Discrete Mathematics 347 (2024) 113951

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Frobenius-König theorem for classes of $(0, \pm 1)$-matrices

Richard A. Brualdi [a], Geir Dahl [b,*]

[a] Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA
[b] Department of Mathematics, University of Oslo, Norway

ARTICLE INFO

Article history:
Received 30 June 2023
Received in revised form 7 February 2024
Accepted 20 February 2024
Available online xxxx

Keywords:
Frobenius-König theorem
Combinatorial matrix theory
$(0, \pm 1)$-matrices

ABSTRACT

The Frobenius-König Theorem has a central role in combinatorial matrix theory; it characterizes when a $(0, 1)$-matrix X contains a permutation matrix P (meaning $P \le X$ entrywise). Our goal is to investigate similar questions for $(0, \pm 1)$-matrices, and a main result is a Frobenius-König Theorem for the class of $(0, \pm 1)$-matrices with all row and column sums being 1. Moreover, some related results are shown for alternating sign matrices.

1. Introduction

One of the most quoted and useful theorems in combinatorial matrix theory (CMT) [4] is known as the **Frobenius-König Theorem**:

Given an $n \times n$ $(0, 1)$-matrix X, there exists an $n \times n$ permutation matrix $P \le X$ (entrywise inequality) if and only if X does not have an $r \times s$ zero submatrix with $r + s = n + 1$.

This theorem asserts when an $n \times n$ $(0, 1)$-matrix contains within it a $(0, 1)$-matrix of very special type, namely a permutation matrix. Although there are other theorems of this type in CMT, none are as celebrated as this theorem. We refer to [11] for a historical account of the Frobenius-König Theorem and some related theorems. An original statement of the (main part of the) theorem is, that if all the $n!$ terms of the determinant of an $n \times n$ matrix X, as given in the Leibniz formula, are zero, then X has an $r \times s$ zero submatrix with $r + s = n + 1$.

Recently $(0, \pm 1)$-matrices have played an increasing role in CMT, for instance in the context of *alternating sign matrices* (ASMs). An ASM is an $n \times n$ $(0, \pm 1)$-matrix such that the ± 1's in each row and column alternate in sign, starting and ending with a 1. For some of the developments of ASMs, see the book [3] and e.g. [2,5–9,12,13] and the references contained therein.

The goal of this paper is to investigate similar questions as in the Frobenius-König Theorem, but for classes of $(0, \pm 1)$-matrices. A main contribution is a Frobenius-König Theorem for the class of $(0, \pm 1)$-matrices with all row and column sums being 1. Moreover, some related results are shown for ASMs.

Since permutation matrices are the ASMs without any -1's, it seems natural in the context of the Frobenius-König Theorem, to replace the set of $n \times n$ (0,1)-matrices with the set $\mathcal{X}_n$ of $n \times n$ $(0, \pm 1)$-matrices and to replace the set $\mathcal{P}_n$ of $n \times n$ permutation matrices with the set $\mathcal{A}_n$ of $n \times n$ ASMs resulting in the following

Basic Question: Given an $n \times n$ $(0, \pm 1)$-matrix, when does it contain an $n \times n$ ASM?

* Corresponding author.
E-mail addresses: brualdi@math.wisc.edu (R.A. Brualdi), geird@math.uio.no (G. Dahl).

https://doi.org/10.1016/j.disc.2024.113951

In this generality, there is probably no simple answer as we have in the Frobenius-König theorem; see the discussion and results in Section 4. But first one has to decide what 'contain' means in this context.

First we note that if A is an $n \times n$ ASM then, unlike permutation matrices, A can contain an $r \times s$ zero submatrix with $r+s=n+1$. For instance, the 3×3 ASM

$$A = \left[\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline 1 & -1 & 1 \\ \hline 0 & 1 & 0 \end{array}\right]$$

contains a 2×2 zero submatrix with $2+2=4$.

In [6] a partial order, called the *pattern-plus partial order* and denoted by $\preceq_{pp}$, was defined on the class $\mathcal{A}_n$ of $n \times n$ ASMs. We extend this partial order to the more inclusive set $\mathcal{X}_n$ of $n \times n$ $(0, \pm 1)$-matrices and adopt a more suggestive name and notation. Let A and X be $n \times n$ $(0, \pm 1)$-matrices. We define $A \trianglelefteq X$ (read: A is *subordinate* to X) to mean that A can be obtained from X by replacing some of the ± 1's of X with 0's (just as in the Frobenius-König Theorem where we replace some of the 1's with 0's). We observe that for a $(0, \pm 1)$-matrix X, $X = X(1) + X(-1)$ where $X(1)$ is the matrix obtained from X by replacing its -1's with 0's and $X(-1)$ is obtained from X by replacing its 1's with 0's.

The notion of maximality of an ASM in the partial order $\trianglelefteq$ was considered in [5] using the notion of an ASM extension and the following theorem (restated in our language here) was proved.

Theorem 1.1. [5] *If an $n \times n$ $(0, \pm 1)$-matrix is a maximal ASM in the $\trianglelefteq$-partial order, then it is a permutation matrix.*

Not every permutation matrix is maximal. The permutation matrix

$$\left[\begin{array}{c|c|c|c} & & 1 & \\ \hline 1 & & & \\ \hline & 1 & & \\ \hline & & & 1 \end{array}\right]$$

is maximal as is easily verified, while the permutation matrix below is not maximal as shown:

$$\left[\begin{array}{c|c|c|c} & 1 & & \\ \hline 1 & & & \\ \hline & & & 1 \\ \hline & & 1 & \end{array}\right] \xrightarrow{\trianglelefteq} \left[\begin{array}{c|c|c|c} & 1 & & \\ \hline 1 & -1 & 1 & \\ \hline & 1 & -1 & 1 \\ \hline & & 1 & \end{array}\right].$$

It is easily verified that the identity matrix I_n is maximal in the partial order $\trianglelefteq$. In our matrices we now usually write $+$ in place of 1's and $-$ in place of -1's; empty positions are assumed to contain 0's.

In [5,6] permutation matrices which are maximal in the $\trianglelefteq$-partial order are characterized as follows. Let $\sigma = i_1 i_2 \cdots i_n$ be a permutation of $\{1, 2, \ldots, n\}$. Then σ contains a 2143 pattern provided that it has a subsequence $j_1 j_2 j_3 j_4$ in the same relative order as 2143: otherwise, σ is 2143-*avoiding*. A permutation matrix is 2143-*avoiding* provided that the corresponding permutation is 2143-avoiding. In a similar way we define 3412-*avoiding* where we note that 3412 is the reverse of 2143.

Theorem 1.2. [5,6] *An $n \times n$ permutation matrix is a maximal ASM in the $\trianglelefteq$-partial order if and only if it is both* 2143*-avoiding and* 3412*-avoiding.*

There are other similar questions that one might consider in the case of $(0, \pm 1)$-matrices:

Question 1.3.

(I) Given an $n \times n$ $(0,1)$-matrix $X \in \mathcal{X}_n$, when does there exist an ASM $A \in \mathcal{A}_n$ obtained by replacing some of the 0's of X with -1's, that is, for which $A(1) = X$?

(II) Given an $n \times n$ $(0,-1)$-matrix $X \in \mathcal{X}_n$, when does there exist an ASM $A \in \mathcal{A}_n$ obtained by replacing some of the 0's of X with 1's, that is, for which $A(-1) = X$?

We do not pursue these questions here. Our motivation for this paper is to answer our basic question by placing some restrictions on the $(0, \pm 1)$-matrix X. Our goal would be to do this in such a way as to obtain the Frobenius-König Theorem as a special case. In this connection, we define an $n \times n$ $(0, \pm 1)$-matrix to be a *near-permutation matrix*, abbreviated NPM, provided every row and column sum equals 1. Thus a NPM need not have the alternating property of ASMs. The *NPM-class* $\mathcal{C}_n$ consists of all $n \times n$ NPMs. The class $\mathcal{C}_n$ is general enough so that we have the inclusions

$$\mathcal{P}_n \subseteq \mathcal{A}_n \subseteq \mathcal{C}_n \subseteq \mathcal{X}_n.$$

We first give an answer to our basic question when X is an arbitrary $n \times n$ $(0, \pm 1)$-matrix in $\mathcal{X}_n$ and the class $\mathcal{A}_n$ is replaced with the more general class $\mathcal{C}_n$. Note that while $\mathcal{A}_n$ for $n \geq 3$ is not invariant under row and column permutations, $\mathcal{C}_n$ is. We also allow X to be restricted in some precise way resulting in the existence of an ASM $A \trianglelefteq X$.

To illustrate some of these ideas developed in this introductory section, we now discuss some examples.

Example 1.4. Let

$$X = \begin{bmatrix} & + & + & \\ + & & - & + \\ & + & + & - \\ & + & & \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} & & + & \\ + & & - & + \\ & & + & \\ & + & & \end{bmatrix}$$

is an ASM and satisfies $A \trianglelefteq X$. In this example there is no permutation matrix P with $P \trianglelefteq X$; in fact, X has a 3×2 submatrix without any 1's (rows 1, 3 and 4, and columns 1 and 4). In contrast, let

$$X = \begin{bmatrix} + & + & + & + & + \\ + & - & + & + & + \\ + & - & + & + & + \\ + & + & + & - & + \\ + & + & + & + & + \end{bmatrix}.$$

Then

$$P = \begin{bmatrix} & + & & & \\ + & & & & \\ & & & & + \\ & & + & & \\ & & & + & \end{bmatrix}$$

is a permutation matrix and satisfies $P \trianglelefteq X$. □

Example 1.5. Let

$$X = \begin{bmatrix} & & + & & & \\ & & & + & & \\ & + & & & + & \\ + & & & & & + \\ & & & + & & \\ & & + & & & \end{bmatrix}.$$

Then the ASMs A equal to

$$\begin{bmatrix} & & + & & & \\ & & & + & & \\ & + & - & & + & \\ + & & & - & & + \\ & & & + & & \\ & & + & & & \end{bmatrix} \text{ and } \begin{bmatrix} & & + & & & \\ & & & + & & \\ & + & & - & + & \\ + & & - & & & + \\ & & & + & & \\ & & + & & & \end{bmatrix}$$

have $A(1) = X$ but different $A(-1)$'s. The two ASMs

$$\begin{bmatrix} & + & & & \\ & & + & & \\ + & & - & & + \\ & & + & & \\ & & & + & \end{bmatrix} \text{ and } \begin{bmatrix} & & + & & \\ + & & & & \\ & + & - & + & \\ & & & & + \\ & & + & & \end{bmatrix}$$

have the same $A(-1)$ but different $A(1)$'s. Obviously, $A(1)$ and $A(-1)$ together always determine A. □

The remaining part of this paper is organized as follows. In Section 2 we consider our class $\mathcal{C}_n$ of $(0, \pm 1)$-matrices with all line sums 1, and show our main result, a Frobenius-König Theorem for the class $\mathcal{C}_n$. Our proof is based on a construction involving circulations in a certain directed graph. The class $\mathcal{C}_n$ is studied further in Section 3 and we characterize the possible sizes of zero submatrices in this class. Section 4 concerns other classes of $(0, \pm 1)$-matrices and whether a matrix in such a class can contain an ASM.

Notation: For a matrix A we let $\sigma(A)$ denote the sum of its entries. If A is an $n \times n$ matrix and $I, J \subseteq \{1, 2, \ldots, n\}$, then $A[I, J]$ denotes the submatrix consisting of the entries with rows in I and columns in J.

2. The Frobenius-König theorem for class $\mathcal{C}_n$

In this section we answer our basic question for arbitrary $n \times n$ $(0, \pm 1)$-matrices X but by replacing the set $\mathcal{A}_n$ of ASMs with the larger, less-restricted set $\mathcal{C}_n$ of $n \times n$ NPMs. We use the following notation for an $n \times n$ $(0, \pm 1)$-matrix $X = [x_{ij}]$: $\sigma_+(X)$ (resp. $\sigma_-(X)$) is the number of 1's (resp. -1's) in X. The complement of a subset S is denoted by $\bar{S}$ (where the ground set is clear).

We first derive inequalities that are satisfied by an $n \times n$ $(0, \pm 1)$-matrix $X \in \mathcal{C}_n$. Let $I, J \subseteq \{1, 2, \ldots, n\}$ and consider a partition of X as indicated by

$$\left[\begin{array}{c|c} X[I, J] & X[I, \bar{J}] \\ \hline X[\bar{I}, J] & X[\bar{I}, \bar{J}] \end{array}\right]$$

after row and column permutations. Let $a \geq 0$ be the number of 1's in $X[I, J]$ and let $b \geq 0$ be the number of -1's in its complementary submatrix $X[\bar{I}, \bar{J}]$. We claim that

$$|I| + |J| - n \leq a + b. \tag{1}$$

Since the row sums corresponding to the rows with index in I equal 1, the submatrix $X[I, \bar{J}]$ must have at least $(|I| - a)$ 1's (this number may be negative). Since the column sums corresponding to the columns indexed by $\bar{J}$ equal 1, the number b of -1's in $X[\bar{I}, \bar{J}]$ is at least $(|I| - a) - |\bar{J}|$ so that

$$b \geq (|I| - a) - |\bar{J}|,$$

that is

$$|I| - |\bar{J}| \leq a + b,$$

equivalently,

$$|I| + |J| - n \leq a + b.$$

We now show that given an arbitrary $n \times n$ $(0, \pm 1)$-matrix X, (1) is equivalent to the existence of a matrix $A \in \mathcal{C}_n$ with $A \trianglelefteq X$.

Theorem 2.1. *Let $X = [x_{ij}] \in \mathcal{X}_n$ be an $n \times n$ $(0, \pm 1)$-matrix. Then there exists a matrix $A \in \mathcal{C}_n$ such $A \trianglelefteq X$ if and only if*

$$|I| + |J| - n \leq \sigma_+(X[I, J]) + \sigma_-(X[\bar{I}, \bar{J}]) \tag{2}$$

for all subsets $I, J \subseteq \{1, 2, \ldots, n\}$ with $|I| + |J| \geq n + 1$.

Proof. We transform the problem into a (network flow) circulation problem in a certain directed graph. Let $D = (V, E)$ be a directed graph with vertex set

$$V = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n, t\},$$

of $2n + 1$ vertices. The set E of directed edges consists of

(a) an edge $e_{ij} = (v_i, w_j)$ whenever $x_{ij} = \pm 1$ for all $1 \leq i, j \leq n$,
(b) an edge $f_i = (t, v_i)$ and an edge $g_i = (w_i, t)$ for all $1 \leq i \leq n$.

Note that there is no edge associated with the pair v_i, w_j when $x_{ij} = 0$. Moreover we define two functions $L, U : E \to \mathbb{Z}$ as follows:

- if $x_{ij} = 1$, then $L(e_{ij}) = 0$ and $U(e_{ij}) = 1$,
- if $x_{ij} = -1$, then $L(e_{ij}) = -1$ and $U(e_{ij}) = 0$ for $1 \le i, j \le n$,
- $L(f_i) = U(f_i) = L(g_i) = U(g_i) = 1$ for $1 \le i \le n$.

For a vertex v, let $\delta^-(v)$ denote the set of edges entering v (having head v), and let $\delta^+(v)$ denote the set of edges leaving v (having tail v). An (L,U)-*circulation* (in D) is a function $y : E \to \mathbb{R}$ satisfying

$$\begin{array}{ll} \sum_{e\in\delta^-(v)} y(e) = \sum_{e\in\delta^+(v)} y(e) & \text{for all } v \in V, \\ L(e) \le y(e) \le U(e) & \text{for all } e \in E. \end{array}$$

In particular, $y(f_i) = y(g_i) = 1$ for $1 \le i \le n$. Associated with such a function y is a matrix $A(y) = [a_{ij}]$ given by $a_{ij} = y(e_{ij})$ for every edge e_{ij} and $a_{ij} = 0$ otherwise. Then $A(y)$ has every row and column sum equal to 1 and, if y is integral, then $A(y)$ is a $(0, \pm 1)$-matrix satisfying $A(y) \trianglelefteq X$. Conversely, a $(0, \pm 1)$-matrix $A = [a_{ij}]$ with $A \trianglelefteq X$ determines an integral (L,U)-circulation y by defining $y(f_i) = y(g_i) = 1$ for $1 \le i \le n$ and $y(e_{ij}) = a_{ij}$ for every edge e_{ij}.

By a theorem by Hoffman (see [4,10]) an (L,U)-circulation y exists if and only if

$$\sum_{e\in\delta^-(S)} L(e) \le \sum_{e\in\delta^+(S)} U(e) \quad \text{for all } S \subseteq V, \tag{3}$$

where $\delta^-(S)$ (resp. $\delta^+(S)$) denotes the set of edges entering S (resp. leaving S). Moreover, as both L and U are integer-valued, there exists an integer-valued y (whenever (3) holds).

We now determine what the conditions (3) are in our context. Let $S \subset V$, and let $I = \{i : v_i \in S\}$ and $J = \{j : w_j \notin S\}$. For simplicity we write $i \notin I$ to mean that $i \in \{1, 2, \ldots, n\} \setminus I$, and similarly for $j \notin J$.

Case 1: $t \notin S$. Then $\delta^-(S)$ consists of the edges (t, v_i) for $i \in I$ and (v_i, w_j) for $i \notin I$, $w_j \notin J$. So

$$\sum_{e\in\delta^-(S)} L(e) = |I| + \sum_{i\notin I, j\notin J} L(e_{ij}) = |I| - \sigma_-(X(\bar{I}, \bar{J})).$$

Moreover, $\delta^+(S)$ consists of the edges (w_j, t) for $j \notin J$ and (v_i, w_j) for $i \in I$, $w_j \in J$. So

$$\sum_{e\in\delta^+(S)} U(e) = |\bar{J}| + \sum_{i\in I, j\in J} U(e_{ij}) = |\bar{J}| + \sigma_+(X(I, J)) = n - |J| + \sigma_+(X(I, J)).$$

Therefore (3) becomes

$$|I| + |J| \le n + \sigma_+(X(I, J)) + \sigma_-(X(\bar{I}, \bar{J})).$$

Case 2: $t \in S$. Then $\delta^-(S)$ consists of the edges (w_j, t) for $j \in J$ and (v_i, w_j) for $i \notin I$, $w_j \notin J$. So

$$\sum_{e\in\delta^-(S)} L(e) = |J| + \sum_{i\notin I, j\notin J} L(e_{ij}) = |J| - \sigma_-(X(\bar{I}, \bar{J})).$$

Moreover, $\delta^+(S)$ consists of the edges (t, v_i) for $i \notin I$ and (v_i, w_j) for $i \in I$, $w_j \in J$. So

$$\sum_{e\in\delta^+(S)} U(e) = |\bar{I}| + \sum_{i\in I, j\in J} U(e_{ij}) = |\bar{I}| + \sigma_+(X(I, J)) = n - |I| + \sigma_+(X(I, J)).$$

Therefore (3) becomes

$$|I| + |J| \le n + \sigma_+(X(I, J)) + \sigma_-(X(\bar{I}, \bar{J})).$$

Thus, in both cases we obtain the same inequalities (2), as desired. Finally, when $|I| + |J| \le n$ the inequality is redundant (as the right hand side is nonnegative). $\square$

Example 2.2. Let $n = 5$ and consider

$$X = \left[\begin{array}{c|c|c||c|c} & & & - & + \\ \hline & & & & + \\ \hline & & & + & \\ \hline\hline & - & + & + & - \\ \hline + & + & & + & - \end{array}\right], \quad A = \left[\begin{array}{c|c|c||c|c} & & & & + \\ \hline & & & & + \\ \hline & & & + & \\ \hline\hline & & + & & \\ \hline + & + & & & - \end{array}\right].$$

Then $A \in \mathcal{C}_n$ and $A \trianglelefteq X$. Let $I = J = \{1, 2, 3\}$. The inequality (2) becomes

$$1 = |I| + |J| - n \leq \sigma_+(X(I, J)) + \sigma_-(X(\bar{I}, \bar{J})) = 0 + 2 = 2.$$

Note that there does not exist a permutation matrix P with $P \trianglelefteq X$ as X has a 3×3 zero submatrix.

Next, consider the modification X' of X obtained by replacing the -1's in positions $(4, 5)$ and $(5, 5)$ by 0. Then $\sigma_-(X'(\bar{I}, \bar{J})) = 0$ and the corresponding inequality (2) is violated. Thus there does not exist a matrix $A' \in \mathcal{C}_n$ with $A' \trianglelefteq X'$. □

Corollary 2.3. *Let $X = [x_{ij}]$ be an $n \times n$ $(0, \pm 1)$-matrix. Assume that there exists an ASM $A \in \mathcal{A}_n$ such that $A \trianglelefteq X$. Then the inequalities (2) hold.*

Proof. This follows from Theorem 2.1 as $\mathcal{A}_n \subseteq \mathcal{C}_n$. □

However, the inequalities (2) are not sufficient for the existence of an ASM $A \in \mathcal{A}_n$ such that $A \trianglelefteq X$. This is illustrated in Example 2.2 where (2) holds, but there is no ASM A with $A \trianglelefteq X$: each -1 can be eliminated and, as mentioned in Example 2.2, there is no suitable permutation matrix.

Corollary 2.4. *Let $X = [x_{ij}] \in \mathcal{X}_n$ be an $n \times n$ $(0, \pm 1)$-matrix. Then one can check in polynomial time if there exists an $A \in \mathcal{C}_n$ such that $A \trianglelefteq X$ and, if so, find such a matrix A. In fact, this can be checked in "nearly linear time" as a max-flow problem.*

Proof. This follows from the construction in Theorem 2.1 as a general circulation problem with lower/upper bounds can be solved efficiently by network flow algorithms. In fact, let y be an (L, U)-circulation (as defined in the proof of Theorem 2.1). Define $x = y - L$, so $O \leq x \leq C$ where $x = (x(e) : e \in E)$, O is the zero vector and $C := U - L$. Then

$$\sum_{e \in \delta^-(v)} (L(e) + x(e)) = \sum_{e \in \delta^+(v)} (L(e) + x(e)) \text{ for all } v \in V,$$

i.e.,

$$\sum_{e \in \delta^+(v)} x(e) - \sum_{e \in \delta^-(v)} x(e) = b(v) \text{ for all } v \in V,$$

where $b(v) = \sum_{e \in \delta^-(v)} L(e) - \sum_{e \in \delta^+(v)} L(e)$ for each vertex v. Thus, one can determine if an (L, U)-circulation exists as a flow feasibility problem: determine if there is a flow x in the graph with divergence b and capacities $O \leq x \leq C$. This problem can be solved as a max-flow problem in a modified graph with two extra vertices (source and sink) and edges with capacities corresponding to C, one for each vertex. For details on this (standard) construction, see [1]. □

Next, we show how the Frobenius-König Theorem can be derived from Theorem 2.1 by letting X be a $(0, 1)$-matrix.

Theorem 2.5. (***Frobenius-König Theorem***) *Let X be an $n \times n$ $(0, 1)$-matrix. Then there exists an $n \times n$ permutation matrix $P \leq X$ if and only if X does not have an $r \times s$ zero submatrix O_{rs} with $r + s = n + 1$.*

Proof. We consider Theorem 2.1 in the special case when X is a $(0, 1)$-matrix. A $(0, \pm 1)$-matrix $A \in \mathcal{C}_n$ with $A \trianglelefteq X$ is necessarily a permutation matrix. Thus, we only need to show that (2) is equivalent to the Frobenius-König condition that every $r \times s$ zero submatrix O_{rs} of X satisfies $r + s \leq n$. Clearly (2) implies this condition since, when $X[I, J] = O_{rs}$, the right hand side in (2) is zero, so $|I| + |J| \leq n$.

Conversely, assume the Frobenius-König condition holds, and consider an inequality (2) associated with I and J. Choose an $r \times s$ zero submatrix $X[I', J']$ of $X[I, J]$ with $r + s$ maximal (we may have $r = s = 0$). The Frobenius-König condition then gives $r + s \leq n$, so

$$\sigma_+(X[I, J]) \geq |I \setminus I'| + |J \setminus J'| = |I| + |J| - (r + s) \geq |I| + |J| - n$$

where the first inequality holds because of the maximality of the zero submatrix $X[I', J']$. Thus, inequality (2) associated with I and J holds (as there are no negative entries). □

Finally, in this section, we comment on Hoffman's Theorem on circulations. Chapter 6 in [4] discusses several existence theorems for matrices and uses the Max-flow min-cut Theorem as a starting point for the proofs. This is possible by considering an entry in a matrix as a flow in an edge in a (directed) graph with a bipartition. In [4] Hoffman's Theorem is derived from the Max-flow min-cut Theorem, but the opposite derivation is also possible. Frobenius' original proof is quite complicated, and we refer to [11] for a discussion of that proof as well as a later proof by König.

3. The class $\mathcal{C}_n$ and zero submatrices

We consider again the NPM class $\mathcal{C}_n$ consisting of all $n \times n$ $(0, \pm 1)$-matrices with every row and column sum equal to 1. In view of the role of zero submatrices in the Frobenius-König Theorem, we determine how large (sum of the number of rows and the number of columns) a zero submatrix can be in a matrix in $\mathcal{C}_n$. Since $\mathcal{C}_n$ is invariant under row and column permutations, we may restrict our attention to the case when the zero submatrix is a leading submatrix. Therefore, consider a matrix $A \in \mathcal{C}_n$ of the form

$$A = \left[\begin{array}{c|c} O & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \tag{4}$$

where the leading zero submatrix has size $r \times s$ with $1 \le r, s < n$. We may assume that all rows of A_{21} and all columns of A_{12} are nonzero. We assume throughout this section that $n \ge 2$ (otherwise there is no zero submatrix).

An *interchange* applied to a $(0, \pm 1)$-matrix A is the addition of

$$\pm \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

to a 2×2 submatrix of A provided the result is also a $(0, \pm 1)$ matrix. The following lemma shows an essential property of $\mathcal{C}_n$.

Lemma 3.1. *Assume that $\mathcal{C}_n$ contains a matrix A as described above in (4). Then A can be transformed by interchanges into another matrix in $\mathcal{C}_n$ of the same form (4) where now both the corresponding submatrices A_{12} and A_{21} are $(0, 1)$-matrices having nonzero rows and nonzero columns, respectively, thus implying that each of the first r rows and the first s columns of A now contain exactly one 1. In particular, $r + s \ge n$.*

Proof. If $A = [a_{ij}]$ itself has the desired form, there is nothing to prove. Assume A_{21} contains a -1. Since each column sum is 1, there are i, k and j such that $a_{ij} = -1$ and $a_{kj} = 1$ where both the positions (i, j) and (k, j) lie in the submatrix A_{21}. Since rows i and k in A each have row sum 1, there must be a $l \ne j$ such that $a_{il} > a_{kl}$. Consider the submatrix A' of A defined by rows i, k and columns j, l (in this order). Then A' equals one of the following three matrices:

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}.$$

In each case we can perform an interchange resulting, respectively, in

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Clearly the updated matrix A lies in $\mathcal{C}_n$ and in each case the number of -1's is strictly reduced. So, we can repeat this procedure until there are no -1's in the submatrix A_{21}. If A_{21} now has a zero row we can include it with the O before we deal with A_{12} in a similar way. If we then get a zero column in A_{12}, we can include it with the O. This may then introduce a -1 in the new A_{21} and we could repeat. Thus eventually we obtain the situation in which A_{21} has no -1's and no zero rows (and a similar situation holds for A_{12}). Thus the number of 1's in A_{21} equals s. Since all column sums equal 1, A_{21} cannot contain two 1's in the same column. Hence A_{21} is an $(n-r) \times s$ $(0, 1)$-matrix with all column sums equal to 1. Thus $s \ge n - r$ and so $r + s \ge n$. □

Note the obvious fact that if a matrix in $\mathcal{C}_n$ contains an $r \times s$ zero submatrix, then it also contains an $r' \times s'$ submatrix for any $r' \le r$ and $s' \le s$.

Theorem 3.2. *Let r, s, n be integers with $1 \le r, s < n$. The class $\mathcal{C}_n$ contains a matrix A with an $r \times s$ zero submatrix as in (4) if and only if*

$$r + s - n \le (n - r)(n - s). \tag{5}$$

Proof. Assume first $\mathcal{C}_n$ contains a matrix A of the form (4) with the conditions given in Lemma 3.1 satisfied. Thus A_{21} contains exactly s 1's. Since each of the last $n - r$ rows sum to 1, there must be (at least) $s - (n - r)$ -1's in the $(n - r)(n - s)$ submatrix A_{22}. Hence $s - (n - r) \le (n - r) \times (n - s)$ and (5) holds; this also follows directly from (2).

Conversely, assume that (5) holds. We construct a matrix $A \in \mathcal{C}_n$ of the form (4). We take for A_{22} any $(0, -1)$-matrix with $k = \max\{0, r + s - n\}$ -1's. We then choose the 1's in A_{21} and A_{12} in staircase patterns in order to achieve that the last $n - r$ rows and last $n - s$ columns have sum 1. This is possible in A_{21} as the number of -1's in A_{22} is k and $r + s - n < s$, where s is the number of columns in A_{21}. Similarly, concerning A_{12}, we have $r + s - n < r$, so it is possible to complete this matrix as desired. □

Example 3.3. Let $n=8$ and consider the three cases (i) $r=5$, $s=4$, (ii) $r=6$, $s=5$, and (iii) $r=6$, $s=6$. The construction in the proof of Theorem 3.2 gives the matrices A_1, A_2 and A_3 in these cases, respectively.

$$A_1=\left[\begin{array}{cccc||cccc} & & & & + & & & \\ & & & & + & & & \\ & & & & & + & & \\ & & & & & & + & \\ & & & & & & & + \\ \hline + & + & & & - & & & \\ & & + & & & & & \\ & & & + & & & & \end{array}\right],\quad A_2=\left[\begin{array}{ccccc||ccc} & & & & & + & & \\ & & & & & + & & \\ & & & & & & + & \\ & & & & & & + & \\ & & & & & & & + \\ & & & & & & & + \\ \hline + & + & + & + & & - & - & - \\ & & & & + & & & \end{array}\right],$$

$$A_3=\left[\begin{array}{cccccc||cc} & & & & & & + & \\ & & & & & & + & \\ & & & & & & + & \\ & & & & & & & + \\ & & & & & & & + \\ & & & & & & & + \\ \hline + & + & + & & & & - & - \\ & & & + & + & + & - & - \end{array}\right]. \quad \square$$

Theorem 3.4. *The maximum ξ_n of $r+s$ such that $\mathcal{C}_n$ contains a matrix A with an $r\times s$ zero submatrix is*

$$\xi_n=\max\{2\lfloor n+1-\sqrt{n+1}\rfloor, 2\lfloor (1/2)\big(2n+3-\sqrt{4n+5}\big)\rfloor-1\}.$$

Proof. By Theorem 3.2, $\mathcal{C}_n$ contains a matrix A with an $r\times s$ zero submatrix as in (4) if and only if (5) holds. We now analyze this condition on (r,s). We call (r,s) a *maximizer* if (r,s) satisfies (5) and $r+s=\xi_n$. Clearly, a maximizer exists.

Claim: There exists a maximizer (r,s) with $r=s$ or $r=s+1$.

Proof of Claim: Let (r,s) be a maximizer. By symmetry in (5) we observe that also (s,r) is a maximizer. We may therefore assume $r\ge s$. If $r=s$, we are done, so assume $r\ge s+1$. Let $(r',s')=(r-1,s+1)$. Then

$$r'+s'=r+s=\xi_n$$

and

$$(n-r')(n-s')+n=(n-r)(n-s)+n+(r-s-1)\ge (n-r)(n-s)+n.$$

Therefore, (r',s') also satisfies (5) and must be a maximizer. If r' equals s' or $s'+1$ we are done, otherwise we can repeat this "shifting" procedure until the desired property holds.

Based on the claim, we are left with a one-variable problem, with two cases.

Case 1*:* $r=s$. Then (5) becomes $2r\le (n-r)^2+n$. By simple algebra this inequality holds if and only if $r\le n+1-\sqrt{n+1}$. This gives the bound

$$r\le \lfloor n+1-\sqrt{n+1}\rfloor$$

Case 2*:* $s=r-1$. Then (5) becomes $2r-1\le (n-r)(n-r+1)+n$. Again by some algebra we obtain the bound

$$r\le \lfloor (1/2)\big(2n+3-\sqrt{4n+5}\big)\rfloor.$$

Using the relationship between r and s in the two cases leads to the desired expression for ξ_n. $\square$

The following table shows the values of ξ_n up to $n=15$, based on Theorem 3.2.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ξ_n	2	4	5	7	8	10	12	13	15	17	18	20	22	24

(As pointed out by a referee, this agrees with the sequence $A123663$ in the Online Encyclopedia of Integer Sequences (OEIS)). For instance, when $n=8$, we get $\xi_8=12$ and a maximizer is shown in Example 3.3, where $r=s=6$. We also note that, for large n, ξ_n is close to $2n$. For instance, for $n=1000$, we have $\xi_{1000}=1938$, so the zero submatrix is very large.

4. Special case: X is a restricted $(0, \pm 1)$-matrix

In this final section we collect a number of additional facts concerning the existence of ASMs that are subordinate to a $(0, \pm 1)$-matrix. The Frobenius-König Theorem concerns the existence of a permutation matrix $P \leq X$ where X is an $n \times n$ $(0, 1)$-matrix, equivalently, an ASM $A \trianglelefteq X$ when X has no -1's. For an arbitrary $n \times n$ $(0, \pm 1)$-matrix $X \in \mathcal{X}_n$, let $\mathcal{A}_n(X)$ be the set of ASMs A with $A \trianglelefteq X$.

Next we obtain a Frobenius-König Theorem for ASMs in the special case of $(0, \pm 1)$-matrices X having a $(0, -1)$-submatrix (not necessarily of consecutive rows and consecutive columns) containing all the 0's and -1's of X.

Theorem 4.1. *Let $X = [x_{ij}]$ be an $n \times n$ $(0, \pm 1)$-matrix such that X has an $r \times s$ $(0, -1)$-submatrix and all other entries of X equal 1. Then there exists an ASM A with $A \trianglelefteq X$ if and only if $r + s \leq n$, in which case there is actually a permutation matrix $P \trianglelefteq X$. If $r + s = n$, then every such A is a permutation matrix.*

Proof. First assume that there exists an ASM A with $A \trianglelefteq X$. If $r + s > n$, then there are $n - r$ rows and $n - s$ columns of X, all of whose entries equal 1 and that contain all the 1's of X, where $(n - r) + (n - s) = 2n - (r + s) \leq n - 1$. Since each of the specified $(n - r)$ rows and $(n - s)$ columns can contain at most one 1 in the ASM $A \trianglelefteq X$, A has at most $(n - 1)$ 1's, a contradiction.

If $r + s \leq n$, then, since all the 0's of X are contained in an $r \times s$ submatrix with $r + s \leq n$, it follows from the Frobenius-König Theorem that there is a permutation matrix, thus ASM, $P \trianglelefteq X$. If $r + s = n$, the calculation above shows that A has n 1's, so it must be a permutation matrix. □

Lemma 4.2. *There does not exist an $n \times n$ ASM with an $r \times s$ submatrix E with $r + s \geq n + 1$ the sum of whose entries is nonpositive and whose complementary submatrix B has nonnegative sum.*

Proof. Suppose there is such an ASM A. After row and column permutations (for convenience only),

$$A = \left[\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right],$$

where the sum of the entries of E is nonpositive and the sum of the entries of B is nonnegative. Then, since $\sigma(B) \geq 0$ and $\sigma(E) \leq 0$, the sum of the entries of A satisfies

$$\sigma(A) \leq (\sigma(B) + \sigma(C)) + \sigma(D) \leq (n - r) + (n - s) = 2n - (r + s) \leq n - 1,$$

a contradiction, since the sum of the entries of an $n \times n$ ASM equals n. □

Theorem 4.3. *Let X be a $(0, \pm 1)$-matrix with an $r \times s$ $(0, -1)$-submatrix E with $r + s \geq n + 1$ and such that E contains all the -1's in X. Then there is no ASM A with $A \trianglelefteq X$.*

Proof. An ASM A with $A \trianglelefteq X$ cannot exist by Lemma 4.2. □

Note that as a special case this contains the easy part of the Frobenius-König Theorem. In the case of a (0,1)-matrix, the converse also holds. But the converse of Lemma 4.2 does not hold in general for ASMs.

Example 4.4. Let $n = 3$ and

$$X = \left[\begin{array}{c|c|c} - & + & - \\ \hline + & - & + \\ \hline - & + & - \end{array}\right].$$

Then

$$A = \left[\begin{array}{c|c|c} & + & \\ \hline + & - & + \\ \hline & + & \end{array}\right]$$

is an ASM and $A \trianglelefteq X$. However, X has an $r \times s$ $(0, -1)$ submatrix with $r + s > n$, namely the 2×2 submatrix of X corresponding to the first and last row and column, and $r + s = 4 > 3 = n$. Now, modify X by the replacing the entry in position $(2, 2)$ with a 0, so

$$X' = \left[\begin{array}{c|c|c} - & + & - \\ \hline + & & + \\ \hline - & + & - \end{array}\right].$$

Then there is no ASM A' with $A' \trianglelefteq X'$ because we would have to replace all the -1's by 0, and then by the Frobenius-König Theorem there cannot exist a permutation matrix P satisfying $P \leq X'$. $\square$

From this example we conclude that the possible existence of an ASM does not only depend on the properties of individual submatrices; the signs of the entries outside the submatrix play a role. We conclude with a question.

Question 4.5. Let X be an $n \times n$ (± 1)-matrix whose -1's form an $r \times s$ submatrix with $r + s \leq n$. What is the maximum number $\eta(X)$ of -1's in an ASM A with $A \trianglelefteq X$? If $r + s = n$, then as shown in Theorem 4.1, $\eta(X) = 0$. Another possibility to consider is the minimum of $\eta(X)$ taken over all $n \times n$ (± 1)-matrices whose -1's form an $r \times s$ submatrix with $r + s \leq n$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors thank two referees for their careful reading of the paper and several very useful comments that improved it.

References

[1] R.K. Ahuja, T.L. Magnanti, J.B. Orlin, Network Flows: Theory, Algorithms, and Applications, Prentice-Hall, Englewood Cliffs, New Jersey, 1993.
[2] A. Ayyer, R.E. Behrend, Extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order, J. Comb. Theory, Ser. A 165 (2019) 78–105.
[3] D. Bressoud, Proofs and confirmations: the story of the alternating sign matrix conjecture, in: MAA Spectrum, Mathematical Association of America, Cambridge University Press, Washington, DC, Cambridge, 1999.
[4] R.A. Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.
[5] R.A. Brualdi, K.P. Kiernan, S.A. Meyer, M. Schroeder, Patterns of alternating sign matrices, Linear Algebra Appl. 438 (2013) 3967–3990.
[6] R.A. Brualdi, G. Dahl, Alternating sign and sign-restricted matrices: representations and partial orders, Electron. J. Linear Algebra 37 (2021) 613–639.
[7] R.A. Brualdi, G. Dahl, Alternating sign matrices and hypermatrices and a generalization of Latin squares, Adv. Appl. Math. 95 (2018) 116–151.
[8] R.A. Brualdi, M.W. Schroeder, Alternating sign matrices and their Bruhat order, Discrete Math. 340 (2017) 1996–2019.
[9] D. Heuer, J. Striker, Partial permutations and alternating sign matrix polytope, SIAM J. Discrete Math. 36 (4) (2022) 2863–2888.
[10] A.J. Hoffman, Some Recent Applications of the Theory of Linear Inequalities to Extremal Combinatorial Analysis, Proc. Symp. Appl. Math., vol. 10, Combinatorial Analysis, Providence, 1960, pp. 113–127.
[11] H. Schneider, The concepts of irreducibility and full indecomposability of a matrix in the works of Frobenius, König and Markov, Linear Algebra Appl. 18 (2) (1977) 139–162.
[12] J. Striker, The alternating sign matrix polytope, Electron. J. Comb. 16 (2009) 41.
[13] D. Zeilberger, Proof of the alternating sign matrix conjecture, Electron. J. Comb. 3 (1976) 13.