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# Some interesting birational morphisms of smooth affine quadric 3-folds\*

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## Abstract

We study a family of birational maps of smooth affine quadric 3-folds, over the complex numbers, of the form  $x_1x_4 - x_2x_3 = \text{constant}$ , which seems to have some (among many others) interesting/unexpected characters: (a) they are cohomologically hyperbolic, (b) their second dynamical degree is an algebraic number but not an algebraic integer, and (c) the logarithmic growth of their periodic points is strictly smaller than their algebraic entropy. These maps are restrictions of a polynomial map on  $\mathbb{C}^4$  preserving each of the quadrics. The study in this paper is a mixture of rigorous and experimental ones, where for the experimental study we rely on Bertini which is a reliable and fast software for expensive numerical calculations in complex algebraic geometry.

Keywords: algebraic numbers, birational maps, dynamical degrees, equidistribution, log-concavity, numerical algebraic geometry, periodic points

Mathematics Subject Classification numbers:

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\* To the late Nessim Sibony.

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## 1. Introduction

A main theme in Complex Dynamics is that of equidistribution for periodic points. Roughly speaking, it is expected that if a rational map (more generally, meromorphic map)  $f: X \dashrightarrow X$  of a complex projective manifold (more generally, compact Kähler manifold) satisfies a certain type of hyperbolicity, then it has an equilibrium measure with no mass on proper Zariski subsets, and to which the average of Dirac measures on hyperbolic periodic points converges. In general, one expects that the logarithmic growth of (hyperbolic) isolated periodic points of a map is a lower bound for its topological entropy (a very fundamental dynamical invariant). These are characteristics of the complexity of a map.

To make precise the above statement, we first recall the definition of dynamical degrees of a map. Let  $X$  be a compact Kähler manifold of dimension  $m$ , and  $f: X \dashrightarrow X$  a dominant meromorphic map (i.e.: there is a proper Zariski subset  $I(f) \subset X$ —called the indeterminacy set of  $f$ —so that  $f$  is given as a holomorphic map  $f: X \setminus I(f) \rightarrow X$  and the image of  $X \setminus I(f)$  is dense in  $X$ ). Let  $\omega$  be the cohomological class of a Kähler form on  $X$ . For a number  $0 \leq j \leq m$ , the following limit exists and is independent of the choice of the Kähler form (see [57] when  $X$  is a projective space, [31, 32] for the general case)

$$\lim_{n \rightarrow \infty} \|(f^n)^* (\omega^j)\|_{H^{2j}(X, \mathbb{C})}^{1/n}.$$

Here,  $f^n = f \circ f \circ \dots \circ f$  ( $n$  times) is the  $n$ th iterate of  $f$ , and  $\|\cdot\|$  is any norm on the finite dimensional vector space  $H^{2j}(X, \mathbb{C})$ . We call the above limit the  $j$ th dynamical degree of  $f$ , and denote by  $\lambda_j(f)$ . In the case  $X$  is a complex projective manifold, one can choose  $\omega$  to be the class of an ample divisor and replace  $H^{2j}(X, \mathbb{C})$  by  $N_{\mathbb{R}}^j(X)$  the space of algebraic cycles of codimension  $j$  (with real coefficients) modulo the numerical equivalence. One always has  $\lambda_0(f) = 1$  while  $\lambda_m(f)$  is the topological degree of  $f$  (and hence is 1 when  $f$  is bimeromorphic). It is known that  $\lambda_j(f)$ 's are bimeromorphic invariants ([31, 32]), that is if  $\pi: Y \dashrightarrow X$  is a bimeromorphic map, and  $f_Y$  is the lifting of  $f$  to  $Y$  (i.e.  $f_Y = \pi^{-1} \circ f \circ \pi$ ), then  $\lambda_j(f_Y) = \lambda_j(f)$  for all  $j$ . There are also arithmetic analogs of these dynamical degrees, where the above mentioned results are largely unknown, and there are many challenging conjectures around (see [18, 50]). Dynamical degrees can also be defined for maps on fields of positive characteristics [16, 62], and the version for correspondence in [62] thus can be used to provide a generalization of Weil's Riemann hypothesis [47, 48, 63]. Dynamical degrees provide a useful way to compute topological entropy of holomorphic selfmaps of compact Kähler manifolds [40, 66]. Inspired by this fact, for a general meromorphic map, the logarithm of the maximum of its dynamical degrees is named its **algebraic entropy** in [9], and this is widely practiced in the literature. Moreover, they provide an upper bound for the topological entropy of dominant meromorphic maps of compact Kähler manifolds [31, 32], and the same upper bound holds in the Berkovich space setting [35, 36]. Besides being of interest in pure mathematics, birational maps appear naturally in some physical models (in lattice statistical mechanics), and their dynamical degrees are an indication of the complexity of these models, see e.g. [2, 3, 8–10, 61].

A dominant meromorphic map  $f: X \dashrightarrow X$  is cohomologically hyperbolic if there is one index  $j$  so that  $\lambda_j(f) > \max_{i \neq j} \lambda_i(f)$ . Cohomological hyperbolicity is a cohomological version of the well known notion of hyperbolic dynamics in differentiable dynamical systems, where many results around periodic points, equilibrium measures and topological entropy are known, see a survey in [42]. The rough idea is that since algebraic dynamical systems are more rigid than smooth dynamical systems, hyperbolicity of an algebraic dynamical system may be detected by the easier invariants on cohomology groups.

With this preparation, an explicit statement of a major folklore conjecture, which attracts a lot of attention and work, is the following:

**Conjecture 1.1 (folklore conjecture)** Let  $X$  be a compact Kähler manifold of dimension  $m$ , and  $f: X \dashrightarrow X$  a dominant meromorphic map. Assume that  $f$  is cohomologically hyperbolic. Then the following are true:

- 1) There is a probability measure  $\mu$  on  $X$  with no mass on proper analytic subsets (hence, one can push it forward by  $f$ ) which is invariant by  $f$ , i.e.  $f_*\mu = \mu$ .
- 2) Let  $\text{HP}_n(f)$  be the set of hyperbolic periodic points of period  $n$  of  $f$ . Then:
  - a) The exponential growth of  $\#\text{HP}_n(f)$  is  $\lambda(f)$ , where  $\lambda(f) = \max_j \lambda_j(f)$ . This means that

$$\lim_{n \rightarrow \infty} \frac{\log \#\text{HP}_n(f)}{n} = \log \lambda(f).$$

- b) The hyperbolic periodic points of  $f$  equidistribute to  $\mu$ . That is,

$$\lim_{n \rightarrow \infty} \frac{1}{\#\text{HP}_n(f)} \sum_{x \in \text{HP}_n(f)} \delta_x = \mu.$$

Here,  $\delta_x$  is the Dirac measure at  $x$ .

- c) The number of other isolated periodic points of  $f$  of order  $n$  is negligible compared to that of  $\#\text{HP}_n(f)$ .

A stronger version (which also involves the topological entropy and Lyapunov exponents) of this conjecture was stated in [41]. A few representatives from the large known results in the literature resolving conjecture 1.1 in the affirmative are: dimension 1 (see [14, 38, 54]); Hénon maps in dimension 2 (see [4, 5]); automorphisms of K3 surfaces (see [15]); birational maps of surfaces satisfying a certain condition on Green currents (see [33], which relies on a precise estimate for the number of isolated periodic points: the lower bound is obtained therein by using laminar currents, and the upper bound is later provided by [24, 49]); a large class of maps of surfaces satisfying an energy condition (see [21]); meromorphic maps of compact Kähler manifolds for which  $\lambda_m > \lambda_{m-1}$  (see [25]); and analogs of Hénon-maps in higher dimensions (see [30]). For a comprehensive survey, see [29], where other topics of equidistribution—besides that of periodic points—are also discussed. Despite all of these partial results and many efforts, conjecture 1.1 is still largely open.

There are some indications in the literature that conjecture 1.1 may be too strong to be true. For example, one related conjecture is that  $\lambda_1(f)$  is an algebraic number [8]. However, recent work [6, 7] shows the existence of maps (can be chosen to be birational maps of  $\mathbb{P}^d$  where  $d \geq 3$ ) for which  $\lambda_1(f)$  is a transcendental number, and hence the mentioned related conjecture does not hold. On the other hand, it is not known if the mentioned maps in [6, 7] provide counter-examples to conjecture 1.1, because these papers treat only the first dynamical degrees and concern neither periodic points nor equilibrium measures.

It is noteworthy that for the various maps in the literature where dynamical degrees can be actually computed (for a birational maps of surfaces a general procedure using point blowups has been given in [22], but besides that one must in general treat case by case), the dynamical degrees are either **algebraic integers** (i.e. roots of a polynomial  $p(t)$  whose coefficients are integers and the leading coefficient is 1) or **transcendental numbers** (i.e. not a root of any

polynomial with integer coefficients). For some special maps, there are even particular speculations about that the dynamical degrees are algebraic integers of special types. Therefore, it is natural to ask what is the situation for the numbers in between these two types, that is algebraic numbers which are not algebraic integers. For example, recall that a real number  $\lambda$  is a weak Perron number if it is an algebraic integer and any of its Galois conjugate  $\mu$  satisfies  $|\mu| \leq \lambda$ . There is the following conjecture [12] (for the first dynamical degree) and [17] (for all dynamical degrees), concerning the very actively studied polynomial maps of affine spaces:

**Conjecture 1.2.** If  $f: \mathbb{C}^d \rightarrow \mathbb{C}^d$  is a polynomial map, then the dynamical degrees (of the extension of  $f$  to  $\mathbb{P}^d$ ) are all weak Perron numbers (in particular, algebraic integers).

For the first dynamical degree, the above conjecture holds in dimension 2 (see [34, 39]) and some automorphisms in  $\mathbb{C}^3$  (see [55]), and some special automorphisms in higher dimensions ([12, 13] and an unpublished result by Mattias Jonsson cited in these papers). For proper polynomials of  $\mathbb{C}^m$  so that  $\lambda_1^2 > \lambda_2$ , [17] shows that  $\lambda_1(f)$  is an algebraic number of degree at most  $m$ . For all dynamical degrees, one non-trivial case is that of a monomial map  $f$  (see [37, 53]) where all of its dynamical degrees are known to be the product of absolute values of some eigenvalues of the integer matrix defining  $f$ , and hence are all algebraic integers (recall that algebraic integers form a ring).

The purpose of this paper is to present a family of birational maps of smooth quadric 3-folds which are a candidate counter-example for conjecture 1.1. At the same time, these birational maps come from polynomial maps on  $\mathbb{C}^4$  whose second and third dynamical degrees seem to be an algebraic number but not an algebraic integer (and hence, can also be a counter-example for conjecture 1.2). As such, the mentioned methods in the last part of the previous paragraph cannot be used to compute the conjectured second dynamical degree, and new ideas have to be developed for this task.

A disadvantage of the statement of conjecture 1.1 is that some parts of it depend on an unspecified birational map  $\pi: Y \dashrightarrow X$  and an unspecified equilibrium measure  $\mu$ . As will be seen in section 3, the following conjecture is a consequence of conjecture 1.1, and concerns only the exponential growth of the set of isolated periodic points (which in theory can be explicitly found) and is independent of the birational model of a given map.

**Conjecture 1.3.** Let  $X$  be a smooth complex projective variety of dimension  $d$ , and let  $f: X \dashrightarrow X$  be a dominant rational map. Assume that  $f$  is cohomologically hyperbolic. Define  $\lambda(f) = \max_j \lambda_j(f)$  and  $\text{IsoPer}_n(f)$  the set of isolated periodic points of period  $n$  of  $f$  (multiplicities accounted). Let  $Z \subset X$  be any Zariski open dense set. Then

$$\limsup_{n \rightarrow \infty} \frac{\log \#(\text{IsoPer}_n(f) \cap Z)}{n} = \log \lambda(f).$$

Note that one part of the above conjecture is known [24] (valid in the Kähler setting, and the proof therein uses the theory of tangent currents developed in [28]): we always have

$$\limsup_{n \rightarrow \infty} \frac{\log \#(\text{IsoPer}_n(f) \cap Z)}{n} \leq \log \lambda(f).$$

(On the other hand, on differentiable dynamical systems, there are no such upper bounds on the number of periodic points, see [51].)

Hence, to disprove conjecture 1.1 it suffices to display a counter-example to conjecture 1.3. We next present our candidate counter-examples for conjectures 1.3 (and hence 1.1) and 1.2, which are strikingly simple. We start with a polynomial  $F: \mathbb{C}^4 \rightarrow \mathbb{C}^4$  given by:

$$F(x_1, x_2, x_3, x_4) = (x_2, -x_4, x_1 - x_1x_2^2, -x_3 + x_1x_2x_4).$$

This is a birational map, with inverse

$$F^{-1}(x_1, x_2, x_3, x_4) = \left( \frac{-x_3}{-1 + x_1^2}, x_1, \frac{x_1 x_2 x_3 + x_4 - x_1^2 x_4}{-1 + x_1^2}, -x_2 \right).$$

**Remark 1.4.** The above map  $F$  is an element of a more general family:

$$G(x_1, x_2, x_3, x_4) = (x_2, -x_4, x_1 - x_2 P(x_1, x_2, x_3, x_4), -x_3 + x_4 P(x_1, x_2, x_3, x_4)). \tag{1}$$

Here  $P(x_1, x_2, x_3, x_4)$  is a polynomial (or more generally a rational function) of the form

$$P(x_1, x_2, x_3, x_4) = x_1 Q_1(x_2, x_4) + x_3 Q_3(x_2, x_4) + R(x_2, x_4).$$

Some special automorphisms of  $\mathbb{C}^4$  in the family  $G$  have been studied in [11].

Let  $\phi : \mathbb{C}^4 \rightarrow \mathbb{C}$  be the map  $\phi(x_1, x_2, x_3, x_4) = x_1 x_4 - x_2 x_3$ . It can be checked that the fibres of  $\phi$  are invariant by  $f$ , that is  $\phi = \phi \circ F$ . In other words, via  $\phi$ ,  $F$  is semi-conjugate to the identity map on  $\mathbb{C}$ . For a generic  $c \in \mathbb{C}$ , let  $Z_c = \phi^{-1}(c) = \{x_1 x_4 - x_2 x_3 = c\} \subset \mathbb{C}^4$  be the fibre of  $\phi$  over  $c$ . Let  $X_c$  be the closure in  $\mathbb{P}^4$  of  $Z_c$  (hence, in homogeneous coordinates  $[x_1 : x_2 : x_3 : x_4 : z]$  of  $\mathbb{P}^4$ ,  $X_c$  is given by  $x_1 x_4 - x_2 x_3 = cz^2$ ). Then both  $X_c$  and  $Z_c$  are smooth, invariant by  $F$ , and  $Z_c$  is a Zariski open dense subset of  $X_c$ . The remaining of this paper is to study the following conjecture for the map  $f_c = F|_{Z_c} : Z_c \rightarrow Z_c$ . Note that since  $f_c$  is a birational map in dimension 3, we have  $\lambda_0(f_c) = \lambda_3(f_c) = 1$ . Hence, the only non-trivial dynamical degrees of  $f_c$  are  $\lambda_1(f_c), \lambda_2(f_c)$ .

**Conjecture 1.5.** Let  $f_c$  be the above map, for a generic value  $c \in \mathbb{C}$ , and  $\widehat{f}_c$  its extension to  $X_c$ . Then we have:

- 1) The first dynamical degree  $\lambda_1(\widehat{f}_c)$  is the largest root  $\zeta_1$  of the polynomial  $t^3 - t^2 - t - 1$ , and  $\zeta_1$  is approximately 1.8393.
- 2) The second dynamical degree  $\lambda_2(\widehat{f}_c)$  is the largest root  $\zeta_2$  of the polynomial  $2t^3 - 3(t^2 - 1) - 4t$ . This polynomial is irreducible over  $\mathbb{Q}$ , and hence  $\lambda_2(f_c)$  is an algebraic number but not an algebraic integer. Here,  $\zeta_2$  is approximately 2.1108.
- 3) Moreover,

**Stronger estimate:**

$$\limsup_{n \rightarrow \infty} \frac{\log(\#\text{IsoFix}_n(f_c))}{n} \leq \log 2.108.$$

**Weaker estimate:**

$$\limsup_{n \rightarrow \infty} \frac{\log(\#\text{IsoFix}_{2n+1}(f_c))}{2n+1} \leq \log 2.108.$$

Part 1 of the above conjecture is solved in lemma 3.4. If  $f_c$  indeed satisfies conjecture 1.5, then it is a primitive map, see section 3 for detail. Conjecture 1.5, if holds, implies that conjectures 1.3 (and hence also conjecture 1.1) and 1.2 do not hold, see section 3. Indeed, the experimental results in section 3 seem to indicate that the limit

$$\lim_{n \rightarrow \infty} \frac{\log(\#\text{IsoFix}_n(f_c))}{n}$$

exists, and moreover is contained in the interval  $[\log 2.0890, \log 2.1071]$ . That the polynomial  $2t^3 - 3(t^2 - 1) - 4t$  in part 2 of conjecture 1.5 is irreducible over  $\mathbb{Q}$  can be easily checked by

many means. (For example, if it were to be reducible over  $\mathbb{Q}$ , then it would have at least one root in  $\mathbb{Q}$ , which then must be half of an integer. Using computer softwares such as Mathematica, one finds that the given polynomial has 3 real roots lying between  $-2.5$  and  $2.5$ , and no half integer in this interval is a root of the polynomial.)

## 2. Method

The method in this paper is a mixture of rigorously theoretical and experimental ones.

We use the current techniques in complex Dynamical Systems to prove the mentioned relations between the conjectures in the Introduction section. We also use rigorously theoretical arguments to show that, provided the pattern observed in our experiments hold for all iterate, then our maps (on the smooth affine quadric 3-folds) are primitive, and to reduce the computation of dynamical degrees on the quadric 3-folds to that on the affine 4-space.

On the other hand, the current techniques in complex Dynamical Systems are not enough for to resolve conjecture 1.5, see the Discussion section and the Conclusion section for more detail. Hence, we need to rely on computational methods. For systems of polynomial equations over the field of complex numbers, there are symbolic methods such as Gröbner basis which have strong theoretical guarantees but usually cannot find individual solutions to the system (which is what we need here since we want to check whether a periodic point is hyperbolic or not) and usually is too slow, in particular for the systems of equations related to computing periodic points of our map. Hence, we need to utilise numerical methods, more specifically here the software Bertini [1].

Here is a brief summary of the idea for using experimental results reported later in this paper to give support to the validity of conjecture 1.5. To check part 2 of conjecture 1.5, we compare the degree sequences of the iterates of  $F$  and some sequences related to the linear recurrence  $w_n = 3/2 \cdot (w_{n-1} - w_{n-3}) + 2w_{n-2}$ , whose exponential growth is  $\zeta_2$ . For to check the Weaker estimate in part 3 of conjecture 1.5, we test if the sequence  $(\#\text{IsoFix}_{2n+1}(f_c))^{1/(2n+1)}$  is **decreasing**, which is related to the log concavity phenomenon in proposition 2.1 (see next), and to observe that for  $n = 4, 5$  the number  $(\#\text{IsoFix}_{2n+1}(f_c))^{1/(2n+1)}$  is  $\leq 2.108$ . For to check the Stronger estimate in part 3 of conjecture 1.5, we compare the two sequences  $(\#\text{IsoFix}_{2n+1}(f_c))^{1/(2n+1)}$  and  $(\#\text{IsoFix}_{2n}(f_c))^{1/(2n)}$  and use the Weaker estimate.

The next result provides a heuristic support for the mentioned decreasing phenomenon of the sequence  $(\#\text{IsoFix}_{2n+1}(f_c))^{1/(2n+1)}$ . To see why, we recall that roughly speaking (under the assumption that  $g$  has only isolated fixed points) the Lefschetz number  $L(g)$  is the sum of fixed points (with multiplicities) of  $g$ . Hence, the observed decreasing phenomenon would be a consequence of the next result, if it were the case that all fixed points of  $\widehat{f_c}^n$  were in  $Z_c$  and isolated.

**Proposition 2.1.** *Let  $X \subset \mathbb{P}^{2m}$  be a smooth quadric. Let  $g : X \dashrightarrow X$  be a dominant rational map. Then the Lefschetz numbers  $L(g^n)$  (i.e. the intersection number between the graph of  $g^n$  and the diagonal of  $X$ ) are all positive, and satisfy log concavity, i.e.*

$$L(g^{n+n'}) \leq L(g^n)L(g^{n'})$$

for all  $n, n' \geq 0$ . In particular, the sequence  $\{L(g^n)^{1/n}\}_{n=1,2,\dots}$  is decreasing.

Similarly, for every  $0 \leq j \leq 2m - 1$ , the degree sequence  $\{\| (g^n)^* |_{H^{2j}(X)} \| \}_{n=1,2,\dots}$  (where  $\| \cdot \|$  is a given norm on  $H^{2j}(X)$ ) satisfies the log concavity.

### 3. Results

In this section we present our results around the map  $f$  and conjecture 1.5. These results consist of both rigorously theoretically proven ones and experimental ones. Based on these results, we propose a road map toward resolving conjecture 1.5 in the affirmative.

Let  $F: \mathbb{C}^4 \rightarrow \mathbb{C}^4$ ,  $X_c$ ,  $Z_c$  and  $f_c: Z_c \rightarrow Z_c$  be defined as in the introductory section. Throughout this section, we use the following notations:

- $\widehat{F}$  = the extension of  $F$  to the projective space  $\mathbb{P}^4$ ;
- $\widehat{f}_c$  = the extension of  $f_c$  to  $X_c$ ;
- $d_n^{(1)}$  = the first degree of  $F^n$  (the  $n$ th iterate of  $F$ ): it is defined as the degree of the inverse image by  $\widehat{F}^n$  of a generic linear 3-dimensional subspace  $H$  in  $\mathbb{P}^4$ ;
- $d_n^{(2)}$  = the second degree of  $F^n$ : it is defined as the degree of the inverse image by  $\widehat{F}^n$  of a generic linear 2-dimensional subspace  $H^2$  in  $\mathbb{P}^4$ ;
- $d_n^{(3)}$  = the third degree of  $F^n$ : it is defined as the degree of the inverse image by  $\widehat{F}^n$  of a generic linear 1-dimensional subspace  $H^3$  in  $\mathbb{P}^4$  (it is indeed the same as the first degree of the inverse map  $F^{-n}$ );
- $b_n$  is the sequence satisfying the linear recurrence  $b_n = 3/2 \cdot (b_{n-1} - b_{n-3}) + 2b_{n-2}$ , with the initial terms  $b_1 = 3$ ,  $b_2 = 7$  and  $b_3 = 17$ ;
- $c_n$  is the sequence satisfying the linear recurrence  $c_n = 3/2 \cdot (c_{n-1} - c_{n-3}) + 2c_{n-2}$ , with the initial terms  $c_1 = 5$ ,  $c_2 = 9$  and  $c_3 = 25$ ;
- $\text{IsoFix}_n$  = the set of isolated fixed points of  $f_c^n$ ;
- $C \subset Z_c$ : the curve defined by the ideal  $\langle x_2 - x_1^2 x_2 - x_3, x_1 + x_4, x_1 x_4 - x_2 x_3 - c \rangle$ ;
- $D_1 \subset Z_c$ : the curve with 2 components defined by the ideals  $\langle x_2 - x_1^2 x_2 - x_3, x_1 + x_4, x_1 x_4 - x_2 x_3 - c \rangle$  and  $\langle -x_2 + x_1^2 x_2 - x_3, x_1 - x_4, x_1 x_4 - x_2 x_3 - c \rangle$ ;
- $D_2 \subset Z_c$ : the curve with 2 components defined by the ideals  $\langle x_2, x_3, x_1 x_4 - x_2 x_3 - c \rangle$  and  $\langle x_1, x_4, x_1 x_4 - x_2 x_3 - c \rangle$ .

Here are some useful comments concerning the above notations. First, by the definition of dynamical degrees, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} [d_n^{(1)}]^{1/n} &= \lambda_1(\widehat{F}), \\ \lim_{n \rightarrow \infty} [d_n^{(2)}]^{1/n} &= \lambda_2(\widehat{F}), \\ \lim_{n \rightarrow \infty} [d_n^{(3)}]^{1/n} &= \lambda_3(\widehat{F}). \end{aligned}$$

Since  $\widehat{F}$  is birational, we have  $\lambda_0(\widehat{F}) = \lambda_4(\widehat{F}) = 1$ . Similarly, we have  $\lambda_0(f_c) = \lambda_3(f_c) = 1$ . The relation between the dynamical degrees of  $\widehat{F}$  and those of  $\widehat{f}_c$  will be given in the next Subsection.

From our experiments, to be presented later in this section, the fixed point set of  $f_c^n$  seems to have the following structure:

- The fixed point set of  $f_c^{4n+2}$  is the union of the curve  $C$  and  $\text{IsoFix}_{4n+2}(f_c)$ ;
- The fixed point set of  $f_c^n$  is the union of the curve  $D$  (which is the union of  $D_1$  and  $D_2$ —of different multiplicities) and  $\text{IsoFix}_{4n}(f_c)$ ;
- The fixed point set of  $f_c^{2n+1}$  is  $\text{IsoFix}_{2n+1}(f_c)$ ;
- All isolated periodic points of  $f_c$  are hyperbolic, i.e.  $\text{IsoFix}_n(f_c) = \text{HP}_n(f_c)$ .

#### 3.1. Theoretical results

We first start with some relations between conjectures 1.1, 1.2, 1.3 and 1.5.



**Lemma 3.1.** 1) If conjecture 1.1 holds, then conjecture 1.3 holds.

2) Assume that parts 1 and 2 of conjecture 1.5 hold.

a) If the Stronger estimate of part 3 of conjecture 1.5 holds, then conjecture 1.3 does not hold.

b) If the Weaker estimate of part 3 of conjecture 1.5 holds, then conjecture 1.1 does not hold.

3) If part 2 of conjecture 1.5 holds, then conjecture 1.2 does not hold.

**Proof.** 1) Assume that conjecture 1.1 holds. Let  $f : X \dashrightarrow X$  be a dominant rational map which is cohomologically hyperbolic. Let  $\pi : Y \dashrightarrow X$  be the birational map given by conjecture 1.1.

Let  $Z \subset X$  be a Zariski open dense set. Since  $\pi : Y \dashrightarrow X$  is birational, there is a Zariski open dense set  $U \subset Z$  and a Zariski open dense set  $V \subset Y$  so that  $\pi$  induces an isomorphism between  $V$  and  $U$ .

By conjecture 1.1,  $HP_n(Y)$  equidistributes to  $\mu$  and where  $\mu$  has no mass on proper analytic subsets. Since  $Y \setminus V$  is a proper analytic subset of  $Y$ , it follows that  $HP_n(f_Y) \cap V$  equidistributes to  $\mu$  as well, and the exponential growth of  $\# [HP_n(f_Y) \cap V]$  is also  $\lambda(f_Y)$ . Since  $\pi : V \rightarrow U$  is an isomorphism, it follows that  $HP_n(f_Y) \cap V = HP_n(f) \cap U$ . Since  $\lambda(f_Y) = \lambda(f)$  by the birational invariance of dynamical degrees, we obtain:

$$\lim_{n \rightarrow \infty} \frac{\log \# [HP_n(f) \cap U]}{n} = \log \lambda(f).$$

Since  $HP_n(f) \cap U \subset \text{IsoFix}_n(f) \cap Z$ , we get:

$$\liminf_{n \rightarrow \infty} \frac{\log \# [\text{IsoFix}_n(f) \cap Z]}{n} \geq \log \lambda(f).$$

On the other hand, by [24], we have

$$\limsup_{n \rightarrow \infty} \frac{\log \# \text{IsoFix}_n(f)}{n} \leq \log \lambda(f).$$

Combining the above two inequalities, we have finally

$$\lim_{n \rightarrow \infty} \frac{\log \# [\text{IsoFix}_n(f) \cap Z]}{n} = \log \lambda(f).$$

This means that conjecture 1.3 holds.

2) We prove for part a) only, the proof of part b) is similar. Assume that parts 1 and 2 of conjecture 1.5 hold, and also that the Stronger estimate of part 3 of conjecture 1.5 holds. Since  $X_c$  is of dimension 3, and the dynamical degrees of  $\widehat{f}_c$  are:  $\lambda_0(\widehat{f}_c) = 1$ ,  $\lambda_1(\widehat{f}_c) = \zeta_1 \sim 1.8393$ ,  $\lambda_2(\widehat{f}_c) = \zeta_2 \sim 2.1108$ , and  $\lambda_3(\widehat{f}_c) = 1$ , the map  $\widehat{f}_c$  is cohomologically hyperbolic. We also have that  $Z_c$  is a Zariski open dense set of  $X_c$ , and  $\widehat{f}_c|_{Z_c} = f_c$ . Hence,  $\text{IsoFix}_n(\widehat{f}_c) \cap Z_c = \text{IsoFix}_n(f_c)$ . By conjecture 1.5 we have

$$\limsup_{n \rightarrow \infty} \frac{\log \# \text{IsoFix}_n(f_c)}{n} \leq \log 2.108 < \log \zeta_2 = \log \lambda(f_c).$$

This contradicts conjecture 1.3.

3) Assume that part 2 of conjecture 1.5 holds. Then we have  $\lambda_2(\widehat{f}_c) = \zeta_2$  is an algebraic number, but not an algebraic integer. By lemma 3.3, we have  $\lambda_3(\widehat{F}) = \lambda_2(\widehat{f}_c)$ , hence  $\lambda_3(\widehat{F})$  is not an algebraic integer. In this case, also  $\lambda_2(\widehat{F}) = \zeta_2$  is not an algebraic integer. Thus the polynomial map  $F : \mathbb{C}^4 \rightarrow \mathbb{C}^4$  is a counter-example to conjecture 1.2.  $\square$

We recall that [67] a dominant rational map  $f: X \dashrightarrow X$  is primitive, if there **do not exist** a variety  $W$  of dimension  $1 \leq \dim(W) \leq \dim(X) - 1$ , a dominant rational map  $\pi: X \dashrightarrow W$ , and a dominant rational map  $g: W \dashrightarrow W$  so that  $\pi \circ f = g \circ \pi$ . Primitive maps can be viewed as building blocks from which all maps can be constructed. It is clear from the definition that being primitive is a birational invariant. We have the following result.

**Lemma 3.2.** *Assume that parts 1 and 2 of conjecture 1.5 hold. Then the map  $f_c$  is primitive.*

**Proof.** Let  $\widehat{f}_c^{-1}$  be the inverse of  $\widehat{f}_c$ . If conjecture 1.5 holds, then

$$\lambda_1(\widehat{f}_c^{-1}) = \lambda_2(\widehat{f}_c) > \lambda_1(\widehat{f}_c) = \lambda_2(\widehat{f}_c^{-1}).$$

This inequality implies that  $\widehat{f}_c^{-1}$  is primitive, [56]. Hence  $f_c^{-1}$  (the inverse of  $f_c$ ) is also primitive.

From this, we will show that  $f_c$  is primitive. Assume by contradiction that  $f_c$  is not primitive. Then there are  $\pi: X \dashrightarrow W$  and  $g: W \dashrightarrow W$  dominant rational maps (with  $1 \leq \dim(W) \leq 2$ ) so that  $\pi \circ f_c = g \circ \pi$ . Since  $f_c$  is a birational map, it is easy to see that  $g$  is also a birational map. Then, it follows that  $\pi \circ f_c^{-1} = g^{-1} \circ \pi$ , which contradicts the fact that  $f_c^{-1}$  is primitive.  $\square$

We next relate dynamical degrees of  $\widehat{F}$  and those of  $\widehat{f}_c$ .

**Lemma 3.3.** *We have*

- 1)  $\lambda_1(\widehat{f}_c) = \lambda_1(\widehat{F})$ , and  $\lambda_2(\widehat{f}_c) = \lambda_3(\widehat{F})$ .
- 2)  $\lambda_2(\widehat{F}) = \max\{\lambda_1(\widehat{F}), \lambda_3(\widehat{F})\}$ .

**Proof.** We know from the introductory section that  $F$  is semi-conjugated to the identity map  $id_{\mathbb{C}}$  on the curve  $\mathbb{C}$ , via the map  $\phi: \mathbb{C}^4 \rightarrow \mathbb{C}$  with  $\phi(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$ . The dynamical degrees of (the extension to  $\mathbb{P}^1$  of)  $id_{\mathbb{C}}$  are  $\lambda_0(id_{\mathbb{C}}) = \lambda_1(id_{\mathbb{C}}) = 1$ .

Let  $\lambda_j(\widehat{F}|\phi)$  ( $j = 0, 1, 2, 3$ ) be the relative dynamical degrees w.r.t.  $\phi$  of  $\widehat{F}$  [26, 27]. Since  $F$  preserves the fibres of  $\phi$ , it follows that  $\lambda_j(\widehat{F}|\phi) = \lambda_j(\widehat{f}_c)$ .

By [26, 27], we have

$$\begin{aligned} \lambda_1(\widehat{F}) &= \max\{\lambda_0(id_{\mathbb{C}})\lambda_1(\widehat{f}_c), \lambda_1(id_{\mathbb{C}})\lambda_0(\widehat{f}_c)\} = \lambda_1(\widehat{f}_c), \\ \lambda_3(\widehat{F}) &= \max\{\lambda_0(id_{\mathbb{C}})\lambda_3(\widehat{f}_c), \lambda_1(id_{\mathbb{C}})\lambda_2(\widehat{f}_c)\} = \lambda_2(\widehat{f}_c), \\ \lambda_2(\widehat{F}) &= \max\{\lambda_0(id_{\mathbb{C}})\lambda_2(\widehat{f}_c), \lambda_1(id_{\mathbb{C}})\lambda_1(\widehat{f}_c)\} \\ &= \max\{\lambda_1(\widehat{f}_c), \lambda_2(\widehat{f}_c)\} = \max\{\lambda_1(\widehat{F}), \lambda_3(\widehat{F})\}. \end{aligned}$$

$\square$

Next, we compute  $\lambda_1(\widehat{f}_c)$ .

**Lemma 3.4.**  $\lambda_1(\widehat{f}_c) = \zeta_1$ , the largest root of the polynomial  $t^3 - t^2 - t - 1$ .

**Proof.** By lemma 3.3, we have  $\lambda_1(\widehat{f}_c) = \lambda_1(\widehat{F})$ . Thus we only need to show that  $\lambda_1(\widehat{F}) = \zeta_1$ . Recall that  $\lambda_1(\widehat{F}) = \lim_{n \rightarrow \infty} [d_n^{(1)}]^{1/n}$ . Hence, the proof is finished if we can show that the degree sequence  $d_n^{(1)}$  of  $F$  satisfies the linear recurrence:

$$d_n^{(1)} = d_{n-1}^{(1)} + d_{n-2}^{(1)} + d_{n-3}^{(1)},$$

for all  $n$ .

The leading term of  $F(x_1, x_2, x_3, x_4) = ((F)_1, (F)_2, (F)_3, (F)_4)$  in terms of degree is  $x_1x_2x_4$  in  $(F)_4$ . Moreover,  $(F)_3$  has a unique leading monomial, and  $x_1x_2x_4$  is the unique leading monomial in  $(F)_4$ . By direct calculation, we find that  $F^2 = ((F^2)_1, (F^2)_2, (F^2)_3, (F^2)_4)$  is given by:

$$(-x_4, x_3 - x_1x_2x_4, x_2 - x_2x_4^2, -x_1 + x_1x_2^2 + x_2x_3x_4 - x_1x_2^2x_4^2).$$

Hence, again, the leading term of  $F^2$  is contained in  $(F^2)_4$ , and  $(F^2)_4$  has a unique leading monomial. One can directly check the same phenomenon for  $F^3$  and  $F^4$ .

We will prove by induction that the leading term of  $f^n = ((f^n)_1, (f^n)_2, (f^n)_3, (f^n)_4)$  is contained in  $(f^n)_4$ , and moreover  $(f^n)_4$  has a unique leading monomial. In addition the concerned linear recurrence for the degree sequence holds. Assume by induction that this claim holds until  $n - 1$ . We will show that it also holds for  $n$ . Since  $F^n(x_1, x_2, x_3, x_4)$  is

$$\begin{aligned} & \left( (F^{n-1})_2, - (F^{n-1})_4, (F^{n-1})_1 - (F^{n-1})_1 \cdot (F^{n-1})_2^2, - (F^{n-1})_3 \right. \\ & \left. + (F^{n-1})_1 \cdot (F^{n-1})_2 \cdot (F^{n-1})_4 \right) \end{aligned}$$

and since

$$(F^{n-1})_2 = - (F^{n-2})_4$$

and

$$(F^{n-1})_1 = (F^{n-2})_2 = - (F^{n-3})_4$$

it follows that the leading degree of  $F^n$  is that of  $(F^{n-1})_4 \cdot (F^{n-2})_4 \cdot (F^{n-3})_4$  which is contained in  $(f^n)_4$ , and that  $(f^n)_4$  has a unique leading monomial, which is the product of the unique leading monomials in  $(F^{n-1})_4$ ,  $(F^{n-2})_4$  and  $(F^{n-3})_4$ . From this, we have immediately the linear recurrence:

$$d_n^{(1)} = d_{n-1}^{(1)} + d_{n-2}^{(1)} + d_{n-3}^{(1)}.$$

□

Finally, we prove proposition 2.1.

**Proof of proposition 2.1.** First, we have the known fact that the cohomology groups of  $X$  come from the pullback on cohomology of the embedding  $\iota : X \hookrightarrow \mathbb{P}^{2m}$ . [For the convenience of the readers, we briefly recall the arguments. By the Lefschetz hyperplane theorem, for  $j \leq 2m - 2$ , the pullback  $\iota^* : H^j(\mathbb{P}^{2m}) \rightarrow H^j(X)$  is an isomorphism. For  $j \geq 2m + 2$  one can use Poincare duality to determine the cohomology group. It remains to show that  $H^{2m-1}(X) = 0$ . This can be done by computing the Euler characteristic of  $X$ , and the latter can be done by using the normal bundle sequence and Whitney sum formula.]

In particular, this means the following:  $H^{2j+1}(X) = 0$  for all  $j$ ; if  $2j$  is even and  $\leq 2m - 2$  then  $H^{2j}(X)$  is generated by  $h^j$  where  $h = \iota^*(H)$  is the pullback of a hyperplane  $H$  in  $\mathbb{P}^{2m}$ , while the remaining groups are computed using Poincare duality. Hence  $H^*(X)$  is generated by  $h$ , and we have  $h^{2m-1} = H^{2m-1}.X = 2$ .

Thus, if  $\pi_1, \pi_2 : X \times X \rightarrow X$  are the two canonical projections, then the cohomology class of the diagonal of  $X$  is

$$\{\Delta_X\} = \frac{1}{2} \sum_{j=0}^{2m-1} \pi_1^*(h^j) \cdot \pi_2^*(h^{2m-1-j}).$$

Indeed, by Kunnetth’s formula and the description of the cohomology group of  $X$  from the previous paragraph, we have  $\{\Delta_X\} = \sum_{j=0}^{2m-1} a_j \pi_1^*(h^j) \cdot \pi_2^*(h^{2m-1-j})$ , for constants  $a_j$ . To determine  $a_i$ , we can do as follows. By the definition of the diagonal, we have  $(\pi_1)_*(\{\Delta_X\} \cdot \pi_2^*(h^i)) = h^i$  for all  $0 \leq i \leq 2m - 1$ . On the other hand, from the formula for  $\{\Delta_X\}$ , by the projection formula, we obtain easily  $(\pi_1)_*(\{\Delta_X\} \cdot \pi_2^*(h^i)) = (\pi_1)_*(a_i \pi_1^*(h^i) \cdot \pi_2^*(h^{2m-1-i}) \cdot \pi_2^*(h^i)) = a_i (h^{2m-1}) \cdot h^i$ . Since  $h^{2m-1}$  is 2, it follows that  $2a_i = 1$  for all  $i$ , and hence  $a_i = 1/2$  for all  $i$ .

From this, we get the following formula for the Lefschetz number of a map  $g : X \rightarrow X$ :

$$L(g) = \sum_{j=0}^{2m-1} g^* \left( \frac{1}{2} h^j \right) \cdot h^{2m-1-j}.$$

In fact, by the previous formula for  $\{\Delta_X\}$ , we have

$$L(g) = \{\Gamma_g\} \cdot \{\Delta_X\} = \sum_{j=0}^{2m-1} \frac{1}{2} \{\Gamma_j\} \cdot \pi_1^*(h^j) \cdot \pi_2^*(h^{2m-1-j}).$$

For each  $j$ , we have

$$\frac{1}{2} \{\Gamma_j\} \cdot \pi_1^*(h^j) \cdot \pi_2^*(h^{2m-1-j}) = \frac{1}{2} (\pi_2)_* (\{\Gamma_j\} \cdot \pi_1^*(h^j)) \cdot h^{2m-1-j}.$$

By definition, we have  $(\pi_2)_*(\{\Gamma_j\} \cdot \pi_1^*(h^j)) = g^*(h^j)$ , thus we obtain the claimed expression for  $L(g)$ . Each of the summands in the right hand side is  $> 0$ , and hence  $L(g) > 0$ . We denote by  $d_j(g) := g^*(\frac{1}{2}h^j) \cdot h^{2m-1-j}$  the  $j$ th degree of  $g$ .

Now, we will show that for any two dominant rational maps  $g_1, g_2 : X \dashrightarrow X$ , and any  $j \leq 2m - 1$  then  $d_j(g_1 \circ g_2) \leq d_j(g_1)d_j(g_2)$ . To this end, we follow the ideas in [32, 57] of using automorphisms of  $X$  to regularise positive closed currents. We recall that the quadric  $X$  is a homogeneous space, whose automorphism group is the subgroup of the linear automorphisms of  $\mathbb{P}^{2m}$  preserving the quadratic form. Hence, by using a convolution process with the aid of the Haar measure of the automorphism group of  $X$ , for any positive closed  $(j, j)$  current  $T$  on  $X$ , there is a sequence of positive closed smooth  $(j, j)$  forms  $T_\epsilon$  weakly converging to  $T$  so that (recall that the cohomology of  $X$  is generated by  $h$ ) in cohomology  $\{T_\epsilon\} = \{T\}$ .

Now we recall how  $d_j(g_1 \circ g_2)$  can be computed. There is a Zariski open set  $Z \subset X$  so that if we choose generic algebraic varieties in  $X$ :  $V_j$  of codimension  $j$  representing the cohomology class  $h^j$  and  $W_{2m-j}$  of codimension  $2m - j$  representing the cohomology class  $h^{2m-j}$ , then

$$d(g_1 \circ g_2) = (g_2|_Z)^{-1} (g_1^{-1}(V_j/2)) \cdot W_{2m-j}.$$

Here  $g_1|_Z$  is a proper map of finite fibres, and hence the preimage of  $g_1|_Z$  of any variety is again a variety of the same dimension. Moreover, we have the following property: If  $T_\epsilon$  is a sequence of positive closed smooth forms weakly converging to the current of integration over  $g_1^{-1}(V_j)/2$  and being of the same cohomology class as that of  $g_1^{-1}(V_j)/2$  (which is proven in the above), then

$$(g_2|_Z)^{-1} (g_1^{-1}(V_j/2)) \cdot W_{2m-j} \leq \lim_{\epsilon \rightarrow 0} g_2^* (\{T_\epsilon\}) \cdot h^{2m-j},$$

because  $g_2^* (\{T_\epsilon\}) \cdot h^{2m-j}$  is the cohomology class of  $g_2^*(T_\epsilon) \cdot W_{2m-j}$ , and the latter is a positive closed current for all  $\epsilon$ , and any cluster point of  $\{g_2^*(T_\epsilon) \cdot W_{2m-j}\}$  (when  $\epsilon$  decreases to 0) will coincide with  $(g_2|_Z)^{-1} (g_1^{-1}(V_j/2)) \cdot W_{2m-j}$  when restricted to  $Z$ . Since

$$\begin{aligned} \{T_\epsilon\} &= \{g_1^*(V_j/2)\} = g_1^*(h^j/2) \\ &= [g_1^*(h^j/2) \cdot h^{2m-j}] / 2 \cdot h^j = d_j(g_1) \frac{h^j}{2}, \end{aligned}$$

we obtain

$$(g_2|_Z)^{-1}(g_1^{-1}(V_j/2)) \cdot W_{2m-j} \leq d_j(g_1) g_2^*(h^j/2) \cdot h^{2m-j} = d_j(g_1) d_j(g_2).$$

Therefore, the degree sequences are log concave. From this, we obtain

$$\begin{aligned} L(g_1 \circ g_2) &= \sum_j d_j(g_1 \circ g_2) \leq \sum_j d_j(g_1) d_j(g_2) \\ &\leq \left( \sum_j d_j(g_1) \right) \left( \sum_j d_j(g_2) \right) = L(g_1) L(g_2). \end{aligned}$$

Applying for  $g_1 = g^n$  and  $g_2 = g^{n'}$  we obtain the log concavity  $L(g^{n+n'}) \leq L(g^n)L(g^{n'})$  needed. From this log concavity property, it is well known that the sequence  $n \mapsto L(g^n)^{1/n}$  is decreasing.  $\square$

**Remark 3.5.** One can also prove proposition 2.1 by a purely algebraic proof, which replaces regularisation of currents by Chow’s moving lemma and which is valid on any algebraically closed field, see [62].

### 3.2. Experimental results

In this subsection we calculate the fixed point sets of the iterates  $f_c^n$ , as well as the degree sequences  $d_n^{(1)}$ ,  $d_n^{(2)}$  and  $d_n^{(3)}$ , for as large as possible  $n$ ’s, to help study conjecture 1.5. We also study how close the degree sequence  $d_n^{(2)}$  and  $d_n^{(3)}$  are to the linear recurrence  $v_n = 3/2 \cdot (v_{n-1} - v_{n-3}) + 2v_{n-2}$  related to the polynomial  $t^3 - 3/2 \cdot (t^2 - 1) - 2t$  (for which  $\zeta_2$  is the largest root). These calculations will be used in the next subsection, where we propose an approach towards solving conjecture 1.5. Here we recall the relevant relations from lemmas 3.3 and 3.4:

$$\begin{aligned} \lim_{n \rightarrow \infty} [d_n^{(1)}]^{1/n} &= \zeta_1 = \lambda_1(\widehat{F}) = \lambda_1(\widehat{f}_c), \\ \lim_{n \rightarrow \infty} [d_n^{(3)}]^{1/n} &= \lambda_3(\widehat{F}) = \lambda_2(\widehat{f}_c), \\ \lim_{n \rightarrow \infty} [d_n^{(2)}]^{1/n} &= \lambda_2(\widehat{F}) = \max \{ \lambda_1(\widehat{F}), \lambda_3(\widehat{F}) \} = \max \{ \lambda_1(\widehat{f}_c), \lambda_2(\widehat{f}_c) \}. \end{aligned}$$

Formal computer algebra techniques (such as Gröbner basis routine on the softwares Mathematica and Maple) can only compute up to about  $N = 5$ . Hence, we utilize regeneration [44–46] and the trace test [43, 52, 59] implemented in Bertini [1]. Although numerical routines, the trace test provides a high level of confidence with reliable performance. Moreover, parallel computing can be used to speed up the computations. We note that the computations for finding the periodic points are more expensive and difficult than those for calculating the degree sequences.

The first table reports on the calculations for periodic points of  $f_c$  (up to period  $n = 12$ ), as well as the exponential growth of isolated periodic points. All the isolated periodic points are hyperbolic.

$N$	Fixed points on general fiber	$[\#\text{IsoFix}_N(f_c)]^{1/N}$
1	4	<b>4</b>
2	$C$ (occurring with multiplicity 1)	0
3	10	<b>2.15443469003</b>
4	$D_1$ (multiplicity 1) & $D_2$ (multiplicity 2)	0
5	44	<b>2.13152551327</b>
6	$C$ (multiplicity 1) AND 12 points	1.51308574942
7	186	<b>2.10967780991</b>
8	$D_1$ (multiplicity 1) & $D_2$ (multiplicity 2) AND 128 points	1.83400808641
9	820	<b>2.10744910267</b>
10	$C$ (multiplicity 1) AND 1440 points	2.06936094886
11	3634	<b>2.10703309279</b>
12	$D_1$ (multiplicity 1) & $D_2$ (multiplicity 2) AND 6908 points	2.08903649661

The next table computes the degree sequence for the iterates  $F^n$  (up to  $n = 14$ ), as well as their exponential growth:

$N$	$d_N^{(1)}$	$d_N^{(2)}$	$d_N^{(3)}$	$[d_N^{(1)}]^{1/N}$	$[d_N^{(2)}]^{1/N}$	$[d_N^{(3)}]^{1/N}$
1	3	5	3	3	5	3
2	5	9	7	2.2360679775	3	2.64575131106
3	9	25	17	2.08008382305	2.92401773821	2.57128159066
4	17	49	37	2.03054318487	2.64575131106	2.46632571456
5	31	109	79	1.98734075466	2.55555539674	2.39621299048
6	57	225	167	1.96175970274	2.46621207433	2.34667391139
7	105	477	353	1.94420174432	2.41348988334	2.3118934527
8	193	1005	745	1.93061049898	2.37285258221	2.28570160944
9	355	2117	1571	1.92025412137	2.34166378698	2.26532588341
10	653	4465	3311	1.91201510161	2.31729938473	2.24903346712
11	1201	9401	6977	1.90527844956	2.29719383004	2.23575612581
12	2209	19817	14701	1.8996910486	2.28079626154	2.22473817189
13	4063	41741	30975	1.89497551023	2.2668767672	2.2154523255
14	7473	87961	65263	1.89094202127	2.25508846088	2.2075207175

We can see from the above two tables that, as predicted by conjecture 1.5, the exponential growth of the degree sequence is much larger than that of the isolated periodic points. From the table for the degree sequence above, we can readily check that the sequence  $d_n^{(1)}$ 's indeed satisfies the linear recurrence  $w_n = w_{n-1} + w_{n-2} + w_{n-3}$ , as proven in lemma 3.4. If conjecture 1.5 holds, then we must have  $\lambda_2(\widehat{F}) = \lambda_3(\widehat{F})$ , and a first idea towards actually showing that  $\lambda_2(\widehat{F}) = \lambda_2(\widehat{F}) = \zeta_2$  is to show that both sequences  $d_n^{(2)}$ 's and  $d_n^{(3)}$ 's satisfy the linear recurrence  $w_n = 3/2 \cdot (w_{n-1} - w_{n-3}) + 2w_{n-2}$ . It turns out that neither of these sequences satisfies this linear recurrence, but the next two tables show that the differences to this linear recurrence are relatively small (in comparison to the size of the concerned degree sequences).

$N$	$d_N^{(2)}$	$3/2 \cdot (d_{N-1}^{(2)} - d_{N-3}^{(2)}) + 2d_{N-2}^{(2)}$	$d_N^{(2)} - [3/2 \cdot (d_{N-1}^{(2)} - d_{N-3}^{(2)}) + 2d_{N-2}^{(2)}]$
1	5		
2	9		
3	25		
4	49	48	1
5	109	110	-1
6	225	224	1
7	477	482	-5
8	1005	1002	3
9	2117	2124	-7
10	4465	4470	-5
11	9401	9424	-23
12	19817	19856	-39
13	41741	41830	-89
14	87961	88144	-183

  

$N$	$d_N^{(3)}$	$3/2 \cdot (d_{N-1}^{(3)} - d_{N-3}^{(3)}) + 2d_{N-2}^{(3)}$	$d_N^{(3)} - [3/2 \cdot (d_{N-1}^{(3)} - d_{N-3}^{(3)}) + 2d_{N-2}^{(3)}]$
1	3		
2	7		
3	17		
4	37	35	2
5	79	79	0
6	167	167	0
7	353	353	0
8	745	745	0
9	1571	1573	-2
10	3311	3317	-6
11	6977	6991	-14
12	14701	14731	-30
13	30975	31039	-64
14	65263	65399	-136

3.3. Further analysis & a road map towards an affirmative answer to conjecture 1.5

In this Subsection we further analyze the experimental findings in the previous Subsection, in connection to conjecture 1.5. Since part 1 of conjecture 1.5 is solved by lemma 3.4, it remains to treat parts 2 and 3 of conjecture 1.5.

3.3.1. An approach towards establishing the upper bound  $\lambda_2(\hat{f}_c) \leq \zeta_2$ . As seen from before, this is equivalent to establishing that

$$\lim_{n \rightarrow \infty} [d_n^{(3)}]^{1/n} \leq \zeta_2.$$

Here is an approach to showing this. From the last 2 tables in the previous Subsection, it seems very plausible that for  $n \geq 5$ , we should have  $d_n^{(3)} \leq 3/2 \cdot (d_{n-1}^{(3)} - d_{n-3}^{(3)}) + 2d_{n-2}^{(3)}$ . Also, it seems very evident that  $d_{n+1}^{(3)} \geq 2d_n^{(3)}$  for all  $n \geq 1$ . This is indeed enough to proving the desired upper bound, as seen in the next lemma.

**Lemma 3.6.** Assume that for all  $n \geq 5$ , we have

$$d_n^{(3)} \leq 3/2 \cdot (d_{n-1}^{(3)} - d_{n-3}^{(3)}) + 2d_{n-2}^{(3)},$$

$$d_n^{(3)} \geq 2d_{n-1}^{(3)}.$$

Then  $\lambda_3(\widehat{F}) \leq \zeta_2$ .

**Proof.** Because  $[d_n^{(3)}]^{1/n}$  is a decreasing sequence (see [57], and see also proposition 2.1), we have  $d_n^{(3)}/d_{n-1}^{(3)} \leq [d_{n-1}^{(3)}]^{1/(n-1)}$  for all  $n \geq 2$ . Therefore, from the assumption that  $d_n^{(3)} \geq 2d_{n-1}^{(3)}$  for all  $n \geq 5$ , we obtain

$$2 \leq \limsup_{n \rightarrow \infty} \frac{d_n^{(3)}}{d_{n-1}^{(3)}} \leq \lim_{n \rightarrow \infty} [d_{n-1}^{(3)}]^{1/(n-1)} = \lambda_3(\widehat{F}).$$

Hence, by lemmas 3.3 and 3.4, we have that  $\lambda_3(\widehat{F}) = \lambda_2(\widehat{F}) > \lambda_1(\widehat{F})$ . Since  $\lambda_1(\widehat{F^{-1}}) = \lambda_3(\widehat{F})$  and  $\lambda_2(\widehat{F^{-1}}) = \lambda_2(\widehat{F})$ , we have  $\lambda_1(\widehat{F^{-1}})^2 > \lambda_2(\widehat{F^{-1}})$ . Applying theorem 1 in [17] and the fact that the first degree sequence for  $F^{-1}$  is the same as the third degree sequence for  $F$ , it follows easily that for all  $j \geq 0$  we have

$$\lim_{n \rightarrow \infty} \frac{d_{n+j}^{(3)}}{d_n^{(3)}} = \lambda_3(\widehat{F})^j.$$

Assume that  $d_n^{(3)} \leq 3/2 \cdot (d_{n-1}^{(3)} - d_{n-3}^{(3)}) + 2d_{n-2}^{(3)}$  for all  $n \geq 5$ . Dividing  $d_{n-3}^{(3)}$  on both side of the inequality, and taking limit when  $n \rightarrow \infty$ , we obtain

$$\lambda_3(\widehat{F})^3 \leq 3/2 \cdot (\lambda_3(\widehat{F})^2 - 1) + 2\lambda_3(\widehat{F}).$$

Looking at the graph of the function  $t \mapsto t^3 - 3/2 \cdot (t^2 - 1) + 2t$  (for  $t \in [1, \infty)$ ), recalling that dynamical degrees of a map are  $\geq 1$ ), we see that the above inequality holds only if  $\lambda_3(\widehat{F}) \leq \zeta_2$ . □

Hence, one promising approach towards establishing  $\lambda_2(\widehat{f}_c) \leq \zeta_2$  is to answer in the affirmative the following question:

**Question 1.**

- a) Is it true that  $d_n^{(3)} \leq 3/2 \cdot (d_{n-1}^{(3)} - d_{n-3}^{(3)}) + 2d_{n-2}^{(3)}$  for  $n$  large enough?
- b) Is it true that  $d_n^{(3)} \geq 2d_{n-1}^{(3)}$  for  $n$  large enough?

**3.3.2. Two approaches towards establishing the lower bound  $\lambda_2(\widehat{f}_c) \geq \zeta_2$ .** Again, to prove that  $\lambda_2(\widehat{f}_c) \geq \zeta_2$  is the same as proving that  $\lambda_3(\widehat{F}) \geq \zeta_2$ , and also is the same as proving that  $\lambda_2(\widehat{F}) \geq \zeta_2$ .

To this end, we have two approaches. One is again to base on the last two tables in the previous Subsection, while the other is based on a new viewpoint. This new viewpoint gives even more support to that we should have  $\lambda_2(\widehat{f}_c) \geq \zeta_2$ .



**Approach 1:** As we mentioned, the last two tables seem to show that while  $d_n^{(3)}$  (as well as  $d_n^{(2)}$ ) does not satisfy the linear recurrence  $w_n = 3/2 \cdot (w_{n-1} - w_{n-3}) + 2w_{n-2}$ , it is very close to satisfying the linear recurrence. More precisely, the difference is relatively small, in the sense that:

$$\frac{d_n^{(3)} - 3/2 \cdot (d_{n-1}^{(3)} - d_{n-3}^{(3)}) - 2d_{n-2}^{(3)}}{d_n^{(3)}}$$

is small. This prompts us to ask the following question:

**Question 2.** Is it true that

$$\limsup_{n \rightarrow \infty} \frac{d_n^{(3)} - 3/2 \cdot (d_{n-1}^{(3)} - d_{n-3}^{(3)}) - 2d_{n-2}^{(3)}}{d_n^{(3)}} \geq 0?$$

We have the following result.

**Lemma 3.7.** Assume that question 2 has an affirmative answer. Then  $\lambda_3(\widehat{F}) \geq \zeta_2$ .

**Proof.** Since  $\lambda_1(\widehat{F}) > 1$ , it follows by properties of dynamical degree i.e. log-concavity) that also  $\lambda_3(\widehat{F}) > 1$ . Therefore, there cannot be a subsequence  $n_k$  of the set of positive integers so that  $d_{n_k-1}^{(3)} - d_{n_k-3}^{(3)} \leq 0$  for all  $k$ . From this and the assumption that the answer to question 2 is affirmative, we obtain

$$\limsup_{n \rightarrow \infty} \frac{d_n^{(3)}}{d_{n-2}^{(3)}} \geq 2.$$

Then we have  $\lambda_3(\widehat{F}) \geq \sqrt{2}$ . By lemmas 3.3 and 3.4,  $2 > \lambda_1(\widehat{F})$  and  $\lambda_2(\widehat{F}) = \max\{\lambda_1(\widehat{F}), \lambda_3(\widehat{F})\}$ . Therefore  $\lambda_3(\widehat{F})^2 > \lambda_2(\widehat{F})$ . As in the proof of lemma 3.6, we then obtain  $\lambda_3(\widehat{F})^3 \geq 3/2 \cdot (\lambda_3(\widehat{F})^2 - 1) + 2\lambda_3(\widehat{F})$ . Since the polynomial  $t^3 - 3/2 \cdot (t^2 - 1) - 2t$  has 3 real roots with approximate values  $-1.202, 0.591$  and  $\zeta_2 \sim 2.1108$ , while  $\lambda_3(\widehat{F}) \geq 1$  by definition, we conclude that we must have  $\lambda_3(\widehat{F}) \geq \zeta_2$ .  $\square$

Hence, a promising approach to establishing the lower bound  $\lambda_3(\widehat{F}) \geq \zeta_2$  is to solve in the affirmative question 2. However, Approach 2 below seems to have more evidence to support than this Approach 1.

**Approach 2:** In this approach, we compare the degree sequence  $d_n^{(3)}$ 's (respectively  $d_n^{(2)}$ 's) with the sequence  $b_n$ 's (correspondingly  $c_n$ 's). The sequence  $b_n$ 's satisfies the linear recurrence  $w_n = 3/2 \cdot (w_{n-1} - w_{n-3}) + 2w_{n-2}$  and has the first 3 initial values the same as that for the sequence  $d_n^{(3)}$ 's. Similarly, the sequence  $c_n$ 's satisfies the linear recurrence  $w_n = 3/2 \cdot (w_{n-1} - w_{n-3}) + 2w_{n-2}$  and has the first 3 initial values the same as that for the sequence  $d_n^{(2)}$ 's. From the experimental results in the previous Subsection, we get the following two tables.

$N$	$d_N^{(3)} = 3\text{rd degree of } F$	$b_N = 3/2 \cdot (b_{N-1} - b_{N-3}) + 2b_{N-2}$	Difference = $d_N^{(3)} - b_N$
1	3	3	0
2	7	7	0
3	17	17	0
4	37	35	2
5	79	76	3
6	167	158.5	8.5
7	353	337.25	15.75
8	745	708.875	36.125
9	1571	1500.0625	70.9375
10	3311	3161.96875	149.03125
11	6977	6679.765625	297.234375
12	14701	14093.4921875	607.5078125
13	30975	29756.81640625	1218.18359375
14	65263	62802.0560546875	2460.9439453125015

$N$	$d_N^{(2)} = 2\text{nd degree of } F$	$c_N = 3/2 \cdot (c_{N-1} - c_{N-3}) + 2c_{N-2}$	Difference = $d_N^{(2)} - c_N$
1	5	5	0
2	9	9	0
3	25	25	0
3	49	48	1
5	109	108.5	0.5
6	225	221.25	3.75
7	477	476.875	0.125
8	1005	995.0625	9.9375
9	2117	2114.46875	2.53125
10	4465	4446.515625	18.484375
11	9401	9406.1171875	-5.1171875
12	19817	19830.50390625	-13.50390625
13	41741	41888.216796875	-147.216796875
14	87961	88384.1572265625	-423.1572265625

It seems very evident that we should have  $d_n^{(3)} \geq b_n$  for all  $n \geq 1$ . This prompts us the following question.

**Question 3.** Is it true that we have  $d_n^{(3)} \geq b_n$  for all  $n \geq 1$ ? (In the proof, we only need  $d_n^{(3)} \geq \epsilon b_n$  for all  $n \geq 1$  and a constant  $\epsilon > 0$ .)

We have the following result.

**Lemma 3.8.** Assume that question 3 has an affirmative answer. Then  $\lambda_3(\hat{f}_c) \geq \zeta_2$ .

**Proof.** This follows easily from the fact that  $\lim_{n \rightarrow \infty} b_n^{1/n} = \zeta_2$ . □

(Note that, on the other hand, it seems that for  $n \geq 11$  then  $d_n^{(2)} \leq c_n$ . Using the latter inequality, we obtain only the upper bound  $\lambda_2(F) \leq \zeta_2$ , which is already discussed in the previous Subsubsection and not the lower bound wanted. However, one can ask whether  $d_n^{(2)} \geq \epsilon c_n$  for all  $n \geq 1$  and a constant  $\epsilon > 0$ . This inequality seems to be supported by the data, and is also enough to deduce that  $\lambda_3(\hat{f}_c) \geq \zeta_2$ .)

Hence, a promising approach towards showing  $\lambda_2(\widehat{f}_c) \geq \zeta_2$  is to solve question 3 in the affirmative.

3.3.3. *An approach to part 3 of conjecture 1.5.* We divide this into two tasks: one concerning the Weaker estimate (which provides a counter-example to conjecture 1.1) and one concerning the Stronger estimate (which provides a counter-example to conjecture 1.3).

**An approach towards the Weaker estimate in part 3 of conjecture 1.5:**

From the table for the number of isolated periodic points, we find that the sequence  $[\#\text{IsoFix}_{2n+1}(f_c)]^{1/(2n+1)}$  (for  $0 \leq n \leq 5$ ) is a **decreasing sequence**: 4, 2.15443469003, 2.13152551, 2.10967780991, 2.10744910267, and 2.10703309279. Also, we find that the fixed point set of  $f_c^{2n+1}$  (for  $0 \leq n \leq 5$ ) consists of isolated points only. These facts do not look like a random coincidence, and hence naturally lead to the following question:

**Question 4.** Is it true that the sequence  $[\#\text{IsoFix}_{2n+1}(f_c)]^{1/(2n+1)}$  (for  $n = 0, 1, 2, \dots$ ) is a decreasing sequence?

If question 4 has an affirmative answer, then since  $[\#\text{IsoFix}_9]^{1/9} = 2.1074\dots < 2.108$ , we obtain right away a proof of the Weaker estimate in part 3 of conjecture 1.5.

Here is a heuristic explanation for why question 4 can have an affirmative answer: We know from proposition 2.1 that the Lefschetz numbers  $\{L(\widehat{f}_c^n)^{1/n}\}$  for the map  $\widehat{f}_c : X_c \dashrightarrow X_c$  is decreasing. It seems that in this special case, we can localise this property to the map  $f_c = \widehat{f}_c|_{Z_c}$ . If this is so, and if we can show that the fixed point set of  $f_c$  consists of only isolated points (as seen in the experiments), then question 4 is solved in the affirmative (since then the Lefschetz number is the same as the number of fixed points).

**An approach towards the Stronger estimate in part 3 of conjecture 1.5:**

This part is probably more difficult to establish. Our clue is that from the table we observe the following phenomenon: for all  $0 \leq n \leq 5$ , we have  $[\#\text{IsoFix}_{2n+1}(f_c)]^{1/2n+1} \geq [\#\text{IsoFix}_{2n+2}(f_c)]^{1/2n+2}$ . Thus comes another question:

**Question 5.** Is it true that  $[\#\text{IsoFix}_{2n+1}(f_c)]^{1/2n+1} \geq [\#\text{IsoFix}_{2n+2}(f_c)]^{1/2n+2}$  for all  $n = 0, 1, 2, \dots$ ?

If question 5 has an affirmative answer, and if moreover the Weaker estimate in part 3 of conjecture 1.5 holds, then the Stronger estimate in part 3 of conjecture 1.5 follows readily.

Why question 5 should have an affirmative answer could be again contributed to a localisation of proposition 2.1. While the log concavity of the sequence  $\text{IsoFix}_n(f_c)$  is violated (see the next paragraph), still some parts could be preserved, allowing us to have affirmative answers to both questions 4 and 5.

In stead of question 5, the following variant is also enough for our purpose

**Question 6.** Is it true that  $\#\text{IsoFix}_{2n+1}(f_c) \geq \#\text{IsoFix}_{2n}(f_c)$  for all  $n = 1, 2, 3, \dots$ ?

3.3.4. *A speculation on the exponential growth of the isolated periodic points.* Another curious phenomenon, which we do not need in resolving conjecture 1.5, is that the sequence  $[\#\text{IsoFix}_{2n+2}(f_c)]^{1/2n+2}$  (for  $n = 0, 1, 2, \dots$ ) seems to be **increasing**. We do not know of a possible explanation for this interesting phenomenon.

However, if this increasing phenomenon is true, and the phenomena mentioned in questions 4 and 5 are also true, then we will obtain

$$\log 2.0890 \leq \lim_{n \rightarrow \infty} \frac{\log \#\text{IsoFix}_{2n}(f_c)}{2n} \leq \lim_{n \rightarrow \infty} \frac{\log \#\text{IsoFix}_{2n+1}(f_c)}{2n+1} \leq \log 2.1071,$$

and it is then reasonable to speculate that indeed

$$\lim_{n \rightarrow \infty} \frac{\log \#\text{IsoFix}_n(f_c)}{n}$$

also exists. (In this case, the limit must be contained in the interval  $[\log 2.0890, \log 2.1071]$ .)

#### 4. Discussion

Our paper is the first one to display a potential counter-example to a major folklore conjecture, conjecture 1.1, in complex Dynamical Systems. At the same time, it presents explicitly the first time a potential example of a polynomial birational map on affine 4-spaces whose dynamical degrees are algebraic numbers but not algebraic integers, thus is also a potential counter-example to another actively studied conjecture, that is conjecture 1.3. Even for the larger class of all dominant rational selfmaps of projective varieties, or compact Kähler manifolds, our maps are also the first potential counter-examples to the mentioned conjectures.

The methods in previous work are not applicable to analyse our maps, at least for the moment. There are three common approaches toward computing dynamical degrees, which involve establishing a linear recurrence between the degrees of the iterates. The first one (for the first dynamical degree  $\lambda_1$ ) is to observe directly a linear recurrence between the degrees of iterates of  $f$ . The second one is to construct a birational (or bimeromorphic) map  $\pi : Y \rightarrow X$  for which the lifting map  $f_Y$  satisfies a so-called algebraic stability under which  $\lambda_j(f)$  is the spectral radius of the pullback map  $f_Y^* : H^{j,j}(Y) \rightarrow H^{j,j}(Y)$ . In the current literature, all of these approaches are only able to show that the concerned is a dynamical degree integer, which is not the case of our map. (A generalization of this is the recent work [17] which shows that the pullback map on an associated Banach space  $RZ(Y)$  is stable. However, unlike on the finite dimensional cohomology groups, there is no guarantee that the eigenvalues on  $RZ(Y)$  are algebraic integers. See the next paragraph for more discussion on this method.) Finally, the third one is to use specialities of toric varieties and toric maps. A more recent method  $t$  in [6, 7] is to use Diophantine approximation, but this can only prove that the concerned dynamical degree is a transcendental number, again not the case for our maps.

Now we discuss more why the results in [17], which is a very strong result on computing dynamical degrees, are not yet applicable to our maps. For example, the proof of theorem 2 in [17]—which asserts that for a proper polynomial map of the complex affine space  $\mathbb{C}^d$  for which  $\lambda_1^2 > \lambda_2$ , we always have that  $\lambda_1$  is an algebraic number—cannot be used to establish part 2 of conjecture 1.5 for two reasons. First, while it seems true that  $\lambda_3(\widehat{F}) = \lambda_1(\widehat{F^{-1}})^2 > \lambda_2(\widehat{F^{-1}})$  (for example, if one can show that  $\lambda_3(\widehat{F}) > \sqrt{2}$ ), the map  $F^{-1}$  is not a polynomial map and hence the arguments used in theorem 6.1 in [17]—which rely on the valuation theory on the affine space—cannot be used. Second, even if the issue mentioned in the previous sentence can be resolved, it can be difficult to actually construct a matrix with rational coefficients whose spectral radius is exactly  $\zeta_2$  and  $\lambda_1(\widehat{F^{-1}})$  is also an eigenvalue.

Previous work only concentrated on confirming conjecture 1.1, and hence the developed techniques cannot be used to prove conjecture 1.5 which aims to disprove conjecture 1.1.

Also, as mentioned, symbolic methods as Gröbner basis can be too slow and not having enough features needed to analyse our maps.

To overcome the above disadvantages of the current techniques, we use the software Bertini instead of symbolic methods. Also, we observe that while it seems that the second degree

sequence of  $f_c$  does not satisfy any linear recurrence, there are some good comparisons (both lower and upper bounds) between the sequence and relevant linear recurrences. Since previous work has no mention on how to proceed if the number of periodic points of a map on a Zariski open set seems not as many as expected, we have to get around by looking whether the log-concavity phenomenon in proposition 2.1 still holds for a specific map and a specific Zariski open set.

## 5. Conclusions

In this paper, we presented a simple family of birational maps on smooth affine quadric 3-folds, coming from a polynomial map on  $\mathbb{C}^4$ , which seem to be cohomologically hyperbolic while having less periodic points than expected. Moreover, the second dynamical degree of these maps seem to be an algebraic number, but not an algebraic integer.

This kind of maps requires the development of new tools/ideas stronger than those in the current literature, see the previous Discussion section. Among the theoretical tools, it needs the development of more effective non-proper intersections of varieties (not only for projective varieties, but also for affine varieties), which help to establish various log concavity phenomena. It also requires taking into account periodic points located in proper analytic subvarieties. In dimension 2, only the case of bimeromorphic self-maps of surfaces has been dealt with in the literature [24, 49, 58]. In higher dimensions, for holomorphic maps or more general meromorphic maps of compact Kähler manifolds satisfying certain technical conditions (including a so-called algebraic stability and an assumption on the sequence of currents of integration associated to the graphs of iterates of the map), there have been work in [23, 64]. Note that since the first dynamical degree of our map  $\widehat{F}^{-1}$  is conjectured to be not an algebraic integer, there will be no birational model of its which is 1-algebraic stable.

Besides conjecture 1.3, another consequence of conjecture 1.1 is the following:

**Conjecture 5.1.** Let  $X$  be a smooth complex projective variety, and  $f: X \dashrightarrow X$  a dominant rational map. If  $f$  is cohomologically hyperbolic, then the set of isolated periodic points of  $f$  is Zariski dense.

Conjecture 5.1 is independent of the birational model of  $f$ . Besides the cases where conjecture 1.1 is known to be valid, one nontrivial case where conjecture 5.1 is solved in the affirmative is when  $X$  is of dimension 2 and  $f$  is a birational map [65]. (In this case,  $f$  being cohomologically hyperbolic is equivalent to  $\lambda_1(f) > 1$ , and the latter fact can be checked by a finite algorithm following the paper [22] mentioned above. Indeed, there is a sufficient criterion for  $\lambda_1(f) > 1$  in terms of some inequalities involving only several first terms in the degree sequence for  $f$ , see [65].) In light of the map considered in conjecture 1.5, it is interesting to check whether conjecture 5.1 holds for every cohomologically hyperbolic birational maps in dimension 3.

Concerning the equality for the Gromov–Yomdin’s and Dinh–Sibony’s bounds on the topological entropy of a map  $f$  in terms of its dynamical degrees, i.e.  $h_{\text{top}}(f) \leq \max \log \lambda_j(f)$ , [19] shows that a generic birational map of  $\mathbb{C}^d$  will obtain this bound. On the other hand, the proof therein uses the assumption that the maps concerned are algebraic stable (which is not satisfied for our map). The construction of the Banach space in [17] makes use of the Néron–Severi groups of birational models of the given map, and for topological entropy a similar consideration using topological entropies of birational models of the given map was proposed in [20]. It is interesting to see whether the ideas in [20] work for our map  $\widehat{f}_c$  (or  $\widehat{F}$ ). One can also try

to check if the more general approach of using étale covers of the map (but then, must work with correspondences in general) in [60] can work.

Among the experimental tools, one needs faster and less expensive (while still being highly reliable) methods to deal with numerical calculations for solving systems of equations, in particular those specially designed for compositions of simple maps. Indeed, the calculations presented in section 2 are more or less at the upper limit of what current calculation methods can afford us (even for the very simple map  $F$ ).

Since dynamical degrees can be defined over any algebraically closed field [16, 62], it is also meaningful to extend the study mentioned above to arbitrary algebraically closed fields.

### Data availability statement

There is no associated data with this paper. The data that support the findings of this study are available upon reasonable request from the authors.

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