UiO : Department of Mathematics
University of Oslo

Using the density method for finding delta of option prices in Brownian and normal inverse Gaussian markets

Jon Elstad

Master's Thesis, Autumn 2023

This master's thesis is submitted under the master's programme *Mathematics*, with programme option *Mathematics for applications*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 30 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Contents

1 Lévy theory

Our primary goal is to find analytic expressions for delta of option prices, where the underlying asset's price is modeled with an exponential normal inverse Gaussian(NIG) process. In section (3) we will use the density method, starting with a Black-Scholes style price, then delta is the derivative with respect to the initial price. We will move the derivative into the expectation used in the pricing by Leibniz' rule. We will also do this with Brownian price processes in section (2), where we also consider spread options. We will end with numerical implementations in section (4) of both price and delta for NIG and Brownian models, and compare these with each other and a Black-Scholes style solution. Both Brownian motion and NIG processes are Lévy processes, hence we start by looking at some general theory for Lévy processes in this section, which primarily will help with finding an integrability and a martingale conditon for the exponential NIG process.

1.1 Infinitely divisible distributions

Before anything else we will define convolution of measures, as this will be used in results in a later subsection.

Definition 1 (Convolution of measures[App09]). Let μ_1 and μ_2 be probability Borel measures on $\mathbb{R}^d$ and A a measurable set, then the convolution of the measures is defined as

$$(\mu_1 * \mu_2)(A) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_A(x+y)\mu_1(dx)\mu_2(dy).$$

Before getting to Lévy processes, we will define infinitely divisible distributions, as we will see later they are closely linked to Lévy processes.

Definition 2 (Infinite divisibility[App09]). Let X be a random variable. We say that X is infinitely divisible if for each $n \in \mathbb{N}$ there exists $X_1^{(n)}, \dots, X_n^{(n)}$ such that

$$X \stackrel{d}{=} X_1^{(n)}, \dots, X_n^{(n)}.$$

We have some equivalent characterizations of infinitely divisible distributions.

Proposition 3 ([App09]). *The following are equivalent:*

- *X is infinitely divisible,*
- *μ_X has a convolution n-th root (something that convoluted with itself n times equals μ_X), for each $n \in \mathbb{N}$,*
- *the characteristic function ϕ_X has an n-th root which is itself the characteristic function of some random variable, for each $n \in \mathbb{N}$.*

Before considering the next theorem, we require another concept.

Definition 4 (Lévy measure[App09])**.** Let ν be a Borel measures on $\mathbb{R}^d \setminus \{0\}$. We say that it is a Lévy measure if

$$\int_{\mathbb{R}^d \setminus \{0\}} min(|x|^2, 1)\nu(dx) < \infty.$$

We can now look at one of the most useful results for working with infinitely divisible distributions. We will employ this later to find both integrability and martingale conditions.

Theorem 5 (Lévy-Khintchine formula[App09])**.** *Let μ be probability Borel measures on $\mathbb{R}^d$, it is infinitely divisible if there exists $b \in \mathbb{R}^d$, a positive definite symmetric $d \times d$ matrix A, and a Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$ such that for all $u \in \mathbb{R}^d$*

$$\phi_\mu(u) = \exp\Big\{ i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d \setminus \{0\}} e^{i(u,y)} - 1 - i(u,y)1_{B_1(0)}(y)\nu(dy) \Big\}, \tag{1}$$

where $\phi_\mu(u)$ is the characteristic function of μ, and $B_1(0)$ the ball of radius 1 *centered at 0. Conversely any mapping of the form (1) is the characteristic function of an infinitely divisible probability measure on $\mathbb{R}^d$.*

We may refer to infinitely divisible distributions by the parameters in the Lévy-Khintchine formula.

Definition 6 (Characteristic triplet[App09])**.** If μ is infinitely divisible, and we find the Lévy-Khintchine formula of the form (1), then we call (b, A, ν) the characterisitc triplet of X.

As the Lévy-Khintchine formula linked the characteristic function to an exponential, we may sometimes want to talk about the exponent.

Definition 7 (Lévy symbol[App09])**.** If μ is infinitely divisible, and we find the Lévy-Khintchine formula of the form (1), we write

$$\phi_\mu(u) = e^{\eta(u)}, \tag{2}$$

where

$$\eta(u) = i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d \setminus \{0\}} e^{i(u,y)} - 1 - i(u,y)1_{B_1(0)}(y)\nu(dy),$$

and we call $\eta : \mathbb{R}^d \to \mathbb{C}$ the Lévy symbol of μ.

Let us now consider some examples, which includes some very well known distributions. We will show that they are infinitely divisible and find their characteristic triplets, this is simply done with Lévy-Khintchine formula.

Example 8. If $X \sim \mathcal{N}(\mu, \sigma^2)$ then its known that

$$\mathrm{E}[e^{iuX}] = e^{i\mu u - \frac{1}{2}\sigma^2 u^2},$$

then by (1) it has an infinitely divisible distribution, which has characteristic triplet $(\mu, \sigma^2, 0)$.

Example 9. Consider X to be Poisson distributed with density

$$\frac{\lambda^k e^{-\lambda}}{k!}.$$

It has characteristic function

$$\begin{aligned}\mathrm{E}[e^{iuX}] &= \sum_{k=0}^{\infty} e^{iuk}\frac{\lambda^k e^{-\lambda}}{k!},\\ &= e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^{iu})^k}{k!}.\end{aligned}$$

Using the Taylor series of e^x

$$\begin{aligned}&= e^{-\lambda}e^{\lambda e^{iu}},\\ &= e^{\lambda(e^{iu}-1)},\\ &= \exp\Big(\int_{\mathbb{R}\setminus\{0\}}(e^{iux}-1)\lambda\delta_1(dx)\Big),\end{aligned}$$

where δ_1 is the Dirac delta centered in 1. We recognize this as a Lévy-Khintchine formula, thus X is infinitely distributed with characteristic triplet $(0, 0, \lambda\delta_1(dx))$.

Example 10. Let $N \sim Poi(\lambda)$, and let $Z_i, i \in \mathbb{N}$ be i.i.d. and independent of N. Then a compound Poisson distribution can be defined as

$$X = \sum_{i=0}^{N} Z_i.$$

We consider the characteristic function

$$\mathrm{E}[e^{iuX}] = \mathrm{E}\Big[\exp\Big(iu\sum_{i=0}^{N} Z_i\Big)\Big],$$

by the law of total expectation we get

$$= \sum_{k=0}^{n}\mathrm{E}\Big[\exp\Big(iu\sum_{i=0}^{N} Z_i\Big)|N = k\Big]\frac{\lambda^k e^{-\lambda}}{k!},$$

since Z_i's are i.i.d.

$$\begin{aligned}&= \sum_{k=0}^{n}\mathrm{E}\Big[\exp\Big(iuZ\Big)\Big]^k\frac{\lambda^k e^{-\lambda}}{k!},\\ &= e^{-\lambda}\sum_{k=0}^{n}\frac{(\lambda\,\mathrm{E}\Big[\exp\Big(iuZ\Big)\Big])^k}{k!},\\ &= e^{\lambda(\mathrm{E}[\exp(iuZ)]-1)},\\ &= \exp\Big(\int_{\mathbb{R}^d}(e^{iux}-1)\lambda\mu_Z(dx)\Big),\end{aligned}$$

thus is infinitely divisible with characteristic triplet $(0, 0, \lambda\mu_Z(dx))$.

1.2 Lévy processes

We are now ready for Lévy processes. One could keep in mind the definition of Brownian motion when looking through the next definition, as we will see in a later subsection that Brownian motion turns out to be a Lévy process, and the definitions have similarities.

Definition 11 (Lévy process[App09]). Let $X = X_t, t \geq 0$ be a stochastic process of a probability space $(\Omega, \mathcal{F}, P)$. We say that it has independent increments if for any $0 < t_1 < t_2 < \cdots < t_n < \infty$ the random variables $X_{t_{j+1}} - X_{t_j}, 1 \leq j \leq n-1$ are independent, and that it has stationary increments if each $X_{t_{j+1}} - X_{t_j} \overset{d}{=} X_{t_{j+1}-t_j} - X_0$. We say that X is a Lévy process if:

1. $X(0) = 0$,

2. X has independent and stationary increments,

3. X is stochastically continuous,

where by stochastic continuity we mean that for all $a > 0$ and for all $s \geq 0$

$$\lim_{t \to s} P(|X_t - X_s| > a) = 0.$$

We have one of many results which shows a link between Lévy processes.

Proposition 12 ([App09]). *If X is a Lévy process, then $X(t)$ is infinitely divisible for each $t \geq 0$.*

The following gives a relationship between the characteristic functions of Lévy processes and corresponding infinitely divisible distributions.

Theorem 13 ([App09]). *If X is a Lévy process, then*

$$\phi_{X(t)}(u) = e^{t\eta(u)}$$

for $u \in \mathbb{R}^d$, $t \geq 0$, where η is the Lévy symbol of $X(1)$. We may write η_X, and define the Lévy symbol of a Lévy process to be the Lévy symbol of $X(1)$.

From this we see that if X is a Lévy process, then X_t is infinitely divisible with characteristic triplet $(tb, tA, t\nu(dx))$. An important intuition for Lévy processes is that b represents drift, A a continuous movement (like in a Brownian motion), and $\nu(dx)$ counts jumps (like in a Poisson or compound Poisson process). We will introduce some notation for the exponent in the previous theorem.

Definition 14 (Lévy symbol of a Lévy process[App09]). We define the Lévy symbol of a Lévy process X to be the Lévy symbol of $X(1)$, and may write it as η_X.

1.3 Subordination

In this subsection we will see one way to construct new Lévy processes from existing ones, by subordination.

Definition 15 (Subordinator[App09]). A subordinator is a non decreasing one dimensional Lévy process.

The intuition about subordinators is that they are random models of time. If $T_t, t \geq 0$ is a subordinator then

$$T_t \geq 0, \tag{3}$$

for $t > 0$, and

$$T_{t_1} \leq T_{t_2},$$

for $t_1 \leq t_2$.

Theorem 16 ([App09]). *Let T be a subordinator, then it has Lévy symbol of the form*

$$\eta(u) = ibu + \int_0^\infty (e^{iuy} - 1)\lambda(dy), \tag{4}$$

where $b \geq 0$, and λ is a Lévy measure that satisfies

$$\lambda((-\infty, 0)) = 0,$$

and

$$\int_0^\infty min(y, 1)\lambda < \infty.$$

Conversely if we have a symbol of the form (4) it is the Lévy symbol of a subordinator.

The following example presents the inverse Gaussian process, which is a subordinator, which is used to construct the normal inverse Gaussian process which we are interested in.

Example 17 ([App09]). Let W_t be the standard Brownian motion, and let $B_t = \gamma t + W_t$. Then the inverse Gaussian process is

$$T_t = inf\{s > 0 | B_t = \delta t\},$$

where $\delta > 0$. We have that

$$\mathrm{E}[e^{-uT_t}] = \exp\Big(- t\delta(\sqrt{2u + \gamma^2} - \gamma)\Big),$$

and T_t has density

$$f_{T_t}(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t \gamma} s^{-3/2} \exp\Big(- \frac{1}{2}(t^2\delta^2 s^{-1} + \gamma^2 s)\Big),$$

for each $t, s \geq 0$. A function that has the density of T_1 is said to be inverse Gaussian distributed, which we write $IG(\delta, \gamma)$.

We have a name for the exponent we found in the previous example, as we will see later it becomes useful to state some properties of subordinators.

Definition 18 (Laplace exponent of subordinator[App09])**.** Let T be a subordinator, and let

$$\mathrm{E}[e^{-uT_t}] = e^{-t\psi(u)},$$

then we call ψ the Laplace exponent of T.

We have

$$\psi(u) = -\eta_T(iu) = bu + \int_0^\infty (1 - e^{-uy})\lambda(dy),$$

for each $u > 0$.

The following theorem shows how we can use subordination to change the time of Lévy processes.

Theorem 19 ([App09])**.** *Let X be a Lévy process, and T a subordinator on the same probability space as X, such that X and T are independent. We can define a new process*

$$Z_t = X(T_t)$$

for each $t \geq 0$, such that for each $\omega \in \Omega$, $Z_t(\omega) = X_{T_t(\omega)}(\omega)$. Then $Z = Z_t, t \geq 0$ is a Lévy process.

If we know the Laplace exponent and the subordinator and the Lévy symbol of the subordinated process we can quickly find the Lévy symbol of the new process with the following result.

Proposition 20 ([App09])**.** *Let X, T, Z be as in (19), then*

$$\eta_Z = -\psi_T \circ (-\eta_X).$$

We demonstrate the previous result in this example, where we also get a first look at the NIG process we will work with in later sections.

Example 21 ([App09])**.** For each $t \geq 0$ let $Z_t = B_{T_t} + \mu t$, where $B_t = \beta t + W_t$, where W_t is the standard Brownian motion, and $\beta \in \mathbb{R}$. Let T be a subordinator independent of B, such that T_1 is $IG(\delta, \sqrt{\alpha^2 - \beta^2})$, where $\alpha \in \mathbb{R}$ and $\alpha^2 \geq \beta^2$. Then from (19) Z is a Lévy process. Further, using 4 we have

$$\eta_Z(u) = -\psi_T(u) \circ (-\eta_B(u)) + iu\mu t,$$

using (17) and (8)

$$\begin{aligned}
&= -\,(t\delta(\sqrt{2u + \alpha^2 - \beta^2} - \sqrt{\alpha^2 - \beta^2}) \circ (-iu\beta + \frac{1}{2}u^2) + iu\mu t,\\
&= iu\mu t + \delta t(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - \beta^2 + u^2 - 2iu\beta}),\\
&= iu\mu t + \delta t(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}).
\end{aligned}$$

We will see more of this in section 3.

1.4 Convolution semigroups and canonical Lévy processes

We had a result that says that given a Lévy process we have an infinitely divisible distribution for each $t \geq 0$, and singled out the one for $t = 1$. In this subsection we will see a way we can start with an infinitely divisible distribution and find a Lévy process. We will first need some concepts defined.

Definition 22 ([App09])**.** We write $C_b(\mathbb{R}^d)$ to denote the continuous bounded functions from $\mathbb{R}^d$ to $\mathbb{R}$. Let $p_t, t \geq 0$ be a family of probability measures. We say that it converge weakly to δ_0 (the dirac delta function) if

$$\lim_{t \to 0^+} \int_{\mathbb{R}^d} f(y) p_t(dy) = f(0),$$

for all $f \in C_b(\mathbb{R}^d)$.

In the definition of Lévy processes we had stochastic continuity, the following result links this to the previous definition.

Proposition 23 ([App09])**.** *If X is a stochastic process such that $X(t)$ has law p_t for each $t \geq 0$ and $X(0) = 0$, then $p_t, t \geq 0$ is weakly convergent to δ_0 if and only if X is stochastically continuous at* 0*.*

The following definition is for the objects we will turn into Lévy processes.

Definition 24 ([App09])**.** We call a family of probability measures $p_t, t \geq 0$ with $p_0 = 0$ a convolution semigroup if

$$p_{t+s} = p_t * p_s$$

for all $t, s \geq 0$. We say that it is weakly continuous if it is weakly convergent to δ_0

Let us look at an example of such a weakly continuous semigroup.

Proposition 25. *Let*

$$\mu_t(dx) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx,$$

then $\mu_t, t \geq 0$ forms a weakly continuous semigroup. This is the Gauss semigroup, and we recognize it as the law of the standard Brownian motion.

Proof. We start by checking directly that $(\mu_t * \mu_s)(A) = \mu_{t+s}(A)$.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1_A(x+y)}{2\pi\sqrt{ts}} e^{-\frac{1}{2}\frac{x^2}{t} - \frac{1}{2}\frac{y^2}{t}} dx dy,$$

we make the substitution $x = z - y$ with $dx = dz$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1_A(z)}{2\pi\sqrt{ts}} e^{-\frac{1}{2}\frac{(z-y)^2}{t} - \frac{1}{2}\frac{y^2}{t}} dz dy, \\
&= \int_{-\infty}^{\infty} 1_A(z) \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{ts}} e^{-\frac{1}{2}\frac{(z-y)^2}{t} - \frac{1}{2}\frac{y^2}{t}} dy dz.
\end{aligned}$$

We examine the exponent

$$
\begin{aligned}
&-\frac{1}{2t}(z^2-2zy+y^2)-\frac{1}{2s}y^2,\\
=&-\frac{1}{2ts}(sz^2-2szy+sy^2+ty^2),\\
=&-\frac{1}{2ts}((t+s)y^2-2szy+sz^2),\\
=&-\frac{1}{2ts}((t+s)y^2-2szy+\frac{s(t+s)z^2}{t+s}),\\
=&-\frac{1}{2ts}((t+s)y^2-2szy+\frac{s^2z^2}{t+s}+\frac{tsz^2}{t+s}),\\
=&-\frac{1}{2ts}(\sqrt{t+s}y-\frac{sz}{\sqrt{t+s}})^2-\frac{1}{2(t+s)}z^2,\\
=&-\frac{1}{2\frac{ts}{t+s}}(y-\frac{sz}{t+s})^2-\frac{1}{2(t+s)}z^2.
\end{aligned}
$$

We return to the integral

$$
\begin{aligned}
&\int_{-\infty}^{\infty}1_A(z)\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{ts}}e^{-\frac{1}{2\frac{ts}{t+s}}(y-\frac{sz}{t+s})^2-\frac{1}{2(t+s)}z^2}dydz,\\
=&\int_{-\infty}^{\infty}1_A(z)e^{-\frac{1}{2(t+s)}z^2}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{ts}}e^{-\frac{1}{2\frac{ts}{t+s}}(y-\frac{sz}{t+s})^2}dydz,
\end{aligned}
$$

the integral in y is a Gaussian integral

$$
\begin{aligned}
=&\int_{-\infty}^{\infty}1_A(z)e^{-\frac{1}{2(t+s)}z^2}\frac{\sqrt{2\pi}\sqrt{\frac{ts}{t+s}}}{2\pi\sqrt{ts}}dz,\\
=&\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{t+s}}1_A(z)e^{-\frac{1}{2(t+s)}z^2}dz,\\
=&\,\mu_{t+s}(A).
\end{aligned}
$$

Hence μ_t is closed under convolution. We need that μ_t is weakly convergent to δ_0. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and bounded, i.e. $f\in C_b(\mathbb{R})$. Let $|f|$ be bounded by C. Fix an arbitrary $\epsilon>0$, and pick $\epsilon_1>0$ and $\epsilon_2>0$ such that $\epsilon_1+C\epsilon_2<\epsilon$. We have

$$
\begin{aligned}
|\int_{\mathbb{R}}f(y)\mu_t(dy)-f(0)|&=|\int_{\mathbb{R}}f(y)\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy-f(0)\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy|.\\
&\le\int_{\mathbb{R}}|f(y)-f(0)|\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy
\end{aligned}
$$

From continuity we have that $\exists\delta_1 > 0$ such that $|f(0)-f(x)| < \epsilon_1$ when $|x| < \delta_1$. We also have that $\exists\delta_2 > 0$ such that if $t < \delta_2$ then $\int_{\mathbb{R}\setminus(-\delta_1,\delta_1)} \mu_t(dy) < \epsilon_2$(see appendix 6). Let $0 \le t < \delta_2$ then

$$\begin{aligned}
|\int_{\mathbb{R}} f(y)\mu_t(dy) - f(0)| \le & \int_{(-\delta_1,\delta_1)} |f(y)-f(0)|\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy \\
& + \int_{\mathbb{R}\setminus(-\delta_1,\delta_1)} |f(y)-f(0)|\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy, \\
\le & \int_{(-\delta_1,\delta_1)} |f(y)-f(0)|\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy + C\epsilon_2, \\
\le & \epsilon_1 \int_{(-\delta_1,\delta_1)} \frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy + C\epsilon_2, \\
\le & \epsilon_1 + C\epsilon_2, \\
< & \epsilon.
\end{aligned}$$

Thus we have that for all $\epsilon > 0$ then there exists $\delta > 0$ such that if $0 \le t < \delta$ then

$$|\int_{\mathbb{R}} f(y)\mu_t(dy) - f(0)| < \epsilon,$$

hence

$$\lim_{t\to 0^+} \int_{\mathbb{R}} f(y)\mu_t(dy) = f(0),$$

and we are done. ∎

And another example of a convolution semigroup.

Proposition 26. *Let*

$$\mu_t(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y t\sqrt{1-\rho^2}} \exp\Big(-\frac{1}{2(1-\rho^2)}\Big[\Big(\frac{x}{\sigma_X\sqrt{t}}\Big)^2 - 2\rho\frac{xy}{\sigma_X\sigma_Y} + \Big(\frac{y}{\sigma_Y\sqrt{t}}\Big)^2\Big]\Big)dxdy,$$

where $\rho = corr(X,Y)$, then $\mu_t, t \ge 0$ is a convolution semigroup. This is the law of a two dimensional Brownian motion with correlation.

Proof. We start by checking directly that $(\mu_t * \mu_s)(A) = \mu_{t+s}(A)$.

$$\begin{aligned}
(\mu_t * \mu_s)(A) = & \int_{\mathbb{R}^4} \frac{1_A(x+u,y+v)}{(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2})^2 ts} \exp\Big(-\frac{1}{2(1-\rho^2)}\Big[\frac{x^2}{\sigma_X^2 t} - 2\rho\frac{xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2 t} \\
& + \frac{u^2}{\sigma_X^2 s} - 2\rho\frac{uv}{\sigma_X\sigma_Y} + \frac{v^2}{\sigma_Y^2 s}\Big]\Big)(dxdy)(dudv),
\end{aligned}$$

We substitute $x = z-u, y = w-v$ and examine the exponent

$$\begin{aligned}
-\frac{1}{2(1-\rho^2)ts}\Big[& \frac{s(z-u)^2}{\sigma_X^2} - 2\rho\frac{s(z-u)(w-v)}{\sigma_X\sigma_Y} + \frac{s(w-v)^2}{\sigma_Y^2} \\
& + \frac{tu^2}{\sigma_X^2} - 2\rho\frac{tuv}{\sigma_X\sigma_Y} + \frac{tv^2}{\sigma_Y^2}\Big],
\end{aligned}$$

we treat the mixed terms

$$\begin{aligned}
&s(z-u)(w-v)+t(uv),\\
=&s(zw-zv-uw+uv)+tuv,\\
=&(t+s)uv-s(zv+uw)+\frac{s(t+s)zw}{t+s},\\
=&(t+s)uv-s(zv+uw)+\frac{s^2zw}{t+s}+\frac{tszw}{t+s},\\
=&(t+s)\Big(uv-s\frac{zv}{t+s}-s\frac{uw}{t+s}+s^2\frac{zw}{(t+s)^2}\Big)+\frac{tszw}{t+s},\\
=&(t+s)\Big(u-\frac{sz}{t+s}\Big)\Big(v-\frac{sw}{t+s}\Big)+\frac{tszw}{t+s},
\end{aligned}$$

which combined with the proof in (25) for the non mixed terms of the exponent, the exponent becomes

$$\begin{aligned}
-\frac{1}{2(1-\rho^2)ts}\Big[&\frac{(t+s)}{\sigma_X^2}\Big(u-\frac{sz}{t+s}\Big)^2+\frac{st}{\sigma_X^2(t+s)}z^2\\
&-2\rho\frac{t+s}{\sigma_X\sigma_Y}\Big(u-\frac{sz}{t+s}\Big)\Big(v-\frac{sw}{t+s}\Big)-2\rho\frac{1}{\sigma_X\sigma_Y}\frac{tszw}{t+s}\\
&+\frac{(t+s)}{\sigma_Y^2}\Big(v-\frac{sw}{t+s}\Big)^2+\frac{st}{\sigma_Y^2(t+s)}w^2\Big],
\end{aligned}$$

we split out the pure z, w part and define

$$E_1:=-\frac{1}{2(1-\rho^2)}\Big[\frac{z^2}{\sigma_X^2(t+s)}-2\rho\frac{zw}{\sigma_X\sigma_Y(t+s)}+\frac{w^2}{\sigma_Y^2(t+s)}\Big],$$

and

$$E_2:=-\frac{1}{2(1-\rho^2)\frac{ts}{t+s}}\Big[\frac{\Big(u-\frac{sz}{t+s}\Big)^2}{\sigma_X^2}-2\rho\frac{\Big(u-\frac{sz}{t+s}\Big)\Big(v-\frac{sw}{t+s}\Big)}{\sigma_X\sigma_Y}+\frac{\Big(v-\frac{sw}{t+s}\Big)^2}{\sigma_Y^2}\Big].$$

Then the original expression becomes

$$(\mu_t*\mu_s)(A)=\int_{\mathbb{R}^2}1_A(z,w)e^{E_1}\int_{\mathbb{R}^2}\frac{1}{(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2})^2ts}e^{E_2}(dudv)(dzdw),$$

we recognize E_2 as the exponent of a binormal distribution, we know that $\int_{\mathbb{R}^2}e^{E_2}=2\pi\sqrt{1-\rho^2}\frac{ts}{t+s}$

$$\begin{aligned}
=&\int_{\mathbb{R}^2}1_A(z,w)\frac{e^{E_1}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}(t+s)}dzdw,\\
=&\mu_{t+s}(A)
\end{aligned}$$

Hence μ_t is closed under convolution. ■

Given a Lévy process the following result finds a weakly convergent semigroup.

Proposition 27 ([App09]). *If X is a Lévy process such that X_t has law p_t for each $t \geq 0$, then $p_t, t \geq 0$ is a weakly convergent semigroup.*

Finally we see a converse result that lets us find a Lévy process given a weakly convergent semigroup.

Theorem 28 (Canonical Lévy process[App09]). *If $p_t, t \geq 0$ is a weakly continuous semigroup of measures, then there exists a Lévy process X such that X_t has law p_t for each $t \geq 0$. We call this the canonical Lévy process.*

The previous result implies that the weakly convergent semigroup we found in (25) can give such a canonical Lévy process, which is the Brownian motion. There are other ways to show that Brownian motion is Lévy, for example just using the definition, but it is a good illustration of turning a family of measures into a Lévy process. Next we finally have the result that guarantees a Lévy process from an infinitely divisible distribution.

Corollary 29 ([App09]). *If μ is an infinitely divisible probability measure on $\mathbb{R}^d$, then there exists a Lévy process X such that μ is the law of $X(1)$.*

1.5 Exponential moments and Martingality

Since it is common to consider exponential processes in financial maths results that guarantee integrability and Martingality of such exponentiated Lévy processes are desirable. We give such results, under somewhat strict assumptions.

Proposition 30. *Let $(L_t)_{t\geq 0}$ be a Lévy process. Fix any $t \geq 0$. If*

$$\int_{\mathbb{R}\setminus(-1,1)} e^x \nu(dx) < \infty,$$

then

$$\mathrm{E}[e^{L_t}] < \infty.$$

Proof. Let ϕ be the characteristic function as in (1) for L_t. Then we have

$$\begin{aligned} \mathrm{E}[e^{L_t}] =& e^{t\psi(-i)}, \\ =& e^{t\left(\frac{1}{2}A+\gamma+\int_{-\infty}^{\infty} e^x-1-x\mathbf{1}_{|x|\leq 1}\nu(dx)\right)}, \end{aligned}$$

using the Taylor series of e^x with error term of second degree, we have $e^x-1-x = Cx^2$ on $(-1,1)$, for some constant C (dependent on the interval, in this case $(-1,1)$), we also have $e^x - 1 \leq e^x$, then

$$\begin{aligned} \leq& e^{t\left(\frac{1}{2}A+\gamma+\int_{\mathbb{R}\setminus(-1,1)} e^x\nu(dx)+\int_{-1}^{1} Cx^2\nu(dx)\right)}, \\ =& De^{t\int_{\mathbb{R}\setminus(-1,1)} e^x\nu(dx)} e^{Ct\int_{-1}^{1} x^2\nu(dx)}, \\ <& \infty, \end{aligned}$$

where $D > 0$ is a constant (dependent on t). ■

And for Martingality we have the following result that utilizes the integrability.

Proposition 31. *Let $(L_t)_{t\geq 0}$ be a Lévy process, and assume that*

$$\int_{\mathbb{R}\setminus(-1,1)} e^x \nu(dx) < \infty,$$

and

$$\frac{1}{2}A + \gamma + \int_{-\infty}^{\infty} e^x - 1 - \mathbf{1}_{|x|\leq 1}\nu(dx) = r,$$

then $e^{-rt}e^{L_t}$ is a martingale.

Proof. Using the previous proposition we have that

$$\mathrm{E}[e^{-rt}e^{L_t}] = e^{-rt}\,\mathrm{E}[e^{L_t}] < \infty,$$

for all $t \geq 0$. Further, for $t \geq s$,

$$\begin{aligned}
\mathrm{E}[e^{-rt}e^{L_t}|\mathcal{F}_s] =& e^{-st}e^{L_S}e^{-r(t-s)}\,\mathrm{E}[e^{L_t-L_s}|\mathcal{F}_s],\\
=& e^{-st}e^{L_S}e^{-r(t-s)}\,\mathrm{E}[e^{L_t-L_s}],\\
=& e^{-st}e^{L_S}e^{-r(t-s)}\,\mathrm{E}[e^{L_{t-s}}],\\
=& e^{-st}e^{L_S}e^{-r(t-s)}e^{(t-s)\left(\frac{1}{2}A+\gamma+\int_{-\infty}^{\infty} e^x - 1 - \mathbf{1}_{|x|\leq 1}\nu(dx)\right)},
\end{aligned}$$

by using the independence and stationarity of increments for Lévy processes, and the Lèvy-Khinchin representation. We need that

$$e^{-r(t-s)}\,\mathrm{E}[e^{L_{t-s}}] = 1,$$

which happens when

$$\frac{1}{2}A + \gamma + \int_{-\infty}^{\infty} e^x - 1 - \mathbf{1}_{|x|\leq 1}\nu(dx) = r.$$

■

When considering option pricing, we will take the expectation of our price process after applying a payoff function. hence we would also like to guarantee integrability of such compositions.

Proposition 32. *Fix $t \geq 0$, and let $\phi : \mathbb{R} \to \mathbb{R}$ be continuous. If $\mathrm{E}[e^{L_t}] < \infty$, and $|\phi(x)| \leq a|x| + b$ for some constants $a, b \geq 0$, then*

$$\mathrm{E}[|\phi(e^{L_t})|] < \infty.$$

Proof.

$$\begin{aligned}
\mathrm{E}[|\phi(e^{L_t})|] =& \int_{-\infty}^{\infty} |\phi(x)|P_{e^{L_t}}(x)dx,\\
\leq& a\int_{-\infty}^{\infty} |x|P_{e^{L_t}}(x)dx + b\int_{-\infty}^{\infty} P_{e^{L_t}}(x)dx,\\
=& a\,\mathrm{E}[e^{L_t}] + b,\\
<& \infty,
\end{aligned}$$

using that since $e^{L_t} \geq 0$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |x| P_{e^{L_t}}(x) dx &= \int_0^{\infty} x P_{e^{L_t}}(x) dx - \int_{-\infty}^{0} x P_{e^{L_t}}(x) dx, \\
&= \int_0^{\infty} x P_{e^{L_t}}(x) dx, \\
&= \mathrm{E}[e^{L_t}].
\end{aligned}$$

■

2 Brownian Market

Our goal is to apply the density method to NIG market options, but we will first look at the same process for Brownian models. The basic idea of the density method is to move a differentiation into an expectation, hence we will first look at a classic result, Leibniz' rule.

2.1 Leibniz' rule

Leibniz' rule is exactly what we need to differentiate an integral. There are different formulations depending on whether one is doing real analysis, multivariable analysis, etc. , while we will state it as a general measure theoretic result.

Lemma 33. *Let $f(\omega, x) : (\Omega, \mathbb{R}) \to \mathbb{R}$ be continuous in the x-variable, differentiable in the x-variable, and integrable over Ω, and let*

$$|\frac{\partial}{\partial x} f(\omega, x)| \leq g(\omega),$$

for some integrable function $g(\omega)$, for all x, then

$$\frac{d}{dx}\int_\Omega f(\omega, x)d\omega = \int_\Omega \frac{\partial}{\partial x} f(\omega, x)d\omega.$$

Proof.

$$\begin{aligned} \frac{d}{dx}\int_\Omega f(\omega, x)d\omega &= \lim_{h\to 0} \frac{\int_\Omega f(\omega, x+h)d\omega - \int_\Omega f(\omega, x)d\omega}{h}, \\ &= \lim_{h\to 0} \int_\Omega \frac{f(\omega, x+h) - f(\omega, x)}{h} d\omega, \\ &= \lim_{h\to 0} \int_\Omega \frac{\partial}{\partial x} f(\omega, x)|_{x=c_h} d\omega, \end{aligned}$$

using the mean value theorem to get $c_h \in (x, x+h)$. Now since $\frac{\partial}{\partial x} f(\omega, x)|_{x=c_h} \to \frac{\partial}{\partial x} f(\omega, x)$ as $h \to 0$ we may use Lebesgue dominated convergence to get

$$\frac{d}{dx}\int_\Omega f(\omega, x)d\omega = \int_\Omega \frac{\partial}{\partial x} f(\omega, x)d\omega.$$

■

2.2 Delta of options in Brownian markets

We will say that the price of an option with strike time T will be of the form

$$C(S_t^x, t, x) = e^{-rT}\,\mathrm{E}[\phi(S_T^x)],$$

where ϕ is a payoff function, x the initial stock price, and S_T^x the price process. We will use this form of the price even in the case that the discounted price process is not a Martingale. For now we will consider the price process

$$S_t^x = xe^{\sigma W_t + \theta t},$$

where W_t is a standard Brownian motion, and $r > 0$ is the risk free interest rate. We are interested in delta, which is the derivative of the price w.r.t. initial price, as well as gamma, which is the second derivative w.r.t. initial price.

Theorem 34. *Let $C(S_T^x, T, x)$ be the premium of an option with strike time $T > 0$, with a continuous payoff function $\phi : \mathbb{R} \to \mathbb{R}$, assume that $\phi(S_T^X)$ has finite expectation, and assume that*

$$|\phi(e^y)\frac{\partial}{\partial x}f_N(y - \ln(x); \theta T, \sigma\sqrt{T})| \leq g_1(y),$$

for all $x > 0$, for some integrable function $g_1 : \mathbb{R} \to \mathbb{R}$, where $f_N(y; \theta, \sigma)$ is the normal density

$$f_N(y; \theta, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-\theta)^2}{2\sigma^2}}.$$

Then

$$C'(S_t^x, t, x) = e^{-rT}\,\mathrm{E}[\phi(S_T^x)\rho_1(S_T^x, x)],$$

where

$$\rho_1(S_T^x, x) = \frac{\ln(S_T^x) - (\ln(x) + \theta T)}{x\sigma^2 T}.$$

Further if we also have that

$$|\phi(e^y)\frac{\partial^2}{\partial x^2}f_N(y - \ln(x); \theta T, \sigma\sqrt{T})| \leq g_2(y),$$

for all $x > 0$, for some integrable function $g_2 : \mathbb{R} \to \mathbb{R}$, then

$$C''(S_t^x, t, x) = e^{-rT}\,\mathrm{E}[\phi(S_T^x)\rho_2(S_T^x, x)],$$

where

$$\begin{aligned}\rho_2(S_T^x, x) =& \Big(\frac{\ln(S_T^x) - (\ln(x) + \theta T)}{x\sigma^2 T}\Big)^2 \\ &- \frac{\ln(S_T^x) - (\ln(x) + \theta T)}{x^2\sigma^2 T} - \frac{1}{x^2\sigma^2 T}.\end{aligned}$$

Proof. We will start by differentiating the density once

$$\begin{aligned}\frac{\partial}{\partial x}f_N(y - \ln(x); \theta T, \sigma\sqrt{T}) =& \frac{\partial}{\partial x}\frac{1}{\sqrt{2\pi\sigma^2 T}}e^{-\frac{(y-(\ln(x)+\theta T))^2}{2\sigma^2 T}}, \\ =& f_N(y - \ln(x); \theta T, \sigma\sqrt{T})\frac{-1}{2\sigma^2 T}\frac{\partial}{\partial x}(y - (\ln(x) + \theta T))^2, \\ =& f_N(y - \ln(x); \theta T, \sigma\sqrt{T})\frac{-y + (\ln(x) + \theta T)}{\sigma^2 T}\frac{-1}{x}, \\ =& f_N(y - \ln(x); \theta T, \sigma\sqrt{T})\frac{y - (\ln(x) + \theta T)}{x\sigma^2 T},\end{aligned}$$

and twice

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} f_N(y-\ln(x);\theta T,\sigma\sqrt{T}) =& \frac{\partial}{\partial x} f_N(y-\ln(x);\theta T,\sigma\sqrt{T}) \frac{y-(\ln(x)+\theta T)}{x\sigma^2 T},\\
=& \frac{y-(\ln(x)+\theta T)}{x\sigma^2 T} \frac{\partial}{\partial x} f_N(y-\ln(x);\theta T,\sigma\sqrt{T})\\
&+ f_N(y-\ln(x);\theta T,\sigma\sqrt{T}) \frac{\partial}{\partial x} \frac{y-(\ln(x)+\theta T)}{x\sigma^2 T},\\
=& f_N(y-\ln(x);\theta T,\sigma\sqrt{T})\Big(\Big(\frac{y-(\ln(x)+\theta T)}{x\sigma^2 T}\Big)^2\\
&+ \frac{\partial}{\partial x}\frac{y-(\ln(x)+\theta T)}{x\sigma^2 T}\Big),\\
=& f_N(y-\ln(x);\theta T,\sigma\sqrt{T})\Big(\Big(\frac{y-(\ln(x)+\theta T)}{x\sigma^2 T}\Big)^2\\
&+ \frac{-1}{x^2}\frac{y-(\ln(x)+\theta T)}{\sigma^2 T}\\
&+ \frac{1}{x}\frac{\partial}{\partial x}\frac{y-(\ln(x)+\theta T)}{\sigma^2 T}\Big),\\
=& f_N(y-\ln(x);\theta T,\sigma\sqrt{T})\Big(\Big(\frac{y-(\ln(x)+\theta T)}{x\sigma^2 T}\Big)^2\\
&- \frac{y-(\ln(x)+\theta T)}{x^2\sigma^2 T} - \frac{1}{x^2\sigma^2 T}\Big).
\end{aligned}
$$

The price is

$$
\begin{aligned}
C(S_t^x,t,x) &= e^{-rT}\,\mathrm{E}[\phi(S_T^x)],\\
&= e^{-rT}\int_\Omega \phi(S_T^x)d\omega,\\
&= e^{-rT}\int_{-\infty}^{\infty}\phi(e^y) f_N(y-\ln(x);\theta T,\sigma\sqrt{T})dy.
\end{aligned}
$$

Then we have, using (33), that

$$
\begin{aligned}
C'(S_t^x,t,x) &= \frac{d}{dx}e^{-rT}\int_{-\infty}^{\infty}\phi(e^y) f_N(y-\ln(x);\theta T,\sigma\sqrt{T})dy,\\
&= e^{-rT}\int_{-\infty}^{\infty}\phi(e^y)\frac{\partial}{\partial x} f_N(y-\ln(x);\theta T,\sigma\sqrt{T})dy,\\
&= e^{-rT}\int_{-\infty}^{\infty}\phi(e^y)\frac{(y-(\ln(x)+\theta T))}{x\sigma^2 T} f_N(y-\ln(x);\theta T,\sigma\sqrt{T})dy,\\
&= e^{-rT}\,\mathrm{E}[\phi(S_T^x)\frac{\sigma W_T}{x\sigma^2 T}].
\end{aligned}
$$

Further we also have

$$\begin{aligned}
C''(S_t^x,t,x) =& \frac{d^2}{dx^2} e^{-rT} \int_{-\infty}^{\infty} \phi(e^y) f_N(y-\ln(x);\theta T,\sigma\sqrt{T})dy,\\
=& e^{-rT} \int_{-\infty}^{\infty} \phi(e^y) \frac{\partial^2}{\partial x^2} f_N(y-\ln(x);\theta T,\sigma\sqrt{T})dy,\\
=& e^{-rT} \int_{-\infty}^{\infty} \phi(e^y) \frac{\partial}{\partial x} \frac{(y-(\ln(x)+\theta T))}{x\sigma^2 T} f_N(y-\ln(x);\theta T,\sigma\sqrt{T})dy,\\
=& e^{-rT} \int_{-\infty}^{\infty} \phi(e^y) \Big(\Big(\frac{(y-(\ln(x)+\theta T))}{x\sigma^2 T} \Big)^2 \\
&- \frac{(y-(\ln(x)+\theta T))}{x^2\sigma^2 T} - \frac{1}{x^2\sigma^2 T} \Big) f_N(y-\ln(x);\theta T,\sigma\sqrt{T})dy,\\
=& e^{-rT}\,\mathrm{E}\Big[\phi(S_T^x)\Big(\Big(\frac{\sigma W_T}{x\sigma^2 T}\Big)^2 - \frac{\sigma W_T}{x^2\sigma^2 T} - \frac{1}{x^2\sigma^2 T}\Big)\Big].
\end{aligned}$$

We finish up by noting that

$$\sigma W_T = \ln(S_T^x) - (\ln(x) + (r - \frac{1}{2}\sigma^2)T).$$

∎

As an example we may consider a long position for a European call option, for which the payoff is $max(S_T^x - K)$.

Proposition 35. *Let $C(S_T^x,t,x)$ be the premium of an option, with the payoff function $max(S_T^x - K)$, where K is the strike price, and the initial price $x \in (0,\infty)$. Then*

$$C'(C(S_T^x,t,x) = \mathrm{E}[max(S_T^x - K)\rho(S_T^x,x)],$$

where

$$\rho(S_T^x,x) = \frac{\ln(S_T^x) - (\ln(x) + (r - \frac{1}{2}\sigma^2)T)}{x\sigma^2 T}.$$

Proof. We need that

$$f(x,y) := \max\{e^y - K, 0\} \frac{\partial}{\partial x} P_{ln(S_T^x)}(y)$$

is integrable over y and continuous in x. We know, using some calculations from the proof of (34), that for fixed x, and after some translation, $f(x,y)$ behaves like

$$max(c_1 y e^{c_2 y - c_3 y^2}, 0),$$

with $c_1 > 0$, which is integrable since

$$\begin{aligned}
\int_0^{\infty} c_1 y e^{c_2 y - c_3 y^2} dy =& \int_0^{\infty} c_1 y e^{c_2 y - c_3 y^2} - \frac{c_1 c_2}{2c_3} e^{c_2 y - c_3 y^2} dy + \frac{c_1 c_2}{2c_3} \int_0^{\infty} e^{c_2 y - c_3 y^2} dy,\\
=& -\frac{c_1}{2c_3} e^{2y - y^2}\Big|_{y=0}^{\infty} + \frac{c_1 c_2}{2c_3} \int_0^{\infty} e^{2y-y^2} dy,\\
=& 0 + \frac{c_1}{2c_3} + \frac{c_1 c_2}{2\sqrt{c_3}} e^{\frac{c_2^2}{4c_3}} \int_0^{\infty} e^{-z^2} dz,\\
<& \infty.
\end{aligned}$$

Now for fixed y, $f(x,y)$ behaves like

$$\begin{aligned}&c_1\frac{1}{x}e^{-c_2\ln(x)-c_3(\ln(x))^2+c_4\ln(x)},\\=&c_1e^{-(1+c_2-c_4)\ln(x)-c_3(\ln(x))^2},\end{aligned}$$

which is continuous for $x \in (0,\infty)$, and moves continuously to 0 as $x \to 0^+$. Then the result follows by applying 34. ∎

We will remind ourselves of the Martingale condition for a Brownian price process, as this will be used in a later section.

Proposition 36. *Assume that*

$$\theta = r - \frac{1}{2}\sigma^2,$$

then $e^{-rt}e^{\sigma W_t+\theta t}$ is a Martingale.

Proof. Let $t > s \geq 0$, we check the Martingale condition

$$\mathrm{E}[e^{-rt}e^{\sigma W_t+\theta t}|\mathcal{F}_s] = e^{-rt}\,\mathrm{E}[e^{\sigma(W_t-W_s)+\theta(t-s)}e^{\sigma W_s+\theta s}|\mathcal{F}_s],$$

now since W_s is $\mathcal{F}_s$ measurable

$$=e^{-rt}e^{\sigma W_s+\theta s}e^{\theta(t-s)}\,\mathrm{E}[e^{\sigma(W_t-W_s)}|\mathcal{F}_s],$$

and since $W_t - W_s$ is independent of W_s, and thus of $\mathcal{F}_s$, we have

$$\begin{aligned}&=e^{-rt}e^{\sigma W_s+\theta s}e^{\theta(t-s)}\,\mathrm{E}[e^{\sigma(W_t-W_s)}],\\&=e^{-rt}e^{\sigma W_s+\theta s}e^{\theta(t-s)}\,\mathrm{E}[e^{\sigma\sqrt{t-s}Z}],\end{aligned}$$

where Z is standard normal, then using what is known about the moment generating function for the normal distribution we get

$$\begin{aligned}&=e^{-rt}e^{\sigma W_s+\theta s}e^{\theta(t-s)}e^{\frac{\sigma^2\sqrt{t-s}^2}{2}},\\&=e^{-rs}e^{\sigma W_s+\theta s}e^{-r(t+s)+\theta(t-s)+\frac{1}{2}\sigma^2(t-s)},\end{aligned}$$

which we want to equal $e^{-rs}e^{\sigma W_s+\theta s}$, hence we require

$$-r(t-s)+\theta(t-s)+\frac{1}{2}\sigma^2(t-s)=0,$$

or

$$\theta = r - \frac{1}{2}\sigma^2,$$

as desired. Lastly we consider integrability

$$\begin{aligned}\mathrm{E}[e^{-rt}e^{\sigma W_t+\theta t}] &= e^{(\theta-r)t}\,\mathrm{E}[e^{\sigma W_t}],\\&=e^{(\theta-r)t}\,\mathrm{E}[e^{\sigma\sqrt{t}Z}],\end{aligned}$$

where Z is standard normal, then again using what is known about the moment generating function for the normal distribution we get

$$\begin{aligned}&=e^{(\theta-r)t}e^{\frac{\sigma^2 t}{2}},\\&<\infty.\end{aligned}$$

∎

2.3 Margrabe's formula

We will now consider a spread option.

Proposition 37. *Let*

$$S_{1,t}^{x_1} = x_1 e^{\sigma_1 W_{1,t} + \theta_1 t},$$

and

$$S_{2,t}^{x_2} = x_2 e^{\sigma_2 W_{2,t} + \theta_2 t},$$

be price processes where $W_{i,t}$ *are Brownian motions with* $\rho = corr(\ln(S_{1,t}^{x_1}), \ln(S_{1,t}^{x_1}))$. *The logarithms of the price processes have joint probability density*

$$P_t(y_1, y_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}t} e^{-\frac{1}{2(1-\rho^2)}\Big(\frac{(y_1-\mu_1)^2}{2\sigma_1^2 t} - \frac{2\rho(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2 t} + \frac{(y_2-\mu_2)^2}{2\sigma_2^2 t}\Big)},$$

where $\mu_i = \ln(x_i) + \theta_i t$. *We let*

$$C(x_1, x_2) = e^{-rT}\,\mathrm{E}[max(S_{1,T}^{x_1} - S_{2,T}^{x_2} - K, 0)]$$

be the premium of a spread with strike time T, *and* $K = K_1 - K_2$ *where* K_1 *and* K_2 *are strike prices of the two options the spread consists of. Then we have*

$$\begin{aligned}
\frac{\partial}{\partial x_i} C(x_1, x_2) &= e^{-rT}\,\mathrm{E}[max(S_{1,t}^{x_1} - S_{2,t}^{x_2} - K, 0)\rho_1(S_{1,t}^{x_1}, S_{2,t}^{x_2})],\\
\frac{\partial^2}{\partial x_i x_j} C(x_1, x_2) &= e^{-rT}\,\mathrm{E}[max(S_{1,t}^{x_1} - S_{2,t}^{x_2} - K, 0)\rho_2(S_{1,t}^{x_1}, S_{2,t}^{x_2})],\\
\frac{\partial^2}{\partial x_i^2} C(x_1, x_2) &= e^{-rT}\,\mathrm{E}[max(S_{1,t}^{x_1} - S_{2,t}^{x_2} - K, 0)\rho_3(S_{1,t}^{x_1}, S_{2,t}^{x_2})],
\end{aligned}$$

where

$$\begin{aligned}
\rho_1(S_{1,t}^{x_1}, S_{2,t}^{x_2}) &= \frac{1}{(1-\rho^2)x_i}\Big(\frac{\ln(S_{i,t}^{x_i}) - \mu_i}{\sigma_i^2 T} - \frac{\rho(\ln(S_{j,t}^{x_j}) - \mu_j)}{\sigma_1\sigma_2 T}\Big),\\
\rho_2(S_{1,t}^{x_1}, S_{2,t}^{x_2}) &= \frac{1}{(1-\rho^2)^2 x_1 x_2}\Big(\Big(\frac{\ln(S_{1,t}^{x_1}) - \mu_1}{\sigma_1^2 T} - \frac{\rho(\ln(S_{2,t}^{x_2}) - \mu_2)}{\sigma_1\sigma_2 T}\Big)\\
&\quad\cdot\Big(\frac{\ln(S_{2,t}^{x_2}) - \mu_2}{\sigma_2^2 T} - \frac{\rho(\ln(S_{1,t}^{x_1}) - \mu_1)}{\sigma_1\sigma_2 T}\Big) + \frac{\rho}{\sigma_1\sigma_2 T}\Big),\\
\rho_3(S_{1,t}^{x_1}, S_{2,t}^{x_2}) &= \frac{1}{(1-\rho^2)x_i^2}\Big(\frac{1}{(1-\rho^2)}\Big(\frac{\ln(S_{i,t}^{x_i}}{\sigma_i^2 T} - \frac{\rho(\ln(S_{j,t}^{x_j})}{\sigma_1\sigma_2 T}\Big)^2\\
&\quad-\Big(\frac{\ln(S_{i,t}^{x_i}}{\sigma_i^2 T} - \frac{\rho(\ln(S_{j,t}^{x_j})}{\sigma_1\sigma_2 T}\Big) - \frac{1}{\sigma_i^2 T}\Big).
\end{aligned}$$

Proof. We will assume that we can always move derivatives into integrals. We start by differentiating the density

$$\begin{aligned}
\frac{\partial}{\partial x_i} P_t(y_1, y_2) =& P_t(y_1, y_2)\Big(-\frac{1}{2(1-\rho^2)}\Big)\Big(\Big(\frac{2(y_i-\mu_i)}{\sigma_i^2 t}\Big)\Big(-\frac{1}{x_i}\Big)\\
&-\frac{2\rho}{\sigma_1\sigma_2 t}(y_j - \mu_j)\Big(-\frac{1}{x_i}\Big)\Big),\\
=& P_t(y_1, y_2)\frac{1}{(1-\rho^2)x_i}\Big(\frac{(y_i-\mu_i)}{\sigma_i^2 t} - \frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2 t}\Big),
\end{aligned}$$

then proceeding as we did in the proof of (34)

$$\begin{aligned}\frac{\partial}{\partial x_i}C(x_1,x_2) =&\frac{\partial}{\partial x_i}e^{-rT}\,\mathrm{E}[max(S^{x_1}_{1,T}-S^{x_2}_{2,T}-K,0)],\\ =&\frac{\partial}{\partial x_i}e^{-rT}\iint_{\mathbb{R}^2} max(e^{y_1}-e^{y_2}-K,0)P_T(y_1,y_2)dy_1dy_2,\\ =&e^{-rT}\iint_{\mathbb{R}^2} max(e^{y_1}-e^{y_2}-K,0)\frac{\partial}{\partial x_i}P_T(y_1,y_2)dy_1dy_2,\\ =&e^{-rT}\iint_{\mathbb{R}^2} max(e^{y_1}-e^{y_2}-K,0)\frac{1}{(1-\rho^2)x_i}\Big(\frac{(y_i-\mu_i)}{\sigma_i^2T}-\frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2T}\Big)P_T(y_1,y_2)dy_1dy_2,\\ =&e^{-rT}\,\mathrm{E}[max(S^{x_1}_{1,T}-S^{x_2}_{2,T}-K,0)\frac{1}{(1-\rho^2)x_i}\Big(\frac{\sigma_iW_{i,T}}{\sigma_i^2T}-\frac{\rho\sigma_jW_{j,T}}{\sigma_1\sigma_2T}\Big)].\end{aligned}$$

We then find the mixed second derivative of the density

$$\begin{aligned}\frac{\partial^2}{\partial x_1x_2}P_t(y_1,y_2) =&\frac{\partial}{\partial x_j}P_t(y_1,y_2)\frac{1}{(1-\rho^2)x_i}\Big(\frac{(y_i-\mu_i)}{\sigma_i^2t}-\frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2t}\Big),\\ =&P_t(y_1,y_2)\Big(\frac{1}{(1-\rho^2)^2x_1x_2}\Big(\frac{(y_1-\mu_1)}{\sigma_1^2t}-\frac{\rho(y_2-\mu_2)}{\sigma_1\sigma_2t}\Big)\Big(\frac{(y_2-\mu_2)}{\sigma_2^2t}-\frac{\rho(y_1-\mu_1)}{\sigma_1\sigma_2t}\Big)\\ &+\frac{1}{(1-\rho^2)x_i}\Big(\frac{\rho}{\sigma_1\sigma_2t}\frac{1}{x_j}\Big)\Big),\\ =&\ P_t(y_1,y_2)\frac{1}{(1-\rho^2)^2x_1x_2}\Big(\Big(\frac{(y_1-\mu_1)}{\sigma_1^2t}-\frac{\rho(y_2-\mu_2)}{\sigma_1\sigma_2t}\Big)\Big(\frac{(y_2-\mu_2)}{\sigma_2^2t}-\frac{\rho(y_1-\mu_1)}{\sigma_1\sigma_2t}\Big)\\ &+\frac{\rho}{\sigma_1\sigma_2t}\Big),\end{aligned}$$

then, proceeding as earlier while skipping some steps, the mixed second derivative of the permium is

$$\begin{aligned}\frac{\partial^2}{\partial x_ix_j}C(x_1,x_2) =&\frac{\partial^2}{\partial x_ix_j}e^{-rT}\,\mathrm{E}[max(S^{x_1}_{1,T}-S^{x_2}_{2,T}-K,0)],\\ =&e^{-rT}\,\mathrm{E}[max(S^{x_1}_{1,T}-S^{x_2}_{2,T}-K,0)\frac{1}{(1-\rho^2)^2x_1x_2}\\ &\cdot\Big(\Big(\frac{\sigma_1W_{1,T}}{\sigma_1^2T}-\frac{\rho\sigma_2W_{2,T}}{\sigma_1\sigma_2T}\Big)\Big(\frac{\sigma_2W_{2,T}}{\sigma_2^2T}-\frac{\rho\sigma_1W_{1,T}}{\sigma_1\sigma_2T}\Big)+\frac{\rho}{\sigma_1\sigma_2T}\Big)].\end{aligned}$$

Lastly we differentiate the density twice in the same initial value

$$
\begin{aligned}
\frac{\partial^2}{\partial x_i^2}P_t(y_1,y_2) =& \frac{\partial}{\partial x_i}P_t(y_1,y_2)\frac{1}{(1-\rho^2)x_i}\Big(\frac{(y_i-\mu_i)}{\sigma_i^2 t}-\frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2 t}\Big),\\
=& P_t(y_1,y_2)\Big(\frac{1}{(1-\rho^2)^2x_i^2}\Big(\frac{(y_i-\mu_i)}{\sigma_i^2 t}-\frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2 t}\Big)^2\\
&+\frac{\partial}{\partial x_i}\frac{1}{(1-\rho^2)x_i}\Big(\frac{(y_i-\mu_i)}{\sigma_i^2 t}-\frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2 t}\Big)\Big),\\
=& P_t(y_1,y_2)\Big(\frac{1}{(1-\rho^2)^2x_i^2}\Big(\frac{(y_i-\mu_i)}{\sigma_i^2 t}-\frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2 t}\Big)^2\\
&+\frac{-1}{(1-\rho^2)x_i^2}\Big(\frac{(y_i-\mu_i)}{\sigma_i^2 t}-\frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2 t}\Big)\\
&+\frac{1}{(1-\rho^2)x_i}\frac{-1}{\sigma_i^2 t x_i}\Big),\\
&= P_t(y_1,y_2)\frac{1}{(1-\rho^2)x_i^2}\Big(\frac{1}{(1-\rho^2)}\Big(\frac{(y_i-\mu_i)}{\sigma_i^2 t}-\frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2 t}\Big)^2\\
&-\Big(\frac{(y_i-\mu_i)}{\sigma_i^2 t}-\frac{\rho(y_j-\mu_j)}{\sigma_1\sigma_2 t}\Big)-\frac{1}{\sigma_i^2 t}\Big),
\end{aligned}
$$

and the second derivative of the premium is

$$
\begin{aligned}
\frac{\partial^2}{\partial x_i^2}C(x_1,x_2) =& \frac{\partial^2}{\partial x_i^2}e^{-rT}\,\mathrm{E}[max(S_{1,T}^{x_1}-S_{2,T}^{x_2}-K,0)],\\
=& e^{-rT}\,\mathrm{E}[max(S_{1,T}^{x_1}-S_{2,T}^{x_2}-K,0)\frac{1}{(1-\rho^2)x_i^2}\\
&\cdot\Big(\frac{1}{(1-\rho^2)}\Big(\frac{\sigma_i W_{i,T}}{\sigma_i^2 T}-\frac{\rho\sigma_j W_{j,T}}{\sigma_1\sigma_2 T}\Big)^2-\Big(\frac{\sigma_i W_{i,T}}{\sigma_i^2 T}-\frac{\rho\sigma_j W_{j,T}}{\sigma_1\sigma_2 T}\Big)-\frac{1}{\sigma_i^2 T}\Big)].
\end{aligned}
$$

■

3 NIG Market

We will now consider option pricing using NIG price processes, but first we will look at the distribution of such processes, which starts by learning a bit about Bessel functions.

3.1 Some Bessel function theory

We consider some theory for the modified Bessel function. We follow the use in the appendix of [TC03]. The modified Bessel function of the first kind is the function that solves

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}-(z^2+\nu^2)w=0, \tag{5}$$

while being bounded as $z \to 0$, for $z \geq 0$ and $\nu \geq 0$. The modified Bessel function of the second kind solves (5) while being bounded when $z \to \infty$, for $z \geq 0$ and $\nu \geq 0$. We will Write $K_\nu(z)$ for the Bessel function of the second kind, of order ν. Note that some literature, like [BN97], and [Ben03] refer to this as the modified Bessel function of the third kind instead. In MATLAB this function is implemented as "besselk" and is documented under modified Bessel function of the second kind. We have some useful properties

Proposition 38. *1. For all orders ν we have $K_{-\nu}(z) = K_\nu(z)$.*

2. The Sommerfeld integral representation:

$$K_\nu(z)=\frac{1}{2}\Big(\frac{z}{2}\Big)^\nu\int_0^\infty e^{-t-\frac{z^2}{4t}}t^{-\nu-1}dt.$$

3. Useful computation result

$$2\Big(\frac{\alpha}{\beta}\Big)^\nu K_\nu(\beta\alpha)=\int_0^\infty e^{-\frac{\alpha^2t}{2}-\frac{\beta^2}{2t}}\frac{dt}{t^{1+\nu}}.$$

4. Derivative for $z > 0$:

$$K'_\nu(z)=\frac{\nu K_\nu(z)}{z}-K_{\nu+1}(z),$$

and

$$K'_\nu(z)=\frac{-\nu K_\nu(z)}{z}-K_{\nu-1}(z).$$

Proof. The results 1., 2., and 3. are taken from the appendix of [TC03]. Using 2. we get

$$\begin{aligned}K'_\nu(z)=&\nu\frac{1}{2}\Big(\frac{z}{2}\Big)^{\nu-1}\int_0^\infty e^{-t-\frac{z^2}{4t}}t^{-\nu-1}dt\\&+\frac{1}{2}\Big(\frac{z}{2}\Big)^\nu\frac{d}{dz}\int_0^\infty e^{-t-\frac{z^2}{4t}}t^{-\nu-1}dt,\end{aligned}$$

we move the differential in by (33), since the new integrand is integrable as we will see by the end

$$
\begin{aligned}
=&\frac{\nu K_\nu(z)}{z} + \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{d}{dz} e^{-t-\frac{z^2}{4t}} t^{-\nu-1} dt, \\
=&\frac{\nu K_\nu(z)}{z} + \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{-z}{2t}\frac{d}{dz} e^{-t-\frac{z^2}{4t}} t^{-\nu-1} dt, \\
=&\frac{\nu K_\nu(z)}{z} - \frac{1}{2}\left(\frac{z}{2}\right)^{\nu+1} \int_0^\infty \frac{d}{dz} e^{-t-\frac{z^2}{4t}} t^{-\nu-2} dt, \\
=&\frac{\nu K_\nu(z)}{z} - K_{\nu+1}(z),
\end{aligned}
$$

which also justified our use of Leibniz' rule. Next we use property 1.

$$
\begin{aligned}
K'_\nu(z) =&K'_{-\nu}(z), \\
=&\frac{(-\nu)K_{-\nu}(z)}{z} - K_{-\nu+1}(z), \\
=&\frac{-\nu K_\nu(z)}{z} - K_{\nu-1}(z),
\end{aligned}
$$

and we are done. ■

The appendix of [TC03] contains some more results for both functions of the first and second kind, but we will not make use of these.

3.2 The normal inverse Gaussian distribution

Before looking at NIG markets we will first reproduce an expression for the NIG distribution. Following [BN97] we may parametrize the inverse Gaussian distribution $IG(\delta, \gamma)$ by

$$
f_{IG}(x; \delta, \gamma) = \frac{\delta}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{\delta\gamma} e^{-\frac{1}{2}(\delta^2 \frac{1}{x} + \gamma^2 x)},
$$

with mean δ/γ and variance δ/γ^3. Then

Proposition 39. *If $Z \sim IG(\delta, \sqrt{\alpha^2 - \beta^2})$, and $X|Z \sim N(\mu + \beta z, z)$, with $0 \le |\beta| \le \alpha$, $\mu \in \mathbb{R}$, and $0 < \delta$, then X is normal inverse Gaussian distributed, with density*

$$
f_{NIG}(x; \alpha, \beta, \mu, \delta) = k e^{\beta(x-\mu)} \frac{K_1(\alpha\sqrt{(x-\mu)^2 + \delta^2})}{\sqrt{(x-\mu)^2 + \delta^2}},
$$

where

$$
k = \frac{\delta\alpha}{\pi} e^{\delta\sqrt{\alpha^2 - \beta^2}},
$$

and K_1 is the modified Bessel function of the second kind of order 1.

Proof. We consider the moment generating function of X

$$
\begin{aligned}
\mathrm{E}[e^{uX}] &= \int_{-\infty}^{\infty} \mathrm{E}[e^{uX}|Z=z] f_{IG}(z;\delta,\sqrt{\alpha^2-\beta^2})dz,\\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{ux}\frac{1}{\sqrt{2\pi z}} e^{-\frac{1}{2}\left(\frac{(x-\mu-\beta z)^2}{z}\right)}\\
&\quad \cdot \frac{\delta}{\sqrt{2\pi}} z^{-\frac{3}{2}} e^{\delta\sqrt{\alpha^2-\beta^2}} e^{-\frac{1}{2}(\delta^2\frac{1}{z}+(\alpha^2-\beta^2)z)} dz dx,\\
&= \int_{-\infty}^{\infty} e^{ux}\frac{e^{\delta\sqrt{\alpha^2-\beta^2}}}{2\pi}\int_{0}^{\infty} e^{-\frac{1}{2z}(x-\mu-\beta z)^2-\frac{1}{2}((\alpha^2-\beta^2)z+\frac{\delta^2}{z})}\frac{dz}{z^2}dx,\\
&= \int_{-\infty}^{\infty} e^{ux}\frac{e^{\delta\sqrt{\alpha^2-\beta^2}}}{2\pi}\int_{0}^{\infty} e^{-\frac{1}{2}\left(\frac{(x-\mu)^2}{z}-2(x-\mu)\beta+\beta^2 z+(\alpha^2-\beta^2)z+\frac{\delta^2}{z}\right)}\frac{dz}{z^2}dx,\\
&= \int_{-\infty}^{\infty} e^{ux}\frac{e^{\delta\sqrt{\alpha^2-\beta^2}}}{2\pi} e^{(x-\mu)\beta}\int_{0}^{\infty} e^{-\frac{(x-\mu)^2+\delta^2}{2z}-\frac{\alpha^2 z}{2}}\frac{dz}{z^2}dx,
\end{aligned}
$$

now using result 3 of (38) with $\nu = 1$ we get

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} e^{ux}\frac{e^{\delta\sqrt{\alpha^2-\beta^2}}}{2\pi} e^{(x-\mu)\beta} 2\frac{\alpha}{\sqrt{(x-\mu)^2+\delta^2}} K_1(\alpha\sqrt{(x-\mu)^2+\delta^2})dx,\\
&= \int_{-\infty}^{\infty} e^{ux}\frac{\delta\alpha}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}} e^{(x-\mu)\beta}\frac{K_1(\alpha\sqrt{(x-\mu)^2+\delta^2})}{\sqrt{(x-\mu)^2+\delta^2}}dx.
\end{aligned}
$$

Now since the moment generating function uniquely determines a distribution we conclude that X must have distribution

$$
f_{NIG}(x;\alpha,\beta,\mu,\delta) = \frac{\delta\alpha}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}} e^{\beta(x-\mu)}\frac{K_1(\alpha\sqrt{(x-\mu)^2+\delta^2})}{\sqrt{(x-\mu)^2+\delta^2}},
$$

as desired. ■

According to [Ben03] we have that if X is NIG distributed as above then we have that

$$
\mathrm{E}[X] = \mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}, \tag{6}
$$

and

$$
V[X] = \frac{\delta\alpha^2}{(\alpha^2-\beta^2)^{3/2}}, \tag{7}
$$

which agrees with [BN97] after rewriting. According to [BN97] we have that if $\beta = 0$, $\alpha \to \infty$, and $\delta/\alpha = \sigma^2$, then $X \sim \mathcal{N}(\mu, \sigma^2)$. Also according to [Ben03] if $beta > 0$ the distribution will skew to the right, if $beta < 0$ the distribution will skew to the left, and we see from (6) that it will be centered around μ. [Ben03] also says that δ plays a similar role to the standard deviation for the normal distribution, and that α models tail heaviness. In [Ben03] it is also stated that if $X \sim NIG(\alpha, \beta, \mu_X, \delta_X)$, and $Y \sim NIG(\alpha, \beta, \mu_Y, \delta_Y)$ independent of X, then $X + Y \sim NIG(\alpha, \beta, \mu_X + \mu_Y, \delta_X + \delta_Y)$. This implies that $p_t(dx), t \geq 0$, where $p_t(dx) = f_{NIG}(x; \alpha, \beta, \mu t, \delta t)dx$, is a convolution semigroup, since it is easy to see that $p_0 = 0$, and since the measure of $X + Y$ is the convolution of the measures of X and Y.

3.3 Normal inverse Gaussian market

We are now ready to consider option prices of NIG distributed markets, and reach our main result. We consider the price process

$$S_t^x = xe^{L_t}$$

with initial value x, where L_t is normal inverse Gaussian with density

$$f_{NIG}(y; \alpha, \beta, \mu t, \delta t),$$

where f_{NIG} is as in (39).

Proposition 40. *Let $C(x)$ be the premium of an option, with a continuous payoff function ϕ, assume that $\phi(S_T^x)$ has finite expectation,and assume that*

$$|\phi(e^y)\frac{\partial}{\partial x} f_{NIG}(y - \ln(x); \alpha, \beta, \mu T, \delta T)| \leq g(y),$$

for all x, for some integrable function $g : \mathbb{R} \to \mathbb{R}$. Then

$$C'(x) = e^{-rT}\,\mathrm{E}[\phi(S_T^x)\frac{1}{x}\Big(- \beta + \frac{\alpha(L_t - \mu)}{\sqrt{(L_T - \mu)^2 + \delta^2 T^2}} \frac{K_2(B\sqrt{(L_T - \mu)^2 + \delta^2 T^2})}{K_1(B\sqrt{(L_T - \mu)^2 + \delta^2 T^2})}\Big)].$$

Alternatively we have

$$\begin{aligned} C'(x) =& e^{-rT}\,\mathrm{E}[\phi(S_T^x)\frac{1}{x}\Big(- \beta + \frac{2(L_t - \mu)}{(L_t - \mu)^2 + \delta^2 T^2} \\ &+ \frac{\alpha(L_t - \mu)}{\sqrt{(L_T - \mu)^2 + \delta^2 T^2}} \frac{K_2(B\sqrt{(L_T - \mu)^2 + \delta^2 T^2})}{K_1(B\sqrt{(L_T - \mu)^2 + \delta^2 T^2})}\Big)] \end{aligned}$$

Proof. We will start by finding the derivative of the distribution with respect to the initial price x. We define $p(z) = \frac{d}{dz} f_{NIG}(z + \mu; \alpha, \beta, \mu, \delta)$ and differentiate this

$$\begin{aligned} p(z) =& \beta p(z) + ke^{\beta z} \frac{d}{dz} \frac{K_1(\alpha\sqrt{z^2 + \delta^2})}{\sqrt{z^2 + \delta^2}}, \\ =& \beta p(z) + ke^{\beta z} \frac{(\frac{d}{dz} K_1(\alpha\sqrt{z^2 + \delta^2}))\sqrt{z^2 + \delta^2}}{z^2 + \delta^2} \\ &- ke^{\beta z} \frac{K_1(\alpha\sqrt{z^2 + \delta^2})\frac{d}{dz}\sqrt{z^2 + \delta^2}}{z^2 + \delta^2}, \end{aligned}$$

we apply result 4 of (38), and $\frac{d}{dz}\sqrt{z^2+\delta^2} = \frac{z}{\sqrt{z^2+\delta^2}}$ to get

$$
\begin{aligned}
=&\beta p(z) + ke^{\beta z}\frac{\left(\frac{K_1(\alpha\sqrt{z^2+\delta^2})}{\alpha\sqrt{z^2+\delta^2}} - K_2(\alpha\sqrt{z^2+\delta^2})\right)\left(\frac{d}{dz}\alpha\sqrt{z^2+\delta^2}\right)\sqrt{z^2+\delta^2}}{z^2+\delta^2} \\
&- ke^{\beta z}\frac{K_1(\alpha\sqrt{z^2+\delta^2})z}{(z^2+\delta^2)\sqrt{z^2+\delta^2}}, \\
=&\beta p(z) + \frac{z}{z^2+\delta^2}p(z) - ke^{\beta z}\frac{\alpha z K_2(\alpha\sqrt{z^2+\delta^2})}{z^2+\delta^2} - \frac{z}{z^2+\delta^2}p(z), \\
=&p(z)\Big(\beta - \frac{\alpha z}{\sqrt{z^2+\delta^2}}\frac{K_2(\alpha\sqrt{z^2+\delta^2})}{K_1(\alpha\sqrt{z^2+\delta^2})}\Big)
\end{aligned}
$$

Alternatively we could have used the second form of 4 of (38) to get

$$
p(z)\Big(\beta - 2\frac{z}{z^2+\delta^2} - \frac{\alpha z}{\sqrt{z^2+\delta^2}}\frac{K_2(\alpha\sqrt{z^2+\delta^2})}{K_1(\alpha\sqrt{z^2+\delta^2})}\Big)
$$

instead. Further

$$
\begin{aligned}
\frac{d}{dz}f_{NIG}(z;\alpha,\beta,\mu,\delta) &= \frac{d}{dz}p(z-\mu), \\
&= p'(z-\mu),
\end{aligned}
$$

and for the process we may simply replace μ and δ with μT and δT. Next we evaluate

$$
\begin{aligned}
\frac{\partial}{\partial x}\,\mathrm{E}[\phi(S_T^x)] =&\frac{\partial}{\partial x}e^{-rT}\int_{-\infty}^{\infty}\phi(e^{\ln(x)+y})f_{NIG}(y;\alpha,\beta,\mu T,\delta T)dy, \\
=&\frac{\partial}{\partial x}e^{-rT}\int_{-\infty}^{\infty}\phi(e^{z})f_{NIG}(z-\ln(x);\alpha,\beta,\mu T,\delta T)dz,
\end{aligned}
$$

we use (33)

$$
\begin{aligned}
=&e^{-rT}\int_{-\infty}^{\infty}\phi(e^{z})\frac{\partial}{\partial x}f_{NIG}(z-\ln(x);\alpha,\beta,\mu T,\delta T)dz, \\
=&e^{-rT}\int_{-\infty}^{\infty}\phi(e^{z})\frac{-1}{x}\Big(\\
&\beta - \frac{\alpha(z-\ln(x)-\mu)}{\sqrt{(z-\ln(x)-\mu)^2+\delta^2T^2}}\frac{K_2(B\sqrt{(z-\ln(x)-\mu)^2+\delta^2T^2})}{K_1(B\sqrt{(z-\ln(x)-\mu)^2+\delta^2T^2})} \\
&\Big)f_{NIG}(z-\ln(x);\alpha,\beta,\mu T,\delta T)dz, \\
=&e^{-rT}\int_{-\infty}^{\infty}\phi(e^{\ln(x)+y})\Big(-\beta + \frac{\alpha(y-\mu)}{\sqrt{(y-\mu)^2+\delta^2T^2}}\frac{K_2(B\sqrt{(y-\mu)^2+\sigma^2T^2})}{K_1(B\sqrt{(y-\mu)^2+\delta^2T^2})} \\
&\Big)\frac{f_{NIG}(y;\alpha,\beta,\mu T,\delta T)}{x}dy, \\
=&e^{-rT}\,\mathrm{E}[\phi(xe^{L_T})\frac{1}{x}\Big(-\beta + \frac{\alpha(L_t-\mu)}{\sqrt{(L_T-\mu)^2+\delta^2T^2}}\frac{K_2(B\sqrt{(L_T-\mu)^2+\delta^2T^2})}{K_1(B\sqrt{(L_T-\mu)^2+\delta^2T^2})}\Big)],
\end{aligned}
$$

and the alternative from

$$\begin{aligned}\frac{\partial}{\partial x}\,\mathrm{E}[\phi(S_T^x)] =& e^{-rT}\,\mathrm{E}[\phi(xe^{L_T})\frac{1}{x}\Big(-\beta+\frac{2(L_t-\mu)}{(L_t-\mu)^2+\delta^2T^2}\\ &+\frac{\alpha(L_t-\mu)}{\sqrt{(L_T-\mu)^2+\delta^2T^2}}\frac{K_2(B\sqrt{(L_T-\mu)^2+\delta^2T^2})}{K_1(B\sqrt{(L_T-\mu)^2+\delta^2T^2})}\Big)],\end{aligned}$$

is reached in the same manner, and we are done. ■

In [TC03] they use an alternative parametrization of the NIG distribution. If we instead consider $\bar{L}_t$ to have the parametrization

$$P_{\bar{L}_t}(y) = Ce^{Ay}\frac{K_1(B\sqrt{y^2+t^2\sigma^2/\kappa})}{\sqrt{y^2+t^2\sigma^2/\kappa}},$$

where

$$\begin{aligned}A =& \frac{\theta}{\sigma^2},\\ B =& \frac{\sqrt{\theta^2+\sigma^2/\kappa}}{\sigma^2},\\ C =& \frac{t}{\pi}e^{t/\kappa}\sqrt{\frac{\theta^2}{\kappa\sigma^2}+\frac{1}{\kappa^2}},\end{aligned}$$

where σ is volatility and θ drift of the Brownian motion, and κ the variance of the inverse Gaussian subordinator, we have the same result.

Proposition 41. *Let $C(x)$ be the premium of an option, with a continuous payoff function ϕ, assume that $\phi(\bar{S}_t^x)$ has finite expectation,and assume that*

$$|\phi(e^y)\frac{\partial}{\partial x}P_{\bar{L}_t}(y-\ln(x))| \leq g(y),$$

for all x, for some integrable function $g:\mathbb{R}\to\mathbb{R}$, where $P_{\bar{L}_t}$ is the density

$$P_{\bar{L}_t}(y) = Ce^{Ay}\frac{K_1(B\sqrt{y^2+t^2\sigma^2/\kappa})}{\sqrt{y^2+t^2\sigma^2/\kappa}}.$$

Then

$$C'(x) = e^{-rT}\,\mathrm{E}[\phi(\bar{S}_t^x)\frac{1}{x}\Big(-A+\frac{B\bar{L}_T}{\sqrt{\bar{L}_T^2+T^2\sigma^2/\kappa}}\frac{K_2(B\sqrt{\bar{L}_T^2+T^2\sigma^2/\kappa)}}{K_1(B\sqrt{\bar{L}_T^2+T^2\sigma^2/\kappa)}}\Big)].$$

Alternatively we have

$$\begin{aligned}C'(x) =& e^{-rT}\,\mathrm{E}[\phi(\bar{S}_t^x)\frac{1}{x}\Big(-A+2\frac{\bar{L}_T}{\bar{L}_T^2+\sigma^2T/\kappa}\\ &+\frac{B\bar{L}_T}{\sqrt{\bar{L}_T^2+t^2\sigma^2/\kappa}}\frac{K_2(B\sqrt{\bar{L}_T^2+T^2\sigma^2/\kappa)}}{K_1(B\sqrt{\bar{L}_T^2+T^2\sigma^2/\kappa)}}\Big)].\end{aligned}$$

From [TC03] we know the first three cumulants for this parametrization to be

$$\begin{aligned} \mathrm{E}[\bar{L}_t] &= \theta t, \\ Var[\bar{L}_t] &= \sigma^2 t + \theta^2 \kappa t, \\ c_3 &= 3\sigma^2\theta\kappa t + 3\theta^3\kappa^2 t, \end{aligned}$$

but for the distribution to be symmetric we need

$$c_3 = 0,$$

but then

$$\begin{aligned} 3\sigma^2\theta\kappa t + 3\theta^3\kappa^2 t &= 0, \\ 3\theta\kappa(\sigma^2 t + \theta^2\kappa t) &= 0, \\ 3\theta\kappa Var[\bar{L}_t] &= 0. \end{aligned}$$

This gives three cases

- $\kappa = 0$
- $Var[L_t] = 0$
- $\theta = 0$

We divide by κ in this parametrization, and besides it is the variance of the inverse Gaussian used to construct the process, so it should not be 0. Next, the variance of our process being 0 is another degenerate case. That leaves $\theta = 0$ as the special case where we may have a symmetric process. Because of this we favor the parametrization we found in (39).

We now consider the price process

$$S_t^x = xe^{L_t}$$

with initial value x and where L_t has distribution

$$f_{NIG}(x; \alpha, \beta, \mu t, \delta t).$$

We would like a specific Martingale condition for implementations.

Proposition 42. *Assume that* $\mathrm{E}[S_t] < \infty$ *for all* $t \geq 0$*, and that* $\alpha \geq |\beta+1|$*. If*

$$\mu = r - \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2}),$$

then $e^{-rt}S_t^x$ *is a Martingale.*

Proof. For $\mathrm{E}[e^{L_t}]$ to be a Martingale, based on the proof of (31), we require

$$e^{-r(t-s)}\,\mathrm{E}[e^{L_{t-s}}] = 1 \tag{8}$$

for all $t \geq s \geq 0$. From [BN97] we know that with the parametrization used, the moment generating function of L_t is

$$M(u; \alpha, \beta, \mu t, \delta t) = e^{\delta t(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+u)^2}) + \mu t u},$$

which equals $\phi(-iu)$, where ϕ is the characteristic function of the NIG process found in 21. Combining this with (8) we see that for $t > s$ we require

$$\delta(t-s)(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+1)^2})+\mu(t-s)-r(t-s)=0,$$
$$\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+1)^2})+\mu-r=0,$$

hence to achieve Martingality we may choose

$$\mu = r-\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+1)^2}).$$

In the case $t = s$, Martingality is trivially true. Now scaling by the initial value is just multiplication by a constant, which preserves Martingality, hence we are done. ■

Notice that we now assume $\alpha \geq |\beta+1|$ rather than $\alpha \geq |\beta|$ as we did for an arbitrary NIG distribution. We see from the proof that we could have other Martingale conditions

$$\mu = \frac{r-\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+1)^2})}{u},$$

where $u \neq 0$ and $\alpha \geq |\beta+u|$. Next we have a result that guarantees the integrability.

Proposition 43. *If $\alpha \geq |1+\beta|$ then*

$$\int_{\mathbb{R}\setminus(-1,1)} e^x \nu(dx) < \infty,$$

where ν is the Lévy measure for a NIG process with $f_{NIG}(x;\alpha,\beta,\mu t,\delta t)$ as its distribution.

Proof. According to [BN97] the Lévy measure of the NIG process parametrized as in (39) is

$$\nu(dx) = \frac{\delta\alpha}{\pi|x|}e^{\beta x}K_1(\alpha|x|)\nu(dx). \tag{9}$$

Further according to [BN97] the asymptotic behaviour as $z \to \infty$ of the modified Bessel function is

$$K_1(z) \sim \frac{\sqrt{2\pi}}{\sqrt{z}}e^{-z},$$

that is

$$\lim_{z\to\infty}\frac{K_1(z)}{\frac{\sqrt{2\pi}}{\sqrt{z}}e^{-z}} = 1.$$

Then multiplying by $1 = \frac{\frac{\delta\alpha}{\pi|x|}e^{\beta x}}{\frac{\delta\alpha}{\pi|x|}e^{\beta x}}$ we also have

$$\lim_{x\to\infty}\frac{\frac{\delta\alpha}{\pi|x|}e^{(1+\beta)x}K_1(\alpha|x|)}{e^{(1+\beta)x}\frac{\delta\sqrt{2}}{\sqrt{\pi}|x|\sqrt{x}}e^{-\alpha|x|}} = 1. \tag{10}$$

From the result (2) of (38) we see that the modified Bessel is always positive, combined with the second form of (4) we conclude that $K_1'(z) < 0$ for $z > 0$, i.e. it is positive and monotonically decreasing. Then the enumerators and denominators of (10) are also positive an monotonically decreasing for $x \geq 0$. Then we may use the integral test for convergence, which gives that

$$\int_1^\infty \frac{\delta\alpha}{\pi|x|} e^{(1+\beta)x} K_1(\alpha|x|)dx,$$

is finite if and only if

$$\sum_{n=1}^\infty \frac{\delta\alpha}{\pi|n|} e^{(1+\beta)n} K_1(\alpha|n|) \tag{11}$$

converges. We also have that

$$\int_1^\infty e^{(1+\beta)x-\alpha|x|} \frac{\delta\sqrt{2}}{\sqrt{\pi}|x|\sqrt{x}} dx$$

is finite if and only if

$$\sum_{n=1}^\infty e^{(1+\beta)n} \frac{\delta\sqrt{2}}{\sqrt{\pi}|n|\sqrt{n}} e^{-\alpha|n|} \tag{12}$$

converges. Now both of these two series' consists of positive elements, thus we may apply the limit comparison test for (11) and (12), which gives that both either converge or diverge if

$$\lim_{n\to\infty} \frac{\frac{\delta\alpha}{\pi|n|} e^{(1+\beta)n} K_1(\alpha|n|)}{e^{(1+\beta)n} \frac{\delta\sqrt{2}}{\sqrt{\pi}|n|\sqrt{n}} e^{-\alpha|n|}} \tag{13}$$

exists, is finite, and non zero, but we know this to equal 1 from (10). Thus

$$\int_1^\infty \frac{\delta\alpha}{\pi|x|} e^{(1+\beta)x} K_1(\alpha|x|)dx,$$

is finite if and only if

$$\int_1^\infty e^{(1+\beta)x-\alpha|x|} \frac{\delta\sqrt{2}}{\sqrt{\pi}|x|\sqrt{x}} dx$$

is finite. We have

$$\int_1^\infty e^{(1+\beta)x-\alpha|x|} \frac{\delta\sqrt{2}}{\sqrt{\pi}|x|\sqrt{x}} dx$$

which is finite if $(1+\beta)x - \alpha|x| \leq 0$. Hence we have that

$$\int_1^\infty \frac{\delta\alpha}{\pi|x|} e^{(1+\beta)x} K_1(\alpha|x|)dx < \infty,$$

if $\alpha|x| \geq (1+\beta)x$, where $x > 0$. Next we consider

$$\int_{-\infty}^{-1} \frac{\delta\alpha}{\pi|x|} e^{(1+\beta)x} K_1(\alpha|x|)dx \leq \int_{-\infty}^{-1} \frac{\delta\alpha}{\pi|x|} e^{|1+\beta||x|} K_1(\alpha|x|)dx,$$
$$= \int_{1}^{\infty} \frac{\delta\alpha}{\pi|x|} e^{|1+\beta||x|} K_1(\alpha|x|)dx < \infty,$$

if $\alpha|x| \geq |1+\beta||x|$, by replacing $(1+\beta)$ with $|1+\beta|$ in the previous case. All together we have

$$\int_{\mathbb{R}\setminus(-1,1)} \frac{\delta\alpha}{\pi|x|} e^{(1+\beta)x} K_1(\alpha|x|)dx < \infty,$$

if $\alpha \geq |1+\beta|$, but since (9)

$$\int_{\mathbb{R}\setminus(-1,1)} e^x \nu(dx) < \infty,$$

if $\alpha \geq |1+\beta|$. ∎

We may combine the two previous results.

Theorem 44. *Let $L_t, t \geq 0$ be a NIG process with distribution $f_{NIG}(x;\alpha,\beta,\mu t,\delta t)$, as defined in (39). If $\alpha \geq |1+\beta|$, and*

$$\mu = r - \delta(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+1)^2}),$$

then $\mathrm{E}[e^{L_t}] < \infty$, and $e^{-rt}e^{L_t}$ is a Martingale.

Proof. From (43) we get that

$$\int_{\mathbb{R}\setminus(-1,1)} e^x \nu(dx) < \infty,$$

then from (30) we get that $\mathrm{E}[e^{L_t}] < \infty$. Then we may apply (42) with $x = 1$ in S_t^x, and we have that $e^{-rt}e^{L_t}$ is a Martingale. ∎

4 Numerical implementation

We have expressions for option prices as well as the related delta in NIG markets, as well as a Martingality condition. The next step is to test out numerical implementation and compare these with the Brownian market case. We have expressions for the Brownian market case from earlier sections, but we would also like to implement a Black-Scholes solution, and its finite differences to compare with as well. Hence we start by finding a Black-Scholes style solution, although we do this for a non Martingale price process to be more general.

4.1 Black Scholes style solution

Proposition 45. *Let $S_t^x = xe^{\theta T+\sigma W_t}$ be a price process, where W_t is the standard Brownian motion. Then for an option with strike price K at time T is*

$$C_{BSs} = e^{-rT} S_T^x \Phi(L - \sigma\sqrt{T}) - K\Phi(L),$$

where

$$L = \frac{\ln(K) - \ln(x) - \theta T}{\sigma\sqrt{T}}.$$

Proof.

$$\begin{aligned} C_{BSs} =& e^{-rT}\,\mathrm{E}[max(S_T^x - K, 0)], \\ =& e^{-rT}\,\mathrm{E}[max(e^{\ln(x)+\theta T+\sigma W_T} - K, 0)], \end{aligned}$$

Let Z be a standard normal random variable

$$\begin{aligned} =& e^{-rT}\,\mathrm{E}[max(e^{\ln(x)+\theta T+\sigma\sqrt{T}Z} - K, 0)], \\ =& e^{-rT}\int_{-\infty}^{\infty} max(e^{\ln(x)+\theta T+\sigma\sqrt{T}y} - K, 0)e^{-\frac{1}{2}y^2}\frac{dy}{2\pi}, \end{aligned}$$

Let L be such that $max(e^{\ln(x)+\theta T+\sigma\sqrt{T}L} - K, 0) > 0$, then since $max(e^{\ln(x)+\theta T+\sigma\sqrt{T}y} - K, 0)$ is increasing

$$\begin{aligned} =& e^{-rT}\int_{L}^{\infty} e^{\ln(x)+\theta T+\sigma\sqrt{T}y-\frac{1}{2}y^2}\frac{dy}{2\pi} - Ke^{-rT}\int_{L}^{\infty} e^{-\frac{1}{2}y^2}\frac{dy}{2\pi}, \\ =& e^{\ln(x)+\theta T-rT}\int_{L}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{T})^2+\frac{1}{2}\sigma^2 T}\frac{dy}{2\pi} - Ke^{-rT}\int_{L}^{\infty} e^{-\frac{1}{2}y^2}\frac{dy}{2\pi}, \end{aligned}$$

make the substitution $z = y - \sigma\sqrt{T}$, we get $dz = dy$ and a new lower limit of $L - \sigma\sqrt{(T)}$

$$\begin{aligned} =& e^{\ln(x)+\theta T+\frac{1}{2}\sigma^2 T-rT}\int_{L-\sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}z^2}\frac{dz}{2\pi} - Ke^{-rT}\int_{L}^{\infty} e^{-\frac{1}{2}y^2}\frac{dy}{2\pi}, \\ =& e^{-rT} S_T^x \Phi(L - \sigma\sqrt{T}) - K\Phi(L), \end{aligned}$$

where Φ is the cumulative function for the standard normal distribution. Now to determine L. We have

$$\begin{aligned} e^{\ln(x)+\theta T+\sigma\sqrt{T}L} - K &= 0, \\ \ln(x) + \theta T + \sigma\sqrt{T}L &= \ln(K), \\ L &= \frac{\ln(K) - \ln(x) - \theta T}{\sigma\sqrt{T}}. \end{aligned}$$

■

Note that when we use the Martingale condition $\theta = (r - \frac{1}{2}\sigma^2)T$ we get the familiar Black-Scholes solution

$$C_{BS} = x\Phi(L_-) - K\Phi(L_+),$$

where

$$L_- = \frac{\ln(K) - \ln(x) - rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T},$$

and

$$L_+ = \frac{\ln(K) - \ln(x) - rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

4.2 NIG vs Gaussian distribution

For numerical implementation we will consider three cases, the Brownian model, the NIG model, and at times the Black-Scholes style solution from (45). We will implement the densities

$$f(y; x, \theta t, \sigma^2 t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-\ln(x)-\theta t)^2}{2\sigma^2}},$$

for the Brownian case, and

$$f(y; x, \alpha, \beta, \mu, \delta) = \frac{\delta\alpha}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}} e^{\beta(y-\ln(x)-\mu)} \frac{K_1(\alpha\sqrt{(y-\ln(x)-\mu)^2+\delta^2})}{\sqrt{(y-\ln(x)-\mu)^2+\delta^2}},$$

for the NIG model. We will use (6) and (7) to set the mean variance to be equal for the two models. We need

$$\theta = \mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}},$$

and

$$\sigma^2 = \frac{\delta\alpha^2}{(\alpha^2-\beta^2)^{3/2}}.$$

We start by comparing the two distributions at various times. We will use the values in table (4.2) for the NIG distribution, and then use the conditions above for the Brownian distribution so they have equal mean and variance. The implementation can be found in appendix (7) In figure (1) we can see the densities when the time is 0.1, 1, 10, and 100. We see that the differences in

α	β	μ	δ
150	0	0	0.015

Table 1: Test parameters.

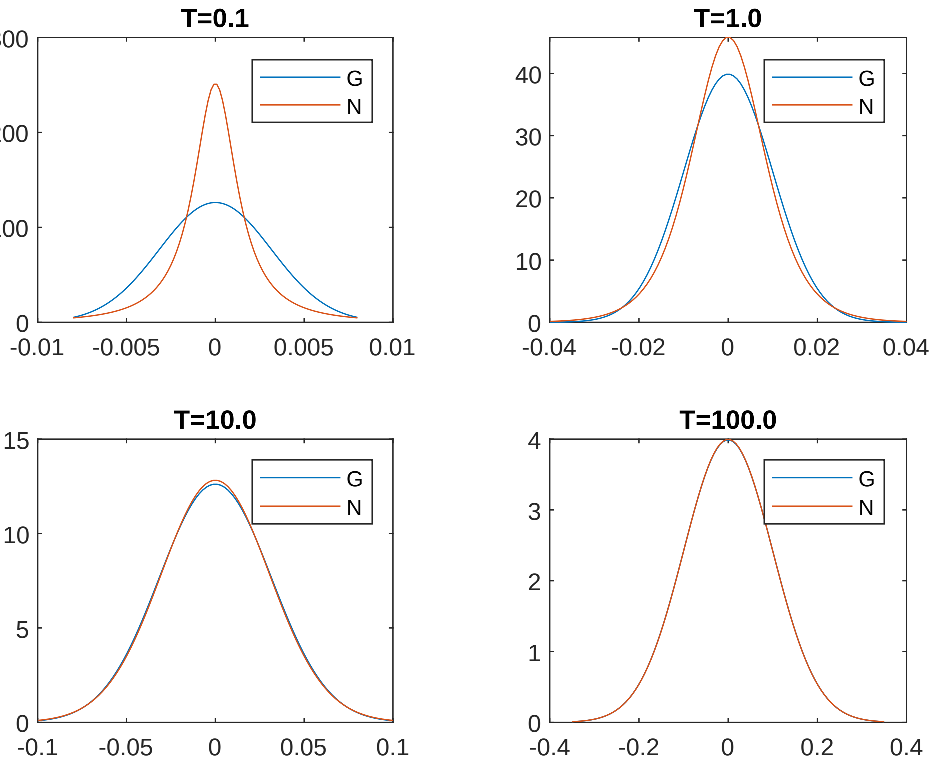

Figure 1: Comparing the NIG and Gaussian densities for different strike times.

the distribution vanish as time increases. Note that this is for a symmetrical NIG process, and neither process has drift, since $\beta = 0$ and $\mu = 0$.
Next we will implement the values found in table (4.2), these are from [Ben03], and are fitted for daily change in the FTSE index. We will start by calculating option premium and delta in a Martingale case, this time with strike price of $K = 50$, and strike time $T = 50$, for initial prices ranging from 1 to 100, with zero interest rate. The implementation is simply using built in numerical integrators in MATLAB on the densities found in (34) and (39) with payoff function $\phi(y) = max(e^y - K, 0)$, we also calculate the Black-Scholes solution and use finite difference on these, the implementation can be found in appendix (8). The result can be seen in figure (2) and figure (3). To achieve Martingality

α	β	μ	δ
105	3.0	-0.0005	0.012

Table 2: FTSE daily parameters from [Ben03].

we first apply the parameters in table (4.2), then override μ by

$$\mu = r - \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2}),$$

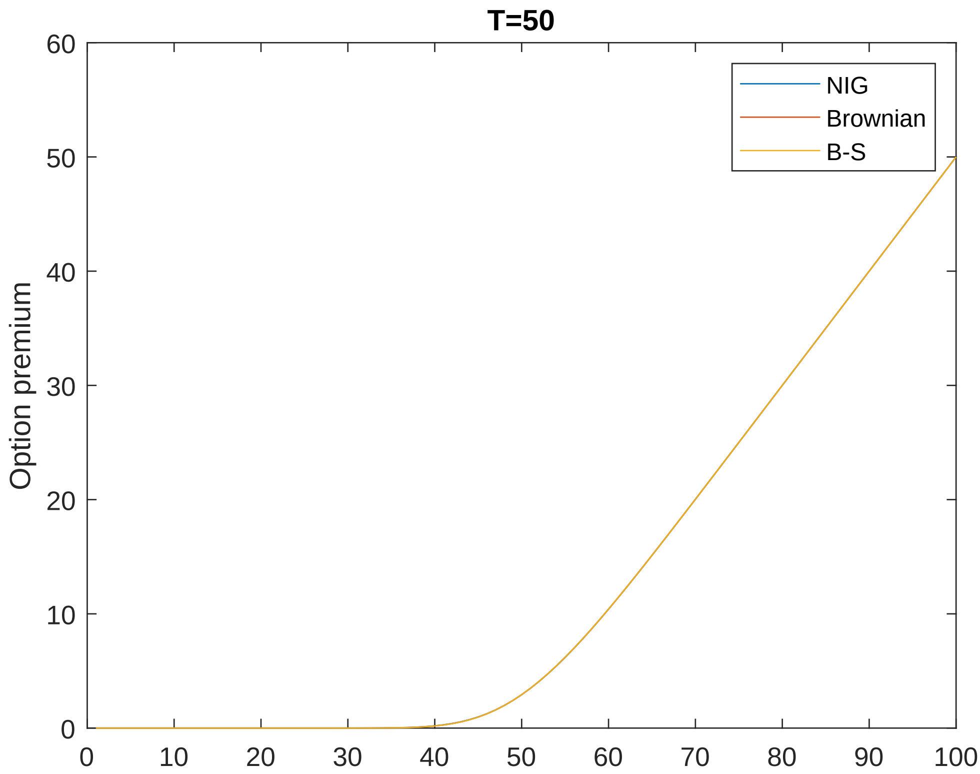

Figure 2: Option premium, comparing NIG and Brownian models with a Black-Scholes style solution.

the condition found in (42) (we fullfill the condition $\alpha \geq |\beta + 1|$), then set the parameters for the Brownian model such that they have equal mean and variance, then apply the condition

$$\theta = r - \frac{1}{2}\sigma^2,$$

to override θ. We observe little difference between the Brownian and NIG models. We proceed by calculating the same at different times. This can be seen in figure (4), where we have zoomed in on the area near the strike price, since this is where the difference is most noticeable. We see that the difference seems to shrink as time increases, which fits well with what we saw in figure (1). We can also see that the range where the function changes growth rate grows with time, this is more clearly seen in (5), since this is simply the derivative. When $T = 0.1$ Delta goes from 0 to 1 for $x \in [49, 51]$, while at $T = 100$ it takes a wider range than $x \in [40, 60]$.
To be a bit more precise we should also calculate the relative error, this time excluding a Black-Scholes type solution, and at $T = 10$, the result can found in figure (6), and the implementation is appendix (8) modified with appendix (9). We see that the natural log of the ratio between the premium from the two models is greatest when x is near 1. This is the results from the values of the

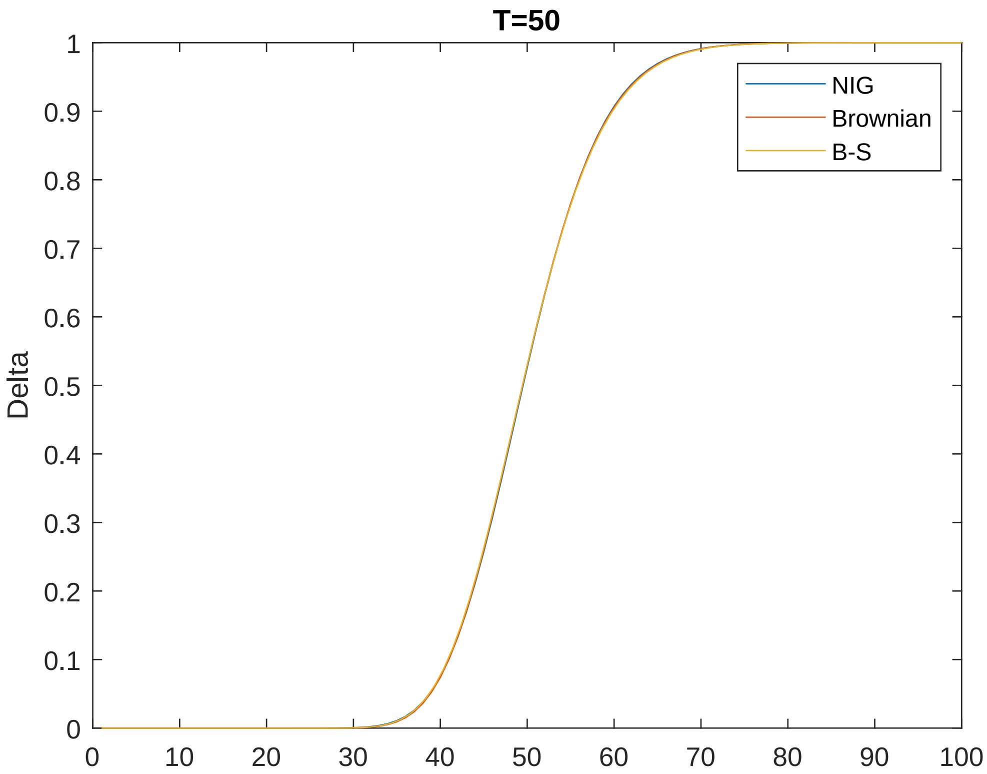

Figure 3: Option delta, comparing NIG and Brownain models with finite difference of a Black-Scholes style solution.

functions being near zero, but slightly different, in fact we need about $x \geq 40$ for the absolute value of the two price functions to reach 0.01 (one cent if we wanted to apply it to a real market). Thus we have also plotted the ratio (not log ratio) for the two price functions for $x \in [40, 60]$ as well. We see the same pattern, where the error is greater for lower x values, where the price is still quite small, while by $x = 45$ the ratio is almost 1, which it reaches and remains at.

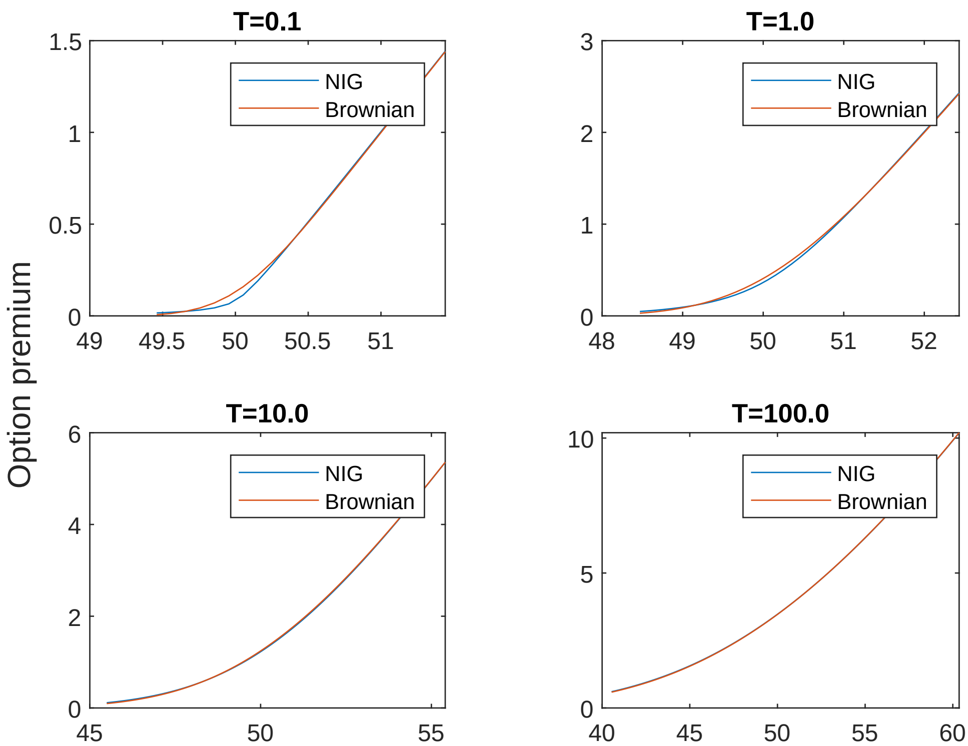

Figure 4: Up close of option premiums for NIG and Brownian models when the initial price is close to the strike price.

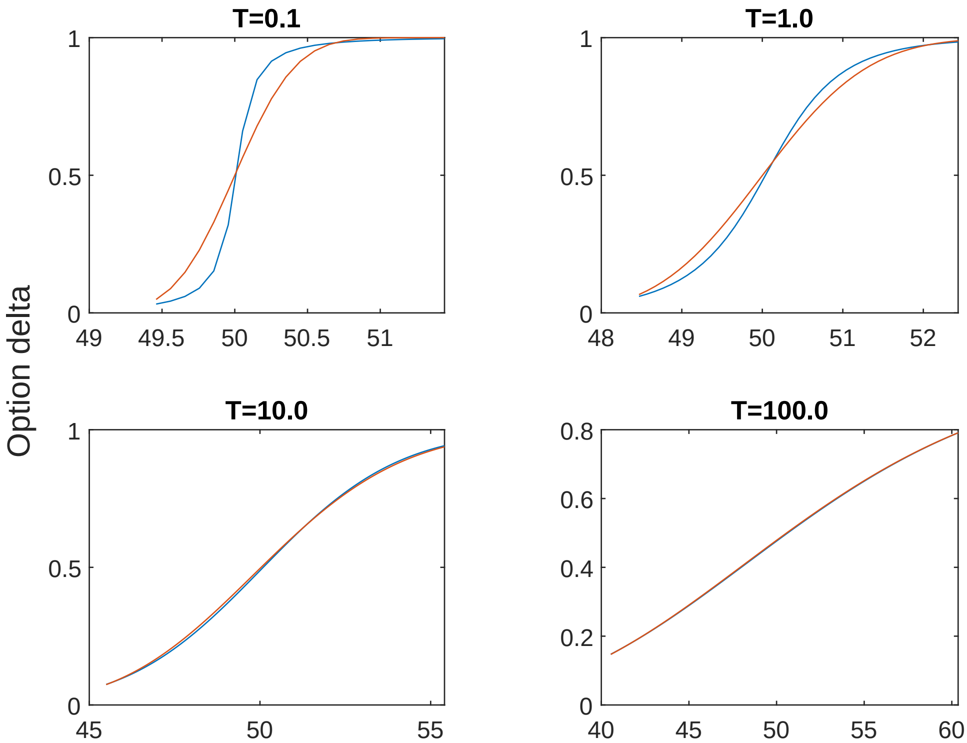

Figure 5: Up close of option deltas for NIG and Brownian models when the initial price is close to the strike price.

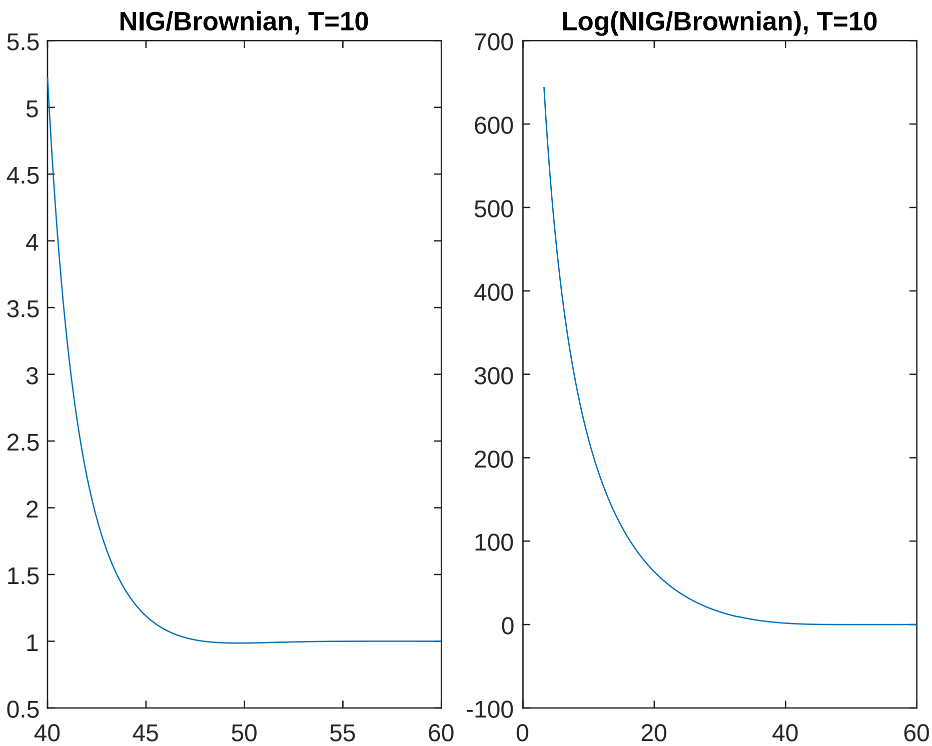

Figure 6: Comparing relative and log10 relative difference between NIG and Brownian option premium.

5 Conclusions and outlook

We have managed to find analytic expressions for the delta of option prices, for NIG and Brownian price processes, as well as for Brownian spread options. We also found analytic expressions for the gamma (second derivative with respect to initial price) for the Brownian cases (normal options and spreads). We found an exponential integrability condition for NIG processes, as well as a Martingale condition for the corresponding discounted price process. We did numerical implementation of the price and delta, and compared these for the Brownian and NIG models, using "reasonable" parameters for real markets, yielding that the difference, in both price and delta, was small except for options with very short strike times. Further study could look at more extreme option cases, to see if there is greater difference when using Brownian versus NIG models. Another extension could be to find analytic expressions for different greeks for options using NIG price processes.

6 Appendix A

Proposition 46. *For $\epsilon > 0$ and $\delta_1 > 0$, $\exists \delta_2 > 0$ such that if $t < \delta_2$ then $\int_{\mathbb{R}\setminus(-\delta_1,\delta_1)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx < \epsilon$.*

Proof. The proof follows a classic proof of the Gaussian integral using polar coordinates. Define

$$I(t) := \int_{\mathbb{R}\setminus(-\delta_1,\delta_1)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx,$$

then

$$\begin{aligned} I(t)^2 &= \int_{\mathbb{R}\setminus(-\delta_1,\delta_1)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \int_{\mathbb{R}\setminus(-\delta_1,\delta_1)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy, \\ &= \iint_{(\mathbb{R}\setminus(-\delta_1,\delta_1))^2} \frac{1}{2\pi t} e^{-\frac{x^2+y^2}{2t}} dxdy, \end{aligned}$$

since the integrand is positive we have

$$\leq \iint_{\mathbb{R}^2\setminus B_{\delta_1}(0)} \frac{1}{2\pi t} e^{-\frac{x^2+y^2}{2t}} dxdy,$$

where $B_{\delta_1}(0)$ is the open ball centered at 0 with radius δ_1. Then we make a change to polar coordinates

$$\begin{aligned} &= \int_0^{2\pi} \int_{\delta_1}^{\infty} \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} drd\theta, \\ &= \int_{\delta_1}^{\infty} \frac{1}{2\pi t} 2\pi e^{-\frac{r^2}{2t}} dr, \end{aligned}$$

we make the substitution $s = -r^2$

$$\begin{aligned} &= \frac{1}{t} \int_{-\infty}^{-\delta_1^2} e^{\frac{s}{2t}} \frac{ds}{2}, \\ &= \frac{1}{2t} [2te^{\frac{s}{2t}}]_{s=-\infty}^{-\delta_1^2}, \\ &= e^{-\frac{\delta_1^2}{2t}}, \end{aligned}$$

then

$$I(t) \leq \sqrt{e^{-\frac{\delta_1^2}{2t}}} = e^{-\frac{\delta_1^2}{4t}},$$

which goes to 0 as $t \to 0^+$. ∎

7 Appendix B

```
%Values for T and x-axis for plot
A = [0.1,1,10,100];
B = [0.008,0.04,0.1,0.35];

%Set up for multiple plots in one figure
tiledlayout('flow')

for i = 1:4

%Interest rate
r = 0;
%Time
T = A(i);
%Strike price
K = 50;

%NIG parameters
delta = 0.015;
mu = 0;
alpha = 150;
beta = 0;

%Gaussian paramateres equal NIG mean and variance
theta = mu+delta*beta/sqrt(alpha^2-beta^2);
sigma = alpha*sqrt(delta)/((alpha^2-beta^2)^(3/4));

%NIG parameters must be scaled by time
mu = mu*T;
delta = delta*T;

k = delta*alpha/pi*exp(delta*sqrt(alpha^2-beta^2));

%Gaussian density
f = @(y) 1./sqrt(2.*pi.*(sigma.^2).*T)...
.*exp(-(((y-theta.*T)./(sigma.*sqrt(T))).^2)./2);

%NIG density
g = @(y) k.*exp(beta.*(y-mu))...
.*besselk(1,alpha.*sqrt(delta.^2+(y-mu).^2))./sqrt(delta.^2+(y-mu).^2);

Y = linspace(-B(i),B(i),100);

%Advance panel in figure
nexttile
plot(Y,f(Y),Y,g(Y))
VarTitle = sprintf('T=%.1f',A(i));
title(VarTitle);
labels = {'G','N'};
```

```
legend(labels)

end
```

8 Appendix C

```
%Interest rate
r = 0;
%Strike time
T = 50;
%Strike price
K = 50;

%Set parameters based on Benth, daily
alpha = 42.3;
beta = 3.8;
mu = -0.0021;
delta = 0.018;

%Set the same variance and mean for the Brownian model
theta = mu+delta*beta/sqrt(alpha^2-beta^2);
sigma = alpha*sqrt(delta)/((alpha^2-beta^2)^(3/4));

%Set for Martingality
mu = r-delta*((alpha^2-beta^2)^(1/2)-(alpha^2-(1+beta)^2)^(1/2));
theta = r-(1/2)*sigma^2;

%NIG parameteres must be scaled with time
mu = mu*T;
delta = delta*T;

k = delta*alpha/pi*exp(delta*sqrt(alpha^2-beta^2));

%Density for delta and premium respectively, for Brownian
f1 = @(y,x) (y-log(x)-theta.*T)/(x.*(sigma.^2).*T).*max(exp(y)-K,0)...
./sqrt(2.*pi.*(sigma.^2).*T).*exp(-(((y-log(x)-theta.*T)...
./(sigma.*sqrt(T))).^2)./2);
f2 = @(y,x) max(exp(y)-K,0)./sqrt(2.*pi.*(sigma.^2).*T)...
.*exp(-(((y-log(x)-theta.*T)./(sigma.*sqrt(T))).^2)./2);

%Density for delta and premium respectively, for NIG
g1 = @(y,x) max(exp(y)-K,0).*k.*exp(beta.*(y-log(x)-mu))...
.*besselk(1,alpha.*sqrt(delta.^2+(y-log(x)-mu).^2))...
./sqrt(delta.^2+(y-log(x)-mu).^2)./x...
.*(-beta+alpha.*(y-log(x)-mu)./sqrt(delta.^2+(y-log(x)-mu).^2)...
.*besselk(2,alpha.*sqrt(delta.^2+(y-log(x)-mu).^2))...
./besselk(1,alpha.*sqrt(delta.^2+(y-log(x)-mu).^2)));
g2 = @(y,x) max(exp(y)-K,0).*k.*exp(beta.*(y-log(x)-mu))...
.*besselk(1,alpha.*sqrt(delta.^2+(y-log(x)-mu).^2))...
./sqrt(delta.^2+(y-log(x)-mu).^2);

maxPrice = 100;
X = linspace(1,maxPrice,maxPrice);
```

```
I1 = zeros(1,maxPrice);
I2 = zeros(1,maxPrice);
I3 = zeros(1,maxPrice);
I4 = zeros(1,maxPrice);
I5 = zeros(1,maxPrice-1);
I6 = zeros(1,maxPrice);
for i = 1:maxPrice
I1(i) = integral(@(y) g1(y,X(i)),-10,10);
I2(i) = integral(@(y) g2(y,X(i)),-10,10);
I3(i) = integral(@(y) f1(y,X(i)),-10,10);
I4(i) = integral(@(y) f2(y,X(i)),-10,10);

%Black-Scholes C
Lplus = (log(K)-log(X(i))-theta*T)/(sigma*sqrt(T));
Lminus = (log(K)-log(X(i))-theta*T)/(sigma*sqrt(T))-sigma*sqrt(T);
I6(i) = exp(log(X(i))+theta*T ...
+1/2*T*sigma^2)*normcdf(-Lminus)-K*normcdf(-Lplus);
end
for i = 1:maxPrice-1
%Black-Scholes C', simply finite difference, step length is 1
I5(i) = I6(i+1)-I6(i);
end
X2 = linspace(1,maxPrice,maxPrice-1);

%{
plot(X,I2,X,I4,X,I6)
VarTitle = sprintf('T=%.0f',T);
title(VarTitle);
labels = {'NIG','Brownian','B-S'};
legend(labels)
xlabel('Initial stock price')
ylabel('Option premium')
%}

%%{
plot(X,I1,X,I3,X2,I5)
VarTitle = sprintf('T=%.0f',T);
title(VarTitle);
labels = {'NIG','Brownian','B-S'};
legend(labels)
xlabel('Initial stock price')
ylabel('Delta')
%%}
```

9 Appendix D

```
%Plot log relative error/relative error
t = tiledlayout(1,2,'TileSpacing','Compact');
offset = 400;
Y = linspace(offset*1/N*100,100*0.6,N-offset+1-0.4*N);
rel1 = I2(offset:N-0.4*N);
rel2 = I4(offset:N-0.4*N);
nexttile
plot(Y,rel1./rel2)
title1 = sprintf('NIG/Brownian, T=%.f',T);
title(title1);

offset = 1;
Y = linspace(offset*1/N*100,100*0.6,N-offset+1-0.4*N);
rel1 = I2(offset:N-0.4*N);
rel2 = I4(offset:N-0.4*N);
nexttile
plot(Y,log(rel1./rel2))
title2 = sprintf('Log(NIG/Brownian), T=%.f',T);
title(title2);

xlabel(t, 'Initial stock price')
```

Bibliography

[App09] Applebaum, D. *Lévy processes and stochastic calculus.* Cambridge university press, 2009.

[Ben03] Benth, F. E. *Option theory with stochastic analysis: an introduction to mathematical finance.* Springer Science & Business Media, 2003.

[BN97] Barndorff-Nielsen, O. E. 'Normal inverse Gaussian distributions and stochastic volatility modelling'. In: *Scandinavian Journal of statistics* vol. 24, no. 1 (1997), pp. 1–13.

[TC03] Tankov, P. and Cont, R. *Financial modelling with jump processes.* CRC press, 2003.