Cosmological Perturbation Theory and Gravitational Entropy

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Chapter 1

Introduction

The aim of this thesis is to investigate how the entropy of a self-gravitating system evolves with time. From classical thermodynamics we know that gases that are inhomogeneous evolve towards being more homogeneous. This is explained by the second law of thermodynamics, which states that the entropy of a closed system tends towards a maximum. For a gas the entropy is maximal when it is homogeneous.

Consider a cosmological gas. If we introduce an inhomogeneity to such a gas, the pull of gravity will result in matter streaming away from the regions in the gas that are under-dense and towards the over-dense regions. This means that the gas becomes more inhomogeneous, which appears to contradict the second law of thermodynamics. The reason for this is that the effects of gravity are not taken into account when one calculates the classical entropy. By adding an additional term to the classical entropy that takes such effects into account, one imagines that the evolution of this general entropy quantity would be in accordance with the second law of thermodynamics. The classical entropy would then have to be replace by a total entropy quantity which is a sum of the ordinary entropy and a *gravitational entropy*. One of the first physicists who suggested this idea was Roger Penrose. He postulated a measure of the gravitational entropy in form of a mathematical quantity that was determined by the geometry of the space-time.

Our task in this thesis is to investigate how this postulated measure of gravitational entropy evolves with time for a perturbed Friedmann-Robertson-Walker (FRW) model. We introduce a localized inhomogeneity to an otherwise flat and homogeneous universe model and investigate how the total entropy evolves with time when this perturbation grows. As far as we know, no one has made such an analysis in terms of perturbed FRW models. We hope therefore that our analysis from this viewpoint will contribute to give a better understanding of the concept of gravitational entropy.

This thesis is diveded in two parts. The first part deals with perturbation theory while the second deals with gravitational entropy. In order to make the proposed analysis of perturbed FRW models we will need to have a good understanding of the theory of cosmological perturbations. We start therefore by introducing this theory in chapter 2. We look at important concepts such as the classification of the perturbations into scalar, vector and tensor perturbations and gauge dependence, and conclude the chapter by finding the equations that determine the perturbations. In chapter 3 we look at some special solutions for the perturbations for one-component, ideal gases. In chapter 4 we introduce Penrose's measure for gravitational entropy and also further motivation for why the concept of gravitational entropy is important. Finally, in chapter 5 we use the results from the first three chapters to determine the time evolution of both the classical and the gravitational entropy in a perturbed flat matter dominated FRW model for a special type of perturbations. We end this thesis with a summary and conclusions in chapter 6.

Part I

Cosmological Perturbation Theory

Chapter 2

Theory of cosmological perturbations

In this chapter we will present a thorough treatment on the theory of cosmological perturbations. Important topics such as gauge invariance and gauge choices are presented and explained.

We start by looking at the most general forms of gauges and perturbations. Later we'll specialize to the conformal Newtonian gauge, which is the most relevant one.

2.1 Introduction

The idea of the theory of cosmological perturbations is to describe the physical universe as a FRW universe plus a small perturbation.

The FRW universes are homogeneous and isotropic and give therefore a good description of the Universe at a large scale. But a homogeneous and isotropic Universe cannot explain the formation of structures such as stars and galaxies.

Cosmological perturbation theory is a theory which explains how such structures can be formed from very small inhomogeneities in an otherwise homogeneous universe. One assumes the universe to be homogeneous and isotropic to the zeroth order, i.e. that it obeys the Friedmann-Robertson-Walker line element to this order,

$$ds^2 = a^2(\eta)(d\eta^2 - \delta_{ij}dx^i dx^j)\,, \tag{2.1}$$

where η is conformal time and we have used units so that the speed of light $c = 1$. The conformal time relates to the usual comoving time, t, in the following way

$$a^2(\eta)d\eta^2 = dt^2 \quad \Rightarrow \quad t = \int_0^\eta a(\eta')d\eta' . \tag{2.2}$$

In the expression above, we have assumed a flat FRW universe. The reason for us not including the open and the closed FRW universes, is that recent experimental cosmological observations have pretty much confirmed that the geometry of the Universe is indeed flat to a high degree of accuracy. Data from the BOOMERanG balloon experiment [1], and also from the more recent WMAP satellite [2] both support this conclusion.

Inhomogeneities are introduced as a first order perturbation to this metric, $\delta g_{\mu\nu}$. Thus, the physical, inhomogeneous line element can be written as

$$ds^2 = ({}^{(0)}g_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu , \tag{2.3}$$

where ${}^{(0)}g_{\mu\nu}$ is the flat FRW metric.

The theory of cosmological perturbations was studied first by Lifshitz [3] in 1946. A comprehensive review of his work in English can be found in [4]. Our approach in this thesis will be based, first and foremost, on [5], [6] and [7]. Futher useful and more recent references on cosmological perturbations are [8], [9] and especially [10], which is based on the standard reference [5].

2.2 Classification of the metric perturbations

The line element (2.3) can be split into a time-time part, a time-space part and a space-space part,

$$ds^2 = a^2(\eta)\left\{(1+2\phi)d\eta^2 - 2w_i d\eta dx^i - (\delta_{ij} - h_{ij})dx^i dx^j\right\} . \tag{2.4}$$

This line element is split further into parts which are called *scalar*, *vector* and *tensor* components. The names given to the components tell us how they can be obtained. The scalar components can be obtained from a scalar function, the vector components from a vector function, while the tensor components cannot be obtained from either.

2.2.1 Decomposition of vectors and tensors

The decomposition of the perturbations into scalar, vector and tensor components is based on the mathematical fact that any three-vector can be split into a divergence-free part and a non-rotational part. Let $\mathbf{V}$ be some three-vector. Then we can write this as

$$\mathbf{V} = \mathbf{V}^{\parallel} + \mathbf{V}^{\perp} \quad \text{where} \quad \nabla\times\mathbf{V}^{\parallel} = \nabla\cdot\mathbf{V}^{\perp} = 0 . \tag{2.5}$$

Since $\mathbf{V}^{\|}$ has a vanishing curl, we can write it as the divergence of some scalar field, ϕ_V. Thus, any vector field can be written as the sum of a part which can be obtained from a scalar field and a part which cannot,

$$\mathbf{V} = \nabla\phi_V + \mathbf{V}^{\perp}\,. \tag{2.6}$$

The scalar part is also called the longitudinal part of the vector, while the divergence-free part is called the transverse or the vector part. The latter is called the vector part since it can be obtained as the curl of some vector potential.

One can perform a similar splitting of a trace-less, symmetric tensor. Let S_{ij} be such a tensor, then it can be written as [7, 11]

$$S_{ij} = S_{ij}^{\|} + S_{ij}^{\perp} + S_{ij}^{T}\,, \tag{2.7}$$

where the different parts satisfy the following constraints

$$\epsilon_{ijk}\partial_j\partial_l S_{lk}^{\|} = 0\,, \quad \partial_i\partial_j S_{ij}^{\perp} = 0\,, \quad \partial_i S_{ij}^{T} = 0\,. \tag{2.8}$$

We will not give a mathematical proof for either the splitting or the constraints, but we see immediately that they have a form which we would expect by applying the results we obtained for the three-vector on both of the indices of the tensor S_{ij}. This tells us that we should expect the tensor to be split into three different parts, namely one in which both indices are longitudinal ($S^{\|}$), one in which one index is longitudinal and the other transverse ($S^{\perp}$), and finally into one in which both indices are transverse (S^{T}).

Using the constraints (2.8), $S^{\|}$ and $S^{\perp}$ can be written as

$$\begin{aligned} S_{ij}^{\|} &= (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\mu\,, \\ S_{ij}^{\perp} &= \partial_i A_j + \partial_j A_i\,, \quad \partial_i A_i = 0\,, \end{aligned} \tag{2.9}$$

where μ is a scalar, while A_i is a vector quantity. The last term in the splitting (2.7), S^T, cannot be obtained from either a scalar nor a vector. This is therefore called the *tensor* part, hence the superscript "T". The first two parts are understandably called the scalar and vector parts.

2.2.2 Scalar, vector and tensor perturbations

We can now use the general results (2.6) and (2.9) for vectors and traceless tensors to decompose the metric (2.4) into scalar, vector and tensor perturbations. Such a decomposition was first proposed by Lifshitz [3].

The time-time component of the perturbation is already written as a scalar, so we can just leave it as it is. The time-space components are given by some three-vector, w_i. Using (2.6), we can split this into a scalar component and a vector component,

$$w_i = \partial_i B + S_i \,, \tag{2.10}$$

where B is some scalar functions and S_i is a divergence-free vector field.

In order to use the results in (2.9) on the space-space components of the metric perturbation, we must first separate these into a traceless part and a trace part,

$$h_{ij} = \frac{1}{3} h \delta_{ij} + h^{\mathrm{tl}}_{ij} \,, \tag{2.11}$$

where $h = \mathrm{Tr}\, h = \sum_i h_{ii}$ and h^{tl}_{ij} is traceless. (The superscript 'tl' stands for 'traceless'.) We can now use (2.7) and (2.9) on h^{tl}_{ij},

$$h^{\mathrm{tl}}_{ij} = (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\mu + \partial_i F_j + \partial_j F_i + h^T_{ij} \,, \tag{2.12}$$

where μ is some scalar function, F_i some divergence-free vector field and $\partial_i h^T_{ij} = 0$. The total space-space component of the metric perturbations is

$$h_{ij} = \frac{1}{3}(h - \nabla^2 \mu)\delta_{ij} + \mu_{,ij} + F_{i,j} + F_{j,i} + h^T_{ij} \,. \tag{2.13}$$

In order to be in agreement with the standard reference [5], we define two new scalar functions, ψ and E,

$$\frac{1}{3}(h - \nabla^2 \mu) \equiv 2\psi \quad \text{and} \quad \mu \equiv -2E \,. \tag{2.14}$$

Thus, the scalar metric perturbations are

$$\delta g^{\mathrm{scalar}}_{\mu\nu} = a^2(\eta) \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E_{,ij}) \end{pmatrix} \,, \tag{2.15}$$

the vector perturbations are

$$\delta g^{\mathrm{vector}}_{\mu\nu} = -a^2(\eta) \begin{pmatrix} 0 & S_i \\ S_i & F_{i,j} + F_{j,i} \end{pmatrix} \,, \tag{2.16}$$

while the tensor perturbations are

$$\delta g^{\mathrm{tensor}}_{\mu\nu} = a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h^T_{ij} \end{pmatrix} \tag{2.17}$$

How many degrees of freedom are there in the total metric perturbation? In the scalar perturbations there are four scalar functions and therefore four degrees of freedom. The vector perturbations have four degrees of freedom since they consist of two divergence-free three-vectors, and finally, there are two degrees of freedom in the tensor perturbations, since they are made up of a symmetric three-tensor of rank two with a vanishing three-divergence. Thus, there are ten degrees of freedom in all, just as we would expect.

Now that we've completed the decomposition of the perturbations into scalar, vector and tensor perturbations, one might ask why we do this. There are two good reasons for doing so, one is mathematical while the other is physical.

Considering the mathematical first, it turns out that the perturbed Einsteinian equations decouple into a scalar equation, vector equations and tensor equations. Each part evolves independently of the others, at least to the first order, and we need therefore only consider one at a time. If, for example, we are interested in how the scalar part of the perturbations evolve, we can simply set the vector and tensor perturbations equal to zero and get equations which determine the scalar functions completely.

Physically, there is also the advantage that the scalar, the vector and the tensor perturbations have different physical interpretations. The scalar perturbations are the only ones which affect the dynamics of the energy in the universe, and they are the only ones which can give gravitational collapse. The physical effect of the vector perturbations is that they give rise to vorticity. In an expanding universe they will always decay with time. Finally, the tensor perturbations give rise to gravitational waves.

Since our ultimate goal is to examine the gravitational entropy of a collapsing gas, we are only interested in those perturbations that yield gravitational collapse. Thus, we can disregard the vector and tensor perturbations. The perturbed metric of interest to us is therefore the perturbed scalar metric,

$$ds^2 = a^2(\eta)\left\{(1+2\phi)d\eta^2 - 2B_{,i}d\eta dx^i - [(1-2\psi)\delta_{ij} + 2E_{,ij}]dx^i dx^j\right\} . \tag{2.18}$$

2.3 Gauge dependence and transformations

In cosmological perturbation theory one deals with two different space-times or manifolds, one being the unperturbed background space-time, while the other is the perturbed, physical space-time. The quantities which we seek to find, namely the perturbed metric, the perturbed energy density and the

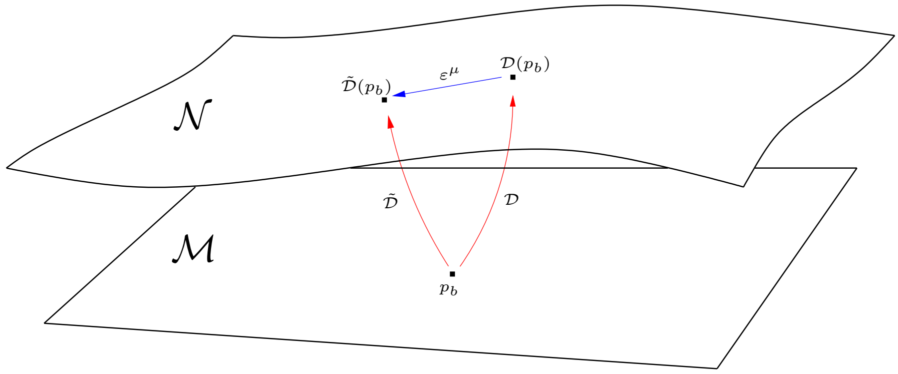


Figure 2.1: *A gauge transformation induces a coordinate transformation in the physical space-time* $\mathcal{N}$

perturbed four-velocity are expressed as differences in quantities in these two space-times. In order to relate quantities defined in these two different space-times, we must first define how points in the physical space-time relate to points in the background space-time. Such a definition of a correspondence of points in the background space-time to points in the physical space-time is called a *choice of gauge*. Mathematically, choosing a gauge means defining a diffeomorphism between the two manifolds which represent the two space-times. Equipped with such a diffeomorphism, we can now define what is meant by a perturbation of a quantity defined on these two manifolds.

Let $\mathcal{M}$ be the unperturbed space-time and $\mathcal{N}$ the perturbed space-time. Furthermore, let x_b^μ be a set of coordinates defined on $\mathcal{M}$. Any diffeomorphism from $\mathcal{M}$ into $\mathcal{N}$, $\mathcal{D}\colon\mathcal{M}\mapsto\mathcal{N}$, will induce a set of coordinates, $x^\mu=\mathcal{D}(x_b^\mu)$, on $\mathcal{N}$.

Let $p_b\in\mathcal{M}$ and Q some physical quantity defined in $\mathcal{N}$. Define ${}^{(0)}Q$ to be the same physical quantity in $\mathcal{M}$, then, per definition, the perturbation of Q is

$$\delta Q(p_b)=Q(p)-{}^{(0)}Q(p_b)\,,\quad p=\mathcal{D}(p_b)\,. \tag{2.19}$$

If we choose a different diffeomorphism, $\widetilde{\mathcal{D}}\colon\mathcal{M}\mapsto\mathcal{N}$, we'll induce a new set of coordinates on $\mathcal{N}$. The perturbation will also be different,

$$\delta\widetilde{Q}(p_b)=\widetilde{Q}(\tilde{p})-{}^{(0)}Q(p_b)\,,\quad \tilde{p}=\widetilde{\mathcal{D}}(p_b)\,. \tag{2.20}$$

In figure 2.1 we have illustrated such a change in diffeomorphism, which is usually called a *gauge transformation*. We see that this transformation induces a coordinate transformation in $\mathcal{N}$,

$$x^\mu\to x'^\mu=x^\mu+\varepsilon^\mu\,. \tag{2.21}$$

This can in turn be viewed as a coordinate transformation in $\mathcal{M}$ for a fixed diffeomorphism. Assume that we have chosen the diffeomorphism $\mathcal{D}$ as the correspondence between the two manifolds, i.e. $x^\mu = \mathcal{D}(x_b^\mu)$. A coordinate transformation in $\mathcal{M}$ will result in a coordinate transformation in $\mathcal{N}$,

$$x_b \to x_b' = x_b + \varepsilon_b \Rightarrow x \to x' = \mathcal{D}(x_b + \varepsilon_b) \approx x + \varepsilon_b \mathcal{D}'(x_b) \equiv x + \varepsilon \,. \quad (2.22)$$

Thus, if we want to study gauge transformations we need simply to look at infinitesimal coordinate changes in the unperturbed space-time $\mathcal{M}$, without having to bother with dealing with diffeomorphisms between different space-times.

The change in the perturbed quantity δQ under the coordinate transformation (2.21) is

$$\Delta \delta Q(p_b) = \delta \widetilde{Q}(p_b) - \delta Q(p_b) = \widetilde{Q}(p_b) - Q(p_b) \equiv -\mathcal{L}_\varepsilon Q \,, \quad (2.23)$$

where $\mathcal{L}_\varepsilon$ denotes the Lie derivative along the vector ε^μ. The minus sign in (2.23) arises from the fact that the Lie derivative is defined to be the change in a tensor quantity under the inverse coordinate transformation to (2.21),

$$x^\mu \to \tilde{x}^\mu = x^\mu - \xi^\mu \;\Rightarrow\; Q \to Q + \mathcal{L}_\xi Q \,. \quad (2.24)$$

In appendix A we calculate the Lie derivative of a general tensor of rank two. This is an expression which we will need a little later.

2.3.1 Scalar coordinate transformations

Since we restrict ourselves to studying scalar perturbations, we must make sure that the gauge transformations (2.21) only induce scalar changes in the metric. The infinitesimal change in coordinates, ε^μ, can be written as

$$\varepsilon^\mu = (\varepsilon^0, \varepsilon^i) \,, \quad (2.25)$$

where ε^i is some three-vector that can be decomposed into a scalar and a vector part,

$$\varepsilon_i = \partial_i \varepsilon + \varepsilon_i^{\parallel} \,. \quad (2.26)$$

The coordinate transformation (2.21) induces a change in the metric that is linear in ε^μ and partial derivatives of this (A.8). Thus, if we want the metric to preserve its scalar property after such a transformation, we must demand that $\varepsilon_i^{\parallel} = 0$. This leaves us with the following general scalar metric transformation

$$\eta \to \eta' = \eta + \varepsilon^0(\eta, x) \,, \; x^i \to x'^i = x^i + \delta^{ij} \varepsilon_{,j}(\eta, x) \,, \quad (2.27)$$

where ε^0 and ε are general, infinitesimal scalar functions. The new perturbed metric induced by this coordinate transformations is

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} - \mathcal{L}_\varepsilon g_{\mu\nu}\,, \tag{2.28}$$

where

$$\mathcal{L}_\varepsilon g_{\mu\nu} = g_{\mu\lambda}\varepsilon^\lambda_{,\nu} + g_{\lambda\nu}\varepsilon^\lambda_{,\mu} + g_{\mu\nu,\lambda}\varepsilon^\lambda\,. \tag{2.29}$$

The components are calculated in appendix B. The values are

$$[\mathcal{L}_\varepsilon g]_{00} = 2a^2\dot{\varepsilon}^0 + 2a\dot{a}\varepsilon^0 \tag{2.30}$$

$$[\mathcal{L}_\varepsilon g]_{0i} = a^2\left(\varepsilon^0 - \dot{\varepsilon}\right)_{,i} \tag{2.31}$$

$$[\mathcal{L}_\varepsilon g]_{ij} = -2a^2(\varepsilon_{,ij} + \frac{\dot{a}}{a}\varepsilon^0\delta_{ij})\,, \tag{2.32}$$

Inserting these components into (2.28) with $\mu\nu = 00, 0i$ and ij respectively, we get

$$2a^2\widetilde{\phi} = 2a^2\phi - 2a^2\dot{\varepsilon}^0 - 2a\dot{a}\varepsilon^0 \tag{2.33}$$

$$-a^2\widetilde{B}_{,i} = -a^2 B_{,i} - a^2\left(\varepsilon^0 - \dot{\varepsilon}\right)_{,i} \tag{2.34}$$

$$2a^2(\widetilde{\psi}\delta_{ij} - \widetilde{E}_{,ij}) = 2a^2(\psi\delta_{ij} - E_{,ij}) + 2a^2(\varepsilon_{,ij} + \frac{\dot{a}}{a}\varepsilon^0\delta_{ij})\,. \tag{2.35}$$

Integration of these three equations yields the transformation of the scalar components of the metric under a scalar coordinate transformation. The result is the following transformation equations

$$\widetilde{\phi} = \phi - \dot{\varepsilon}^0 - \frac{\dot{a}}{a}\varepsilon^0\,,\ \widetilde{B} = B + \varepsilon^0 - \dot{\varepsilon}\,,\ \widetilde{\psi} = \psi + \frac{\dot{a}}{a}\varepsilon^0\,,\ \widetilde{E} = E - \varepsilon \tag{2.36}$$

In deriving these equations, we have set all integration constants equal to zero, which means physically that we choose the two coordinate systems to coincide at the initial time.

From the gauge dependent quantities which appear in (2.36), we can construct two gauge independent quantities, Φ and Ψ,

$$\Phi = \phi + \frac{1}{a}\frac{\partial}{\partial\eta}[(B - \dot{E})a]\,,\ \Psi = \psi - \frac{\dot{a}}{a}(B - \dot{E})\,. \tag{2.37}$$

The gauge independence of these quantities can be easily verified by using the transformation rules in (2.36).

2.3.2 Freedom of gauge choice

In the scalar metric (2.18) there are four perturbing functions, ϕ, ψ, B and E. But these are not all uniquely determined. As we have shown above, these change when we perform a scalar coordinate transformation (2.27). Since the functions that appear in this coordinate transformation, ε^0 and ε, are arbitrary, we can put two constraints on the metric perturbations by choosing the coordinate transformation appropriately. Different gauges are characterized by different choices of constraints on the metric perturbations. Later we will discuss two such gauges, namely the *synchronous gauge* and the *conformal Newtonian gauge.*

There is an other approach to cosmological perturbation theory which doesn't require one to choose a gauge. Instead, one works directly with gauge invariant quantities. The reason for doing this is that the metric perturbations are generally gauge dependent. This is analogous to potentials in electromagnetism: The potential ϕ and vector potential $\mathbf{A}$ are gauge dependent, while the magnetic field, $\mathbf{B}$, and the electric field, $\mathbf{E}$, which are derived from these potentials, are gauge independent. The reason for this is that the electric and the magnetic field are physical quantities, while the potentials are not. All quantities that correspond to some physical, measurable property must be gauge independent. In cosmology, the metric perturbations do not constitute some physical property, and are therefore gauge dependent. But by arranging the metric perturbations appropriately into quantities that can be interpreted physically, we get quantities that are gauge independent. In gauge invariant perturbation theory one works therefore only with quantities that have a physical interpretation. This guarantees that they are gauge independent and that the results one gets are unique.

There have been several attempts to formulate a gauge invariant perturbation theory over the past fourty year, e.g. by Hawking [12] and Olson [13]. But it was Bardeen [6] who first formulated the complete theory of gauge invariant cosmological perturbations in 1980.

When doing gauge invariant perturbation theory, one must find a sufficient set of gauge invariant quantities and then reformulate the equations using only these quantities. Any solution of this new set of equations will then automatically be gauge invariant.

So, how does one find these gauge invariant quantities? They are quantities that remain unchanged when we make the infinitesimal coordinate transformation (2.21). Using our definition of the Lie derivative, we know that a general tensor quantity is changed by a quantity equal to the Lie derivative along the vector field $-\varepsilon^\mu$ under the coordinate transformation (2.24). This leads us to **Stewart's lemma**: *A general tensor quantity is gauge invariant* *if and only if it has a vanishing Lie derivative along every infinitesimal vector field.*

Two such gauge invariant quantities are the Bardeen potentials, Φ_A and Φ_H. These quantities were introduced by Bardeen in [6], and are, up to a minus sign, equal to the quantities we introduced in (2.37),

$$\Phi = \Phi_A \quad \text{and} \quad \Psi = -\Phi_H \,. \tag{2.38}$$

We shift now our attention to the gauge dependent theory and take a closer look at two particular gauges.

2.3.3 The synchronous gauge

This was the gauge used by Lifshitz and the first cosmologists who dealt with the theory of cosmological perturbations. It is defined by the following two constraints on the scalar perturbations

$$\phi = B \equiv 0 \,. \tag{2.39}$$

The drawback of this gauge is that it is not uniquely defined by this requirement. There is still the freedom to make a further transformation and still stay within this gauge. In other words, the metric perturbations are not defined uniquely in this gauge. Thus, it is not clear what metric perturbations are real, physical perturbations and what are simply coordinate artifacts.

We shall now show this coordinate dependence explicitly. Let (η, x^i) and $(\widetilde{\eta}, \widetilde{x}^i)$ be two sets of synchronous coordinates. The synchronous gauge is determined uniquely if and only if these two sets of coordinates are equal.

The first constraint of the synchronous gauge requires that ϕ and $\widetilde{\phi}$ are equal to zero. Inserting this into (2.36) gives

$$\frac{\dot{a}}{a}\varepsilon^0 = -\dot{\varepsilon}^0 \;\Rightarrow\; -\frac{da}{a} = \frac{d\varepsilon^0}{\varepsilon^0} \,.$$

Integration of this expression yields

$$\varepsilon^0 = \frac{C_1(\mathbf{x})}{a} \,, \tag{2.40}$$

where $C_1(\mathbf{x})$ is an arbitrary function of spatial coordinates only. The second constraint dictates that we put $B = \widetilde{B} = 0$ in (2.36), which gives the following expression

$$\dot{\varepsilon} = \varepsilon^0 \;\Rightarrow\; \varepsilon = \int \varepsilon^0 d\eta \,.$$

We insert ε^0 from (2.40) and get

$$\varepsilon = C_1(\mathbf{x}) \int \frac{d\eta}{a} + C_2(\mathbf{x}) , \tag{2.41}$$

where, again, $C_2(\mathbf{x})$ is an arbitrary function of spatial coordinates only. Thus, the relation between the two sets of coordinates, (η, x^i) and $(\widetilde{\eta}, \widetilde{x}^i)$, is

$$\widetilde{\eta} = \eta + \frac{C_1(\mathbf{x})}{a} , \ \widetilde{x}^i = x^i + \delta^{ij} \left[C_{1,j}(\mathbf{x}) \int \frac{d\eta}{a} + C_{2,j}(\mathbf{x}) \right] . \tag{2.42}$$

This shows that the two sets of coordinates are not necessarily equal, and that by making an appropriate coordinate transformation we get another set of synchronous coordinates from an already existing one. This proves our claim that the synchronous gauge is not defined uniquely.

2.3.4 The conformal Newtonian gauge

The conformal Newtonian gauge is defined by the two constraints

$$B = E \equiv 0 . \tag{2.43}$$

Is this gauge uniquely defined or is it possible, just as in the synchronous gauge, to make a further coordinate transformation within it? To answer this question, we again define two sets of coordinates, (η, x^i) and $(\widetilde{\eta}, \widetilde{x}^i)$, and take these to be conformal Newtonian coordinates. The constraint $E = \widetilde{E} = 0$ determines ε uniquely,

$$\varepsilon = 0 , \tag{2.44}$$

while the other constraint, $B = \widetilde{B} = 0$, gives us

$$\varepsilon^0 = \dot{\varepsilon} = 0 . \tag{2.45}$$

Thus, the two sets of coordinates are identical,

$$\widetilde{\eta} = \eta , \ \widetilde{x}^i = x^i . \tag{2.46}$$

The Newtonian coordinates, and hence also the metric perturbations, are determined uniquely. This means that there are no coordinate effects in the resulting perturbations. All solutions found when working in this gauge are pure, physical solutions. This can also be seen directly by realizing that the two remaining metric perturbations, ϕ and ψ, are equal to the gauge independent quantities Φ and Ψ in this gauge. Because of this property of invariance of metric perturbations, we will be using the conformal Newtonian gauge from now on. The line element which we will be using will then take the following form

$$ds^2 = a^2(\eta) \left\{ (1 + 2\Psi) d\eta^2 - (1 - 2\Phi) \delta_{ij} dx^i dx^j \right\} . \tag{2.47}$$

2.4 The energy-momentum tensor

So far we have only considered perturbations in the metric tensor, i.e. geometric perturbations. However, geometry and energy are closely related to each other via Einstein's field equations of general relativity,

$$G^\mu_\nu = 8\pi G T^\mu_\nu \,, \tag{2.48}$$

where T^μ_ν is the energy-momentum tensor, and $G^\mu_\nu = R^\mu_\nu - \frac{1}{2} g^\mu_\nu R$ is the Einstein tensor. Thus, a perturbation in the metric must be matched by a similar perturbation in the energy-momentum tensor. Or, in other words, a perturbation in the geometry of space must be matched by a perturbation in the matter or energy that occupies that space.

The energy-momentum tensor that is used in cosmology is that of a hydrodynamical medium. For a perfect fluid without any anisotropic stress, this can be written as

$$T^\mu_\nu = (\rho + p) u^\mu u_\nu - p\delta^\mu_\nu \,, \tag{2.49}$$

where ρ is the energy density, p the pressure and u^μ is the four-velocity of the medium. Since the metric is just the FRW-metric to the zeroth order, which is co-moving to the medium, the spatial components of the four-velocity vanish to this order,

$$^{(0)}u^i = 0 \,. \tag{2.50}$$

The zeroth component of the unperturbed four velocity can be obtained from the line element (2.47) by setting $dx^i = 0$ and $\Phi = \Psi = 0$, which gives us

$$^{(0)}u^0 = \frac{d\eta}{ds} = a^{-1} \,. \tag{2.51}$$

We can use the Krönecker delta to write the total unperturbed four-velocity into one expression,

$$^{(0)}u^\mu = a^{-1}\delta^\mu_0 \,. \tag{2.52}$$

The total four-velocity can be written as a perturbation to this non-perturbed velocity,

$$u^\mu = a^{-1}\delta^\mu_0 + \delta u^\mu \,. \tag{2.53}$$

This expression has to satisfy the four-velocity identity, $g_{\mu\nu}u^\mu u^\nu = 1$, which puts a constraint on the perturbed components. Insertion into the four-velocity identity gives

$$\begin{aligned}
g_{\mu\nu}u^\mu u^\nu &= g_{\mu\nu}(a^{-1}\delta_0^\mu + \delta u^\mu)(a^{-1}\delta_0^\nu + \delta u^\nu) \\
&= g_{\mu\nu}(a^{-2}\delta_0^\mu\delta_0^\nu + a^{-1}\delta_0^\mu\delta u^\nu + a^{-1}\delta_0^\nu\delta u^\mu) \\
&= a^{-2}g_{00} + a^{-1}g_{o\nu}\delta u^\nu + a^{-1}g_{\mu 0}\delta u^\mu = a^{-2}g_{00} + 2a^{-1}g_{\mu 0}\delta u^\mu \\
&= a^{-2}g_{00} + 2a^{-1}g_{00}\delta u^0 + 2a^{-1}g_{0i}\delta u^i = a^{-2}g_{00} + 2a^{-1}g_{00}\delta u^0 \\
&= 1 + 2\Phi + 2a\delta u^0 \overset{!}{=} 1
\end{aligned}$$

Thus, the constraint on the four-velocity reads

$$\delta u^0 = -a^{-1}\Phi\,. \tag{2.54}$$

This expression doesn't involve the spatial components of the four-velocity, which therefore remain as parameters that have to be determined by use of Einstein's field equations. Finally, we are left with the following expression for the total four-velocity

$$u^0 = a^{-1}(1-\Phi) \quad \text{and} \quad u^i = \delta u^i\,. \tag{2.55}$$

Equipped with the four-velocity of the energy/matter content of the Universe, we can easily calculate the components of the energy-momentum tensor. Keeping only terms up to the first order, the time-time component becomes

$$\begin{aligned}
T_0^0 &= (\rho+p)u^0u_0 - p = (\rho+p)u^0 g_{0\mu}u^\mu - p \\
&= (\rho+p)a^{-1}(1-\Phi)g_{00}u^0 - p \\
&= (\rho+p)a^{-1}(1-\Phi)a^2(1+2\Phi)a^{-1}(1-\Phi) - p \\
&= (\rho+p)a^{-1}(1-\Phi)a(1+\Phi) - p = \rho\,.
\end{aligned}$$

The time-space components are

$$\begin{aligned}
T_i^0 &= (\rho+p)u^0u_i = (\rho+p)a^{-1}(1-\Phi)g_{i\mu}u^\mu \\
&= (\rho+p)a^{-1}(1-\Phi)g_{ij}u^j \\
&= (\rho+p)a^{-1}(1-\Phi)\left[-a^2\left((1-2\Phi)\delta_{ij} + 2E_{,ij}\right)\delta u^j\right] \\
&= -(\rho+p)a(1-\Phi)\delta u^i = -(\rho+p)a\delta u^i \\
&= -(\rho_0+p_0)a\delta u^i\,,
\end{aligned}$$

where we again have kept terms only up to the first order, and ρ_0 and p_0 are the unperturbed energy density and pressure, respectively.

Finally, the space-space components are

$$T^i_j = (\rho + p)u^i u_j - p\delta^i_j = (\rho + p)\delta u^i \delta u_j - p\delta^i_j \simeq -p\delta^i_j\,.$$

In summary, the energy-momentum tensor split in zeroth and first order parts is

$$^{(0)}T^0_0 = \rho_0\,, \qquad \delta T^0_0 = \delta\rho \tag{2.56}$$

$$^{(0)}T^0_i = 0\,, \qquad \delta T^0_i = -(\rho_0 + p_0)a\delta u^i \tag{2.57}$$

$$^{(0)}T^i_j = -p_0\delta^i_j\,, \qquad \delta T^i_j = -\delta p\delta^i_j\,. \tag{2.58}$$

2.4.1 The equation of state

The pressure, which appears in the energy-momentum, is determined by the equation of state of the medium. This is an equation which gives the pressure as a function of other physical quantities. In a general hydrodynamical medium, the pressure is a function of two quantities, namely the energy density, ρ, and the entropy per particle, S,

$$p = p(\rho, S)\,. \tag{2.59}$$

Fluctuations in the pressure will then arise from fluctuations in the energy density and the entropy per particle,

$$\delta p = \frac{\partial p}{\partial \rho}\delta\rho + \frac{\partial p}{\partial S}\delta S \overset{\text{def.}}{=} c_s^2\delta\rho + \frac{\partial p}{\partial S}\delta S\,, \tag{2.60}$$

where $c_s^2 = \frac{\partial p}{\partial \rho}$ is interpreted as the speed of sound in the medium when $\frac{\partial p}{\partial \rho} > 0$.

In a one-component ideal gas there are no perturbations in the entropy per particle. Such perturbations arise only as a result of interactions between different components of a multi-component gas. In this thesis we will only consider one-component ideal gases, namely either pure matter universes, pure radiation universes or a universe with only vacuum energy. We can therefore put $\delta S = 0$. These types of perturbations where the entropy per particle doesn't change are called adiabatic perturbations or, sometimes, curvature perturbations. Perturbations which arise from perturbations in the entropy per particle are called entropy perturbations or isocurvature perturbations. The latter form of perturbations will generally be present in a multi-component fluid.

The equation of state of an ideal gas is

$$p = w\rho\,, \tag{2.61}$$

where w takes the values 0, $\frac{1}{3}$ and -1 for matter, radiation and vacuum energy, respectively. Since there are no entropy perturbations, the speed of sound is simply $c_s^2 = w$, and the pressure perturbation is

$$\delta p = w\delta\rho\,. \tag{2.62}$$

2.5 Einstein's field equations

2.5.1 The Einstein tensor

In order to use Einstein's field equations to get the desired differential equations that govern the evolution of the perturbed quantities, we must first calculate the Einstein tensor. The Einstein tensor is expressed through the Ricci tensor R^μ_ν, which in turn is expressed through the Christoffel symbols $\Gamma^\mu_{\nu\lambda}$. The definitions of these two quantities are

$$R_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu}\,, \tag{2.63}$$

and

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\sigma}(g_{\nu\sigma,\lambda} + g_{\lambda\sigma,\nu} - g_{\nu\lambda,\sigma})\,. \tag{2.64}$$

We see that there are a lot calculations involved if we want to determine the Einstein tensor. Instead of doing this be hand, we will let a computer program do that for us. The program we will use for this is "Maple" with an additional package called "GRTensorII"[1]. This package is developed especially for performing calculations in general relativity. In appendix C the reader is guided through the steps taken in order to determine the components of the Einstein tensor up to the first order, in a universe described by a conformal Newtonian metric.

The computer program gives us the following zeroth order components

$$^{(0)}G^0_0 = \frac{3}{a^2}\mathcal{H}^2\,, \tag{2.65}$$

$$^{(0)}G^0_i = 0\,, \tag{2.66}$$

$$^{(0)}G^i_j = \frac{1}{a^2}\left[\mathcal{H}^2 + 2\dot{\mathcal{H}}\right]\delta^i_j \tag{2.67}$$

where $\mathcal{H}$ is a "Hubble type" parameter,

$$\mathcal{H} = \frac{\dot{a}}{a}\,. \tag{2.68}$$

[1]http://grtensor.phy.queensu.ca/

The first order components are

$$\delta G^0_0 = \frac{2}{a^2}\left[\nabla^2\psi - 3\mathcal{H}(\dot{\Psi} + \Phi\mathcal{H})\right]. \quad (2.69)$$

$$\delta G^0_i = \frac{2}{a^2}\left[\dot{\Psi} + \mathcal{H}\Phi\right]_{,i}. \quad (2.70)$$

$$\delta G^i_j = -\frac{2}{a^2}\left[\Big([\mathcal{H}^2 + 2\dot{\mathcal{H}}]\Phi + \ddot{\Psi} + 2\mathcal{H}\dot{\Psi} + \mathcal{H}\dot{\Phi} + \frac{1}{2}\nabla^2 D\Big)\delta^i_j - \frac{1}{2}D_{,ij}\right]. \quad (2.71)$$

where $D = \Phi - \Psi$.

2.5.2 Simplification of the equations, $\Phi = \Psi$

The tensor components we have calculated above can be simplified greatly by realizing that the two metric perturbations Φ and Ψ are equal. We will show that this will always be the case when the spatial part of the perturbed energy-momentum tensor is diagonal, i.e. when $\delta T^i_j \propto \delta^i_j$, which is the case when there is no shear in the hydrodynamical medium.

Consider the ij component of perturbed Einstein tensor (2.71) with $i \neq j$. According to Einstein's field equations, this must be proportional to δT^i_j, which vanishes for $i \neq j$. The off-diagonal elements of the spatial part of the perturbed Einstein tensor are $\delta G^i_j \propto D_{,ij}$ $(i \neq j)$. Thus, we get the following equation for D,

$$D_{,ij} = 0\,. \quad (2.72)$$

The solution to this homogeneous partial differential equation can be expanded in Fourier modes,

$$D(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} D(\mathbf{k}) e^{-i\mathbf{kx}}\,. \quad (2.73)$$

Double differentiation with respect to the coordinates x^i and x^j yields

$$\frac{\partial^2 D(\mathbf{x})}{\partial x^i \partial x^j} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} D(\mathbf{k})\,(-k_i k_j)\,e^{-i\mathbf{kx}}\,. \quad (2.74)$$

This expression vanishes if and only if each of the Fourier modes vanishes,

$$-k_i k_j D(\mathbf{k}) \stackrel{!}{=} 0\,, \quad (2.75)$$

which implies that

$$D(\mathbf{k}) = 0\,. \quad (2.76)$$

Insertion back into the expression (2.73) leads us to the desired result,

$$D(\mathbf{x}) = 0\,. \tag{2.77}$$

Thus, we get the important result that

$$\Phi(\mathbf{x}, \eta) = \Psi(\mathbf{x}, \eta)\,. \tag{2.78}$$

2.5.3 Simplified Einstein tensor

With the result (2.78), the line element can be written by use of only one perturbing function,

$$ds^2 = a^2(\eta)\left\{(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j\right\}. \tag{2.79}$$

The same goes for the perturbed Einstein tensor, which now simplifies to

$$\delta G^0_0 = \frac{2}{a^2}\left\{\nabla^2\Phi - 3\mathcal{H}(\dot{\Phi} + \mathcal{H}\Phi)\right\}, \tag{2.80}$$

$$\delta G^0_i = \frac{2}{a^2}\left\{\dot{\Phi} + \mathcal{H}\Phi\right\}_{,i}, \tag{2.81}$$

$$\delta G^i_j = -\frac{2}{a^2}\left\{(\mathcal{H}^2 + 2\dot{\mathcal{H}})\Phi + \ddot{\Phi} + 3\mathcal{H}\dot{\Phi}\right\}\delta^i_j. \tag{2.82}$$

2.5.4 The Einstein equations

We now have everything we need in order to compute the Einstein equations for our model. The energy-momentum tensor is listed in (2.56)-(2.58), while the components of the Einstein tensor can be found in (2.65)-(2.67) and (2.80)- (2.82). Using Einstein's field equations (2.48), we get the differential equations that govern the evolution of the energy density and metric perturbations.

We start with the zeroth order equations, which should simply yield the Friedmann equations. The time-time component is

$$\mathcal{H}^2 = \frac{8}{3}\pi G a^2 \rho_0\,. \tag{2.83}$$

The space-space components are

$$\mathcal{H}^2 + 2\dot{\mathcal{H}} = -8\pi G a^2 p_0 = -8\pi G a^2 w \rho_0\,, \tag{2.84}$$

where we have expressed the pressure in terms of the energy density through the equation of state (2.61).

Next, we consider the first order Einstein equations. The time-time component is

$$\nabla^2\Phi - 3\mathcal{H}(\dot{\Phi} + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho\,. \tag{2.85}$$

If we take the Newtonian limit of this equation, i.e we let $a \rightarrow 1$ and $\mathcal{H} \rightarrow 0$, it reduces to

$$\nabla^2\Phi = 4\pi G \delta\rho\,. \tag{2.86}$$

We recognize this equation as the Poisson equation for ordinary Newtonian gravity, where Φ is the gravitational potential due to the mass-inhomogeneity $\delta\rho$. This explains why this gauge is called the conformal Newtonian gauge.

Instead of using the perturbed energy density, it is often more convenient to use the quantity known as the density contrast, δ. This quantity is defined as

$$\delta = \frac{\delta\rho}{\rho_0}\,. \tag{2.87}$$

In terms of the density contrast, the time-time component of the Einstein equations can be written as

$$\nabla^2\Phi - 3\mathcal{H}(\dot{\Phi} + \mathcal{H}\Phi) = 4\pi G a^2 \rho_0 \delta\,. \tag{2.88}$$

We can use the zeroth order equation (2.83) to eliminate the unperturbed energy density and the scale factor from the right hand side of this equations. This leaves us with the following equation

$$\nabla^2\Phi - 3\mathcal{H}(\dot{\Phi} + \mathcal{H}\Phi) = \frac{3}{2}\mathcal{H}^2\delta\,. \tag{2.89}$$

The time-space components are

$$\left\{\dot{\Phi} + \mathcal{H}\Phi\right\}_{,i} = -\frac{3}{2}\mathcal{H}^2(1+w)a\delta u^i\,, \tag{2.90}$$

while the space-space components are

$$(\mathcal{H}^2 + 2\dot{\mathcal{H}})\Phi + \ddot{\Phi} + 3\mathcal{H}\dot{\Phi} = \frac{3}{2}\mathcal{H}^2 w\delta\,. \tag{2.91}$$

2.5.5 Solutions to the zeroth order equations

In order to solve the first order equations, which are the equations that really are of interest to us, we must first find the zeroth order quantities $\mathcal{H}$, a and ρ_0. These are determined by the zeroth order equations (2.83) and (2.84), which are simply the Friedmann equations for a flat universe model expressed in conformal time. We will now solve these equations.

A linear combination of (2.83) and (2.84) eliminates $\mathcal{H}^2$ from these equations,

$$\dot{\mathcal{H}} = -\frac{4}{3}\pi G\rho_0 a^2(1+3w)\,. \tag{2.92}$$

Further, we can use (2.83) to express the right hand side of this equation in terms of only $\mathcal{H}$,

$$\dot{\mathcal{H}} = -\frac{1}{2}\mathcal{H}^2(1+3w)\,. \tag{2.93}$$

An integration of both sides of this equation yields

$$\mathcal{H}^{-1} = \frac{1}{2}(1+3w)\eta + C_0\,, \tag{2.94}$$

where C_0 is a constant of integration which will be determined below.

The next step is to determine the scale factor, a. This is done by use of (2.94) along with the defining equation for $\mathcal{H}$, (2.68). An integration of the latter gives us the scale factor,

$$\int \frac{da}{a} = \int \mathcal{H} d\eta\,. \tag{2.95}$$

This is a rather simple separable first order differential equation, which yields

$$a = C_1\left(\frac{1}{2}(1+3w)\eta + C_0\right)^{\frac{2}{1+3w}}\,, \tag{2.96}$$

where, again, C_1 is some constant of integration. These constants can be determined by imposing appropriate normalization and boundary condition. For a Universe model where $w \neq -1$, we can impose the condition that a vanishes at $\eta = 0$. This implies that $C_0 = 0$. Furthermore, we impose the normalization that $a = 1$ when $\eta = \eta_0$. This allows us to write the scale factor and the "Hubble parameter" in the following simple form

$$a = \left(\frac{\eta}{\eta_0}\right)^{\frac{2}{1+3w}} \quad \text{and} \quad \mathcal{H} = \frac{2}{1+3w}\frac{1}{\eta}\,, \quad w \neq -1\,. \tag{2.97}$$

We can use these two expressions along with equation (2.83) to find the unperturbed energy density ρ_0. The result is

$$\rho_0(\eta) = \frac{3\eta_0^{\frac{4}{1+3w}}}{2\pi G}\eta^{-\frac{6(1+w)}{1+3w}}\,. \tag{2.98}$$

It is often more common to express cosmological quantities such as these in co-moving time rather than conformal time. We will therefore derive an expression that relates the co-moving time to conformal time. This expression is given by the integral equation in (2.2),

$$t = \int_0^\eta a(\eta')d\eta' = \eta_0^{-\frac{2}{1+3w}} \int_0^\eta \eta'^{\frac{2}{1+3w}} d\eta' = \frac{1+3w}{3(1+w)} \eta_0^{-\frac{2}{1+3w}} \eta^{\frac{3(1+w)}{1+3w}} . \tag{2.99}$$

The comoving time that corresponds to η_0 is

$$t_0 = \frac{1+3w}{3(1+w)} \eta_0 . \tag{2.100}$$

Using this expression, we can write (2.99) as

$$t = t_0 \left(\frac{\eta}{\eta_0}\right)^{\frac{3(1+w)}{1+3w}} , \tag{2.101}$$

or, if we instead want to express the conformal time as a function of comoving time,

$$\eta = \eta_0 \left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}} . \tag{2.102}$$

We insert this expression into (2.97) and arrive at an expression for the scale factor expressed in comoving time,

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} . \tag{2.103}$$

Having used expression (2.97), these results are valid only for Universe models where $w \neq -1$, i.e. they are not valid for a Universe which is dominated by vacuum energy. For such models we have to go back to expression (2.96), and choose an other value for C_0. We can no longer demand that $a(\eta = 0) = 0$. Setting $w = -1$ in (2.96) gives the following scale factor

$$a = \frac{C_1}{C_0 - \eta} , \tag{2.104}$$

while $\mathcal{H}$ becomes

$$\mathcal{H} = \frac{1}{C_0 - \eta} . \tag{2.105}$$

If we divide (2.105) with (2.104) we will get a constant. According to equation (2.83), this implies that ρ_0 is constant. Define the following constant

$$H_\Lambda = \sqrt{\frac{8\pi G \rho_0}{3}} . \tag{2.106}$$

Equation (2.83) can now be written as

$$\dot{a} = a^2 H_\Lambda \,. \tag{2.107}$$

This is a fairly simple equation to integrate. Again, we choose the normalization condition $a(\eta_0) = 1$. The result is

$$a = \frac{1}{1 - H_\Lambda(\eta - \eta_0)} \,. \tag{2.108}$$

Next, we find the relation between conformal and comoving time. This is given by the integral

$$\int_{t_0}^{t} dt' = \int_{\eta_0}^{\eta} a(\eta')d\eta' \,. \tag{2.109}$$

Inserting from (2.108) for $a(\eta)$ and performing the integration, we get the following result

$$t - t_0 = -\frac{1}{H_\Lambda} \ln\left(1 - H_\Lambda(\eta - \eta_0)\right) . \tag{2.110}$$

Inverting this expression, we arrive at

$$1 - H_\Lambda(\eta - \eta_0) = e^{-H_\Lambda(t - t_0)} \,. \tag{2.111}$$

Finally, we insert this expression into (2.108), which gives us the following scale factor for a vacuum dominated universe expressed in comoving time

$$a(t) = e^{H_\Lambda(t - t_0)} \,. \tag{2.112}$$

We can summerize the results for the three universe models which we consider in this thesis into the following simple expressions. The relation between conformal and comoving time is

$$\eta = \begin{cases} \eta_0 + \frac{1}{H_\Lambda}\left(1 - e^{-H_\Lambda(t - t_0)}\right) & \text{for } w = -1 \\ \eta_0 \left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}} & \text{for } w = 0 \,, \frac{1}{3} \end{cases} \tag{2.113}$$

The scale factor is

$$a(t) = \begin{cases} e^{H_\Lambda(t - t_0)} & \text{for } w = -1 \\ \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} & \text{for } w = 0 \,, \frac{1}{3} \end{cases} \tag{2.114}$$

2.6 Conservation of four-momentum

In addition to the field equations (2.89)- (2.91), it is often useful to find the set of equations that define the conservation of four-momentum. However, the Einstein equations automatically satisfy four-momentum conservation. Thus, the latter set of equations is not a new set of dynamic equations that have to be satisfied in addition to the Einstein equations. The reason that we want to derive these equations is that their form is simpler than that of the Einstein equations. This allows us to substitute some of the Einstein equations with an appropriate amount of four-momentum conservation equations, which results in a simpler set of equations to solve. Also, we can use these equations to verify whether a calculated solution to the Einstein equations is correct.

The condition for four-momentum conservation is stated by the requirement that the energy-momentum tensor must be divergence-free,

$$T^{\nu}_{\mu;\nu} = 0\,. \tag{2.115}$$

We can write out the left hand side of this equation by using the familiar formula for the covariant derivative of a tensor of rank two,

$$T^{\nu}_{\mu;\nu} = T^{\nu}_{\mu,\nu} + T^{\lambda}_{\mu}\Gamma^{\nu}_{\lambda\nu} - T^{\nu}_{\lambda}\Gamma^{\lambda}_{\mu\nu}\,. \tag{2.116}$$

The Christoffel symbols are calculated using the GRTensorII package for the computer program Maple. A transcript of these calculations can be found in appendix D.

2.6.1 Conservation of energy

The zeroth component of (2.115) is the conservation equation for energy,

$$T^{\nu}_{0;\nu} = T^{\nu}_{0,\nu} + T^{\lambda}_{0}\Gamma^{\nu}_{\lambda\nu} - T^{\nu}_{\lambda}\Gamma^{\lambda}_{0\nu} \equiv T_1 + T_2 + T_3\,. \tag{2.117}$$

The first term of this expression is

$$T_1 = T^{\nu}_{0,\nu} = T^{0}_{0,0} + T^{i}_{0,i}\,. \tag{2.118}$$

Using (2.56)-(2.58) and (2.61)-(2.62), we write this as

$$T_1 = \dot{\rho}_0(1+\delta) + \rho_0\dot{\delta} + a\rho_0(1+w)\delta u^{i}_{,i}\,. \tag{2.119}$$

The second term in (2.117) is

$$T_2 = T^{\lambda}_{0}\Gamma^{\nu}_{\lambda\nu} = T^{0}_{0}\Gamma^{0}_{00} + T^{0}_{0}\Gamma^{i}_{0i} + T^{i}_{0}\Gamma^{0}_{i0} + T^{i}_{0}\Gamma^{j}_{ij}\,. \tag{2.120}$$

The last two terms in this expression are of order two, and we can therefore disregard them. We end up with

$$\begin{aligned} T_2 &= T^0_0\Gamma^0_{00} + T^0_0\Gamma^i_{0i} \\ &= \rho_0(1+\delta)(\mathcal{H}+\dot\Phi) + 3(1+\delta)(\mathcal{H}-\dot\Phi) \\ &= 2\rho_0(2\mathcal{H}+2\mathcal{H}\delta-\dot\Phi) \end{aligned} \tag{2.121}$$

The third term in (2.117) is

$$T_3 = -T^\nu_\lambda\Gamma^\lambda_{0\lambda} = -\left\{T^0_0\Gamma^0_{00} + T^0_i\Gamma^i_{00} + T^i_0\Gamma^0_{0i} + T^i_j\Gamma^j_{0i}\right\} . \tag{2.122}$$

The terms in the middle of this expression can be disregarded since they are of second order. Thus, we are left with the following expression

$$\begin{aligned} T_3 &= -\left\{T^0_0\Gamma^0_{00} + T^i_j\Gamma^j_{0i}\right\} \\ &= -\Big\{\rho_0(1+\delta)(\mathcal{H}+\dot\Phi) - w\rho_0(1+\delta)\delta^i_j\delta^j_i(\mathcal{H}-\dot\Phi)\Big\} \\ &= -\rho_0(\mathcal{H}+\mathcal{H}\delta+\dot\Phi) + 3w\rho_0(\mathcal{H}+\mathcal{H}\delta-\dot\Phi) . \end{aligned} \tag{2.123}$$

Finally, we arrive at the expression for energy conservation by adding the three terms calculated above

$$\begin{aligned} T^\nu_{0;\nu} =& \dot\rho_0(1+\delta) + \rho_0\dot\delta + a\rho_0(1+w)\delta u^i_{,i} \\ &+ \rho_0(3\mathcal{H}+3\mathcal{H}\delta-3\dot\Phi) + 3w\rho_0(\mathcal{H}+\mathcal{H}\delta-\dot\Phi) = 0 . \end{aligned} \tag{2.124}$$

The zeroth order part of this equation is

$$\dot\rho_0 + 3\mathcal{H}\rho_0(1+w) = 0 , \tag{2.125}$$

while the first order part is

$$\dot\delta + a(1+w)\delta u^i_{,i} - 3\dot\Phi(1+w) = 0 . \tag{2.126}$$

We recognize the zeroth order equation as the energy conservation equation for the FRW models, as we would expect.

2.6.2 Conservation of momentum

The spatial components of equation (2.115) express conservation of momentum. We divide this expression into three terms and calculate each term independently,

$$T^\nu_{i;\nu} = T^\nu_{i,\nu} + T^\lambda_i\Gamma^\nu_{\lambda\nu} - T^\nu_\lambda\Gamma^\lambda_{i\nu} \equiv \widetilde{T}_1 + \widetilde{T}_2 + \widetilde{T}_3 . \tag{2.127}$$

The first term in this equation is

$$\widetilde{T}_1 = T^{\nu}_{i,\nu} = T^{0}_{i,0} + T^{j}_{i,j}\,, \tag{2.128}$$

which, when written out, produces the following expression

$$\begin{aligned}\widetilde{T}_1 &= -(1+w)(\dot{\rho}_0 a\delta u^i + \rho_0\dot{a}\delta u^i + \rho_0 a\dot{\delta u}^i) - w\rho_0\delta_{,i}\\ &= -(1+w)a(\dot{\rho}_0\delta u^i + \rho_0\mathcal{H}\delta u^i + \rho_0\dot{\delta u}^i) - w\rho_0\delta_{,i}\,.\end{aligned} \tag{2.129}$$

Next, we calculate the second term in (2.127). Symbolically, this can be written as

$$\widetilde{T}_2 = T^{\lambda}_{i}\Gamma^{\nu}_{\lambda\nu} = T^{0}_{i}\Gamma^{0}_{00} + T^{0}_{i}\Gamma^{j}_{0j} + T^{j}_{i}\Gamma^{0}_{j0} + T^{j}_{i}\Gamma^{k}_{jk}\,. \tag{2.130}$$

We insert the explicit components of the energy-momentum tensor and the Christoffel symbols into this expression, which yields

$$\begin{aligned}\widetilde{T}_2 &= -\rho_0 a(1+w)\delta u^i(\mathcal{H}+\dot{\Phi}) - 3\rho_0 a(1+w)\delta u^i(\mathcal{H}-\dot{\Phi})\\ &\quad - w\rho_0(1+\delta)\delta^j_i\Phi_{,j} - w\rho_0(1+\delta)\delta^j_i(\Phi_{,j}\delta_{jk} - \Phi_{,j}\delta_{kk} - \Phi_{,k}\delta_{kj})\\ &= 2w\rho_0\Phi_{,i} - 4\rho_0 a\mathcal{H}(1+w)\delta u^i\,.\end{aligned} \tag{2.131}$$

The last term in (2.127) is

$$-\widetilde{T}_3 = T^{\nu}_{\lambda}\Gamma^{\lambda}_{i\nu} = T^{0}_{0}\Gamma^{0}_{i0} + T^{0}_{j}\Gamma^{j}_{i0} + T^{j}_{0}\Gamma^{0}_{ij} + T^{j}_{k}\Gamma^{k}_{ij}\,. \tag{2.132}$$

When we write out this expression, we get

$$\begin{aligned}-\widetilde{T}_3 &= \rho_0(1+\delta)\Phi_{,i} - \rho_0 a(1+w)\delta u^j\delta^j_i(\mathcal{H}-\dot{\Phi})\\ &\quad + a\rho_0(1+w)\delta u^j\delta_{ij}(\mathcal{H} - 4\mathcal{H}\Phi - \dot{\Phi}) - w\rho_0\delta^j_k(\Phi_{,k}\delta_{ij} - \Phi_{,i}\delta_{kj} - \Phi_{,j}\delta_{ik})\\ &= \rho_0\Phi_{,i} + 3w\rho_0\Phi_{,i}\,.\end{aligned} \tag{2.133}$$

Going back to equation (2.127), we can write the equation for momentum conservation as the sum of the three terms calculated above. This gives us an expression where all terms contain a common factor that is either the zeroth order density or the derivative of this. If we use the zeroth order energy conservation equation, we can write the derivative of the zeroth order density in terms of the zeroth order density and the Hubble parameter. Thus, we get a common factor, ρ_0, in the expression for momentum conservation,

$$\begin{aligned}T^{\nu}_{i;\nu} &= -(1+w)a\left\{\rho_0\mathcal{H}(2-3w)\delta u^i + \rho_0\dot{\delta u}^i\right\}\\ &\quad - (1+w)\rho_0\Phi_{,i} - w\rho_0\delta_{,i} = 0\,.\end{aligned} \tag{2.134}$$

We see that this expression is a pure first order expression. Momentum conservation is satisfied trivially to the zeroth order.

The common factor ρ_0 can be factored out, which allows us to write the momentum conservations in a little more compact form,

$$a\mathcal{H}(1+w)(2-3w)\delta u^i + a(1+w)\dot{\delta u}^i + (1+w)\Phi_{,i} + w\delta_{,i} = 0\,. \tag{2.135}$$

Chapter 3

Solutions of the first order equations

In the previous chapter we derived the differential equations that determine the perturbed quantities δ, Φ and δu^i in the conformal Newtonian gauge. This was done in a general sense, i.e. no symmetries were assumed for the perturbations. The three perturbed quantities mentioned above are determined by the three coupled partial differential equations (2.89)-(2.91). Our goal in this chapter is to solve these differential equations.

We shall consider three types of hydrodynamical media. These are media which consist of only dust/matter, only radiation energy or only vacuum energy. We'll obtain the differential equations equivalent to (2.89)-(2.91) for each of these cases and solve them analytically where possible.

3.1 General solutions in the case of a dust dominated model

Although the solutions to the equations (2.89)-(2.91) will generally depend on the symmetries imposed on the perturbations, it turns out that in the case of the dust dominated model, a general solution can be found regardless of any symmetries. We will therefore start out by considering a dust dominated model, i.e. a model in which $w = 0$. This gives us the following scale factor and Hubble parameter

$$a = \left(\frac{\eta}{\eta_0}\right)^2 \quad \text{and} \quad \mathcal{H} = \frac{2}{\eta}\,. \tag{3.1}$$

We see immediately that the differential equation (2.91) decouples from the other differential equations, and we are left with a homogeneous differential equation for only the metric perturbation Φ. Inserting for for a and $\mathcal{H}$, the equation simplifies to

$$\ddot{\Phi} + \frac{6}{\eta}\dot{\Phi} = 0\,. \tag{3.2}$$

First, we integrate this differential equation with respect to $\dot{\Phi}$,

$$\int \frac{d\dot{\Phi}}{\dot{\Phi}} = -\int 6\frac{d\eta}{\eta}\,. \tag{3.3}$$

The solution to this integral equation is

$$\dot{\Phi} = -5C_2(\mathbf{x})\frac{1}{\eta^6}\,, \tag{3.4}$$

where is $C_2(\mathbf{x})$ is an integration constant with respect to η. A further integration of this expression yields the desired result

$$\Phi(\mathbf{x},\eta) = C_2(\mathbf{x})\frac{1}{\eta^5} + C_1(\mathbf{x})\,, \tag{3.5}$$

where, again, $C_1(\mathbf{x})$ is an integration constant with respect to η.

Next, we use (2.89) to determine the density contrast. For a dust dominated universe, this expression can be written as

$$\delta = \frac{\eta^2}{6}\left(\nabla^2\Phi - \frac{6}{\eta}\dot{\Phi} - \frac{12}{\eta^2}\Phi\right)\,. \tag{3.6}$$

When we insert the expression (3.5) for Φ, we get the following result

$$\delta(\mathbf{x},\eta) = \frac{1}{6}\left(\frac{1}{\eta^3}\nabla^2 C_2(\mathbf{x}) + \frac{18}{\eta^5}C_2(\mathbf{x}) + \eta^2\nabla^2 C_1(\mathbf{x}) - 12C_1(\mathbf{x})\right)\,. \tag{3.7}$$

The remaining perturbed quantity, δu^i, is determined by using the expression (2.90),

$$\delta u^i = -\frac{\eta_0^2}{6}\left[\dot{\Phi} + \frac{2}{\eta}\Phi\right]_{,i}\,. \tag{3.8}$$

This gives us the following expression for the perturbed four-velocity

$$\delta u^i = \eta_0^2\left[\frac{1}{2}\frac{\partial C_2(\mathbf{x})}{\partial x^i}\frac{1}{\eta^6} - \frac{1}{3}\frac{\partial C_1(\mathbf{x})}{\partial x^i}\frac{1}{\eta}\right]\,. \tag{3.9}$$

If we now look at the expression for the density contrast (3.7), we see that this consists of two types of solutions. The first type is those in which C_2 is zero, while C_1 is not. These solution grow with time, and hence, are called growing solutions, or the growing modes if we expand the solutions into Fourier modes and look at each mode separately.

The other type of solutions is those in which $C_1 = 0$ and $C_2 \neq 0$. These decay with time, and are therefore called the decaying solutions, or decaying modes. Since we are interested mainly in perturbations that produce gravitational clustering, we need only consider the growing modes. Thus, the solutions that are of interest to us simplify to the following

$$\Phi(\mathbf{x}, \eta) = C_1(\mathbf{x}) \,, \tag{3.10}$$

$$\delta(\mathbf{x}, \eta) = \frac{1}{6}\nabla^2 C_1(\mathbf{x})\eta^2 - 2C_1(\mathbf{x}) \,, \tag{3.11}$$

$$\delta u^i(\mathbf{x}, \eta) = -\frac{\eta_0^2}{3}\frac{\partial C_1(\mathbf{x})}{\partial x^i}\frac{1}{\eta} \,. \tag{3.12}$$

We can at this point draw a very important conclusion: The first of these expressions tells us that in a matter dominated model in which there is a clustering of matter or energy, the metric perturbation is constant in time. Since we have shown that the metric perturbation can be interpreted as the gravitational potential due to clustering, we can state equivalently that the gravitational potential remains constant when matter clusters together. This can also be seen from examining the acceleration, $\mathbf{g}$, of an object falling freely under the influence of gravity. In the Newtonian limit, the value of this quantity is given by the following Christoffel symbol [14]

$$g^i = -\Gamma^i_{\ 00} \,. \tag{3.13}$$

Inserting from (D.5), we get

$$g^i = -\Phi_{,i} \,. \tag{3.14}$$

Thus, a constant Φ leads to a vanishing gravitational acceleration and therefore; a constant gravitational potential.

Let's examine what these solutions look like when expressed in co-moving time instead of conformal time. We use the relation (2.113) and find that in a dust dominated universe, the co-moving time relates to the conformal time in the following manner: $\eta \propto t^{\frac{1}{3}}$. Inserted into expression (3.11), we find that the growing solution of δ increases proportionally to $t^{2/3}$.

3.1.1 Fourier decomposition

Often one is only interested in perturbations of a length scale that is either much smaller or much lager than the Hubble length. These two extremes can be examined by decomposing the solutions into Fourier modes and then limit one self to those modes which have a wavelength that is either much larger or much smaller than the Hubble length. The Hubble length in a dust dominated FRW model is

$$L_{\mathrm{H}} = \mathcal{H}^{-1} = \frac{\eta}{2} \sim \eta \,. \tag{3.15}$$

Consider the expression (3.11). We expand both the density contrast and the function $C_1(\mathbf{x})$ into Fourier modes,

$$\delta(\mathbf{x}, \eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta_{\mathbf{k}}(\eta) e^{-i\mathbf{k}\mathbf{x}} \tag{3.16}$$

and

$$C_1(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} C_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}} \,, \tag{3.17}$$

where $\mathbf{k}$ is the wave vector.

Using these expansions, the expression (3.11) can be written as a series of expressions for each of the Fourier modes. For a given mode $\mathbf{k}$, the Fourier mode for the density contrast is

$$\delta_{\mathbf{k}}(\eta) = -2C_{\mathbf{k}} \left[1 + \frac{k^2}{12} \eta^2 \right] . \tag{3.18}$$

The wavenumber k is inversely proportional to the co-moving wavelength of the mode,

$$k = \frac{2\pi}{\lambda} \sim \lambda^{-1} \,. \tag{3.19}$$

From equation (3.15) and (3.19) we see that

$$k\eta \sim \frac{L_{\mathrm{H}}}{\lambda} \,. \tag{3.20}$$

This tells us that the length scales of the Fourier modes are given by the expression $k\eta$. We see that $k\eta \gg 1$ is equivalent to $L_{\mathrm{H}} \gg \lambda$ and similarly $k\eta \ll 1$ implies that $L_{\mathrm{H}} \ll \lambda$. This means that modes with a large $k\eta$ are well inside the Hubble length, and conversely, modes with small $k\eta$ are much larger than the Hubble length.

Consider modes that are much smaller than the Hubble length. These will have a $k\eta$ that is much larger than unity. The expression in the brackets in (3.18) tends to $\sim \eta^2$ for these modes. While for modes that are much larger than the Hubble length, for which $k\eta \ll 1$, the same expression tends to a constant. Thus, the density contrast has the following asymptotic behaviour

$$\delta_{\mathbf{k}}(\eta) \propto \begin{cases} \text{constant} & , \text{for } k\eta \ll 1 \quad (\lambda \gg L_{\mathrm{H}}) \\ \eta^2 = t^{2/3} & , \text{for } k\eta \gg 1 \quad (\lambda \ll L_{\mathrm{H}}) \end{cases} , \tag{3.21}$$

while the metric perturbation is constant for all scales,

$$\Phi_{\mathbf{k}}(\eta) \propto \text{constant}, \; \forall\, k\eta\,. \tag{3.22}$$

3.1.2 Time evolution of a particular mode

We will now illustrate how a particular mode of the density contrast evolves with time. Consider a mode with a wave number k and which at a time η_0 is a pure sine wave,

$$\delta(\mathbf{x}, \eta_0) = \delta_0 \sin \mathbf{kx}\,. \tag{3.23}$$

To simplify matters somewhat, we shall restrict ourselves to a plane symmetric wave, i.e. we let $\mathbf{x} \to x$. The results we get apply also to e.g. cylindrical and spherical waves if we instead let $\mathbf{x} \to r$. We assume that we can write $C_1(x)$ on a similar form,

$$C_1(x) = A_k \sin kx\,. \tag{3.24}$$

The constant A_k is determined by use of the expression (3.7) with η equal to η_0, which gives us

$$A_k = -\frac{\delta_0}{2(1 + \frac{k^2}{12}\eta_0^2)}\,. \tag{3.25}$$

Finally, we arrive at the expression for the time evolution of the density contrast in this particular case by reinserting the expressions (3.24) and (3.25) into (3.11). The answer is

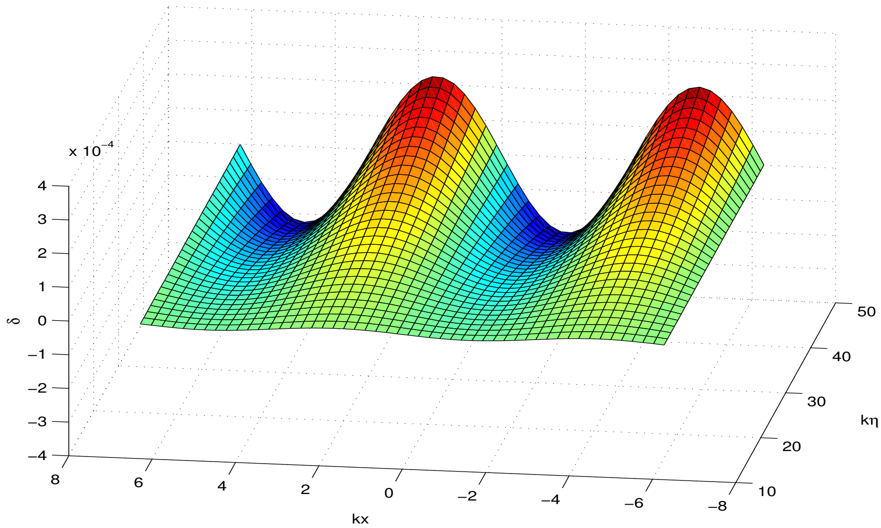
$$\delta(x, \eta) = \frac{\delta_0}{1 + \frac{k^2}{12}\eta_0^2}\left[1 + \frac{k^2}{12}\eta^2\right] \sin kx\,. \tag{3.26}$$

We can find a two-dimensional plot of this density perturbation in figure 3.1. The constants chosen in order to obtain this plot were $\delta_0 = 10^{-5}$ for the amplitude of the density perturbation, and $k\eta_0 = 10$ for the length scale. This choice of length scale insures that the solution is a time growing one, and not constant.

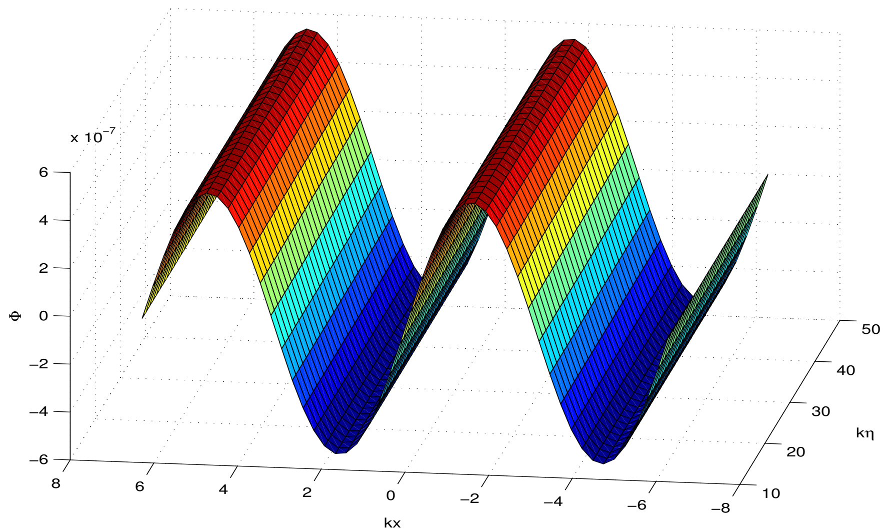
The metric perturbation that corresponds to the density perturbation (3.26) is

$$\Phi(x, \eta) = -\frac{\delta_0}{2(1 + \frac{k^2}{12}\eta_0^2)} \sin kx\,. \tag{3.27}$$

This too is plotted in figure 3.1.



(a)



(b)

Figure 3.1: (a) *A plot of the evolution of the density contrast in a matter dominated universe. The initial perturbation is a plane wave with amplitude* $\delta_0 = 10^{-5}$ *and* $k\eta_0 = 10$. (b) *The corresponding metric perturbation*

3.2 Pure vacuum energy perturbations

For vacuum energy, the ratio of pressure to energy density is $w = -1$. The zeroth order parameters that will appear in the Einstein equation (2.89)-(2.91) are

$$a = \frac{1}{1 - H_\Lambda(\eta - \eta_0)}\,, \tag{3.28}$$

$$\mathcal{H} = -\frac{H_\Lambda}{1 - H_\Lambda(\eta - \eta_0)}\,, \tag{3.29}$$

and

$$\dot{\mathcal{H}} = \frac{H_\Lambda^2}{\left(1 - H_\Lambda(\eta - \eta_0)\right)^2}\,. \tag{3.30}$$

The three Einstein equations can now be written as

$$\frac{1}{H_\Lambda^2}\left(1 - H_\Lambda(\eta - \eta_0)\right)^2 \nabla^2\Phi - \frac{3}{H_\Lambda}\left(1 - H_\Lambda(\eta - \eta_0)\right)\dot{\Phi} - 3\Phi = \frac{3}{2}\delta\,, \tag{3.31}$$

$$\frac{\partial}{\partial x^i}\left(\dot{\Phi} - \frac{H_\Lambda}{1 - H_\Lambda(\eta - \eta_0)}\Phi\right) = 0\,, \tag{3.32}$$

and

$$\frac{1}{H_\Lambda^2}\left(1 - H_\Lambda(\eta - \eta_0)\right)^2 \ddot{\Phi} - \frac{3}{H_\Lambda}\left(1 - H_\Lambda(\eta - \eta_0)\right)\dot{\Phi} + 3\Phi = -\frac{3}{2}\delta\,. \tag{3.33}$$

We can in principle solve these coupled differential equations, but in this particular case there is a much easier way to obtain the solutions. We know that any solution to these equations must also solve the equations for the four-momentum conservation. For a pure vacuum model these equations take a particularly simple form. The energy conservation equation (2.126) becomes

$$\dot{\delta} = 0\,, \tag{3.34}$$

while the momentum conservation equation (2.135) reduces to

$$\delta_{,i} = 0\,. \tag{3.35}$$

Equation (3.34) tells us that the density contrast cannot depend on time. Using the second equation (3.35), we must conclude that the density contrast cannot depend on spatial coordinates either. Thus, it must be a constant in both time and space,

$$\delta(\mathbf{x}, \eta) = \delta_0\,. \tag{3.36}$$

The total energy density, which can be written as

$$\rho(\mathbf{x},\eta) = \rho_0(\eta)(1+\delta(\mathbf{x},\eta))\,, \tag{3.37}$$

must then be homogeneous. In other words, the solution is the same as the vacuum dominated FRW model. The conclusion we are left with, is that there cannot be any inhomogeneous perturbations to the first order in a homogeneous universe model which contains only vacuum energy.

3.3 Perturbations of a radiation dominated universe model

In this section we turn our attention to the radiation dominated model. The equation of state for radiation expressed as a hydrodynamic fluid states that $w = \frac{1}{3}$. This value of w gives us the following zeroth order parameters

$$a = \frac{\eta}{\eta_0}\,, \quad \mathcal{H} = \frac{1}{\eta}\,, \quad \dot{\mathcal{H}} = -\frac{1}{\eta^2}\,. \tag{3.38}$$

Next, we insert these parameters into the general Einstein equations (2.89)-(2.91). The time-time component can be written as

$$\eta^2\nabla^2\Phi - 3\eta\dot{\Phi} - 3\Phi = \frac{3}{2}\delta\,, \tag{3.39}$$

while the space-time components are

$$\frac{\partial}{\partial x^i}\left(\eta\dot{\Phi} + \Phi\right) = -\frac{2}{\eta_0}\delta u^i\,. \tag{3.40}$$

Finally, the space-space components are

$$\eta^2\ddot{\Phi} + 3\eta\dot{\Phi} - \Phi = \frac{1}{2}\delta\,. \tag{3.41}$$

These equations cannot be solved generally in a closed, analytical form. We must instead expand the solutions into Fourier modes and study each mode separately. In terms of time-dependent Fourier modes, the three perturbations δ, Φ and δu^i are

$$\delta(\mathbf{x},\eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\delta_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}}\,, \tag{3.42}$$

$$\Phi(\mathbf{x},\eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Phi_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}}\,, \tag{3.43}$$

$$\delta u^i(\mathbf{x},\eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\delta u^i_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\mathbf{x}}\,. \tag{3.44}$$

We insert these into equations (3.39) -(3.41), which gives us the following equations for the Fourier modes

$$-\eta^2 k^2 \Phi_{\mathbf{k}} - 3\eta\dot{\Phi}_{\mathbf{k}} - 3\Phi_{\mathbf{k}} = \frac{3}{2}\delta_{\mathbf{k}}\,, \tag{3.45}$$

$$ik_i\eta\dot{\Phi}_{\mathbf{k}} + ik_i\Phi_{\mathbf{k}} = \frac{2}{\eta_0}\delta u^i_{\mathbf{k}}\,, \tag{3.46}$$

$$\eta^2\ddot{\Phi}_{\mathbf{k}} + 3\eta\dot{\Phi}_{\mathbf{k}} - \Phi_{\mathbf{k}} = \frac{1}{2}\delta_{\mathbf{k}}\,. \tag{3.47}$$

These equations in momentum space are simple ordinary differential equations which can be solved analytically. The first and the third equation can be combined into a differential equation for $\Phi_{\mathbf{k}}$ only. This new equation can be written as

$$\ddot{\Phi}_{\mathbf{k}} + \frac{4}{\eta}\dot{\Phi}_{\mathbf{k}} + \omega^2\Phi_{\mathbf{k}} = 0\,, \tag{3.48}$$

where we have defined the constant ω as

$$\omega^2 = \frac{k^2}{3}\,. \tag{3.49}$$

The differential equation (3.48) can be solved analytically, either by hand or by using a computer algebra program such as Maple. The latter gives the following general solution

$$\Phi_{\mathbf{k}}(\eta) = \frac{C_1}{\eta^3}(\omega\eta\cos\omega\eta - \sin\omega\eta) + \frac{C_2}{\eta^3}(\omega\eta\sin\omega\eta + \cos\omega\eta)\,, \tag{3.50}$$

where C_1 and C_2 are integration constants.

With Φ_k determined, we can simply use equations (3.47) and (3.46) to determine the Fourier components of the density contrast and the perturbed velocity. The density contrast is

$$\begin{aligned}\delta_{\mathbf{k}}(\eta) =& \frac{4}{\eta^3}C_1\left\{(\omega^2\eta^2 - 1)\sin\omega\eta + \omega\eta(1 - \frac{1}{2}\omega^2\eta^2)\cos\omega\eta\right\}\\ &+ \frac{4}{\eta^3}C_2\left\{(1-\omega^2\eta^2)\cos\omega\eta + \omega\eta(1 - \frac{1}{2}\omega^2\eta^2)\sin\omega\eta\right\}\,,\end{aligned} \tag{3.51}$$

while the perturbed velocity can be written as

$$\begin{aligned}\delta u^i_{\mathbf{k}} =& -\frac{i\eta_0 k_i}{\eta^3}C_1\left\{\omega\eta\cos\omega\eta + (\frac{1}{2}\omega^2\eta^2 - 1)\sin\omega\eta\right\}\\ &- \frac{i\eta_0 k_i}{\eta^3}C_2\left\{(1 - \frac{1}{2}\omega^2\eta^2)\cos\omega\eta + \omega\eta\sin\omega\eta\right\}\,.\end{aligned} \tag{3.52}$$

If we look at the expression for the density contrast (3.51), we see that this consists of terms that are proportional to 1, η^{-1}, η^{-2} and η^{-3}, times a trigonometric function. In contrast to when we had a matter dominated model, it is generally not possible to classify the solutions in growing and decreasing modes. Such a distinction can, however, be made in the asymptotic limits of the wavelength.

Consider a solution in which $C_2 = 0$ and $C_1 \neq 0$. Call this solution δ^+,

$$\delta^+ = \frac{4}{\eta^3} C_1 \left\{ (\omega^2\eta^2 - 1) \sin\omega\eta + \omega\eta(1 - \frac{1}{2}\omega^2\eta^2) \cos\omega\eta \right\} . \tag{3.53}$$

The product $\omega\eta$ is a measure of the length scales of the Fourier modes, just as $k\eta$ was for matter dominated model. If we use the fact that $\omega \sim k \sim \lambda^{-1}$ and that the Hubble length in a radiation dominated universe is of the same order as that in a matter dominated universe, $L_{\mathrm{H}} \sim \eta$, we can write

$$\omega\eta \sim k\eta \sim \frac{L_{\mathrm{H}}}{\lambda} . \tag{3.54}$$

In the long wavelength limit, i.e. when $\omega\eta \ll 1$, we can expand this solution in polynomials of $\omega\eta$. The lowest non-vanishing order in this expansion is found to be the third, which gives the following leading term behaviour

$$\delta^+ \simeq \frac{2}{3} C_1 \omega^3 . \tag{3.55}$$

The solution δ^+ is, in other words, constant in the long wavelength limit, or equivalently: it is constant for scales that are much larger than the Hubble length.

Now, consider the "opposite" solution, i.e. the solution in which $C_1 = 0$ and $C_2 \neq 0$. This we call δ^-,

$$\delta^- = \frac{4}{\eta^3} C_2 \left\{ (1 - \omega^2\eta^2) \cos\omega\eta + \omega\eta(1 - \frac{1}{2}\omega^2\eta^2) \sin\omega\eta \right\} . \tag{3.56}$$

Just as we did for δ^+, we examine how this solution behaves in the long wavelength limit by expanding it in polynomials of $\omega\eta$, and keeping only those terms which are of the leading non-vanishing order. Carrying out this expansion, we see that the lowest non-vanishing order is the zeroth. This allows us to write the solution as

$$\delta^- = \frac{4}{\eta^3} C_2 . \tag{3.57}$$

In physical time, we can write this as

$$\delta^- \sim C_2 t^{-\frac{3}{2}} , \tag{3.58}$$

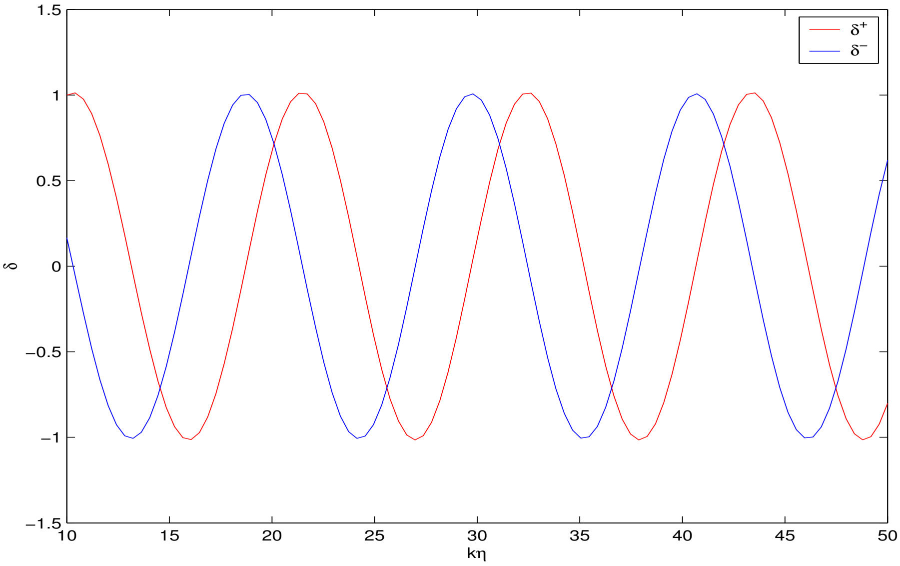


Figure 3.2: *A plot of the both types of solutions, δ^+ and δ^-, for $k\eta > 10$, i.e. inside the Hubble length. These solutions have been normalized to an amplitude of order one.*

where we have used (2.113), which gives the relation between physical and conformal time. For radiation this states that $\eta \sim \sqrt{t}$. Thus, we conclude that the the set of solutions which we call δ^- behave as decreasing solutions in the long wavelength limit, while the solutions δ^+ are non-decreasing solutions in the same limit. In the small wavelength limit and for scales up to the order of the Hubble length, neither of the solutions exhibit any particular behaviour that distinguishes them from each other. They will both be oscillating solutions. This is seen in figure 3.2, which is a plot of both $\delta^+_{\mathbf{k}}(\eta)$ and $\delta^-_{\mathbf{k}}(\eta)$ for $k\eta > 10$. In summary, the two solutions behave in the following way

$$\delta^+ \sim \begin{cases} \text{constant}\,, & \text{for } \omega\eta \ll 1 \quad (\lambda \gg L_{\mathrm{H}}) \\ \text{oscillating}\,, & \text{otherwise} \end{cases}\,, \tag{3.59}$$

and

$$\delta^- \sim \begin{cases} \eta^{-3}\,, & \text{for } \omega\eta \ll 1 \quad (\lambda \gg L_{\mathrm{H}}) \\ \text{oscillating}\,, & \text{otherwise} \end{cases}\,. \tag{3.60}$$

These expressions lead us to the important conclusion that in a radiation dominated universe model, perturbations in the density of radiation do not grow with time. Outside the Hubble horizon they either remain constant or

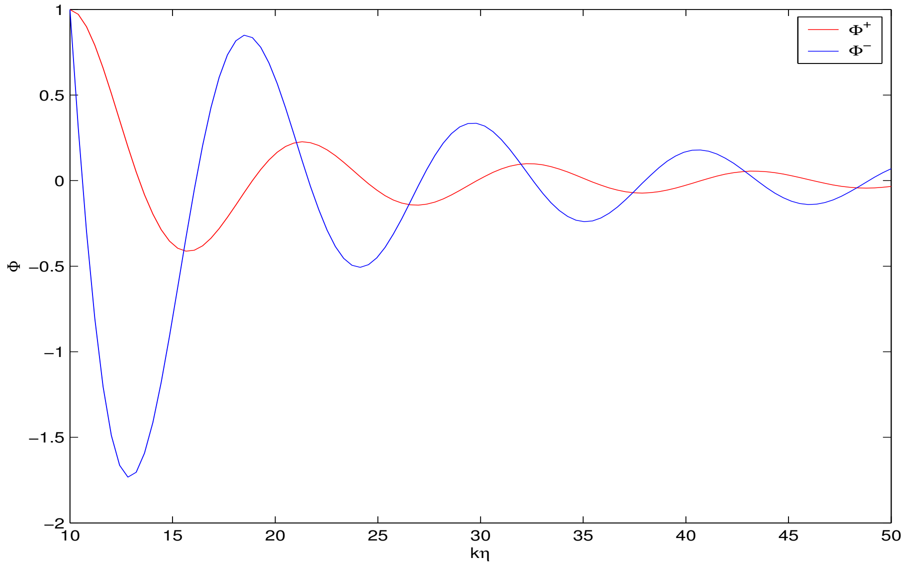


Figure 3.3: *A plot of the two types of metric perturbations inside the Hubble length. The graphs have been scaled to a maximal amplitude of order one.*

decrease with time, while inside the Hubble horizon they oscillate. With this information in hand, we make the choice to work with only δ^+ from now on, since the other solution is either decaying (long wavelength limit) or more or less the same (small wavelength limit and up to the Hubble scale).

Next, we investigate how the corresponding metric perturbations Φ^+ and Φ^- behave. In the long wavelength limit, i.e. when $\omega\eta \ll 1$, we make a Taylor expansion of the two solutions and keep only those terms that are of the leading order, just as we did when considering the density contrast. The result is the following

$$\Phi^+ \sim \text{constant}\,, \quad \text{and} \quad \Phi^- \sim \eta^{-3}\,. \tag{3.61}$$

In the small wavelength limit, Φ^+ and Φ^- will both be oscillating, decreasing functions. This is illustrated in fig 3.3, which shows a plot of both solutions for $k\eta > 10$.

In summary, the Fourier modes of the metric perturbation behave in the following way

$$\Phi^+ \sim \begin{cases} \text{constant}, & \text{for } \omega\eta \ll 1 \quad (\lambda \gg L_{\rm H}) \\ \text{oscillating and decreasing}, & \text{otherwise} \end{cases}\,, \tag{3.62}$$

and

$$\Phi^- \sim \begin{cases} \eta^{-3}\,, & \text{for } \omega\eta \ll 1 \quad (\lambda \gg L_{\mathrm{H}}) \\ \text{oscillating and decreasing,} & \text{otherwise} \end{cases} \tag{3.63}$$

3.3.1 A particular solution

We will now look at a particular solution of the type δ^+ with a given spatial configuration at a given time. Just as we did in the case of the matter dominated model, we consider a solution that is a pure sine wave with wave vector $\mathbf{k}$ at a time η_0 with amplitude δ_0,

$$\delta(\eta_0, \mathbf{x}) = \delta_0 \sin \mathbf{kx}\,. \tag{3.64}$$

For simplicity, we choose to examine perturbations that are plane symmetric. The results we obtain below will be equally applicable to perturbations that are not plane symmetric. This simplification allows us to skip the vector notation in the spatial part of the perturbations,

$$\sin(\mathbf{kx}) \to \sin(kx)\,. \tag{3.65}$$

Since the "sum" in the Fourier expansion (3.42) spans over both positive and negative values of the wave vector $\mathbf{k}$, there are two terms that contribute to a wave with a certain wave number $k = \|\mathbf{k}\|$,

$$\delta(\eta, x) = \delta_k(\eta) e^{-ikx} + \delta_{-k}(\eta) e^{ikx}\,. \tag{3.66}$$

In order to get a pure sine wave, the following condition must be satisfied

$$\delta_k(\eta) = \delta_{-k}(\eta)\,. \tag{3.67}$$

If we, in addition, demand that (3.64) is satisfied, we end up with the following solution for the density contrast

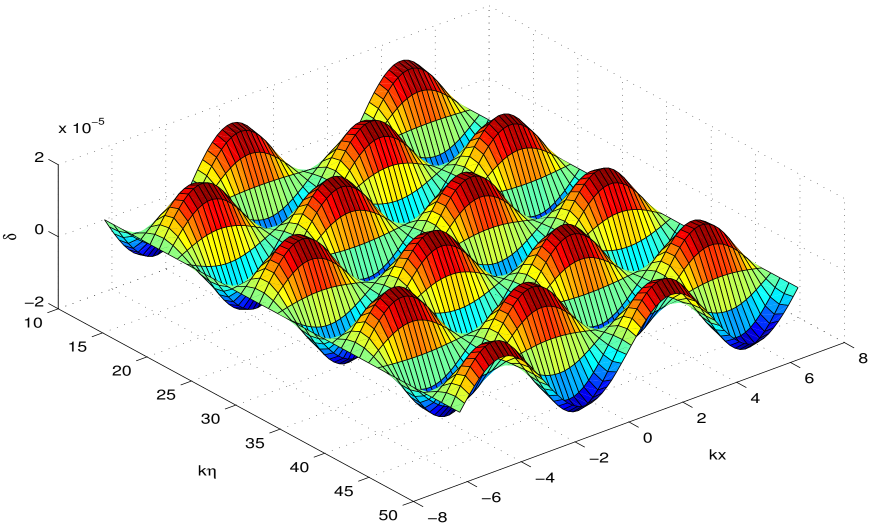
$$\delta(\eta, x) = \delta_0 \left(\frac{\eta_0}{\eta}\right)^3 \frac{(\omega^2\eta^2 - 1)\sin\omega\eta + \omega\eta(1 - \frac{1}{2}\omega^2\eta^2)\cos\omega\eta}{(\omega^2\eta_0^2 - 1)\sin\omega\eta_0 + \omega\eta_0(1 - \frac{1}{2}\omega^2\eta_0^2)\cos\omega\eta_0} \sin kx\,. \tag{3.68}$$

The metric perturbation that corresponds to this density contrast is obtained by use of (3.50) with $C_2 = 0$, and adding the two terms that contribute to the sine wave. The result is

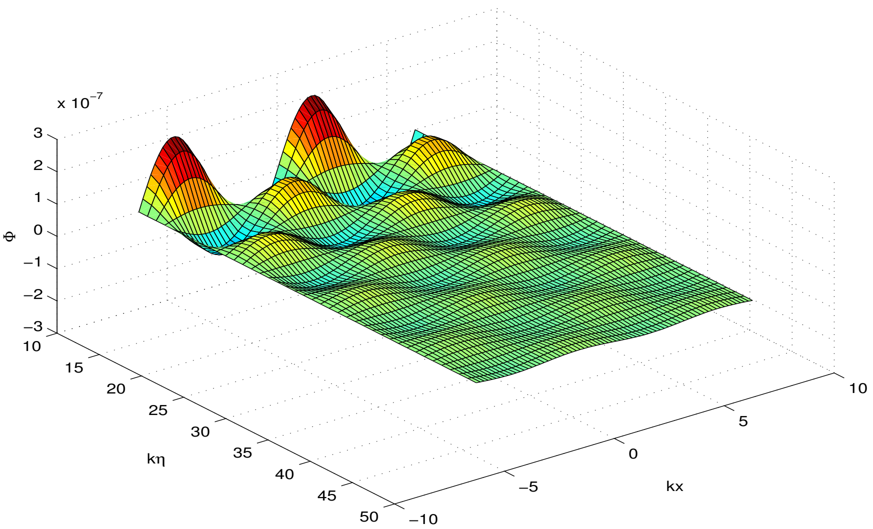
$$\Phi(\eta, x) = \frac{\delta_0}{4} \left(\frac{\eta_0}{\eta}\right)^3 \frac{\omega\eta\cos\omega\eta - \sin\omega\eta}{(\omega^2\eta_0^2 - 1)\sin\omega\eta_0 + \omega\eta_0(1 - \frac{1}{2}\omega^2\eta_0^2)\cos\omega\eta_0} \sin kx\,. \tag{3.69}$$

Both the density contrast and the metric perturbation for this sine solution are plotted in figure 3.4. In this plot we have chosen $k\eta_0 = 10$, which means that the perturbation is inside the Hubble length and oscillating. Furthermore, the amplitude of the perturbation is chosen to be $\delta_0 = 10^{-5}$. These values are the same as those which were chosen for the earlier considered matter dominated solution, which allows us to compare the two sets of solutions.

We choose a fixed spatial coordinate in our solution and examine how the density contrast and the metric perturbation evolve at this particular point, compared to the same point in the corresponding perturbations in the matter dominated model. A plot showing this comparative behaviour of the two universe models can be seen in figure 3.5.

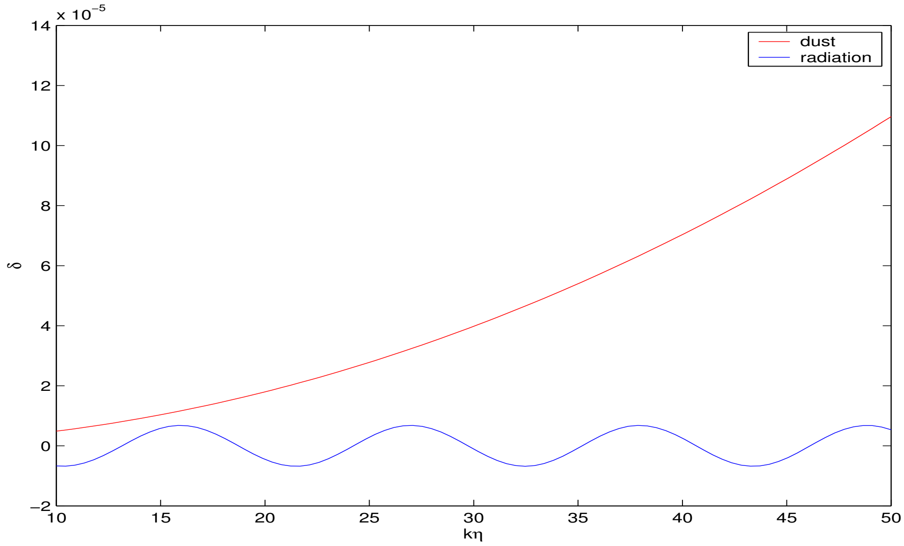


(a)

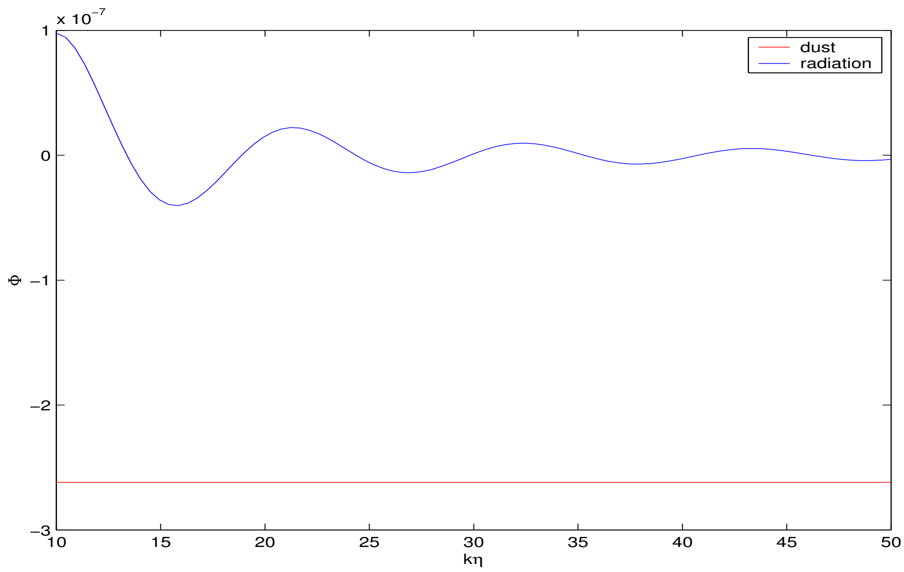


(b)

Figure 3.4: (a) *Time evolution of the density contrast in a radiation dominated universe. The initial perturbation is a plane wave with amplitude* $\delta_0 = 10^{-5}$ *at an initial conformal time* $k\eta_0 = 10$*.* (b) *The corresponding metric perturbation.*



(a)



(b)

Figure 3.5: *A comparison of The time evolution of the density contrast (a) and the metric perturbation (b) at a fixed point in space for the dust dominated model versus the radiation dominated model.*

Part II

Gravitational Entropy

Chapter 4

The Weyl Curvature Hypothesis

In this chapter we will the discuss the need to introduce a quantity which must account for an intrinsic entropy of the gravitational field in addition to the usual thermodynamic entropy.

This quantity along with a specific choice of initial conditions imposed on the Universe, known as the Weyl Curvature Hypothesis, appears to explain a number of fundamental physical questions.

4.1 Introduction

Today there is a broad consensus among cosmologists that the initial configuration of energy in the early Universe was very homogeneous and isotropic. The "gas" which initially filled up space was very hot and with almost the same temperature throughout. In terms of thermodynamics, this translates into the statement that the Universe was in (almost) thermal equilibrium. Using the usual definition of entropy in thermodynamics, this means that the Universe must have been in a state of (near) maximal entropy.

This claim is strengthened further by considering the quantity which measures entropy per baryon in the Universe. This quantity is related directly to the ratio of photon density to baryons density in the Universe, and is therefore measurable. The measured value of the latter is $\sim 10^9$. MacCallum [15] argues that this value imposes a strong constraint on the degree of anisotropy and inhomogeneity in the early Universe, and therefore also on the entropy of the initial state of the Universe. The ratio has remained constant since the creation-annihilation of matter and anti-matter finished [16], and constitutes the largest contribution to entropy in the Universe. Penrose [17] argues that this numeric value is very high and that this in turn points toward a high entropy initial Universe.

Furthermore, Penrose [17] notes that if the initial matter in the Universe was not in thermal equilibrium, we could not reproduce theoretically the correct helium abundance in the Universe.

Thus, we are led to the conclusion that the initial state of the Universe was one of maximal entropy. According to the second law of thermodynamics, which states that a physical system evolves toward a state of maximal entropy, the Universe cannot evolve beyond the initial isotropic and homogeneous state since any such evolution would mean a reduction in entropy. But, nevertheless, we know that the matter eventually "breaks up" due to gravitational attraction and ends up forming structures such as galaxies, stars, planets, planetary clouds etc. The temperatures of these objects varies over a wide range, which means that the Universe no longer is in thermal equilibrium, which in turn means that the entropy must have reduced in contradiction to the second law of thermodynamics.

It appears, therefore, as if the evolution of the Universe breaks the second law of thermodynamics, which is considered a fundamental law in physics. How can this be? One of the most plausible explanation to this problem was suggested by Penrose [18] in 1977 in terms of a hypothesis which goes under the name *The Weyl Curvature Hypothesis* and which introduces the concept of gravitational entropy.

4.2 Time Asymmetry

Penrose argued that a series of outstanding issues within cosmology and physics generally could be solved by postulating an initial Universe of very low entropy instead of high entropy. He argues [18, 17, 19] that such an initial Universe, among other things, presents a possible answer to the question of the *arrow of time.* The issue of this so-called arrow of time has been discussed by a whole host of authors over the years, and it relates to the time direction in which the entropy of a physical system grows. We can state the problem as follows: the laws of physics are stated via mathematical equations which are symmetric in time. For every solution of these equations, we get another solution by substituting the time parameter, t, with $-t$. But, nevertheless, all macroscopic observations are time-asymmetric. Evolution of any macroscopic, physical state is always in the direction of increasing time.

The time direction in which the entropy of a physical state grows defines the arrow of time. Since the physical laws are time symmetric, we would not expect there to be any arrow of time. We would, in other words, expect there not to be any preferred time direction in which the entropy grows. But Nature, for some unknown reason, seems to have chosen a specific arrow of time, namely that entropy grows with increasing time. For extensive treatments of the topic of time asymmetry and the arrow of time, the interested reader is referred to a couple of works by P.C.W Davies [20, 21].

Penrose notes that by imposing an initial state of the Universe which has a much lower entropy than the "final state", the arrow of time is automatically explained. Since the "final state" of the Universe has higher entropy than the initial state, this must mean that entropy grows with time. This solves the question of the arrow of time, according to Penrose, by imposing boundary conditions on the Universe. What Penrose has done is, in effect, to move the focus point away from the laws of physics themselves and over to boundary conditions imposed on the very same laws. It still remains an open question as to why the initial state of the Universe must be a low-entropy state.

4.3 Gravitational Entropy

We have seen that by imposing a particular initial condition on the Universe, namely that of a low-entropy initial state, the arrow of time "falls out" as a direct consequence. But, contrary to this, as we explained at the start of this chapter, it looks as if the initial state was one of very high entropy instead of low. This problem is cured by postulating an additional entropy quantity which we call *gravitational entropy*. The ordinary entropy quantity, which is the thermodynamical entropy, doesn't account for gravitational forces. It applies only for physical systems where gravity has no effect, i.e. over time periods which are small compared to the relaxation time of the gravitation forces between the different constituents of the physical system. Therefore, for an ordinary laboratory system this additional entropy quantity can be disregarded. Whereas for large systems in which gravity plays an effect, such as the Universe itself, it must be considered.

4.3.1 Black hole entropy

The proposition of intrinsic entropy in gravitational fields is not a new one. Such a quantity was already proposed by Bekenstein for the special case of a black hole [22, 23, 24]. After works by Floyd and Penrose [25] and Christodoulou [26] it was already known that the area of black holes increased under most transformations. Later Hawking [27] proved generally that the area cannot decrease under *any* transformation of the black hole. This prompted Bekenstein to draw a comparison between the surface area of black holes and entropy in thermodynamics. He postulated a quantity which he called the entropy of black holes. This quantity is now referred to as the Bekenstein-Hawking formula for black hole entropy and reads as follows

$$S_{BH} = A\frac{k_B c^3}{8\pi\hbar G}\ln 2\,, \tag{4.1}$$

where k_B is Boltzmann's constant, c is the speed of light in vacuum, G is Newton's gravitational constant, and A is the area of the horizon of the black hole.

Penrose used this expression to illustrate that the initial entropy of the Universe must indeed have been much smaller than that at late stages. He considers a closed universe and argues that it is plausible to assume that the natural course of all matter in this universe is to be collected together in a final, gigantic black hole. This is a state in which there cannot be any further gravitational clumping, and we can therefore used it as a possible final state of the Universe. Now, by using the Bekenstein-Hawking formula for black hole entropy, he arrives at a final entropy of $\sim 10^{123}$.

On the other hand, the entropy in the initial state of this universe can be calculated using the measured value of entropy per baryon. The initial state is a state of thermal equilibrium with no black holes. Thus, there is no contribution to the entropy from black holes, and the initial entropy is simply the thermodynamical entropy. Assuming there to be $\sim 10^{80}$ baryons in this closed universe, one arrives at an initial entropy of the order of $\sim 10^{89}$. The derivation which gives us this value can be found in e.g. [28]. This value is, obviously, much smaller than the value which was found for the possible final state. This shows that when one takes into consideration some measure of gravitational entropy in addition to the usual thermodynamical entropy, the initial state can still be one of very low entropy even if it has maximal thermal entropy.

4.3.2 Gravitational entropy and the Weyl curvature hypothesis

The formula for gravitational entropy which we used in the previous section is valid only for black holes. What we seek is a quantity which applies for any gravitational field. Through the nature of gravity and the requirement that the *total* entropy must increase with time, we can deduce qualitatively how this entropy of gravitational fields must behave. It is clear that the more clumping of matter there is, the greater the gravitational entropy must be. This means that it must be minimal for isotropic and homogeneous distributions of matter, and that it must increase the farther away the matter distribution moves from homogeneity and isotropy. A candidate for such a quantity is the Weyl tensor. Penrose notes in [29] the following properties of the Weyl tensor:

> "...In terms of spacetime curvature, the absence of clumping corresponds to the absence of Weyl conformal curvature (since absence of clumping implies spatial-isotropy, and hence no gravitational principal null-directions). When clumping takes place, each clump is surrounded by a region of nonzero Weyl curvature. As the clumping gets more pronounced owing to gravitational contraction, new regions of empty space appear with Weyl curvature of greatly increased magnitude.."

Thus, the Weyl tensor has the desired properties which one would want in a quantity which describes gravitational entropy. But in order for the initial entropy to be small, the initial gravitational entropy must be very small or vanish. This would then allow for a minimal initial entropy, since if the gravitational entropy grows faster than the thermodynamical decreases due to gravitational collapse, the initial total entropy will be minimal even if the thermodynamical is maximal. This is the very essence of the Weyl Curvature Hypothesis, which can now be stated rather precisely as:

> The universe models that describe the evolution of our Universe are those that have a vanishing initial gravitational entropy.

If we by gravitational entropy understand Weyl tensor, then this hypothesis tells us that we can only allow for those universe models which have a Weyl tensor which tends to zero as time goes to zero. The hypothesis is, in other words, a hypothesis of boundary conditions.

4.4 The Weyl tensor

It has become evident over the previous sections of this chapter that the Weyl tensor will play an important role in this topic. In this section we will examine this mathematical quantity more closely, and seek to understand the geometric meaning of it.

The Weyl tensor is a tensor of rank four, and is related to the curvature of a space-time manifold. It is defined for manifolds of dimension higher than two, however, for a three dimensional manifold it vanishes identically. Following Weinberg [30], we write the Weyl tensor for an n-dimensional manifold as follows

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{2}{2-n}\left(g_{\mu[\alpha}R_{\beta]\nu} - g_{\nu[\alpha}R_{\beta]\mu}\right) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\alpha}g_{\beta]\nu}\,, \tag{4.2}$$

where the brackets are understood to mean anti-symmetric combination of the bracketed indices. The other symbols have their usual meaning.

Because of its close relation to the Riemann tensor, the symmetries which are present in the latter are also present in the Weyl tensor. Thus, the Weyl tensor satisfies the following three symmetries

$$\begin{aligned} C_{\mu\nu\alpha\beta} &= C_{\alpha\beta\mu\nu} \\ C_{\mu\nu\alpha\beta} &= -C_{\mu\nu\beta\alpha} \\ C_{\mu\nu\alpha\beta} &= -C_{\nu\mu\alpha\beta}\,, \end{aligned} \tag{4.3}$$

and also the the Bianchi identity

$$C_{\mu\nu\alpha\beta} + C_{\mu\beta\nu\alpha} + C_{\mu\alpha\beta\nu} = 0\,. \tag{4.4}$$

But in addition to these symmetries, the Weyl tensor also satisfies the following symmetry

$$C^{\mu}_{\nu\mu\beta} \equiv 0\,. \tag{4.5}$$

Using this symmetry along with (4.3) and (4.4), we can deduce that the Weyl tensor must vanish for any pair of contracted indices. Thus, we say that the Weyl tensor is the "trace-less" part of the Riemann tensor.

The additional symmetry of vanishing trace reduces the number of independent components of the Weyl tensor compared to the Riemann tensor. For an n-dimensional manifold, the number of independent components of the Weyl tensor, C_n, is

$$C_n = \frac{1}{12}N(N+1)(N+2)(N-3)\,. \tag{4.6}$$

For a four-dimensional space-time, the number of independent components is ten, which is half the number of independent components of the Riemann tensor in the same space-time.

The Weyl tensor is the part of the Riemann tensor which gives curvature in empty space. Consider a section of space-time in which there is no energy density, e.g. the exterior Schwarzschild solution of the Einstein equations. Although there is no energy density there, space-time is nevertheless curved outside such a solution. Absence of energy density translates via the field equations of gravity to a vanishing Ricci tensor. From (4.2) we can write the Riemann tensor in vacuum as

$$R^{\text{vac}}_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta}\,. \tag{4.7}$$

Thus, one often says that it is the Weyl tensor which allows gravity to propagate through empty space.

A very important property of the Weyl tensor is its invariance under conformal transformations. In fact, it was introduced by Weyl [31] as an aid in studying conformal Riemannian manifolds and it was referred to as *the conformal curvature tensor* by Weyl.

A conformal transformation of an manifold is a transformation which leaves the metric unchanged up to a conformal factor. Let $g_{\mu\nu}$ be the metric in the non-transformed manifold. Under a conformal transformation the metric transforms to a new metric $\bar{g}_{\mu\nu}$, which can be written as

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \,, \tag{4.8}$$

where $\Omega(x)$ is a non-vanishing, differentiable function of space-time. Geometrically, a conformal transformation is a transformation that preserves angles between vectors. As mentioned above, the Weyl tensor is invariant under such a transformation, i.e.

$$\bar{C}_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} \,, \tag{4.9}$$

where the bar denotes the components of the conformally transformed tensor. Thus, if a space-time is conformally flat, i.e. conformal to the Minkowski space-time, it will have a vanishing Weyl tensor. In fact, the converse can also be shown to be true, which leads us the following theorem:

Theorem. *A space-time is conformally flat if and only if it has a vanishing Weyl tensor.*

A consequence of this theorem is that all the FRW models have a vanishing Weyl tensor, in agreement with the Weyl curvature hypothesis, since the FRW models are homogeneous and isotropic.

4.5 Measures of gravitational entropy

As we have seen in the previous sections, the Weyl tensor satisfies the properties which one would assign qualitatively to a quantity which measures gravitational entropy. But entropy is a scalar quantity, which means that if we wish to construct a measure of entropy from the Weyl tensor, we have to use some scalar composition of it. In this section we will consider three such candidates. The first one is after B.L. Hu [32] and is the simplest scalar expression which can be obtained from the Weyl tensor alone. Due to the extensive amount of symmetries which are present in the Weyl tensor, there are just four non-vanishing, independent scalar expressions which can be obtained from the Weyl tensor alone. The simplest is the "square" of the Weyl tensor, which is defined as

$$C^2 \equiv \; C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} \, . \tag{4.10}$$

By using the definition of the Weyl tensor, we can write this quantity in a simple form in terms of the Ricci and the Riemann tensor,

$$C^2 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \, . \tag{4.11}$$

This quantity depends on the space-time coordinates and must therefore be interpreted as an entropy density. In order to obtain the total gravitational entropy, one must perform an integration of (4.10) over the whole Universe. The gravitational entropy can thus be written as

$$S_G = \int d^3x\sqrt{|h|}\, C^2 \, , \tag{4.12}$$

where h is the determinant of the spatial projection of the metric, and $dV = d^3x\sqrt{|h|}$ is the spatial integration measure.

Another quantity which has been used as a measure of gravitational entropy is the quantity which was proposed by Goode et al. in 1985. They proposed [33] that one use the ratio of the square of the Weyl tensor to the square of the Ricci tensor as a measure of the entropy,

$$P^2 = \frac{C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}}{R_{\lambda\sigma}R^{\lambda\sigma}} \, . \tag{4.13}$$

A further analysis on the behaviour of this quantity can be found in an article by the same authors [34] and Coley from 1991.

In fact, the possibility of using such a quantity was proposed by Penrose himself in [17]. Penrose notes that requiring the Weyl tensor to vanish at the initial singularity might be too strict a constraint, and he goes further on to suggest that it might suffice to let the Weyl tensor be dominated by the Ricci tensor.

The quantity (4.13) has a spatial dependence and must again be interpreted as a local entropy or entropy density. The total gravitational entropy is the integral of this quantity over the whole space. Grøn and Hervik suggested in [35, 36] that one consider the square root of the quantity (4.13) integrated over a small co-moving volume as a more appropriate measure of entropy,

$$\mathcal{S} = \int_V P\sqrt{|h|}d^3x \, . \tag{4.14}$$

When the integration volume is sufficiently small, the integrand can be taken to be constant, which yields the following measure of gravitational entropy in a small comoving volume

$$\mathcal{S} = \int_V P\sqrt{|h|}d^3x \approx P\sqrt{|h|}\,. \qquad (4.15)$$

It is the quantity (4.14) which will be of greatest interest to us, and which will be devoted the most attention in the next chapter where we examine how the gravitational entropy evolves in our previously introduced perturbed FRW model.

Chapter 5

Cosmological Entropy

In this chapter we will examine how both the thermodynamic and the gravitational entropy evolve with time in a perturbed, flat matter dominated FRW model. We will look specifically at perturbation which have a Gauss-like form. In order to simplify the expressions and the actual calculation of the quantities mentioned above, we will impose certain symmetries on these perturbation in addition to having this special form. The symmetries which we consider are plane symmetries, cylindrical symmetries and spherical symmetries.

5.1 Thermodynamic entropy of a cosmological ideal gas

The universe model which we consider is that of a matter dominated, flat FRW, or a so-called Einstein-de Sitter model. The medium in such a model is a pressureless hydrodynamical gas. Since there is no pressure in this medium, there is also no interaction between the different particles of which the gas consists. This is equivalent to the ideal gas in thermodynamics, which per definition is a non-interacting gas. Thus, we can use the well-known expression for the entropy of an ideal gas as the entropy in our cosmological gas. This expression can be found in most introductory textbooks on thermodynamics such as [37]. For an ideal gas which consists of N particles it reads as

$$S_{\text{ideal}} = k_B N \ln \left[\frac{V}{N} \left(\frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} e^{5/2} \right], \tag{5.1}$$

where m is the mass of the gas particles, V is the volume occupied by the gas, and T is the temperature of the gas. This expression is valid for a gas which consists of distinct particles. However, we wish to use an expression which applies for a continuous gas distribution, instead of a distinct distribution. Consider an ideal gas within a small volume element dV. The number of particles dN within this volume element can be written as the total mass within the volume element divided by the mass of of each individual particle,

$$dN = \frac{\rho dV}{m} . \tag{5.2}$$

We can use expression (5.1) to calculate the entropy dS within this volume element,

$$dS = k_B \frac{\rho}{m} \ln \left[\frac{m}{\rho} \left(\frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} e^{5/2} \right] dV \equiv \mathcal{S}_T dV \, , \tag{5.3}$$

where $\mathcal{S}_T$ can be interpreted as the entropy density of the ideal, continuous gas,

$$\mathcal{S}_T = k_B \frac{\rho}{m} \ln \left(\frac{m}{\rho} \kappa_T \right) . \tag{5.4}$$

The parameter κ_T is defined as

$$\kappa_T = \left(\frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} e^{5/2} . \tag{5.5}$$

The energy density which appears in these formulas is the the sum of the homogeneous, unperturbed and the perturbed, inhomogeneous energy densities, $\rho = \rho_0(1+\delta)$. We substitute this expression for the density into the expression for the entropy density (5.4), which gives us

$$\mathcal{S}_T = k_B \frac{\rho_0}{m} (1+\delta) \ln \left(\frac{m\kappa_T}{\rho_0(1+\delta)} \right) \approx k_B \frac{\rho_0}{m} (1+\delta) \ln \left(\frac{m\kappa_T}{\rho_0} (1-\delta) \right) . \tag{5.6}$$

Keeping terms only up to the first order, we can write this as

$$\mathcal{S}_T \approx k_B \frac{\rho_0}{m} \left\{ \ln \frac{m\kappa_T}{\rho_0} + \delta \left(\ln \frac{m\kappa_T}{\rho_0} - 1 \right) \right\} . \tag{5.7}$$

In arriving at this expression we have made use of the following formula

$$\ln (1-\delta) = -\delta + \mathcal{O}(2) . \tag{5.8}$$

For a flat, matter dominated universe model, the energy density is that which we get from expression (2.98) with $w = 0$. This reads

$$\rho_0(\eta) = \frac{3\eta_0^4}{2\pi G} \eta^{-6} . \tag{5.9}$$

We wish to write this expression in physical or co-moving time t instead of conformal time η. According to (2.113), the relation between conformal and physical time in a matter dominated universe is

$$\eta^2 = \eta_0^2 \left(\frac{t}{t_0}\right)^{2/3} , \tag{5.10}$$

where t_0 is the physical time that corresponds to η_0. It is the initial time of the perturbations. If we assume that the perturbations which we describe are the actual initial perturbations in our physical universe, then the value of t_0 is constrained by observations. It is the time when the universe became transparent. According to the latest data from measurements of the CMB anisotropies, which currently are the WMAP data, it is equal to $t_0 \approx 380\,000$ years after Big Bang.

We introduce a new dimensionless time parameter τ, which measures time after t_0 relative to t_0, i.e.

$$\tau = \frac{t - t_0}{t_0} . \tag{5.11}$$

Thus, the square of the conformal time as a function of this new time parameter is

$$\eta^2 = \eta_0^2 (1+\tau)^{2/3} . \tag{5.12}$$

Next, we insert this expression into (5.9), which allows us to write the unperturbed energy density as a function of τ,

$$\rho_0(\tau) = \frac{3}{2\pi G \eta_0^2} (1+\tau)^{-2} . \tag{5.13}$$

Finally, we wish to substitute the physical initial time t_0 for η_0 in this expression. According to (2.100), t_0 and η_0 relate to each other in the following manner

$$t_0 = \frac{\eta_0}{3} . \tag{5.14}$$

The final expression for the unperturbed energy density is then

$$\rho_0(\tau) = \frac{1}{6\pi G t_0^2} (1+\tau)^{-2} . \tag{5.15}$$

Using this expression, we can write the thermodynamic entropy density (5.7) as

$$\mathcal{S}_T = \frac{k_B \alpha_0}{(1+\tau)^2} \left\{ \ln \frac{\kappa_T (1+\tau)^2}{\alpha_0} + \delta \left(\ln \frac{\kappa_T (1+\tau)^2}{\alpha_0} - 1 \right) \right\} , \tag{5.16}$$

where we have defined the following constant

$$\alpha_0 = \frac{1}{6\pi G t_0^2 m}\,. \tag{5.17}$$

Boltzmann's constant k_B appears in this expression as a multiplicative constant, just as it does in most quantities that measure some kind of entropy. Instead of carrying this constant in all our calculations, we define an entropy density relative to Boltzmann's constant,

$$\sigma_T \equiv \frac{\mathcal{S}_T}{k_B} = \frac{\alpha_0}{(1+\tau)^2}\left\{\ln\frac{\kappa_T(1+\tau)^2}{\alpha_0} + \delta\left(\ln\frac{\kappa_T(1+\tau)^2}{\alpha_0} - 1\right)\right\}\,. \tag{5.18}$$

The constant κ_T depends on the temperature of gas, which in turn depends on the scale factor. In a matter dominated universe it can be shown [20] that the temperature varies like

$$T \propto a^{-2}\,. \tag{5.19}$$

With the value of a_0 chosen earlier to be equal to unity, we can write the temperature as

$$T = T_0 a^{-2} = T_0(1+\tau)^{-4/3}\,, \tag{5.20}$$

where T_0 is the temperature of the gas at the initial time t_0. We insert this expression for the temperature into the definition of the parameter κ_T (5.5),

$$\kappa_T = \left(\frac{m k_B T_0 (1+\tau)^{-4/3}}{2\pi\hbar^2}\right)^{3/2} = \kappa_{T_0}(1+\tau)^{-2}\,. \tag{5.21}$$

We see immediately that the explicit time dependence inside the parenthesis in expression (5.18) cancels out,

$$\sigma_T = \frac{\alpha_0}{(1+\tau)^2}\left\{\ln\frac{\kappa_{T_0}}{\alpha_0} + \delta\left(\ln\frac{\kappa_{T_0}}{\alpha_0} - 1\right)\right\}\,. \tag{5.22}$$

The thermodynamic entropy inside a small volume element dV is equal to the entropy density inside the same volume element times the volume element itself,

$$ds_T = \sigma_T dV\,, \tag{5.23}$$

where we use a small instead of a capital "s" as a symbol for the entropy to mark that it is an entropy relative to k_B. The expression for the volume element is determined by the spatial metric via the usual expression

$$dV = \sqrt{|g^{\mathrm{sp}}|}\prod_{i=1}^{3} dx^i\,. \tag{5.24}$$

The superscript "sp" marks that the determinant must be taken over only the spatial components of the metric. We use the line element (2.47) to calculate this determinant. The result is

$$|g^{\mathrm{sp}}| = a^6(1-6\Phi)\,. \tag{5.25}$$

Substituting for a and η from (3.1) and (5.12), the volume element can be written as

$$dV = (1+\tau)^2(1-3\Phi)\prod_{i=1} dx^i\,. \tag{5.26}$$

Multiplication of this quantity with the entropy density in (5.22) yields the thermodynamic entropy element,

$$ds_T = \alpha_0\left[(1-3\Phi)\ln\frac{\kappa_{T_0}}{\alpha_0} + \delta(\ln\frac{\kappa_{T_0}}{\alpha_0} - 1)\right]\prod_{i=1} dx^i\,. \tag{5.27}$$

For the unperturbed model the thermodynamic entropy inside a co-moving volume V is

$$s_T^{\mathrm{unpert}} = \int_V \alpha_0 \ln\frac{\kappa_{T_0}}{\alpha_0}\prod_{i=1} dx^i = \alpha_0 V \ln\frac{\kappa_{T_0}}{\alpha_0}\,. \tag{5.28}$$

We see that using (5.27) as a measure of thermodynamic entropy density of the cosmological gas, the entropy inside a co-moving volume in a non-perturbed matter dominated FRW universe is constant. This is what we would expect, since it is common knowledge that the thermodynamic entropy inside co-moving regions of space is constant in all FRW models. Thus, we conclude that (5.27) appears to be a good measure of thermodynamic entropy for a pressureless cosmological gas.

5.2 Gravitational entropy of a perturbed flat FRW model

As a measure of entropy density we will use the quantity P which is defined in (4.13). To determine this quantity, we will need to calculate both the Weyl tensor and the Ricci tensor. We let GRTensor perform these calculation for plane symmetric, cylindrically symmetric and spherically symmetric perturbations. The results can be written as

$$P = \frac{1}{3\sqrt{3}}Q[\Phi]\eta^2\,, \tag{5.29}$$

where $Q[\Phi]$ is a functional of the metric perturbation Φ and which depends on the symmetries of the geometry,

$$Q[\Phi] = \begin{cases} \frac{\partial^2\Phi}{\partial x^2} & \text{for plane symmetry} \\ \frac{1}{r}\sqrt{(\frac{\partial\Phi}{\partial r})^2 - r\frac{\partial\Phi}{\partial r}\frac{\partial^2\Phi}{\partial r^2} + r^2(\frac{\partial^2\Phi}{\partial r^2})^2} & \text{for cylinder symmetry} \\ \frac{\partial^2\Phi}{\partial r^2} - \frac{1}{r}\frac{\partial\Phi}{\partial r} & \text{for spherical symmetry} \end{cases} \tag{5.30}$$

The quantity P is dimensionless. If we want to use it as a measure of gravitational entropy density, we must therefore multiply it with some appropriate constants in order to get the right dimensions. Since entropy has dimension equal to Boltzmann's constant, we must therefore multiply it with k_B times some arbitrary constant χ, which has dimension equal to inverse length cubed, m^{-3}. Thus, the gravitational entropy density of a perturbed, flat FRW model can be written as

$$\mathcal{S}_G = \frac{k_B\chi}{3\sqrt{3}}\,|Q[\Phi]|\,\eta^2\,. \tag{5.31}$$

Note that we have taken the absolute value of the expression on the left hand side in order to insure that the gravitational entropy remains positive.

We see from the definition of $Q[\Phi]$ that the right hand side of (5.31) vanishes to the zeroth order, i.e. $Q[\Phi] \sim \mathcal{O}(1)$. This means that the unperturbed FRW model has a vanishing gravitational entropy. This is exactly what we would expect, since $\mathcal{S}_G$ is constructed to measure deviations form homogeneity, and we know that the FRW models are isotropic and homogeneous at all times.

Just as we did in the previous section in the case of thermodynamic entropy, we can define a new entropy density quantity σ_G which measures gravitational entropy density relative to Boltzmann's constant. Furthermore, we use relation (5.12) to express the time dependence through the dimensionless time parameter τ,

$$\sigma_G = \frac{\chi}{3\sqrt{3}}\,|Q[\Phi]|\,\eta_0^2(1+\tau)^{2/3}\,. \tag{5.32}$$

The gravitational entropy ds_G inside a small volume dV can now be calculated by multiplying σ_G with the volume element which can be found in (5.26). When discarding all terms of order higher than one, we can write the result as

$$ds_G = \frac{\chi}{3\sqrt{3}}\,|Q[\Phi]|\,\eta_0^2(1+\tau)^{8/3}\prod_i dx^i\,. \tag{5.33}$$

The gravitational entropy inside a co-moving region of space is the integral of this expression over the volume of that region. Since Φ and hence also

$Q[\Phi]$ are time independent, we can write the gravitational entropy as

$$s_G \propto (1+\tau)^{8/3}\,. \tag{5.34}$$

5.3 Gauss-like density perturbations

So far we have not specified any special form on the perturbations. We have demanded that they posses certain symmetries, but we have not said anything on their functional form. We are free to choose any form on either the density perturbation or the metric perturbation at a certain time, as long as they evolve according to equations (3.10) and (3.11).

We choose to limit ourselves to density perturbations that are localized around the origin of our coordinate system at the initial time t_0, and which vanish quickly as we move away from the origin. A possible function which has this behaviour is the Gauss function. Furthermore, we wish that the total mass of the universe at the initial time to be equal to that in the unperturbed universe. By demanding this, we say effectively that we don't introduce any new mass at the time t_0. We simply rearrange the mass which we already have present. In order for this to be satisfied, the initial density perturbation must satisfy the following integral

$$\int_V \delta(\mathbf{x},\eta_0)dV = 0\,. \tag{5.35}$$

It is clear that the Gauss function alone cannot satisfy this relation since it is positive everywhere. We will therefore modify it slightly by multiplying it with a polynomial. The choice of polynomial depends on whether the density perturbation is plane symmetric, cylindrically symmetric of spherically symmetric. We will call such perturbations for Gauss-like perturbations.

5.3.1 Plane symmetric density perturbations

By a plane symmetric perturbation we understand a perturbation which depends on only one spatial coordinate. We choose this coordinate to be the x coordinate. In this case we can use the following form on the initial density perturbation

$$\delta(x,\eta_0) = \delta_0\left[1-2\left(\frac{x}{L}\right)^2\right]e^{-(\frac{x}{L})^2}\,. \tag{5.36}$$

The constant δ_0 is the amplitude of the initial density perturbation. According to CMB measurements, the amplitude of the real, physical density perturbations around the time when the universe became transparent is of the order $\sim 10^{-5}$. We will therefore choose $\delta_0 = 10^{-5}$. The second constant L is a measure of the length scale of the perturbation. For $|x| >> L$ the size of the perturbation is effectively zero.

Define a new dimensionless coordinate $\hat{x}$,

$$\hat{x} \equiv \frac{x}{L} \,. \tag{5.37}$$

When we use this coordinate, the initial density perturbation can be written as

$$\delta(\hat{x}, \eta_0) = \delta_0(1 - 2\hat{x}^2)e^{-\hat{x}^2} \,. \tag{5.38}$$

To calculate how this perturbation evolves with time we must use equation (3.11). Expressed using the coordinates (5.37) and (5.11), we can write this equation as

$$\delta(\hat{x}, \tau) = \frac{1}{6}\left(\frac{\eta_0}{L}\right)^2 (1+\tau)^{2/3} \frac{d^2\Phi(\hat{x})}{d\hat{x}^2} - 2\Phi(\hat{x}) \,. \tag{5.39}$$

If we know the metric perturbation Φ, we can use this equation to determine δ at all times τ. Since Φ is time independent it can be determined by considering equation (5.39) at the time $\tau = 0$, which is the time when we know δ. We get the following non-homogeneous second order differential equation for Φ

$$\frac{1}{6}\left(\frac{\eta_0}{L}\right)^2 \frac{d^2\Phi(\hat{x})}{d\hat{x}^2} - 2\Phi(\hat{x}) = \delta_0(1 - 2\hat{x}^2)e^{-\hat{x}^2} \,. \tag{5.40}$$

This equation cannot be solved analytically in a closed form. We will therefore have to solve it numerically. But before we can do this, we must first specify the order of the length scale L. In chapter 2 we showed that density perturbations of a size that is much larger than the Hubble length L_H will remain almost constant. We, on the other hand, are interested in density perturbations that grow over the time period which we consider. We must therefore choose the length scale of the perturbations to be smaller than the Hubble length. Since $L_H \sim \eta$ we must choose $\frac{\eta_0}{L} > 1$. Let therefore L be such that $\frac{\eta_0}{L} = 10$.

The differential equation (5.40) is solved using a standard finite element method. The interested reader is referred to [38] for a presentation of such numerical methods.

In appendix E.1.1 the reader can find the source code for a C++ program which implements a finite element algorithm to solve this explicit differential equation. The solution for Φ which the program outputs is valid in the range $-10 < \hat{x} < 10$, i.e. $|x| < 10L$. Outside this range the function (5.38) will have decreased sufficiently that we can assume that both Φ and $\delta(\hat{x}, \tau)$ are identically zero.

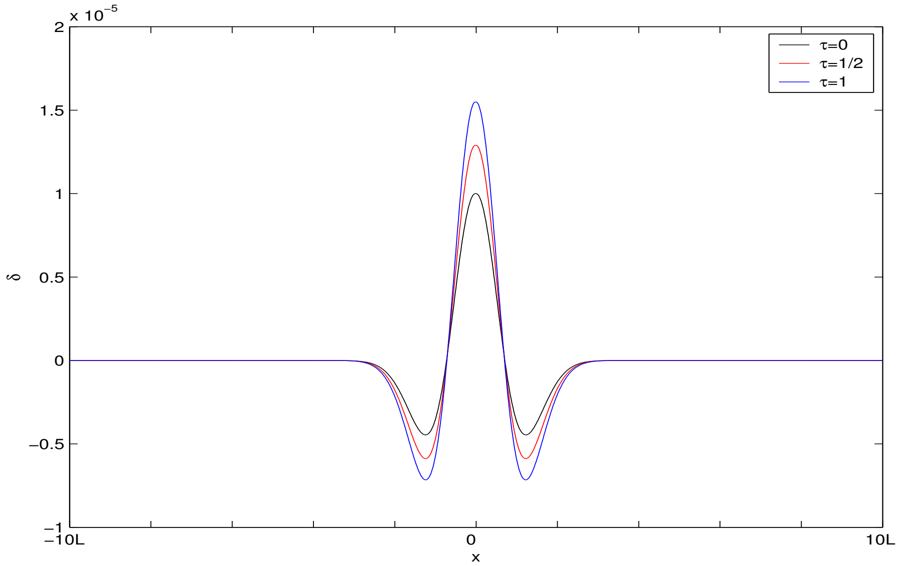


Figure 5.1: *A plot of the Gauss-like, plane symmetric density perturbation at three different times.*

We can now insert this solution for Φ into equation (5.39) in order to obtain the solution for $\delta(\hat{x}, \tau)$. This too is done by the computer program in appendix (E.1.1). In figures 5.1 and 5.2 we can see a plot of respectively the density and the metric perturbation. The density perturbation is plotted at three different times $\tau = 0, \frac{1}{2}, 1$ to illustrate how it changes with time. The metric perturbation is constant in time so it suffices to plot it for $\tau = 0$.

Now that both the density and the metric perturbations are determined we can move on to our main target, which is to determine the time evolution of the thermodynamic and the gravitational entropy. These quantities are given by the volume integrals of the expressions in (5.27) and (5.33). The Cartesian volume element which appears in these will because of the plane symmetry reduce to

$$\prod_i dx^i \rightarrow A dx \,, \tag{5.41}$$

where A is a constant which arises from the integration over the two spatial coordinated of which the perturbations do not depend. This constant doesn't contain any interesting physics and we will therefore set it equal to unity.

Using the definition in (5.30) and the dimensionless spatial coordinate $\hat{x}$, we see that the function $Q[\Phi]$, which appears in the expression for the

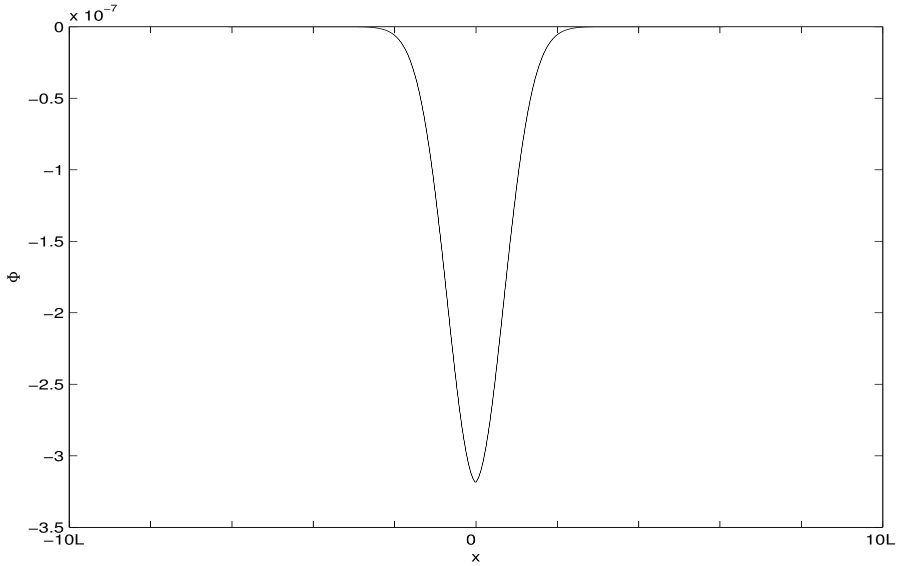


Figure 5.2: *A plot of the metric perturbation which yields the density perturbation in figure 5.1*

gravitational entropy, can be written as

$$Q[\Phi] = \frac{1}{L^2}\frac{\partial^2\Phi}{\partial\hat{x}^2} \, . \tag{5.42}$$

Finally, we can now write down the formal expression for the two entropy types. These are

$$s_T = \alpha_0 \int \left\{ (1-3\Phi)\ln\frac{\kappa_{T_0}}{\alpha_0} + \delta(\ln\frac{\kappa_{T_0}}{\alpha_0} - 1) \right\} d\hat{x} \, , \tag{5.43}$$

for the perturbed thermodynamic entropy, and

$$s_G = \frac{\chi}{3\sqrt{3}}\left(\frac{\eta_0}{L}\right)^2 (1+\tau)^{8/3} \int \left| \frac{\partial^2\Phi}{\partial\hat{x}^2} \right| d\hat{x} \, , \tag{5.44}$$

for the gravitational entropy. The limits in the integrals run formally from $-\infty$ to ∞. However, since the perturbations are almost identically equal to zero outside the range $-L < \hat{x} < L$, the contribution from these integrals outside this range is simply a constant. In our analysis we are not really interested in absolute entropies, but merely in relative entropies. We will therefore disregard such constant shifts of the entropies, and thus carry out the integrals over the finite range defined above instead of over the infinite range.

In the expression for the thermodynamic entropy in (5.43) there are two constants, α_0 and κ_{T_0}, which need to be determined before we can get a numerical result. α_0 is defined in (5.17). It depends on the initial time of the perturbations t_0 and on the mass of the particles of which the gas consists. We have already given a numerical value for t_0, namely $t_0 = 380\,000$ years. In regards to the particle type, it is natural to assume that the gas consists of protons, i.e. $m = 1.67 \cdot 10^{-27}$ kg. This gives the following value for α_0

$$\alpha_0 = 3.32 \cdot 10^9 \, \mathrm{m}^{-3} \, . \tag{5.45}$$

The second constant, κ_{T_0}, which is defined in (5.5), depends on the temperature in the gas at the initial time. It is a well-known fact that the temperature in the background radiation in the universe today is about 2.7 K. This corresponds to a temperature of the order $\sim 3.0{\cdot}10^3$ K at the time of transparency. Using this value for the temperature in our ideal gas at the initial time gives the following value for κ_{T_0}

$$\kappa_{T_0} = 3.79{\cdot}10^{32} \, \mathrm{m}^{-3} \, . \tag{5.46}$$

We substitute these values into the expression (5.44) and perform the integration numerically using the computer program MATLAB. A transcript of the MATLAB code with does this can be found in appendix E.2.1.

As stated earlier, we are only interested in relative entropies, i.e. entropies relative to the unperturbed entropy. In analogy with the density contrast, we will therefore define a dimensionless quantity Δ which we shall refer to as the *entropy contrast*, and which measures entropy relative to the unperturbed entropy,

$$\Delta \text{ ”=” } \frac{\text{entropy - unperturbed entropy}}{\text{unperturbed entropy}} \, . \tag{5.47}$$

In figure 5.3 we can find a plot of the thermodynamic entropy contrast Δ_T. We see that it does indeed decrease with time, just as we argued from a physical point of view in the previous chapter. The question is whether the total entropy contrast Δ_{tot}, which is the entropy contrast we get when we consider both the thermodynamic and the gravitational entropy, grows with time. The gravitational entropy in itself will of course grow with time since it was constructed to do just that, and which can also be seen from the general expression (5.34). However, the sum of this and the thermodynamic entropy will only grow with time if the growth in s_G is sufficiently large to overcome the decrease in s_T. This in turn depends on the hitherto unspecified constant χ.

We calculate s_G numerically using MATLAB (see appendix E.2.1) and find that the total entropy contrast will grow when χ is larger than a certain

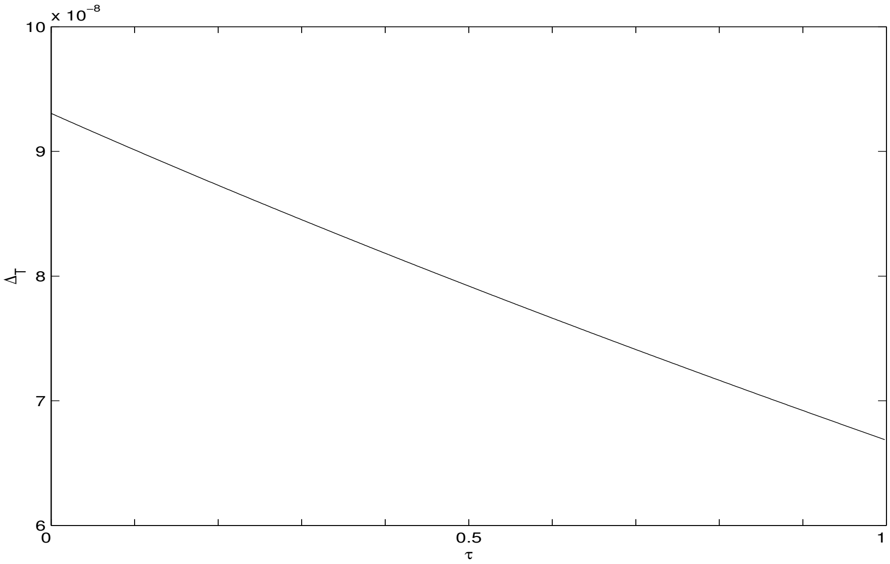


Figure 5.3: *The thermodynamic entropy contrast which results from the Gauss-like, plane symmetric density perturbation.*

minimal value,

$$\chi_{\text{plane}} > 2.0 \cdot 10^9 \, \text{m}^{-3} \, . \tag{5.48}$$

The subscript "plane" is to indicate that this value is calculated for the Gauss-like, plane symmetric density perturbation. Later, when we consider other symmetries, we will find that the values of the minimal χ are slightly different.

Thus, we can conclude that the total entropy will indeed grow with time and hence "rescue" the second law of thermodynamics *if* we choose an appropriate value for χ. We have illustrated this in figure 5.4 by plotting the total entropy contrast for $\chi = 5.0 \cdot 10^9 \, \text{m}^{-3}$, which is a value which produces a growing entropy contrast.

5.3.2 Cylindrically symmetric density perturbations

In this section we turn our attention to density perturbations that are cylindrically symmetric. Using cylinder coordinates (r, θ, z) instead of Cartesian coordinates allows us to write the perturbations as a function of only the radial coordiante r. An initial density perturbation which satisfies (5.35) is

$$\delta(r, \eta_0) = \delta_0 \left[1 - (\frac{r}{L})^2 \right] e^{-\left(\frac{r}{L}\right)^2} \, . \tag{5.49}$$

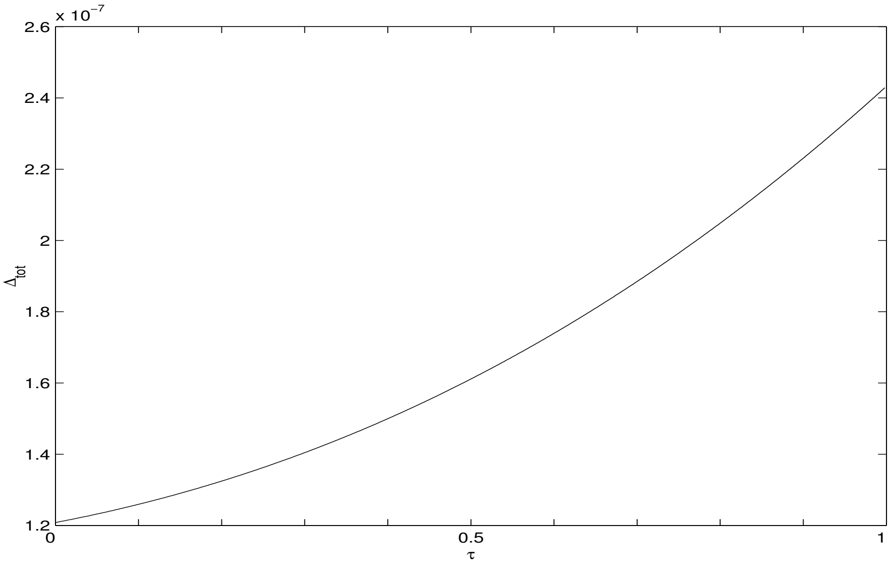


Figure 5.4: *The total entropy contrast for the Gauss-like, plane symmetric density perturbation with* $\chi = 5.0{\cdot}10^9\, m^{-3}$.

We use dimension-less coordinates $\hat{r}$ and τ just as we did in the previous section. τ is defined in the same way as earlier, while $\hat{r}$ is

$$\hat{r} \equiv \frac{r}{L}\,. \tag{5.50}$$

The equation which determines the time evolution of the density perturbation (3.11) expressed in these coordinates is

$$\delta(\hat{r},\tau) = \frac{1}{6}\left(\frac{\eta_0}{L}\right)^2 (1+\tau)^{2/3}\left\{\frac{d^2\Phi(\hat{r})}{d\hat{r}^2} + \frac{1}{\hat{r}}\frac{d\Phi(\hat{r})}{d\hat{r}}\right\} - 2\Phi(\hat{r})\,. \tag{5.51}$$

To solve this equation we follow the same procedure as in the last section. That is, we find Φ by solving the differential equation we get by putting $\tau = 0$, and then reinserting this solution back into equation (5.51). This was done numerically (see appendix E.1.2). In figure 5.5 we can find a plot of the density perturbation at three different times, while in figure 5.6 we have plotted the metric perturbation for the same solution. Next, we move on to determine the thermodynamic and the gravitational entropy. The Cartesian volume element which appears in the integrals (5.27) and (5.33) reduces to

$$\prod_i dx^i \to r dr\,, \tag{5.52}$$

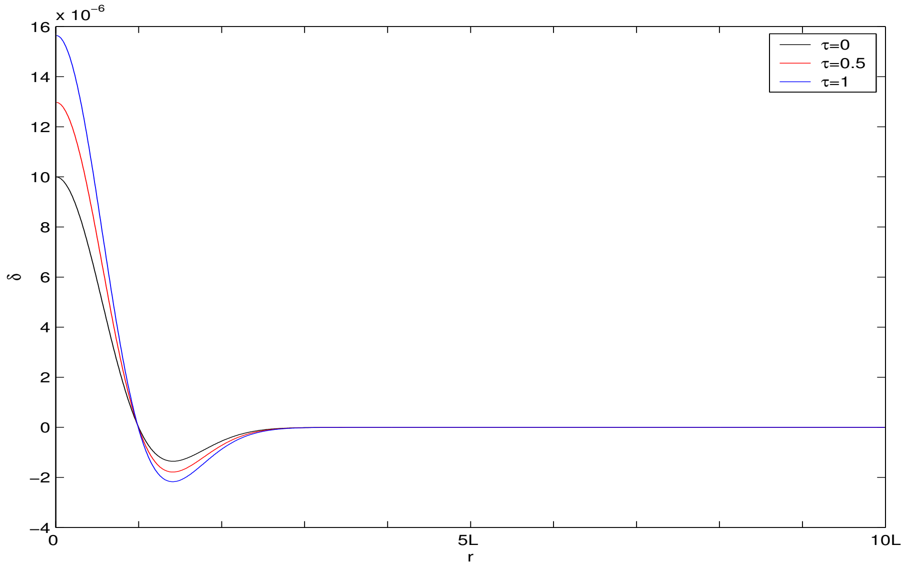


Figure 5.5: *Illustration of the time evolution of the Gauss-like, cylindrically symmetric density perturbation. The plot shows the form of the density perturbation at three different times.*

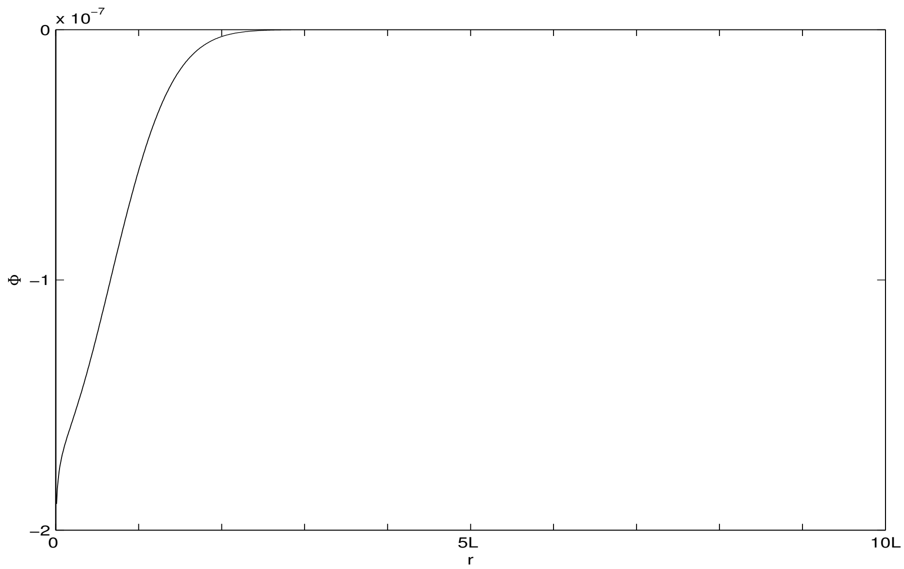


Figure 5.6: *The metric perturbation that produces the Gauss-like, cylindrically symmetric density perturbation.*

when we disregard any constants due to the cylindrical symmetry. The function $Q[\Phi]$ will in this case be

$$Q[\Phi] = \frac{1}{L^2}\frac{1}{\hat{r}}\sqrt{(\frac{\partial\Phi}{\partial\hat{r}})^2 - \hat{r}\frac{\partial\Phi}{\partial\hat{r}}\frac{\partial^2\Phi}{\partial\hat{r}^2} + \hat{r}^2(\frac{\partial^2\Phi}{\partial\hat{r}^2})^2}\,. \tag{5.53}$$

Thus, the expression for the thermodynamic entorpy will be

$$s_T = \alpha_0 \int \left\{(1-3\Phi)\ln\frac{\kappa_{T_0}}{\alpha_0} + \delta(\ln\frac{\kappa_{T_0}}{\alpha_0} - 1)\right\}\hat{r}d\hat{r}\,, \tag{5.54}$$

while the expression for the gravitational entropy becomes

$$s_G = \frac{\chi}{3\sqrt{3}}\left(\frac{\eta_0}{L}\right)^2(1+\tau)^{8/3}\int\left|\sqrt{(\frac{\partial\Phi}{\partial\hat{r}})^2 - \hat{r}\frac{\partial\Phi}{\partial\hat{r}}\frac{\partial^2\Phi}{\partial\hat{r}^2} + \hat{r}^2(\frac{\partial^2\Phi}{\partial\hat{r}^2})^2}\right|\hat{r}d\hat{r}\,. \tag{5.55}$$

We use the same values for the constant α_0, κ_{T_0} and $\frac{\eta_0}{L}$ as we used for the plane symmetric perturbations, and determine these integrals numerically using MATLAB (see appendix E.2.2). The results for the first integral are illustrated in figure 5.7, which shows a plot of the thermodynamic entropy contrast. As expected, this is a decreasing quantity, just as it was in the plane symmetric case. The gravitational entropy will of course always increasing with time, but the sum of the thermodynamic and the gravitational entropy will only increase for those χ which satisfy the follwing relation

$$\chi_{\text{cyl}} > 1.8\cdot 10^9\,\text{m}^{-3}\,. \tag{5.56}$$

In figure 5.8 illustrate such a growing entropy contrast by plotting Δ_{tot} for $\chi = 5.0\cdot 10^9\,\text{m}^{-3}$.

5.3.3 Spherically symmetric perturbations

Spherically symmetric perturbations are perturbation that depend on only the radial coordinate r when expressed in spherical coordinates. As the initial density perturbation we can choose the following function

$$\delta(r,\eta_0) = \delta_0\left[1 - \frac{2}{3}\left(\frac{r}{L}\right)^2\right]e^{-\left(\frac{r}{L}\right)^2}\,, \tag{5.57}$$

which satisfies the integral equation (5.35). We follow the same procedures as in the previous sections and introduce a dimensionless spatial coordinate,

$$\hat{r} \equiv \frac{r}{L}\,. \tag{5.58}$$

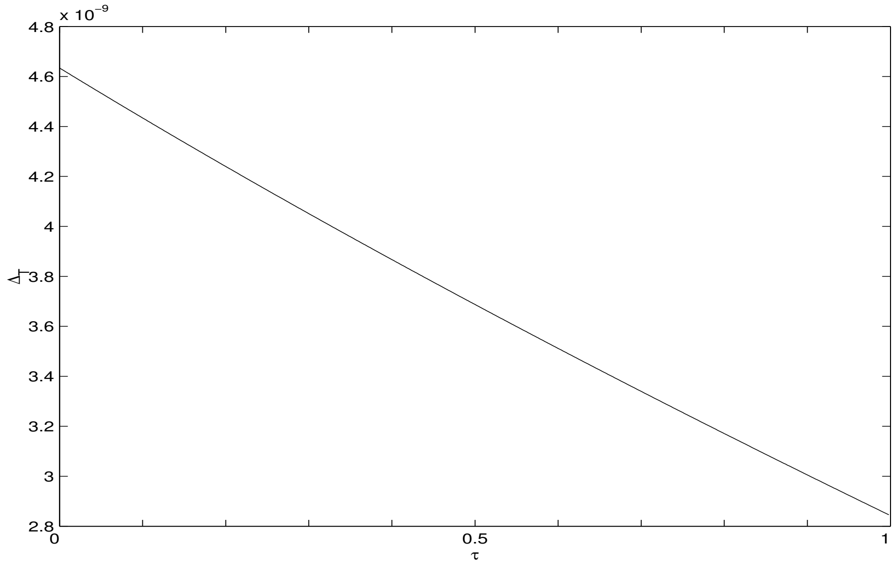


Figure 5.7: *A plot of the time evolution of the thermodynamic entropy contrast for the Gauss-like, cylindrically symmetric density perturbation.*

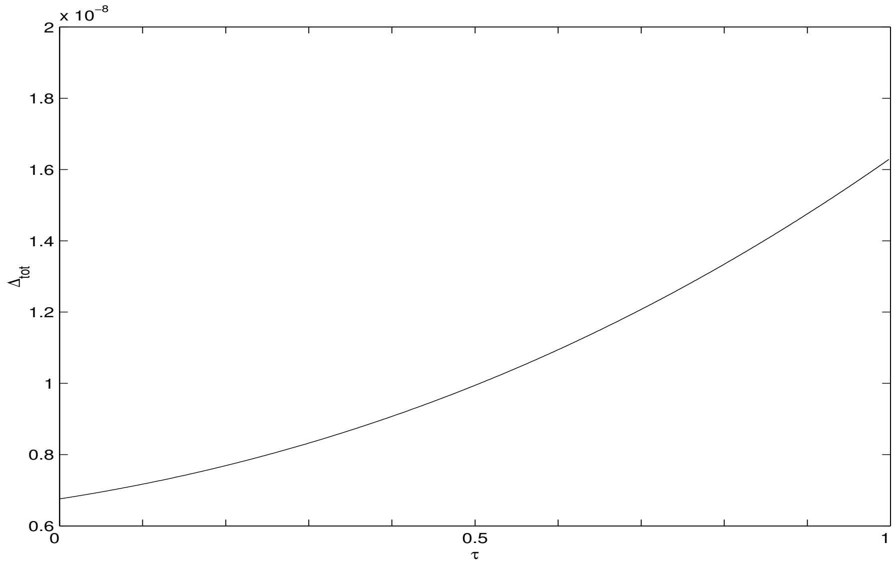


Figure 5.8: *A plot of the time evolution of the total entropy contrast for the Gauss-like, cylindrically symmetric density perturbation. The value of* χ *is chosen to be* $\chi = 5.0{\cdot}10^9\ m^{-3}$.

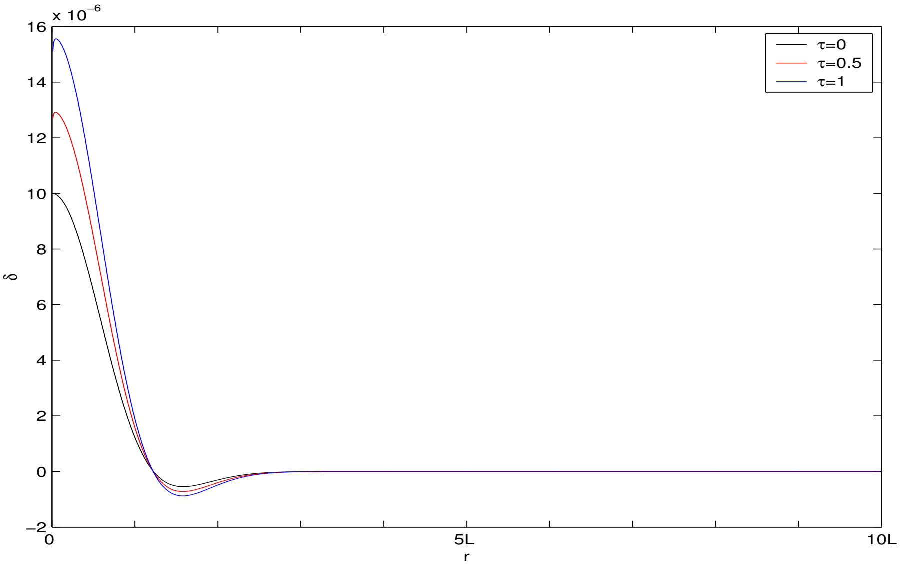


Figure 5.9: *The Gauss-like, spherically symmetric density perturbation plotted at three different times,* $\tau = 0$*,* $\tau = 0.5$ *and* $\tau = 1.0$*.*

Furhermore, we use the dimensionless time coordinate τ instead of conformal time. Equation (3.11) takes the following form in this case

$$\delta(\hat{r}, \tau) = \frac{1}{6}\left(\frac{\eta_0}{L}\right)^2 (1+\tau)^{2/3} \left\{ \frac{d^2\Phi(\hat{r})}{d\hat{r}^2} + \frac{2}{\hat{r}}\frac{d\Phi(\hat{r})}{d\hat{r}} \right\} - 2\Phi(\hat{r})\,. \tag{5.59}$$

We solve this equation numerically for Φ at the initial time $\tau = 0$, and then reinsert the solution into the same equation for general τ in order to obtain the time evolution of the density perturbation. The source code for the numerical program can be found in appendix E.1.3. Again, we illustrate the time evolution of the density perturbation by plotting it in figure 5.9 for three different values of τ. The corresponding metric perturbation can be found in figure 5.10.

The volume element which appears in the integrals which define the thermodynamic and the gravitational entropy, will in spherically symmetric coordinates be

$$\prod_i dx^i \to r^2 dr \tag{5.60}$$

up to a multiplicative constant. The function $Q[\Phi]$ can in this case be written as

$$Q[\Phi] = \frac{1}{L^2}\left\{ \frac{\partial^2\Phi}{\partial\hat{r}^2} - \frac{1}{\hat{r}}\frac{\partial\Phi}{\partial\hat{r}} \right\}\,. \tag{5.61}$$

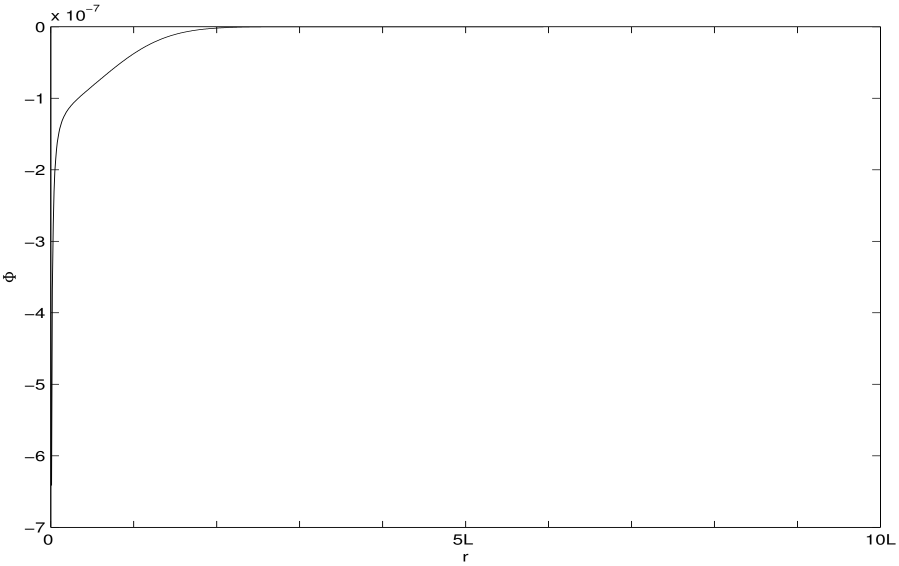


Figure 5.10: *Plot of the metric perturbation that corresponds to the Gauss-like, spherically symmetric density perturbation.*

Substituting (5.60) and (5.61) into the definitions (5.27) and (5.33), we end up with the expressions

$$s_T = \alpha_0 \int \left\{ (1 - 3\Phi) \ln \frac{\kappa_{T_0}}{\alpha_0} + \delta(\ln \frac{\kappa_{T_0}}{\alpha_0} - 1) \right\} \hat{r}^2 d\hat{r} \tag{5.62}$$

for the thermodynamic entropy, and

$$s_G = \frac{\chi}{3\sqrt{3}} \left(\frac{\eta_0}{L}\right)^2 (1+\tau)^{8/3} \int \left| \frac{\partial^2 \Phi}{\partial \hat{r}^2} - \frac{1}{\hat{r}} \frac{\partial \Phi}{\partial \hat{r}} \right| \hat{r}^2 d\hat{r} \tag{5.63}$$

for the gravitational entropy. These integrals were calculated numerically using MATLAB in a similar way as in the two previous cases. The physical constant α_0, κ_{T_0} and $\frac{\eta_0}{L}$ were chosen as earlier. The script which performs the necessary numerics can be found in appendix E.2.3.

The time evolution of the thermodynamic entropy can be seen in figure 5.11. As expected, it decreases with time as the density perturbation grows. The sum of the thermodynamic entropy and the gravitational entropy increases when χ is chosen so that

$$\chi_{\text{sphere}} > \chi = 5.9 \cdot 10^8 \,\text{m}^{-3} \,. \tag{5.64}$$

In figure 5.12 we can find a plot of such a growing total entropy with $\chi = 5.0 \cdot 10^9 \,\text{m}^{-3}$.

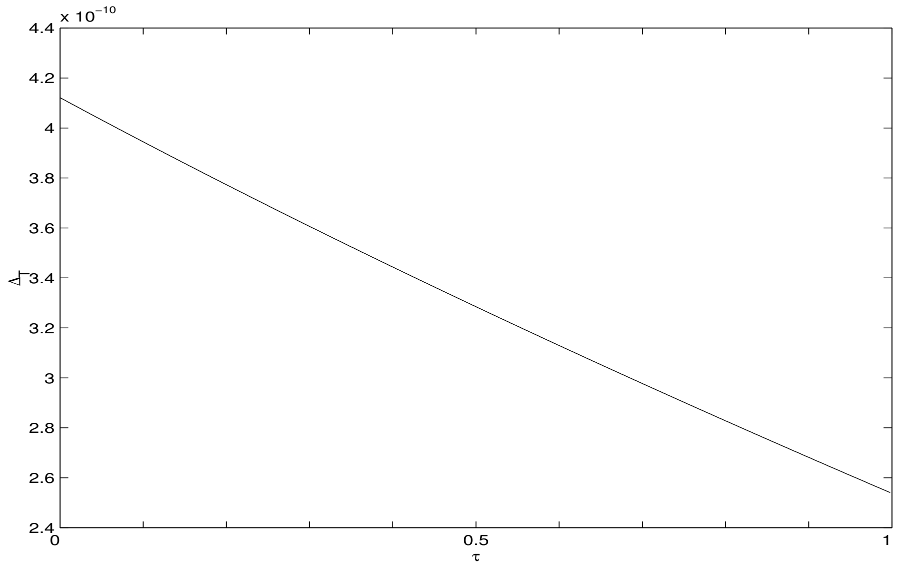


Figure 5.11: *An illustration of the time evolution of the thermodynamic entropy contrast for the Gauss-like, spherically symmetric density perturbation.*

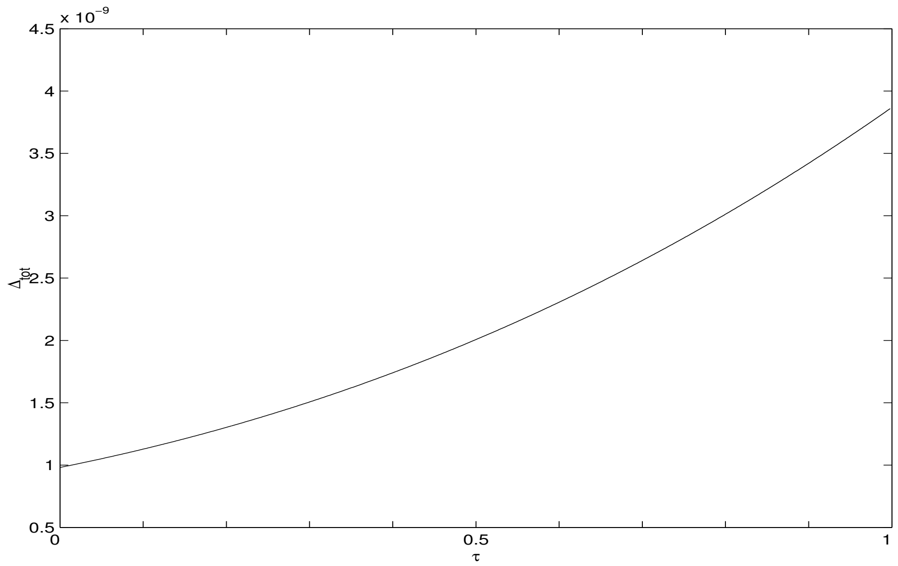


Figure 5.12: *A plot of the total entropy contrast for the Gauss-like, spherically symmetric density perturbation. The value of* χ *is chosen to be* $\chi = 5.0 \cdot 10^9\ m^{-3}$.

Chapter 6

Conclusion and summary

The main objective of this thesis was to consider the concept of gravitational entropy from a perturbative point of view. A flat, matter dominated universe model was subjected to a linear perturbation, and the entropic properties of the resulting universe were investigated.

We confirmed that the classical thermodynamic entropy of such a universe decreases with time as the perturbation grows, which appears to be a breach of the second law of thermodynamics (SLT). According to the *Weyl curvature hypothesis*, this can be rectified by adding an extra term to the entropy called the gravitational entropy. This term arises from the geometry of space-time and takes into account the attractive nature of the gravitational force. The sum of the classical thermodynamic and the gravitational entropy defines a general entropy. It is this general entropy quantity, and not the classical entropy alone, that must satisfy the SLT.

We considered a special type of perturbations which were products of second order polynomials and Gauss functions, and showed that the SLT is indeed satisfied if the constant χ, which appears in the expression for the gravitational entropy, is chosen larger than a certain value. However, this minimal value appears to depend on the form of the perturbations. There does not appear to exist a definite value of χ which makes the entropy of *all* types of perturbations grow. We saw for example that if we chose $5.9 \cdot 10^8 \mathrm{m}^{-3} < \chi < 2.0 \cdot 10^9 \mathrm{m}^{-3}$ the total entropy of the spherically symmetric perturbation would grow, whereas the total entropy of the plane symmetric perturbation would not. Thus, since the minimal value of χ which makes the total entropy grow seems to depend on the perturbation, it is natural to assume that for every value of χ we can construct a perturbation for which the total entropy doesn't grow. There is, in other words, no χ which preserves the SLT for all types of perturbations.

Our conclusion is therefore that adding (5.31) as a measure of gravitational entropy to the classical entropy doesn't preserve the SLT in the general case. But if we accept that χ can take different values depending on the perturbations, then the SLT is preserved individually for each type of perturbation.

The cosmological fluid which was used in our calculations in this thesis was the simples possible fluid, i.e. a one-component, pressureless fluid. This simplified the calculations greatly and allowed us to write simple expression for the various quantites which had to be determined. For a more realistic model we would have had to include radiation in equilibrium with the matter. In addition to the entropy quantities which we calculated in this thesis, we would then have had to include entropy due to the radiation. Since radiation contributes to entropy in the universe with a factor that is about $\sim 10^7$ greater than the contribution of baryonic matter, it is quite possible that the behaviour of the entropy quantities could differ greatly from that in the model which was considered. But a similar analysis of such complex models would certainly be much more difficult and time consuming than for the simple model which we considered. Therefore we will not pursue this idea any further, and instead conclude this thesis here.

Appendix A

The Lie derivative of a tensor of rank two

Let $A_{\mu\nu}(x)$ be some covariant tensor of rank two. Consider the infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu \,. \tag{A.1}$$

The transformation matrices between the two sets of coordinates are

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu - \frac{\partial \xi^\mu}{\partial x^\nu}, \tag{A.2}$$

and

$$\begin{aligned}\frac{\partial x^\nu}{\partial x'^\mu} &= \delta^\nu_\mu + \frac{\partial \xi^\nu}{\partial x'^\mu} = \delta^\nu_\mu + \frac{\partial \xi^\nu}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial x'^\mu} = \delta^\nu_\mu + \frac{\partial \xi^\nu}{\partial x^\lambda}\left\{\delta^\lambda_\mu + \frac{\partial \xi^\lambda}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial x'^\mu}\right\}\\ &= \delta^\nu_\mu + \frac{\partial \xi^\nu}{\partial x^\mu} + \mathcal{O}(2)\,.\end{aligned}$$

Up to first order, the last expression can be written as

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu + \frac{\partial \xi^\nu}{\partial x^\mu}\,. \tag{A.3}$$

The coordinate transformation (A.1) changes the tensor A relative to the new coordinates x'^μ. The new tensor is related to the old one via the usual transformation rule for tensors,

$$A'_{\mu\nu}(x') = A_{\lambda\sigma}(x)\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\,. \tag{A.4}$$

Next, we make a Taylor expansion of $A'_{\mu\nu}(x)$ about $A'_{\mu\nu}(x')$,

$$A'_{\mu\nu}(x) = A'_{\mu\nu}(x'+\xi) \simeq A'_{\mu\nu}(x') + \frac{\partial A'_{\mu\nu}(x')}{\partial x'^\lambda}\xi^\lambda \simeq A'_{\mu\nu}(x') + \frac{\partial A'_{\mu\nu}(x')}{\partial x^\lambda}\xi^\lambda\,. \tag{A.5}$$

In this expression, we can rewrite $A'_{\mu\nu}(x')$ in terms of $A_{\mu\nu}(x)$ and ξ^μ by using (A.4) and (A.3),

$$\begin{aligned} A'_{\mu\nu}(x') &= A_{\lambda\sigma}(x)\left(\delta^\lambda_\mu + \frac{\partial\xi^\lambda(x)}{\partial x^\mu}\right)\left(\delta^\sigma_\nu + \frac{\partial\xi^\sigma(x)}{\partial x^\nu}\right) \\ &\simeq A_{\mu\nu}(x) + A_{\mu\lambda}(x)\frac{\partial\xi^\lambda(x)}{\partial x^\nu} + A_{\lambda\nu}(x)\frac{\partial\xi^\lambda(x)}{\partial x^\nu}. \end{aligned} \tag{A.6}$$

We insert this expression into the right hand side of (A.5) and keep terms up to the first order, which yields

$$A'_{\mu\nu}(x) = A_{\mu\nu}(x) + A_{\mu\lambda}(x)\frac{\partial\xi^\lambda(x)}{\partial x^\nu} + A_{\lambda\nu}(x)\frac{\partial\xi^\lambda(x)}{\partial x^\mu} + \frac{\partial A_{\mu\nu}(x)}{\partial x^\lambda}\xi^\lambda \tag{A.7}$$

Thus, using the defintion of the Lie derivative (2.24), we get

$$\mathcal{L}_\xi A_{\mu\nu}(x) = A_{\mu\lambda}(x)\xi^\lambda_{,\nu} + A_{\lambda\nu}(x)\xi^\lambda_{,\mu} + A_{\mu\nu,\lambda}(x)\xi^\lambda\,. \tag{A.8}$$

Appendix B

The Lie derivative of the scalar metric

Using (2.29), we get

$$
\begin{aligned}
[\mathcal{L}_\varepsilon g]_{00} &= g_{0\lambda}\varepsilon^\lambda_{,0} + g_{\lambda 0}\varepsilon^\lambda_{,0} + g_{00,\lambda}\varepsilon^\lambda = 2g_{0\lambda}\varepsilon^\lambda_{,0} + g_{00,\lambda}\varepsilon^\lambda \\
&= 2g_{00}\varepsilon^0_{,0} + 2g_{0i}\varepsilon^i_{,0} + g_{00,0}\varepsilon^0 + g_{00,i}\varepsilon^i \\
&= 2a^2(1+2\phi)(\varepsilon^0)' - 2a^2 B_{,i}\partial^i\varepsilon' + [a^2(1+2\phi)]'\varepsilon^0 + 2a^2\phi_{,i}\partial^i\varepsilon \\
&\simeq 2a^2(\varepsilon^0)' + [2aa'(1+2\phi) + 2a^2\phi']\varepsilon^0 \\
&\simeq 2a^2(\varepsilon^0)' + 2aa'\varepsilon^0 \,, \qquad \text{(B.1)} \\
[\mathcal{L}_\varepsilon g]_{0i} &= g_{0\lambda}\varepsilon^\lambda_{,i} + g_{\lambda i}\varepsilon^\lambda_{,0} + g_{0i,\lambda}\varepsilon^\lambda \\
&= g_{00}\varepsilon^0_{,i} + g_{0j}\varepsilon^j_{,i} + g_{0i}\varepsilon^0_{,0} + g_{ji}\varepsilon^j_{,0} + g_{0i,0}\varepsilon^0 + g_{0i,j}\varepsilon^j \\
&= a^2(1+2\phi)\varepsilon^0_{,i} - a^2 B_{,j}\partial^j\varepsilon_{,i} - a^2 B_{,i}(\varepsilon^0)' \\
&\quad - a^2[(1-2\psi)\delta_{ij} + 2E_{,ij}]\partial^j\varepsilon' - (a^2 B_i)'\varepsilon^0 - a^2 B_{i,j}\partial^j\varepsilon \\
&= a^2(\varepsilon^0 - \varepsilon')_{,i} \,, \qquad \text{(B.2)} \\
[\mathcal{L}_\varepsilon g]_{ij} &= g_{i\lambda}\varepsilon^\lambda_{,j} + g_{\lambda j}\varepsilon^\lambda_{,i} + g_{ij,\lambda}\varepsilon^\lambda \\
&= g_{i0}\varepsilon^0_{,j} + g_{ik}\varepsilon^k_{,j} + g_{0j}\varepsilon^0_{,i} + g_{kj}\varepsilon^k_{,i} + g_{ij,0}\varepsilon^0 + g_{ij,k}\varepsilon^k \\
&= -a^2 B_{,i}\varepsilon^0_{,j} - a^2[(1-2\psi)\delta_{ij} + 2E_{,ik}]\varepsilon_{,kj} - a^2 B_{,j}\varepsilon^0_{,i} \\
&\quad - a^2[(1-2\psi)\delta_{kj} + 2E_{,kj}]\varepsilon_{,ki} - \left(a^2[(1-2\psi)\delta_{ij} + 2E_{,ij}]\right)'\varepsilon^0 \\
&\quad - \left(a^2[(1-2\psi)\delta_{ij} + 2E_{,ij}]\right)_{,k}\varepsilon_{,k} \\
&\simeq a^2\varepsilon_{,ij} + a^2\varepsilon_{,ji} + 2aa'\varepsilon^0\delta_{ij} = 2a^2\left(\varepsilon_{,ij} + \frac{a'}{a}\varepsilon^0\delta_{ij}\right) \,. \qquad \text{(B.3)}
\end{aligned}
$$

Appendix C

Calculation of the Einstein tensor using Maple

C.1 Maple commands and output

The command 'grtw()' loads the GRTensorII package in Maple in the computers at the theory group. This is a special function which is written spesifically for use at the theory group. In an ordinary computer the package can be loaded by typing the following two commands: 'readlib(grii):' and 'grtensor()'.

```
> grtw():
```

GRTensorII Version 1.79 (R4)
6 February 2001
Developed by Peter Musgrave, Denis Pollney and Kayll Lake
Copyright 1994 − 2001 by the authors.
Latest version available from : http : //grtensor.phy.queensu.ca/

The GRTensorII package is now loaded. We must then spesify which metric we want to use. There is a whole host of metrics that are predefined and which can be loaded into the memory by using the 'qload' command. For example, if we wish to use the Kerr metric, we simply type 'qload(kerr):'. In our case, we wish to define a new metric, namely the metric for conformal Newtonian perturbations. Then we must use the 'makeg' command, which tells Maple that we wish to construct our own metric. We give the name of the metric as argument to this command.

```
> makeg(conf_newt):
```

```
Makeg 2.0: GRTensor metric/basis entry utility

To quit makeg, type 'exit' at any prompt.

Do you wish to enter a 1) metric [g(dn,dn)],
                       2) line element [ds],
                       3) non-holonomic basis [e(1)...e(n)], or
                       4) NP tetrad [l,n,m,mbar]?
```

There are several ways in which we can type in the metric. We choose to type in the line element.

```
> 2:

Enter coordinates as a LIST (eg. [t,r,theta,phi]):
> [eta,x,y,z]:

Enter the line element using d[coord] to indicate differentials.

(for example,  r^2*(d[theta]^2 + sin(theta)^2*d[phi]^2)

[Type 'exit' to quit makeg]

 ds^2 =
> a(eta)^2*((1+2*alpha*phi(eta,x,y,z))*d[eta]^2-(1-2*alpha
*psi(eta,x,y,z))*(d[x]^2+d[y]^2+d[z]^2)):


If there are any complex valued coordinates, constants or functions
for this spacetime, please enter them as a SET ( eg. { z, psi } ).

Complex quantities [default={}]:
> {}:
```

The values you have entered are :

$$Coordinates = [\eta,\, x,\, y,\, z]$$

Metric :

$$g_{a\ \ b} = \begin{bmatrix} \mathrm{a}(\eta)^2 + 2\,\mathrm{a}(\eta)^2\,\alpha\,\phi(\eta,\,x,\,y,\,z) & 0 & 0 & 0 \\ 0 & \%1 & 0 & 0 \\ 0 & 0 & \%1 & 0 \\ 0 & 0 & 0 & \%1 \end{bmatrix}$$

$$\%1 := -\mathrm{a}(\eta)^2 + 2\,\mathrm{a}(\eta)^2\,\alpha\,\psi(\eta,\,x,\,y,\,z)$$

```
You may choose to 0) Use the metric WITHOUT saving it,
                  1) Save the metric as it is,
                  2) Correct an element of the metric,
                  3) Re-enter the metric,
                  4) Add/change constraint equations,
                  5) Add a text description, or
                  6) Abandon this metric and return to Maple.

> 1;

Information written to: "grii/conf_newt.mpl"
Do you wish to use this spacetime in the current session?
(1=yes [default], other=no):

> 1;

Initializing: conf_newt

Calculated ds for conf_newt (0.000000 sec.)
```

$$\mathit{Default\ spacetime} = \mathit{conf\_newt}$$

$$\mathit{For\ the\ conf\_newt\ spacetime} :$$

$$\mathit{Coordinates}$$

$$\mathrm{x}(\mathit{up})$$

$$x^{\ a} = [\eta,\ x,\ y,\ z]$$

$$\mathit{Line\ element}$$

$$ds^2 = (\mathrm{a}(\eta)^2 + 2\,\mathrm{a}(\eta)^2\,\alpha\,\phi(\eta,\ x,\ y,\ z))\ \ d\,\eta^2\ \ + \%1\ \ d\,x^2\ \ + \%1\ \ d\,y^2\ \ + \%1\ \ d\,z^2$$

$$\%1 := -\mathrm{a}(\eta)^2 + 2\,\mathrm{a}(\eta)^2\,\alpha\,\psi(\eta,\ x,\ y,\ z)$$

```
makeg() completed.
```

The construction of the metric is now finished and we are ready to calculate the components of any tensor we want. Notice that in our definition of the metric we have included a parameter 'alpha' wherever there is a first order quantity. This is done so that we can expand the quantities we calculate to the first order in this parameter and thereby getting rid of all terms with an order higher that this. The tensor we wish to calculate is the Einstein tensor with mixed components, with the first index contravariant ('up') and the second covariant ('dn').

```
> grcalc(G(up,dn)):

Created definition for G(up,dn)
Calculated detg for conf_newt (0.000000 sec.)
```

```
Calculated g(up,up) for conf_newt (.019000 sec.)
Calculated g(dn,dn,pdn) for conf_newt (.010000 sec.)
Calculated Chr(dn,dn,dn) for conf_newt (0.000000 sec.)
Calculated Chr(dn,dn,up) for conf_newt (.011000 sec.)
Calculated R(dn,dn) for conf_newt (.070000 sec.)
Calculated Ricciscalar for conf_newt (.010000 sec.)
Calculated G(dn,dn) for conf_newt (.069000 sec.)
Calculated G(up,dn) for conf_newt (.030000 sec.)
```

$$CPU\ \ Time\ = .289$$

The tensor is now caluculated, but Maple doesn't display it unless we tell it to do so explicitly. The command that makes Maple display calculated quantities is 'grdisplay'. But if we were to display the Einstein tensor now, it will be done so to an arbitrary order. We will therefore first tell Maple to expand the components to first order. This is done by first making a Taylor expansion to first order, and then telling it to convert the answer to a polynomial. Finally, we put the expansion parameter 'alpha' equal to one. The commands that we use are the standard Maple commands 'taylor', 'convert', 'polynom' and 'subs'. But these must be used in junction with the GRTensorII command 'grmap', which tells Maple to apply the ordinary Maple commands to all components of the tensor.

```
> grmap(G(up,dn),taylor,'x',alpha=0,2);

Applying routine taylor to G(up,dn)

> grmap(G(up,dn),convert,'x',polynom);

Applying routine convert to G(up,dn)

> grmap(G(up,dn),subs,alpha=1,'x');

Applying routine subs to G(up,dn)

> grmap(G(up,dn),simplify,'x');

Applying routine simplify to G(up,dn)
```

The result is now displayed by typing the command 'grdisplay'.

```
> grdisplay(G(up,dn));
```

$$For\ the\ conf\_newt\ spacetime:$$

$$G(up,dn)$$

$$\mathrm{G}(up,\ dn)$$

$$G^{\eta}{}_{\eta} = (3\,(\tfrac{\partial}{\partial \eta}\,\mathrm{a}(\eta))^2 + 2\,(\tfrac{\partial^2}{\partial x^2}\,\psi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2 + 2\,(\tfrac{\partial^2}{\partial z^2}\,\psi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2$$
$$- 6\,(\tfrac{\partial}{\partial \eta}\,\mathrm{a}(\eta))\,\mathrm{a}(\eta)\,(\tfrac{\partial}{\partial \eta}\,\psi(\eta,\,x,\,y,\,z)) + 2\,(\tfrac{\partial^2}{\partial y^2}\,\psi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2$$
$$- 6\,\phi(\eta,\,x,\,y,\,z)\,(\tfrac{\partial}{\partial \eta}\,\mathrm{a}(\eta))^2)\Big/\mathrm{a}(\eta)^4$$

$$G^{x}{}_{\eta} = -2\,\frac{(\frac{\partial}{\partial x}\,\phi(\eta,\,x,\,y,\,z))\,(\frac{\partial}{\partial \eta}\,\mathrm{a}(\eta)) + \mathrm{a}(\eta)\,(\frac{\partial^2}{\partial x\,\partial \eta}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^3}$$

$$G^{y}{}_{\eta} = -2\,\frac{(\frac{\partial}{\partial y}\,\phi(\eta,\,x,\,y,\,z))\,(\frac{\partial}{\partial \eta}\,\mathrm{a}(\eta)) + \mathrm{a}(\eta)\,(\frac{\partial^2}{\partial y\,\partial \eta}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^3}$$

$$G^{z}{}_{\eta} = -2\,\frac{(\frac{\partial}{\partial z}\,\phi(\eta,\,x,\,y,\,z))\,(\frac{\partial}{\partial \eta}\,\mathrm{a}(\eta)) + \mathrm{a}(\eta)\,(\frac{\partial^2}{\partial z\,\partial \eta}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^3}$$

$$G^{\eta}{}_{x} = 2\,\frac{(\frac{\partial}{\partial x}\,\phi(\eta,\,x,\,y,\,z))\,(\frac{\partial}{\partial \eta}\,\mathrm{a}(\eta)) + \mathrm{a}(\eta)\,(\frac{\partial^2}{\partial x\,\partial \eta}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^3}$$

$$G^{x}{}_{x} = (-(\tfrac{\partial}{\partial \eta}\,\mathrm{a}(\eta))^2 + 2\,\mathrm{a}(\eta)\,(\tfrac{\partial^2}{\partial \eta^2}\,\mathrm{a}(\eta)) + 2\,\phi(\eta,\,x,\,y,\,z)\,(\tfrac{\partial}{\partial \eta}\,\mathrm{a}(\eta))^2$$
$$- 4\,\mathrm{a}(\eta)\,(\tfrac{\partial^2}{\partial \eta^2}\,\mathrm{a}(\eta))\,\phi(\eta,\,x,\,y,\,z) - 2\,\mathrm{a}(\eta)^2\,(\tfrac{\partial^2}{\partial \eta^2}\,\psi(\eta,\,x,\,y,\,z))$$
$$+ (\tfrac{\partial^2}{\partial z^2}\,\psi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2 - (\tfrac{\partial^2}{\partial y^2}\,\phi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2 + (\tfrac{\partial^2}{\partial y^2}\,\psi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2$$
$$- 4\,(\tfrac{\partial}{\partial \eta}\,\mathrm{a}(\eta))\,\mathrm{a}(\eta)\,(\tfrac{\partial}{\partial \eta}\,\psi(\eta,\,x,\,y,\,z)) - (\tfrac{\partial^2}{\partial z^2}\,\phi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2$$
$$- 2\,\mathrm{a}(\eta)\,(\tfrac{\partial}{\partial \eta}\,\phi(\eta,\,x,\,y,\,z))\,(\tfrac{\partial}{\partial \eta}\,\mathrm{a}(\eta)))\Big/\mathrm{a}(\eta)^4$$

$$G^{y}{}_{x} = -\frac{-(\frac{\partial^2}{\partial y\,\partial x}\,\phi(\eta,\,x,\,y,\,z)) + (\frac{\partial^2}{\partial y\,\partial x}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^2}$$

$$G^{z}{}_{x} = -\frac{(\frac{\partial^2}{\partial z\,\partial x}\,\psi(\eta,\,x,\,y,\,z)) - (\frac{\partial^2}{\partial z\,\partial x}\,\phi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^2}$$

$$G^{\eta}{}_{y} = 2\,\frac{(\frac{\partial}{\partial y}\,\phi(\eta,\,x,\,y,\,z))\,(\frac{\partial}{\partial \eta}\,\mathrm{a}(\eta)) + \mathrm{a}(\eta)\,(\frac{\partial^2}{\partial y\,\partial \eta}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^3}$$

$$G^{x}{}_{y} = -\frac{-(\frac{\partial^2}{\partial y\,\partial x}\,\phi(\eta,\,x,\,y,\,z)) + (\frac{\partial^2}{\partial y\,\partial x}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^2}$$

$$G^{y}{}_{y} = (-(\tfrac{\partial}{\partial\eta}\,\mathrm{a}(\eta))^2 + 2\,\mathrm{a}(\eta)\,(\tfrac{\partial^2}{\partial\eta^2}\,\mathrm{a}(\eta)) + 2\,\phi(\eta,\,x,\,y,\,z)\,(\tfrac{\partial}{\partial\eta}\,\mathrm{a}(\eta))^2$$
$$-\,4\,\mathrm{a}(\eta)\,(\tfrac{\partial^2}{\partial\eta^2}\,\mathrm{a}(\eta))\,\phi(\eta,\,x,\,y,\,z) - 2\,\mathrm{a}(\eta)^2\,(\tfrac{\partial^2}{\partial\eta^2}\,\psi(\eta,\,x,\,y,\,z))$$
$$+\,(\tfrac{\partial^2}{\partial z^2}\,\psi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2 - (\tfrac{\partial^2}{\partial x^2}\,\phi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2 + (\tfrac{\partial^2}{\partial x^2}\,\psi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2$$
$$-\,4\,(\tfrac{\partial}{\partial\eta}\,\mathrm{a}(\eta))\,\mathrm{a}(\eta)\,(\tfrac{\partial}{\partial\eta}\,\psi(\eta,\,x,\,y,\,z)) - (\tfrac{\partial^2}{\partial z^2}\,\phi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2$$
$$-\,2\,\mathrm{a}(\eta)\,(\tfrac{\partial}{\partial\eta}\,\phi(\eta,\,x,\,y,\,z))\,(\tfrac{\partial}{\partial\eta}\,\mathrm{a}(\eta)))\Big/\mathrm{a}(\eta)^4$$

$$G^{z}{}_{y} = \frac{(\frac{\partial^2}{\partial z\,\partial y}\,\phi(\eta,\,x,\,y,\,z)) - (\frac{\partial^2}{\partial z\,\partial y}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^2}$$

$$G^{\eta}{}_{z} = 2\,\frac{(\frac{\partial}{\partial z}\,\phi(\eta,\,x,\,y,\,z))\,(\frac{\partial}{\partial\eta}\,\mathrm{a}(\eta)) + \mathrm{a}(\eta)\,(\frac{\partial^2}{\partial z\,\partial\eta}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^3}$$

$$G^{x}{}_{z} = -\frac{(\frac{\partial^2}{\partial z\,\partial x}\,\psi(\eta,\,x,\,y,\,z)) - (\frac{\partial^2}{\partial z\,\partial x}\,\phi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^2}$$

$$G^{y}{}_{z} = \frac{(\frac{\partial^2}{\partial z\,\partial y}\,\phi(\eta,\,x,\,y,\,z)) - (\frac{\partial^2}{\partial z\,\partial y}\,\psi(\eta,\,x,\,y,\,z))}{\mathrm{a}(\eta)^2}$$

$$G^{z}{}_{z} = (-(\tfrac{\partial}{\partial\eta}\,\mathrm{a}(\eta))^2 + 2\,\mathrm{a}(\eta)\,(\tfrac{\partial^2}{\partial\eta^2}\,\mathrm{a}(\eta)) + 2\,\phi(\eta,\,x,\,y,\,z)\,(\tfrac{\partial}{\partial\eta}\,\mathrm{a}(\eta))^2$$
$$-\,4\,\mathrm{a}(\eta)\,(\tfrac{\partial^2}{\partial\eta^2}\,\mathrm{a}(\eta))\,\phi(\eta,\,x,\,y,\,z) + (\tfrac{\partial^2}{\partial y^2}\,\psi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2$$
$$+\,(\tfrac{\partial^2}{\partial x^2}\,\psi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2 - (\tfrac{\partial^2}{\partial x^2}\,\phi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2 - (\tfrac{\partial^2}{\partial y^2}\,\phi(\eta,\,x,\,y,\,z))\,\mathrm{a}(\eta)^2$$
$$-\,4\,(\tfrac{\partial}{\partial\eta}\,\mathrm{a}(\eta))\,\mathrm{a}(\eta)\,(\tfrac{\partial}{\partial\eta}\,\psi(\eta,\,x,\,y,\,z)) - 2\,\mathrm{a}(\eta)^2\,(\tfrac{\partial^2}{\partial\eta^2}\,\psi(\eta,\,x,\,y,\,z))$$
$$-\,2\,\mathrm{a}(\eta)\,(\tfrac{\partial}{\partial\eta}\,\phi(\eta,\,x,\,y,\,z))\,(\tfrac{\partial}{\partial\eta}\,\mathrm{a}(\eta)))\Big/\mathrm{a}(\eta)^4$$

C.2 Simplified expressions

We now wish to simplify the expressions that Maple yields for the Einstein tensor. The time-time component is

$$G_0^0 = \frac{2}{a^2}\left[\frac{3}{2}\mathcal{H}^2 + \nabla^2\Psi - 3\mathcal{H}\dot{\Psi} - 3\Phi\mathcal{H}^2\right].$$

The zeroth and first order parts of this component are

$$ {}^{(0)}G^0_0 = \frac{3}{a^2}\mathcal{H}^2 \,, \tag{C.1} $$

$$ \delta G^0_0 = \frac{2}{a^2}\left[\nabla^2\psi - 3\mathcal{H}(\dot\Psi + \Phi\mathcal{H})\right]. \tag{C.2} $$

Since the line element is symmetric in the spatial coordinates, we need only to consider the '$0x$' component when we wish to find the general expression for the '$0i$' components of the Einstein tensor. According to the Maple output above, the '$0x$' component is

$$ G^0_x = \frac{2}{a^3}\left[a\dot\Psi_{,x} + \dot a\Phi_{,x}\right] = \frac{2}{a^2}\left[\dot\Psi + \mathcal{H}\Phi\right]_{,x} . $$

Thus, the general time-space components of the Einstein tensor are

$$ G^0_i = \frac{2}{a^2}\left[\dot\Psi + \mathcal{H}\Phi\right]_{,i} . $$

We see that these components vanish to the zeroth order,

$$ {}^{(0)}G^0_i = 0 \,, \tag{C.3} $$

$$ \delta G^0_i = \frac{2}{a^2}\left[\dot\Psi + \mathcal{H}\Phi\right]_{,i} . \tag{C.4} $$

The remaining components that we have to consider are the space-space components. Again, we can collect all the space-space components that Maple has calculated into one expression by using the symmetry of the line element with respect to the spatial components. But before we can do this, we must first consider the G^i_j components with $i = j$ and with $i \neq j$ separately. The G^x_x component is

$$ G^x_x = -\frac{1}{a^4}\Big[\dot a^2 - 2a\ddot a - 2\Phi\dot a^2 + 4a\ddot a\Phi + a^2(\Phi-\Psi)_{,yy} + a^2(\Phi-\Psi)_{,zz} + \\ 2a^2\ddot\Psi + 4a\dot a\dot\Psi + 2a\dot a\dot\Phi\Big] . $$

Defining $D = \Phi - \Psi$ and using the del operator, we can rewrite this as

$$ G^x_x = -\frac{1}{a^4}\Big[\dot a^2 - 2a\ddot a - 2\Phi\dot a^2 + 4a\ddot a\Phi + a^2\nabla^2 D - a^2 D_{,xx} \\ 2a^2\ddot\Psi + 4a\dot a\dot\Psi + 2a\dot a\dot\Phi\Big] . $$

This expression leads us to the more general G^i_j,

$$G^i_j = -\frac{1}{a^4}\Big[\dot{a}^2 - 2a\ddot{a} - 2\Phi\dot{a}^2 + 4a\ddot{a}\Phi + a^2\nabla^2 D - a^2 D_{,ij}$$
$$2a^2\ddot{\Psi} + 4a\dot{a}\dot{\Psi} + 2a\dot{a}\dot{\Phi}\Big], \quad \text{for } i \neq j\,.$$

If $i \neq j$ we can consider, for example, the G^x_y component. This is simply

$$G^x_y = \frac{1}{a^2}D_{,xy}\,,$$

or more generally

$$G^i_j = \frac{1}{a^2}D_{,ij}\,.$$

This can be combined with the result above to give us the general space-space component of the Einstein tensor,

$$G^i_j = \frac{1}{a^2}\Bigg[\Big\{\mathcal{H}^2 + 2\dot{\mathcal{H}} - 2\big([\mathcal{H}^2 + 2\dot{\mathcal{H}}]\Phi + \ddot{\Psi} + 2\mathcal{H}\dot{\Psi} + \mathcal{H}\dot{\Phi} + \frac{1}{2}\nabla^2 D\big)\Big\}\delta^i_j$$
$$+ D_{,ij}\Bigg]\,.$$

The two first terms in this expression are zeroth order, while the remaining are first order,

$$^{(0)}G^i_j = \frac{1}{a^2}\left[\mathcal{H}^2 + 2\dot{\mathcal{H}}\right]\delta^i_j\,, \tag{C.5}$$
$$\delta G^i_j = -\frac{2}{a^2}\left[\Big([\mathcal{H}^2 + 2\dot{\mathcal{H}}]\Phi + \ddot{\Psi} + 2\mathcal{H}\dot{\Psi} + \mathcal{H}\dot{\Phi} + \frac{1}{2}\nabla^2 D\Big)\delta^i_j - \frac{1}{2}D_{,ij}\right]. \tag{C.6}$$

Appendix D

Calculation of the Christoffel symbols

```
> restart:
> grtw():
```

$$\textit{GRTensorII Version 1.79 (R4)}$$
$$\textit{6 February 2001}$$
$$\textit{Developed by Peter Musgrave, Denis Pollney and Kayll Lake}$$
$$\textit{Copyright 1994 - 2001 by the authors.}$$
$$\textit{Latest version available from: http://grtensor.phy.queensu.ca/}$$

Instead of defining the metric from scratch, we simply load the previously defined metric by using the 'qload' command.

```
> qload(conf_newt):
```

```
Calculated ds for conf_newt (0.000000 sec.)
```

$$\textit{Default spacetime} = \textit{conf\_newt}$$
$$\textit{For the conf\_newt spacetime:}$$
$$\textit{Coordinates}$$
$$\mathrm{x}(up)$$
$$x^{\ a} = [\eta,\, x,\, y,\, z]$$
$$\textit{Line element}$$

$$ds^2 = (\mathrm{a}(\eta)^2 + 2\,\mathrm{a}(\eta)^2\,\alpha\,\phi(\eta,\, x,\, y,\, z))\ \ d\,\eta^2\ \ + \%1\ \ d\,x^2\ \ + \%1\ \ d\,y^2\ \ + \%1\ \ d\,z^2$$
$$\%1 := -\mathrm{a}(\eta)^2 + 2\,\mathrm{a}(\eta)^2\,\alpha\,\phi(\eta,\, x,\, y,\, z)$$

We wish to calculate the Christoffel symbols. There is a built-in definition of the Christoffel symbols in the GrTensorII package, but this differs from the usual definition of the Christoffel symbols. For example, the built-in Christoffel symbols are not symmetric in the two lower indices. Instead, we must define our own Christoffel symbols. We use the built-in function grdef, which defines tensors from pre-existing ones, and call the 'new' Christoffel symbols Gamma.

```
>  grdef(`Gamma{^a b c} := 1/2*g{^a ^d}*(g{b d, c}
+g{c d, b}-g{b c, d})`);

Created definition for Gamma(up,dn,dn)

>  grcalc(Gamma(up,dn,dn)):

Calculated detg for conf_newt (0.000000 sec.)
Calculated g(up,up) for conf_newt (0.000000 sec.)
Calculated g(dn,dn,pdn) for conf_newt (0.000000 sec.)
Calculated Gamma(up,dn,dn) for conf_newt (.010000 sec.)
```

$$CPU\ \ Time\ = .010$$

```
>  grmap(Gamma(up,dn,dn),taylor,'x',alpha=0,2);

Applying routine taylor to Gamma(up,dn,dn)

>  grmap(Gamma(up,dn,dn),convert,'x',polynom);

Applying routine convert to Gamma(up,dn,dn)

>  grmap(Gamma(up,dn,dn),subs,alpha=1,'x');

Applying routine subs to Gamma(up,dn,dn)
```

In order to simplify the answers, we tell the program to replace $\frac{1}{a(\eta)}\frac{da(\eta)}{d\eta}$ with the Hubble parameter H.

```
>  grmap(Gamma(up,dn,dn),subs,'diff(a(eta),eta)=
a(eta)*H(eta)','x');

Applying routine subs to Gamma(up,dn,dn)

>  grmap(Gamma(up,dn,dn),simplify,'x');

Applying routine simplify to Gamma(up,dn,dn)

>  grdisplay(Gamma(up,dn,dn));
```

$$For\ the\ conf\_newt\ spacetime:$$

$$Gamma(up,\, dn,\, dn)$$

$$\Gamma^{\eta}{}_{\eta\eta} = \mathrm{H}(\eta) + (\frac{\partial}{\partial\eta}\,\phi(\eta,\, x,\, y,\, z))$$

$$\Gamma^{x}{}_{\eta\eta} = \frac{\partial}{\partial x}\phi(\eta, x, y, z)$$
$$\Gamma^{y}{}_{\eta\eta} = \frac{\partial}{\partial y}\phi(\eta, x, y, z)$$
$$\Gamma^{z}{}_{\eta\eta} = \frac{\partial}{\partial z}\phi(\eta, x, y, z)$$
$$\Gamma^{\eta}{}_{x\eta} = \frac{\partial}{\partial x}\phi(\eta, x, y, z)$$
$$\Gamma^{x}{}_{x\eta} = \mathrm{H}(\eta) - (\frac{\partial}{\partial \eta}\phi(\eta, x, y, z))$$
$$\Gamma^{\eta}{}_{y\eta} = \frac{\partial}{\partial y}\phi(\eta, x, y, z)$$
$$\Gamma^{y}{}_{y\eta} = \mathrm{H}(\eta) - (\frac{\partial}{\partial \eta}\phi(\eta, x, y, z))$$
$$\Gamma^{\eta}{}_{z\eta} = \frac{\partial}{\partial z}\phi(\eta, x, y, z)$$
$$\Gamma^{z}{}_{z\eta} = \mathrm{H}(\eta) - (\frac{\partial}{\partial \eta}\phi(\eta, x, y, z))$$
$$\Gamma^{\eta}{}_{\eta x} = \frac{\partial}{\partial x}\phi(\eta, x, y, z)$$
$$\Gamma^{x}{}_{\eta x} = \mathrm{H}(\eta) - (\frac{\partial}{\partial \eta}\phi(\eta, x, y, z))$$
$$\Gamma^{\eta}{}_{xx} = \mathrm{H}(\eta) - 4\,\mathrm{H}(\eta)\,\phi(\eta, x, y, z) - (\frac{\partial}{\partial \eta}\phi(\eta, x, y, z))$$
$$\Gamma^{x}{}_{xx} = -(\frac{\partial}{\partial x}\phi(\eta, x, y, z))$$
$$\Gamma^{y}{}_{xx} = \frac{\partial}{\partial y}\phi(\eta, x, y, z)$$
$$\Gamma^{z}{}_{xx} = \frac{\partial}{\partial z}\phi(\eta, x, y, z)$$
$$\Gamma^{x}{}_{yx} = -(\frac{\partial}{\partial y}\phi(\eta, x, y, z))$$
$$\Gamma^{y}{}_{yx} = -(\frac{\partial}{\partial x}\phi(\eta, x, y, z))$$
$$\Gamma^{x}{}_{zx} = -(\frac{\partial}{\partial z}\phi(\eta, x, y, z))$$
$$\Gamma^{z}{}_{zx} = -(\frac{\partial}{\partial x}\phi(\eta, x, y, z))$$
$$\Gamma^{\eta}{}_{\eta y} = \frac{\partial}{\partial y}\phi(\eta, x, y, z)$$
$$\Gamma^{y}{}_{\eta y} = \mathrm{H}(\eta) - (\frac{\partial}{\partial \eta}\phi(\eta, x, y, z))$$
$$\Gamma^{x}{}_{xy} = -(\frac{\partial}{\partial y}\phi(\eta, x, y, z))$$
$$\Gamma^{y}{}_{xy} = -(\frac{\partial}{\partial x}\phi(\eta, x, y, z))$$
$$\Gamma^{\eta}{}_{yy} = \mathrm{H}(\eta) - 4\,\mathrm{H}(\eta)\,\phi(\eta, x, y, z) - (\frac{\partial}{\partial \eta}\phi(\eta, x, y, z))$$
$$\Gamma^{x}{}_{yy} = \frac{\partial}{\partial x}\phi(\eta, x, y, z)$$
$$\Gamma^{y}{}_{yy} = -(\frac{\partial}{\partial y}\phi(\eta, x, y, z))$$
$$\Gamma^{z}{}_{yy} = \frac{\partial}{\partial z}\phi(\eta, x, y, z)$$

$$\Gamma^{y}\ {}_{z\,y} = -(\tfrac{\partial}{\partial z}\,\phi(\eta,\,x,\,y,\,z))$$
$$\Gamma^{z}\ {}_{z\,y} = -(\tfrac{\partial}{\partial y}\,\phi(\eta,\,x,\,y,\,z))$$
$$\Gamma^{\eta}\ {}_{\eta\,z} = \tfrac{\partial}{\partial z}\,\phi(\eta,\,x,\,y,\,z)$$
$$\Gamma^{z}\ {}_{\eta\,z} = \mathrm{H}(\eta) - (\tfrac{\partial}{\partial \eta}\,\phi(\eta,\,x,\,y,\,z))$$
$$\Gamma^{x}\ {}_{x\,z} = -(\tfrac{\partial}{\partial z}\,\phi(\eta,\,x,\,y,\,z))$$
$$\Gamma^{z}\ {}_{x\,z} = -(\tfrac{\partial}{\partial x}\,\phi(\eta,\,x,\,y,\,z))$$
$$\Gamma^{y}\ {}_{y\,z} = -(\tfrac{\partial}{\partial z}\,\phi(\eta,\,x,\,y,\,z))$$
$$\Gamma^{z}\ {}_{y\,z} = -(\tfrac{\partial}{\partial y}\,\phi(\eta,\,x,\,y,\,z))$$
$$\Gamma^{\eta}\ {}_{z\,z} = \mathrm{H}(\eta) - 4\,\mathrm{H}(\eta)\,\phi(\eta,\,x,\,y,\,z) - (\tfrac{\partial}{\partial \eta}\,\phi(\eta,\,x,\,y,\,z))$$
$$\Gamma^{x}\ {}_{z\,z} = \tfrac{\partial}{\partial x}\,\phi(\eta,\,x,\,y,\,z)$$
$$\Gamma^{y}\ {}_{z\,z} = \tfrac{\partial}{\partial y}\,\phi(\eta,\,x,\,y,\,z)$$
$$\Gamma^{z}\ {}_{z\,z} = -(\tfrac{\partial}{\partial z}\,\phi(\eta,\,x,\,y,\,z))$$

These componets can be summerized in the following seven expressions

$$\Gamma^0_{00} = \mathcal{H} + \dot{\Phi} \tag{D.1}$$
$$\Gamma^i_{00} = \Phi_{,i} \tag{D.2}$$
$$\Gamma^0_{0i} = \Phi_{,i} \tag{D.3}$$
$$\Gamma^i_{j0} = (\mathcal{H} - \dot{\Phi})\delta^i_j \tag{D.4}$$
$$\Gamma^0_{ij} = (\mathcal{H} - 4\mathcal{H}\Phi - \dot{\Phi})\delta_{ij} \tag{D.5}$$
$$\Gamma^i_{jk} = \Phi_{,i}\delta_{jk} - \Phi_{,j}\delta_{ik} - \Phi_{,k}\delta_{ij} \tag{D.6}$$

Appendix E

Numerical calculations

E.1 C++ source code

This part of this appendix contains the source code for the C++ programs which were written to determine the perturbations $\delta(\hat{x}, \tau)$ and $\Phi(\hat{x})$ and also the function $Q[\Phi]$.

E.1.1 Plane symmetry

```
-----------------------------------------------------------------

#include <math.h>
#include <iostream>
#include <fstream>
#include <stdlib.h>

const double d0 = 1E-5;
const double x_range = 10.0;
const double t_range = 1.0;
const int Nx = 500;
const int Nt = 500;
const double dx = x_range/Nx;
const double dt = t_range/Nt;
const double eta0L = 10.0;

double phi[Nx];
double delta[Nx][Nt];
double Q[Nx];
```

```
void findpert();
void writetofile();
void findQ();
double initial_delta(double x);

using namespace std;

int main(){
   findpert();
   findQ();
   writetofile();
   return 0;
}

void findpert(){
   double x,t;
   phi[Nx-1]=0.0;
   phi[Nx-2]=0.0;
   for(int i=Nx-1;i≥0;i--){
      x = i*dx;
      if(i>0&&i<(Nx-1)){
         phi[i-1]=2*phi[i]-
phi[i+1]+6*dx*dx*(initial_delta(x)+2*phi[i])/(eta0L*eta0L);
      }
      for(int j=0;j<Nt;j++){
         t = j*dt;
         delta[i][j]=(initial_delta(x)+2*phi[i])*pow(1+t,2.0/3.0)-2*phi[i];
      }
   }
}

void findQ(){
   for(int i=0;i<Nx;i++){
      double x=i*dx
         Q[i]=6.0*(initial_delta(x)+2*phi[i])/(eta0L*eta0L);
   }
}

double initial_delta(double x){
   return d0*(1-2*x*x)*exp(-x*x);
}
```

```
void writetofile(){
  ofstream fut1("delta_plane.dat");
  ofstream fut2("phi_plane.dat");
  ofstream fut3("pos_plane.dat");
  ofstream fut4("time_plane.dat");
  ofstream fut5("Q_plane.dat");

  for(int i=0;i<2*Nx;i++){
    for(int j=0;j<Nt;j++){
      if(i<Nx){
        fut1 << delta[Nx-i-1][j] << " ";
      }
      else{
        fut1 << delta[i-Nx][j] << " ";
      }
    }
    fut1 << endl;
  }
  fut1.close();

  for(int i=0;i<2*Nx;i++){
    for(int j=0;j<Nt;j++){
      if(i<Nx){
        fut2 << phi[Nx-i-1] << " ";
      }
      else
        fut2 << phi[i-Nx] << " ";
    }
    fut2 << endl;
  }
  fut2.close();

  for(int i=0;i<2*Nx;i++){
    double x = i*dx-x_range;
    fut3 << x << endl;
  }
  fut3.close();

  for(int j=0;j<Nt;j++){
    double t = j*dt;
```

```
      fut4 << t << endl;
    }
    fut4.close();

    for(int i=0;i<2*Nx;i++){
        if(i<Nx){
          fut5 << Q[Nx-i-1] << endl;
        }
        else{
          fut5 << Q[i-Nx] << endl;
        }
    }
    fut5.close();

}
```

E.1.2 Cylinder symmetry

```
#include <math.h>
#include <iostream>
#include <fstream>
#include <stdlib.h>

const double d0 = 1E-5;
const double r_range = 10.0;
const double t_range = 1.0;
const int Nr = 1000;
const int Nt = 500;
const double dr = r_range/Nr;
const double dt = t_range/Nt;
const double eta0L = 10.0;

double phi[Nr];
double delta[Nr][Nt];
double Q[Nr];

void findpert();
void writetofile();
```

```
void findQ();
double initial_delta(double);

using namespace std;

int main(){
   findpert();
   findQ();
   writetofile();
   return 0;
}

void findpert(){
   double r,t;
   phi[Nr-1]=0.0;
   phi[Nr-2]=0.0;
   for(int i=Nr-1;i≥1;i--){
      r = i*dr;
      if(i>1&&i<(Nr-1)){
         phi[i-1]=(2*dr*dr*r/(2*r-
dr))*((6.0/(eta0L*eta0L))*(initial_delta(r)+2.0*phi[i])
                                          +2.0*phi[i]/(dr*dr)-
(2*r+dr)*phi[i+1]/(2*dr*dr*r));

      }
      for(int j=0;j<Nt;j++){
         t = j*dt;
         delta[i][j]=(initial_delta(r)+2*phi[i])*pow(1+t,2.0/3.0)-2*phi[i];
      }
   }
}

void findQ(){
   double r, diffphi, diff2phi;
   diffphi=(phi[2]-phi[1])/dr;
   diff2phi=(phi[3]+phi[1]-2*phi[2])/(dr*dr);
   Q[1]=sqrt(diffphi*diffphi-dr*diffphi*diff2phi+
                  dr*dr*diff2phi*diff2phi)/dr;
   for(int i=2;i<(Nr-1);i++){
      r=i*dr;
      diffphi=(phi[i+1]-phi[i-1])/(2.0*dr);
```

```
        diff2phi=(phi[i+1]+phi[i-1]-2.0*phi[i])/(dr*dr);
        Q[i]=sqrt(diffphi*diffphi-dr*diffphi*diff2phi+
                    dr*dr*diff2phi*diff2phi)/r;
    }
    int k=Nr-1;
    r=k*dr;
    diffphi=(phi[k]-phi[k-1])/dr;
    diff2phi=(phi[k]-2*phi[k-1]+phi[k-2])/(dr*dr);
    Q[k]=sqrt(diffphi*diffphi-dr*diffphi*diff2phi+
                    dr*dr*diff2phi*diff2phi)/r;
}

double initial_delta(double r){
    double tmp=d0*(1-r*r)*exp(-r*r);
    return tmp;
}

void writetofile(){
    ofstream fut1("delta_cyl.dat");
    ofstream fut2("phi_cyl.dat");
    ofstream fut3("pos_cyl.dat");
    ofstream fut4("time_cyl.dat");
    ofstream fut5("Q_cyl.dat");

    for(int i=1;i<Nr;i++){
        for(int j=0;j<Nt;j++){
                fut1 << delta[i][j] << " ";
        }
        fut1 << endl;
    }
    fut1.close();

    for(int i=1;i<Nr;i++){
        for(int j=0;j<Nt;j++){
            fut2 << phi[i] << " ";
        }
        fut2 << endl;
    }
    fut2.close();

    for(int i=1;i<Nr;i++){
```

```
    double r = i*dr;
    fut3 << r << endl;
  }
  fut3.close();

  for(int j=0;j<Nt;j++){
    double t = j*dt;
    fut4 << t << endl;
  }
  fut4.close();

  for(int i=1;i<Nr;i++){
    fut5 << Q[i] << endl;
  }
  fut5.close();
}
```

E.1.3 Spherical symmetry

```
#include <math.h>
#include <iostream>
#include <fstream>
#include <stdlib.h>

const double d0 = 1E-5;
const double r_range = 10.0;
const double t_range = 1.0;
const int Nr = 1000;
const int Nt = 500;
const double dr = r_range/Nr;
const double dt = t_range/Nt;
const double eta0L = 10.0;

double phi[Nr];
double delta[Nr][Nt];
double Q[Nr];
```

```
void findpert();
void writetofile();
void findQ();
double initial_delta(double);

using namespace std;

int main(){
   findpert();
   findQ();
   writetofile();
   return 0;
}

void findpert(){
   double r,t;
   phi[Nr-1]=0.0;
   phi[Nr-2]=0.0;
   for(int i=Nr-1;i≥1;i--){
      r = i*dr;
      if(i>1&&i<(Nr-1)){

         phi[i-1]=(dr*dr*r/(r-dr))*((2.0/(dr*dr))*phi[i]-
phi[i+1]*(1/(dr*dr)+1/(dr*r))+
                                          6.0*(initial_delta(r)+2.0*phi[i])/(eta0L*eta0L));
         //cout << i-1 << "  " << r << "  " << phi[i-1] << endl;
      }
      for(int j=0;j<Nt;j++){
         t = j*dt;
         delta[i][j]=(initial_delta(r)+2*phi[i])*pow(1+t,2.0/3.0)-2*phi[i];
      }
   }
}

void findQ(){
   double r=0;
   Q[1]=(phi[3]-3*phi[2]+2*phi[1])/(dr*dr);
   for(int i=2;i<(Nr-1);i++){
      r=i*dr;
      Q[i]=(phi[i+1]+phi[i-1]-2*phi[i])/(dr*dr)
         -(phi[i+1]-phi[i-1])/(2*dr*r);
```

```
  }
  int k=Nr-1;
  r=k*dr;
  Q[k]=(phi[k-2]-2*phi[k-1]+phi[k])/(dr*dr)
    -(phi[k]-phi[k-1])/(r*dr);
}

double initial_delta(double r){
  double tmp=d0*(1-(2.0/3.0)*r*r)*exp(-r*r);
  return tmp;
}

void writetofile(){
  ofstream fut1("delta_sphere.dat");
  ofstream fut2("phi_sphere.dat");
  ofstream fut3("pos_sphere.dat");
  ofstream fut4("time_sphere.dat");
  ofstream fut5("Q_sphere.dat");

  for(int i=1;i<Nr;i++){
    for(int j=0;j<Nt;j++){
        fut1 << delta[i][j] << " ";
    }
    fut1 << endl;
  }
  fut1.close();

  for(int i=1;i<Nr;i++){
    for(int j=0;j<Nt;j++){
      fut2 << phi[i] << " ";
    }
    fut2 << endl;
  }
  fut2.close();

  for(int i=1;i<Nr;i++){
    double r = i*dr;
    fut3 << r << endl;
  }
  fut3.close();
```

```
  for(int j=0;j<Nt;j++){
    double t = j*dt;
    fut4 << t << endl;
  }
  fut4.close();

  for(int i=1;i<Nr;i++){
    fut5 << Q[i] << endl;
  }
  fut5.close();
}
```

E.2 MATLAB code

This part of the appendix contains scripts for the computer program MATLAB. These scripts read the data for the perturbations that was produced by the C++ programs in the last section of this appendix, and calculates the perturbed and unperturbed thermodynamic entropy and the gravitational entropy.

E.2.1 Plane symmetry

```
%% The data which was created by the C++ program is read into
%% MATLAB and renamed into more appropriate names
load delta_plane.dat
load phi_plane.dat
load pos_plane.dat
load time_plane.dat
load Q_plane.dat
x=pos_plane;
t=time_plane';
delta=delta_plane;
phi=phi_plane;
Q=Q_plane.dat;
clear pos_plane;
clear time_plane;
```

```
clear delta_plane;
clear phi_plane;
clear Q_plane;
%% The constants are given numerical value, and the integrals are
%% carried out using the trapezoidal algorithm for numerical intgration.
M=length(x);
K=length(t);
alpha0=3.32E9;
chi=5E9;
eta0=3.3E4;
lnKTdivA0=53.09;
Q=abs(Q);
dST=alpha0*(lnKTdivA0*(1-3*phi)+delta*(lnKTdivA0-1));
dSTun=alpha0*lnKTdivA0*ones(M,K);
dSG=chi/3/sqrt(3)*eta0L*eta0L*(Q*(1+t).^(8/3));
for i=1:K
    ST(i)=trapz(x,dST(:,i));
    STun(i)=trapz(x,dSTun(:,i));
    SG(i)=trapz(x,dSG(:,i));
end
%% ST is the thermal entropy, STun is the unperturbed thermal
%% entropy, while SG is the gravitational entropy.
```

E.2.2 Cylindrical symmetry

```
load delta_cyl.dat
load phi_cyl.dat
load pos_cyl.dat
load time_cyl.dat
load grfunc_cyl.dat
x=pos_cyl;
t=time_cyl';
delta=delta_cyl;
phi=phi_cyl;
Q=Q_cyl;
clear pos_cyl;
clear time_cyl;
clear delta_cyl;
```

```
clear phi_cyl;
clear Q_cyl;
M=length(x);
K=length(t);
alpha0=3.32E9;
chi=5E9;
eta0L=10;
lnKTdivA0=53.09;
Q=abs(Q);
dST=2*pi*alpha0*(x*ones(1,K)).*(lnKTdivA0*(1-
3*phi)+delta*(lnKTdivA0-1));
dSTun=2*pi*alpha0*lnKTdivA0*(x*ones(1,K));
dSG=2*pi*eta0L*eta0L*chi/3/sqrt(3)*(Q.*x)*((1+t).^(8/3));
for i=1:K
    ST(i)=trapz(x,dST(:,i));
    STun(i)=trapz(x,dSTun(:,i));
    SG(i)=trapz(x,dSG(:,i));
end
```

E.2.3 Spherical symmetry

```
load delta_sphere.dat
load phi_sphere.dat
load pos_sphere.dat
load time_sphere.dat
load Q_sphere.dat
x=pos_sphere;
t=time_sphere';
delta=delta_sphere;
phi=phi_sphere;
Q=Q_sphere;
clear Q_sphere;
clear pos_sphere;
clear time_sphere;
clear delta_sphere;
clear phi_sphere;
M=length(x);
K=length(t);
```

```
alpha0=3.32E9;
chi=5E9;
eta0L=10;
lnKTdivA0=53.09;
Q=abs(Q);
dST=4*pi*alpha0*((x.*x)*ones(1,K)).*(lnKTdivA0*(1-
3*phi)+delta*(lnKTdivA0-1));
dSTun=4*pi*alpha0*lnKTdivA0*((x.*x)*ones(1,K));
dSG=4*pi*eta0L*eta0L*chi/3/sqrt(3)*(Q.*x.*x)*((1+t).^(8/3));
for i=1:K
    ST(i)=trapz(x,dST(:,i));
    STun(i)=trapz(x,dSTun(:,i));
    SG(i)=trapz(x,dSG(:,i));
end
```

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