements along the diagonal[51]. Remember that Σ_{11} is invertible and thus has σ_{ii}^{-1} (*i = 1…m*) as eigenvalues[52]. Knowing that the inverse of a upper triangular matrix is also upper triangular, it is clear that σ_{ii}^{-1} is placed along the diagonal of Σ_{11}^{-1}. Multiplying $\Sigma_{11}{}^{-1}$ with Φ_{11} yields a new upper triangular matrix with $\sigma_{ii}^{-1}\phi_{ii}$ along its diagonal:

$$\Sigma_{11}{}^{-1}\Phi_{11}=\begin{bmatrix}\sigma_{11}^{-1} & * & \cdots & * \\ 0 & \sigma_{22}^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \sigma_{mm}^{-1}\end{bmatrix}\begin{bmatrix}\phi_{11} & \phi_{12} & \cdots & \phi_{1m} \\ 0 & \phi_{22} & \cdots & \phi_{2m} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \phi_{mm}\end{bmatrix}=\begin{bmatrix}\sigma_{11}^{-1}\phi_{11} & * & \cdots & * \\ 0 & \sigma_{22}^{-1}\phi_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \sigma_{mm}^{-1}\phi_{mm}\end{bmatrix} \quad (A1.43)$$

* = unspecified element (not relevant here)

Calculating $\left(\Sigma_{11}{}^{-1}\Phi_{11}\right)^n$ in the same fashion yields an upper triangular matrix with $\left(\sigma_{11}^{-1}\phi_{11}\right)^n$ as elements along the main diagonal. Accordingly, $\left(\sigma_{11}^{-1}\phi_{11}\right)^n$ (*i = 1,…,m*) are also the eigenvalues. Since $\left|\phi_{ii}/\sigma_{ii}\right|<1$ (*i = 1,…,m*), it is clear that the resulting matrix has stable eigenvalues.

Strang (1980), pg. 202 shows that a difference equation of the type of (A1.42) is stable if and only if *all* eigenvalues of the coefficient matrix are less than one in absolute value. I.e.:

$$\left|\phi_{ii}/\sigma_{ii}\right|<1 \Rightarrow \lim_{n\to\infty} w(P^n x)=\lim_{n\to\infty}\left(\Sigma_{11}{}^{-1}\Phi_{11}\right)^n w(x)=0 \Leftrightarrow \lim_{n\to\infty} P^n x=0 \quad \forall x$$

$$(i=1,\dots,m) \qquad (A1.44)$$

(A1.44) must hold for *any x*. In other words we can conclude that *P* is stable if and only if at least *m* of the generalized eigenvalues of *Δ* and *Ξ* have absolute value less than one. Remember that (A1.26) was an alternative way of writing the original difference equation

[51] Cf. e.g. Strang (1980), pg. 187
[52] Cf. e.g. Sydsæter et. al (1998), sect. 21.6.

(A1.19). This leads to a more intuitive interpretation of the stability requirements. One can conclude that the solution P of (A1.26) is stable if the *coefficient matrices* $\mathit{\Delta}$ and $\mathit{\Xi}$ have at least as many stable generalized eigenvalues[53] as the dimension of P (m).

If one or more of the m lowest eigenvalues are unit roots, i.e. take the value 1, this does not necessarily mean that the model should be rejected. It means that at least one of the endogenous state variables and possibly one or more of the other endogenous variables follow a random walk. The first thing to realize is that such a model can have no unique steady state. There is no way to know in advance what value that should initially be chosen for a random walk variable. However if the variable mentioned e.g. denotes a price, this should be no problem at all. It is usual in many models to just choose one price to be the numeraire, and there would be no real changes to the model if this price initially would have been set to some other number. This section will not discuss the problem of unit roots formally – but rather deal with the problem by examining impulse responses closely to determine which variables that are random walks. If only prices (price indices, exchange rates etc.) are affected, the unit roots are totally unproblematic. If also real variables are affected, the model results (impulse responses, simulations etc.) may depend heavily on the initial steady state chosen. In the model presented in chapters 3 - 7, it turns out that *bond holdings*, *output* and *consumption* all follow a random walk. It is simply assumed a symmetric steady state, where agents initially hold no bonds – leading to the same level of per capita output and consumption in the two countries. Recall from chapter 6.3 that a money shock leads to a redistribution of wealth, and that a new steady state is reached. Other initial steady states could have been possible, and the asymmetric initial wealth would have affected the shock analysis. This random walk behaviour is no problem in the model though; the redistribution of wealth is one of the main conclusions. Cf. chapter 6 for a thorough discussion on this matter.

Another problem occurs if there are more than m generalized eigenvalues. Then there is in principle no way to determine what eigenvalues that should be chosen. In such a case,

[53] Note that Uhlig (2003b) calls the generalized eigenvectors of the difference equation (A1.26) the *roots* of the equation.

due to the ordering of eigenvalues in the QZ decomposition in (A1.22), the lowest m eigenvalues in absolute value are chosen. This is however not necessarily a solution that should be trusted. All combinations of m stable generalized eigenvalues on the upper left part of the main diagonal of Δ and Ξ are possible solutions. Models of this kind may be of a type with multiple equilibria and where self-fulfilling expectations take part in determining which equilibrium that will be realized[54]. Methods on how to reach a unique solution in this case will not be discussed further.

[54] Farmer and Guo (1994) claim that business cycle movements may not only occur due to agents responding optimally to economic variables, but also due to agents responding to beliefs about fluctuations of non-economic variables, called *sunspot* variables. If an agent believes that other agents respond to sunspots, it is optimal for him also to respond. This gives rise to self-fulfilling beliefs. Adding a structure of this kind to the model, might resolve the uniqueness problem. Such assumptions introduce psychology into formation of beliefs and breaks with the rational expectations tradition. The idea is however disputed.

APPENDIX II - Calculating equilibrium conditions

To solve for the price index and the demand function, minimize expenditure to buy one unit of the consumption basket:

$$\min_{c_t(z)} \int_0^1 p_{t-1}(z)c_t(z)dz \quad s.t. \quad C_t = (\int_0^1 c_t(z)^{\frac{\theta-1}{\theta}} dz)^{\frac{\theta}{\theta-1}} = 1$$

$$\Leftrightarrow L = \int_0^1 p_{t-1}(z)c_t(z)dz - \lambda\left[(\int_0^1 c_t(z)^{\frac{\theta-1}{\theta}} dz)^{\frac{\theta}{\theta-1}} - 1\right]$$

$$FOC: \ \frac{\partial L}{\partial c_t(z)} = p_{t-1}(z) - \lambda\left(\frac{\theta}{\theta-1}\right)\left[\int_0^1 c_t(z)^{\frac{\theta-1}{\theta}} dz\right]^{\frac{\theta}{\theta-1}-1}\left(\frac{\theta-1}{\theta}\right)c_t(z)^{\frac{\theta-1}{\theta}-1} = 0$$

$$\Leftrightarrow p_{t-1}(z) - \lambda\left[\int_0^1 c_t(z)^{\frac{\theta-1}{\theta}} dz\right]^{\frac{1}{\theta-1}} c_t(z)^{\frac{-1}{\theta}} = 0$$

use the def. of C_t :

$$\Leftrightarrow \frac{\partial L}{\partial c_t(z)} = p_{t-1}(z) - \lambda C_t^{\frac{1}{\theta}} c_t(z)^{-\frac{1}{\theta}} = 0 \qquad (A2.1)$$

Use that C_t equals 1, then multiply the first-order condition with $c_t(z)$ on both sides and finally take integrals;

$$p_{t-1}(z)c_t(z) = \lambda c_t(z)^{\frac{\theta-1}{\theta}}$$

$$\Leftrightarrow \int_0^1 p_{t-1}(z)c_t(z)dz = \lambda\int_0^1 c_t(z)^{\frac{\theta-1}{\theta}} dz = \lambda \qquad (A2.2)$$

The left hand side is exactly equal to the definition of the price index. Thus we can conclude that $P_t = \lambda$.

Solve the FOC for $c_t(z)$:

$$c_t(z) = \left(\frac{p_{t-1}(z)}{P} \right)^{-\theta} \tag{3.3}$$

which is the solution for the demand function;

Plug the solution for $c_t(z)$ in the expression for C_t;

$$C_t = \left(\int_0^1 \left(\frac{p_{t-1}(z)}{P} \right)^{-\theta(\frac{\theta-1}{\theta})} dz \right)^{\frac{\theta}{\theta-1}} = 1$$

$$\Leftrightarrow \int_0^1 p_{t-1}(z)^{1-\theta} P_t^{\theta-1} dz = 1$$

$$\Leftrightarrow P_t = \left[\int_0^1 p_{t-1}(z)^{1-\theta} dz \right]^{\frac{1}{1-\theta}} \tag{3.2}$$

which is the price index. To solve for equations (3.20)-(3.22), set up the Lagrangian for the agent's problem:

$$\begin{aligned}\max\ L = E_t \Bigg\{ \sum_{s=t}^{\infty} \beta^{s-t} \Bigg(\ln C_s + \chi \ln\left(\frac{M_s}{P_s} \right) - \frac{\kappa_s}{2} y_s(p_{t-1})^2 \\ -\lambda_t \left[F_s + \frac{M_s}{P_s} - (1+r_{s-1})F_{s-1} - \frac{M_{s-1}}{P_s} - \frac{p_{t-1}}{P_t} y_s(p_{t-1}) + C_s + T_s \right] \Bigg) \Bigg\} \\ \text{w.r.t. } \{C_s\},\{M_s\},\{p_t\},\{F_s\}\end{aligned} \tag{A2.3}$$

y_s is written as a function of p_{t-1}, meaning that work effort depends on product price via the total demand function (3.11). This must be taken into account when derivating wrt. p_{t-1}. Expect for that, this problem yields the first order conditions (3.15)-(3.18) in a straight forward way.

Plug (3.15) into (3.16)-(3.18) to substitute out for λ_t, plug (3.18) into (3.16) to substitute out for $E_t\{\lambda_{t+1}\}$ and rearrange slightly:

$$(3.16) \Rightarrow \frac{\chi}{M_t} = \frac{1}{P_t C_t} - \beta E_t \left\{ \frac{1}{\beta(1+r_t)P_{t+1}C_t} \right\}$$

$$\Leftrightarrow \frac{\chi}{M_t} = \frac{1}{C_t}\left[\frac{1}{P_t} - E_t\left\{ \frac{1}{(1+r_t)P_{t+1}} \right\} \right]$$

$$\Leftrightarrow \frac{\chi}{M_t} = \frac{1}{C_t} E_t \left\{ \frac{(1+r_t)P_{t+1} - P_t}{(1+r_t)P_{t+1}P_t} \right\}$$

$$\Leftrightarrow \frac{M_t}{P_t} = \chi C_t E_t \left\{ \frac{(1+r_t)\frac{P_{t+1}}{P_t}}{(1+r_t)\frac{P_{t+1}}{P_t} - 1} \right\} \tag{3.20}$$

$$(3.17) \Rightarrow p_t(h) = \frac{\theta}{\theta - 1} E_t\{\kappa_{t+1} y_{t+1} P_{t+1} C_{t+1}\} \tag{3.21}$$

$$(3.18) \Rightarrow E_t\{C_{t+1}\} = \beta(1+r_t)C_t \tag{3.22}$$

To solve for equations (3.23) and (3.24), use that domestic prices, indexed from 0 to n, all are equal to $p_{t-1}(h)$ measured in domestic currency and that foreign prices, indexed from n to 1, all are equal to $p_{t-1}^*(f)$ measured in foreign currency, the price index (3.2) can be rewritten as:

$$P_t = \left(\int_0^n p_{t-1}(h)^{1-\theta} dz + \int_n^1 \left[\varepsilon_t p_{t-1}^*(f)\right]^{1-\theta} dz \right)^{\frac{1}{1-\theta}}$$

$$\Leftrightarrow P_t = \left[n p_{t-1}(h)^{1-\theta} + (1-n)\left(p_{t-1}^*(f)\varepsilon_t\right)^{1-\theta} \right]^{\frac{1}{1-\theta}} \tag{3.23}$$

$$P_t^* = \left(\int_0^n \left[\frac{p_{t-1}(h)}{\varepsilon_t} \right]^{1-\theta} dz + \int_n^1 p_{t-1}^*(f)^{1-\theta} dz \right)^{\frac{1}{1-\theta}}$$

$$\Leftrightarrow P_t^* = \left[n\left(\frac{p_t(h)}{\varepsilon_t} \right)^{1-\theta} + (1-n) p_t^*(f)^{1-\theta} \right]^{\frac{1}{1-\theta}} \tag{3.24}$$

To solve for (E12), the total demand for a domestic product; plug (3.3) and (3.5) into (3.10). Thereafter, use the definitions (3.8) and (3.9). (E15), the total demand for a foreign product, is the foreign version. Note that $p_{t-1}*(h)$ and $p_{t-1}(f)$ denote the foreign currency price of a domestic product and the domestic currency price of a foreign product, respectively.

$$y_t = n\left[\left(\frac{p_{t-1}(h)}{P_t}\right)^{-\theta} C_t + \left(\frac{p_{t-1}(h)}{P_t}\right)^{-\theta} G_t\right] + (1-n)\left[\left(\frac{p^*_{t-1}(h)}{P^*_t}\right)^{-\theta} C^*_t + \left(\frac{p^*_{t-1}(h)}{P^*_t}\right)^{-\theta} G^*_t\right]$$

$$\Leftrightarrow y_t = \left(\frac{p_{t-1}(h)}{P_t}\right)^{-\theta} n\left(C_t + G_t\right) + (1-n)\left(\frac{\varepsilon_t p_{t-1}(h)}{\varepsilon_t P_t}\right)^{-\theta} (1-n)\left(C^*_t + G^*_t\right)$$

$$\Leftrightarrow y_t = \left(\frac{p_{t-1}(h)}{P_t}\right)^{-\theta} \left(C^w_t + G^w_t\right) \qquad \text{(E12)}$$

Equivalently for (E13):

$$y^*_t = n\left[\left(\frac{p_{t-1}(f)}{P_t}\right)^{-\theta} C_t + \left(\frac{p_{t-1}(f)}{P_t}\right)^{-\theta} G_t\right] + (1-n)\left[\left(\frac{p^*_{t-1}(f)}{P^*_t}\right)^{-\theta} C^*_t + \left(\frac{p^*_{t-1}(f)}{P^*_t}\right)^{-\theta} G^*_t\right]$$

$$\Leftrightarrow y_t = \left(\frac{\dfrac{p^*_{t-1}(f)}{\varepsilon_t}}{\dfrac{P^*_t}{\varepsilon_t}}\right)^{-\theta} n\left(C_t + G_t\right) + (1-n)\left(\frac{p^*_{t-1}(f)}{P^*_t}\right)^{-\theta} (1-n)\left(C^*_t + G^*_t\right)$$

$$\Leftrightarrow y^*_t = \left(\frac{p^*_{t-1}(f)}{P^*_t}\right)^{-\theta} \left(C^w_t + G^w_t\right) \qquad \text{(E15)}$$

APPENDIX III - Log-linearization of the equilibrium equations

(E1) becomes:

$$(\text{E1}) \Rightarrow E_t\left\{\overline{C}e^{\hat{C}_{t+1}}\right\} = \beta(1+\overline{r}e^{\frac{\tilde{\hat{r}}_t}{\overline{r}}})\overline{C}e^{\hat{C}_t}$$

$$\Leftrightarrow E_t\left\{\overline{C}(1+\hat{C}_{t+1})\right\} = \beta\overline{C}e^{\hat{C}_t} + \beta\overline{r}\,\overline{C}e^{\frac{\tilde{\hat{r}}_t}{\overline{r}}+\hat{C}_t}$$

$$\Leftrightarrow E_t\left\{\overline{C}(1+\hat{C}_{t+1})\right\} = \beta\overline{C}(1+\hat{C}_t) + \beta\overline{r}\,\overline{C}(1+\frac{\tilde{\hat{r}}_t}{\overline{r}}+\hat{C}_t)$$

$$\Leftrightarrow \overline{C} + E_t\left\{\overline{C}\hat{C}_{t+1}\right\} = \beta\overline{C}(1+\overline{r}) + \beta\overline{C}\hat{C}_t + \beta\overline{r}\,\overline{C}(\frac{\tilde{\hat{r}}_t}{\overline{r}}+\hat{C}_t)$$

apply (S1)

$$\Leftrightarrow E_t\left\{\hat{C}_{t+1}\right\} = \beta(1+\overline{r})\hat{C}_t + \beta\tilde{\hat{r}}_t$$

apply (3.30): $\overline{r} = \dfrac{1-\beta}{\beta}$

$$\Leftrightarrow E_t\left\{\hat{C}_{t+1}\right\} = \hat{C}_t + \frac{1}{1+\overline{r}}\tilde{\hat{r}}_t \qquad \text{(L1)}$$

(L2) follows equivalently.

(E3) becomes:

multiply both sides of (E3) with $E_t\left\{\left(\chi C_t(1+r_t)\dfrac{P_{t+1}}{P_t}-1\right)P_t^2\right\}$:

$$\Rightarrow M_t(1+r_t)E_t\{P_{t+1}\} - M_tP_t = \chi C_t(1+r_t)E_t\{P_{t+1}\}P_t$$

$$\Leftrightarrow \overline{M}e^{\hat{M}_t}(1+\overline{r}e^{\frac{\tilde{\hat{r}}_t}{\overline{r}}})e^{E_t\{\hat{P}_{t+1}\}} - \overline{M}e^{\hat{M}_t+\hat{P}_t} = \chi\overline{C}e^{\hat{C}_t}(1+\overline{r}e^{\frac{\tilde{\hat{r}}_t}{\overline{r}}})e^{E_t\{\hat{P}_{t+1}\}}e^{\hat{P}_t}$$

$$\Leftrightarrow \overline{M}e^{\hat{M}_t+E_t\{\hat{P}_{t+1}\}} + \overline{M}\overline{r}e^{\hat{M}_t+\frac{\tilde{\hat{r}}_t}{\overline{r}}+E_t\{\hat{P}_{t+1}\}} - \overline{M}e^{\hat{M}_t+\hat{P}_t} =$$

$$\chi\overline{C}e^{\hat{C}_t+E_t\{\hat{P}_{t+1}\}+\hat{P}_t} + \chi\overline{C}\overline{r}e^{\hat{C}_t+\frac{\tilde{\hat{r}}_t}{\overline{r}}+E_t\{\hat{P}_{t+1}\}+\hat{P}_t}$$

$$\Leftrightarrow \overline{M}(1+\hat{M}_t+E_t\{\hat{P}_{t+1}\}) + \overline{M}\overline{r}(1+\hat{M}_t+\frac{\tilde{\hat{r}}_t}{\overline{r}}+E_t\{\hat{P}_{t+1}\}) - \overline{M}(1+\hat{M}_t+\hat{P}_t) =$$

$$\chi\overline{C}(1+\hat{C}_t+E_t\{\hat{P}_{t+1}\}+\hat{P}_t) + \chi\overline{C}\overline{r}(1+\hat{C}_t+\frac{\tilde{\hat{r}}_t}{\overline{r}}+E_t\{\hat{P}_{t+1}\}+\hat{P}_t)$$

$$\Leftrightarrow \overline{M}\overline{r} + \overline{M}(\hat{M}_t + E_t\{\hat{P}_{t+1}\}) + \overline{M}\overline{r}(\hat{M}_t + \frac{\tilde{\hat{r}}_t}{\overline{r}} + E_t\{\hat{P}_{t+1}\}) - \overline{M}(\hat{M}_t + \hat{P}_t) =$$

$$\chi\overline{C}(1+\overline{r}) + \chi\overline{C}(\hat{C}_t + E_t\{\hat{P}_{t+1}\} + \hat{P}_t) + \chi\overline{C}\overline{r}(\hat{C}_t + \frac{\tilde{\hat{r}}_t}{\overline{r}} + E_t\{\hat{P}_{t+1}\} + \hat{P}_t)$$

Apply (S3)

$$\Leftrightarrow \overline{M}(\hat{M}_t + E_t\{\hat{P}_{t+1}\}) + \overline{M}\overline{r}(\hat{M}_t + \frac{\tilde{\hat{r}}_t}{\overline{r}} + E_t\{\hat{P}_{t+1}\}) - \overline{M}(\hat{M}_t + \hat{P}_t) =$$

$$\chi\overline{C}(\hat{C}_t + E_t\{\hat{P}_{t+1}\} + \hat{P}_t) + \chi\overline{C}\overline{r}(\hat{C}_t + \frac{\tilde{\hat{r}}_t}{\overline{r}} + E_t\{\hat{P}_{t+1}\} + \hat{P}_t)$$

Apply (3.31): $\overline{M} = \chi\overline{C}\left(\frac{1+\overline{r}}{\overline{r}}\right)$

$$\Leftrightarrow \chi\overline{C}(1+\overline{r})\left[\frac{1}{\overline{r}}(\hat{M}_t + E_t\{\hat{P}_{t+1}\} - \hat{M}_t - \hat{P}_t) + \hat{M}_t + \frac{\tilde{\hat{r}}_t}{\overline{r}} + E_t\{\hat{P}_{t+1}\}\right] =$$

$$\chi\overline{C}\left(\hat{C}_t + E_t\{\hat{P}_{t+1}\} + \hat{P}_t + \overline{r}(\hat{C}_t + \frac{\tilde{\hat{r}}_t}{\overline{r}} + E_t\{\hat{P}_{t+1}\} + \hat{P}_t)\right)$$

$$\Leftrightarrow (1+\overline{r})\left[\frac{1+\overline{r}}{\overline{r}}E_t\{\hat{P}_{t+1}\} - \frac{1}{\overline{r}}\hat{P}_t + \hat{M}_t + \frac{\tilde{\hat{r}}_t}{\overline{r}}\right] = (1+\overline{r})\left(\hat{C}_t + E_t\{\hat{P}_{t+1}\} + \hat{P}_t\right) + \tilde{\hat{r}}_t$$

$$\Leftrightarrow (1+\overline{r})\left[\frac{1}{\overline{r}}E_t\{\hat{P}_{t+1}\} - \left(\frac{1}{\overline{r}}+1\right)\hat{P}_t + \hat{M}_t - \hat{C}_t\right] = -\frac{1}{\overline{r}}\tilde{\hat{r}}_t$$

$$\Leftrightarrow \hat{M}_t - \hat{P}_t = \hat{C}_t - \frac{1}{\overline{r}(1+\overline{r})}\tilde{\hat{r}}_t - \frac{1}{\overline{r}}\left(E_t\{\hat{P}_{t+1}\} - \hat{P}_t\right) \quad \text{(L3)}$$

(L4) follows equivalently.

(E5) becomes:

$$\overline{C}^w e^{\hat{C}_t^w} = n\overline{C}e^{\hat{C}_t} + (1-n)\overline{C}^* e^{\hat{C}_t^*}$$

$$\Leftrightarrow \overline{C}^w(1+\hat{C}_t^w) = n\overline{C}(1+\hat{C}_t) + (1-n)\overline{C}^*(1+\hat{C}_t^*)$$

apply (S5)

$$\Leftrightarrow \overline{C}^w\hat{C}_t^w = n\overline{C}\hat{C}_t + (1-n)\overline{C}^*\hat{C}_t^* \Leftrightarrow \hat{C}_t^w = n\hat{C}_t + (1-n)\hat{C}_t^* \quad \text{(L5)}$$

(E6) becomes:

$$\overline{G}^{w} e^{\hat{G}_t^w} = n\overline{G} e^{\hat{G}_t} + (1-n)\overline{G}^*$$
$$\Leftrightarrow \overline{G}^{w}(1+\hat{G}_t^w) = n\overline{G}(1+\hat{G}_t) + (1-n)\overline{G}^*$$
$$\text{apply (S6)}$$
$$\Leftrightarrow \overline{G}^{w}\hat{G}_t^w = n\overline{G}\hat{G}_t \Leftrightarrow \hat{G}_t^w = n\hat{G}_t \qquad \text{(L6)}$$

(E7) becomes:

$$e^{\hat{P}_t} = e^{\hat{\varepsilon}_t} e^{\hat{P}_t^*} = e^{\hat{\varepsilon}_t + \hat{P}_t^*} \Leftrightarrow \hat{P}_t = \hat{\varepsilon}_t + \hat{P}_t^* \qquad \text{(L7)}$$

(E8) becomes:

$$\text{(E8)} \Rightarrow \qquad P_t^{1-\theta} = \left[n p_t(h)^{1-\theta} + (1-n)\left(p_t^*(f)\varepsilon_t\right)^{1-\theta} \right]$$
$$\Leftrightarrow e^{(1-\theta)\hat{P}_t} = n e^{(1-\theta)\hat{p}_t(h)} + (1-n) e^{(1-\theta)\hat{p}_t^*(f)} e^{(1-\theta)\hat{\varepsilon}_t} =$$
$$\Leftrightarrow e^{(1-\theta)\hat{P}_t} = n e^{(1-\theta)\hat{p}_t(h)} + (1-n) e^{(1-\theta)\hat{p}_t^*(f) + (1-\theta)\hat{\varepsilon}_t}$$
$$\Leftrightarrow 1 + (1-\theta)\hat{P}_t = n\left(1 + (1-\theta)\hat{p}_t(h)\right) + (1-n)\left(1 + (1-\theta)\hat{p}_t^*(f) + (1-\theta)\hat{\varepsilon}_t\right)$$
$$\Leftrightarrow \hat{P}_t = n\hat{p}_t(h) + (1-n)\hat{p}_t^*(f) + (1-n)\hat{\varepsilon}_t \qquad \text{(L8)}$$

(L9) follows equivalently.

Define $\tilde{\hat{F}}_t \equiv \lim\limits_{\overline{F}\to 0}\left(\frac{\overline{F}}{\overline{y}}\hat{F}_t\right)$. (E10) then becomes:

$$\lim_{\overline{F}\to 0}\left(\overline{F} e^{\hat{F}_t}\right) = \lim_{\overline{F}\to 0}\left((1 + \overline{r} e^{\frac{\tilde{\hat{r}}_{t-1}}{\overline{r}}})\overline{F} e^{\hat{F}_{t-1}} + \frac{e^{\hat{p}_t(h)}}{e^{\hat{P}_t}} \overline{y} e^{\hat{y}_t} - \overline{C} e^{\hat{C}_t} - \overline{G} e^{\hat{G}_t} \right)$$
$$\Leftrightarrow \lim_{\overline{F}\to 0}\left(\overline{F} e^{\hat{F}_t}\right) = \lim_{\overline{F}\to 0}\left(\overline{F} e^{\hat{F}_{t-1}} + \overline{r}\,\overline{F} e^{\frac{\tilde{\hat{r}}_{t-1}}{\overline{r}} + \hat{F}_{t-1}} + \overline{y} e^{\hat{p}_t(h) + \hat{y}_t - \hat{P}_t} - \overline{C} e^{\hat{C}_t} - \overline{G} e^{\hat{G}_t} \right)$$
$$\Leftrightarrow \lim_{\overline{F}\to 0}\left[\overline{F}(1+\hat{F}_t)\right] = \lim_{\overline{F}\to 0}\left[\overline{F}(1+\hat{F}_{t-1}) + \overline{r}\,\overline{F}(1 + \frac{\tilde{\hat{r}}_{t-1}}{\overline{r}} + \hat{F}_{t-1}) + \right.$$
$$\left. \overline{y}(1 + \hat{p}_t(h) + \hat{y}_t - \hat{P}_t) - \overline{C}(1+\hat{C}_t) - \overline{G}(1+\hat{G}_t) \right]$$
$$\text{Apply (S10)}$$

$$\Leftrightarrow \lim_{\overline{F}\to 0}\left(\overline{F}\hat{F}_t\right)=\lim_{\overline{F}\to 0}\left\lfloor \overline{F}(1+\overline{r})\hat{F}_{t-1}+\overline{F}\tilde{\hat{r}}_{t-1}+\overline{y}\hat{p}_t(h)-\overline{y}\hat{P}_t+\overline{y}\hat{y}_t-\overline{C}\hat{C}_t-\overline{G}\hat{G}_t\right\rfloor$$

$$\Leftrightarrow \lim_{\overline{F}\to 0}\left(\frac{\overline{F}}{\overline{y}}\hat{F}_t\right)_t=\lim_{\overline{F}\to 0}\left[(1+\overline{r})\frac{\overline{F}}{\overline{y}}\hat{F}_{t-1}+\frac{\overline{F}}{\overline{y}}\tilde{\hat{r}}_{t-1}+\hat{p}_t(h)-\hat{P}_t+\hat{y}_t-\overline{c}\hat{C}_t-\overline{g}\hat{G}_t\right]$$

$$\Leftrightarrow \tilde{\hat{F}}_t=(1+\overline{r})\tilde{\hat{F}}_{t-1}+\hat{p}_t(h)-\hat{P}_t+\hat{y}_t-\overline{c}\hat{C}_t-\overline{g}\hat{G}_t \qquad \text{(L10)}$$

(L11) follows equivalently.

(E12) becomes:

$$\overline{y}e^{\hat{y}_t}=\left(\frac{e^{\hat{p}_t(h)}}{e^{\hat{P}_t}}\right)^{-\theta}\left(\overline{C}^{w}e^{\hat{C}_t^w}+\overline{G}^{w}e^{\hat{G}_t^w}\right)$$

$$\Leftrightarrow \overline{y}e^{\hat{y}_t}=e^{\theta\hat{P}_t-\theta\hat{p}_t(h)}\left(\overline{C}^{w}e^{\hat{C}_t^w}+\overline{G}^{w}e^{\hat{G}_t^w}\right)$$

$$\Leftrightarrow \overline{y}e^{\hat{y}_t}=\overline{C}^{w}e^{\theta\hat{P}_t-\theta\hat{p}_t(h)+\hat{C}_t^w}+\overline{G}^{w}e^{\theta\hat{P}_t-\theta\hat{p}_t(h)+\hat{G}_t^w}$$

$$\Leftrightarrow \overline{y}(1+\hat{y}_t)=\overline{C}^{w}(1+\theta\hat{P}_t-\theta\hat{p}_t(h)+\hat{C}_t^w)+\overline{G}^{w}(1+\theta\hat{P}_t-\theta\hat{p}_t(h)+\hat{G}_t^w)$$

Apply (S12)

$$\Leftrightarrow \overline{y}\hat{y}_t=\overline{C}^{w}(\theta\hat{P}_t-\theta\hat{p}_t(h)+\hat{C}_t^w)+\overline{G}^{w}(\theta\hat{P}_t-\theta\hat{p}_t(h)+\hat{G}_t^w)$$

$$\Leftrightarrow \overline{y}\hat{y}_t=\theta\left(\overline{C}+\overline{G}\right)\left(\hat{P}_t-\hat{p}_t(h)\right)+\overline{C}\hat{C}_t^w+\overline{G}\hat{G}_t^w$$

Apply (3.34)

$$\Leftrightarrow \hat{y}_t=\theta\left(\hat{P}_t-\hat{p}_t(h)\right)+\overline{c}\hat{C}_t^w+\overline{g}\hat{G}_t^w \qquad \text{(L12)}$$

(L13) follows equivalently.

(E14) becomes:

$$\text{(E14)}\Rightarrow y_t^{\theta+1}=C_t^{-\theta}\kappa_t^{-\theta}\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(C_t^w+G_t^w\right)$$

$$\Leftrightarrow \overline{y}^{\theta+1}e^{(\theta+1)\hat{y}_t}=\overline{C}^{-\theta}e^{-\theta\hat{C}_t}e^{-\theta\hat{\kappa}_t}\left(\frac{\theta-1}{\theta\overline{\kappa}}\right)^{\theta}\left(\overline{C}^{w}e^{\hat{C}_t^w}+\overline{G}^{w}e^{\hat{G}_t^w}\right)$$

$$\Leftrightarrow \overline{y}^{\theta+1}e^{(\theta+1)\hat{y}_t}=\left(\frac{\theta-1}{\theta\overline{\kappa}}\right)^{\theta}\left(\overline{C}^{-\theta}\overline{C}^{w}e^{-\theta\hat{C}_t-\theta\hat{\kappa}_t+\hat{C}_t^w}+\overline{C}^{-\theta}\overline{G}^{w}e^{-\theta\hat{C}_t-\theta\hat{\kappa}_t+\hat{G}_t^w}\right)$$

$$\Leftrightarrow \bar{y}^{\theta+1}\left[1+(\theta+1)\hat{y}_t\right]=\left(\frac{\theta-1}{\theta\bar{\kappa}}\right)^{\theta}\Big[\bar{C}^{1-\theta}(1-\theta\hat{C}_t-\theta\hat{\kappa}_t+\hat{C}_t^{w})+$$
$$\bar{C}^{-\theta}\bar{G}(1-\theta\hat{C}_t-\theta\hat{\kappa}_t+\hat{G}_t^{w})\Big]$$

Apply (S14)

$$\Leftrightarrow \bar{y}^{\theta+1}(\theta+1)\hat{y}_t=\left(\frac{\theta-1}{\theta\bar{\kappa}}\right)^{\theta}\Big[\bar{C}^{1-\theta}(-\theta\hat{C}_t-\theta\hat{\kappa}_t+\hat{C}_t^{w})+$$
$$\bar{C}^{-\theta}\bar{G}(-\theta\hat{C}_t-\theta\hat{\kappa}_t+\hat{G}_t^{w})\Big]$$

$$\Leftrightarrow \bar{y}^{\theta+1}(\theta+1)\hat{y}_t=\left(\frac{\theta-1}{\theta\bar{\kappa}}\right)^{\theta}\left[-\bar{C}^{-\theta}\theta\bar{y}\hat{C}_t-\bar{C}^{-\theta}\theta\bar{y}\hat{\kappa}_t+\bar{C}^{1-\theta}\hat{C}_t^{w}+\bar{C}^{-\theta}\bar{G}\hat{G}_t^{w}\right]$$

Apply that $\bar{y}\equiv\left(\dfrac{\theta-1}{\theta\bar{\kappa}\bar{C}}\right)$ and (3.33)

$$\Leftrightarrow \bar{y}^{\theta+1}(\theta+1)\hat{y}_t=\bar{y}^{\theta}\left[-\theta\bar{y}\hat{C}_t-\theta\bar{y}\hat{\kappa}_t+\bar{C}\hat{C}_t^{w}+\bar{G}\hat{G}_t^{w}\right]$$

$$\Leftrightarrow (\theta+1)\hat{y}_t=-\theta(\hat{C}_t+\hat{\kappa}_t)+(\bar{c}\hat{C}_t^{w}+\bar{g}\hat{G}_t^{w}) \qquad \text{(L14)}$$

(L15) follows equivalently.

(E16) becomes:

$$\text{(E16)}\Rightarrow \overline{\text{W}}\text{e}^{\hat{W}_t}=\ln(\bar{C}e^{\hat{C}_t})+\chi\ln\left(\frac{\bar{M}}{\bar{P}}e^{\hat{M}_t-\hat{P}_t}\right)-\frac{\bar{\kappa}\bar{y}^2}{2}e^{\hat{\kappa}_t+2\hat{y}_t}$$

$$\Leftrightarrow \overline{\text{W}}(1+\hat{W}_t)=\ln\bar{C}+\hat{C}_t+\chi\ln\left(\frac{\bar{M}}{\bar{P}}\right)+\chi(\hat{M}_t-\hat{P}_t)-\frac{\bar{\kappa}\bar{y}^2}{2}(1+\hat{\kappa}_t+2\hat{y}_t)$$

Apply (S16):

$$\Leftrightarrow \overline{\text{W}}\hat{W}_t=\hat{C}_t+\chi(\hat{M}_t-\hat{P}_t)-\bar{\kappa}\bar{y}^2(\frac{\hat{\kappa}_t}{2}+\hat{y}_t)$$

Plug (3.40) into (3.39) to get the following expression:

$$\bar{y}=\left(\frac{\theta-1}{\theta\bar{\kappa}\bar{c}\bar{y}}\right)\Leftrightarrow\bar{\kappa}\bar{y}^2=\left(\frac{\theta-1}{\theta\bar{c}}\right)$$

Plug into the log-linearized equation:

$$\overline{\text{W}}\hat{W}_t=\hat{C}_t+\chi(\hat{M}_t-\hat{P}_t)-\left(\frac{\theta-1}{\theta\bar{c}}\right)(\frac{\hat{\kappa}_t}{2}+\hat{y}_t) \qquad \text{(L16)}$$

(L17) follows equivalently.

APPENDIX IV - MatLab Source code

```
% February 2th 2004, COPYRIGHT O. LIEN
% REDUX.M calculates through a modified version the intertemporal dynamic
% optimization model with monopolistic competition and sticky prices
% in M. Obstfeld & K. Rogoff (1995a), "Exchange Rate Dynamics Redux".
% First, parameters are set and the steady state is calculated. Next, the matrices are
% declared.  In the last line, the model is solved and analyzed by calling DO_IT.M
%
% Cf. Lien, Ole Chr. (2004), "Sticky prices and the macroeconomy - a quantitative linear
% approximation analysis" for further details.
%
% TOOLKIT:
% Copyright: H. Uhlig.  Feel free to copy, modify and use at your own risk.
% However, you are not allowed to sell this software or otherwise impinge
% on its free distribution.

disp('Obstfeld & Rogoff intertemporal monopolistic competition and sticky prices
disp('model,');
disp('see Obstfeld M., & Rogoff, K., "Exchange Rate Dynamics Redux"');
disp('Journal of Political Economy, (1995a), pp 624-659.');

disp('Hit any key when ready...');
pause;

% Clear memory just in case...:
clear all;

% Setting parameters and steady state values:
if Country == 1
    n=0.32;
    c_=0.77;
    r_=0.046;
    chi=0.046;
else
    n=0.006;
    c_=0.71;
    r_=0.026;
    chi=0.027;
end;

g_=1-c_;
th=6; % (theta)

if Shock_type == 1
    rho_g=1;
    rho_K=1; % (kappa)
    rho_m=1;
end;
if Shock_type == 2
    rho_g=0;
    rho_K=0; % (kappa)
    rho_m=0;
end;
if Shock_type == 3
    rho_g=0.9;
    rho_K=0.95; % (kappa)
    if Country == 1
        rho_m=0.49;
    else
        rho_m=0.15;
    end;
end;
if Country == 1
   sigma_g=1.01;
   sigma_m=1.39;
else
    sigma_g=1.37;
    sigma_m=2.53;
```

```
end;
sigma_K=1.4;
W_=1;

% some definitions; let t be short for tilda
rho_mt=1/(1+r_-rho_m);
rho_gt=rho_g/(1+r_-rho_g);
rho_Kt=rho_K/(1+r_-rho_K);

% some shortcuts for some LONG coefficients...
coef1=-(c_*(1+r_)/r_)-((th-1)/((th+1)*r_));
coef2=-((th-1)/(th*c_));
coef3=g_*(rho_gt+1);
coef4=rho_Kt*(th-1)/(th+1);
coef5=g_*rho_g/(c_+1);
coef6=rho_K/(c_+1);
coef7=-((th-1)/(th*2*c_));

% Declaring the matrices.
VARNAMES = ['bonds                ',  % x1
             'price dif.           ',  % x2
            'world price          ',  % x3   3 endogenous state variables - vector x(t)
            'exchange rate        ',  % y1
            'consumption dif.     ',  % y2
            'world consumption    ',  % y3
            'output dif           ',  % y4
            'world output         ',  % y5
            'interest rate        ',  % y6
            'domestic consumption ',  % y7
            'foreign consumption  ',  % y8
            'domestic output      ',  % y9
            'foreign output       ',  % y10
            'domestic price index ',  % y11
            'foreign price index  ',  % y12
            'domestic welfare     ',  % y13
            'foreign welfare      ',  % y14 %14 other endogenous variables - vector y(t)
            'domestic gov. spending',  % z1
            'foreign gov. spending ',  % z2
            'domestic technology  ',  % z3
            'foreign technology   ',  % z4
            'domestic money       ',  % z5
            'foreign money        ']; % z6   6 stochastic processes - vector z(t)

% Translating into coefficient matrices.
%
% Find matrices for format:
% 0 = AA x(t) + BB x(t-1) + CC y(t) + DD z(t)
% 0 = E_t [ FF x(t+1) + GG x(t) + HH x(t-1) + JJ y(t+1) + KK y(t) + LL z(t+1) + MM z(t)]
% z(t+1) = NN z(t) + epsilon(t+1) with E_t [ epsilon(t+1) ] = 0,
%
% Equations without expectation:
% for k(t):
% order:F,       pd, pw
AA = [  -1/(1-n),0,  0   %M1
        0,       0,  0   %M2
        0,       0,  0   %M3
        0,       0,  0   %M4
        0,       0,  0   %M5
        0,       0,  0   %M6
        0,       0,  0   %M7
        0,       0,  0   %M8
        0,       0,  0   %M9
        0,       0,  0   %M10
        0,       0,  0   %M11
        0,       0,  1   %M12
        0,       0,  1   %M13
        0,       0,  0   %M14
        0,       0,  0]; %M15

% for k(t-1):
% order:F,           pd,     pw
```

```
BB = [  (1+r_)/(1-n),1,      0   %M1
        0,           -th, 0   %M2
        0,           0,      0   %M3
        0,           0,      0   %M4
        (1+r_)/(1-n),1-th,0   %M5
        0,           0,      0   %M6
        0,           0,      -1  %M7
        0,           0,      0   %M8
        0,           0,      0   %M9
        0,           0,      0   %M10
        0,           0,      0   %M11
        0,           0,      0   %M12
        0,           0,      0   %M13
        0,           0,      0   %M14
        0,           0,      0]; %M15

% for y(t):
% order:e,   cd,   cw,yd, yw,r,       c, c*,y,    y*,   P,   P*,  W,  W*
CC = [  -1,  -c_,  0, 1,  0, 0,       0, 0, 0,    0,    0,   0,   0,  0     %M1
        th,  0,    0, -1, 0, 0,       0, 0, 0,    0,    0,   0,   0,  0     %M2
        0,   0,    c_,0,  -1,0,       0, 0, 0,    0,    0,   0,   0,  0     %M3
        -1,  -1,   0, 0,  0, 0,       0, 0, 0,    0,    0,   0,   0,  0     %M4
        th-1,coef1,0, 0,  0, 0,       0, 0, 0,    0,    0,   0,   0,  0     %M5
       0,   0,    1, 0,  0, 1/(1+r_),0, 0, 0,    0,    0,   0,   0,  0     %M6
           0,   0,    -1,0,  0, 0,       0, 0, 0,    0,    0,   0,   0,  0     %M7
        0,   1-n,  1, 0,  0, 0,       -1,0, 0,    0,    0,   0,   0,  0     %M8
        0,   -n,   1, 0,  0, 0,       0, -1,0,    0,    0,   0,   0,  0     %M9
        0,   0,    0, 1-n,1, 0,       0, 0, -1,   0,    0,   0,   0,  0     %M10
        0,   0,    0, -n, 1, 0,       0, 0, 0,    -1,   0,   0,   0,  0     %M11
        1-n, 0,    0, 0,  0, 0,       0, 0, 0,    0,    -1,  0,   0,  0     %M12
        -n,  0,    0, 0,  0, 0,       0, 0, 0,    0,    0,   -1,  0,  0     %M13
        0,   0,    0, 0,  0, 0,       1, 0, coef2,0,    -chi,0,   -W_,0     %M14
        0,   0,    0, 0,  0, 0,       0, 1, 0,    coef2,0,   -chi,0,  -W_]; %M15

% for z(t):
% order:G,      G*          kappa,  kappa*      M,         M*
DD = [  -g_,    g_,         0,      0           0,         0              %M1
        0,      0           0,      0           0,         0              %M2
        g_*n,   g_*(1-n),   0,      0           0,         0              %M3
        0,      0           0,      0           r_*rho_mt, -r_*rho_mt     %M4
        -coef3, coef3,      -coef4, coef4,      0,         0              %M5
        n*coef5,(1-n)*coef5,n*coef6,(1-n)*coef6,0,         0              %M6
        0,      0           0,      0           n*r_*rho_m,(1-n)*r_*rho_m %M7
        0,      0           0,      0           0,         0              %M8
        0,      0           0,      0           0,         0              %M9
        0,      0           0,      0           0,         0              %M10
        0,      0           0,      0           0,         0              %M11
        0,      0           0,      0           0,         0              %M12
        0,      0           0,      0           0,         0              %M13
        0,      0,          coef7,  0,          chi,       0              %M14
        0,      0,          0,      coef7,      0,         chi];          %M15

% Equations with expectations:
% for x(t+1):
% order:F,pd,pw
FF = [  0,0, 0   %M15
        0,0, 0]; %M16

% for x(t):
% order:F,pd,pw
GG = [  0,-1,0   %M15
        0,0, 0]; %M16

% for x(t-1):
% order:F,pd, pw
HH = [  0,0,  0   %M15
        0,0,  0]; %M16

% for y(t+1):
% order:E,cd,cw,yd,yw,r,c,c*,y,y*,P,P*,W,W*
JJ = [  1,0, 0, 1, 0, 0,0,0, 0,0, 0, 0,0,0   %M15
```

```
        0,0, 1, 0, 1, 0,0,0, 0,0, 0, 0,0,0]; %M16

% for y(t):
% order:E,cd,cw,yd, yw,r,c,c*,y, y*,P,P*
KK = [  0, 1, 0,0,  0, 0,0,0, 0, 0, 0, 0,0,0    %M15
        0, 0, 0,0,  0, 0,0,0, 0, 0, 0, 0,0,0];  %M16

% for z(t+1):
% order:G,G*,kappa,kappa*,M,M*
LL = [  0,0, 1,    -1,    0,0   %M15
        0,0, n,    1-n,   0,0]; %M16

% for z(t):
% order:G,G*,kappa,kappa*,M,M*
MM = [  0,0, 0,    0,     0,0   %M15
        0,0, 0,    0,     0,0]; %M16

% AR1-processes
% for z(t):
% order:G,     G*,   kappa,    kappa*,   M,    M*
NN = [  rho_g, 0,    0,        0,        0,    0       %A1
        0,     rho_g,0,        0,        0,    0       %A2
        0,     0,    rho_K,    0,        0,    0       %A3
        0,     0,    0,        rho_K,    0,    0       %A4
        0,     0,    0,        0,        rho_m,0       %A5
        0,     0,    0,        0,        0,    rho_m]; %A6

% the variance-covariance matrix of the error terms of the AR(1) processes
% Sigma(i,j) gives the covariance between the error term i and j, where i and j
% correspond to indexes of the vector z(t)
% order:  G,        G*,       kappa,    kappa*,   M,        M*  | order:
Sigma = [ sigma_g^2,0,        0,        0,        0,        0   % G
          0,        sigma_g^2,0,        0,        0,        0   % G*
          0,        0,        sigma_K^2,0,        0,        0   % kappa
          0,        0,        0,        sigma_K^2,0,        0   % kappa*
          0,        0,        0,        0,        sigma_m^2,0   % M
          0,        0,        0,        0,        0,sigma_m^2]; % M*

% Setting the options: (cf. the file OPTIONS.m included in Unhlig's Toolkit for details)
[l_equ,m_states] = size(AA); % k, l, m and n defined as in chapter 2
[l_equ,n_endog ] = size(CC);
[l_equ,k_exog  ] = size(DD);

PERIOD = 1;  % number of periods per year, 1 for annually
IMP_SELECT = 1:(m_states+n_endog+k_exog); % a vector containing the indices of the
% variables to be plotted
SELECT_SHOCKS = 1 : k_exog; % select the shocks to which impulse responses should be
% plotted.
DO_NO_ZERO_RESPONSE = 0; % also plot variables that do not respond to shocks
DO_STATE_RESP = 0; % do not analyse shocks in state variables
DO_IMPRESP=1; % calculate impulse responses
HORIZON = 16; % calcualte impulse responses for 16 periods (years)
DO_MOMENTS = 0; % do not calculate frequency-domain based moments

DO_SIMUL=1; % run simulations
SIM_MODE=2; % calculate simulation based moments
SIM_GRAPH=1; % create plots of the simulated series
DO_HP_FILTER = 0; % do not HP-filter the simulated time series before calculating
               % correlations (since they are already detrended)
SIM_SINGLE=1; each simulated series should be plotted in one graph
SIM_SUBPLOT=0; Set =1, if simulated series should be plotted in a subplot
SIM_SUB_FONT=9; % font size for
SIM_LENGTH=100; % simulation length
SIM_RANDOM_START=0; % start from steady state
GNP_INDEX = 12; % the index of domestic output (used to calculate crosscorrelations)

DO_QZ = 1; % use the QZ-method to solve the model

% Starting the calculations:
do_it;
```

APPENDIX V - The HP-filter

This appendix gives a brief overview of the Hodrick-Prescott filter (HP-filter) and shows one example related to the model in the main sections of the paper. The theoretical background of the appendix follows the frameworks in Ravn and Uhlig (1997) and Uhlig (2003a), both of which build on Hodrick and Prescott (1980). The filter is used in chapter 5 and 7 to remove long-run trends from real data – making it possible to compare the data with simulated data from the main model. As claimed by Ravn and Uhlig (1997): "The HP-filter has become the standard method for removing long-run movements from the data in business cycles literature."

The objective of the non-linear HP-filter is to remove a long run trend from a time series y_t. The smooth trend τ_t is found by the following formula:

$$\min \sum_{t=1}^{T} \left[(y_t - \tau_t)^2 + \lambda \left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right]^2 \right] \qquad (A4.1)$$

The HP-filter takes two objectives into account, where λ determines their relative importance. The first objective is that the trend τ_t should be close to real data – close in the sense that the square of the deviations from the trend should be minimized ceteris paribus. This is accounted for in the first term of (A4.1). The second objective is that trend should be smooth. More precisely this is measured as the square of the change of the slope of the trend. These two objectives are more easily understood when considering two extreme cases:

1. $\lambda = 0$. The second term disappears and we are left with the first objective. The first objective states that the trend should be as close to data as possible. The trend found will in fact be the data itself. This case has of course no practical importance.
2. $\lambda \to \infty$. First of all, minimizing (A4.1) must imply finding a trend such that the second term is zero. The second term is zero only when the slope of the trend

never changes, i.e. a straight line. Among the set of all straight lines, the trend is found by minimizing the first term of (A4.1) (given that the trend must be a straight line). We are then in fact left with the OLS-regression. The trend will be the straight line that minimizes the square of the deviations from the trend (regression line).

An intermediate value for λ, will find a non-linear trend that is a compromise between the two objectives. In time series data one is very often interested in finding out what components of the data represent a long run trend and what components represent short- or medium run deviations from the trend (e.g. cyclical movement around the trend). There is no reason why the long run trend should be linear – and the HP-filter takes this into account.

Uhlig (2003a) defines business cycles as *recurrent medium-run movements around a smooth long-run trend in production, investment, consumption and employment*. With an appropriate choice for λ one can remove this trend and find the business cycle components. For quarterly data it has become standard to choose $\lambda = 1600$. Ravn and Uhlig (1997) are of the view that this choice is nothing more than a definition of business cycles – that movements of the data that end up in the business cycles component $y_t - \tau_t$ are per definition business cycles – and that other movements belong to the long-run trend τ_t. However, Hodrick and Prescott (1980) based their choice of $\lambda = 1600$ on estimations of the volatility in business cycles and the long-run trend, respectively.

When going from quarterly to annual data, it is clear that λ should be adjusted in such a way that the component considered to be business cycles in quarterly data also should be considered the same in annual data. Based on this idea Ravn and Uhlig (1997) find that λ should be adjusted with the change in frequency to the 4th power. For annual data this implies a λ of $\left(1/4\right)^4 \cdot 1600 = 6.25$. This value is used in HP-filtering the data in chapter 5 and 7.

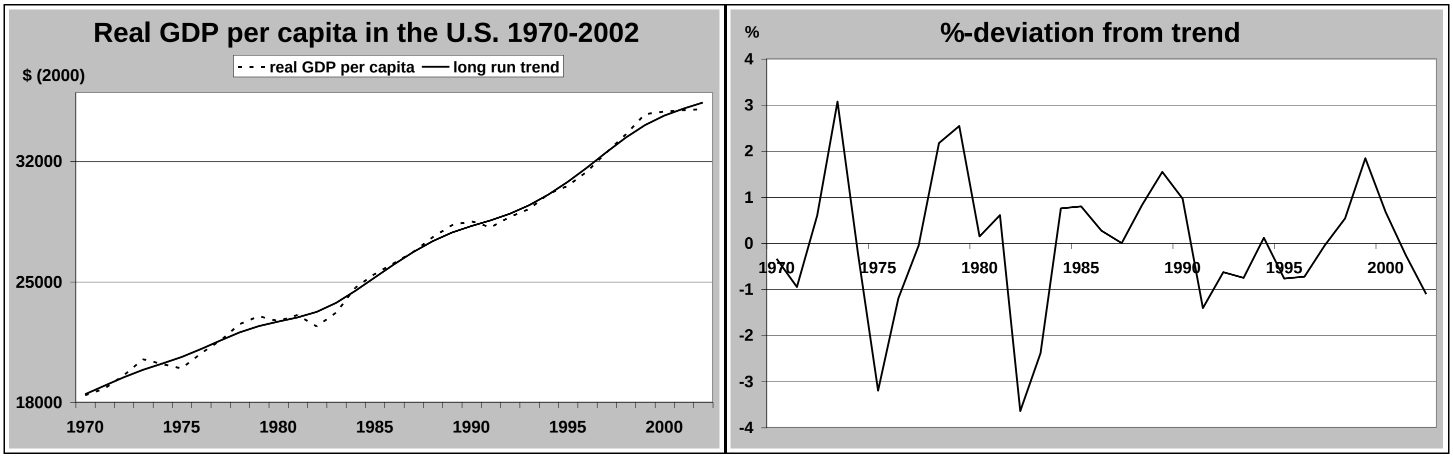

Figure A.5.1 Real GDP in the U.S. 1970-2002, the HP-filtered trend and percentage deviations from trend

Figure A4.1 displays an example - the HP-filter applied on a time series for U.S. real per capita GDP in the period 1970-2002 (cf. appendix V), where $\lambda = 6.25$ has been used.

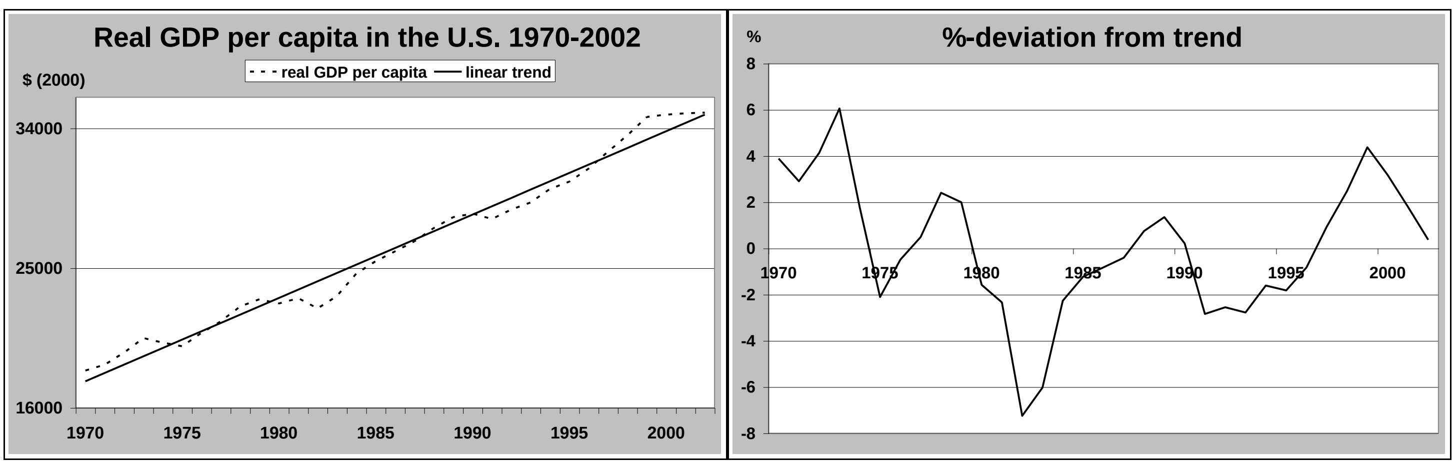

Figure A.5.2 Real GDP per capita in the U.S. 1970-2002 (Source: BEA), the OLS-regression and percentage deviations from trend

As a comparison an OLS regression of the same time series has been added, assuming for a moment that the data series was stationary for the period 1970-2002. But one also indirectly assumes that GDP evolves according to a linear growth trend. If this is not the case, one will in some time periods overestimate business cycles and in other periods underestimate. Notice that the deviations from trend are much higher with the linear filter than with the HP-filter – up to 7 %. Figure A4.2 illustrates how a purely linear filter might be misleading to filter out business cycles. If we were to trust this filter, one could misconclude that U.S. GDP in 1973 was 6 % above trend, and in 1982 more than 7 % below trend.

Certainly there could be other ways of filtering the data than the HP-filter. An OLS regression could possibly be more suitable if one computed the natural logaritm of the data before calculating the regression. There are also other non-linear filters that could have been applied. However, this paper will not dive deeper into the litterature on this field.

APPENDIX VI – Data series

year	GDP (mill. USD)	private consumption (mill. USD)	government spending (mill. USD)	current account (mill. USD)	population (in 1000's)	nominal int. rate (in %)	M3 (mill. USD)	CPI (2000=100)
1970	1038545	648465	233752	2331	205052	7.91	638317	27.54
1971	1127118	701868	246461	-1433	207661	5.73	732817	28.92
1972	1238291	770603	263452	-5795	209896	5.25	832650	30.17
1973	1382704	852417	281688	7140	211909	8.03	943667	31.85
1974	1499978	933426	317949	1962	213854	10.81	1036192	34.72
1975	1638338	1034394	357748	18116	215973	7.86	1122283	38.01
1976	1825267	1151914	383024	4295	218035	6.84	1243283	40.20
1977	2030945	1278609	414089	-14335	220239	6.83	1395375	42.76
1978	2294706	1428535	453584	-15143	222585	9.06	1563375	45.76
1979	2563325	1592215	500781	-285000	225055	12.67	1736808	49.55
1980	2789504	1757133	566175	2317	227224	15.26	1900658	54.06
1981	3128435	1941060	627521	5030	229466	18.87	2132542	59.13
1982	3255011	2077268	680491	-5536	231665	14.85	2368708	62.74
1983	3536665	2290556	733459	-38691	233792	10.79	2591483	65.21
1984	3933173	2503287	796994	-94344	235825	12.04	2854317	67.66
1985	4220262	2720305	878985	-118155	237924	9.93	3108983	69.72
1986	4462825	2899724	949313	-147177	240133	8.33	3366442	71.27
1987	4739471	3100234	999471	-160655	242289	8.21	3602817	73.20
1988	5103790	3353615	1038987	-121153	244499	9.32	3830458	75.71
1989	5484351	3598496	1099056	-99486	246819	10.87	4003558	78.57
1990	5803067	3839937	1180150	-78965	249464	10.01	4123533	81.61
1991	5995926	3986066	1234442	3747	252153	8.46	4196808	84.46
1992	6337744	4235265	1270950	-48013	255030	6.25	4222542	86.40
1993	6657407	4477887	1291179	-81989	257783	6.00	4232542	88.39
1994	7072228	4743287	1325468	-117678	260327	7.15	4304183	90.27
1995	7397651	4975787	1369221	-105217	262803	8.83	4507600	92.12
1996	7816861	5256832	1416002	-117203	265229	8.27	4811750	93.86
1997	8304344	5547400	1468704	-127684	267784	8.44	5209408	95.42
1998	8746997	5879482	1518325	-204691	270248	8.35	5748425	96.48
1999	9268412	6282474	1620798	-290846	272691	8.00	6251875	97.87
2000	9816972	6739378	1721602	-411458	281421	9.23	6841567	100.00
2001	10100772	7045362	1814712	-393745	282178	6.91	7621992	102.38
2002	10480821	7385315	1932534	-480861	287974	4.67	8232908	103.95

Table A6.1 Data series for the U.S. in the period 1970-2002[55]

[55] **Sources:** *GDP*, *private consumption*, *government spending* and CPI: the data series *gross domestic product*, *private consumption expenditures*, *government consumption expenditures and gross investment* and *price indices for gross domestic product* from Bureau of Economic Analysis (BEA). Current account: the data series *net lending/saving* from OECD. Population: estimates from U.S. Census Bureau. Nominal int. rate and M3: the data series *bank prime loan rate* and *broad monetary aggregate M3* from the Federal Reserve.

year	GDP (mill. NOK)	private consumption (mill. NOK)	government spending (mill. NOK)	current account (mill. NOK)	population (in 1000's)	nominal int. rate (in %)	M3 (mill. NOK)	CPI (1920=100)
1970	91100	47048	14791	-1860	3863.221	5.15	43193	232.47
1971	101825	52806	17348	-3701	3888.305	5.28	48209	248.05
1972	112821	57952	19555	-573	3917.773	5.33	53885	264.94
1973	127974	64389	22351	-2216	3948.235	5.43	60345	285.71
1974	148322	72620	26120	-6363	3972.99	6.13	66994	311.69
1975	169896	85086	31530	-12916	3997.525	6.48	76748	348.05
1976	193812	97840	37374	-20771	4017.101	6.86	90209	380.52
1977	218484	112991	42587	-26773	4035.202	7.00	105782	415.58
1978	239951	119892	48010	-8202	4051.208	8.16	117791	449.35
1979	264802	131481	51771	-5183	4066.134	8.82	133402	470.13
1980	314363	146664	59773	7380	4078.9	9.84	151274	522.08
1981	358176	165794	69220	12715	4092.34	10.72	168045	592.21
1982	396186	186189	77821	3559	4107.063	11.48	185472	659.74
1983	439023	205619	86318	18016	4122.511	11.64	203460	715.58
1984	494457	225601	92926	26774	4134.353	11.46	244765	761.04
1985	547286	261243	101211	26221	4145.854	11.70	278449	803.90
1986	561842	292660	110944	-34641	4159.187	13.58	284127	861.04
1987	613157	312868	127327	-30008	4175.521	13.58	328606	936.36
1988	643375	325167	134538	-27139	4198.289	13.04	345081	998.70
1989	686034	338778	142703	-1504	4220.686	11.58	374771	1044.16
1990	726799	357100	154193	17590	4233.116	11.34	395097	1087.01
1991	769782	378939	167619	27481	4249.83	10.88	442720	1124.68
1992	790300	396793	179707	26716	4273.634	11.02	479220	1150.65
1993	830416	416228	187473	24765	4299.167	7.44	476015	1176.62
1994	873410	435350	193832	25264	4324.815	7.18	501305	1193.51
1995	937445	462262	202144	32182	4348.41	6.82	530257	1223.38
1996	1026924	498965	214675	69892	4369.957	6.08	564364	1237.66
1997	1111349	527135	227490	68615	4392.714	5.46	578538	1270.13
1998	1132134	554540	247435	-327	4417.599	7.66	605329	1298.70
1999	1233039	584272	263730	65020	4445.329	6.32	670121	1328.60
2000	1469075	625501	281117	228002	4478.497	7.39	731843	1370.10
2001	1526601	656990	309566	237701	4503.436	7.20	795218	1411.70
2002	1520728	685179	332450	200190	4524.066	7.17	855390	1430.10

Table A6.2 Data series for Norway in the period 1970-2002 [56]

[56] **Sources:** *GDP*, *private consumption*, *government spending* and *population*: the data series *gross domestic product*, *consumption expenditure of households and NPISHs (Non-Profit Institutions Serving Households)*, *final consumption expenditure of general government* and *population* from Statistics Norway. Current account: the data series net lending/saving from OECD. *Nominal int. rate*, *M3* and *CPI*: the data series bank *interest rate indicator, broad monetary aggregate M2* and *CPI* from Norges Bank.