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# Bivariate frequency analysis of flood characteristics in Glomma and Gudbrandsdalslågen

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# Abstract

The spring flood is a multivariate event that can be characterized by the flood characteristics peak discharge, duration and volume. Traditionally these flood characteristics have been modelled separately, but by modelling the spring flood as a multivariate event, information about the joint probabilistic behaviour is obtained. In this thesis the bivariate logistic extreme value distribution with GEV margins are used to model asymptotically dependent flood characteristics, and the bivariate log normal distribution with log normal margins are used for modelling asymptotically independent flood characteristics. A new method is made for the assessment of the duration of the spring flood. The bivariate models and the method for assessment of duration are tested in the Glomma basin in the rivers Glomma and Gudbrandsdalslågen at stations Elverum and Losna, respectively. At Elverum the method for assessment of duration gave reasonable results, whereas at Losna the method gave poor results. For the bivarate models there were good agreement between the models and observations, and the models were found appropriate for modelling the corresponding flood characteristics. The bivarate joint distributions of the flood characteristics were used for constructing bivariate return periods.

# Contents

# Chapter 1

# Introduction

## 1.1 Background

The river has always been important for the development of civilization. Almost every large city in the world is located near a river, in fact the start of civilization was situated between the rivers Eufrat and Tigris. But being located near a river has both advantages and disadvantages, and a major disadvantage is the risk of floods. Floods can cause severe damage and can lead to loss of lives of both humans and livestock. In addition there are limited posibilities for protection against extreme floods. Every year there are several extreme floods in the world that lead to severe damage. One of the last extreme floods in Europe took place in the river Elbe in the summer of 2002. The economical loss for the countries Germany and the Czech-Republic caused by this flood summed up to the amount of $12 billiones (Rüschlikon, 2004). On top of this comes the loss of irreplaceable cultural inheritage. In the last decade there has been more focus on risk areas for floods when spatial planning areas are located, and also on assessment of flood risk damage in developed areas. Bakkehøi (2003) performed a flood risk analysis for the cities Hamar and Lillestrøm in Norway.

A flood can be defined in many ways, e.g. U.S. Geological Survey uses the following definition: "*An overflow or inundation that comes from a river or other body of water and causes or threatens damage* (USGS, 2004). Any relatively high streamflow overtopping the natural or artificial banks in any reach of a stream". A norwegian definition of flood used by Otnes and Ræstad (1978) is given as " *a river has a flood if the discharge exceeds the mean-discharge of the river"*. This is a rather wide definiton of floods, and with this definition most of the norwegian rivers will be in a flood situation on average 1/3'rd of the year. In this thesis the term spring-flood is used. A spring flood is a flood that occurs in the spring or early summer, and is caused by snowmelt or a combination of snowmelt and precipitation. Even though the spring flood occurs every year in regions with stable winter conditions, it does not nessecerily have to be associated with damage. Most years the spring flood pass through without causing any damage, but occationally damage occur. The largest known flood in Norway, the "storofsen" in 1789, was for instance a spring flood. In Norway the spring-flood is typically found in the mountain regime ($H_1L_1$) and in the inland regime ($H_2L_1$), but it can also be found in the transition regime ($H_2L_2$). Both the mountain and the inland regimes are regions with stable winter conditions. For more information on regimes, see Gottschalk et al. (1979).

The size of a flood is often given as a return period. A return period is defined as how many years in average it takes before a certain discharge is equalled or exceeded. The probability for a flood with a given return period to occur each year is assumed to be constant, e.g. the probability of a 50 year flood to occur once every year is 0.05. This means that if a 50 year flood occured one year, it is just as likely for a 50 year flood to occur the next year.

A flood event is a multivariate event that can be characterized by the flood characteristrics peak-discharge, duration and volume. These flood characteristics have traditionally been modeled separately, e.g. in Norway Sælthun et al. (1997) performed a regional flood frequency analysis of norweigan basins using peak-discharge and Solberg (1998) modeled the regional distribution of spring-flood volumes. The n-year return period of flood characteristics is often used in hydrological design as a criterion for the constructional requirements. Separate modeling of the flood characteristics are sufficient in hydrological design if the constructional requirements only depend on one flood characteristic. On the other hand if the constructional requirements depends on several flood characteristics, multivariate modelling of the flood characteristics is prefrable. For instance, for a reservoir spillway the n-year return period of the peak-discharge is an important constructional requirement. But if the inflow in the reservoir is large compared to the storage capacity, the n-year return period of volume is also important. Since the peak-discharge and volume are associated, a joint return period constructed from the joint distribution of peak-discharge and volume would be a better constructional requirement.

In hydrology there has been some attempts to model extreme events by using bivariate distributions. Ashkar (1980) investigated floods and derived a relationship between the peak discharge, the duration and the volume by using a simplified bivariate model. Buishand (1984) was concerned with the joint distribution of maxima at two different sites, which is importnant for the application of the station year method. For modelling Buishand applied the bivariate logistic distribution with standard Gumbel distributions as marginal distributions. Goel et al. (1998) applied a bivariate normal distribution for modeling flood peak discharge and flood volume, and Yue et al. (1999) applied the Gumbel mixed model for a flood frequency analysis. Yue (2000) applied a bivariate log normal distribution to model a multivariate flood episode and Yue and Rasmussen (2002) discussed some useful concepts of bivariate frequency analysis in hydrological applications.

During the last years there has been a thriving development in the branch of multivariate extreme value analysis, especially the statistical group lead by Jonathan Tawn has been active. This thesis will adapt some of the new techniques that have been developed that are not yet applied in hydrology.

Especially the paper written by Coles et al. (1999) is important for this thesis.

## 1.2 Objectives

In Norwegian basins the largest flood each year is typically the spring flood. Since the spring flood is a joint event that can be characterized by the flood characteristics peak discharge, duration and volume, multivariate modeling of the flood characteristics would lead to a gain in information of the spring flood compared to a separate modeling of the flood characteristics. By modeling the spring flood with a multivariate distribution, there is also a potential for the different flood characteristics to inform inferences on each other. This potential increases with increasing association between the variables.

The main objective of this thesis is to evaluate the applicability of bivariate frequency analysis on spring floods. By using a bivariate model for modeling the spring flood, the following combinations of flood characteristics are needed in order to characterize the spring flood; the peak discharge and volume, the duration and volume and peak discharge and duration. The bivariate analysis will especially focus on the selection of a bivariate distribution for modeling the different pairs of flood characteristics, and on bivariate return periods constructed for different design criteria.

In addtion to the main objective, a method to assess the duration from a hydrograph is required.

## 1.3 Thesis outline

This thesis consist of five chapters. Chapter one is the introduction of the thesis. Chapter two starts with a general background of the Glomma basin, and then focus is put on the selected hydrological stations. Important physiographic data of the seleced stations and a short discussion of the criteria for the selection of the stations are given. Finally arguments are given for performing a frequency analysis even though the series are regulated. In chapter three the methods for the assessment of flood characeristics are described. Then the theoretical background for modeling extremevalues are given. The theory is first given for univariate modeling, and then generalised to bivariate modeling. Both parts consist of a preliminary analysis for checking the model assumptions, and a part where the data are modeled. Chapter four consist of a presentation and a discusion of the results from the analysis. In chapter five a conclusion of the results are given. The appendix consist of the univariate theory that was not placed in chapter 3 and plots of the bivariate return periods for the flood characteristics at Losna.

# Chapter 2

# Data

## 2.1 General background of the Glomma basin

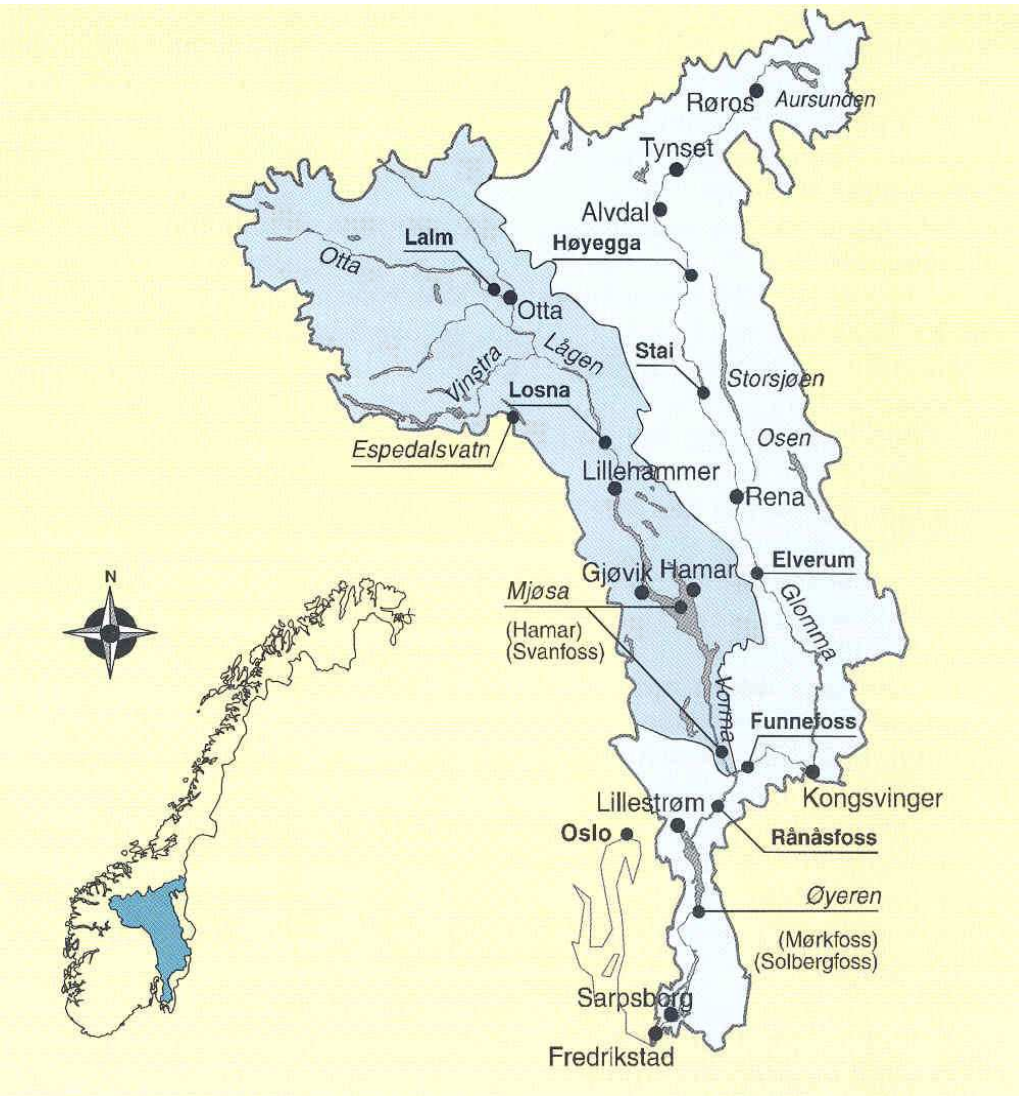


Figure 2.1: The Glomma basin.

### 2.1.1 Physiography

The Glomma basin is the largest basin in Norway and covers an area of 41922 km$^2$. It is situated near Røros in the North to Fredrikstad in the South, and from Grotli in the West to just across the Swedish border in the East. The basin consist of two main branches, the Glomma river in Østerdalen and river Gudbrandsdalslågen(Lågen) in Gudbransdalen. Downstream Funnefoss the two rivers join, and the name Glomma is kept downstream the confluence. The elvation of the basin range between sea level and 2469 m.a.s.l. (Galdhøpiggen). About 30 percent of the basin area is located above 1000 m.a.s.l., and about 30 percent is located below 500 m.a.s.l. There are large differences in the area elevation distribution within the basin, e.g. in the Oppland County 57 percent of the area is located above 900 m.a.s.l., whereas in Hedemark county only 21 percent is above.

Some fair-sized lakes are found in the basin. Lake Mjøsa is the largest lake in Norway with and area of 362 km$^2$. Other lakes are e.g. Storsjøen and Osensjøen. The presence of lakes in a basin will have an influence on the floods that occur in the basin, and will result in damped floods downstream the lake.

The total mean runoff of the basin is 705 m$^3$/s or 16.8 l/s km$^2$. These values have been determined from a runoff map that has been corrected with discharge values from the basin. There are many hydropower regulations in the basin, but the regulations only constitute 16 percent of the total mean runoff, which is a low regulation percentage. Still, the regulations will to some extent influence both floods and the low flow, and as a result the low flow is higher during winter and floods are generally smaller.

### 2.1.2 Climate

There are large variations in the precipitation and temperature in the basin, e.g. due to rain shadow from high mountains, the precipitation in the areas located around Lom is as low as 250 mm. In the west of the basin near the water divide to the Western part of Norway, the precipitation is beyond 1500 mm. In winter, the temperature is generally low in areas with continental climate, whereas in areas with a more maritime climate the winter temperature is more moderate. There is also a large temperature gradient between higher and lower parts of the basin.

In the Glomma basin several hydrological runoff regimes are found. The area upstream the confluence of Glomma and Lågen mainly consist of the mountain regime ($H_1L_1$) and inland regime ($H_1L_2$). Downstream the confluence the transition regime ($H_2L_2$) is mainly found.

## 2.2 Hydrological data

The hydrological series used in this thesis are taken from the HYDRA II database at NVE. This database consists of the series from all the hydrometric stations in Norway. At the hydrometric stations it is usually the water level and not the discharge that is measured, and the water level has to be converted to discharge. This is done by using the hydrograph of the corresponding hydrometric station. The hydrograph is the relation between water level and discharge, and is determined by a series of consecutive measurements of water level and discharge. For high water levels the hydrograph become increasingly uncertain due to changes in the profile and few discharge measurements.

In this thesis series from the hydrometric stations Elverum and Losna are chosen, and below some background information about these stations are given.

**Elverum**
The measurements at Elverum started in 1871 and are the longest continuous discharge series in Norway. It is located in the Glomma river, and has the following physiographical data; the station is located and 177 m.a.s.l, the maximum elevation difference is 2001. In figure 2.3 the hypsographic curve of the station is given. The area coverage is 15426 km$^2$, the glacier percentage is 0.01

**Losna**
The hydrometric station is located in lake Losna in the river Gudbrandsdalslågen. It is also a long series, and the measurements started in 1896. The station has the following physiographical data; the station is located 180 m.a.s.l., the maximum elevation difference is 2290 m. In figure 2.3 the hypsographic curve of the station is given. The area coverage is 11087 km$^2$, the glassier percentage is 2.98

The reasons for selecting these two stations for the bivariate analysis were, in the first place to use series were the spring floods behave differently, and secondly that uncertainty of the hydrograph is low at high levels. At Losna a larger part of the area percentage is located at higher elevation than Elverum, and the glacier percentage is larger at Losna compared to Elverum. As a consequence, the spring flood starts later and lasts longer than the spring flood at Glomma. Also, the quality of both data and the hydrograph at high levels are approved by the hydrometric department at NVE (The Norwegian Water Resources and Energy Directorate).

In this thesis the period 1961-2000 is chosen for both Elverum and Losna. There do not exist naturalized series for these stations for this period, thus the bivariate analysis is performed on the regulated series. Figure 2.2 shows that, even if the regulation percentage is low, the AMS values are to some extent influenced by the regulations. But since the regulation percentage is low for both stations, the AMS values are only to some extent influenced by the regulations (figure 2.2). And as long as the regulations lead to storage in the reservoirs, a frequency analysis on regulated data gives a more real picture of the expected discharge downstream the regulation compared to if a naturalized series was used.

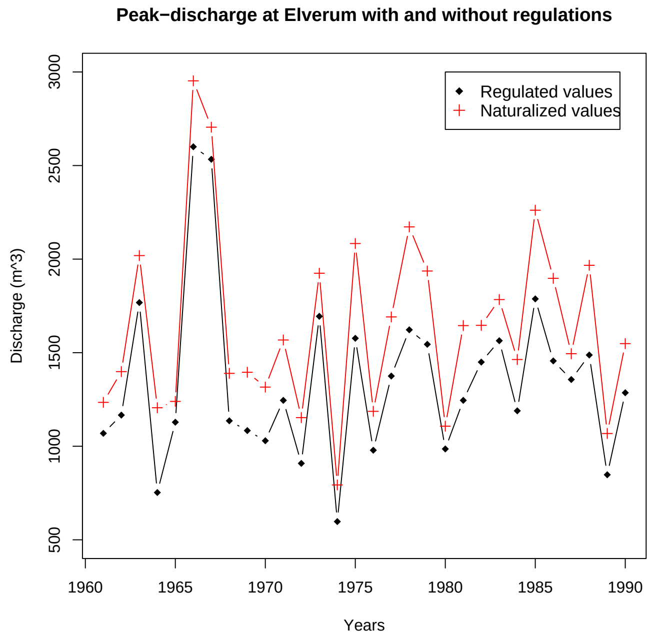


Figure 2.2: Peak-discharge at Elverum with and without regulations. The black line indicates the regulated values and red the red line indicates peak discharge values from a naturalized series of Elverum.

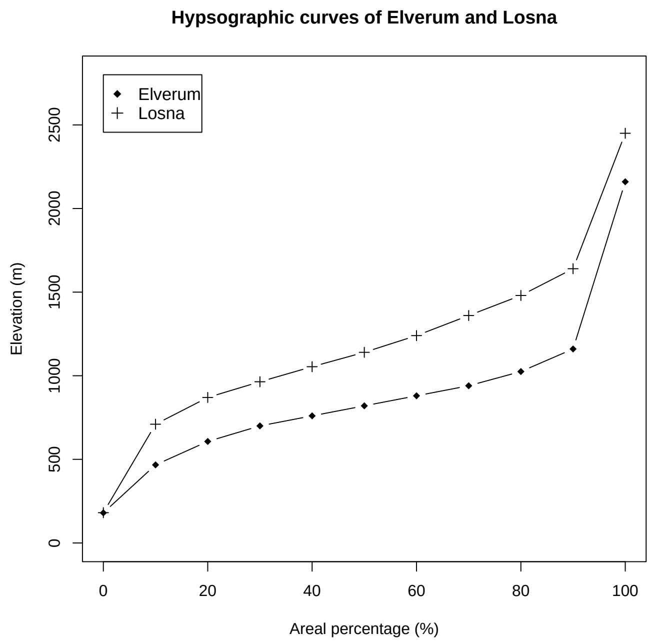


Figure 2.3: Hypsographic curves of Elverum and Losna.

# Chapter 3

# Methods and theory

## 3.1 Flood characteristics

A flood can be described by a set of flood characteristics defined from the discharge of a river at a certain measuring site. Important characteristics are the peak discharge, duration and volume. During a flood each of these characteristics will have different practical implications on the surroundings, the peak discharge, which is the maximum instantaneous discharge during the flood, determines what will be flooded, the duration determines the time effect on damage and the volume can be used for determining the volume that needs to be retained or stored in order to avoid flood damages.

When analysing the flood characteristics, the maximum values (or extreme values) of each characteristic are often of interest due to the application of extreme value methods. If the flood characteristics are evaluated separately, the largest value of each characteristic does not necessarily belong to the same flood event, e.g. at station Elverum in Glomma river, the maximum flood duration and volume usually appear in the spring mainly due to snowmelt, but occasionally the maximum peak discharge appear in the autumn due to precipitation (see chapter 2 for more information).

Previous works by different authors like Goel et al. (1998) and Yue et al. (1999) have shown that the flood characteristics are associated. Yue found that the peak discharge and volume, and the duration and volume were correlated, but argued that from a physical point of view, the peak discharge and duration should not be correlated. The last argument requires that the volume is random in each flood event. If, on the other hand, the volume does not vary too much, the peak discharge and duration is expected to be negatively correlated, which in fact is found in this thesis. Flood events with associated flood characteristics can be regarded as multivariate events. Then, instead of focusing on one flood characteristic, which has been done traditionally, the flood event can now be modelled using a joint distribution of several flood characteristics.

### 3.1.1 Assessment of duration

The assessment of the duration of a flood can be a difficult task. A flood is often a complex and diverse event, and no formal definition of the duration can be found. By using an informal definition given by Yue et al. (1999) the start of a flood is identified as an abrupt increase in the discharge from the base flow, and the end of a flood by the flattening of the recession curve of the discharge. The flood duration is then found as the difference between the start date and the end date. Even though this definition seems quite straight forward, it can be difficult to determine the end of a flood, especially spring floods. The spring flood is often a combination of different events like snowmelt, precipitation and discharge from glaciers (if present), which can lead to rather complex hydrographs. In addition, if the drainage basin analysed is large and there is a large altitude gradient, the snow in the higher part of the drainage basin will melt later than the snow in the lower part, and the hydrographs can become event more complex, especially if the altitude gradient is not smooth. Another problem is that summer rain is sometimes included in the spring flood. The recession is often interrupted by precipitation events, and it can be difficult to determine whether or not the event belong to the spring flood. Figure 3.1 shows a hydrograph for the river Gudbrandsdalslågen at the station Losna affected by the difficulties described above. The identification of the flattening of the recession curve and separation of summer precipitation in this hydrograph is almost impossible. Hence, in view of this example, an alternative method for the determination of the end of a flood is required.

The method used in this thesis is a modified verson of Yue‘s method where instead of identifying the end of the flood by using the natural recession curve, a master recession curve constructed from a recession analysis is used. A recession analysis is a way of modelling the stream outflow in a drainage basin given a climatic input (see appendix A.1 for more information). On this master recession curve the end of the flood is chosen to be the value of the 70-th percentile from the flow duration curve. By using a predestined value for determining the end of the flood instead of using the flattening of the reseccion curve, no subjective decisions needs to be made for the determination of the end. The reason for choosing the 70-th percentile is that in the hydrological regions in Norway where the spring flood occurs, discharge below this level is often considered to be base flow. A difficulty with this model is to determine when to initiate the use of the master recession curve. Generally, this is when the discharge starts to rise again due to summer precipitation after a general fall in the discharge values when there is no snow left. More specifically, one possibility is to identify the first precipitation event that belongs to the summer precipitation on the hydrograph. This can be done by the help of meteorological data from the drainage basin investigated. The initiation point for the master recession curve is then the point where this precipitation event starts, given that the discharge has not reached the 70-th percentile. If it is difficult to determine whether or not the precipitation event belongs to the summer precipitation, approximative methods can be used for determination of the initiation point.

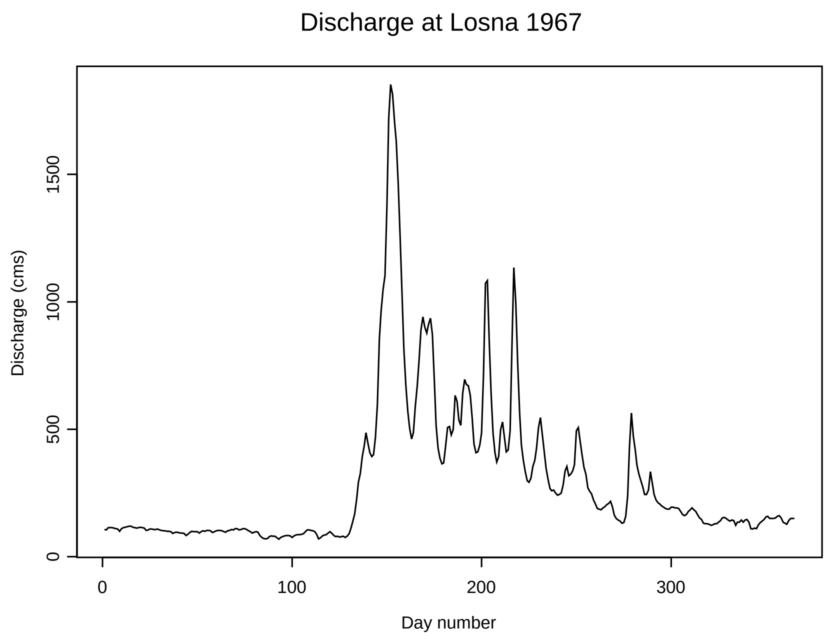


Figure 3.1: Discharge at Losna 1967.

These methods are subjective and to some extent individual for each station and are therefore explained in section 4.1.1, where the assessment of flood duration is undertaken. Finally, when all unknown factors for the assessment of the duration is determined, the duration can be found by calculating the difference between the dates of the initiation of the master recession and the start of the flood, and the difference between the dates of the end of the flood and the initiation of the master recession, and then add these differences.

### 3.1.2 Assessment of volume

The volume of a flood is defined as the volume of the discharge in the duration period of the flood. For assessment of this volume the area under the graph of the hydrograph in the duration period can be estimated. Due to the fact that the unit of the discharge is given in CMS, the estimated area must be multiplied with 86400 (the number of seconds in 24 hours) in order to obtain the correct unit for the estimated volume. The area under the graph can be estimated using the trapezoidal rule, and is given by

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right), \tag{3.1}$$

where $f(x_0)$ is the discharge at the start of the flood and $f(x_n)$ is the discharge at the end of the flood. The trapezoidal rule can be found in any calculus book, e.g. Linstrøm (1995).

The assessment of the volume of a flood does not involve any subjective considerations. Still, since the assessment of the volume depends on the duration, this must be evaluated before the volume is regarded as being obtained subjective or objective.

## 3.2 Modeling of extreme values

Section 3.2 is based on Coles (2001).
When dealing with extreme events like floods, extreme values from the processes that characterizes the flood are required in order to perform a (flood) frequency analysis. If the distributions of these processes are examined, the extreme values will appear in the tails of the distributions. Since, by definition, the observations in the tail are scarce, and most of the data are found around the centre of the distribution, estimation of the tail is difficult. If standard statistical estimation procedures where applied to estimate tail probabilities, it could lead to major discrepancies. Standard models like the normal distribution uses central values ($\mu$ and $\sigma$) in both assessing the model and estimating probabilities. This can be a good model for estimating probabilities on the body of a distribution, but due to the lack of model fit on the tails, estimating tail probabilities based on central values is generally not a good method. This is especially the case when the distribution is extrapolated beyond observed values. Thus an alternative model, which is capable to model the tail of a distribution, is required.

A part of the extreme value analysis is concerned with how often extreme events appear and how large those events are expected to be. These concepts are termed return period and return level (see section A.2.3) and the analysis is termed frequency analysis. Frequency analysis is very important in hydrological design, both for preventing catastrophes and for optimal construction. The result is obvious if the resistance of a construction is to low, but on the other hand, it is very expensive if a construction can resist much more than needed.

Frequency analysis is carried out by first selecting a method for the selection of extremes. There are three methods available, namely block maxima, partial duration series (pot) and the r largest order method. All methods are defined is section A.2.2. A distribution is then fitted to the extremes, preferably an extreme value distribution. The return period and the return level can then be constructed from this distribution.

### 3.2.1 Selection of extremes

The most common approaches for selecting extremes are the block maxima and the partial duration series (PDS). In the block maxima model the data are partitioned into blocks of the same length, and the greatest value from each block is put together in a new series. In Norwegian regions a block size of one year is often selected when modelling floods due to bias and variance considerations Coles (2001). The block maxima model is then called annual maximum series (AMS). The GEV distribution is appropriate for modelling block maxima. When using PDS, all values over a predefined truncation level are used as extremes. For modelling PDS the generalized Pareto distribution is used. Both of these approaches are actually special cases of the point process characterisation of extreme values. A definition of this approach can be found in Coles (2001).

The choice of approach depends on the data available and the sort of analysis to be undertaken. When the selected extremes are applied in a statistical model, there are some assumptions that need to be fulfilled in order to obtain satisfactory results. These assumptions are investigated in section A.4, and the most important assumption is that the extremes are independent and identically distributed (iid). In addition there must be a sufficient amount of data to stabilize the limit distribution in the extremmal types theorem (equation A.3). The choice of approach can influence both these issues. In the AMS approach the iid assumption is automatically fulfilled, given that the underlying distribution is stationary (see section A.4.1). For the PDS approach the choice of the truncation level is important for the iid assumption. If the truncation level is chosen too low, too many extremes are selected and the extremes become dependent. The selected extremes will then not follow a Poisson process, which is a requirement for this approach. With the PDS approach it is also likely that a larger amount of extreme values are selected compared to the AMS approach due to the fact that this approach uses all values over a predefined truncation level.

Both of these approaches have advantages and disadvantages that need to be taken into consideration when choosing a selection method. Some of the disadvantages by using AMS are that AMS uses only the largest value from each year, thus with this definition the second largest value one year can be greater than the largest value from another year. This can lead to the fact that information can be lost. Another disadvantage is that there is no mechanism verifying if the largest value in a year actually is extreme. Non extreme data included in the analysis can lead to bias when an extreme value distribution is fitted to data. The advantages with the AMS approach are that it is quite easy to carry out compared to PSD, and given stationarity the extremes are automatically iid. The PDS is an approach with greater analytical complexity. Two of the main difficulties are the selection of truncation level and independence of the extremes (discussed above). If the extremes are dependent, bias is expected in the consecutive extreme value analysis. There are methods for both selecting a truncation level and for declustering dependent extremes, see Lang et al. (1999) for more information. The advantaged with PDS is that more extreme values are usually selected during the same time period compared to AMS, and the selected extreme values are actually extreme since no values below the truncation level are used.

In this thesis the AMS approach is selected due to the nature of the spring flood. Since there are only one spring flood event each year, there is no gain in using PDS due to the fact that the amount of extreme values selected by both selection methods are equal.

### 3.2.2 Model requirements and preliminary analysis

Every statistical model has some assumptions that need to be fulfilled in order to obtain satisfactory results. If these assumptions are violated, an increased uncertainty in the results or a rejection of the model is expected. The most important assumption in extreme value models is that the underlying distribution of the extreme values must consist of a sequence of independent random variables. This assumption is almost never fulfilled in hydrological processes. But if stationarity is used instead of the iid assumption, it turns out that the extreme values satisfy the iid assumption. The preliminary analysis consist of different methods to reveal non-stationarity, i.e temporal variation (trend, step change) and persistence. These methods consist of exploratory data analysis (EDA) and statistical tests. For more information about these methods, see appendix A.4.

### 3.2.3 The extreme value distributions

For modelling AMS values, the generalized extreme value distribution (GEV) is appropriate. The GEV distribution was independently derived by von Mises (1936) and Jenkinson (1955), and is given as

$$G(x) = exp\{-[1 + \xi(\frac{x-\mu}{\sigma})]^{-1/\xi}\}, \tag{3.2}$$

where -$\infty \leq \xi \leq \infty$, -$\infty \leq u \leq \infty$ and $\alpha > 0$. The parameter $\xi$ determines the distribution. If $\xi > 0$ it is the frèchet distribution, $\xi < 0$ the Weibull distribution and if $\xi \to 0$ the Gumbel distribution. The parameters u and $\alpha$ are the location and the scale parameters, respectively.

The GEV distribution arise as an asymptotic argument where sample maxima are renormalized with sequences of renormalization constants. As the number of sample maxima increases, the distribution of the renormalized sample maxima converges to the GEV distribution. This is called the extremal types theorem. See appendix A.2.1 for a proper deduction.

### 3.2.4 Return level, return period and reduced plot

The return level $x_p$ is connected to the return period, and is the quantile function of the GEV distribution (equation 3.2) associated with the return period 1/p. Thus, the return level is defined as the expected time before a certain return(quantile)-level is exceeded. The quantile function of GEV and a more thorough examination of the return period is found in appendix A.2.3.

The return level and the return periods are often graphically represented by reduced plots. In these plots an appropriate plotting position is used for the estimation of non-exceedance probability of an empirical distribution. The estimated non-exceedance probabilities on a reduced form can then be plotted against data. If also a theoretical distribution on a reduced form is included, the discrepancy between the estimated non-exceedance probabilities and the theoretical model can be obtained. See appendix A.2.3 for more information.

## 3.3 Non EV-distributions used for modelling extreme values

The extreme value distributions are not always used for modelling extreme values. Extreme hydrological events like peak-discharge and flood volume are generally positively skewed and skew distributions like the log normal and the gamma distribution can give satisfactory results (Yue, 2000). These distributions do not have the same theoretical background for modelling extreme values as the extreme value distributions, and extreme value distributions are often preferred over other distributions.

### 3.3.1 The log normal distribution

The log normal distribution is an asymmetric distribution with a pronounced tail towards high values. It is derived from the normal distribution, and have to parameters $\mu$ and $\sigma$. If X is a log normal distributed random variable, then $Y = \log X$ is normally distributed. The probability density function (PDF) of the log normal distribution is given by

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{1}{2}\left(\frac{\log(x) - \mu_Y}{\sigma_Y}\right)^2\right] \; (X > 0) \; , \qquad (3.3)$$

where $\mu_Y$ and $\sigma_Y$ are the mean and standard deviation of Y, respectively Yue (2000). For estimation of the parameters and calculation of the CDF, see appendix A.3.

## 3.4 Selection of a theoretical distribution and model validation

In extreme value analysis, the selection of a theoretical distribution is particularly important. Different distributions will model the tail of the distribution differently, and since the chosen distribution often is extrapolated far beyond observed values, there can be substantial differences between the models. An incorrect prediction can have dramatic consequences.

In a hydrological setting several distributions have been used to model AMES, including normal, lognormal, gamma and the GEC distribution. In Australia and USA the gamma distribution is chosen as a standard distribution for flood frequency analysis, in Great Britain the GEC distribution is applied. In Norway the GEC distribution is often used, e.g. Sælthun et al. (1997) uses this distribution in regional flood frequency analysis of Norwegian basins.

The distribution chosen should reflect the number of observations available. A distribution with many parameters will fit the observations better, but will have greater uncertainty in the parameters. If there are few observations available, the uncertainty in estimating more parameters will exceed the gain in model fit, and a simpler model is more appropriate (Væringstad, 2001).

When selecting a theoretical distribution, there are methods available helping to decide which distribution is appropriate. These methods are termed model validation methods. There are several different model validation methods, including graphical methods, goodness of fit tests and maximum likelihood methods, like the Akaikes Information Criterion (AIC). The first two methods can be used to measure the discrepancy between the theoretical distribution and an empirical estimated distribution. The maximum likelihood method finds the best model among several candidate models, but give no information about how good the chosen model fits data. In this thesis graphical methods and the AIC is used. The AIC is found in section A.5.2, and the graphical methods are given below.

One of the main objectives of this thesis is to model the spring flood by using bivariate distributions. The selection of a theoretical distribution for univariate models is therefore performed with a view to apply the univariate distributions as marginal distributions in the bivariate distributions. Thus, in addition to satisfying the model requirements of univariate extreme value models (section A.4), the selected distribution must satisfy the model requirements of bivariate extreme value models (section 3.8). As a result, the bivariate preliminary analysis determines whether asymptotically dependent or asymptotically independent bivariate distributions are appropriate, and the univariate model validation methods determine which distribution within the class of asymptotically dependent or asymptotically independent distributions that are appropriate. For instance if AMS series are analysed and the bivariate preliminary analysis indicate asymptotic independce, the bivariate normal or the bivariate log normal distribution is appropriate. The univariate model validation methods can then be used for finding which of the marginal distributions of bivariate normal or bivariate log normal distribution that is appropriate. If the bivariate preliminary analysis indicates asymptotic dependence, the bivariate extreme value distributions is appropriate. In this situation only one marginal distribution is appropriate, namely the GEV distribution. This is due to the fact that in bivariate extreme value distributions the GEV family gives rise to the complete class of marginal limit distributions, and by generalizing the marginal distributions the complete family of the bivariate extreme value distributions can be obtained (see section 3.6.3).

### 3.4.1 Graphical methods

The graphical method is a subjective method where an appropriate plotting position is used for estimation of non-exceedance probabilities. These estimated values are then used together with theoretical distributions in various plots like e.g. histograms, qq-plots and reduced plots. In this way a visual picture of the fit of the different distributions are obtained, and the most appropriate distribution can be selected. This method can also be used for model validation, since a visual picture of the discrepancy between the theoretical distribution and the estimated non-exceedance probabilities is obtained.

## 3.5 Inference

In this thesis the following techniques for parameter estimation in statistical models have been used, l-moments (see appendix A.5.1) for estimation in the univariate extreme value models, methods of moments (see appendix A.3) for estimation of the parameters in the log normal distribution and maximum likelihood (see appendix A.5.2) for estimation in the bivariate models. One of the reasons for choosing maximum likelihood for bivariate parameter estimation is that routines for l-moment estimation has not yet been established. After the parameters are estimated with one of the techniques above, every parameter is re-estimated using a bootstrap routine for increased parameter accuracy. In addition, BCa bootstrap confidence intervals for the parameters are constructed. The bootstrap routine and the BCa confidence interval are given in appendix A.6. From the maximum likelihood estimation the Akaikes information criteria (AIC) can be calculated. This is a model validation method applied for selecting the best model among several candidate models. The AIC is given is appendix A.5.2.

## 3.6 Modelling of bivariate extreme values

### 3.6.1 The copula function

This thesis has focus one the joint distribution of different flood characteristics. Since the flood characteristics have different marginal distributions, it is difficult to make inferences about the dependence structure in the joint distribution. By transforming the marginal distributions into standardized marginals, the marginal structure is removed and the dependence structure can be evaluated. A joint distribution function with standardized margins is called a copula function. If $F_{X,Y}(x, y)$ is any continuous bivariate distribution function with marginal distributions $F_X(x)$ and $F_Y(y)$, the copula function can be expressed as

$$F_{X,Y}(x, y) = C\{F_X(x), F_y(y)\} = C(u, v) \; , \tag{3.4}$$

(Coles et al., 1999).

A bivariate distribution with marginal distributions $F_X(x)$ and $F_Y(y)$ can be transformed and standardised to have any continuous marginal distributions $G_X(x)$ and $G_Y(y)$ by using the probability integral transform

$$\tilde{x} = G_X^{-1}\{F_X(x)\} \text{ and } \tilde{y} = G_Y^{-1}\{F_Y(y)\} \; , \tag{3.5}$$

where $G_X^{-1}$ is the quantile function of $G_X$. The copula function can then be written as

$$C(u, v) = F\left(G_X^{-1}(u), G_Y^{-1}(v)\right) . \tag{3.6}$$

When constructing a bivariate extreme value copula, the marginal distributions are often GEV distributed. But since the dependence structure in the copula remains constant if the marginal distributions are transformed, marginal distributions that give simple theoretical expressions can be chosen. In this thesis the standard Frechet distribution, F(z) = exp(-1/z), z > 0, is used. If $X_1 ... X_n$ are iid standard Fréchet variables, then $M_n = max(X_1 ... X_n)$ satisfies Pr($M_n$/n $\leq$ x) = exp (-1/x) , for all n. Hence if the variables are standard Fréchet distributed, the extremes are also standard Fréchet distributed.

Componentwise maxima often belong to the bivariate extreme value logistic family, F(x,y) = exp{-($x^{-1/\alpha} + y^{-1/\alpha})^{\alpha}$}, x,y > 0 and $\alpha \in (0, 1]$. The copula function for the bivariate logistic extreme value distribution can be found using equations 3.5 on the preceding page and 3.6 on the page before. If the marginal distributions of a physical process originally are GEV distributed, they are transformed to a standard Fréchet distribution. The quantile functions of the standard Fréchet is $\tilde{x} = -1/ln(u)$ and $\tilde{y} = -1/ln(v)$, where U and V are the cdf's of the respective GEV marginal distributions. By inserting in equation 3.6 on the preceding page, one obtains the copula

$$C(u, v) = exp[-\{(-log\ u)^{1/\alpha} + (-log\ v)^{1/\alpha}\}^{\alpha}]. \quad (3.7)$$

For more information about the copula function, see (Joe, 1997).

### 3.6.2 Extremal dependence and limiting dependence behaviour

When performing a multivariate analysis, the variables in the analysis are regarded either dependent or independent of each other. If they are all independent, the multivariate distribution is just the product of the marginal distributions. If they are dependent, the dependence is modelled in the joint distribution, e.g. in the bivariate normal distribution, correlation ($\rho$) is used as a measure of the linear dependence between the variables. Since $\rho$ is constant in the distribution, the dependence is often assumed constant, but a more thorough investigation of the dependence show that the dependence varies with the level of data. For most purposes the approximation of constant dependence is sufficient, but when reaching extreme levels the variation of dependence needs to be taken into consideration. By continuing with the bivariate normal distribution, variables in this distribution with correlation < 1 will have independent independent extremes, given that the extremes are sufficiently high. This is very important when fitting a model to data. If this aspect is not considered, there is a possibility that the model will overestimate the dependence when extrapolating, since the model assumes that the extremes will happen simultaneously.

The dependence at extreme levels is termed extremal dependence, and is found in the limiting dependence behaviour of the extremes. In view of the preceding example there are two situations possible, asymptotic dependent and asymptotic independent extreme values. If (X,Y) is a random pair with an unknown joint distribution function F, unit Fréchet margins and unknown dependence structure, X and Y are asymptotically independent if

$$Pr(Y > t|X > t) \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (3.8)$$

and asymptotically dependent if

$$Pr(Y > t|X > t) \rightarrow c > 0 \text{ as } t \rightarrow \infty \tag{3.9}$$

(Heffernan and Tawn).

Several dependence measures are suggested by different authors for inference and interpreation, and most of them are based on the limiting dependence behaviour. For this thesis the measures $\chi$ and $\overline{\chi}$ from Coles et al. (1999) and the coefficient of tail dependence from Ledford and Tawn (1996) have been chosen. The measures are defined in sections 3.8.1 and 3.9.1.

### 3.6.3 The bivariate extreme value distributions

The bivariate extreme value theory is an extension of the univariate results in section A.2.1. The approach for modelling is also here by the use of sample maxima, but in two dimensions there is one additional problem - there is no natural maximum in a bivariate distribution. Authors like Gumbel and Tawn have chosen componentwise maxima as a definition for maxima in bivariate distributions, and this is also used here. If $(X_1, Y_1), (X_2, Y_2)...(X_n, Y_n)$ is a sequence of independent realizations of a random vector with distribution function F(x,y), and $M_{x,n} = max_{i=1,...,n}\{X_i\}$ and $M_{y,n} = max_{i=1,...,n}\{Y_i\}$, the vector of component maxima can be written

$$\mathbf{M}_n = (M_{x,n}, M_{y,n}). \tag{3.10}$$

Observe that with this definition $\mathbf{M}_n$ need not to be an observed vector in the original series.

In analogy with the univariate theory, the distribution of $\mathbf{M}_n$ can now be found by investigating the asymptotic behaviour of $\mathbf{M}_n$ as $n \rightarrow \infty$. But instead of evaluating the joint distribution of $\mathbf{M}_n$, the components are evaluated separately. $M_{x,n}$ and $M_{y,n}$ are univariate random variables, thus standard univariate extreme value methods can be applied to find the distributions of the variables (see sections A.2.1 and A.2.2). These distributions are the marginal distributions in the joint distribution of $\mathbf{M}_n$.

When the marginal distributions are known, the copula function is helpful for finding the bivariate dependence structure. By transforming the marginal distributions into standardized marginals using the probability integral transform defined in equation 3.5, the marginal structure is removed and the inference for the dependence structure is easier.
The marginal distributions of componentwise maxima are often GEV distributed, but by using the probability integral transform any of the extreme value distributions can be transformed into each other without any loss of information. Thus any of the extreme value distributions can be used for marginal distribution as long as it is standardized. In fact, any standardized continuous distribution can be used for marginal distribution. The choice of marginal distribution is more a question of which distribution is easier to work with. Tawn (1988) uses standardized exponential distributions as marginal distributions, Coles (2001) uses standardized

Fréchet distribution. The reason why distribution different authors are using is that different marginal distributions lies in the underlying marginal distributions $X_i$ and $Y_i$. In some cases, if the underlying distribution is known, the extreme value distribution is also known, e.g. if the underlying distribution is standard Frechet, the extremes are also standard Fréchet distributed given that the extremes are rescaled. This means that if $X_i$ and $Y_i$ are standard Fréchet distributed, then

$$Pr\{M_n/n \le z\} = exp\left(\frac{-1}{x}\right), \quad \text{for all n}, \tag{3.11}$$

where n is the number of observations. By assuming that the underlying marginal distributions are standard Fréchet, the vector of component maxima can then be given as

$$\mathbf{M}_n^* = (\max_{i=1,\dots,n} X_i/n, \max_{i=1,\dots,n} Y_i/n). \tag{3.12}$$

It is now possible to define the bivariate analogy to the extremal types theorem (see equation A.3 for the univariate case). If $\mathbf{M}_n^* = (M_{x,n}^*, M_{y,n}^*)$ is defined as above, then

$$Pr\left\{M_{x,n}^* \le x, M_{y,n}^* \le y\right\} \xrightarrow{\sim} G(x,y), \tag{3.13}$$

where G is a non-degenerate distribution function and is termed the class of bivariate extreme value distributions.

Different authors give the form of G differently. Pickands (1981) introduces a dependence function A(.) in the expression for G (not to be confused with the copula and other dependence functions given in this thesis). Pickands' class of bivariate extreme value distribution is given as

$$G(x,y) = exp\left\{-(x+y)A(\frac{y}{x+y})\right\}, \ \text{x>0, y>0}. \tag{3.14}$$

Coles and Tawn (1991) uses a function V(x,y) termed the exponential measure in G, and is on the form

$$G(x,y) = exp\{-V(x,y)\}, \quad \text{x>0, y>0}. \tag{3.15}$$

Coles notation is followed in this thesis. The reason for mentioning Pickands definition is that it is often used in papers concerning bivariate extreme value analysis, and it can cause great confusion if his dependence function are mixed with other dependence functions. Fore more information about Pickands dependence function and definition of the class of bivariate extreme value distributions, see Pickands (1981).

The function V(x,y) in equation 3.15 is given as

$$V(x,y) = 2\int_0^1 max\Big(\frac{w}{x}, \frac{1-w}{y}\Big) dH(w). \tag{3.16}$$

H is a distribution function on [0,1] that satisfies the constraint

$$\int_0^1 w dH(w) = 1/2. \tag{3.17}$$

The equations 3.16 and 3.17 are derived from a point process characterization of componentwise block maxima. This model is outside the scope of this thesis, but bellow a brief interpretation of the terms w and H introduced in these equations is given.
In the point process model the variables are transformed from Cartesian to pseudo polar coordinates, (x,y)→(r,w), where r = x+y and w = $\frac{x}{x+y}$. The transformed variable r is a measure of distance from the origin and w measures angle on a [0,1] scale. If w = 0, it corresponds to the x axis and if w = 1, the y axis. The distribution function H determines the angular spread of points in the limit Poisson process of the point process characterization. If H is a continuous distribution function with a density h and w measures the relative size of the (x,y) pair, then h can be iterpretated as the relative frequency of events of different relative size. If the extremes are near independent, large values of x/n and small values of y/n or opposite are expected. In this case h(w) is large close to w = 0 and w = 1, and small elsewhere. On the other hand, if there is strong dependence, x/n and y/n are likely to be similar in value and h(w) is large close to 1/2.
The last paragraph is based on Coles (2001).

### 3.6.4 Parametric families for modelling block maxima

In univariate extreme value theory the GEV family gives rise to the complete class of extreme value distributions. By letting the marginal distributions in the bivariate extreme value distribution be GEV distributed, the complete class of bivariate extreme value distributions can be found. But in the bivariate setting there is an additional problem. Any distribution H in equation 3.16 that satisfies equation 3.17 is a valid bivariate extreme value distribution. This is a problem because the class of distributions has no finite parametrization, which again leads to estimation difficulties. One way to overcome this problem is to restrict H by introducing parametric sub families for the distributions of H. Normally this would only lead to a subset of the class of bivariate extreme value distributions, since G is defined by H in equation 3.16. But it is possible to obtain parametric families for H, and hence G, such that every member of the limit class G can be approximated by a member of the sub family generated by the family of H.

Gumbel (1960) was the first to introduce parametric families for modelling bivariate extreme values. In recent years there has been great development in this field, and now there exist several parametric families, including the logistic family, the asymmetric logistic family, the Dirichlet model and the bilogistic model. The logistic family is often used for modelling componentwise block maxima. It is a very flexible family which covers all levels of dependence from independence to perfect dependence. The logistic family is given as

$$G(x,y) = exp\{-(x^{-1/\alpha} + y^{-1/\alpha})^{\alpha}\}, \text{ x,y} > 0, \tag{3.18}$$

where $\alpha$ is the dependence parameter $\in (0,1)$. Independence correspond to $\alpha \to 1$ and dependence to $\alpha \to 0$. A limitation in the model is that the variables (x,y) in equation 3.18 are bound to be exchangeable due to symmetry of the density function of H. If the density of H is given as

$$h(w) = \frac{1}{2}(\alpha^{-1} - 1)\{w(1-w)\}^{-1-1/\alpha}\{w^{-1/\alpha} + (1-w)^{-1/\alpha}\}^{\alpha-2}\ , \quad (3.19)$$

it is possible to show that equation 3.18 is obtained through equation 3.16, but this is rather complex and is not dealt with here.

In some situations the assumption of exchangeability between the variables is not appropriate and a model which allows for asymmetry is needed. Both the asymmetric logistic family and the bilogistic family are models that can be used. The asymmetric logistic family has three parameters $\alpha$, $t_1$ and $t_2$, where $\alpha$ is the dependence parameter and $t_1, t_2$ are asymmetric parameters.

$$G(x,y) = exp\left\{-(1-t_1)y - (1-t_2)y - [(t_1x)^{-1/\alpha} + (t_2y)^{-1/\alpha})]^{\alpha}\right\}\ . \quad (3.20)$$

When $t_1 = t_2$ the model is equal to the logistic family.

The bilogistic distribution has two dependence parameters $\alpha$ and $\beta$, and is given as

$$G(x,y) = \exp\left\{-xq^{1-\alpha} - y(1-q)^{1-\beta}\right\}, \quad (3.21)$$

where $q = q(x,y;\alpha,\beta)$ is the root of the equation

$$(1-\alpha)x(1-q)^{\beta} - (1-\beta)yq^{\alpha} = 0. \quad (3.22)$$

When $\alpha = \beta$ the bilogistic distribution is also equal to the logistic family.

## 3.7 The bivariate log normal distribution

If the assumptions for the bivariate extreme value models are violated, another model with a different dependence structure is needed in order to obtain satisfactory results (see section 3.8). A possible candidate model is the bivariate log normal distribution. One of the main difference between bivariate extreme value distributions and the bivariate log normal distribution is that, while the extremes in bivariate extreme value distributions are associated, the extremes in bivariate lognormal distribution become independent for sufficiently high values (see section 3.6.2). If the random variables $X_1$ and $X_2$ are log normal distributed then $Y_1 = \log(X_1)$ and $Y_2 = \log(X_2)$ are normally distributed. The PDF of the bivariate log normal distribution is given by

$$f(x_1, x_2) = \frac{1}{2\pi x_1 x_2 \sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2}}\exp(-\tfrac{w}{2})$$

$$w = \tfrac{1}{1-\rho^2}\left[\left(\tfrac{\log(x_1)-\mu_{Y_1}}{\sigma_{Y_1}}\right)^2 - 2\rho\left(\tfrac{\log(x_1)-\mu_{Y_1}}{\sigma_{Y_1}}\right)\left(\tfrac{\log(x_2)-\mu_{Y_2}}{\sigma_{Y_2}}\right) + \left(\tfrac{\log(x_2)-\mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right], \quad (3.23)$$

where $\mu_{Y_i}$ and $\sigma_{Y_i}$ (i=1,2) are the parameters of the marginal log normal distribution (see section A.3), and $\rho$ is the correlation between the variables $Y_1$ and $Y_2$.

If the random variables $X_1$ and $X_2$ are log normal distributed, then the conditional log normal distribution can also be defined by

$$f(x_1|x_2) = \frac{f(x_1,x_2)}{f(x_2)} = \frac{1}{x_1\sigma_{Y_1|Y_2}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\log(x_1)-\mu_{Y_1|Y_2}}{\sigma_{Y_1|Y2}}\right)^{\frac{1}{2}}\right], \tag{3.24}$$

where

$$\mu_{Y_1|Y_2} = \mu_{Y_1} + \rho\frac{\sigma_{Y_1}}{\sigma_{Y_2}}\left[\log(x_2) - \mu_{Y_2}\right] \tag{3.25}$$

and

$$\sigma_{Y_1|Y_2} = \sigma_{Y_1}\sqrt{1-\rho^2}. \tag{3.26}$$

For the bivariate log normal distribution there exist no analytical form of the CDF, thus a method is needed to estimate the CDF. Two possible methods that gives satisfactory results are numerical integration of the PDF and Monte Carlo simulation of the conditional probability density function (CPDF). In this thesis the Monte Carlo simulation is chosen. This method is similar to the bootstrap method found in section A.6, but instead of resampling the observations, the data is drawn from a distribution. The idea is that for each time a value is drawn from the chosen distribution, this value is used for estimating/calculating the event of interest. By drawing many values and performing many calculations, a distribution of the result of the calculation is obtained. And as the number of calculations increases the mean of this distribution will converge to the true value of the estimation/calculation. This is equivalent to the bootstrap parameter estimate given in equation A.34. When estimating the bivariate log normal CDF, the different combinations of $x_1$ and $x_2$ values that are of interest are selected. For each $x_1$ value, one million values were drawn from the CDF of $X_2$. Then one million cumulative conditional log normal values were estimated by using the predestined $x_1$ value and the one million drawn values as conditional values. Each estimated cumulative value is then multiplied with an indicator function that is 1 if the drawn value is smaller than the predestined $x_2$ value and 0 if larger. The bivariate lognormal CDF value of the predestined $x_1$ and $x_2$ value can now be found by calculating the mean of the one million cumulative conditional log normal values after the multiplication of the indicator function. This procedure must be repeated for each pair of $x_1$ and $x_2$ value. The entire estimation now can be given as

$$F_{X_1,X_2}(x_1,x_2) = \frac{1}{N}\sum_{i=1}^{N} F(x_1|X_2 = x_i)\ I(x_i < x_2)\ , \tag{3.27}$$

where $I(x_i < x_2)$ is the indicator function and $x_i$ are the values drawn from $F_{X_2}(x_2)$.

## 3.8 Bivariate preliminary analysis

The preliminary analysis of bivariate models for extreme values is an extension of the univariate analysis found in section A.4, and is concerned with the assumptions a bivariate extreme value model needs to fulfil in order to obtain satisfactory results. In bivariate models the model assumptions focus on the dependence structure between the variables of the process modelled. Different models have different dependence structures, thus by obtaining an informal picture of the dependence structure, an indication of the appropriateness of the models can be found. Due to the fact that the model assumptions consist of different aspects regarding the dependence structure, dependence measures defined from the copula function together with an EDA are used for an informal assessment of the dependence structure. The details are given below.

### 3.8.1 Dependent measures $\chi$ and $\overline{\chi}$

From section 3.6.2, multivariate extremes where divided into two classes, dependent and independent extremes. For inference of multivariate dependence the copula function can be used, but for easier inference and interpretation Coles et al. (1999) constructed the two parametric measures $\chi$ and $\overline{\chi}$. They serve as extremal dependence measures for respectively dependent and independent extremes. The idea is that the two measures are complementary, so that both measures are required for assessment of extremal dependence of an arbitrary random vector.

$\chi$ can be found by starting with equations 3.8 and 3.9. By a transformation of the marginal distributions to uniform margins, the equations can be expressed as

$$\chi = \lim_{u \to 1} Pr(V > u | U > u). \tag{3.28}$$

Now, by rewriting equation 3.28, one can obtain

$$\begin{aligned} Pr(V > u|U > u) &= \tfrac{Pr(U>u,V>u)}{Pr(U>u)} \\ &= \tfrac{1-2u+C(u,u)}{1-u} \\ &= 2 - \tfrac{1-C(u,u)}{1-u} \\ &\sim 2 - \tfrac{logC(u,u)}{logu}, \text{ as u} \to 1. \end{aligned} \tag{3.29}$$

It is now possible to define an extremal dependence function

$$\chi(u) = 2 - \frac{logPr(U < u, V < u)}{logP(U < u)} \text{ for } 0 \leq \text{u} \leq 1, \tag{3.30}$$

The function $\chi(u)$ can be interpretated as a quantile dependent measure of dependence. The sign of $\chi(u)$ determines whether the variables are positively or negatively associated at the quantile level u.

From equation 3.30, the dependence measure $\chi$ can be defined as

$$\chi = \lim_{u \to 1} \chi(u), \tag{3.31}$$

which is an asymptotically equivalent function to equation 3.28. $\chi$ is defined beween $0 \leq \chi \leq 1$, where $\chi = 0$ indicates asymptotic independence and $\chi \geq 0$ indicates asymptotic dependence. Generally, an increase in the value of $\chi$ leads to an increase in the strength of extremal dependence in the class of asymptotically dependent extremes.

If equation 3.28 indicates asymptotic independence, the example in section 3.6.2 shows that there can exist dependence at sub-asymptotic levels. Obviously $\chi$ can not be used to assess information about the strength of the dependence, since $\chi = 0$ by definition for asymptotically independent extremes. Thus an other measure is needed for assessment of extremal dependence in that class. By introducing the joint survivor function, $\overline{F}(x, y) = 1 - F_X(x) - F_Y(y) + F(x, y)$, the copula survivor function can be defined as

$$\overline{C}(u, v) = 1 - u - v + C(u, v). \tag{3.32}$$

Now, in the same way as $\chi(u)$ is defined, a new function $\overline{\chi}(u)$ can be defined as

$$\overline{\chi} = \frac{2logPr(U>u)}{logPr(U>u,V>u)} - 1 \tag{3.33}$$

$$= \frac{2log(1-u)}{log\overline{C}(u,u)} - 1, \text{ for } 0 \leq \text{u} \leq 1,$$

and a dependence measure $\overline{\chi}$ analogous to equation 3.31, can be defined as

$$\overline{\chi} = \lim_{u \to 1} \chi(u). \tag{3.34}$$

$\overline{\chi}$ is defined between $-1 \leq \overline{\chi} \leq 1$, where $\overline{\chi} = 1$ indicates asymptotic dependence, and $\overline{\chi} < 1$ indicates asymptotic independece.

With a complete pair of the complementary measures ($\chi$ and $\overline{\chi}$), a summary of the multivariate extremal dependence can be determined.

- ($\chi = 0, \overline{\chi} < 1$) indicates asymptotic independence, and the value of $\overline{\chi}$ determines the strength of the dependence in the respective class.
- ($\chi > 0, \overline{\chi} = 1$) indicates asymptotic dependence, and the value of $\chi$ determines the strength of the dependence in the respective class.

### 3.8.2 Preliminary analysis of bivariate extremal dependence

The preliminary analysis of bivariate extremes is concerned with different aspects regarding the dependence structure between the variables of a process modelled. By performing an EDA, graphical representations of the dependence structure give informal results that can be used as an indication of the appropriateness of different extreme value models. The first plot in the EDA is a plot of the copula function of the investigated process. In this plot the marginal distributions are estimated by the empirical distribution function and then transformed to uniform distributions by the probability integral transform (equation 3.5). In this way the marginal distributions have the same distribution, and data can be regarded as realisations from the associated copula. By plotting the estimated uniform marginals against each other, an informal picture of the dependence at all levels is obtained. The extremal dependence of the process can then be obtained by an examination of the largest values of the marginal distributions. If there is an increased density of points near the point (1,1) in the plot, the most extreme events are likely to be associated. Association of the exteme events are one of the assumptions of the bivariate extreme value models, thus the plot give an indication of whether or not bivariate extreme value models are appropriate. If the association between the extremes are low, it can be difficult to extract information from the plot, and the next plot in the EDA is more appropriate.

The next step in the EDA is to calculate and plot the dependence measures $\chi(u)$ and $\overline{\chi}(u)$ using the estimated marginal distributions from the previous plot. By analysing the limiting behaviour of the dependence measures as the uniform marginal reach supremium (u→1), indications of whether data are asymptotic dependent or independent can be found. The plots of the dependent measures can also give informal results of model validation. Extreme value models are asymptotically dependent models, and $\overline{\chi} = 1$ for these models. Also, $\chi(u)$ is constant for the extreme value models. A violation of these assumptions indicates that extreme value models are not appropriate for modelling the process investigated.

## 3.9 Parametric inference for asymptotic dependence

The parametric inference for the dependence measures $\chi(u)$ and $\overline{\chi}(u)$ is based on a model defined by Ledford and Tawn (1996) called the coefficient of tail dependence. In this model there are two dependece measures that resembles $\chi(u)$ and $\overline{\chi}(u)$, in fact the next section will show that there is a connection between $\chi(u)$, $\overline{\chi}(u)$ and the dependence measures given by Ledford and Tawn. Since the inference is easier for this model due to an introduction of a structure variable (see section 3.9.2), $\chi(u)$ and $\overline{\chi}(u)$ can be found by first estimate the parameters in the coefficient of tail dependence model, and then calculate $\chi(u)$ and $\overline{\chi}(u)$ from the dependence measures in this model.

### 3.9.1 Coefficient of tail dependence model

Ledford and Tawn‘s model is based on a joint survivor function of an arbitrary random pair (X,Y) with unit Fréchet marginals that satisfies the asymptotic condition

$$Pr\{X > z, Y > z\} \sim \ell(z)\{Pr(X > z)\}^{1/\eta}, \text{ for large z.} \tag{3.35}$$

This notation is found in Coles (2001). Here the $\ell(z)$ is a slowly varying function as z→ $\infty$ and $\eta$ is the coefficient of tail dependence. A deduction of the model is outside the scope of this thesis, but an interpretation of the model is given. In the model $\eta$ is a coefficient defined on (0,1] that characterizes the type of limiting dependence, and the function $\ell(z)$ is a measure of the strength of the dependence when a value of $\eta$ given. If $\eta < 1$ the variables are asymptotically independent, if $\eta = 1$ and $\ell(z) \to c > 0$, the variables are asymptotically dependent with a degree c.

The resemblance between $\eta$ and $\overline{\chi}$ and the resemblance between $\ell(z)$ and $\chi$ is now clear, but it remains to show the connection between them. Also here only the results will be given. If $\eta$ is estimated, $\overline{\chi}$ is then given as $\overline{\chi} = 2\eta - 1$. If $\eta = 1$, $\ell(z)$ and $\chi$ are equal for large z.

### 3.9.2 Inference

For the inference of the coefficient of tail dependence model the joint survivor function is reduced to a univariate survivor function T defined by $T = min(X, Y)$. The function T is termed the structure variable, and by inserting this into equation 3.35, it is possible to obtain

$$Pr(T > z) = Pr\{X > z, Y > z\} \sim \ell(z)z^{-1/\eta}, \text{ as z} \to \infty. \tag{3.36}$$

Ledford and Tawn then show that since T is a univariate variable, extreme value threshold methods (PDS) can be used on T. These results can then be used for inference for the tail of coefficient model. If T satisfy equation 3.36, a threshold model of T using the GP distribution can be given as

$$Pr(T > u + t \mid T > u) \sim (1 + \xi t/\sigma)^{-1/\xi} \,. \tag{3.37}$$

In this equation the shape parameter in the GPA distribution is equal to $\eta$ in the coefficient of tail dependence model and $\ell(z)$ is equal to the scale parameter. The dependence parameters $\chi(u)$ and $\overline{\chi}(u)$ can then be found by the conversion equations given in the previous section.

## 3.10 Selection of a theoretical distribution and model validation in bivariate EV-distributions

The selection of a distribution for modelling an extreme bivariate event involve the same considerations as for the univariate case (see section 3.4). In addition bivariate distributions must satisfy an important criterion, namely that the selected distribution needs to reflect the extremal dependence of the observations. This means that if the observations are asymptotically dependent, an asymptotically dependent distribution must be applied for modelling, or vice versa. When extrapolating in the bivariate setting there will be large discrepancies between the predications from the model and the "true" values if a distribution with incorrect exstremal dependence is applied.

There exist many different methods for selecting an appropriate distribution, including the graphical method , gof tests and the AIC (section A.5.2). In this thesis the AIC is used for the selection of bivarate distributions and the graphical method is used for model validation.

### 3.10.1 The graphical method for bivariate models

The graphical method for bivariate models is a generalization of the univariate graphical method, where an appropriate plotting position is used for estimation of non-exceedance joint probabilities. These estimates are then plotted together with a theoretical distribution, and a visual picture of the fit is obtained. The bivariate plotting positions are more complex than the univariate, and it is time-consuming to estimate the non-exceedance joint probabilities. For asymptotically dependent observations, the bivariate Gringorten potting position is applied for the estimation of non-exceedance joint probabilities. It is given by

$$F(p,v) = Pr(P \leq p_i, V \leq v_j = \frac{\sum_{m=1}^{i}\sum_{l=1}^{j} n_{ml} - 0.44}{N + 0.12}, \tag{3.38}$$

where N is the number of observations (pairs) and $n_{ml}$ is the rank of the combinations of $p_i$ and $v_j$. The last term can be found in the following way. A two dimensional table is constructed where the observation of the first variable are arranged in ascending order, and the values of the second variable correspond to the values of the first variable (see figure 4.13). The observations can now be plotted in the order given in the table (order number). When each pair is plotted, the order number of the plotted pair is determined among the already plotted values. In this way two consecutive pairs can have the same order.

For asymptotically independent observations the bivariate Weibull plotting position is applied for the estimation of non-exceedance joint probabilities, and is given by

$$F(p,v) = Pr(P \leq p_i, V \leq v_j) = \frac{\sum_{m=1}^{i}\sum_{l=1}^{j} n_{ml}}{N + 1}, \tag{3.39}$$

where N is the number of observations (pairs), and $n_{ml}$ is the rank of the combinations of $p_i$ and $v_j$.

## 3.11 Bivariate frequency analysis

The bivariate frequency analysis is an extension of the univariate frequency analysis, and is concerned with return levels and return period for bivariate events. An increase in the dimension of an analysis generally raises new issues that need to be considered. In the bivariate frequency analysis one of these issues is that return periods in two dimensions are ambiguous. Another issue is that bivariate analysis‘ are considerably more complex than univariate, and more sophisticated mathematics are needed for calculation of the bivariate return periods. This lead to difficulties in interpreting the bivariate return periods. In the next section the bivariate return periods are defined and some of the issues that arise due to dimensionality are discussed.

### 3.11.1 Bivariate return periods

In one dimension the return period of an event is defined as the average time interval before a certain quantile level (return level) is exceeded (section A.2.3). This is a familiar concept in hydrology which is often used in connection with hydrological design and frequency analysis. The concept of return period can be generalised into two dimensions, but then it is no longer unambiguous and several classes and subclasses of bivariate return periods exist. This arise because there are no natural maxima in a bivariate distribution. In section 3.6.3 this was solved by introducing componentwise maxima, and since bivariate distributions are included in the definition of bivariate return periods, the variables in bivariate return periods are also defined componentwise. In view of this issue three classes of return periods exist, these classes are

- Independent return periods
- Joint return periods
- Conditional exceedance probabilities

Independent return periods are used when the variables are independent or the design criterion is only dependent of one variable. The return period is then equal to the univariate return period of the variable of interest.

Joint return periods are used when the bivariate distribution is a joint distribution. Within this class subclasses that depends of the design criterion exist. The first sub-class is defined by a generalisation of the univariate return period, and is given by

$$T(x,y) = \frac{1}{1 - F(x,y)} , \tag{3.40}$$

where T(x,y) is the joint return period. This return period is valid for events where $X \geq x$ or $Y \geq y$.
The second sub-class of joint return periods is given by

$$T'(x,y) = \frac{1}{1 - F(x) - F(y) + F(x,y)} , \tag{3.41}$$

which is valid for events where $X \geq x$ and $Y \geq y$. Be aware that with the definitions of joint return periods that are used here, different combinations of values of the variables can give the same return period. A comparison of the joint return periods with the univariate return periods based on the same variables show that the following inequalities apply

$$T(x,y) \leq min(T_X, T_Y) \leq max(T_X, T_Y) \leq T'(x,y) \tag{3.42}$$

where $T_X$ and $T_Y$ are the marginal return periods for the variables x and y (Yue and Rasmussen, 2002). This means that by using univariate return periods instead of joint return periods for the first subclass (equation 3.40), the return period for the event of interest will be overestimated. On the other hand by using univariate return periods instead of joint return periods for the second subclass (equation 3.41), the return period for the event of interest will be underestimated. If these result are used for hydrological design, the planned construction is build weaker than intended by the constructional requirement. As a consequence the construction might not withstand the events it was designed for.

If the distributions of the different pairs of flood characteristics are given conditionally, the conditional exceedance probabilities can be calculated. The reason for using exceedance probabilities and not return periods is that with the use of conditional distributions the concept of return period is no longer meaningful. In joint distributions with componentwise block maxima there is one observation of each variable in each block, whereas in conditional distributions only the observations that satisfy the conditional condition are used. Thus in conditional distributions the time interval between the observations are not fixed. Since the time resolution of the return period is defined from the time resolution of the block, the observations used for determining the return period must satisfy the block condition, which is one observation in each block. Clearly, conditional distributions will not satisfy this condition. Also within this class there exist sub-classes depending on the design criterion of the analysis. The first sub-class is defined for the design criterion $X \geq x$ given $Y = y$, and is given by

$$T(x|y) = \frac{1}{1 - F_{X|Y}(x|y)}, \tag{3.43}$$

where $F_{X|Y}(x|y) = Pr[X \leq x|Y = y]$ is the conditional cumulative distribution function, and is defined as

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(u|y)du = \frac{\int_{-\infty}^{x} f(u,y)du}{f_Y(y)} . \tag{3.44}$$

For the event $Y \geq y$ given $X = x$, the conditional return period is given by an equation equivalent to equation 3.43, but with an exchange of the variables x and y.
The second sub-class of conditional return periods are useful in hydrological applications and is valid for the design criterion $X \geq x$ given $Y \geq y$. The exceedance probability for this event can be written as

$$S_{X|Y}(x|y) = S(X > x|Y > y) = \frac{1 - F_X(x) - F_Y(y) + F(x,y)}{1 - F_Y(y)}, \tag{3.45}$$

where S is the survival function. The return period for this event is then given as

$$T^{'}(x|y) = \frac{1}{S_{X|Y}(x|y)} = \frac{1 - F_Y(y)}{1 - F_X(x) - F_Y(y) + F(x,y)} . \tag{3.46}$$

By defining the conditional return period in this way, the value of the conditional variable can be given as an exceedance probability or an n-year univariate return period. This is an advantage because constructional requirements in hydrological design often are given as exceedance probabilities or return periods of the variable or variables of interest. For the event $Y > y$ given $X > x$, the conditional return period is given by an equation equivalent to equation 3.46, but with an exchange of the variables x and y. A comparison of the conditional exceedance probabilities in the second subclass (equation 3.46) with univariate exceedance probabilities based on the same variables show that by using univariate exceedance probabilities the event of interest is overestimated.

For the classes joint return periods and conditional exceedance probabilities there exist more than the two sub-classes in each class given in this theses. Other sub-classes can be defined by changing the design criterion for the analysis. For more information of bivariate return periods, see Yue and Rasmussen (2002).

# Chapter 4

# Results and discussion

## 4.1 Flood characteristics

### 4.1.1 Assessment of flood characteristics

**Elverum**

In sections 3.1.1 and 3.1.2 the objective methods for the assessment of the flood characteristics duration and volume are given. At Elverum it was most years possible to apply objective methods for the assessment of the duration. Those years when objective methods failed, approximative methods were used. These methods are adjustments on the objective methods made individually for each station, and subjective considerations can sometimes be necessary. The first approximate method was applied when the objective methods failed due to difficulties in determining whether precipitation events belonged to the summer rain or should be included in the spring-flood. This led to difficulties in determining the initialisations point for the master recession curve. The approximate method consist of initialising the master recession curve just before the first undefined precipitation event, instead of initialising the master recession curve just before the first precipitation event that belongs to the summer rain. Since these precipitation events are often small, the difference between the objective and the approximate method will also be small. The second approximative method was applied if the objective methods showed that the master recession curve should be initialised beyond the end of July. In these cases the master recession curve was initialised on the 31 of July.

Generally, it is difficult to evaluate the results of the assessment of the duration and volume since the "true" values can not be obtained. Instead the preliminary analysis (section A.4) can reveal if there are irregularities like trends in the assessed values, and the graphical method (section 3.4.1) can be used to investigate the distribution of the assessed values. If no irregularities are found in the preliminary analysis and the distribution seems appropriate, the probability that the results are reasonable increases. The results of duration and volume for Elverum satisfied both of the conditions given above, thus the results seems reasonable. Another issue is the objectivity of the results. For the assessment of the duration approximate methods were only applied a few times, and since the difference between the objective and approximative method are small, the duration values are considered obtained objectively. The assessment of volume does not involve any approximate methods. But since it is dependent of the duration, and the duration is considered obtained objective, so is the volume.

**Losna**
For the assessment of the duration at Losna the objective methods failed. The hydrograph is so complex that it is impossible to determine the initialisations point for the master recession curve, and approximate methods were needed. But it was also difficult to find an approximative method. The chosen method consist of identifying the event that contains the peak discharge, and to initialise the master recession curve just before the first large event after the peak discharge event that leads to a rise in discharge. With this method the flood duration of a sub basin upstream Losna is found, and not the volume of the entire basin. Thus, the results of the assessment cannot be used for hydrological design, since the assessment will underestimate the duration. The bivariate frequency analysis is still carried out for the sub basin.

For assessing the duration of the entire basin of Losna a totally different approach is needed. In the future satellite pictures for determining the snow coverage can be used. The duration of the spring flood can then be ended when it is observed that all the snow has melted or the snow coverage is below a certain predefined value. The results of the assessment of duration will affect the assessment of volume at Losna in the same way as the duration affected the volume at Elverum. Thus, only the volume of a sub basin upstream Losna is found.

For the assessed duration and volume no irregularities were found in the preliminary analysis, but the graphical method revealed that the duration and volume was bimodal distributed (see figure 4.2.3 for the duration). Thus, the chosen approximative method does not manage to separate the duration of the chosen sub basin with other sub basins, and some of the flood durations and volumes of the sub basin will be overestimated.

### 4.1.2 Association of flood characteristics

In order to regard a flood as a multivariate event, the flood characteristics must be associated. For evaluation of the association between the flood characteristics the correlation coefficient is used. In table 4.1 the results of the estimated correlations is given. The results show that there exist a positive correlation between the peak discharge and volume, and duration and volume as expected. But a correlation of 0.23 between the peak discharge and volume at Losna is a rather weak correlation. One possible reason for obtaining the weak correlation can be the assessment methods applied for the assessment of the duration at Losna (section 4.1.1). Since the assessment of the volume is dependent of the duration, poor results from the assessment of the duration will lead to poor results of the volume. It is also seen that there is a negative correlation between peak discharge and duration. This opens for the possibility to model a flood as a trivariate event, but a trivariate model is

Correlation of flood characteristics

| | Elverum | Losna |
|---|---|---|
| Peak vs. volume | 0.48 | 0.23 |
| Duration vs. volume | 0.35 | 0.57 |
| Peak vs. duration | - 0.41 | -0.25 |

Table 4.1: Correlation between the flood characteristics for stations Elverum and Losna

considerably more complex than a bivariate, and both model validation and computation will be more difficult. Another possibility is to model the peak-discharge and duration as a bivariate event, but the bivariate families applied in this thesis are not capable of modelling negatively associated variables. Bivariate families that are appropriate for modelling these events are outside the scope of this thesis. Thus it is chosen to model the spring flood as univariate and bivariate events. In the latter case the flood characteristics peak discharge and volume, and duration and volume are modelled.

## 4.2 Univariate preliminary analysis

In the preliminary analysis the test for revealing short term trends are performed on data from the period 1872-2000. For the other tests data from the period 1961-2000 are used. In the period 1921-1960 there were many regulations in both Glomma and Losna, and the data from this period are inhomogeneous (see section 2.2). After 1961 there have only been minor regulations in the basin, and the influence of these regulations on the flood characteristics are neglectable. Hence, the appropriateness of the period 1961-2000 is investigated. The results of the preliminary analysis are mainly presented for the flood characteristics peak discharge and volume at Elverum. For the other flood characteristics, the results are presented if important aspects are revealed.

### 4.2.1 Trend

In a time series both short term and long term trends can exist, depending on the mechanism that causes the trend. When using linear regression the objective is to reveal a long term trend, thus the entire series should be included in the analysis. The trend is modelled as the linear variation of the mean value of a time series. A constant mean value implies that the regression gradient is zero, and there is no trend. If a long term trend is found, other time series in the same region should be analysed to confirm the trend.

For finding short term trends, a moving average model (gauss filters) can be used. A plot of peak discharge at Elverum in the period 1872-2003 with a 9 and a 27 year gauss filter is found in Figure 4.1. Short term trends are not usually interesting when a series is investigated for stationary, but due to the relatively short time period chosen in this thesis, the 27 year gauss filter is of interest. An evaluation of the plot shows that for the 27 year gauss filter

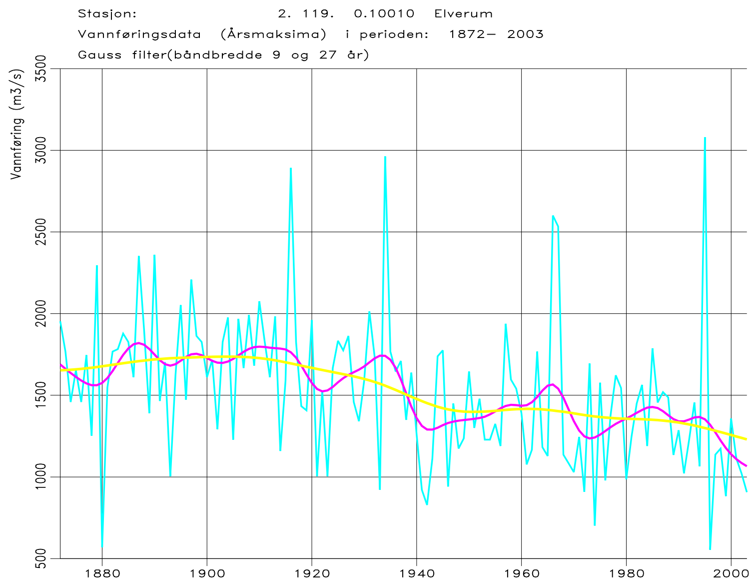


Figure 4.1: Peak discharge at Elverum with a 9 year (pink) and a 27 year (yellow) gauss filter.

trend line, there is an apparent negative trend in the years between 1920 and 1950. This trend is caused by the regulations discussed above. After 1960 a small change is seen, but this must be further investigated before any conclusions can be drawn.

The results from the trend tests with linear regression show that there is no significant trend in any of the time series of the flood characteristics that were investigated, when a 5 percent significance level was used. A plot of the regression line fitted to peak discharge at Elverum is found in Figure 4.2. Even though the regression gradient is negative in the plot, it is far from significant with a t-value of -0.388.

Trend tests using linear regression in combination with bootstrapping is also performed. The original series was resampled 10000 times and the regression gradient was calculated for each sample. If there is a trend in data the distribution of the calculated test statistics will be unsymmetrical, and the mean of the calculated test statistics (the bootstrap result) will differ from the test statistics of the original series. Non of the distributions from the calculated test statistics where unsymmetrical, and there were good accordance between the bootstrap results and the statistics of the original series.

A conclusion of the trend tests is that non of the trend tests revealed any inhomogenities in the time series of the flood characteristics at Elverum and Losna in the period 1961-2000.

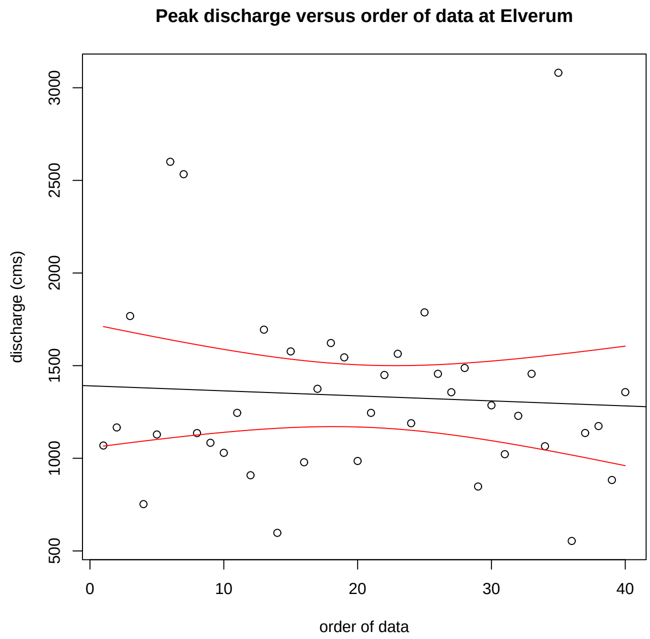


Figure 4.2: Trendplot using linear regression. The black line is the regression line. The red lines indicates the 95 percent confidence interval for the regression line.

### 4.2.2 Independency

In a stationary time series block maxima are approximately independent if they have a large lag in time and have sufficiently high values. To see if this condition is fulfilled the auotcorrelation function can be used. The autocorrelation function has been calculated for all series of flood characteristics, and non of them showed any sign of autocorrelation. All the values where inside the 95 percent confidence limits. A plot of the autocorrelation function of flood volume at Losna is found in Figure 4.3.

The second test for independence applied in this thesis is the runtest. In addition to independency of the fluctuation of data around a given threshold, the test can also to some extent reveal trends. The result of the test indicated that all series of flood characteristics where independent. With a 5 percent significance level, the number of runs from each series where all inside the confidence limits.

### 4.2.3 Selection of a theoretical distribution

In section 3.4 it was argued that both the results from the univaritate and the bivariate preliminary analysis is needed for the selection of a theoretical distribution. The result of the bivariate preliminary analysis (section 4.4.1) show that the peak-discharge and volume at Elverum is asymptotically dependent, and the rest of the pairs of flood characteristics are asymptotically independent. For the peak discharge and volume at

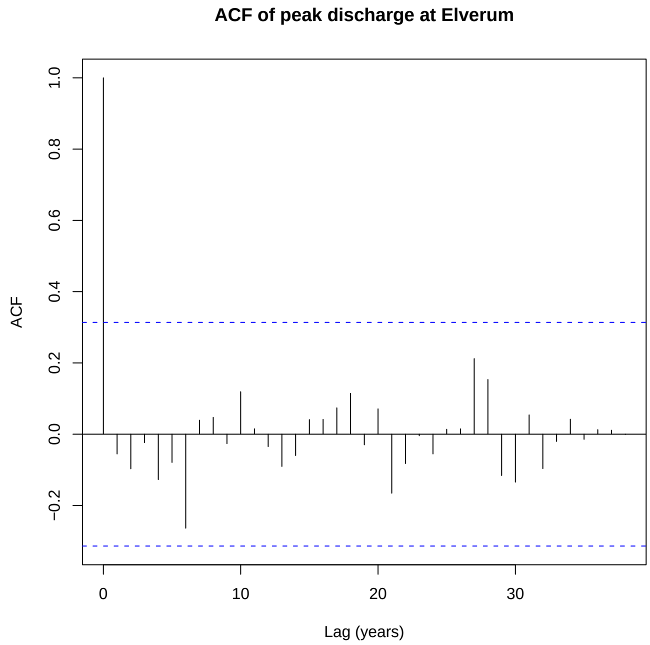


Figure 4.3: Plot of autocorrelation function of flood peak discharge at Elverum with a simple 95 percent confidence interval.

Elverum, the GEV distribution satisfy the bivariate model requirements and graphical methods are used to assess the fit of this distribution. For the asymptotically independent pairs, both normal and lognormal marginal distributions satisfy the bivariate model requirements. The normal distribution is a symmetric distribution and is rarely used in extreme value models due to the fact that the events that are modelled usually are skew (see section 3.3). Thus, the log normal distribution is selected as a potential distribution and the fit is investigated by using graphical methods.

In figure 4.4 a quantile plot, a probability plot and a density plot of the peak discharge at Elverum is given, and in figure 4.6 the reduced Gumbel plot of the same event is found. An evaluation of the plots show that there is generally good agreement between the GEV distribution and the observations. The model underestimates the three largest observations, but these observations lies on the upper 95 percent confidence band in the reduced Gumbel plot, and the discrepancy of the model is acceptable (see section 4.3.1 for a more thorough investigation of the possible outliers). For the volume at Elverum, the reduced Gumbel plot is given in figure 4.4. Also here there is a good fit between the model and the observations. Hence, the GEV distribution is selected as a theoretical distribution for the flood characteristics peak discharge and volume at Elverum.

In figure 4.5 a quantile plot, a probability plot and a density plot of the duration at Losna is given, and in figures 4.7 and 4.8 reduced plots of the duration and volume at Elverum, and the peak discharge, duration and volume at Losna are given. For the duration at Losna, the histogram in the

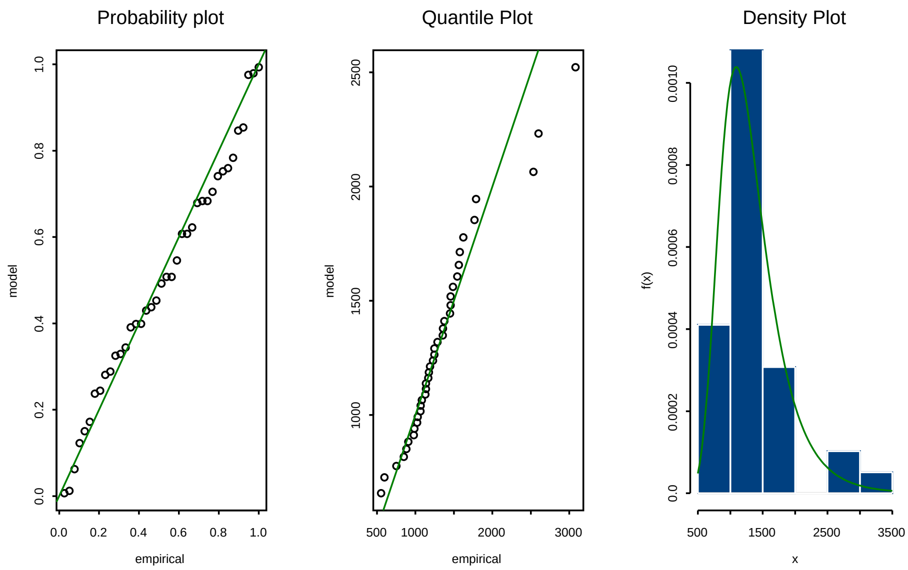


Figure 4.4: Probability plot, quantile-plot and density plot of peak discharge at Losna using the GEV distribution

density plot is bimodal and the log normal distribution will not manage to model this situation in a good way. The bimodal distribution arise due to poor assessment of the duration (see section 4.1.1), and since the duration and volume are dependent, the bimodality will also apply for the volume at Losna. Still, the model manage to some extent to model the largest values, and since this is the best obtainable model, the log normal distribution is selected as a theoretical distribution for the duration and volume at Losna. The problems that arise due to the bimodality of the data are discussed in section 4.3.2. For the duration and volume at Elverum and the peak discharge at Losna, the log normal distribution is found to be appropriate and is selected as a theoretical distribution.

## 4.3 Univariate frequency analysis

The frequency analysis is performed on spring block maxima from the stations Elverum and Losna for peak discharge, duration and volume in the period 1961-2000. But due to a comparison between different bivariate extreme value models in section 4.5, the univariate frequency analysis is performed by using both the GEV and the log normal distribution for all the flood characteristics. Still, the analysis is evaluated in view of the theoretical distributions selected in section 4.2.3. Upstream both stations there are several hydro power plant regulations, but the regulations only constitute of 10 and 13 percent of the mean runoff respectively. Thus, the

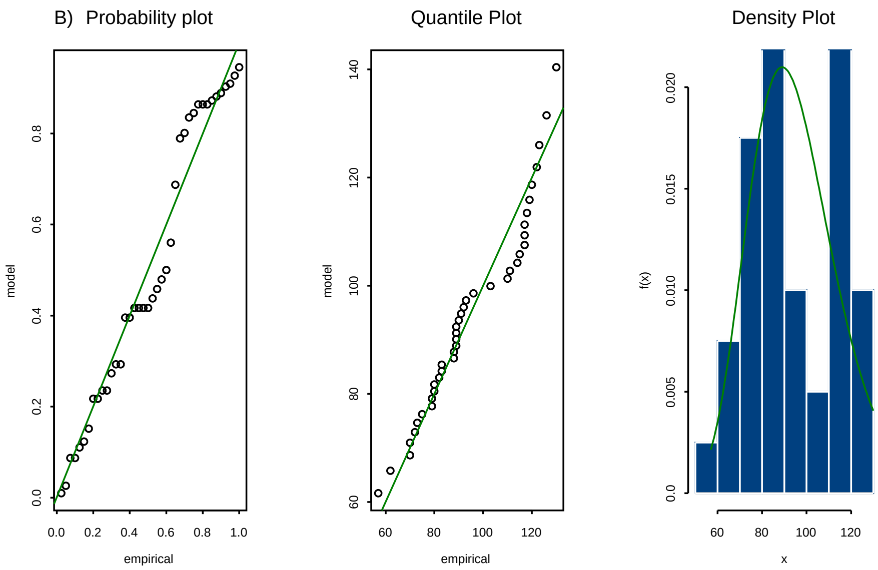


Figure 4.5: Probability plot, quantile-plot and density plot of flood duration at Losna using the log normal distribution.

regulations will to only a small extent influence the extremes (see section 2.2 for more information). The theoretical distributions applied for modelling the flood characteristics are selected due to both univariate and bivariate considerations. If the objective of this thesis was to perform an univariate frequency analysis, the GEV distribution would have been selected for all the flood characteristics instead of using the log normal distribution on some of the flood characteristics. This would have given better results in the analysis.

### 4.3.1 Frequency analysis of flood characteristics at Elverum

For the peak discharge at Elverum the GEV distribution was selected as a theoretical distribution, and in figure 4.6a) the reduced Gumbel plot is given. The estimated theoretical distribution is Frechet distributed with a $\xi$-value of -0.10. Generally, there is a good agreement between the model and the observations, but there are three outliers that to some extent influence the estimation. Block maxima from peak discharge is expected to be Gumbel distributed, and by removing the largest outlier (the 1966 observation) the $\xi$-value become 0.058 and the Gumbel distribution would also be an appropriate distribution. The tree outliers arise from the floods in 1966, 1967 and 1995, and an investigation of the meteorological conditions during the 1995 flood show that a stationary front situated in the north-south direction over the drainage basin lead to continuously large quantities of precipitation in addition to rapid snow melt. In 1967 there was also large quantities of precipitation during the flood, but the meteorological conditions are not known in detail. The meteorological conditions for the 1966 flood has unfortunately not been established due to difficulties in getting hold of data. In this thesis the outliers are included in the analysis despite the possibility that they belong to a different population. There is not sufficient data to model the outliers separately, and it is of interest to model all the extreme observations. Since the outliers lies inside the limits of the 95 percent confidence bands, the discrepancy from the model is acceptable. But one must be aware that there is a possibility that the model will underestimate floods that belong to the same population as the outliers, and to some extent overestimate the population not consisting of outliers.

The log normal distribution was selected as a theoretical distribution for the flood duration at Elverum, and the reduced plot is given in figure 4.7 b). There is a generally good agreement between the model and the observations, especially for the largest values. A factor that can influence the reduced Gumbel plot is the assessment of the duration. In some situations when objective methods failed, approximative methods have been applied (see section 3.1.1 for more information). If the duration values determined by these methods deviate from the "true" value, there is a possibility that the model will overestimate the duration. Unfortunately, it is difficult to examine whether or not deviations are present among the determined values, since the "true" value is not known. If there are only few values that deviates, and the deviations are rather large, there might be some indications of strange observations that need to be investigated in the preliminary analysis (section A.4), but generally, this is difficult to identify. In section 4.1.1 it was shown that approximate methods where used only a few times, and that the difference between approximate and objective methods was small. Thus, the approximate duration values will to only a small extent influence the reduced plot.

The flood volume at Elverum is used in both asymptotically dependent and asymptotically independent bivariate models, and the volume is therefore modelled with both the GEV and the log normal distribution. In the figures 4.6 c) and 4.7 c), the reduced Gumbel plot and the reduced plot for flood volume are given. For both plots all of the observations are inside the 95 percent confidence bands, and there is a good agreement between the models and the observations. The volume can also be influenced by approximative methods for the assessment of the duration due to the dependence between the duration and volume, and the same considerations as discussed for the duration applies for the volume.

The return periods that are given in the reduced plots and the reduced Gumbel plots are estimated using an appropriate plotting position. When the return period is estimated in this way, it is the number of observations that determines the largest value of the return period, e.g. in this analysis there are 39 observations of each flood characteristic. The largest observed return period for each flood characteristic is thus 71.4 (years). A more common way of estimating return periods is to use quantile values from a theoretical distribution fitted to observations (see section 4.2.3) and

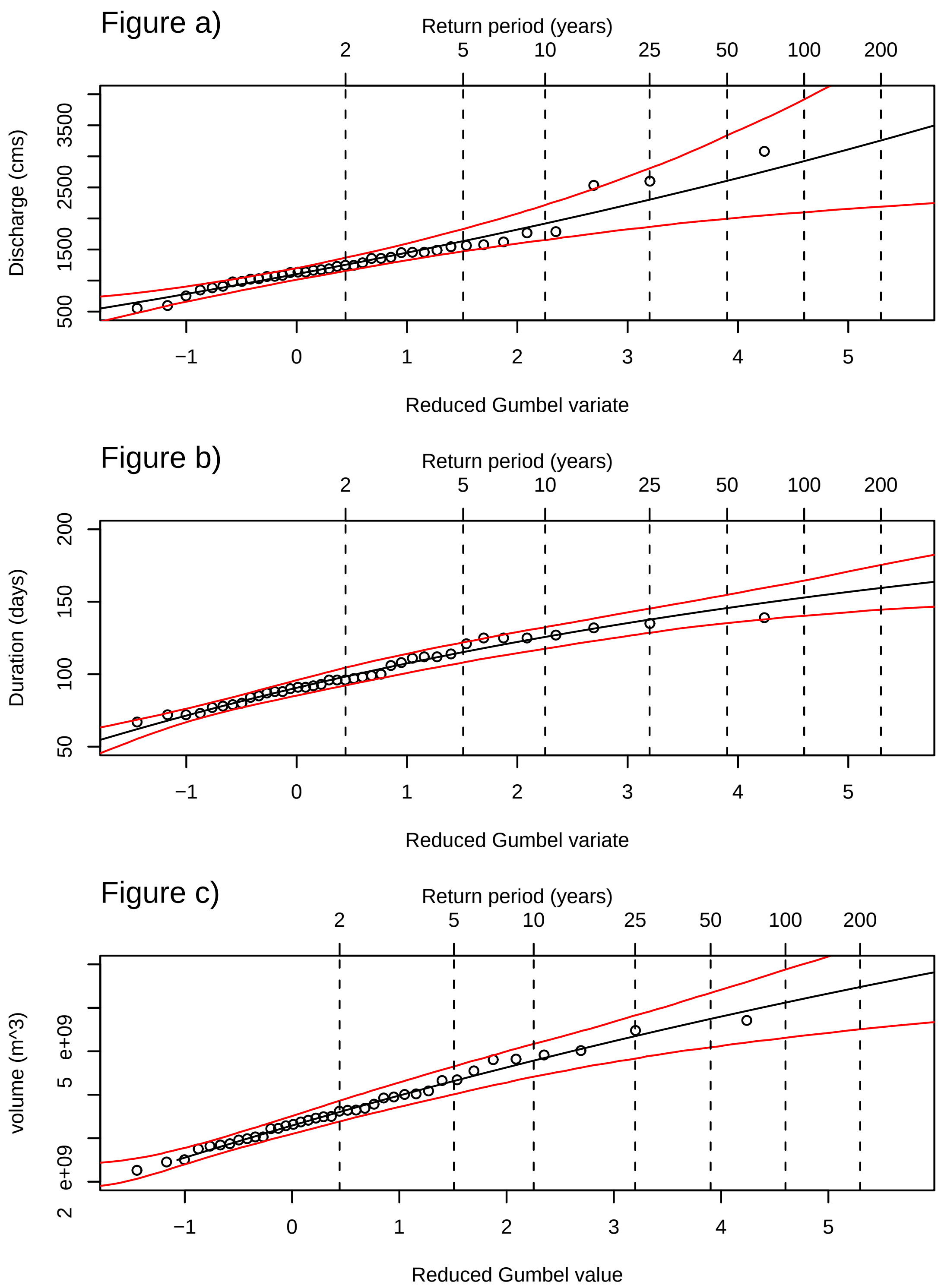


Figure 4.6: a) Reduced Gumbel plot of peak discharge at Elverum, b) Reduced Gumbel plot of flood duration at Elverum, and c) Reduced Gumbel plot of flood volume at Elverum. Red lines are 95 percent confidence intervals, the black line is theoretical distribution (GEV) and the points are observed data.

| Return period (years) | Peak-discharge(cms) | Duration (days) | Volume ($m^3$) |
|---|---|---|---|
| 2 | 1227 | 97 | 2.576e9 |
| 5 | 1652 | 114 | 3.296e9 |
| 10 | 1962 | 124 | 3.757e9 |
| 20 | 2283 | 133 | 4.187e9 |
| 50 | 2735 | 143 | 4.727e9 |
| 100 | 3104 | 151 | 5.120e9 |

Table 4.2: Univariate return periods for peak-discharge and volume at station Elverum. The return periods for the peak discharge and volume are estimated by using the GEV distribution, and the return periods for the duration is estimated by using the log normal distribution.

figure 4.4. The largest observed return period for the peak-discharge at station Elverum with this method is $\approx 96$ (years). When estimating return periods in this thesis, it is the latter method that has been used. In table 4.2 estimated return periods for the peak-discharge, duration and volume at Elverum are given. The return periods for the peak discharge and volume are estimated by using the GEV distribution, and the return period for the duration is estimated by using the log normal distribution.

### 4.3.2 Frequency analysis of flood characteristics at Losna

In section 4.2.3, the log normal distribution was selected as the theoretical distribution for the flood characteristics at Losna, and in figure 4.8 the reduced plot of the different flood characteristics is found. For the peak discharge in figure a) there is good agreement between the model and the observations, with the exception of one outlier. This outlier is due to the same meteorological conditions that caused the 1995 outlier at station Elverum. It is also interesting to notice that the 1966 and 1967 floods, that are possible outliers at station Glomma, are not outliers at this station. Hence, the 1966 and 1967 floods were more local than the 1995 flood.

Figure b) and c) show the reduced plots for the duration and volume. A problem with these plots is that the duration values originate from two different populations. This is the same problem as was seen in sections 4.2.3 and 4.1.1, and occur due to poor assessment of the duration. The hydrograph of the discharge at Losna is so complex that it is hard to find a method that can assess the duration without ending the flood duration far out in the autumn. Other possibilities are to remove the largest population and model with the remaining data, or to identify the values in the largest population and try to find conditions for ending the duration events earlier. Corrections on the data like this must be done with extreme care, so that the data is not altered in view of increasing the fit of the model. Thus, it is chosen to use all the duration values in the estimation of the return periods, and instead take into consideration the increased uncertainty by using these values. The increased uncertainty will also apply for the volume, due to the dependence between the duration and volume. In table 4.3 estimated return periods by using the log normal distribution for the peak-discharge, duration and

| Return period (years) | Peak-discharge(cms) | Duration (days) | Volume ($m^3$) |
|---|---|---|---|
| 2 | 1159 | 93 | 1.987e9 |
| 5 | 1491 | 110 | 2.517e9 |
| 10 | 1702 | 120 | 2.849e9 |
| 20 | 1898 | 130 | 3.155e9 |
| 50 | 2145 | 141 | 3.539e9 |
| 100 | 2328 | 149 | 3.821e9 |

Table 4.3: Univariate return periods by using the log normal distribution for the peak discharge, duration and volume at Losna.

| | | Elverum | | | Losna | | |
|---|---|---|---|---|---|---|---|
| | | $\hat{\xi}$ | $\hat{\sigma}$ | $\hat{\mu}$ | $\hat{\xi}$ | $\hat{\sigma}$ | $\hat{\mu}$ |
| Peak-discharge | Upper CI | 0.092 | 409.9 | 1165.5 | 0.455 | 437.5 | 1133.7 |
| | Lower CI | -0.293 | 241.9 | 955.4 | -0.135 | 228.4 | 935.7 |
| | Boostrap-estimated | -0.104 | 339.6 | 1100.7 | 0.135 | 338.7 | 1070.4 |
| Flood-duration | Upper CI | 0.300 | 21.27 | 94.44 | 0.383 | 22.82 | 92.55 |
| | Lower CI | -0.04 | 13.45 | 83.44 | 0.024 | 14.04 | 79.25 |
| | Boostrap-estimated | 0.098 | 18.09 | 90.16 | 0.200 | 19.76 | 86.95 |
| Flood-volume | Upper CI | 0.234 | 8.155e8 | 2.460e9 | 0.265 | 6.427e8 | 1.945e9 |
| | Lower CI | -0.137 | 4.646e8 | 2.071e9 | -0.058 | 4.223e8 | 1.597e9 |
| | Boostrap-estimated | 0.037 | 6.582e8 | 2.336e9 | 0.101 | 5.582e8 | 1.817e9 |

Table 4.4: Estimated parameters of the GEV distribution for all the flood characteristic using the bootstrap, and estimated 90 percent confidence intervals for these parameters using the BCa method.

volume at Losna are given.

### 4.3.3 Uncertainty

No analysis should be considered complete unless the uncertainty is estimated. In this thesis non-parametric bootstrapping is used for estimation of uncertainty. The advantage of using bootstrapping is that it is not necessary to deduce a theoretical expression for the uncertainty, it is simply estimated by resampling the estimator. In this way it is possible to estimate the uncertainty no matter how complicated the estimator is. The theory of bootstrapping is found is section A.6.

The uncertainty of the parameters in the GEV distribution are estimated by using the bootstrap in combination with l-moment, and the parameters of the log normal distribution are estimated by using the bootstrap in combination with ordinary moments. In table 4.4 the the estimated parameters of the GEV distribution, and in table 4.5 the estimated parameters of the log normal distribution are given. For both figures 90 percent confidence intervals by using the BCa method are made. In the reduced plots and the reduced Gumbel plots in figures 4.6, 4.7 and 4.8, 95 percent local confidence bands are made by using the percentile method.

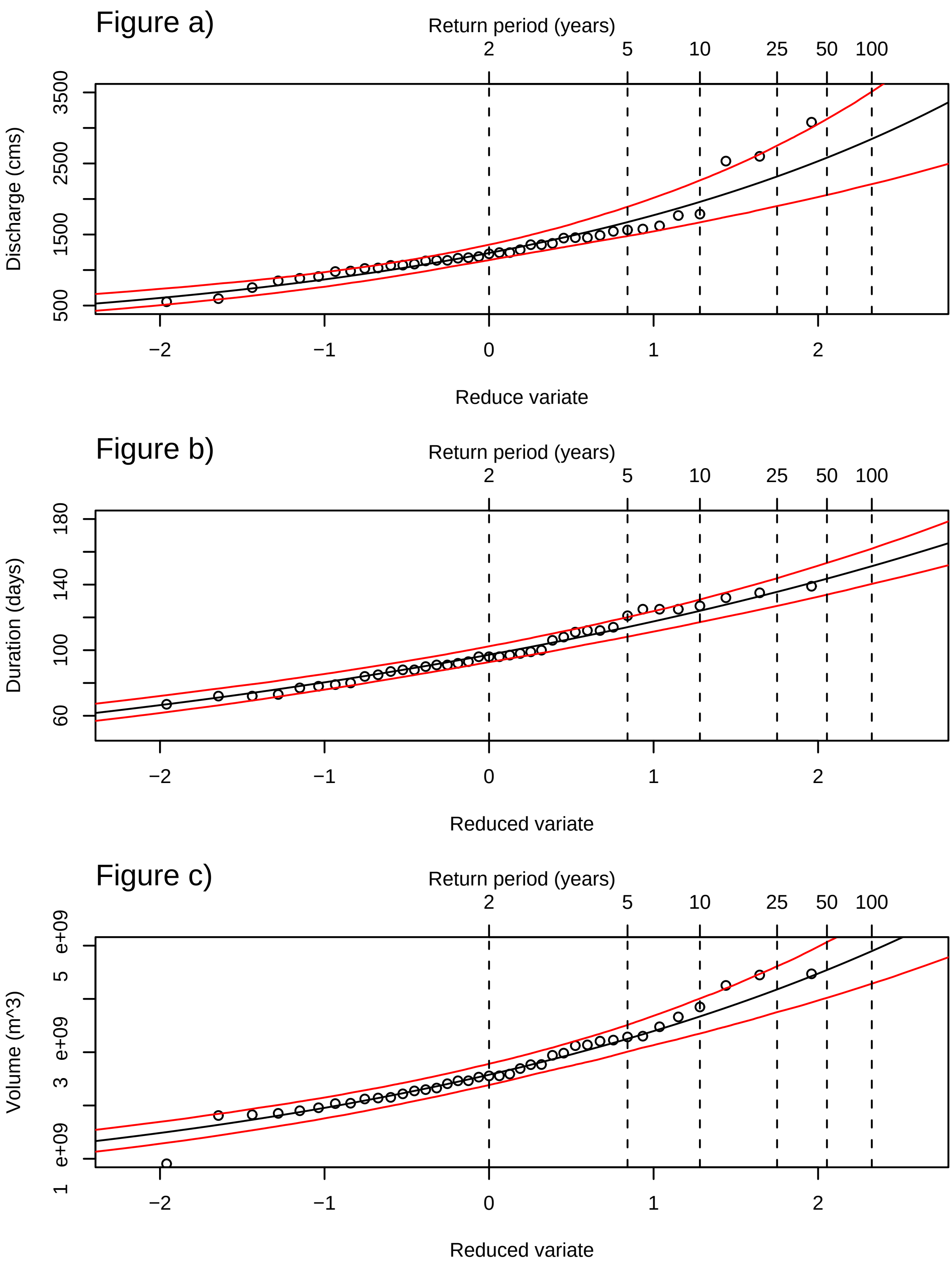


Figure 4.7: a) Reduced plot of peak discharge, b) Reduced Gumbel plot of flood duration, and c) Reduced Gumbel plot of flood volume. Red lines are 95 percent confidence intervals, the black line is theoretical distribution (log normal) and the points are observed data. The plots are valid for station Elverum.

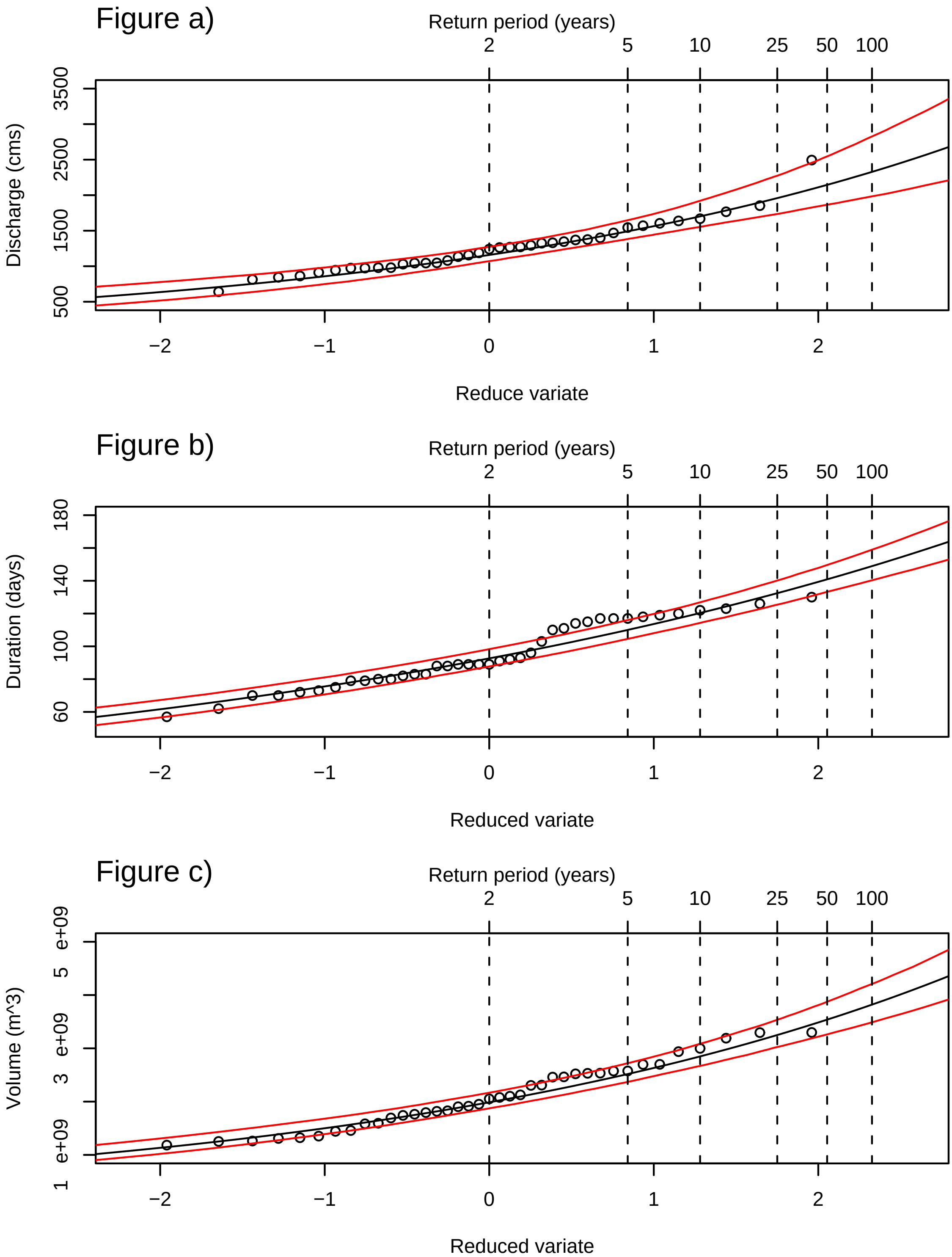


Figure 4.8: a) Reduced plot of peak discharge, b) Reduced Gumbel plot of flood duration, and c) Reduced Gumbel plot of flood volume. Red lines are 95 percent confidence intervals, the black line is theoretical distribution (log normal) and the points are observed data. The plots are valid for station Losna.

| | | Elverum | | Losna | |
|---|---|---|---|---|---|
| | | $\hat{\mu_Y}$ | $\hat{\sigma_Y}$ | $\hat{\mu_Y}$ | $\hat{\sigma_Y}$ |
| Peak-discharge | Upper CI | 7.214 | 0.461 | 7.129 | 0.401 |
| | Lower CI | 7.034 | 0.293 | 6.948 | 0.228 |
| | Boostrap-estimated | 7.123 | 0.357 | 7.055 | 0.300 |
| Flood-duration | Upper CI | 4.627 | 0.223 | 4.582 | 0.232 |
| | Lower CI | 4.526 | 0.170 | 4.473 | 0.179 |
| | Boostrap-estimated | 4.577 | 0.190 | 4.529 | 0.204 |
| Flood-volume | Upper CI | 21.75 | 0.346 | 21.49 | 0.319 |
| | Lower CI | 21.60 | 0.242 | 21.33 | 0.242 |
| | Boostrap-estimated | 21.67 | 0.276 | 21.41 | 0.281 |

Table 4.5: Estimated parameters of the log normal distribution for all the flood characteristic using the bootstrap, and estimated 90 percent confidence intervals for these parameters using the BCa method.

## 4.4 Bivariate extreme value analysis

### 4.4.1 Preliminary analysis of bivariate extremal dependence

The assumptions of the bivariate extreme value models are given in section 3.8.2, and are concerned with different aspects regarding the dependence structure between the variables of the process modelled. An EDA is performed to give informal results and plots of the bivariate dependence structure.

Figure 4.9 shows plots of the copula functions for the different flood characteristics at the stations Elverum and Losna. An increased density of points near the point (1,1) in the plots indicates that the most extreme events are likely to be associated, and thus bivariate extreme value models for modelling the process would be appropriate. For all the plots in figure 4.9, the association between the flood characteristics appears to correspond to the correlations in table 4.1. Peak discharge and volume, and duration and volume are positively associated, and peak discharge and duration are negatively associated. In the plots the association appears generally low, also for the most extreme events. In two of the plots, plots (b) and (e), there is a tendency of increased density near the point(1,1), but the results are not conclusive. To substantiate the results from the plot of the copula function and to gain more information about the dependence structure, the dependence measures $\chi$ and $\overline{\chi}$ were calculated and plotted. In figure 4.10 the dependence measures for peak discharge and volume at Elverum are given. It can be seen on the plot that $\overline{\chi}(u) = 1$ is a plausible value as $u \rightarrow 1$, and is consistent with asymptotic dependence. On the other hand the plot of $\chi(u)$ shows that the dependence structure to some extent varies with the level of u, which is not consistent with extreme value models. But this inconsistency is rather small, and the extreme value model is not rejected due to this deviation. Hence, the bivariate extreme value model is found to be appropriate for modeling the peak discharge and volume at Elverum. The sudden drop of $\overline{\chi}(u)$ at $u \approx 0.9$ is due to an error in the routine for the estimation of $\overline{\chi}(u)$. Unfortunately this error cannot be corrected.

Figure 4.11 shows the plots of the dependence measures peak discharge and volume at station Losna. Here $\overline{\chi} \approx 0.2$, which is consistent with asymptotic independence. Hence, bivariate extreme value models are not appropriate for modelling this event due to the fact that all bivariate extreme value models are asymptotically dependent. For modelling asymptotically independent events, a bivariate normal or bivariate lognormal distribution could be appropriate. The results of the pairs of flood characteristics that have not yet been investigated, the duration and volume at Elverum and the duration and volume at Losna, show that both pairs have dependence structures that resembles the dependence structure found in figure 4.11. Hence, all pairs of flood characteristics that have been analysed are asymptotically independent except for peak discharge and volume at station Elverum.

The exploratory analysis of extremal dependence has not been previously used in hydrological applications. It could therefore be interesting to model some the pairs of flood characteristics with both bivariate lognormal and bivariate extreme value models. In this way the results from the two different models can be compared to see if they model the extremes differently. Thus, both bivariate models are used to model all the pairs of flood characteristics, but the results will of course focus on the models that were found appropriate by the preliminary analysis.

### 4.4.2 Assessment of asymptotic dependence using parametric inference

In section 3.9 it was shown that the inference of the dependence measures $\chi$ and $\overline{\chi}$ goes through another dependence model called the coefficient of tail dependence. By estimating the coefficients in the coefficient of tail dependence model, the dependence measures $\chi$ and $\overline{\chi}$ can also be found. The inference in the coefficient of tail dependence model consist of performing an extreme value threshold analysis (PDS) on the structure variable defined in the model. This can be done because the shape parameter in the GP distribution is equal to the coefficient $\eta$ in the model. A limiting factor of this model is that estimates of $\eta$ will be biased toward asymptotic independence if the threshold is to low. This problem arise when there is little data available, which is often the case when modelling with AMS. If the threshold is chosen sufficiently high in a model with few data , the standard error of the parameters in the model increases, and the parameters become uncertain.

The results of the estimation of $\eta$ for the different flood characteristics are given in table 4.6. By setting the threshold probability to 0.05, none of the pairs of flood characteristics are asymptotically dependent. But due to the low threshold, bias in the results is expected. If the threshold probability is set to 0.5, the estimated value of $\eta$ for the peak discharge and volume at Elverum is 0.814 with a standard error of 0.344, and asymptotic dependence cannot be rejected. The other pairs of flood characteristics

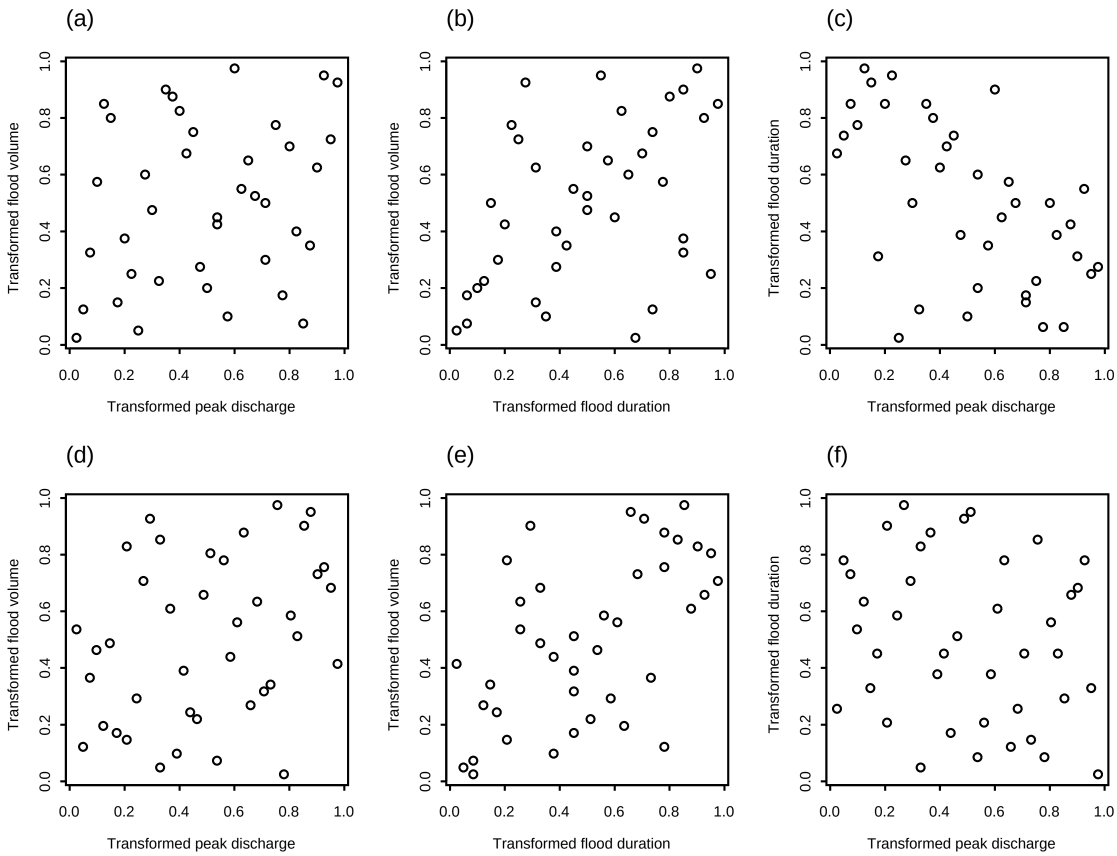


Figure 4.9: Plots of (a) and (d) Peak discharge and flood volume, (b) and (e) flood duration and flood volume, (c) and (f) peak discharge and flood duration. Figures (a)-(c) are from Elverum, figures (d)-(f) are from Losna. The marginal distributions of the flood characteristics have been transformed to uniform distributions before plotted.

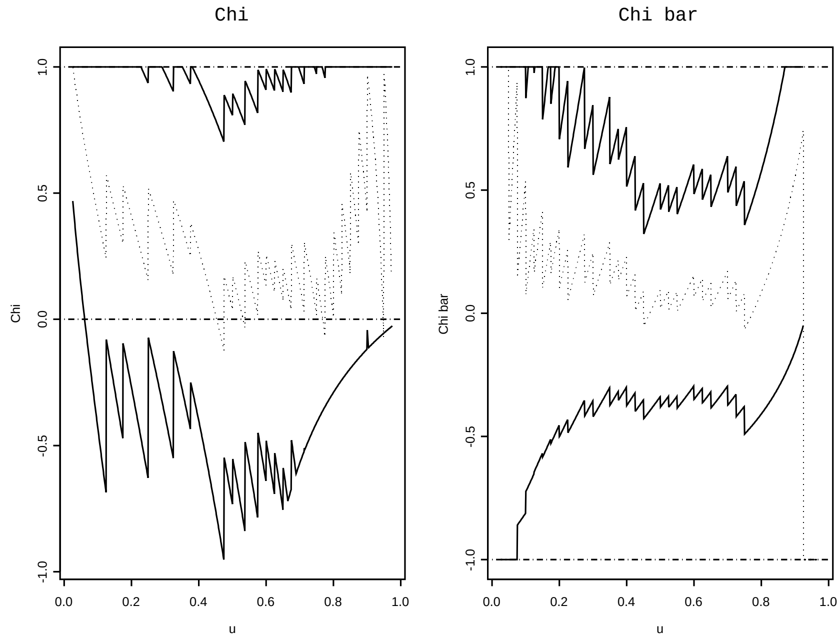


Figure 4.10: The dependent measures $\chi$ and $\overline{\chi}$ for the peak discharge and flood volume at Elverum with 95 percent confidence intervals.

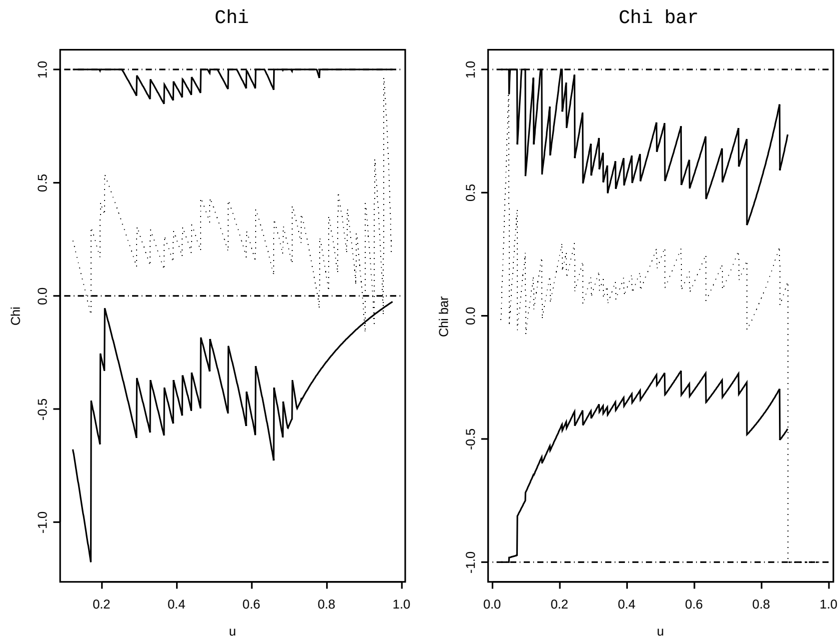


Figure 4.11: The dependent measures $\chi$ and $\overline{\chi}$ for the peak discharge and flood volume at Losna with 95 percent confidence intervals.

Thresholdprobability = 0.6

| | | $\hat{\eta}$ | Se |
|---|---|---|---|
| Elverum | Peak-discharge vs. volume | 1.145 | 0.563 |
| | Duration vs. volume | 0.737 | 0.503 |
| Losna | Peak-discharge vs. volume | 0.316 | 0.333 |
| | Duration vs. volume | -0.383 | 0.290 |

Thresholdprobability = 0.5

| | | $\hat{\eta}$ | Se |
|---|---|---|---|
| Elverum | Peak-discharge vs. volume | 0.814 | 0.344 |
| | Duration vs. volume | 0.512 | 0.335 |
| Losna | Peak-discharge vs. volume | 0.439 | 0.330 |
| | Duration vs. volume | -0.013 | 0.345 |

Thresholdprobability = 0.05

| | | $\hat{\eta}$ | Se |
|---|---|---|---|
| Elverum | Peak-discharge vs. volume | 0.742 | 0.200 |
| | Duration vs. volume | 0.752 | 0.204 |
| Losna | Peak-discharge vs. volume | 0.709 | 0.213 |
| | Duration vs. volume | 0.750 | 0.201 |

Table 4.6: Results of the estimation of the coefficient $\eta$ with standard error for different flood characteristics using different thresholdprobabilities.

have $\eta$ values less than one with the standard error included, and are still asymptotically independent. At threshold probability 0.6, no further information is obtained. By comparing the standard errors between the different threshold probabilities, there is a general increase in the values. This is expected since the amount of data decreases when the threshold is raised.

The conclusion of the evaluation of asymptotic dependence is as follows

- Only the peak-discharge and volume at Elverum is asymptotically dependent, and thus bivariate extreme values models is appropriate. Because the data are componentwise block maxima, and that the values in figure 4.9 are relative symmetric around the line u = v (u and v are the axes of the plots), a logistic bivariate extreme value models is assumed appropriate (see section 3.6.4).
- The rest of the pairs of flood characteristics are asymptotically independent, and thus the bivariate log normal distribution can be appropriate for modelling these events.

A good agreement between the results of asymptotic dependence obtained by the preliminary analysis and the results obtained by parametric inference indicates good results in the preliminary analysis. In this thesis there is a relatively good correspondence between the two different estimation methods. If a more thorough comparison is to be made, the dependence measure $\chi$ can be calculated from $\eta$ using the conversion equation found in section 3.9.1.

### 4.4.3 Modeling flood characteristics using bivariate extreme value models

The bivariate logistic model is often an appropriate model for compnentwise maxima, and is given in equation 3.18. It contains one dependence parameter that needs to be estimated. For estimation of parameters in bivariate distributions maximum likelihood is used (see section A.5.2). Due to the complexity of bivariate models, numerical methods need to be applied. These methods are found in Stephenson (2003) or Coles et al. (1999). The results of the maximum likelihood estimation is given in table 4.7.

The Peak-discharge and volume at Elverum has an estimated dependence parameter of $\hat{\alpha} = 0.788$, which is a relatively weak dependence. When the $\alpha$ parameter is known, it is possible to estimate the dependence measure $\chi(u)$ by the equation $\chi = 2 - 2^{\alpha}$ Coles et al. (1999). This value can then be compared with the value of $\chi$ estimated in the preliminary analysis, to see if the values correspond. In figure 4.10, the $\chi$ value estimated in the preliminary analysis is $\hat{\chi} \approx 0.25$, and the $\chi$ value estimated from parametric inference is $\hat{\chi} = 0.27$. Hence, there is an agreement between the two estimated $\chi$ values, and thus the estimated $\alpha$ value is consistent with the previous results.

The logistic model was also tried to be fitted to the asymptotically independent pairs of flood characteristics due to the comparason of the bivariate models, and here the agreement between the estimated $\chi$ values were more varying. All of the pairs have $\hat{\chi}$ values around 0.25 when estimated in the preliminary analysis, but from the parametric inference the estimated $\hat{\chi}$ values are 0.25, 0.40 and 0.44 for duration and volume at Elverum, peak discharge and volume at Losna, and duration and volume at Losna, respectively. The discrepancies are probably due to the fact that an asymptotically dependent model is used for modelling asymptotically dependent data.

An assumption in the logistic model is that the variables in the model must be exchangeable (section 3.6.4). In figure 4.9 the data appear relatively symmetric around the line (U = V) in the plot, thus this condition seems to be fulfilled. For a more thorough investigation of the exchangeability of the variables, the asymmetry parameters $t_1$ and $t_2$ in the assymetric logistic family (equation 3.20) can be estimated. If any of the asymmetric parameters are $\neq 1$, asymmetry is present and a model which allows asymmetry could be more appropriate. A possible limitation for these models is that they require a lot of data due to the number of parameters that need to be estimated. Thus, because the logistic distribution has less parameters, it can still be the most appropriate model, although models which allows asymmetry fit the observations better. For estimating the parameters in the asymmetric logistic model, a maximum likelihood routine found in Stephenson(2003) is used. The results are given in table 4.8, only the asymmetric and the dependence parameters are given. An evaluation of the results of the estimation clearly shows that there are not enough data available to use this model. The standard error for some of the asymmetric parameters is larger than the

Elverum

| | $\hat{\alpha}$ | Bootstrap $\hat{\alpha}$ | CI up | CI low | AIC |
|---|---|---|---|---|---|
| Peak-discharge vs. volume | 0.788 | 0.778 | 0.994 | 0.606 | 2293.20 |
| Duration vs. volume | 0.682 | 0.685 | 0.890 | 0.518 | 2052.59 |

Losna

| | $\hat{\alpha}$ | Bootstrap $\hat{\alpha}$ | CI up | CI low | AIC |
|---|---|---|---|---|---|
| Peak-discharge vs. volume | 0.810 | 0.798 | 1.000 | 0.705 | 2330.97 |
| Duration vs.volume | 0.645 | 0.639 | 0.799 | 0.540 | 2075.14 |

Table 4.7: Results of the maximun likelihood estimation for the dependence parameter $\alpha$ of the bivariate logistic family with 90 percent confidence intervals estimated by the BCa method and AIC scores

Elverum

| | $\hat{t_1}$ | $\hat{t_2}$ | $\hat{\alpha}$ | Std.err $\hat{t_1}$ | Std.err $\hat{t_2}$ | Std.err $\hat{\alpha}$ |
|---|---|---|---|---|---|---|
| Peak-discharge vs. volume | 0.788 | 0.994 | 0.606 | 2.0e-6 | 1.159 | 0.566 |
| Duration vs. volume | 0.178 | 0.487 | 0.100 | 0.065 | 0.202 | 2.0e-6 |

Losna

| | $\hat{t_1}$ | $\hat{t_2}$ | $\hat{\alpha}$ | Std.err $\hat{t_1}$ | Std.err $\hat{t_2}$ | Std.err $\hat{\alpha}$ |
|---|---|---|---|---|---|---|
| Peak-discharge vs. volume | 0.325 | 0.999 | 0.712 | 1.08 | 2.0e-6 | 0.482 |
| Duration vs.volume | 0.999 | 0.992 | 0.633 | 2.0e-6 | 0.488 | 0.127 |

Table 4.8: Results of the maximum likelihood estimation of the asymmetric parameters $t_1$, $t_2$ and the dependence parameter $\alpha$ of the asymmetric bivariate logistic family with standard errors.

domain of the asymmetric parameter, thus no conclusions can be drawn from the results of this model, and hence the model is rejected.

A third candidate model is the bilogistic model given in equation 3.21. This model has two dependence parameters, $\alpha$ and $\beta$, and if $\alpha = \beta$ the model is equivalent to the logisic model. The result of the estimation is given in table 4.9 together with the AIC scores. For this model the standard errors is considerably smaller than for the asymmetric logistic model, and the model cannot be rejected due to large standard errors. An evaluation of the results show that for the peak-discharge and volume at station Elverum, $\alpha \neq \beta$, indicating that asymmetry is present. But since this model has more parameters than the logistic model, the logistic model can still be the most appropriate model. Thus a method is needed to choose between the two candidate models. The AIC (section A.5.2) is an effective model selection tool. For each candidate model the AIC score is calculated, and the model with the highest score is the preferred model. A comparason of the AIC score for the bilogistic model with the logistic model for the peak discharge and volume at station Elverum, show that the logistic model has the higest score, and thus is the preferred model.

An evaluation of the parameters of the other pairs of flood characteristics show that also here the logistic model is probably the most appropriate. The duration and volume at Elverum have $\alpha$ and $\beta$ parameters that are close in

Elverum

| | $\hat{\alpha}$ | $\hat{\beta}$ | Std.err $\hat{\alpha}$ | Std.err $\hat{\beta}$ | AIC |
|---|---|---|---|---|---|
| Peak-discharge vs. volume | 0.101 | 0.913 | 2.0e-6 | 0.056 | 2291.95 |
| Duration vs. volume | 0.824 | 0.714 | NA | NA | 2054.29 |

Losna

| | $\hat{\alpha}$ | $\hat{\beta}$ | Std.err $\hat{\alpha}$ | Std.err $\hat{\beta}$ | AIC |
|---|---|---|---|---|---|
| Peak-discharge vs. volume | 0.966 | 0.101 | 0.024 | 2.0e-6 | 2331.00 |
| Duration vs.volume | 0.579 | 0.669 | 0.173 | 0.144 | 2077.03 |

Table 4.9: Results of the maximun likelihood estimation of the dependence parameters $\alpha$ and $\beta$ of the bilogistic family with standard errors and AIC scores.

value, indicating a logistic model. But it cannot be concluded due to the lack of standard errors. On the other hand the AIC score for the bilogistic model is the highest. For the peak-discharge and volume at Losna the AIC score is almost identical for both candidate models. The duration and volume at Losna also have parameters that are close in value. By including the standard errors, there is no evidence to state that $\alpha \neq \beta$, and the bilogistic model is equivalent to the logistic model. Hence, even though both models seem appropriate for some of the pairs, it is chosen to use the model with less parameters.

### 4.4.4 Modeling flood characteristics using bivariate lognormal distribution

The bivariate log normal distribution is difficult to work with compared to the bivariate extreme value models. As for the univaiate log normal distribution, analytical solutions for the CDF of the bivariate distribution are not available, and Monte Carlo simulation (section 3.7) is used for the estimation of the CDF. The result of the estimation of peak discharge and volume at Losna is found in figure 4.12. In the plot each CDF value is estimated by using 100 000 simulations. This give good results when the CDF values are plotted in figure 4.12, but when the CDF values are used to estimate bivariate return periods, some noise arise when the return periods are plotted. For reducing the notice, the number of simulations must be increased to 1000 000. But this many simulations are time demanding, and thus for most of the CDF estimations 100 000 simulations are used. In table 4.10 the correlation coefficients for the different pairs of flood characteristics are given. The correlation coefficient is used in the Monte Carlo simulation.

## 4.5 Model validation

In section 4.4.2 the analysis of the extremal dependence showed that only the flood characteristics peak-discharge and volume at station Elverum was asymptotically dependent. The rest of the pairs of flood characteristics were asymptotically independent. Among the possible bivariate extreme value candidate models for modelling the asymptotically dependent flood

Elverum

| | $\hat{\rho}$ | Bootstrap $\hat{\rho}$ | CI up | CI low |
|---|---|---|---|---|
| Peak-discharge vs. volume | 0.48 | 0.453 | 0.704 | 0.202 |
| Duration vs. volume | 0.35 | 0.358 | 0.585 | 0.088 |

Losna

| | $\hat{\rho}$ | Bootstrap $\hat{\rho}$ | CI up | CI low |
|---|---|---|---|---|
| Peak-discharge vs. volume | 0.23 | 0.240 | 0.461 | 0.033 |
| Duration vs.volume | 0.57 | 0.570 | 0.711 | 0.338 |

Table 4.10: Results of the estimation for the correlation coefficient $\rho$ for the bivariate log normal distribution with 90 percent confidence intervals estimated by the BCa method

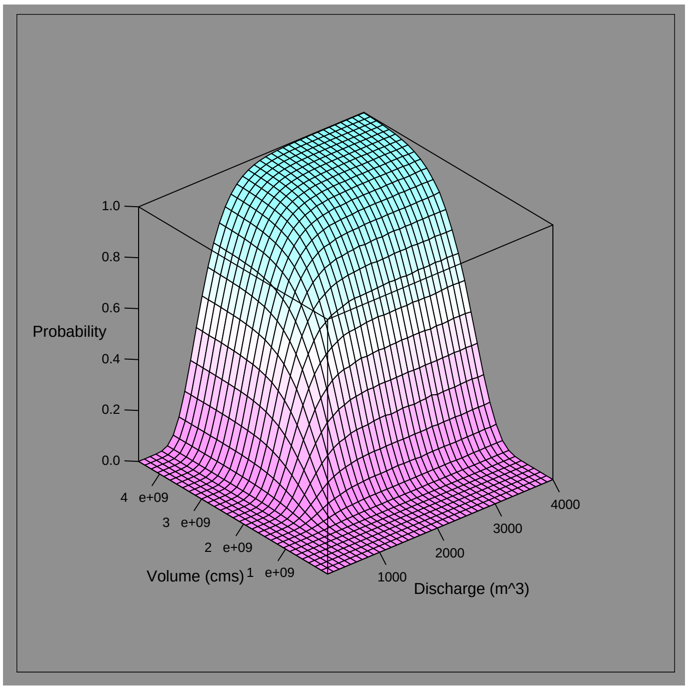


Figure 4.12: .The cumulative bivariate lognormal distribution function for the peak discharge and volume at Losna.

| Order nr. | Year | Peak-discharge (cms) | Volume ($m^3$) | Joint non-exceedance probabilities: Gringorten | Joint non-exceedance probabilities: Theoretical | $n_{ml}$ |
|---|---|---|---|---|---|---|
| 1 | 1996 | 553.8 | 9.06E+08 | 0.014 | 5.73E-06 | 1 |
| 2 | 1974 | 597.5 | 1.90E+09 | 0.040 | 0.002 | 2 |
| 3 | 1964 | 752.7 | 2.27E+09 | 0.065 | 0.027 | 3 |
| 4 | 1989 | 847.7 | 2.69E+09 | 0.091 | 0.083 | 4 |
| 5 | 1999 | 882.8 | 3.48E+09 | 0.117 | 0.134 | 5 |
| 6 | 1972 | 908.5 | 3.29E+09 | 0.117 | 0.150 | 5 |
| 7 | 1976 | 978.7 | 1.96E+09 | 0.040 | 0.062 | 2 |
| 8 | 1980 | 985.6 | 2.33E+09 | 0.117 | 0.122 | 5 |
| 9 | 1991 | 1022 | 2.14E+09 | 0.091 | 0.105 | 4 |
| 10 | 1970 | 1029 | 1.81E+09 | 0.040 | 0.050 | 2 |
| 11 | 1994 | 1065 | 2.77E+09 | 0.219 | 0.239 | 9 |
| 12 | 1961 | 1069 | 2.53E+09 | 0.193 | 0.202 | 8 |
| 13 | 1969 | 1084 | 2.12E+09 | 0.117 | 0.121 | 5 |
| 14 | 1965 | 1129 | 3.85E+09 | 0.347 | 0.383 | 14 |
| 15 | 1968 | 1136 | 3.66E+09 | 0.347 | 0.383 | 14 |
| 16 | 1997 | 1137 | 3.30E+09 | 0.321 | 0.361 | 13 |
| 17 | 1962 | 1166 | 2.98E+09 | 0.296 | 0.349 | 12 |
| 18 | 1998 | 1174 | 3.21E+09 | 0.321 | 0.385 | 13 |
| 19 | 1984 | 1189 | 2.15E+09 | 0.168 | 0.161 | 7 |
| 20 | 1992 | 1229 | 2.04E+09 | 0.117 | 0.138 | 5 |
| 21 | 1971 | 1245 | 2.46E+09 | 0.270 | 0.277 | 11 |
| 22 | 1981 | 1245 | 2.46E+09 | 0.296 | 0.277 | 12 |
| 23 | 1990 | 1286 | 1.85E+09 | 0.065 | 0.091 | 3 |
| 24 | 1987 | 1357 | 4.47E+09 | 0.602 | 0.609 | 24 |
| 25 | 2000 | 1357 | 2.59E+09 | 0.372 | 0.365 | 15 |
| 26 | 1977 | 1375 | 2.94E+09 | 0.449 | 0.478 | 18 |
| 27 | 1982 | 1450 | 2.56E+09 | 0.372 | 0.385 | 15 |
| 28 | 1986 | 1456 | 2.22E+09 | 0.244 | 0.245 | 10 |
| 29 | 1993 | 1456 | 2.56E+09 | 0.398 | 0.386 | 16 |
| 30 | 1988 | 1488 | 3.23E+09 | 0.602 | 0.599 | 24 |
| 31 | 1979 | 1545 | 2.04E+09 | 0.142 | 0.179 | 6 |
| 32 | 1983 | 1564 | 3.12E+09 | 0.602 | 0.609 | 24 |
| 33 | 1975 | 1577 | 2.41E+09 | 0.347 | 0.351 | 14 |
| 34 | 1978 | 1622 | 1.83E+09 | 0.065 | 0.105 | 3 |
| 35 | 1963 | 1768 | 2.30E+09 | 0.347 | 0.320 | 14 |
| 36 | 1985 | 1788 | 2.77E+09 | 0.602 | 0.547 | 24 |
| 37 | 1967 | 2533 | 4.45E+09 | 0.935 | 0.947 | 37 |
| 38 | 1966 | 2600 | 3.14E+09 | 0.730 | 0.742 | 29 |
| 39 | 1995 | 3081 | 4.25E+09 | 0.935 | 0.950 | 37 |

Figure 4.13: Empirical joint non-exceedance probabilities obtained from the Gringorten plotting position and theoretical joint non-exceedance probabilities obtained from the bivariate logistic extreme value distribution. The figure also includes the number of occurrences ($n_{ml}$) of the corresponding combination (order number) of peak-discharge and volume. All values are valid for station Elverum.

characteristics, the bivariate logistic extreme value distribution was in section 4.4.3 found to be the best model among the candidate models by using AIC. For the asymptotically independent flood characteristics, the bivariate log normal distribution was selected before the bivariate normal distribution since the bivariate normal distribution rarely is used to model extreme events. For validation of both for the asymptotically dependent and independent flood characteristics, the graphical method was used (section 3.10). The estimations from the bivariate Gringorten plotting position for the peak discharge and volume at Elverum are given in table 4.13 together with the theoretical values obtained from the bivariate logistic extreme value distribution, and in figure 4.14 the joint non-exceedance probabilities and the theoretical values are plotted. There is generally good agreement between the estimated joint non-exceedance probabilities and the theoretical distribution, especially for the most extreme values. Hence, it can be concluded that the bivariate logistic extreme value distribution is an appropriate model for the peak discharge and volume at Elverum.

In figure 4.16 the joint non-exceedence probabilities estimated with the bivariate Weibull plotting position for the duration and volume at Elverum is plotted together with theoretical values obtained from the bivariate log normal distribution. Also here there is good agreement between the joint non-exceedance probabilities and the theoretical values. For the peak discharge and volume, and the duration and volume at Losna, there were good agreement between the joint non-exceedance probabilities and the theoretical values. Hence, it can be concluded that the bivariate log normal distribution is an appropriate model for all the asymptotically independent pairs of flood characteristics.

### 4.5.1 A comparison of the selected bivariate models

In section 4.4.1 it was decided to model the flood characteristics by using both the bivariate logistic extreme value distribution and the bivariate log normal distribution. In this way a comparison between the two different models can be obtained, and it can be investigated if the models model the extremes differently. The estimated joint non-exceedance probabilities and theoretical values from the bivariate log normal distribution is given for the peak discharge and volume at Elverum are found in figure 4.15, and in figure 4.17 the estimated joint non-exceedance probabilities and theoretical values from the bivariate log normal distribution is given for the duration and volume at Elverum. By comparing these plots with the plots given in figures 4.14 and 4.16, a comparison between the two models is obtained. An evaluation of the plots shows that there are surprisingly small differences between the two models when the same event is modelled. But for the extremes, the bivariate logistic extreme value model is more appropriate for the peak discharge at Elverum, and the logistic distribution is more appropriate for the duration and volume at Elverum, as expected. The differences between the models will be more apparent if the distributions are extrapolated.

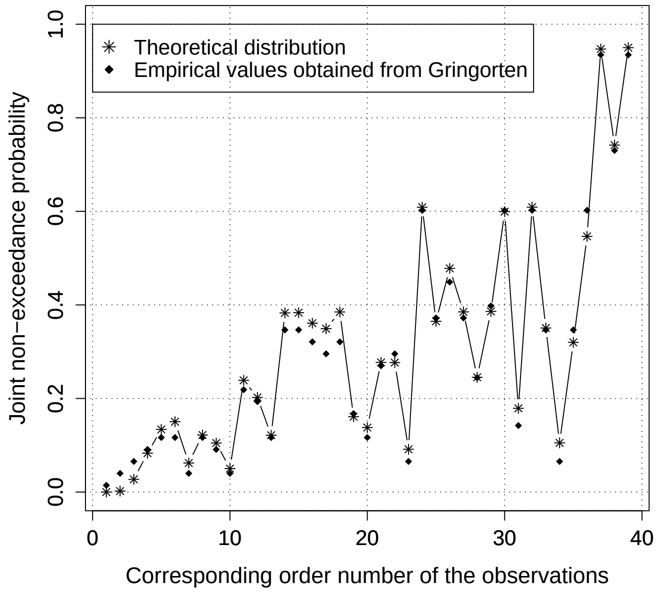


Figure 4.14: Comparison of estimated and theoretical non-exceedence probability of peak-discharge and volume at Elverum. The Theoretical joint non-exceedance probabilities is calculated from the bivariate logistic extreme value distribution.

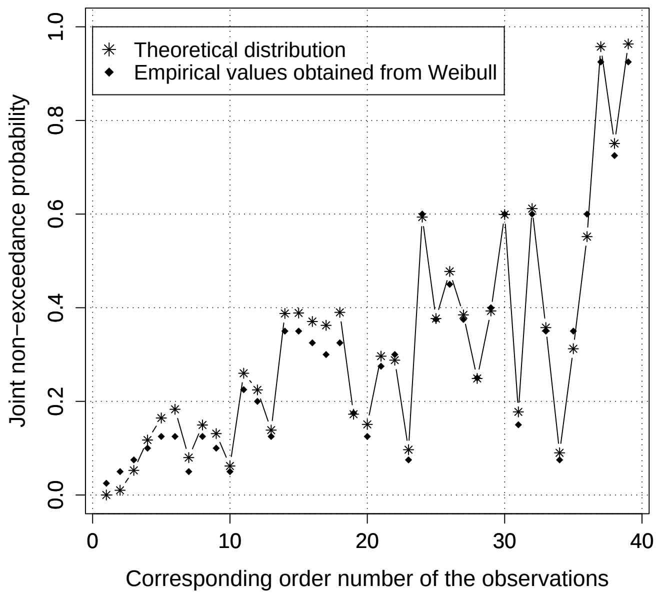


Figure 4.15: Comparison of estimated and theoretical non-exceedence probability of peak-discharge and volume at Elverum. The Theoretical joint non-exceedance probabilities is calculated from the bivariate log normal distribution.

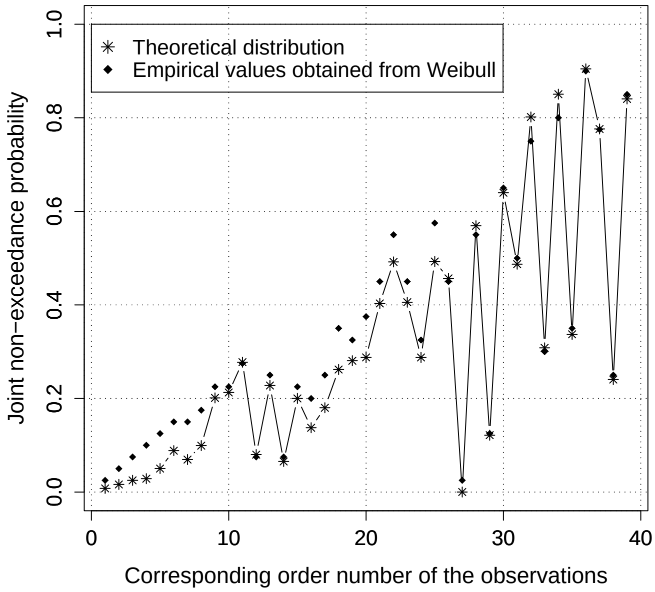


Figure 4.16: Comparison of estimated and theoretical non-exceedence probability of duration and volume at Elverum. The Theoretical joint non-exceedance probabilities is calculated from the bivariate log normal distribution.

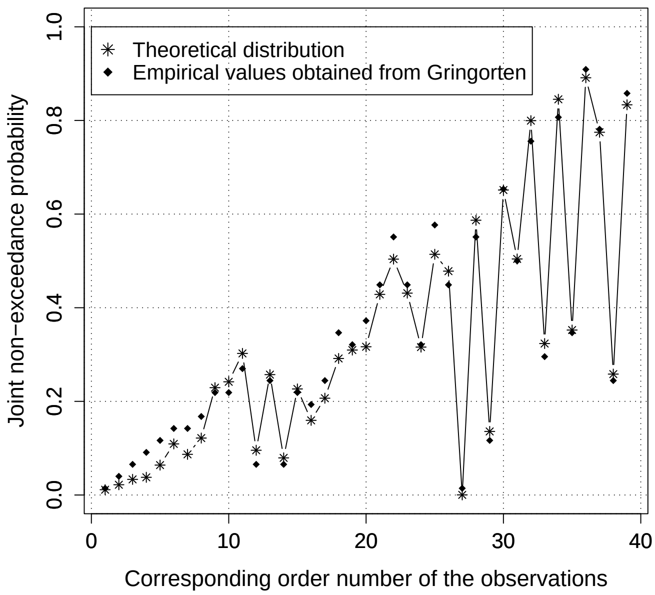


Figure 4.17: Comparison of estimated and theoretical non-exceedence probability of duration and volume at Elverum. The Theoretical joint non-exceedance probabilities is calculated from the bivariate logistic extreme value distribution.

### 4.5.2 Bivariate return periods constructed from the bivariate logistic distribution

When constructing bivariate return periods the design criterion is determinant for the type of return period to use. There is a large difference in value between different types of bivariate return periods for the same event, and choosing the wrong type might have dramatic consequences. In this thesis the bivariate flood frequency analysis is not made for a specific event or design criterion. Instead different bivariate return periods are estimated so that the results can be compared. For this reason all of the bivariate return periods given in this section are estimated for the peak-discharge and volume at station Elverum.

In figures 4.18 and 4.19 the joint return periods are given. Insted of a three-dimensional plot, the joint return periods are plotted as contour lines. By using this type of plot it is easier to both extract information from the plot and to compare the different return periods. The contour lines in the plots represent the different combinations of values of the variables that lead to the associated joint retutn period. A comparison of the two sub classes of the joint return periods shows that the difference between the subclasses is apparent. For the second subclass (Figure 4.19) the contour lines are restricted by the axes in the plot, whereas for the first subclass (figure 4.18) the contour lines are not restricted. Another aspect is that for the same event, the joint return periods are different for the two subclasses, e.g. for the event x = 2000 cms and y = 4.0e9 $m^3$, the joint return periods found in figures 4.18 and 4.19 are approximately 8 years and 40 years, respectively.

In figure 4.20 a plot of the conditional exceedance probability of peak discharge given volume with the design criterion X>x given Y>y is found (equation 3.46). The volumes that are used as conditional values in the plot correspond to the univariate exceedance probabilities for the volumes. This is appropriate for hydrological design because constructional requirements often are given as exceedance probabilities of the variable or variables of interest. In the plot the marginal exceedance probability is also given for comparison. The conditional exceedance probability plot can be used to find one of the three variables (peak discharge, condition exceedance probability and exceedance probability of volume) if the two other variables are given, e.g. if the conditional excedance probability is set to 0.01 and the exeedance probability of the volume is 0.2, the peak-discharge is found to be approximately 4000 cms. A univariate frequency analysis of the same event shows that the peak-discharge is approximately 3100 cms for an exceedance probability of 0.01. Hence, there is a large difference between the conditional and the marginal distribution. An evaluation of the plot shows that for a constant conditional exceedance probability the peak-disharge increases for decreasing exceedance probability of volume, which is expected.

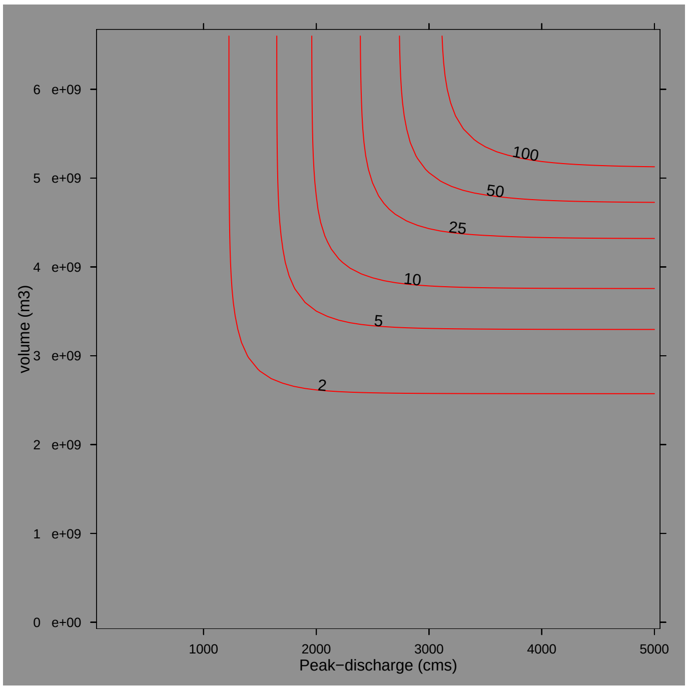


Figure 4.18: Joint return periods for peak-discharge and volume at station Elverum for the event X>x or Y>y constructed from the bivariate logistic distribution

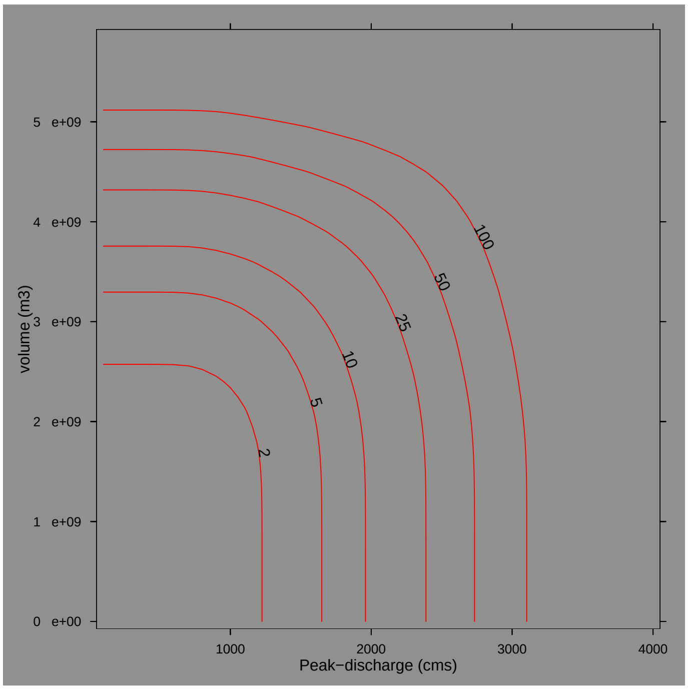


Figure 4.19: Joint return periods for peak-discharge and volume at station Elverum for the event X>x and Y>y constructed from the bivariate logistic distribution.

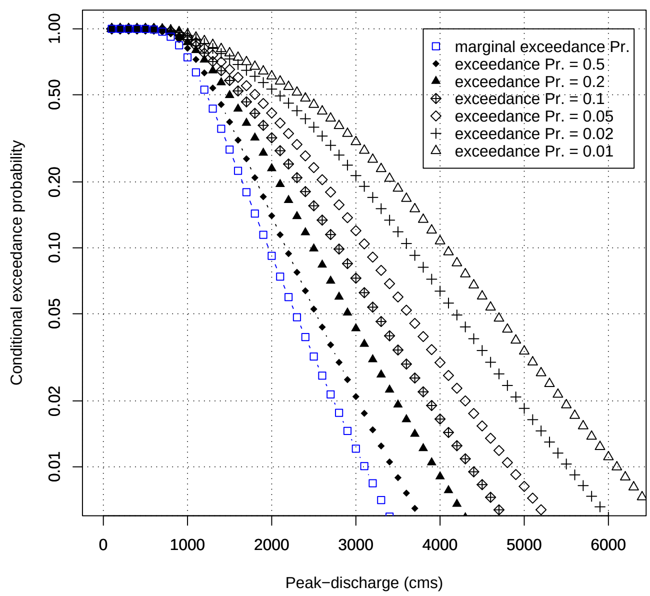


Figure 4.20: Conditional exceedance probabilities for peak discharge given volume at Elverum for the event X>x given Y>y. The marginal exceedance probability for the peak discharge is also included.

### 4.5.3 Bivariate return periods constructed from bivariate log normal distribution

The bivariate return periods constructed from the bivariate log normal distribution are found in figures 4.21 and 4.22. Both of the bivariate return periods are constructed using a Monte Carlo simulation with 100 000 simulations for each value in the plots. In figure 4.22 the values that are larger than the 100 years return period are noise. This noice occur because not enough simulations have been used in the Monte Carlo simulation. By increasing the number of simulations to 1 000 000, the noice is removed, but this is very time-consuming. The bivariate return periods for the flood characteristics at Losna are given in the appendix.

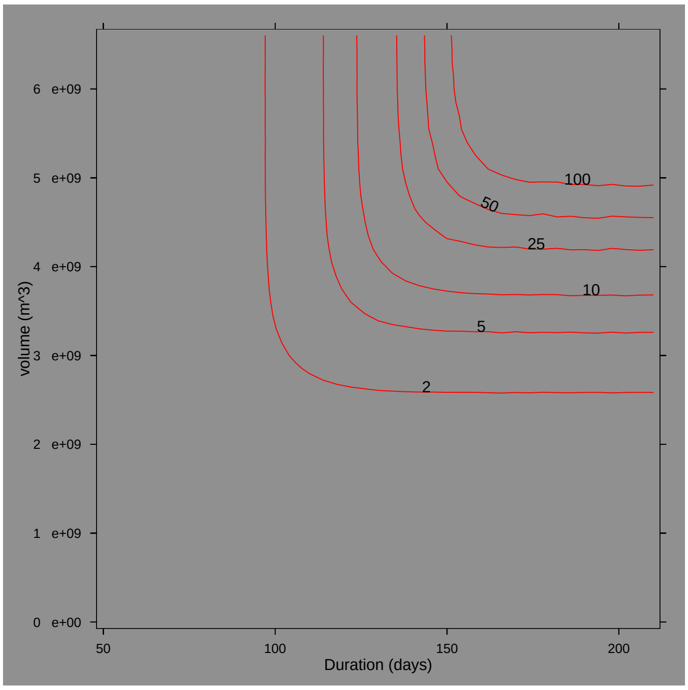


Figure 4.21: Joint return periods for the duration and volume at station Elverum for the event X>x or Y>y, or X>x and Y>y constructed from the bivariate log normal distribution.

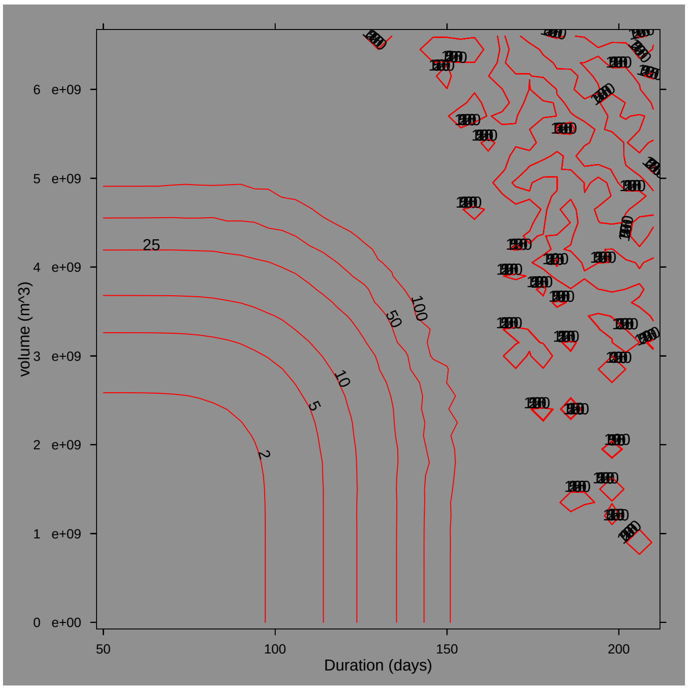


Figure 4.22: Joint return periods for the duration and volume at station Elverum for the event X>x and Y>y constructed from the bivariate log normal distribution.

# Chapter 5

# Conclusion

In this thesis the main objective was to evaluate the applicability of bivariate frequency analysis on spring floods. For modelling spring floods bivariate extreme value methods were applied on the flood characteristics that characterises a spring flood, the peak discharge, duration and volume. In this way the spring flood was modelled by using different pairs of flood characteristics in bivariate models, and joint distributions of the pairs peak discharge and volume, and duration and volume were obtained. The last pair, the peak discharge and duration was not modelled due to a negative association between the flood characteristics, and bivariate extreme value models for negatively associated values are outside the scope of this thesis.

The analysis were undertaken in the Glomma basin for the rivers Glomma and Gudbransdalslågen at the stations Elverum and Losna, respectively. An analysis of the exstremal dependence on the different pairs of flood characteristics showed that only one pair, the peak discharge and volume at Elverum was asymptotically dependent. The other pairs of flood characteristics, the duration and volume at Elverum and Losna, and the peak discharge and volume at Losna were asymptotically independent. For the asymptotic dependent pair, the bivariate logistic extreme value distribution with GEV-distributed marginals was found appropriate. In the case of asymptotic independence, the bivariate log normal distribution with lognormal margins was found appropriate. The model validation showed that for both models there were good agreements between the theoretical bivariate distributions and the observations.

From the obtained bivariate joint distributions different bivariate return periods were constructed and plotted, and it was emphasised that the event or the design criterion is determinant for which of the joint return period to apply in order to obtain correct results. These bivariate return periods can for example be applied for hydrological design and management. Hence, a bivariate frequency analysis provides information about the spring flood that is not obtainable by a univariate analysis of the flood characteristics.

A new method to assess the duration from a hydrograph is found, and when the method was applied on the hydrograph at Elverum the results seemed reasonable. On the other hand the complexity of the hydrograph at Losna lead to difficulties in applying the new method, and an approximative method was used. This approximaive method only assessed the duration of a sub basin of Losna, and not the entire basin. Thus, the duration values cannot be used for hydrological design. The results of the assessment of duration lead to a bimodal distribution of the results, indicating that some of the flood durations of the sub basin of Losna were overestimated.

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# Appendix A

# Univariate theory

## A.1 Streamflow recession

*"A drainage basin may be considered as a dynamic system which transforms a specified climatic input into an output function of stream outflow"* (Gottschalk et al., 1997). The outflow is a combination of surface, unsaturated subsurface and groundwater flow, and is is therefore quite complex to model. One possibility is to use the black box approach. By assuming that the contributions from the different types of outflow originate from each storage, the outflow can be modelled as a compound system of several reservoirs. If there is no input to the system and the output is assumed linear, the outflow can be given as

$$q_t = q_0 e^{-t/k}, \; -\infty < t < \infty \tag{A.1}$$

where $q_0$ is the outflow at time $t = 0$ and k is a time constant.
The constant k can be found by performing a recession analysis. If suitable recession segments from the discharge series analysed are identified, equation A.1 can be fitted to each segment using the least square method, in order to find the recession constant k for each segment. An estimate of k is then the average of the set of recession constants. With this information a master recession curve can be drawn.

## A.2 Modelling extreme values

### A.2.1 The extreme value distributions

One approach for modelling the tail of a distribution is by the use of sample maxima. If $X_1, ...X_n$ is a sequence of iid random variables with distribution function F, and $M_n = max(X_1, ..., X_n)$, the exact distribution of $M_n$ can then be written

$$\begin{aligned} Pr(M_n \leq x) &= Pr(X_1 \leq x, ..., X_n \leq x) \\ &= Pr(X_1 \leq x)...(X_n \leq x) \\ &= \{F(x)\}^n, \end{aligned} \tag{A.2}$$

But since F usually is unknown, the distribution of $M_n$ must be approximated. This can be done using an asymptotic argument. If the distribution of $M_n$ is investigated as n $\rightarrow \infty$, it converges in probability to $sup\{x : F(x) < 1\}$, the upper endpoint of F. This means that the distribution collapses into a point and is degenerate. The same problem is seen in the central limit theorem. It is overcome by introducing a linear renormalization of $M_n$

$$Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) \rightarrow G(x) \tag{A.3}$$

This is called the extremal types theorem, and is based on the theorem of Fisher and Tippet (1928). If $a_n$ and $b_n$ exist, they are sequences of renormalization constants and G is one of the following non degenerate distribution families

$$\text{GUMBEL: } G(x) = \quad exp\{-exp[-(\tfrac{x-b}{a})]\},\ -\infty \leq x \leq \infty \tag{A.4}$$

$$\text{FRÈCHET: } G(x) = \begin{cases} 0 & ,x \leq 0 \\ exp\{-(\frac{x-b}{a})^{\alpha}\} & ,x > b,\ \alpha > 0 \end{cases}$$

$$\text{WEIBULL: } G(x) = \begin{cases} exp\{-[-(\frac{x-b}{a})]^{\alpha} & ,x < b \\ 1 & ,x \geq 0 \end{cases}$$

These distributions are termed the extreme value distributions. For more information about this topic and a proof for the extreme value distributions, see Coles (2001).

### A.2.2 The generalized extreme value distribution (GEV)

When modelling extreme values, it is necessary to choose one of the extreme value distributions before parameter estimation can be accomplished. Alternatively all three distributions can be fitted to data, and the most appropriate distribution can be chosen. Still, a method for choosing a distribution is required. If we instead adapt a parameterisation that is combining the extreme value distributions into one single family, it is not necessary to choose a distribution. The data themselves will decide the distribution, and the inference is simplified.
The GEV distribution was independently derived by von Mises (1936) and Jenkinson (1955), and is given as

$$G(x) = exp\{-[1 + \xi(\frac{x-\mu}{\sigma})]^{-1/\xi}\}, \tag{A.5}$$

where -$\infty \leq \xi \leq \infty$, -$\infty \leq u \leq \infty$ and $\alpha > 0$. The parameter $\xi$ determines the distribution. If $\xi > 0$ it is the frèchet distribution, $\xi < 0$ the Weibull distribution and if $\xi \rightarrow 0$ the Gumbel distribution. The parameters u and $\alpha$ are respectively the location and the scale parameters.

### A.2.3 Return level, return period and reduced plot

The return level $x_p$ can be found by finding the quantile function for the GEV distribution. If the GEV distribution is inverted, the quantile function is given as

$$x_p = \begin{cases} \mu + \frac{\alpha}{\xi}[1 - (-ln(F))]^{\xi}, & \text{for } \xi \neq 0, \\ \mu - \alpha\, ln[-ln(F)], & \text{for } \xi = 0. \end{cases} \tag{A.6}$$

The return period is defined as the expected time before a certain return(quantile)-level is exceeded. It is derived from the geometric distribution, and the expected value for this distribution is given as

$$E(Y) = \frac{1}{p}, \tag{A.7}$$

where p is the probability that an event occurs, and y is the number of trials before success. If the event is defined as exceeding a discharge level x, and the distribution of x is given as $G(x) = Pr(X \leq x)$, the probability of exceeding x is $Pr(X \geq x) = 1 - G(x)$. The expected value can then be written

$$E(y) = \frac{1}{p} = \frac{1}{1 - F(x)} = T(x), \tag{A.8}$$

which in fact is the return period. If X correspond to componentwise block maxima with a block size of one year, and the GEV distribution is used for modelling, T(x) has the resolution years.

One way to graphically represent the return level or return period is by the use of a reduced plot. Here an appropriate plotting position is used for the estimation of non-exceedance probability of an empirical distribution. A plotting possition with a sound theoretical background, and which is appropriate for the GEV distribution is the Gringorten formula

$$p_{(r)} = F(x) = \frac{r - 0,44}{n + 0,12}, \tag{A.9}$$

where n is the number and r is the order of the observations. These plots are called reduced Gumbel plot. For more information about about this topic, see Gottschalk and Krasovskaia (2001). The reduced Gumbel plot is now made by plotting $E[X_{(r)}] = -ln(-ln(p_{(r)}))$ against $x_{(r)}$ (data). It will be a straight diagonal line if the data is Gumbel distributed, a concave curve if Weibull distributed and a convex curve if Frèchet distributed. Hence the Gumbel plot represents both a model presentation and a model validation.

In this thesis the lognormal distribution is also being used to model AMS values. An appropriate plotting possition for this distribution is the Weibull plotting position, given as

$$E[p_{(r)}] = \frac{r}{n + 1}, \tag{A.10}$$

also here r is the order and n is the number of observations. The reduced plot is made by plotting $E[X_{(r)}] = \Phi^{-1}(p_{(r)})$ against $x_{(r)}$ (data), where $\Phi^{-1}$ is the quantile function of the standard normal distribution.

There is possible to obtain even more information of the reduced plot if the return period is included. The connection between $E[X_{(r)}]$ and the return period (T) for the reduced Gumbel plot is given as

$$-ln(-ln(p_{(r)})) = log(T) - \frac{1}{2T}. \tag{A.11}$$

For the reduced plot with log normal distribution the connection between $E[X_{(r)}]$ and the return period (T) is given as

$$\Phi^{-1}(p_{(r)}) = \Phi^{-1}(1 - 1/T), \tag{A.12}$$

where T is the return period.

## A.3 The log normal distribution

The parameters in the log normal distribution can be estimated by using the method of moments. By a rearrangement of the moments of the distribution, the parameters are given by

$$\sigma_Y = \left[\log\left(1 + \frac{\sigma_X^2}{\mu_X^2}\right)\right]^{\frac{1}{2}} \tag{A.13}$$

$$\mu_Y = \log(\mu_X) - \tfrac{1}{2}\sigma_Y^2.$$

(Yue, 2000).

Since the log normal distribution is derived from the normal distribution, there exist no analytical form of the CDF. Calculation of the CDF can be done by a transformation of the lognormal values to normal values, and then estimate the CDF using the normal distribution

$$F(x) = Pr(X \leq x) = Pr(Y \leq log(x) = \Phi\left[\frac{log(x) - \mu_Y}{\sigma_Y}\right], \tag{A.14}$$

where $X_1 > 0$ and $\Phi$ is the CDF of the standard normal distribution. The CDF of the normal distribution is implemented in almost any statistical programme, or can be found in statistical tables.

## A.4 Preliminary analysis

EDA or visual analysis consists of making different graphs to explore and understand data. It is easier to identify and interpret patterns in a graph than from tables of different statistics. In this way characteristics in data that are not consistent with a model can be revealed in an early stage of the analysis. Other aspects possible to reveal with EDA are e.g. temporal variation (trend, step change), seasonal variation, independence, persistence and data error (Robson, 2000).

The statistical tests are a separate part of the preliminary analysis that investigate the same aspects as the EDA, but in a formal way. If the EDA has revealed some possible aspects, the statistical analysis is easier and can be used to test if the revealed aspects are significant. On the other hand, if the EDA fails to reveal an aspect, it is likely to be found in the statistical tests anyhow. This lead to the fact that EDA is not necessary in the preliminary analysis, but it is a helpful tool.

### A.4.1 Stationary conditions

In extreme value models there is an assumption that the underlying process of the extreme values consist of a sequence of independent random variables, (see section A.2.1). When dealing with hydrological processes this assumption is almost never fulfilled, e.g. in river runoff knowledge of the discharge one day influence what the discharge will be the next. This is also the case with extreme values. By following the same example, the discharge level in a flood is often extreme during several consecutive observations. To overcome this problem, the assumptions of the extreme value model is generalised. Instead of independence, stationarity of the underlying process is chosen. Stationarity is a condition where the variables of the process investigated have the same distribution, and that the distributions remain the same in time. So, for example if $X_1$ and $X_8$ are variables in a stationary series, then $X_1$ and $X_8$ are bound to have the same distribution.

Dependence between the variables is a more realistic assumption than independence and will better reflect the stochastic properties of the process, but most important, many of the stationary series satisfy a condition that will limit the dependence for extremes, given that the extremes actually are extreme and that they have a large separation in time. This limitation of dependence is sufficient for not affecting the extremal types theorem given in section A.2.1. Hence maxima of stationary series and independent series follow the same limit laws. A thorough investigation of this topic is given by Leadbetter et al (1983).

### A.4.2 Non-stationary conditions

Even though stationarity is a more general assumption than independence, there are processes that do not satisfy this assumption. For different reasons, some processes tend to vary in time. In a hydrological setting river runoff is often a process investigated. If there are non-stationary conditions in a time series of river runoff, usually changes at the measuring site or in the catchment area has taken place. The changes can be abrupt (step change), gradually variating trends or more complex changes. Some changes are

- Error in the hydrograph. The error is often due to erodation in the profile related to large floods or (slow) mass deposits.
- Hydropower regulation and/or human interference in the river. A regulation often leads to a redistribution of the water during a year. If there have been physical changes in the catcment like urbanizing, drainage of bogs and wet land, deforesting and building of dikes, the discharge distribution could change.

- Long term changes in the climatic conditions.

(Roald, 1999)

If a change is discovered in a time series, there are different methods available to remove the effect of the change in order to regain stationarity. These methods depend on the type of change, e.g. if there is a trend in a time series, the trend can be identified and removed from the the time series. In the presence of a step change, naturalised series can be constructed and used instead of the original time series. A thorough investigation of the topic ˝change in hydrological data˝ can be found in Kundzewicz and Robson (2000).

It is also possible in some situations to model non-stationary time series. If a change in a time series is discovered, the affected parameter in the appropriate distribution can be modelled as time dependent , e.g. if $Z_t$ is the value in year t in a given AM series, the GEV distribution with parameters GEV$(u, \alpha, \xi)$ can be modelled by

$$Z_t \sim GEV\left(u(t), \alpha, \xi\right). \tag{A.15}$$

In this example the location parameter is time dependent and must be modelled in accordance to the type of change. If a linear change is assumed, the location parameter can be modelled by

$$u(t) = \gamma_0 + \gamma_1 t. \tag{A.16}$$

Here $\gamma_0$ is the start value for the location parameter and $\gamma_1$ is the annual change. Other types of change is also possible e.g. a step change can be modelled as

$$u(t) = \begin{cases} u_1 & \text{for } t \leq t_0, \\ u_2 & \text{for } t > t_0, \end{cases} \tag{A.17}$$

where $t_0$ is the point where the change occurs, and $u_1$ and $u_2$ are the respective location parameters before and after the change.

The scale parameter $\alpha$ can be modelled in the same way as the location parameter, but the shape parameter $\xi$ is difficult to model as a smooth function of time or with an abrupt change. Instead a model with different parameters in each season can be adapted. This type of modelling is usually performed using the PDS approach.

### A.4.3 Trendtests

One of the most common trendtest is the linear regression, where a regression line is fitted to data. In the analysis data is the response variable and the order of the data with respect to time is the explanatory variable. The regression gradient is a measure of an possible trend, and a t-test is then used to decide if the trend is significant. Care should be taken when using linear regression. The test assumes that the data is normally distributed, or at least that the residuals are. If these condition are violated, an increased uncertainty in the parameters is expected.

An other test is to use the regression gradient in combination with bootstrapping Robson (2000). The idea is that if there is no trend in the data, the order of the data is not relevant. Hence the regression gradient calculated from the resampeled data should not differ much from the original test statistic.

### A.4.4 Independency tests

Two tests for independency are used in this theses. The first test is the autocorrelation coefficient based on the autocorrelation function (acf). Acf measures the linear correlation of the process itself, with the process being displayed in time. The displacement in time is called lag and often has the same unit as the resolution of the time series. In order to use this test, the time series has to be recorded with constant time spacing.
Formally the autocorrelation is the autocovariance standardized by the variance, and is given as

$$r_\tau = \frac{cov_\tau}{varY} = \frac{[\sum_{t=1+\tau}^{n} Y_t Y_{t-\tau} - (n-\tau)\overline{Y}_t \overline{Y}_{t-\tau}]/(n-\tau-1)}{[\sum_{=1}^{n}(Y_t - \overline{Y})^2]/(n-1)} \tag{A.18}$$

In the equation Y is the time series. Each position in the time series is recognised as $Y_t$, and $Y_{t+\tau}$ is a lag with length $\tau$. The autocorrelation coefficient $r_\tau$ is a number between -1 and 1, where 1 indicates that the time series are exactly identical and -1 exactly opposite. If autocorrelation is present in the time series, data is not independent.
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Runtest is the second independency test. The test examine whether the values in the time series is above or under the a given treshold, e.g. the median. Values above the median are assigned a plus sign, values under a minus. This gives a sequence of plus and minus signs. By defining a run as a sequence of equal signs, it is possible to count the number of runs in the sequence. If the time series is independent, the number of runs is normal distributed. The parameters in the distribution can then be estimated using

$$\mu = \frac{2n_1 n_2}{n_1 + n_2} + 1 \tag{A.19}$$

$$\sigma^2 = \frac{2n_1 n_2(2n_1 n_2 - n_1 - n_2}{(n_1 + n_2)^2(n_1 + n_2 - 1)}, \tag{A.20}$$

where $n_1$ is the number of plus signs and $n_2$ is the number of minus signs. By a simple hypotheses testing it is easy to determine if the time series is independent and to calculate approximate CI-intervals.
The theory of this section is based on Davis (2002)

## A.5 Inference

### A.5.1 Ordered statistic and l-moments

One way of defining a distribution function is by its moments. The moments are functions of the respective function parameters, thus by estimating the moments of a distribution, its parameters can be found. This is the well-known method termed the method of moments. Unfortunately sample estimation of ordinary moments is subjected to bias in small samples due to the fact that estimation of moments $> 1$ involves squaring or cubing of the sample data. In this way large values (outliers) are given too much weight. This was one of the reasons for J.R.M. Hosking to develop l-moments in the middle of the 1980th. L-moments are expectations of linear combinations of ordered statistic, and thus do not involve squaring or cubing of sample data. Hence the bias is considerably reduced.

Order statistic is concerned with statistic and statistical methods that depend on the order of the observations and has applications such as empirical estimation of distribution functions, empirical estimation of probability density functions and graphical representations of empirical distributions (plotting positions). The deduction of the order statistic methods start by assuming a sample of n observations of a stochastic variable X. These observations are arranged in increasing order so that $x_{(1)} \leq x_{(2)} \leq \; ... \; x_{(n)}$. The empirical distribution can now be found by using the well known empirical distribution function given as

$$F_{(r)}(x) = r/n, \text{ where r=0,1,2,...,n }, \tag{A.21}$$

where r is the range number. A limitation with this distribution is that it is impossible for values greater that the largest observed value or lower than the lowest observed value to exist. A more general distribution function is the distribution function of the r-th order statistic of totally n, which in a discrete form is given as

$$F_{(r)}(x) = \sum_{i=1}^{n} \binom{n}{i} F^i(x)[1 - F(x)]^{n-i}. \tag{A.22}$$

The distribution is derived from the binomial distribution where any sequence of i successes occurs with probability $F^i(x)[1 - F(x)]^{n-i}$, and $\binom{n}{i}$ gives the total number of sequences. For the r-th order statistic (as above) $i = r$.

By using the Beta distribution, the previous distribution also exist in a continuous form given by

$$F_{(r)}(x) = \frac{1}{B(r, n - r + 1)} \int_0^{p=F(x)} F^{r-1}(1 - F)^{n-r} dF. \tag{A.23}$$

By rewriting the Beta distribution as $n!/(r - 1)!(n - r)!$ and derive equation A.23, the density function is obtained, and is given as

$$f_{(r)}(x) = \frac{n!}{(r - 1)!(n - r)!} F(x)^{r-1}[1 - F(x)]^{n-r} f(x). \tag{A.24}$$

It is also possible to define moments for order statistic. By keeping the expression for ordinary moments in mind, the k-th moment for the r-th ordered value of n is given by

$$E[X^k_{(r):n}] = \frac{1}{B(r, n-r+1)} \int_0^1 \{F^{-1}\}^k F^{r-1}[1-F]^{n-r} dF, \tag{A.25}$$

where $F^{-1}$ is the quantile function of F. For more information about order statistic, see Gottschalk and Krasovskaia (2001).

After defining the order statistic, focus can now be attended to the L-moments. Hosking (1990) defined the L-moment of order r of a stochastic variable X by

$$\lambda_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E\{X_{r-k:r}\}. \tag{A.26}$$

By combining equatinon A.26 and A.25 the r-th order L-moment can be rewritten as

$$\lambda_r = \int_0^1 F^{-1} P^*_{r-1}(F) dF, \tag{A.27}$$

where

$$P^*_r(F) = \sum_{k=0}^{r} p^*_{r,k} F^k \tag{A.28}$$

and

$$p^*_r = (-1)^{r-k} \binom{r}{k} \binom{r+k}{k}. \tag{A.29}$$

In the equations above $P^*_r(F)$ is the r-th shifted Legendre polynomal, see Hosking, 1990 for more information. The first three L-moments is now given below, but also higher moments and L-moment ratios (e.g. L-CV) exist.

$$\begin{aligned} \lambda_1 &= E[X] = \textstyle\int_0^1 F^{-1} dF \\ \lambda_2 &= \tfrac{1}{2} E[X_{2:2} - X_{1:2}] = \textstyle\int_0^1 F^{-1}(2F-1) dF \\ \lambda_3 &= \tfrac{1}{3} E[X_{3:3} - 2X_{2:3} + X_{1:3}] = \textstyle\int_0^1 F^{-1}(6F^2 - 6F + 1) dF. \end{aligned} \tag{A.30}$$

The L-moments are clearly expectations of linear combinations of order statistic.

A comparison of ordinary moments and L-moments show that they are analog measures of location, scale, skewness etc. The first L-moment are identical with the ordinary moment, thus no explanation is needed for this measure. The second L-moment, the L-scale measures the difference between the observations in a random sample of size two ($X_{2:2} - X_{1:2}$) drawn from a distribution. If the values in the distribution are close to the centre of the distribution, the difference between the two sample observations will tend to be small. If the values in the distribution are further dispersed from the centre, the difference will be larger. In the third L-moment random samples of size three are used for determining the skewness. If a distribution is symmetric, then $X_{3:3} - X_{2:3} \approx X_{2:3} - X_{1:3}$ . This is the same as writing $X_{3:3} - 2X_{2:3} + X_{1:3} \approx 0$. If the distribution is right-skewed the latter expression is positive, and if the distribution is left-skewed the expression is negative.

The previous definition of L-moments is defined for a probability distribution, but when applied on a finite sample with an unknown distribution, estimation is needed. The estimation can be carried out using either order statistic or probability weighted moments (PWM), and the estimation can be found in Gottschalk and Krasovskaia (2001) or Hosking (1990)

### A.5.2 Maximum likelihood estimation and AIC

Maximum likelihood is a method that can be applied for estimation of a set of unknown $\theta_1, \theta_2 ... \theta_n$ in a given family $F$of distributions. The likelihood function is defined as the probability of observing data as a function of the parameters $\theta_1, \theta_2 ... \theta_n$. Different values of the parameters define different candidate distributions within the given family, and parameters with a high likelihood have a high probability of observing the data. The maximum likelihood estimate (mle) are those parameters that makes the observed data "most probable" or "most likely". These parameters are called the maximum likelihood estimates of the parameters and can be estimated by maximizing the likelihood function (Rice, 1995). In this thesis numerical estimation of the maximum likelihood estimates is required. These routines are found in the EVD package Stephenson (2003) or in Coles (2001). The likelihood function is given as

$$L(\theta_1, \theta_2 ... \theta_n) = \prod_{i=1}^{n} f(x_i | \ \theta_1, \theta_2 ... \theta_n), \tag{A.31}$$

where $f(x_i | \ \theta_1, \theta_2 ... \theta_n)$ is the probability density function and $x_1, ..., x_n$ are independent realizations of this PDF. If instead the logarithm of equation A.31 is used, the maximazing of the likelihood function become easier due to the convenience of working with sums instead of products;

$$logL(\theta_1, \theta_2 ... \theta_n) = \sum_{i=1}^{n} \log f(x_i | \ \theta_1, \theta_2 ... \theta_n). \tag{A.32}$$

The maximum likelihood estimation can also be used for model validation. The Akaikes information criterion (AIC) applies the logL value and subtracts the number of parameters for each different candidate model. Since the logL value is the most likely model in a family, different candidate models from different families can be compared. The model with the highest AIC score is the preferred model. The subtraction of the number of parameters is a penalty for the number of parameters fitted.

## A.6 Bootstrapping

The bootstrap is a computer intensive method developed by Ebson and Tibshirani that uses a data based simulation method for statistical inference on the sampling distribution of a parameter. This inference is used for producing different inferences including bootstrap parameter estimates, bootstrap standard errors and bootstrap confidence intervals. The advantage of using the bootstrap is that the method is available no mather how complicated the estimator is. When the bootstrap is applied there are two main techniques, namely parametric bootstraping and non-parametric bootstraping. The difference between these techniques is that the parametric bootstrap samples from a population, while the non-parametric bootstrap resamples from a sample of the population. Parameters estimated using parametric bootstrapping are more accurate and show less variation than parameters estimated with non-parameric bootstraping, but it requires that the deviation between the empirical distribution of the investigated data-sample and the distribution of the population is sufficiently small. If this condition is not fulfilled, the non-parametric bootstrap gives better results. In hydrology this deviation is usually to large, and non-parametric bootstrapping is commonly applied.

### A.6.1 Non-parametric bootstrap estimation of standard error

In many situations the accuracy of an estimated parameter is of interest. Traditionally theoretical expressions for the standard deviation of the parameter have been made, but when the estimator become complicated it can be hard to derive the expression for the standard error. If instead the bootstrap standard error is estimated, only the parameter estimator is needed for the estimation. The method for estimating the bootstrap standard error starts with a random sample $\mathbf{x} = (x_1, x_2..., x_n)$ from an unknown probability distribution F. From this probability distribution the parameter $\theta = t(F)$ estimated from $\mathbf{x}$ is the parameter of interest. An estimate of this parameter is $\hat{\theta} = s(\mathbf{x})$. It is now possible to use bootstrapping to estimate the standard error of $\hat{\theta}$. In the non-parametric bootstrap the empirical distribution $\hat{F}$ is used as an estimation of the probability distribution F. From this empirical distribution the bootstrap samples are made. Each bootstrap sample consist of n observations resampled from $\hat{F}$ with replacement, $\hat{F} \rightarrow (x_1^*, x_2^*, ..., x_n^*)$, where the star notation indicates that the observations are resampled. The number of bootstrap samples needed depends on the parameter of interest, but lies usually between 1000 and 10000 replications. If B bootstrap samples is drawn, they can be represented as $(\mathbf{x}^{*1}, \mathbf{x}^{*2}, ..., \mathbf{x}^{*B})$. The next step in to estimate the parameter in each bootstrap sample,

$$\hat{\theta}^*(b) = s(\mathbf{x}^{*b}),\ b = 1, 2, \dots, B. \tag{A.33}$$

This result in B different estimates of the parameter $\hat{\theta}^*(b)$. The distribution of $\hat{\theta}^{*b}$ is the sampling distribution of the parameter estimator given in equation A.33. Finally the bootstrap parameter estimate $\hat{\theta}^*(\cdot)$ is given by

$$\hat{\theta}^*(\cdot) = \sum_{b=1}^{B} \hat{\theta}^*(b)/B\ . \tag{A.34}$$

When the bootstrap parameter estimate is calculated, the standard error of the parameter can be estimated by

$$\hat{se}_B = \left\{ \sum_{b=1}^{B} [\hat{\theta}^*(b) - \hat{\theta}^*(\cdot)]^2/(B-1) \right\}^{\frac{1}{2}}, \tag{A.35}$$

where $\hat{se}_B$ is the approximation of $se_{\hat{F}}(\hat{\theta})$. This last term is called the ideal bootstrap estimate of standard error of $\hat{\theta}$, and the approximation $\hat{se}_B$ converges asymptotically to the ideal bootstrap estimate as B goes to infinity

$$\lim_{B\to\infty} \hat{se}_B = se_{\hat{F}} = se_{\hat{F}}(\hat{\theta}^*). \tag{A.36}$$

### A.6.2 The BCa confidence interval

When the standard error of a parameter is estimated, the accuracy of a parameter can be represented by a confidence interval. There exist a number of different confidence intervals with different correctness‘ and accuracies. The type of confidence intervals that are used the most are the normal approximated confidence intervals with coverage 1-2$\alpha$, given as

$$\left[\hat{\theta} - z^{(1-\alpha)} \cdot \hat{se},\ \hat{\theta} - z^{\alpha} \cdot \hat{se}\right], \tag{A.37}$$

where $z^{\alpha}$ is the 100$\cdot\alpha$ percentile in a standard normal distribution. The reason for using the standard normal distribution is that if the size of the sample that the parameter is estimated from increases, the sampling distribution of the parameter becomes more and more normally distributed with mean near $\theta$ (the true value of the parameter) and variance near $\hat{se}^2$, $\hat{\theta} \sim N(\theta,\ \hat{se}^2)$. Equivalently this can be written

$$\frac{\hat{\theta} - \theta}{\hat{se}} \sim N(0,1) \tag{A.38}$$

(Efron and Tibshirani, 1997).

A problem with these confidence intervals is that the approximation does not always hold. If the size of the sample the parameter is estimated from is small, the sampling distribution of the parameter can be sqew. The result is that there is a bias in the coverage of the confidence interval. A confidence interval that has better coverage performance is the percentile interval, given by

$$\left[\hat{G}^{-1}(\alpha),\ \hat{G}^{-1}(1-\alpha)\right], \tag{A.39}$$

where $G^{-1}$ is the quantile function of the empirical distribution. But also with this confidence interval there are some limitations (which also applies for normal approximated confidence intervals). Bias in the estimated parameter lead to a biased normal estimate

$$\hat{\theta} \sim N(\theta + \text{ bias},\ \hat{se}^2). \tag{A.40}$$

The only confidence interval that corrects for both skew sampling distributions and bias in the estimated parameter is an improved version of the percentile interval termed the BCa confidence interval. BCa stands for bias corrected and accelerated, where the term acceleration refers to the correction of bias in the coverage of the confidence interval. The BCa confidence interval is far more complicated than the confidence intervals given previously, but quite easy to use in practice. A 1-2$\alpha$ coverage BCa interval is given by

$$\left[\hat{\theta}_{lo},\ \hat{\theta}_{up}\right] = \left[\hat{\theta}^{*(\alpha_1)},\ \hat{\theta}^{*(\alpha_2)}\right], \tag{A.41}$$

where

$$\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z^{(\alpha)}}{1-\hat{a}(\hat{z}_0 + z^{(1-\alpha)})}\right) \tag{A.42}$$

$$\alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z^{(1-\alpha)}}{1-\hat{a}(\hat{z}_0 + z^{(1-\alpha)})}\right).$$

In equation A.42 $\Phi(\cdot)$ is the standard cumulative distribution function, $z^{\alpha}$ is the 100$\cdot\alpha$ percentile in a standard normal distribution and $\hat{z}_0$ and $\hat{a}$ are the correction terms for the bias correction and the acceleration, respectively. If $\hat{z}_0$ and $\hat{a}$ are zero, the BCa interval is equal to the percentile interval. For calculation of $\hat{z}_0$ the following equation can be used

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\#\{\hat{\theta}^*(b) < \hat{\theta}}{B}\right). \tag{A.43}$$

This equation calculates the difference between the median of $\hat{\theta}^*$ and $\hat{\theta}$ in normal units. If this difference is zero, the sampling distribution is symmetrical ($\hat{z}_0 = 0$), and exactly half of the $\hat{\theta}(b)$ values are less than or equal to $\hat{\theta}$. The acceleration $\hat{a}$ is calculated by

$$\hat{a} = \frac{\sum_{i=1}^{n}(\hat{\theta}_{(\cdot)} - \hat{\theta}_{(i)})^3}{6\{\sum_{i=1}^{n}(\hat{\theta}_{(\cdot)} - \hat{\theta}_{(i)})^2\}^{3/2}}, \tag{A.44}$$

where $\hat{\theta}_{(\cdot)}$ is the jackknife estimate of the parameter $\hat{\theta}$ and $\hat{\theta}_{(i)}$ is the parameter estimated from the data sample with the i'th value left out. The acceleration correction is needed due to, as previously stated, bias in the coverage of the confidence interval. This bias arise in normal approximated confidence intervals because the normal approximation assumes that the standard error of $\hat{\theta}$ is constant for all $\theta$. For correction of this bias, the rate of the standard error of $\hat{\theta}$ with respect to the true parameter value $\theta$, measured on a normal scale can be calculated. This correction is the acceleration given in equation A.44.

The theory of section A.6 is based on Efron and Tibshirani (1997).

# Appendix B

# Results

In this chapter the different bivariate return periods for the peak discharge and volume, and the duration and volume at Losna are given.

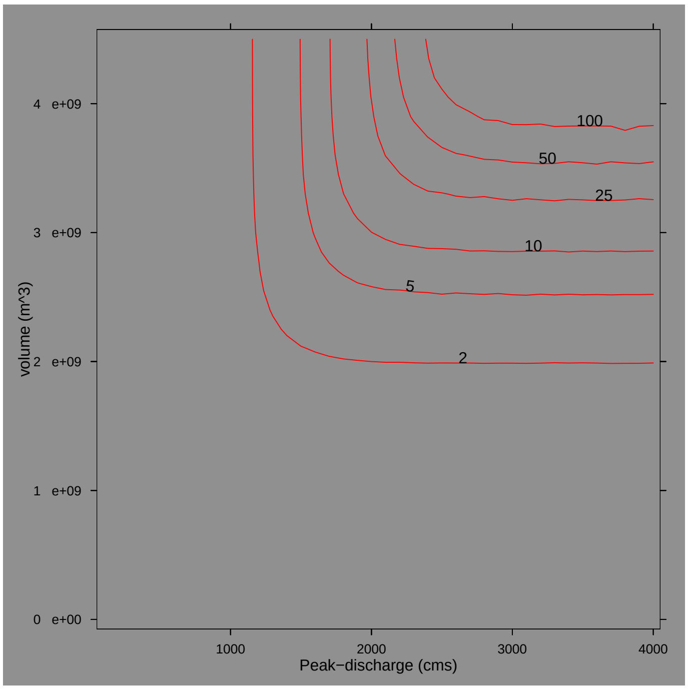


Figure B.1: Joint return periods for peak discharge and volume at Losna for the event X>x or Y>y constructed from the bivariate logistic distribution

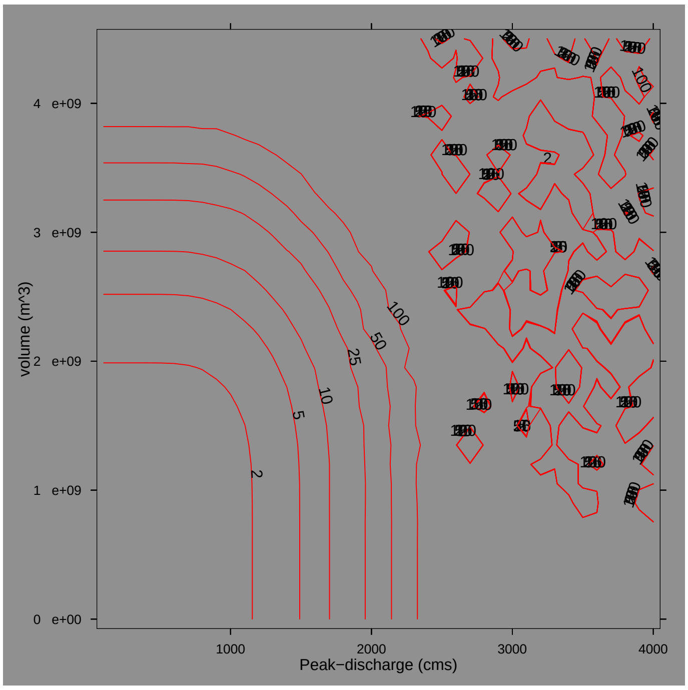


Figure B.2: Joint return periods for the peak discharge and volume at Losna for the event X>x and Y>y, constructed from the bivariate log normal distribution.

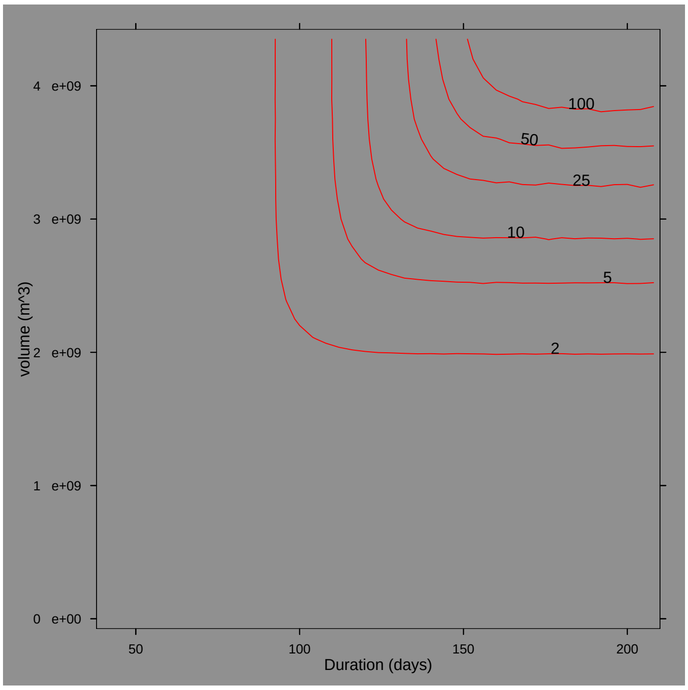


Figure B.3: Joint return periods for duration and volume at Losna for the event X>x or Y>y constructed from the bivariate logistic distribution

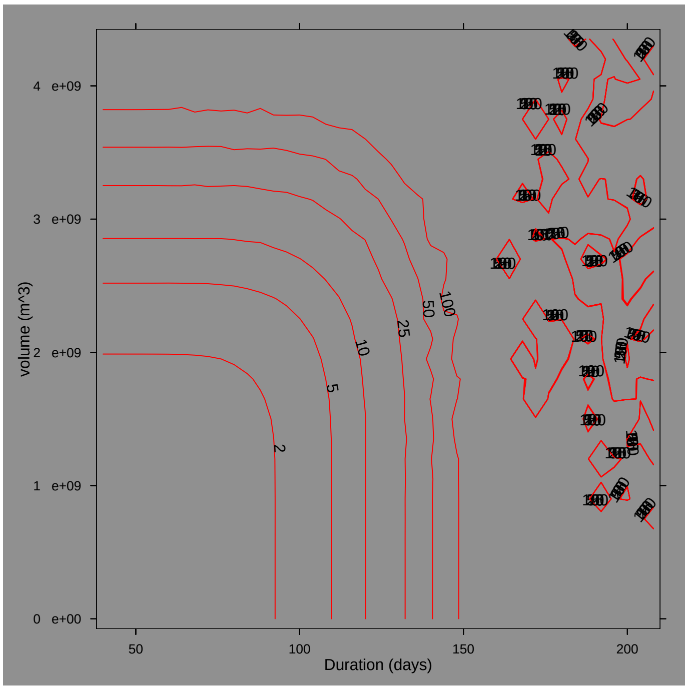


Figure B.4: Joint return periods for the duration and volume at Losna for the event X>x and Y>y, constructed from the bivariate log normal distribution.